Week 14 第二次作业答案.

《线性代数与几何》

7.29

(1) $\because$ A半正定 $\therefore \exists$ Q正交. $Q^TAQ=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$ $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

$$\therefore \det(A+I) = \det Q^T \cdot \det(A+I) \cdot \det Q$$

$$= \det(Q^TAQ + Q^TQ) = \det\left(\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} + I_n\right) = \det\begin{pmatrix}\lambda_1+1 & & \\ & \ddots & \\ & & \lambda_n+1\end{pmatrix}$$

$$= (\lambda_1+1)\cdots(\lambda_n+1) \geq 1$$

(2) $\det(A+I) = 1 \iff \lambda_1 = \cdots = \lambda_n \iff Q^TAQ = 0 \iff A = 0$

7.30. $\because$ A正定 $\therefore A^{-1}$存在且正定, $\det A > 0$

合同变换 $\begin{pmatrix}1 & -x^TA^{-1} \\ 0 & I_n\end{pmatrix}\begin{pmatrix}0 & x^T \\ x & A\end{pmatrix}\begin{pmatrix}1 & 0 \\ -A^{-1}x & I_n\end{pmatrix}$

$$= \begin{pmatrix}-x^TA^{-1}x & \\ & A\end{pmatrix}$$

$\therefore \begin{vmatrix}0 & x^T \\ x^T & A\end{vmatrix} = \begin{vmatrix}-x^TA^{-1}x & 0 \\ 0 & A\end{vmatrix} = \underbrace{|A|}_{>0}\ \underbrace{(-x^TA^{-1}x)}_{\leq 0} \leq 0$ 当且仅当 $x=0$ 取等

$\therefore$ 为负定二次型

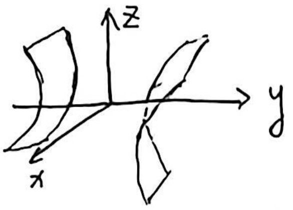
7.33 方程设为 $x^2+y^2+z^2+Dx+Ey+Fz+G=0$

代入四个点 $\begin{cases} G=0 \\ D+E+F+3=0 \\ 9+3E=0 \\ 1+4+1+D+2E-F=0 \end{cases} \Rightarrow \begin{cases} D=0 \\ E=-3 \\ F=0 \\ G=0 \end{cases}$

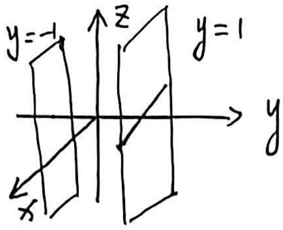
$\therefore$ 方程 $x^2+y^2+z^2-3y=0$

34.

(2) 双曲面

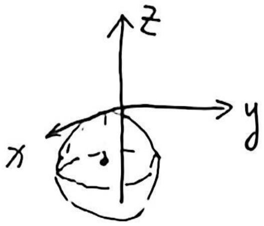


(4) 两平行平面



$y^2=1$.

(5) 球面 $(x-1)^2+y^2+(z+2)^2=2^2$



圆心 $(1,0,-2)$.
半径 2

35. (1) $\begin{cases} 2x^2+3y^2=1 \\ z=1 \end{cases}$ 在 $z=1$ 平面上一个椭圆

(4) $\begin{cases} x^2+y^2+z^2=16 \\ (x-1)^2+y^2+z^2=16 \end{cases}$ 在 $x=\frac{1}{2}$ 平面上一个圆

36. $\begin{cases} x^2+y^2+z^2=36 \\ x^2+y^2=2x \end{cases}$ 在 Oxz 上投影，分别令 $y=0$.

$\Rightarrow \begin{cases} x^2+z^2=36 \\ x^2=2x \end{cases}$ $\Rightarrow z^2=36-2x$ 且 $y^2=2x-x^2\geq 0$ $\therefore x\in[0,2]$

投影曲线 $\underset{x\in[0,2]}{z^2=36-2x}$ 为一段抛物线弧

38. (1) 曲线上 (x,y,z) 到轴距离为原先的 $|z|$ $\therefore$ 用 y^2+z^2 代替 z^2.

$x^2+\frac{y^2}{4}+\frac{z^2}{4}=1$ 椭球面

(2) (x,y,z) 到 y 轴距离与一开始相等. 即 $\sqrt{y}=\sqrt{x^2+z^2}$. 图略.

$\therefore y=x^2+z^2$ 椭圆抛物面（抛物旋转面）.

39. (1) $\frac{x^2}{9}+\frac{y^2}{9}+\frac{z^2}{16}=1$ 椭球面

图略.

(2) $\frac{x^2}{16}-\frac{y^2}{16}+\frac{z^2}{4}=1$ 单叶双曲面

(3) $\frac{z^2}{9}-\frac{4y^2}{9}-\frac{x^2}{9}=1$, 双叶双曲面

(4) $z=\frac{x^2}{18}-\frac{y^2}{8}$ 双曲抛物面.

