

Dec 01 作业

习题一：课本第163-164页习题5.5题1：求下列不定积分

$$(1) \quad \int \frac{1}{(x+1)(x+2)^2} dx; \quad (3) \quad \int \frac{x^3+1}{x^3-5x^2+6x} dx;$$

$$(5) \quad \int \frac{x^4}{x^4+5x^2+4} dx; \quad (7) \quad \int \frac{x^7}{(1-x^2)^5} dx.$$

解(1)：我们先化简被积分式函数，令

$$\frac{1}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2},$$

其中 A, B, C 为待定常数。用 $(x+1)(x+2)^2$ 乘以上述等式得

$$1 = A(x+2)^2 + B(x+1)(x+2) + C(x+1). \quad (1)$$

令 $x = -2$ 得 $C = -1$ 。将式 (1) 右边的项 $C(x+1) = -(x+1)$ 移至式 (1) 左边得

$$x+2 = A(x+2)^2 + B(x+1)(x+2). \quad (2)$$

约去共因子 $x+2$ 得

$$1 = A(x+2) + B(x+1). \quad (3)$$

令 $x = -2$ 得 $B = -1$ 。再令 $x = -1$ 得 $A = 1$ 。于是我们得

$$\int \frac{1}{(x+1)(x+2)^2} dx = \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{(x+2)^2}.$$

故所求不定积分为

$$\int \frac{1}{(x+1)(x+2)^2} dx = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} - \int \frac{dx}{(x+2)^2}$$

$$= \ln|x+1| - \ln|x+2| + \frac{1}{x+2} + C.$$

解(3): 注意被积分式函数是假分式. 我们先化简这个分式:

$$\frac{x^3 + 1}{x^3 - 5x^2 + 6x} = \frac{x^3 - 5x^2 + 6x + 5x^2 - 6x + 1}{x^3 - 5x^2 + 6x} = 1 + \frac{5x^2 - 6x + 1}{x^3 - 5x^2 + 6x}.$$

以下我们将分式 $\frac{5x^2 - 6x + 1}{x^3 - 5x^2 + 6x}$ 化为最简分式. 由于 $x^3 - 5x^2 + 6x = x(x^2 - 5x + 6) = x(x-2)(x-3)$, 故我们可令

$$\frac{5x^2 - 6x + 1}{x^3 - 5x^2 + 6x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3},$$

其中 其中 A, B, C 为待定常数. 用 $x(x-2)(x-3)$ 乘以上述等式得

$$5x^2 - 6x + 1 = A(x-2)(x-3) + Bx(x-3) + Cx(x-2). \quad (4)$$

在式 (4) 中令 $x = 0$ 得 $A = \frac{1}{6}$. 由式 (4) 得

$$\begin{aligned} 5x^2 - 6x + 1 - \frac{1}{6}(x-2)(x-3) &= 5x^2 - 6x + 1 - \frac{1}{6}(x^2 - 5x + 6) \\ &= \frac{29}{6}x^2 - \frac{31}{6}x = Bx(x-3) + Cx(x-2). \end{aligned}$$

约去因子 x 得

$$\frac{29}{6}x - \frac{31}{6} = B(x-3) + C(x-2).$$

令 $x = 3$ 得 $C = \frac{56}{6}$. 再令 $x = 2$ 得 $B = -\frac{9}{2}$. 于是

$$\frac{x^3 + 1}{x^3 - 5x^2 + 6x} = 1 + \frac{5x^2 - 6x + 1}{x^3 - 5x^2 + 6x} = 1 + \frac{1}{6x} - \frac{9}{2(x-2)} + \frac{28}{3(x-3)}.$$

故所求不定积分为

$$\begin{aligned} \int \frac{x^3 + 1}{x^3 - 5x^2 + 6x} dx &= \int x dx + \int \frac{dx}{6x} - \int \frac{9dx}{2(x-2)} + \int \frac{28dx}{3(x-3)} \\ &= x + \frac{1}{6} \ln|x| - \frac{9}{2} \ln|x-2| + \frac{28}{3} \ln|x-3| + C. \end{aligned}$$

解(5): 我们先将分解分式 $\frac{x^4}{x^4+5x^2+4}$:

$$\frac{x^4}{x^4+5x^2+4} = \frac{x^4+5x^2+4-5x^2-4}{x^4+5x^2+4} = 1 - \frac{5x^2+4}{x^4+5x^2+4},$$

由式 $x^4+5x^2+4=(x^2+1)(x^2+4)$, 易得

$$\frac{5x^2+4}{x^4+5x^2+4} = \frac{4x^2+4+x^2}{(x^2+1)(x^2+4)} = \frac{4}{x^2+4} + \frac{x^2}{(x^2+1)(x^2+4)}.$$

往下我们来分解分式 $\frac{y}{(y+1)(y+4)}$. 令

$$\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}.$$

以 $(y+1)(y+4)$ 乘以上式得 $y = A(y+4) + B(y+1)$. 由此得 $A = -\frac{1}{3}$, $B = \frac{4}{3}$. 于是

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{1}{3} \left(\frac{4}{x^2+4} - \frac{1}{x^2+1} \right).$$

综上被积分式被分解为

$$\begin{aligned} \frac{x^4}{x^4+5x^2+4} &= 1 - \frac{5x^2+4}{x^4+5x^2+4} = 1 - \frac{4}{x^2+4} - \frac{1}{3} \left(\frac{4}{x^2+4} - \frac{1}{x^2+1} \right) \\ &= 1 - \frac{1}{3(x^2+1)} - \frac{16}{3(x^2+4)} \end{aligned}$$

因此所求不定积分为

$$\int \frac{x^4}{x^4+5x^2+4} dx = x + \frac{1}{3} \arctan x - \frac{8}{3} \arctan \frac{x}{2} + C.$$

解(7): 由于

$$\int \frac{x^7}{(1-x^2)^5} dx = \frac{1}{2} \int \frac{x^6}{(1-x^2)^5} d(x^2) = \frac{1}{2} \int \frac{y^3 dy}{(1-y)^5},$$

