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Math142 FA19
Homework 1
Due: Mon, Sep 16, 2019

7(a) Prove that similar matrices have the same eigenvalues

Suppose A and B are $n \times n$ similar matrices. Then there is a non-singular matrix M such that

$$M^{-1}AM = B$$

Suppose that λ is an eigenvalue of A with eigenvector ψ, so that $A\psi = \lambda\psi$

We have that both M and M^{-1} are non-singular and so have kernal $\{\vec{0}\}$.

It follows that

$$\psi \neq \vec{0} \quad \Rightarrow \quad M^{-1}\psi \neq \vec{0}$$

in addition

$$\begin{aligned} BM^{-1}\psi &= M^{-1}AMM^{-1}\psi \\ &= M^{-1}A\psi \\ &= M^{=}-1\lambda\psi \\ &= \lambda M^{-1}\psi \end{aligned}$$

So that $M^{-1}\psi$ is an eigenvector of B with eigenvalue λ, showing

$$\lambda \in spectrum(A) \quad \Rightarrow \quad \lambda \in spectrum(B)$$

Next suppose that μ is an eigenvalue of B with eigenvector ϕ. Then as above

$$\phi \neq \vec{0} \quad \Rightarrow \quad M\phi \neq \vec{0}$$

Furthermore

$$M^{-1}AM = B \quad \Rightarrow \quad A = MBM^{-1}$$

so that

$$
\begin{aligned}
AM\phi &= MBM^{-1}M\phi \\
&= MB\phi \\
&= M\mu\phi \\
&= \mu M\phi
\end{aligned}
$$

So that $M\phi$ is an eigenvector of A with eigenvalue μ, showing

$$\mu \in spectrum(B) \quad \Rightarrow \quad \mu \in spectrum(A)$$

We conclude

$$spectrum(A) = spectrum(B)$$

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12(c) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Prove that

$$\forall u, v \in \mathbb{R}^n \ < Lu, Lv > = < u, v > \quad \Leftrightarrow \quad \forall u \in \mathbb{R}^n \ \|Lu\| = \|u\|$$

$\Rightarrow$) We have that

$$\forall u, v \in \mathbb{R}^n \ < Lu, Lv > = < u, v >$$

and so in particular when $u = v$ we get

$$< Lu, Lu > = < u, u > \Rightarrow \sqrt{< Lu, Lu >} = \sqrt{< u, u >}$$

$$\Rightarrow \|Lu\| = \|u\|$$

Since u was arbitrary we have $\quad \forall u \in \mathbb{R}^n \ \|Lu\| = \|u\|$

$\Leftarrow$) We have that given any $w \in \mathbb{R}^n$

$$\|Lw\| = \|w\|$$

which, by squaring both sides, implies that given any $w \in \mathbb{R}^n$

$$< Lw, Lw > = < w, w >$$

So given any $u, v \in \mathbb{R}^n$, if in the above we set $w = u + v$ then we have

$$< L(u+v), L(u+v) > = < u+v, u+v >$$

$$\Rightarrow < Lu, Lu > + 2 < Lu, Lv > + < Lv, Lv > = < u, u > + 2 < u, v > + < v, v >$$

$$\Rightarrow 2 < Lu, Lv > = 2 < u, v > + (< u, u > - < Lu, Lu >) + (< v, v > - < Lv, Lv >)$$

but by assumption

$$(< u, u > - < Lu, Lu >) = 0 \quad \text{and} \quad (< v, v > - < Lv, Lv >) = 0$$

so we have shown

$$\forall u, v \in \mathbb{R}^n \ < Lu, Lv > = < u, v >$$

$\blacksquare$

12(d) Let $L : V \to V$ be an orthogonal linear transformation. show that if λ is an eigenvalue of L, then $|\lambda| = 1$.

Let λ be an eigenvalue of L.

By definition L is orthogonal if and only if

$$\forall u, v \in \mathbb{R}^n \;\; < Lu, Lv >=< u, v >$$

which by problem 12(c) is true if and only if

$$\forall w \in \mathbb{R}^n \; \|Lw\| = \|w\|$$

In particular, if ψ is an eigenvector of L for the eigenvalue λ then we have

$$\|\psi\| = \|L\psi\| = \|\lambda\psi\| = |\lambda|\|\psi\|$$

Now since ψ is an eigenvector we have

$$\psi \neq \vec{0} \quad \Rightarrow \quad \|\psi\| \neq 0$$

and so

$$\|\psi\| = |\lambda|\|\psi\| \quad \Rightarrow \quad |\lambda| = 1$$

as required.

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14(b) Show that all the eigenvalues of a real symmetric matrix are real numbers.

Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of a real symmetric matrix A with corresponding eigenvector ψ; so that

$$A\psi = \lambda\psi$$

A couple of observations:

- A real symmetric $\quad \Rightarrow \quad A^* = A \quad \Rightarrow \quad \forall v \;\; < v, Av >=< Av, v >$
- ψ an eigenvector $\quad \Rightarrow \quad \psi \neq \vec{0} \quad \Rightarrow \quad < \psi, \psi > \neq 0$

The inner product is linear in the first variable and conjugate linear in the second variable. So we have

$$\lambda < \psi, \psi >=< \lambda\psi, \psi >=< A\psi, psi >=< \psi, A\psi >=< \psi, \lambda\psi >= \bar{\lambda} < \psi, \psi >$$

But

$$< \psi, \psi > \neq 0 \quad \text{and} \quad \lambda < \psi, \psi >= \bar{\lambda} < \psi, \psi > \quad \Rightarrow \quad \lambda = \bar{\lambda} \quad \Rightarrow \quad \lambda \in \mathbb{R}$$

as requried

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14(c) Show that if A is a symmetric real matrix, then eigenvectors that belong to distinct eigenvalues of A are orthogonal.

Suppose that

$$\psi \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda$$

and

$$\phi \text{ is an eigenvector of } A \text{ with eigenvalue } \mu$$

and that

$$\lambda \neq \mu$$

then we have from problem 14(b) that

$$\lambda, \mu \in \mathbb{R}$$

Under these circumstances we can make the following two observations

$$\mu < \psi, \phi > = < \psi, \mu\psi >$$

$$< A\psi, \phi > = < \psi, A\phi >$$

but then

$$\lambda < \psi, \phi > = < \lambda\psi, \phi > = < A\psi, \phi > = < \psi, A\phi > = < \psi, \mu\phi > = \mu < \psi, \phi >$$

Now we have that

$$\lambda \neq \mu \quad \Rightarrow \quad \lambda - \mu \neq 0$$

so that

$$\lambda < \psi, \phi > = \mu < \psi, \phi > \quad \Rightarrow \quad (\lambda - \mu) < \psi, \phi > = 0 \quad \Rightarrow \quad < \psi, \phi > = 0$$

demonstrating that ψ and ϕ are orthogonal, as required.

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