Module 3 Assignment

Nathan Shea Ouedraogo

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Dimensional Reduction of a Scalar Variable

A dimensional variable u may be reduced to its non-dimensional form $\tilde{u}$ by re-centering it to the origin by a scalar u_r and dividing it by a scalar factor u_s :

$$\tilde{u} = \frac{u - u_r}{u_s} \tag{1}$$

If we assume that the variable is already centered at the origin, u_r disappears:

$$\tilde{u} = \frac{u}{u_s} \tag{2}$$

Rearranging to put the equation in terms of the dimensional variable:

$$u_s \tilde{u} = u \ \square \tag{3}$$

Dimensional Reduction of a Vector Variable

Given a vector variable $\vec{u} = \langle u_1, u_2, \cdots, u_{n-1} \rangle$, $\vec{u_n} \in \mathbb{R}^n$ with *n* number of dimensional elements its non-dimensional form $\vec{\tilde{u}}$ is:

$$\vec{\tilde{u}} = \langle \frac{u_1 - u_{r_1}}{u_{s_2}}, \frac{u_2 - u_{r_2}}{u_{s_2}}, \cdots, \frac{u_{n-1} - u_{r_{n-1}}}{u_{s_{n-1}}} \rangle \tag{4}$$

If we assume the vector is centered at the origin, the equation becomes:

$$\vec{\tilde{u}} = \langle \frac{u_1}{u_{s_2}}, \frac{u_2}{u_{s_2}}, \cdots, \frac{u_{n-1}}{u_{s_{n-1}}} \rangle \tag{5}$$

If all elements of $\vec{\tilde{u}}$ have the same dimension, we can scale some by a characteristic factor U_0:

$$\vec{\tilde{u}} = \langle \frac{u_1}{U_0}, \frac{u_2}{U_0}, \cdots, \frac{u_{n-1}}{U_0} \rangle \tag{6}$$

$$\vec{\tilde{u}} = \frac{1}{U_0} \langle u_1, u_2, \cdots, u_{n-1} \rangle \tag{7}$$

And finally we can rearrange the equation in terms of the dimensional vector:

$$U_0 \vec{\tilde{u}} = \langle u_1, u_2, \cdots, u_{n-1} \rangle \tag{8}$$

$$U_0 \vec{\tilde{u}} = \vec{u} \ \square \tag{9}$$

i) Navier-Stokes Equation for Incompressible Liquids

$$\rho\left(\frac{\partial\vec{v}}{\partial t}\right)+\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}=-\vec{\nabla}p+\eta\nabla^{2}\vec{v}+\rho\vec{F} \tag{10}$$

ii) Dimensional Reduction of Time and Pressure

Applying Equation (1):

$$\tilde{t}=\frac{t-t_r}{t_s} \tag{11}$$

$$\tilde{p}=\frac{p-p_r}{p_s} \tag{12}$$

Applying Equation (2):

$$\tilde{t}=\frac{t}{t_s} \tag{13}$$

$$\tilde{p}=\frac{p}{p_s} \tag{14}$$

Applying Equation (3):

$$t_s\tilde{t}=t \tag{15}$$

$$p_s\tilde{p}=p \ \ \square \tag{16}$$

iii) Dimensional Reduction of Velocity

The dimensional velocity vector is:

$$\vec{v}=\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix} \tag{17}$$

Applying equation (4) gives the dimensionless velocity vector:

$$\vec{\tilde{v}}=\begin{bmatrix}\dfrac{v_x-v_{r_x}}{v_{s_x}}\\\dfrac{v_y-v_{r_y}}{v_{s_y}}\\\dfrac{v_y-v_{r_y}}{v_{s_y}}\end{bmatrix} \tag{18}$$