我们只需计算上式最右边的积分. 令

$$\frac{y^3}{(1-y)^5} = \frac{A}{1-y} + \frac{B}{(1-y)^2} + \frac{C}{(1-y)^3} + \frac{D}{(1-y)^4} + \frac{E}{(1-y)^5}.$$

用 $(1-y)^5$ 乘以上式得

$$y^3 = A(1-y)^4 + B(1-y)^3 + C(1-y)^2 + D(1-y) + E. \quad (5)$$

令 $y=1$ 即得 $E=1$. 于是由式 (5) 得

$$y^3 - 1 = (y-1)(y^2 + y + 1) = A(1-y)^4 + B(1-y)^3 + C(1-y)^2 + D(1-y).$$

约去因子 $y-1$ 得

$$-(y^2 + y + 1) = A(1-y)^3 + B(1-y)^2 + C(1-y) + D.$$

令 $y=1$ 得 $D=-3$. 由此进一步得

$$\begin{aligned} -D - (y^2 + y + 1) &= 3 - (y^2 + y + 1) = 2 - y^2 - y \\ &= (1-y)(2+y) = A(1-y)^3 + B(1-y)^2 + C(1-y). \end{aligned}$$

约去因子 $1-y$ 得

$$2+y = A(1-y)^2 + B(1-y) + C.$$

令 $y=1$ 得 $C=3$. 由此得 $y-1 = A(1-y)^2 + B(1-y)$. 约去 $y-1$ 得 $1 = A(1-y) + B$. 故得 $B=-1$, $A=0$. 于是我们得到分式分解

$$\frac{y^3}{(1-y)^5} = -\frac{1}{(1-y)^2} + \frac{3}{(1-y)^3} - \frac{3}{(1-y)^4} + \frac{1}{(1-y)^5}.$$

于是

$$\begin{aligned} \int \frac{y^3 dy}{(1-y)^5} &= - \int \frac{dy}{(1-y)^2} + \int \frac{3dy}{(1-y)^3} - \int \frac{3dy}{(1-y)^4} + \int \frac{dy}{(1-y)^5} \\ &= \frac{1}{y-1} + \frac{3}{2(1-y)^2} - \frac{1}{(1-y)^3} + \frac{1}{4(1-y)^4} + C. \end{aligned}$$

故所求不定积分为

$$\int \frac{x^7}{(1-x^2)^5} dx = \frac{1}{2(x^2-1)} + \frac{3}{4(1-x^2)^2} - \frac{1}{2(1-x^2)^3} + \frac{1}{8(1-x^2)^4} + C.$$

习题二：课本第164页习题5.5题2：求下列不定积分

$$(1) \int \frac{\sin^4 x dx}{\cos^3 x}; \quad (3) \int \frac{\sin^2 x dx}{1 + \sin^2 x};$$

$$(5) \int \frac{1 - \tan x}{1 + \tan x} dx; \quad (7) \int \frac{\sin x}{\sin x + \cos x} dx; \quad (9) \int \frac{\cos x}{\sin x + \cos x} dx.$$

解(1)：

$$\begin{aligned} \int \frac{\sin^4 x dx}{\cos^3 x} &= - \int \frac{\sin^3 x}{\cos^3 x} d(\cos x) = \frac{1}{2} \int \sin^3 x d\left(\frac{1}{\cos^2 x}\right) = \frac{1}{2} \frac{\sin^3 x}{\cos^2 x} - \frac{1}{2} \int \frac{3 \sin^2 x \cos x dx}{\cos^2 x} \\ &= \frac{1}{2} \frac{\sin^3 x}{\cos^2 x} - \frac{3}{2} \int \frac{\sin^2 x dx}{\cos x} = \frac{1}{2} \frac{\sin^3 x}{\cos^2 x} - \frac{3}{2} \int \frac{1 - \cos^2 x dx}{\cos x} \\ &= \frac{1}{2} \frac{\sin^3 x}{\cos^2 x} - \frac{3}{2} \int \frac{dx}{\cos x} + \frac{3}{2} \int \cos x dx = \frac{1}{2} \frac{\sin^3 x}{\cos^2 x} + \frac{3}{2} \sin x - \frac{3}{2} \int \frac{dx}{\cos x}. \end{aligned}$$

回忆以前计算过如下积分：

$$\begin{aligned} \int \frac{dx}{\cos x} &= \int \frac{\cos x dx}{\cos^2 x} = \int \frac{d \sin x}{1 - \sin^2 x} \\ &= \frac{1}{2} \int \left(\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} \right) d \sin x = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C_1. \end{aligned}$$

于是

$$\int \frac{\sin^4 x dx}{\cos^3 x} = \frac{1}{2} \frac{\sin^3 x}{\cos^2 x} + \frac{3}{2} \sin x - \frac{3}{4} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C.$$

解(3)：

$$\int \frac{\sin^2 x dx}{1 + \sin^2 x} = \int \frac{1 + \sin^2 x - 1}{1 + \sin^2 x} dx = x - \int \frac{dx}{1 + \sin^2 x}.$$

以下计算上式右边的不定积分：

$$\begin{aligned} \int \frac{dx}{1 + \sin^2 x} &= \int \frac{dx}{2 \sin^2 x + \cos^2 x} = \int \frac{1}{2 \tan^2 x + 1} \cdot \frac{dx}{\cos^2 x} \\ &= \int \frac{d \tan x}{2 \tan^2 x + 1} = \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + C_1. \end{aligned}$$

于是所求不定积分为

$$\int \frac{\sin^2 x dx}{1 + \sin^2 x} = x - \int \frac{dx}{1 + \sin^2 x} = x - \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + C.$$

解(5):

$$\int \frac{1 - \tan x}{1 + \tan x} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx = \int \frac{d(\cos x + \sin x)}{\cos x + \sin x} dx = \ln |\cos x + \sin x| + C.$$

解(7)和(9): 记

$$I = \int \frac{\sin x dx}{\sin x + \cos x}, \quad J = \int \frac{\cos x dx}{\sin x + \cos x},$$

