Linear Algebra 2025 Fall

Sample Solution

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《线性代数入门》Exercise 4.3.8、4.3.9

Exercise 4.3.8 Solution:

1. 将第一行加到第二行知 $\det(B_n) = \det(B_{n-1}) \cdot \det(B_n) = 1$.
2. $\det(A_n) = \det(B_n) + \det(A_{n-1}) \cdot \det(A_n) = n + 1$.

Exercise 4.3.9 行列式中的 Fibonacci 数列

Solution:

1. 将矩阵第一行分解为 $\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$.
2. $\det(S_{n+2}) = 3\det(S_{n+1}) - \det(S_n)$ 二者均满足某个二阶线性递推方程.
3. $t_{n+2} = t_{n+1} + t_n$. 观察前两列的选择情况，分别化归为 $n+1$ 和 n 的情形。

《线性代数与几何》Exercise 1.10(2~6)、1.13、1.14、1.19、1.20

Exercise 1.10

2. **Solution:** $\begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_2 & x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 0 & 0 & \cdots & x & -1 \\ a_n & 0 & 0 & \cdots & 0 & x \end{vmatrix} = \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 + a_1 x & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i=1}^{n-1} a_i x_i^{n-i-1} & 0 & 0 & \cdots & -1 \\ \sum\limits_{i=1}^{n} a_i x_i^{n-i} & 0 & 0 & \cdots & 0 \end{vmatrix}$

$$= (-1)^{n+1} \cdot (-1)^{n-1} \cdot \sum_{i=1}^{n} a_i x^{n-i} = \sum_{k=1}^{n} a_k x^{n-k}$$

3. **Solution:** 记为 D_n，按第一列展开有 $D_n = 2D_{n-1} - D_{n-2}$，由 $D_2 = 2, D_3 = 3$ 可递推得到 $D_n = n + 1$.

4. **Solution:** 记为 D_n，按最后一列展开有 $D_n = 2\cos\theta D_{n-1} - D_{n-2}$，由数学归纳法以及
   $D_k = 2\cos\theta D_{k-1} - D_{k-2} = 2\cos\theta\cos(k-1)\theta - \cos(k-2)\theta$

   $=2\cos\theta\cos(k-1)\theta - \cos((k-1)\theta - \theta) = \cos\theta\cos(k-1)\theta - \sin(k-1)\theta\sin\theta$ 得到结果.

   $=\cos((k-1)\theta + \theta) = \cos k\theta$.

5. **Solution:**

$$\begin{vmatrix} x_1 & a_2 & \cdots & a_n \\ a_1 & x_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & x_n \end{vmatrix} = \begin{vmatrix} x_1 & a_2 & \cdots & a_n \\ a_1 - x_1 & x_2 - a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 - x_1 & 0 & \cdots & x_n - a_n \end{vmatrix}$$

$$= \begin{vmatrix} x_1 + \sum_{i=2}^{n} \frac{x_1 - a_1}{x_i - a_i} a_i & & & \\ 0 & (x_2 - a_2) & & \\ \vdots & & \ddots & \\ 0 & & & (x_n - a_n) \end{vmatrix} = \left(\frac{x_1}{x_1 - a_1} + \sum_{i=2}^{n} \frac{a_i}{x_i - a_i} \right) \prod_{i=2}^{n} (x_i - a_i)$$

$$= \left(1 + \sum_{i=1}^{n} \frac{x_i}{x_i - a_i} \right) \prod_{i=2}^{n} (x_i - a_i)$$

6. **Solution:**

$$= \prod_{i=1}^{n+1} a_i^n \cdot \begin{vmatrix} 1 & \frac{b_1}{a_1} & \cdots & \left(\frac{b_1}{a_1}\right)^n \\ 1 & \frac{b_2}{a_2} & \cdots & \left(\frac{b_2}{a_2}\right)^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{b_2}{a_2} & \cdots & \left(\frac{b_2}{a_2}\right)^n \end{vmatrix} = \prod_{i=1}^{n+1} a_i^n \cdot \prod_{1 \leqslant i < k \leqslant n+1} \left(\frac{b_k}{a_k} - \frac{b_i}{a_i} \right)$$

$$= \left(\prod_{1 \leqslant i < k \leqslant n+1} a_i a_k \right) \cdot \prod_{1 \leqslant i < k \leqslant n+1} \left(\frac{b_k}{a_k} - \frac{b_i}{a_i} \right) = \prod_{1 \leqslant i < k \leqslant n+1} (a_i b_k - a_k b_i)$$

Exercise 1.13 Solution:

1. $p(x) = \prod_{i=1}^{n-1} (a_i - x) \cdot \prod_{1 \leqslant i < j \leqslant n-1} (a_j - a_i)$，为 $(n-1)$ 次多项式

2. 即为 $a_1, a_2, \cdots, a_{n-1}$

Exercise 1.14 **Solution:** 按第一列展开有 $F_n = F_{n-1} + F_{n-2}$，由 $F_1 = 1, F_2 = 2$ 可递推得到前六项为 $1, 2, 3, 5, 8, 13$

Exercise 1.19 **Solution:**

设 $f(x) = ax^3 + bx^2 + cx + d$, 列出线性方程组并求解（可以使用 Cramer 法则，也可以 Gauss 消元）

$$\begin{cases} d + (-c) + b + (-a) = 0 \\ d + c + b + a = 4 \\ d + 2c + 4b + 8a = 3 \\ d + 3c + 9b + 27a = 16 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = -5 \\ c = 0 \\ d = 7 \end{cases} \Rightarrow f(x) = 2x^3 - 5x^2 + 7$$

Exercise 1.20 证明题

Solution:

三条直线交于一点 $\Rightarrow \begin{cases} \alpha x + \beta y = -\gamma \\ yx + \alpha y = -\beta \\ \beta x + yy = -\alpha \end{cases}$ 有唯一解，则将该方程化简为阶梯形后，必有某行为 0，于是有 $\begin{vmatrix} \alpha & \beta & -\gamma \\ \gamma & \alpha & -\beta \\ \beta & \gamma & -\alpha \end{vmatrix} = 0$ ，即 $\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix} = 0.$

$$\begin{aligned} &\Rightarrow (\alpha+\beta+\gamma) \begin{vmatrix} 1 & \beta & \gamma \\ 1 & \alpha & \beta \\ 1 & \gamma & \alpha \end{vmatrix} = (\alpha+\beta+\gamma) \begin{vmatrix} 1 & \beta & \gamma \\ 0 & \alpha-\beta & \beta-\gamma \\ 0 & \gamma-\beta & \alpha-\gamma \end{vmatrix} \\ &= (\alpha+\beta+\gamma)\left[(\alpha-\beta)(\alpha-\gamma) + (\beta-\gamma)^2\right] = (\alpha+\beta+\gamma)\frac{(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\alpha-\gamma)^2}{2} \\ &= 0. \end{aligned}$$

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