Given a characteristic velocity V_0 and using equations (5) & (9):

$$\vec{\tilde{v}} = \frac{1}{V_0}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{19}$$

$$V_0 \vec{\tilde{v}} = \vec{v} \ \square \tag{20}$$

iv) Dimensional Reduction of the Position Vector

The position vector is:

$$\vec{f} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{21}$$

Which by equations (4)-(7) and a characteristic length of L_0 gives the relationship:

$$\vec{f} = L_0 \vec{\tilde{f}} \ = L_0 \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} L_0 \tilde{x} \\ L_0 \tilde{y} \\ L_0 \tilde{z} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{f} \ \square \tag{22}$$

v) Dimensional Reduction of the Pressure Gradient

The dimensional pressure gradient is:

$$-\vec{\nabla} p = - \begin{bmatrix} \dfrac{\partial f\left(x, y, z\right)}{\partial x} p_x \\ \dfrac{\partial f\left(x, y, z\right)}{\partial y} p_y \\ \dfrac{\partial f\left(x, y, z\right)}{\partial z} p_z \end{bmatrix} \tag{23}$$

And its dimensionless form:

$$-\vec{\tilde{\nabla}} \tilde{p} = - \begin{bmatrix} \dfrac{\partial \tilde{f}\left(\tilde{x}, \tilde{y}, \tilde{z}\right)}{\partial \tilde{x}} \tilde{p}_x \\ \dfrac{\partial \tilde{f}\left(\tilde{x}, \tilde{y}, \tilde{z}\right)}{\partial \tilde{y}} \tilde{p}_y \\ \dfrac{\partial \tilde{f}\left(\tilde{x}, \tilde{y}, \tilde{z}\right)}{\partial \tilde{z}} \tilde{p}_z \end{bmatrix} \tag{24}$$

Substituting dimensionless quantities from equations (16), (22) into equation (23) gives:

$$-\vec{\nabla}p = -\begin{bmatrix} \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial L_0 \tilde{x}} p_s \tilde{p}_x \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial L_0 \tilde{y}} p_s \tilde{p}_y \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial L_0 \tilde{z}} p_s \tilde{p}_z \end{bmatrix} \tag{25}$$

Since the characteristic length is a constant, it can be pulled out:

$$-\vec{\nabla}p = -\frac{1}{L_0}\begin{bmatrix} \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{x}} p_{s_x} \tilde{p}_x \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{y}} p_{s_y} \tilde{p}_y \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{z}} p_{s_z} \tilde{p}_z \end{bmatrix} \tag{26}$$

And by equation (24) gives:

$$-\vec{\nabla}p = -\frac{1}{L_0}\vec{\tilde{\nabla}} p_s \tilde{p} \ \square \tag{27}$$

vi) Dimensional Reduction of the Viscosity Function

The viscosity function is given by:

$$\eta \nabla^2 \vec{v} = \eta \begin{bmatrix} \dfrac{\partial^2 f\left(x,y,z\right)}{\partial x^2} v_x \\ \dfrac{\partial^2 f\left(x,y,z\right)}{\partial y^2} v_y \\ \dfrac{\partial^2 f\left(x,y,z\right)}{\partial z^2} v_z \end{bmatrix} \tag{28}$$

And when dimensionally reduced:

$$\eta \tilde{\nabla}^2 \vec{\tilde{v}} = \eta \begin{bmatrix} \dfrac{\partial^2 \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{x}^2} \tilde{v_x} \\ \dfrac{\partial^2 \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{y}^2} \tilde{v_y} \\ \dfrac{\partial^2 \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{z}^2} \tilde{v_z} \end{bmatrix} \tag{29}$$

Substituting equation (22) into equation (28) with a characteristic velocity V_0 gives:

$$\eta \nabla^2 \vec{v} = \eta \begin{bmatrix} \dfrac{\partial^2 \tilde{f}\left(\tilde{x}, \tilde{y}, \tilde{z}\right)}{\partial (L_0 \tilde{x})^2} V_0 \vec{\tilde{v}_x} \\ \dfrac{\partial^2 \tilde{f}\left(\tilde{x}, \tilde{y}, \tilde{z}\right)}{\partial (L_0 \tilde{y})^2} V_0 \vec{\tilde{v}_y} \\ \dfrac{\partial^2 \tilde{f}\left(\tilde{x}, \tilde{y}, \tilde{z}\right)}{\partial (L_0 \tilde{z})^2} V_0 \vec{\tilde{v}_z} \end{bmatrix} \tag{30}$$