则

$$I + J = \int \frac{(\sin x + \cos x) dx}{\sin x + \cos x} = \int dx = x + C_1,$$

$$-I + J = \int \frac{(-\sin x + \cos x) dx}{\sin x + \cos x} = \int \frac{d(\sin x + \cos x)}{\sin x + \cos x} = \ln |\cos x + \sin x| + C_2.$$

根据上述两个等式可解得

$$I = \int \frac{\sin x dx}{\sin x + \cos x} = \frac{1}{2} \left(x - \ln |\cos x + \sin x| \right) + C_3$$

$$J = \int \frac{\cos x dx}{\sin x + \cos x} = \frac{1}{2} \left(x + \ln |\cos x + \sin x| \right) + C_4.$$

习题三: 课本第164页习题5.5题3: 求下列不定积分

$$(1) \quad \int \frac{dx}{\sqrt{x}(\sqrt{x} + \sqrt[3]{x})}; \quad (2) \quad \int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx;$$

$$(3) \quad \int x \sqrt{x+2} dx; \quad (5) \quad \int x \sqrt{x^4 + 2x^2 - 1} dx; \quad (9) \quad \int \frac{1}{(\sqrt{a^2 - x^2})^3} dx.$$

解(1): 对积分作变量代换 $x = t^6$ 得

$$\int \frac{dx}{\sqrt{x}(\sqrt{x} + \sqrt[3]{x})} = \int \frac{6t^5 dt}{t^3(t^3 + t^2)} = 6 \int \frac{dt}{1+t} = 6 \ln |1+t| + C = 6 \ln |1+x^{\frac{1}{6}}| + C.$$

解(2): 化简被积函数得

$$\frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} = \frac{(\sqrt{x+1} - \sqrt{x-1})^2}{x+1 - (x-1)} = x - \sqrt{x^2 - 1}.$$

于是

$$\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx = \int (x - \sqrt{x^2 - 1}) dx = \frac{1}{2}x^2 - \int \sqrt{x^2 - 1} dx.$$

回忆积分公式(证明大意: 先分部积分, 然后利用双曲函数作变量代换计算积分)

$$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln|x + \sqrt{x^2 \pm a^2}| + C.$$

故所求不定积分为

$$\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx = \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2 - 1} + \frac{1}{2}\ln|x + \sqrt{x^2 - 1}| + C.$$

另解:

$$\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx = \int \frac{\sqrt{\frac{x+1}{x-1}} - 1}{\sqrt{\frac{x+1}{x-1}} + 1} dx.$$

令

$$u^2 = \frac{x+1}{x-1},$$

反解得

$$x = \frac{u^2 + 1}{u^2 - 1} = 1 + \frac{2}{u^2 - 1}.$$

于是

$$dx = \frac{-4u}{(u^2 - 1)^2} du.$$

于是原不定积分化为

$$\int \frac{\sqrt{\frac{x+1}{x-1}} - 1}{\sqrt{\frac{x+1}{x-1}} + 1} dx = \int \frac{u-1}{u+1} \frac{-4u}{(u^2 - 1)^2} du = -4 \int \frac{u-1}{u+1} \frac{u}{(u^2 - 1)^2} du = -4 \int \frac{udu}{(u-1)(u+1)^3}.$$

令

$$\frac{u}{(u-1)(u+1)^3} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{C}{(u+1)^2} + \frac{D}{(u+1)^3},$$

其中 A, B, C, D 为待定常数. 上述分解式等价于

$$u = A(u+1)^3 + B(u-1)(u+1)^2 + C(u-1)(u+1) + D(u-1).$$

代入 u 的一些特殊值不难求得 $A = \frac{1}{8}$, $D = -\frac{1}{4}$, $C = \frac{1}{4}$, $B = \frac{1}{8}$. 于是

$$\frac{u}{(u-1)(u+1)^3} = \frac{1}{8(u-1)} + \frac{1}{8(u+1)} + \frac{1}{4(u+1)^2} - \frac{1}{4(u+1)^3}.$$

由此得

$$\begin{aligned} \int \frac{\sqrt{\frac{x+1}{x-1}} - 1}{\sqrt{\frac{x+1}{x-1}} + 1} dx &= -4 \int \frac{udu}{(u-1)(u+1)^3} \\ &= - \int \frac{du}{2(u-1)} - \int \frac{du}{2(u+1)} - \int \frac{du}{(u+1)^2} + \int \frac{du}{(u+1)^3}. \\ &= -\frac{1}{2} \ln|u-1| - \frac{1}{2} \ln|u+1| + \frac{1}{u+1} - \frac{1}{2(u+1)^2} + C. \end{aligned}$$

再将 $u = \sqrt{\frac{x+1}{x-1}}$ 代入即得所求积分.

解(3): 对积分 $\int x\sqrt{x+2}dx$ 作变量代换 $x+2=u^2$ 得

$$\begin{aligned} \int x\sqrt{x+2}dx &= \int (u^2-2)u \cdot 2udu = \int (2u^4-4u^2)du \\ &= \frac{2}{5}u^5 - \frac{4}{3}u^3 + C = \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C. \end{aligned}$$

解(5): 记 $y = x^2$, $u = y+1$, 则

$$\begin{aligned} \int x\sqrt{x^4+2x^2-1}dx &= \frac{1}{2} \int \sqrt{x^4+2x^2-1}d(x^2) = \frac{1}{2} \int \sqrt{x^4+2x^2-1}du \\ &= \frac{1}{2} \int \sqrt{y^2+2y-1}dy = \frac{1}{2} \int \sqrt{u^2-2}du. \end{aligned}$$

回忆积分公式

$$\int \sqrt{x^2 - a^2} dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2 \ln|x + \sqrt{x^2 - a^2}| + C.$$

故

$$\int \sqrt{u^2 - 2} du = \frac{1}{2}u\sqrt{u^2 - 2} - \ln|u + \sqrt{u^2 - 2}| + C$$