(5) $x=y=z=0$, 原点.

(6) $y=\frac{3}{2}x^2+\frac{z^2}{2}$ 椭圆抛物面

(7) $\frac{(x-1)^2}{16}+\frac{9}{16}y^2-\frac{z^2}{4}=1$ 单叶双曲面

(8) $y^2=1 \Rightarrow y=\pm 1$ 两平行平面

40. (1) 二次项 $A=\begin{pmatrix}4&0&0\\0&-6&-2\\0&-2&-6\end{pmatrix} \xrightarrow{\text{正交对角}} A\begin{pmatrix}1&0&0\\0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\0&-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}=Q\begin{pmatrix}4&&\\&-4&\\&&-8\end{pmatrix}$

令 $\begin{pmatrix}x\\y\\z\end{pmatrix}=Q\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$, 原式化为 $4x_1^2-4y_1^2-8z_1^2-4x_1+4(\frac{1}{\sqrt2}y_1+\frac{1}{\sqrt2}z_1)+4(-\frac{1}{\sqrt2}y_1+\frac{1}{\sqrt2}z_1)-5=0$

即 $4(x_1-\frac12)^2-4y_1^2-8(z_1-\frac{\sqrt2}{4})^2=5$

平移得 $\frac45x_2^2-\frac45y_2^2-\frac85z_2^2=1$ 双叶双曲面

(2) 二次项 $A=\begin{pmatrix}1&0&0\\0&-\frac43&-\frac{8}{3\sqrt2}\\0&-\frac{8}{3}\sqrt2&\frac43\end{pmatrix}$ $A\begin{pmatrix}1&0&0\\0&\frac{1}{\sqrt3}&\frac{2\sqrt2}{3}\\0&-\frac{\sqrt2}{\sqrt3}&\frac13\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\frac{1}{\sqrt3}&\frac{2\sqrt2}{3}\\0&\frac{-\sqrt2}{\sqrt3}&\frac13\end{pmatrix}\begin{pmatrix}1&&\\&4&\\&&-4\end{pmatrix}$

令 $\begin{pmatrix}x\\y\\z\end{pmatrix}=Q\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$, 化为 $x_1^2+4y_1^2-4z_1^2+4x_1+\frac83\sqrt6(\frac{1}{\sqrt3}y_1+\frac{2\sqrt2}{3}z_1)+\frac83\sqrt3(-\frac{\sqrt2}{\sqrt3}y_1+\frac13z_1)-1=0$

$\therefore (x_1+2)^2+4y_1^2-4(z_1-\frac{5\sqrt3}{9})^2=\frac{105}{81}$ 平移之后, 单叶双曲面

(3) $A=\begin{pmatrix}4&2&0\\2&1&0\\0&0&-1\end{pmatrix} \xrightarrow{\text{正交对角}} AQ=\begin{pmatrix}\frac{2}{\sqrt5}&\frac{1}{\sqrt5}&0\\\frac{1}{\sqrt5}&-\frac{2}{\sqrt5}&0\\0&0&1\end{pmatrix}\begin{pmatrix}5&&\\&0&\\&&-1\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix}=Q\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$，代入 $5x_1^2-z_1^2-\frac{1}{\sqrt5}\left(\frac{2}{\sqrt5}x_1+\frac{1}{\sqrt5}y_1\right)+\frac{2}{\sqrt5}\left(\frac{1}{\sqrt5}x_1-\frac{2}{\sqrt5}y_1\right)+4z=0$

$5x_1^2-y_1-(z_1-2)^2=-4$ 平移 $y_2=5x_2^2-z_2^2$ 双曲抛物面

(4) $A=\begin{pmatrix}2&&\\&-\frac{3}{5}&-\frac{4}{5}\\&-\frac{4}{5}&\frac{3}{5}\end{pmatrix} \xrightarrow[\text{相似}]{\text{正交}} AQ=\begin{pmatrix}1&0&0\\0&\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\\0&\frac{-2}{\sqrt5}&\frac{1}{\sqrt5}\end{pmatrix}\begin{pmatrix}2&&\\&1&\\&&-1\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix}=Q\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$，代入 $2x_1^2+y_1^2-z_1^2+4x_1+2\sqrt5\left(\frac{1}{\sqrt5}y_1+\frac{2}{\sqrt5}z_1\right)-2\sqrt5\left(\frac{-2}{\sqrt5}y_1+\frac{1}{\sqrt5}z_1\right)+10=0$

$\therefore\ 2(x_1+1)^2+(y_1+3)^2-(z_1+1)^2=0.$

平移有 $z_2^2=2x_2^2+y_2^2$ 锥面