Pulling out the characteristic length and velocity:

$$\eta \nabla^2 \vec{v} = \frac{\eta V_0}{L_0^2} \begin{bmatrix} \dfrac{\partial^2 \tilde{f}\left(\tilde{x}, \tilde{y}, \tilde{z}\right)}{\partial \tilde{x}^2} \vec{\tilde{v}_x} \\ \dfrac{\partial^2 \tilde{f}\left(\tilde{x}, \tilde{y}, \tilde{z}\right)}{\partial \tilde{y}^2} \vec{\tilde{v}_y} \\ \dfrac{\partial^2 \tilde{f}\left(\tilde{x}, \tilde{y}, \tilde{z}\right)}{\partial \tilde{z}^2} \vec{\tilde{v}_z} \end{bmatrix} \tag{31}$$

And finally, substituting equation (29):

$$\eta \nabla^2 \vec{v} = \frac{\eta V_0}{L_0^2} \tilde{\nabla}^2 \vec{\tilde{v}} \ \square \tag{32}$$

vii) The Reynolds Number (Re)

The Reynolds number is a scalar value which modulates the dimensionless velocity gradient:

$$Re = \rho \left(\frac{L_0 V_0}{\eta} \right) \tag{33}$$

In the following section we will see how this number is applied.

viii) Non-Dimensional Navier-Stokes Equation

Substituting equations (15), (16), (20), (27), and (32) into equation (10) gives:

$$\rho\left(\frac{\partial V_0\tilde{\vec{v}}}{\partial t_s\tilde{t}}+\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)\right)=-\frac{1}{L_0}\tilde{\vec{\nabla}}p_s\tilde{p}+\frac{\eta V_0}{L_0^2}\tilde{\nabla}^2\tilde{\vec{v}}+\rho\vec{F} \tag{34}$$

$$\rho\left(\frac{V_0}{t_s}\right)\left(\frac{\partial\tilde{\vec{v}}}{\partial\tilde{t}}\right)+\rho\left(\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)\right)=-\frac{1}{L_0}\tilde{\vec{\nabla}}p_s\tilde{p}+\frac{\eta V_0}{L_0^2}\tilde{\nabla}^2\tilde{\vec{v}}+\rho\vec{F} \tag{35}$$

Dividing by the highest order derivative term and assuming negligible contribution from body forces:

$$\left(\frac{L_0^2}{\eta V_0}\right)\rho\left(\frac{V_0}{t_s}\right)\left(\frac{\partial\tilde{\vec{v}}}{\partial\tilde{t}}\right)+\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)=\left(\frac{L_0^2}{\eta V_0}\right)\left(-\frac{1}{L_0}\tilde{\vec{\nabla}}p_s\tilde{p}\right)+\left(\frac{L_0^2}{\eta V_0}\right)\frac{\eta V_0}{L_0^2}\tilde{\nabla}^2\tilde{\vec{v}} \tag{36}$$

$$\rho\left(\frac{L_0^2}{\eta}\right)\left(\frac{\partial\tilde{\vec{v}}}{\partial\tilde{t}}\right)+\rho\left(\frac{L_0V_0}{\eta}\right)\left(\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)\right)=-\frac{L_0}{\eta V_0}\tilde{\vec{\nabla}}p_s\tilde{p}+\tilde{\nabla}^2\tilde{\vec{v}} \tag{37}$$

And finally applying equation (33) to equation (37):

$$\rho\left(\frac{L_0^2}{\eta}\right)\left(\frac{\partial\tilde{\vec{v}}}{\partial\tilde{t}}\right)+Re\left[\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)\right]=-\frac{L_0}{\eta V_0}\tilde{\vec{\nabla}}p_s\tilde{p}+\tilde{\nabla}^2\tilde{\vec{v}} \tag{38}$$

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Per equation (33) Re is defined as:

$$Re = \rho \left(\frac{L_0 V_0}{\eta} \right)$$

Where L_0 and V_0 are characteristic length (hydrodynamic diameter) and characteristic velocity respectively, and ρ and η are the density and dynamic viscosity of the fluid respectively If pressure and temperature are assumed to be constant and body forces are assumed to be negligible, ρ and η can be assumed to be constants. Re is therefore dependent on two forces; viscosity forces dependent on ρ and η and inertial forces dependent on L_0 and V_0.