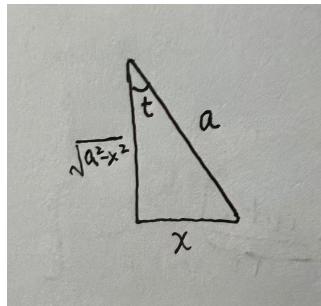
于是所求不定积分为

$$\begin{aligned}\int x\sqrt{x^4 + 2x^2 - 1} dx &= \frac{1}{2} \int \sqrt{u^2 - 2} du = \frac{1}{4}u\sqrt{u^2 - 2} - \frac{1}{2}\ln|u + \sqrt{u^2 - 2}| + C \\ &= \frac{1}{4}(x^2 + 1)\sqrt{x^4 + 2x^2 - 1} - \frac{1}{2}\ln|x^2 + 1 + \sqrt{x^4 + 2x^2 - 1}| + C.\end{aligned}$$

解(9): 对积分 $\int \frac{1}{\sqrt{(a^2-x^2)^3}} dx$ 作变量代换 $x = a \sin t$, $|t| < \frac{\pi}{2}$, 则

$$\begin{aligned}\int \frac{dx}{\sqrt{(a^2-x^2)^3}} &= \int \frac{a \cos t dt}{(a^2 \cos^2 t)^{\frac{3}{2}}} = \frac{1}{a^2} \int \frac{dt}{\cos^2 t} = \frac{1}{a^2} \int d \tan t \\ &= \frac{1}{a^2} \tan t + C = \frac{1}{a^2} \frac{x}{\sqrt{a^2 - x^2}} + C.\end{aligned}$$

由变换公式 $\sin t = \frac{x}{a}$ 可得 $\tan t = \frac{x}{\sqrt{a^2 - x^2}}$, 如图所示.



习题四: 课本第164页习题5.5题3: 求下列不定积分

$$(1) \quad \int \frac{\sqrt{1 + \cos x}}{\sin x}; \quad (3) \quad \int \frac{x}{1 - \cos x} dx; \quad (5) \quad \int \frac{1 - \ln x}{\ln^2 x} dx;$$

$$(7) \quad \int \frac{x^2 - 1}{x^4 + 1} dx; \quad (8) \quad \int \frac{x \ln x}{(x^2 + 1)^2} dx; \quad (9) \quad \int \frac{\arctan x}{x^2(1 + x^2)} dx.$$

解(1): 当 $|x| < \pi$ 时,

$$\begin{aligned} \int \frac{\sqrt{1 + \cos x}}{\sin x} dx &= \int \frac{\sqrt{2 \cos^2 \frac{x}{2}}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \frac{\sqrt{2}}{2} \int \frac{\cos \frac{x}{2} dx}{\sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \frac{\sqrt{2}}{2} \int \frac{dx}{\sin \frac{x}{2}} = 2\sqrt{2} \int \frac{d \frac{x}{4}}{2 \sin \frac{x}{4} \cos \frac{x}{4}} = \sqrt{2} \int \frac{d \tan \frac{x}{4}}{\tan \frac{x}{4}} \\ &= \sqrt{2} \ln \left| \tan \frac{x}{4} \right| + C. \end{aligned}$$

解(3):

$$\begin{aligned} \int \frac{x}{1 - \cos x} dx &= \int \frac{x dx}{2 \sin^2 \frac{x}{2}} = - \int x d \cot \left(\frac{x}{2} \right) = -x \cot \left(\frac{x}{2} \right) + \int \cot \left(\frac{x}{2} \right) dx \\ &= -x \cot \left(\frac{x}{2} \right) + \int \frac{\cos \left(\frac{x}{2} \right)}{\sin \left(\frac{x}{2} \right)} dx = -x \cot \left(\frac{x}{2} \right) + 2 \ln \left| \sin \left(\frac{x}{2} \right) \right| + C. \end{aligned}$$

解(5):

$$\int \frac{1 - \ln x}{\ln^2 x} dx = \int \left(-\frac{x}{\ln x} \right)' dx = -\frac{x}{\ln x} + C.$$

题(7)解法一: 记 $y = x + \frac{1}{x}$, 则

$$\begin{aligned} \int \frac{x^2 - 1}{x^4 + 1} dx &= \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx = \int \frac{d(x + \frac{1}{x})}{(x + \frac{1}{x})^2 - 2} \\ &= \int \frac{dy}{y^2 - 2} = \frac{1}{2\sqrt{2}} \left(\frac{1}{y - \sqrt{2}} - \frac{1}{y + \sqrt{2}} \right) dy = \frac{1}{2\sqrt{2}} \ln \frac{y - \sqrt{2}}{y + \sqrt{2}} + C \\ &= \frac{1}{2\sqrt{2}} \ln \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} + C = \frac{1}{2\sqrt{2}} \ln \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} + C. \end{aligned}$$

题(7)解法二: 由于 $x^4 + 1 = (x^2 + 1 + \sqrt{2}x)(x^2 + 1 - \sqrt{2}x)$, 故可分解分式

$$\frac{x^2 - 1}{x^4 + 1} = \frac{Ax + B}{x^2 + 1 - \sqrt{2}x} + \frac{Cx + D}{x^2 + 1 + \sqrt{2}x},$$

或等价地写作

$$x^2 - 1 = (Ax + B)(x^2 + 1 + \sqrt{2}x) + (Cx + D)(x^2 + 1 - \sqrt{2}x), \quad (6)$$

其中 A, B, C, D 为待定常数. 比较等式 (6) 两边关于幂 $x^3, x^2, x, 1$ 的系数得

$$\begin{aligned} A + C &= 0 \\ B + D + \sqrt{2}A - \sqrt{2}C &= 1 \\ A + C + \sqrt{2}B - \sqrt{2}D &= 1 \\ B + D &= -1 \end{aligned}$$

解上述线性代数方程组得 $B = D = \frac{-1}{2}$, $A = \frac{1}{\sqrt{2}}$, $C = \frac{-1}{\sqrt{2}}$. 于是我们的分式分解

$$\frac{x^2 - 1}{x^4 + 1} = \frac{1}{2\sqrt{2}} \left(\frac{2x - \sqrt{2}}{x^2 + 1 - \sqrt{2}x} - \frac{2x + \sqrt{2}}{x^2 + 1 + \sqrt{2}x} \right).$$

