Linear Algebra 2025 Fall

Sample Solution

TA: Zhuohan Cai *Week 2-1*

《线性代数入门》 Exercise 4.2.3

Exercise 4.2.3 设 $A_n = \begin{bmatrix} -1 & 1 \\ -6 & 4 \end{bmatrix} + nI_2$

1. **Solution:**

$$A_0 = \begin{bmatrix} -1 & 1 \\ -6 & 4 \end{bmatrix},\ \det(A_0) = -4 + 6 = 2$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix},\ \det(A_1) = 0 + 6 = 6$$

$$A_2 = \begin{bmatrix} 1 & 1 \\ -6 & 6 \end{bmatrix},\ \det(A_2) = 6 + 6 = 12$$

$$A_3 = \begin{bmatrix} 2 & 1 \\ -6 & 7 \end{bmatrix},\ \det(A_3) = 14 + 6 = 20$$

2. **Solution:** $\begin{vmatrix} -1+x & 1 \\ -6 & 4+x \end{vmatrix} = x^2 + 3x + 2 = (x+1)(x+2)$

《线性代数与几何》 Exercise 1.6-1.7

Exercise 1.6 计算行列式。

1. **Solution:** $\begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 5 & -5 \\ 0 & -1 & 4 \end{vmatrix} = 15$

2. **Solution:** $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = \begin{vmatrix} 2(x+y) & y & x+y \\ 2(x+y) & x+y & x \\ 2(x+y) & x & y \end{vmatrix} = 2(x+y) \begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix}$

$$= 2(x+y)\begin{vmatrix} 1 & y & x+y \\ 0 & x & -y \\ 0 & x-y & -x \end{vmatrix} = 2(x+y)(-x^2+xy-y^2) = -2(x^3+y^3)$$

3. **Solution:** $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix} = \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ \omega^2+1+\omega & 1 & \omega \\ \omega+\omega^2+1 & \omega^2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & 1 & \omega \\ 0 & \omega^2 & 1 \end{vmatrix} = 0$

Exercise 1.7 证明下列等式。

1. **Solution:** $\begin{vmatrix} a^2 & ab & b^2 \\ 2a & a+b & 2b \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a^2-ab & ab-b^2 & b^2 \\ a-b & a-b & 2b \\ 0 & 0 & 1 \end{vmatrix} = (a^2-ab)(a-b)-(ab-b^2)(a-b)$

   $= (a-b)(a^2-2ab+b^2) = (a-b)^3$

2. **Solution:** $\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1-a_1x^2 & a_1x+b_1 & c_1 \\ a_2-a_2x^2 & a_2x+b_2 & c_2 \\ a_3-a_3x^2 & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix}$

   $= (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

3. **Solution:** 对于三阶行列式 $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$，考虑四阶范德蒙行列式 $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$

   $D_4 = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = D_{30} + dD_{31} + d^2D_{32} + d^3D_{33}$，其中 $D_{31} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

   由于 $D_4 = (d-c)(d-b)(d-a)(c-b)(c-a)(b-a) = (d-c)(d-b)(d-a)D_3$

   $= (d-c)(d-b)(d-a)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = D_{30} + dD_{31} + d^2D_{32} + d^3D_{33} = D_4$

   其中，含 d 的一次项系数对应相等，即：$D_{31} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

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Sample Solution

TA: Zhuohan Cai | *Week 2-2*

《线性代数入门》Exercise 4.3.1-4.3.2

Exercise 4.3.1 计算行列式。

1. **Solution:** $\begin{vmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{vmatrix} = 3$

2. **Solution:** $\begin{vmatrix} 1 & 1 & 10 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -5 \end{vmatrix} = 3$

Exercise 4.3.2 计算行列式。

1. **Solution:** $\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{vmatrix} = -4 \begin{vmatrix} 1 & 0 & 2 \\ 5 & 4 & 3 \\ 2 & 0 & 1 \end{vmatrix} = -4 \times 4 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 48$

2. **Solution:** $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -12 + 10 = -2$

《线性代数与几何》Exercise 1.1-1.5

Exercise 1.1 **Solution:** 可知 $\{j, k\} = \{3, 8\}$:

当 $j = 3, k = 8$ 时，$\tau(127435689) = 5$，为奇排列.

当 $j = 8, k = 3$ 时，$\tau(127485639) = 10$，为偶排列.

故：$j = 3, k = 8$.

Exercise 1.2

1. **Solution:** $\tau(34215) = 5$，负号.

2. **Solution:** $\tau(31425) = 3$，负号.

Exercise 1.3 **Solution:** 每项记为 $a_{1j_1}a_{23}a_{3j_3}a_{4j_4}a_{5j_5}$，则需要满足 $\tau(j_1 3 j_3 j_4 j_5)$ 为奇数，共有以下 12 种情况：

$(j_1, j_3, j_4, j_5) = \{(1,2,4,5),(1,4,5,2),(1,5,2,4),(2,1,5,4),(2,4,1,5),(2,5,4,1),(4,1,2,5),$
$(4,2,5,1),(4,5,1,2),(5,1,4,2),(5,2,1,4),(5,4,2,1)\}$

Exercise 1.4

1. **Solution:** $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (-1)^{\tau(2143)} = 1$

2. **Solution:** $\begin{vmatrix} 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 2 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ n-1 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & n \end{vmatrix} = (-1)^{\tau(n-1,n-2,\cdots,1,n)} \times n! = (-1)^{\frac{(n-2)(n-1)}{2}} \times n!$

$= \begin{cases} n!, \ n \equiv 1,2 \pmod 4 \\ -n!, \ n \equiv 0,3 \pmod 4 \end{cases}$

Exercise 1.5

1. **Solution:** $\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2n-1} & a_{2n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn-1} & a_{nn} \end{vmatrix} = (-1)^{\tau(n,n-1,\cdots,1)} a_{1n}a_{2n-1}\cdots a_{n1} = (-1)^{\frac{n(n-1)}{2}} a_{1n}a_{2n-1}\cdots a_{n1}$

2. **Solution:** $D = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 0 & \cdots & 0 \\ 3 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 0 & 0 & \cdots & n \end{vmatrix} = n! \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix} = n! D_1$

此时将 D_1 中的 $a_{1i}(i \neq 1)$ 全部化为 0，则有 $a_{11} = 1 - 2 - \cdots - n = 2 - \dfrac{n(n+1)}{2}$

此时，$D_1 = a_{11} \times 1^{n-1} = 2 - \dfrac{n(n+1)}{2}$

因此，$D = n!D_1 = (2 - \dfrac{n(n+1)}{2})n!$