We can construct two simplified mental models of the physical effect of Re using our intuition and own experience. We know that viscous fluids like honey have a greater tendency to 'stick' and less tendency to 'flow'. When honey does flow it is slow and smooth with little bubbles or turbulence. However, upon heating honey it becomes much easier to pour which can be modeled as increasing the fluid's velocity. The flow of hot honey is noticeably more chaotic with bubbles and turbulence. Another way to increase this observed chaos would be to increase the diameter of the flowing liquid; honey flowing from the tip of a squeeze bottle is much more smooth than honey flowing from an upturned jar.

In the first model where fluid flow is 'smooth', the system is dominated by viscosity forces ($Re < 1$) and the fluid flow is said to be laminar:

$$\frac{\rho}{\eta} > L_0 V_0 \tag{39}$$

The second model with 'chaotic' flow is dominated by inertial forces (Re>>1) and the flow is said to be turbulent:

$$\frac{\rho}{\eta} << L_0 V_0 \tag{40}$$

Intermediate values are neither fully turbulent nor fully laminar and have contributions by both forces.

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Hagen-Poiseuille Flow between Parallel Plates

Given a cross section perpendicular to the flow of the fluid, $\partial\mathcal{C}$, the flow is said to be under Hagen-Poiseuille conditions if:

1. *The system is translation invariant (i.e. exact position is not needed) along a specified axis*
2. *pressure is unidirectional along the given traslationally invariant axis and zero on the other axis*
3. *no body forces act upon the system*
4. *For a given cross-section of a pipe, $\mathcal{C}$, the velocity at any point along the boundary (wall), $\partial\mathcal{C}$, is zero (no-slip condition)*
5. *Pressure boundary conditions hold:*

$$p(t_i) = p_0 + \Delta p$$
$$p(t_f) = p_0$$

i) Navier-Stokes Equation for Incompressible Liquids in Cartesian Coordinates

The convective term of the Navier Stokes equation (eq. 10) is:

$$\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}$$

Since pressure is unidirectional along the z-axis the equation will expand to:

$$\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = v_x\left(\frac{\partial}{\partial x}v_z\vec{i}\right) + v_y\left(\frac{\partial}{\partial y}v_z\vec{j}\right) + v_z\left(\frac{\partial}{\partial z}v_z\vec{k}\right) \tag{41}$$

Velocity on the xy plane is zero:

$$\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = (0)\left(\frac{\partial}{\partial x}v_z\vec{i}\right) + (0)\left(\frac{\partial}{\partial y}v_z\vec{j}\right) + v_z\left(\frac{\partial}{\partial z}v_z\vec{k}\right) = v_z\left(\frac{\partial}{\partial z}v_z\vec{k}\right) \tag{42}$$

Since the z-axis is our chosen translationally invariant axis, velocity is zero:

$$\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = (0)\,v_z\left(\frac{\partial}{\partial z}v_z\vec{k}\right) = 0 \;\blacksquare \tag{43}$$

By equation (43), a system under Hagen-Poiseuille flow have no convection.

Since convection is zero under Hagen-Poiseuille flow, pressure is the only force which will be acting on our system. Additionally, under Pouisseuille flow pressure will be unidirectional along a single axis. Picking the z-axis, we can apply equation (43) to equation (10):

$$0 = -\vec{\nabla} p + \eta \nabla^2 \vec{v_z} \tag{44}$$

$$\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} = \eta \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \tag{45}$$

$$\frac{\partial p}{\partial z} = \eta \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{46}$$

$$\frac{\partial p}{\partial z} = \eta \left(\frac{\partial^2 v_z}{\partial y^2} \right) \quad \blacksquare \tag{47}$$

ii) Derivation of Fluid Velocity

Given Hagen-Poiseuille flow, the velocity on $\partial \mathcal{C}$ is zero. Moving away from $\partial \mathcal{C}$ towards the center of the cross section $\mathcal{C}$ will increase the velocity with a maxima at the center of $\mathcal{C}$. We will define this center point as $y_0 = 0$. Physically, this change in velocity is due to sheer stress along the pipe walls increasing drag. Since velocity is symmetrical about y_0 the change in velocity with respect to position will be zero. Therefore $\frac{\partial v_z}{\partial y} = 0$. Applying to equation (47) gives a simple second-order ordinary differential equation:

$$\int \eta \frac{\partial^2 v_z}{\partial y^2} dy = \int \frac{\partial p}{\partial z} dy \tag{48}$$

$$\eta \frac{\partial v_z}{\partial y} + C_0 = \frac{\partial p}{\partial z} y \tag{49}$$

$$C_0 = \left(\frac{\partial p}{\partial z} \right) \left(\frac{y_0}{\eta} \right) - \frac{\partial v_{z_0}}{\partial y} \tag{50}$$

$$C_0 = \left(\frac{\partial p}{\partial z} \right) \left(\frac{0}{\eta} \right) - 0 \tag{51}$$

$$C_0 = 0 \tag{52}$$

$$\therefore \frac{\partial p}{\partial z} y = \eta \frac{\partial v_z}{\partial y} \quad \square \tag{53}$$

The no-slip boundary condition states that the velocity on $\partial\mathcal{C}$ is zero. Let us define the vertical position at the boundary as y_1 and the velocity at the boundary as $v_{z_1} = 0$. Applying to equation (53) leaves a first-order ordinary differential equation:

$$\int \eta \frac{\partial v_z}{\partial y} dy = \int \frac{\partial p}{\partial z} y dy \tag{54}$$

$$v_z = \left(\frac{y^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) + C_1 \tag{55}$$

$$C_1 = v_{z_1} - \left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{56}$$

$$C_1 = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{57}$$

$$v_z = \left(\frac{y^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) - \left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{58}$$

$$\therefore v_z = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)\left(1 - \frac{y^2}{y_1^2}\right) \ \blacksquare \tag{59}$$

iii) Compliance with No-Slip Conditions

If the velocity is non-zero on the boundary of $\mathcal{C}$, then the velocity will not be symmetrical. Physically this is due to the fluid experiencing shear stress and drag across the wall which causes non-zero velocity at the wall ('slip') and non-symmetrical velocity. No-slip conditions hold that the velocity at the wall is zero. This is trivial to prove as $y^2 = y_1^2$ on $\partial\mathcal{C}$. Substituting into equation (59) gives:

$$v_z = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)\left(1 - \frac{y_1^2}{y_1^2}\right) \tag{60}$$

$$v_z = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)(1 - 1) \tag{61}$$

$$v_z = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)(0) \tag{62}$$

$$\therefore v_z = 0 \ \blacksquare \tag{63}$$

Hagen-Poiseuille Flow in a Circular Channel

i) Navier-Stokes Equation for Incompressible Liquids in Cylindrical Coordinates

Mapping the cartesian z-component from equation (10) to cylindrical coordinates gives:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\left(\frac{v_\theta}{r}\right)\left(\frac{\partial v_z}{\partial \theta}\right)+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta\nabla^2 v_z+\rho g_z \tag{64}$$

Assuming no-slip conditions:

$$\frac{\partial p}{\partial z}=\eta\nabla^2 v_z \tag{65}$$

Expanding the Laplacian:

$$\frac{\partial p}{\partial z}=\eta\left[\frac{1}{r}\left(\frac{\partial}{\partial r}\right)\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2 v_z}{\partial \theta^2}\right)+\frac{\partial^2 v_z}{\partial z^2}\right] \tag{66}$$

Reducing terms due to no slip conditions and simplifying gives us the Poiseuillian Navier-Stokes equation in cylindrical coordinates:

$$\frac{\partial p}{\partial z}=\frac{\eta}{r}\left[\left(\frac{\partial}{\partial r}\right)\left(r\frac{\partial v_z}{\partial r}\right)\right] \tag{67}$$

$$\left(\frac{r}{\eta}\right)\left(\frac{\partial p}{\partial z}\right)dr=\partial\left(r\frac{\partial v_z}{\partial r}\right)\quad\square \tag{68}$$

ii) Derivation of Velocity

Equations (47) & (68) are analogous, so the method used to derive equation (53) may be used here as well. As with equation (47) let the boundary conditions be $r_0=0$, $v_{z_0}=0$:

$$\int\left(\frac{r}{\eta}\right)\left(\frac{\partial p}{\partial z}\right)dr=\int\partial\left(r\frac{\partial v_z}{\partial r}\right) \tag{69}$$

$$\left(\frac{r^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)=r\left(\frac{\partial v_z}{\partial r}\right)+C_0 \tag{70}$$

$$\because C_0=\left(\frac{r_0^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)-r_0\left(\frac{\partial v_z}{\partial r_0}\right)=0 \tag{71}$$

$$\therefore\left(\frac{r^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)=r\left(\frac{\partial v_z}{\partial r}\right)\quad\square \tag{72}$$

Equations (72) & (53) are analogous, so the same method applies:

$$\int \left(\frac{r^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) dr = \int r \left(\frac{\partial v_z}{\partial r}\right) \tag{73}$$

$$v_z = \left(\frac{r^2}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right) + C_1 \tag{74}$$

$$C_1 = v_{z_1} - \left(\frac{r_1^2}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{75}$$

$$C_1 = -\left(\frac{r_1^2}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{76}$$

$$\therefore v_z\left(r,\theta\right) = -\left(\frac{1}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right)\left(r_1^2 - r^2\right) \quad \blacksquare \tag{77}$$

iii) Compliance with No-Slip Conditions

By setting the radius $\left(r^2\right)$ to the boundary, from equation we get (77):

$$v_z\left(r,\theta\right) = -\left(\frac{1}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right)\left(r_1^2 - r_1^2\right) \tag{78}$$

$$v_z\left(r,\theta\right) = -\left(\frac{1}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right)(0) \tag{79}$$

$$\therefore v_z\left(r,\theta\right) = 0 \quad \blacksquare \tag{80}$$

We can prove no-slip conditions in a circular pipe in cartesian coordinates as well. To convert back we set $\dfrac{\partial p}{\partial z} = \dfrac{\Delta p}{L_0}$, and $r^2 = a^2$. Since $\mathcal{C}$ is a circle, any point on $\partial\,\mathcal{C}$ can be used to derive the radius: $r = a = \sqrt{x_{boundary}^2 + y_{boundary}^2}$. If we make these substitutions into equation (77) it is trivial to prove compliance with no-slip boundary conditions:

$$\vec{v_z}\left(x,y\right) = -\left(\frac{\Delta p}{4\eta L_0}\right)\left(a^2 - \left(\sqrt{x_{boundary}^2 + y_{boundary}^2}\right)^2\right) \tag{81}$$

$$\vec{v_z}\left(x,y\right) = -\left(\frac{\Delta p}{4\eta L_0}\right)\left(a^2 - a^2\right) \tag{82}$$

$$\therefore \vec{v_z}\left(x,y\right) = 0 \quad \blacksquare \tag{83}$$

Hagen-Poiseuille Flow in a Rectangular Channel

Unlike a channel between parallel plates with theoretically 'zero' height and circular channels which are symmetrical in length and width, rectangular channels introduce asymmetry between the x and y axis. Therefore the velocity gradient will now have x and y components. As we have done previously, we will assume pressure is only applied in the z-direction and the z-axis is translationally invariant. Starting from equation (46), our velocity function becomes:

$$\nabla^2 \vec{v_z}\,(x,y) = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} \tag{84}$$

Unfortunately there is no known analytical solution to this differential equation and the best we can do is a Fourier approximation. Per the Fourier transform, any periodic function function $f\,(u)\,,\ u \in [-D, +D]$ can be represented as a series of cosines and sines:

$$f\,(u) = \sum_{n=odd}^{\infty} b_n\,(u) \sin(\frac{n\pi u}{D}) + \sum_{n=even}^{\infty} a_n\,(u) \cos(\frac{n\pi u}{D}) \tag{85}$$