由此得所求不定积分为

$$\begin{aligned} \int \frac{x^2 - 1}{x^4 + 1} dx &= \frac{1}{2\sqrt{2}} \int \left(\frac{2x - \sqrt{2}}{x^2 + 1 - \sqrt{2}x} - \frac{2x + \sqrt{2}}{x^2 + 1 + \sqrt{2}x} \right) \\ &= \frac{1}{2\sqrt{2}} \int \frac{d(x^2 + 1 - \sqrt{2}x)}{x^2 + 1 - \sqrt{2}x} - \int \frac{d(x^2 + 1 + \sqrt{2}x)}{x^2 + 1 + \sqrt{2}x} = \frac{1}{2\sqrt{2}} \ln \frac{x^2 + 1 - \sqrt{2}x}{x^2 + 1 + \sqrt{2}x} + C. \end{aligned}$$

题(8):

$$\begin{aligned} \int \frac{x \ln x}{(x^2 + 1)^2} dx &= \frac{1}{2} \int \frac{\ln x}{(x^2 + 1)^2} d(x^2) = -\frac{1}{2} \int \ln x d\left(\frac{1}{x^2 + 1}\right) \\ &= -\frac{1}{2} \frac{\ln x}{x^2 + 1} + \frac{1}{2} \int \frac{dx}{(x^2 + 1)x}. \end{aligned}$$

令

$$\frac{1}{(x^2 + 1)x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \quad \text{或} \quad 1 = A(x^2 + 1) + (Bx + C)x,$$

其中 A, B, C 待定系数. 不难确定 $A = 1, B = -1, C = 0$. 于是

$$\int \frac{dx}{(x^2+1)x} = \int \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx = \ln|x| - \frac{1}{2} \ln(1+x^2) + C_1. \quad (7)$$

因此所求不定积分为

$$\int \frac{x \ln x}{(x^2+1)^2} dx = -\frac{1}{2} \frac{\ln x}{x^2+1} + \frac{1}{2} \int \frac{dx}{(x^2+1)x} = -\frac{1}{2} \frac{\ln x}{x^2+1} + \frac{1}{2} \ln|x| - \frac{1}{4} \ln(1+x^2) + C.$$

题(9): 注意

$$\frac{1}{x^2(1+x^2)} = \frac{1}{x^2} - \frac{1}{1+x^2},$$

故

$$\begin{aligned} \int \frac{\arctan x}{x^2(1+x^2)} dx &= \int \frac{\arctan x dx}{x^2} - \int \frac{\arctan x dx}{1+x^2} \\ &= - \int \arctan x d\frac{1}{x} - \int \arctan x d\arctan x \\ &= -\frac{1}{2}(\arctan x)^2 - \frac{\arctan x}{x} + \int \frac{dx}{x(x^2+1)}. \end{aligned}$$

根据题(8)解答中的结果 (7) 知

$$\int \frac{dx}{x(x^2+1)} = \ln|x| - \frac{1}{2} \ln(1+x^2) + C.$$

于是所求不定积分为

$$\int \frac{\arctan x}{x^2(1+x^2)} dx = -\frac{1}{2}(\arctan x)^2 - \frac{\arctan x}{x} + \ln|x| - \frac{1}{2} \ln(1+x^2) + C.$$

解答完毕.

Dec 03 作业

习题一. 利用 Stirling 公式求极限

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{n^2} \frac{n!}{n^n \sqrt{n}}.$$

解：根据 Stirling 公式 $n! = \sqrt{2n\pi} \left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{4n}}$, $\theta_n \in (0, 1)$, 可得

$$\left(1 + \frac{1}{n}\right)^{n^2} \frac{n!}{n^n \sqrt{n}} = \left(1 + \frac{1}{n}\right)^{n^2} \sqrt{2n\pi} \left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{4n}} \frac{1}{n^n \sqrt{n}} = \sqrt{2\pi} e^{\frac{\theta_n}{4n}} \left[\frac{\left(1 + \frac{1}{n}\right)^n}{e} \right]^n.$$

以下考虑极限

$$\lim_{n \rightarrow +\infty} \left[\frac{\left(1 + \frac{1}{n}\right)^n}{e} \right]^n.$$

令

$$a_n = \left[\frac{\left(1 + \frac{1}{n}\right)^n}{e} \right]^n,$$

则

$$\ln a_n = n \left[n \ln \left(1 + \frac{1}{n}\right) - 1 \right].$$

由 Taylor 展式 $\ln(1 + x) = x - \frac{1}{2}x^2 + o(x^2)$ 得

$$\ln \left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right).$$

于是

$$\ln a_n = n \left[n \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) - 1 \right] = n \left(-\frac{1}{2n} + o\left(\frac{1}{n}\right) \right) = -\frac{1}{2} + o(1).$$

由此可见 $\ln a_n \rightarrow -\frac{1}{2}$. 故 $a_n \rightarrow \frac{1}{\sqrt{e}}$. 因此所求极限为

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{n^2} \frac{n!}{n^n \sqrt{n}} = \sqrt{\frac{2\pi}{e}}.$$

解答完毕.

习题二：课本第170页习题5.6题1：求下列定积分 (1),(2),(3),(4).

$$(1) \quad \int_0^{2\pi} |\sin x| dx, \quad (2) \quad \int_0^\pi \sqrt{1 - \sin 2x} dx,$$

$$(3) \quad \int_0^2 |(x-1)(x-2)| dx, \quad (4) \quad \int_1^{\sqrt{3}} \frac{dx}{x+x^3}.$$

解(1):

$$\int_0^{2\pi} |\sin x| dx = 4 \int_0^{\frac{\pi}{2}} \sin x dx = 4.$$

解(2):

$$\begin{aligned} \int_0^\pi \sqrt{1 - \sin 2x} dx &= \int_0^\pi \sqrt{\cos^2 x + \sin^2 x - 2 \cos x \sin x} dx = \int_0^\pi |\cos x + \sin x| dx \\ &= \sqrt{2} \int_0^\pi \left| \sin \left(x + \frac{\pi}{4} \right) \right| dx = \sqrt{2} \int_{\frac{\pi}{4}}^{\pi + \frac{\pi}{4}} |\sin u| du = \sqrt{2} \int_0^\pi \sin u du = 2\sqrt{2}. \end{aligned}$$

解(3):

$$\begin{aligned} \int_0^2 |(x-1)(x-2)| dx &= \int_0^1 (1-x)(2-x) dx + \int_1^2 (x-1)(2-x) dx \\ &= \int_0^1 (2-3x+x^2) dx + \int_1^2 (-2-x^2+3x) dx = 2 - \frac{3}{2} + \frac{1}{3} - 2 - \frac{1}{3}(2^3 - 1^3) + \frac{3}{2}(2^2 - 1^2) \\ &= \frac{1}{2} + \frac{1}{3} - 2 - \frac{7}{3} + \frac{9}{2} = 1. \end{aligned}$$

解(4): 注意被积函数 $\frac{1}{x+x^3}$ 可作如下分解

$$\frac{1}{x+x^3} = \frac{1}{x} - \frac{x}{1+x^2}.$$

于是

$$\begin{aligned} \int_1^{\sqrt{3}} \frac{dx}{x+x^3} &= \int_1^{\sqrt{3}} \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx = \int_1^{\sqrt{3}} \frac{dx}{x} - \int_1^{\sqrt{3}} \frac{x dx}{1+x^2} \\ &= \ln \sqrt{3} - \frac{1}{2} (\ln 4 - \ln 2) = \frac{1}{2} \ln 3 - \frac{1}{2} \ln 2 = \frac{1}{2} \ln(3/2). \end{aligned}$$

习题三: 课本第171页习题5.6题2: 求下列定积分 (1)(2)(3)(4).

$$(1) \quad \int_0^\pi (1-2x) \sin x dx, \quad (2) \quad \int_0^1 x^2 e^{-2x} dx,$$

$$(3) \quad \int_0^{\frac{\pi}{2}} x \cos^2 x dx, \quad (4) \quad \int_0^{\sqrt{3}} x \arctan x dx.$$

解(1):

$$\int_0^\pi (1-2x) \sin x dx = \int_0^\pi \sin x dx - 2 \int_0^\pi x \sin x dx = 2 - 2 \int_0^\pi x \sin x dx.$$

用分部积分计算 $\int_0^\pi x \sin x dx$ 得

$$\int_0^\pi x \sin x dx = -x \cos x \Big|_0^\pi + \int_0^\pi \cos x dx = \pi.$$

于是所求积分为

$$\int_0^\pi (1-2x) \sin x dx = 2 - 2\pi.$$

解(2): 使用两次分部积分方法得

$$\begin{aligned} \int_0^1 x^2 e^{-2x} dx &= -\frac{1}{2} \int_0^1 x^2 de^{-2x} = -\frac{1}{2} x^2 e^{-2x} \Big|_0^1 + \int_0^1 e^{-2x} x dx = -\frac{1}{2} e^{-2} - \frac{1}{2} \int_0^1 x de^{-2x} \\ &= -\frac{1}{2} e^{-2} - \frac{1}{2} x e^{-2x} \Big|_0^1 + \frac{1}{2} \int_0^1 e^{-2x} dx = -\frac{1}{2} e^{-2} - \frac{1}{2} e^{-2} - \frac{1}{4} (e^{-2} - 1) \\ &= -e^{-2} + \frac{1}{4} (1 - e^{-2}) = \frac{1}{4} - \frac{5}{4e^2}. \end{aligned}$$

解(3):

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \cos^2 x dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} x (1 + \cos 2x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} x dx + \frac{1}{2} \int_0^{\frac{\pi}{2}} x \cos 2x dx \\ &= \frac{1}{4} \left(\frac{\pi}{2}\right)^2 + \frac{1}{4} x \sin 2x \Big|_0^{\frac{\pi}{2}} - \frac{1}{4} \int_0^{\frac{\pi}{2}} \sin 2x dx = \frac{\pi^2}{16} - \frac{1}{4}. \end{aligned}$$

解(4):

$$\begin{aligned} \int_0^{\sqrt{3}} x \arctan x dx &= \frac{1}{2} \int_0^{\sqrt{3}} \arctan x d(x^2) = \frac{1}{2} x^2 \arctan x \Big|_0^{\sqrt{3}} - \frac{1}{2} \int_0^{\sqrt{3}} \frac{x^2}{1+x^2} dx \\ &= \frac{3}{2} \arctan \sqrt{3} - \frac{1}{2} \int_0^{\sqrt{3}} \left(1 - \frac{1}{1+x^2}\right) dx = \frac{3}{2} \cdot \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{1}{2} \int_0^{\sqrt{3}} \frac{dx}{1+x^2} \end{aligned}$$

$$= \frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{1}{2} \arctan \sqrt{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

习题四：课本第171页习题5.6题3：求下列定积分 (2)(5)(6)(9).

$$(2) \quad \int_0^\pi \sin^5 x dx, \quad (5) \quad \int_0^\pi \sin^2 x \cos^4 x dx,$$

$$(6) \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^4 2x dx, \quad (9) \quad \int_0^{2a} x \sqrt{a^2 - (x-a)^2} dx, \quad a > 0.$$

解(2):

$$\begin{aligned} \int_0^\pi \sin^5 x dx &= - \int_0^\pi \sin^4 x d(\cos x) = - \int_0^\pi (1 - \cos^2 x)^2 d(\cos x) \\ &= - \int_0^\pi (1 - 2\cos^2 x + \cos^4 x) d(\cos x) = - \cos x \Big|_0^\pi + \frac{2}{3} \cos^3 x \Big|_0^\pi - \frac{1}{5} \cos^5 x \Big|_0^\pi \\ &= -(-1 - 1) + \frac{2}{3}((-1)^3 - 1) - \frac{1}{5}((-1)^5 - 1) = \frac{16}{15}. \end{aligned}$$