Let the y-axis be the height of the channel with bounds $0 \le y \le h$, let the x-axis be the width of the channel with bounds $-w_{1/2} \le x \le +w_{1/2}$ and finally let the z-axis be the direction of the flow. The sign of the width terms denotes which side of the midpoint $(x = 0)$ they are on. As we did in the last two sections, this will give boundary conditions $x \in \left[-w_{1/2}, +w_{1/2}\right]$, $y \in [0, h]$. As with the previous examples, velocity is constant across the y-axis (the height of the channel) but variable along the x-axis (the width of the channel). Using equation (85), the general form of the solution for equation (84) will be:

$$\vec{v_z}\,(x,y) = \sum_{n=odd}^{\infty} b_n\,(x) \sin(\frac{n\pi x}{h}) + \sum_{n=even}^{\infty} a_n\,(x) \cos(\frac{n\pi x}{h}) \tag{86}$$

The even expansion satisfies $f\,(-t) = f\,(t)$ whereas the odd expansion satisfies $f\,(-t) = -f\,(t)$. Only the odd terms will be used as they satisfy the boundary conditions:

$$\vec{v_z}\,(x,y) = \sum_{n=odd}^{\infty} b_n\,(x) \sin(\frac{n\pi x}{h}) \tag{87}$$

Subbing equation (87) into equation (84) gives:

$$\nabla^2 \vec{v_z}\left(x,y\right) = \sum_{n=odd}^{\infty} \left(\frac{\partial^2 v_z}{\partial x^2} \left[b_n\left(x\right) \sin(\frac{n\pi x}{h}) \right] + \left[\frac{\partial^2 v_z}{\partial y^2} b_n\left(x\right) \sin(\frac{n\pi x}{h}) \right] \right) \tag{88}$$

$$\nabla^2 \vec{v_z}\left(x,y\right) = \sum_{n=odd}^{\infty} \frac{\partial^2 v_z}{\partial x^2} \left[b_n\left(x\right) \sin(\frac{n\pi x}{h}) \right] + 0 \tag{89}$$

$$\nabla^2 \vec{v_z}\left(x,y\right) = \sum_{n=odd}^{\infty} \frac{\partial v_z}{\partial x} \left[b_n^{'}\left(x\right) \sin(\frac{n\pi x}{h}) + b_n^{'}\left(x\right) \frac{n\pi x}{h} \cos(\frac{n\pi x}{h}) \right] \tag{90}$$

$$\nabla^2 \vec{v_z}\left(x,y\right) = \sum_{n=odd}^{\infty} \left[b_n^{''}\left(x\right) \sin(\frac{n\pi x}{h}) - b_n\left(x\right) \frac{n^2\pi^2 x^2}{h^2} \sin(\frac{n\pi}{h}) \right] + \sum_{n=even}^{\infty} 2 \left[b^{'}\left(x\right) \frac{n\pi x}{h} \cos(\frac{n\pi x}{h}) \right] \tag{91}$$

Since we are only concerned with the odd terms, the equation reduces to:

$$\nabla^2 \vec{v_z}\left(x,y\right) = \sum_{n=odd}^{\infty} \left[b_n^{''}\left(x\right) - b_n\left(x\right) \frac{n^2\pi^2 x^2}{h^2} \right] \sin(\frac{n\pi x}{h}) \;\square \tag{92}$$

4

Experimental Verification of Hagen-Poiseuille's Law

i) Derivation of Hagen-Poiseuille's Law in a Circular Pipe

Flow rate (Q) is obtained by integrating the velocity by the area of our cross-section $\mathcal{C}$. We must define some terms before performing the integration. First, let us define the bounds of integration on $\mathcal{C}$. Given a circular cross-section, the bounds will be from the the minimum radius in the center of $\mathcal{C}$, $r_{min} = 0$, to the maximum radius on $\partial\,\mathcal{C}$, $r_{max} = R$. Next we will define $\frac{\partial p}{\partial z}$ as $\frac{\partial p}{\partial z} = \frac{\Delta p}{L_0}$. Finally, we must define our area of integration and the Jacobian (scaling factor). The area of integration, dA, will be 2π and the Jacobian for cylindrical coordinates is simply $J\,(r, \theta, z) = r$. Now we are ready to derive Q:

$$Q = -\int_{r_{min}}^{r_{max}} v_z J\,(r, \theta, z)\, dA \tag{93}$$

$$= -\int_{r=0}^{r=R} -\left(\frac{1}{4\eta}\right)\left(\frac{\Delta p}{L_0}\right)\left(R^2 - r^2\right) r 2\pi dr \tag{94}$$