解(5):

$$\begin{aligned} \int_0^\pi \sin^2 x \cos^4 x dx &= 2 \int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x dx = 2 \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^4 x dx \\ &= 2 \int_0^{\frac{\pi}{2}} \cos^4 x dx - 2 \int_0^{\frac{\pi}{2}} \cos^6 x dx = 2 \left(\frac{3!!}{4!!} - \frac{5!!}{6!!} \right) \frac{\pi}{2} = \frac{\pi}{16}. \end{aligned}$$

解(6):

$$\begin{aligned} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^4 2x dx &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^4 2x d(2x) = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 u du \\ &= \int_0^{\frac{\pi}{2}} \sin^4 u du = \frac{3!!}{4!!} \frac{\pi}{2} = \frac{3\pi}{16}. \end{aligned}$$

解(9):

$$\int_0^{2a} x \sqrt{a^2 - (x-a)^2} dx = \int_{-a}^a (u+a) \sqrt{a^2 - u^2} du = a \int_{-a}^a \sqrt{a^2 - u^2} du$$

$$\begin{aligned}
&= 2a \int_0^a \sqrt{a^2 - u^2} du = 2a \int_0^{\frac{\pi}{2}} a \sqrt{1 - \sin^2 t} (a \cos t) dt = 2a^3 \int_0^{\frac{\pi}{2}} \cos^2 t dt \\
&= 2a^3 \frac{1}{2!} \frac{\pi}{2} = \frac{\pi a^3}{2}.
\end{aligned}$$

习题五：课本第171页习题5.6题4：设 $f(x)$ 为实轴 \mathbb{R} 上的连续函数。证明

- (1) 若 $f(x)$ 为奇函数，则 $\int_0^x f(t) dt$ 为偶函数；
- (2) 若 $f(x)$ 为偶函数，则 $\int_0^x f(t) dt$ 为奇函数；
- (3) 奇函数所有的原函数均为偶函数；偶函数的原函数中只有一个奇函数。

证 (1). 若 $f(x)$ 为奇函数，则对任意 $x \in \mathbb{R}$,

$$\int_0^{-x} f(t) dt = \int_0^x f(-u) d(-u) = \int_0^x -f(u)(-1) du = \int_0^x f(u) du.$$

即函数 $\int_0^x f(t) dt$ 为偶函数。

证 (2). 若 $f(x)$ 为偶函数，则对任意 $x \in \mathbb{R}$,

$$\int_0^{-x} f(t) dt = \int_0^x f(-u) d(-u) = \int_0^x f(u)(-1) du = - \int_0^x f(u) du.$$

即函数 $\int_0^x f(t) dt$ 为奇函数。

证 (3). 由于 $f(x)$ 是 \mathbb{R} 上的连续函数，故 $F(x)$ 的任意一个原函数 $F(x)$ 均可表示为

$$F(x) = \int_0^x f(t) dt + C.$$

- (i) 当 $f(x)$ 为奇函数时，根据结论 (1) 知函数 $\int_0^x f(t) dt$ 为偶函数，故 $F(x)$ 是偶函数。
- (ii) 当 $f(x)$ 为偶函数时，根据结论 (2) 知函数 $\int_0^x f(t) dt$ 为奇函数。故只有当 $C = 0$ 时，原函数 $F(x) = \int_0^x f(t) dt + C$ 是奇函数。结论得证。证毕。

习题六：课本第171页习题5.6题5：设 $f(x)$ 是 $[-1, 1]$ 上的连续函数。证明

$$\int_0^\pi t f(\sin t) dt = \frac{\pi}{2} \int_0^\pi f(\sin t) dt.$$

证明：对积分 $\int_0^\pi t f(\sin t) dt$ 作变量代换 $t = \pi - u$, 则

$$\begin{aligned} \int_0^\pi t f(\sin t) dt &= \int_\pi^0 (\pi - u) f(\sin(\pi - u)) d(\pi - u) = \int_0^\pi (\pi - u) f(\sin u) du \\ &= \pi \int_0^\pi f(\sin u) du - \int_0^\pi u f(\sin u) du. \end{aligned}$$

由于 $\int_0^\pi t f(\sin t) dt = \int_0^\pi u f(\sin u) du$, 故

$$\int_0^\pi t f(\sin t) dt = \frac{\pi}{2} \int_0^\pi f(\sin t) dt.$$

证毕.

习题七：课本第171页习题5.6题7：设 $f(x)$ 在实轴 \mathbb{R} 上连续. 证明

$$\int_0^x (x-t) f(t) dt = \int_0^x \left(\int_0^t f(s) ds \right) dt. \quad (*)$$

证明：记

$$F(x) \stackrel{\text{def}}{=} \int_0^x (x-t) f(t) dt, \quad G(x) \stackrel{\text{def}}{=} \int_0^x \left(\int_0^t f(s) ds \right) dt.$$

显然 $F(0) = 0$, $G(0) = 0$, 且 $F(x)$ 和 $G(x)$ 均连续可导, 并且

$$\begin{aligned} F'(x) &= \left(\int_0^x (x-t) f(t) dt \right)' = \left(x \int_0^x f(t) dt \right)' - \left(\int_0^x t f(t) dt \right)' \\ &= \int_0^x f(t) dt + x f(x) - x f(x) = \int_0^x f(t) dt, \\ G'(x) &= \left(\int_0^x \left(\int_0^t f(s) ds \right) dt \right)' = \int_0^x f(s) ds, \end{aligned}$$

即 $F'(x) = G'(x)$, $\forall x \in \mathbb{R}$. 因此 $F(x) = G(x)$. 命题得证.

习题八：课本第171页习题5.6题8：计算积分 $\int_0^1 x f(x) dx$, 其中 $f(x) = \int_1^{x^2} e^{-t^2} dt$.

解：由分部积分得

$$\int_0^1 x f(x) dx = \frac{1}{2} \int_0^1 f(x) d(x^2) = \frac{1}{2} x^2 f(x) \Big|_0^1 - \frac{1}{2} \int_0^1 x^2 f'(x) dx.$$

由定义 $f(x) = \int_1^{x^2} e^{-t^2} dt$, 知 $f(1) = 0$, 并且 $f'(x) = 2xe^{-x^4}$. 于是

$$\int_0^1 xf(x)dx = -\frac{1}{2} \int_0^1 x^2 \cdot 2xe^{-x^4} dx = -\int_0^1 x^3 e^{-x^4} dx = \frac{-1}{4} \left(1 - \frac{1}{e}\right) = \frac{1-e}{4e}.$$

解答完毕.

习题九: 课本第172页习题5.6题9(有修改): 设

$$I_n(x) = \int_0^x \frac{t^n dt}{\sqrt{t^2 + a^2}}, \quad a > 0. \quad (*)$$

- (1) 证明递推关系式 $nI_n(x) = x^{n-1}\sqrt{x^2 + a^2} - a^2(n-1)I_{n-2}(x)$, $\forall n \geq 2$;
 (2) 当 $a = 1$ 时, 求 $I_k(1)$, $k = 0, 1, 2, 3$.

解 (1): 设 $n \geq 2$ 为正整数. 对积分 (*) 作分部积分得

$$\begin{aligned} I_n &= \int_0^x \frac{t^n dt}{\sqrt{t^2 + a^2}} = \frac{1}{2} \int_0^x \frac{t^{n-1} d(t^2)}{\sqrt{t^2 + a^2}} = \int_0^x t^{n-1} d\sqrt{t^2 + a^2} \\ &= t^{n-1} \sqrt{t^2 + a^2} \Big|_0^x - (n-1) \int_0^x t^{n-2} \sqrt{t^2 + a^2} dt = x^{n-1} \sqrt{x^2 + a^2} - (n-1) \int_0^x t^{n-2} \sqrt{t^2 + a^2} dt \\ &= x^{n-1} \sqrt{x^2 + a^2} - (n-1) \int_0^x t^{n-2} \frac{t^2 + a^2}{\sqrt{t^2 + a^2}} dt \\ &= x^{n-1} \sqrt{x^2 + a^2} - (n-1) \int_0^x \frac{t^n dt}{\sqrt{t^2 + a^2}} - a^2(n-1) \int_0^x \frac{t^{n-2} dt}{\sqrt{t^2 + a^2}} \\ &= x^{n-1} \sqrt{x^2 + a^2} - (n-1)I_n - a^2(n-1)I_{n-2}. \end{aligned}$$

由此得递推关系式 $nI_n(x) = x^{n-1}\sqrt{x^2 + a^2} - a^2(n-1)I_{n-2}(x)$, $\forall n \geq 2$.

解(2). 当 $a = 1$, $x = 1$ 时, 根据结论 (1) 中的递推关系式得

$$nI_n(1) = \sqrt{2} - (n-1)I_{n-2}(1), \quad \forall n \geq 2. \quad (**)$$

由定义 (*) 知

$$I_0(1) = \int_0^1 \frac{dt}{\sqrt{t^2 + 1}}.$$

对上述积分作变量代换 $t = \sinh u$, 其反函数为 $u = \ln(t + \sqrt{t^2 + 1})$. 当 $t = 1$ 时, $u = \ln(1 + \sqrt{2})$. 于是

$$I_0(1) = \int_0^1 \frac{dt}{\sqrt{t^2 + 1}} = \int_0^{\ln(1+\sqrt{2})} \frac{\cosh u du}{\cosh u} = \ln(1 + \sqrt{2}).$$

我们在来计算 $I_1(1)$. 由定义 (*) 知

$$I_1(1) = \int_0^1 \frac{tdt}{\sqrt{t^2 + 1}} = \frac{1}{2} \int_0^1 \frac{d(t^2)}{\sqrt{t^2 + 1}} = \sqrt{t^2 + 1} \Big|_0^1 = \sqrt{2} - 1.$$

根据递推关系式 (**) 得

$$\begin{aligned} 2I_2(1) &= \sqrt{2} - I_0(1) = \sqrt{2} - \ln(1 + \sqrt{2}), \quad I_2(1) = \frac{1}{2} [\sqrt{2} - \ln(1 + \sqrt{2})], \\ 3I_3(1) &= \sqrt{2} - I_1(1) = \sqrt{2} - (\sqrt{2} - 1) = 1, \quad I_3(1) = \frac{1}{3}. \end{aligned}$$

解答完毕.

习题十: 课本第172页习题5.6题10(有修改): 设

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx.$$

证明

- (1) $I_{n+1} < I_n, \forall n \geq 1;$
- (2) $I_n + I_{n-2} = \frac{1}{n-1}, \forall n \geq 2;$
- (3) 对 $\forall n \geq 2, \frac{1}{n+1} < 2I_n < \frac{1}{n-1}.$

证(1): 由于函数 $\tan x$ 在区间 $[0, \frac{\pi}{4}]$ 上严格单调上升, 故对 $x \in [0, \frac{\pi}{4})$, $\tan x < \tan 1 = 1$. 从而 $\tan^{n+1} < \tan^n x$. 因此

$$I_{n+1} = \int_0^{\frac{\pi}{4}} \tan^{n+1} x dx < \int_0^{\frac{\pi}{4}} \tan^n x dx = I_n.$$

结论 (1) 得证.

证(2). 对 $\forall n \geq 2$,

$$I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n-2} x) dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\tan^2 x + 1) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \frac{dx}{\cos^2 x} = \int_0^{\frac{\pi}{4}} \tan^{n-2} x d(\tan x) = \frac{1}{n-1} \tan^{n-1} x \Big|_0^{\frac{\pi}{4}} = \frac{1}{n-1}.$$

证(3). 根据结论 (1): $I_n < I_{n-1}$, 以及结论 (2): $I_n + I_{n-2} = \frac{1}{n-1}$ 得

$$2I_n < I_n + I_{n-2} = \frac{1}{n-1}, \quad 2I_n > I_{n+2} + I_n = \frac{1}{n+1}.$$

这就证明了结论 (3). 证毕.