$$= \left(\frac{\pi}{2\eta}\right)\left(\frac{\Delta p}{L_0}\right)\int_{r=0}^{r=R}\left(R^2 r - r^3\right) dr \tag{95}$$

$$= \left(\frac{\pi}{2\eta}\right)\left(\frac{\Delta p}{L_0}\right)\left(\frac{R^2 r^2}{2} - \frac{r^4}{4}\right)\Bigg|_{r=0}^{r=R} \tag{96}$$

$$= \left(\frac{\pi}{2\eta}\right)\left(\frac{\Delta p}{L_0}\right)\left(\frac{R^4}{4}\right) \tag{97}$$

$$= \left(\frac{R^4 \pi}{8\eta}\right)\left(\frac{\Delta p}{L_0}\right) \quad \square \tag{98}$$

ii) Hydraulic Resistance in a Circular Pipe

Δp of a system is proportional to the flow rate and how easily the fluid can pass through the medium of $\partial\,\mathcal{C}$. The ability of a fluid to pass through a medium is called 'hydraulic resistance', $R_{hyd} = \frac{8\eta L_0}{R^4 \pi}$. Starting from equation (89):

$$\frac{Q L_0}{\Delta p} = \frac{R^4 \pi}{8\eta} \tag{99}$$

$$\Delta p = \frac{8\eta L_0}{R^4 \pi} Q \tag{100}$$

$$\Delta p = R_{hyd} Q \tag{101}$$

$$R_{hyd} = \frac{\Delta p}{Q} \quad \blacksquare \tag{102}$$

v) Experimental Data

$\boldsymbol{\Delta p}$ **(mbar)**	$\boldsymbol{H_2O}$ **mass (g)**
50	0.0348
250	0.2327
500	0.484
750	0.775
950	0.9061

The data above was collected by applying pressure across a microfluidic device and collecting the mass of water at the outlet for 300s. Given STAP conditions and flow rate 'Q' with units of $\frac{m^3}{s}$, we may derive the hydraulic resistance R_{hyd} which has units of $Pa\frac{s}{m^3}$. Starting with equation (93):

$$R_{hyd} = \left(\frac{\Delta p \times \frac{100Pa}{1mbar}}{\left(mass_{H_2O} \times \frac{m^3}{1000g} \right)(s)} \right) \tag{103}$$

$$R_{hyd} = \left(\Delta p \times \frac{100Pa}{1mbar} \right) \times \left(\frac{1000g \times s}{mass_{H_2O} \times m^3} \right) \quad \square \tag{104}$$

Using values from the first row in equation (95) gives:

$$R_{hyd} = \left(50mbar \times \frac{100Pa}{1mbar} \right) \times \left(\frac{1000g \times 300s}{0.0348g \times m^3} \right)$$

$$\boldsymbol{R_{hyd} \approx 4.3 \times 10^{10} Pa\frac{s}{m^3}}$$

Performing calculations for all experimental results gives the following:

$\boldsymbol{\Delta p}\ (mbar)$	$\boldsymbol{R_{hyd}}\ (Pa)\,(s)\,m^{-3}$
50	4.31×10^{10}
250	3.22×10^{10}
500	3.10×10^{10}
750	2.90×10^{10}
950	3.15×10^{10}

Hydraulic resistance is analogous to electrical resistance. As such, total hydraulic resistance in series is the sum of each 'resistor'and in parallel the resistance is the sum of the reciprocal of each resistor. If we were to add a tube of a specific resistance to the above device, the total resistance would increase by the tube's hydraulic resistance R_{tubes}. Given $R_{tubes} = 1 \times 10^8\,(Pa)\,(s)\,m^{-3}$, the previously calculated resistances would become:

$\boldsymbol{\Delta p}\,(mbar)$	$\boldsymbol{R_{hyd}}\,(Pa)\,(s)\,m^{-3}$
50	4.32$\times 10^{10}$
250	3.23$\times 10^{10}$
500	3.11$\times 10^{10}$
750	2.91 $\times 10^{10}$
950	3.16 $\times 10^{10}$