Module 3 Assignment

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Dimensional Reduction of a Scalar Variable

A dimensional variable u may be reduced to its non-dimensional form $\tilde{u}$ by re-centering it to the origin by a scalar u_r and dividing it by a scalar factor u_s :

$$\tilde{u} = \frac{u - u_r}{u_s} \tag{1}$$

If we assume that the variable is already centered at the origin, u_r disappears:

$$\tilde{u} = \frac{u}{u_s} \tag{2}$$

Rearranging to put the equation in terms of the dimensional variable:

$$u_s \tilde{u} = u \ \square \tag{3}$$

Dimensional Reduction of a Vector Variable

Given a vector variable $\vec{u} = \langle u_1, u_2, \cdots, u_{n-1} \rangle$, $\vec{u_n} \ \in \ \mathbb{R}^n$ with *n* number of dimensional elements its non-dimensional form $\vec{\tilde{u}}$ is:

$$\vec{\tilde{u}} = \langle \frac{u_1 - u_{r_1}}{u_{s_1}}, \frac{u_2 - u_{r_2}}{u_{s_2}}, \cdots, \frac{u_{n-1} - u_{r_{n-1}}}{u_{s_{n-1}}} \rangle \tag{4}$$

If we assume the vector is centered at the origin, the equation becomes:

$$\vec{\tilde{u}} = \langle \frac{u_1}{u_{s_1}}, \frac{u_2}{u_{s_2}}, \cdots, \frac{u_{n-1}}{u_{s_{n-1}}} \rangle \tag{5}$$

If all elements of $\vec{\tilde{u}}$ have the same dimension, we can scale some by a critical factor U_0:

$$\vec{\tilde{u}} = \langle \frac{u_1}{U_0}, \frac{u_2}{U_0}, \cdots, \frac{u_{n-1}}{U_0} \rangle \tag{6}$$

$$\vec{\tilde{u}} = \frac{1}{U_0} \langle u_1, u_2, \cdots, u_{n-1} \rangle \tag{7}$$

And finally we can rearrange the equation in terms of the dimensional vector:

$$U_0 \vec{\tilde{u}} = \langle u_1, u_2, \cdots, u_{n-1} \rangle \tag{8}$$

$$U_0 \vec{\tilde{u}} = \vec{u} \ \square \tag{9}$$

i) Navier-Stokes Equations for Incompressible Liquids

$$\rho\left(\frac{\partial\vec{v}}{\partial t}\right)+\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}=-\vec{\nabla}p+\eta\nabla^2\vec{v}+\rho\vec{F} \tag{10}$$

ii) Dimensional Reduction of Time and Pressure

Applying Equation (1):

$$\tilde{t}=\frac{t-t_r}{t_s} \tag{11}$$

$$\tilde{p}=\frac{p-p_r}{p_s} \tag{12}$$

Applying Equation (2):

$$\tilde{t}=\frac{t}{t_s} \tag{13}$$

$$\tilde{p}=\frac{p}{p_s} \tag{14}$$

Applying Equation (3):

$$t_s\tilde{t}=t \tag{15}$$

$$p_s\tilde{p}=p \ \ \square \tag{16}$$

iii) Dimensional Reduction of Velocity

The dimensional velocity vector is:

$$\vec{v}=\begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix} \tag{17}$$

Applying equation (4) gives the dimensionless velocity vector:

$$\vec{\tilde{v}}=\begin{bmatrix}\dfrac{v_x-v_{r_x}}{v_{s_x}}\\ \dfrac{v_y-v_{r_y}}{v_{s_y}}\\ \dfrac{v_y-v_{r_y}}{v_{s_y}}\end{bmatrix} \tag{18}$$

Given a critical velocity V_0 and using equations (5) & (9):

$$\vec{\tilde{v}} = \frac{1}{V_0}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{19}$$

$$V_0\vec{\tilde{v}} = \vec{v} \ \square \tag{20}$$

iv) Dimensional Reduction of the Position Vector

The position vector is:

$$\vec{f} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{21}$$

Which by equations (4)-(7) and a critical length of L_0 gives the relationship:

$$\vec{f} = L_0\vec{\tilde{f}} \ = L_0\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} L_0\tilde{x} \\ L_0\tilde{y} \\ L_0\tilde{z} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{f} \ \square \tag{22}$$

v) Dimensional Reduction of the Pressure Gradient

The dimensional pressure gradient is:

$$-\vec{\nabla}p = -\begin{bmatrix} \dfrac{\partial f\left(x,y,z\right)}{\partial x}p_x \\ \dfrac{\partial f\left(x,y,z\right)}{\partial y}p_y \\ \dfrac{\partial f\left(x,y,z\right)}{\partial z}p_z \end{bmatrix} \tag{23}$$

And its dimensionless form:

$$-\vec{\tilde{\nabla}}\tilde{p} = -\begin{bmatrix} \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{x}}\tilde{p}_x \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{y}}\tilde{p}_y \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{z}}\tilde{p}_z \end{bmatrix} \tag{24}$$

Substituting dimensionless quantities from equations (16), (22) into equation (23) gives:

$$-\vec{\nabla}p = -\begin{bmatrix} \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial L_0\tilde{x}} p_s\tilde{p}_x \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial L_0\tilde{y}} p_s\tilde{p}_y \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial L_0\tilde{z}} p_s\tilde{p}_z \end{bmatrix} \tag{25}$$

Since the critical length is a constant, it can be pulled out:

$$-\vec{\nabla}p = -\frac{1}{L_0}\begin{bmatrix} \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{x}} p_{s_x}\tilde{p}_x \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{y}} p_{s_y}\tilde{p}_y \\ \dfrac{\partial \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{z}} p_{s_z}\tilde{p}_z \end{bmatrix} \tag{26}$$

And by equation (24) gives:

$$-\vec{\nabla}p = -\frac{1}{L_0}\vec{\tilde{\nabla}} p_s\tilde{p} \ \square \tag{27}$$

vi) Dimensional Reduction of the Dynamic Viscosity Field

The dynamic viscosity field is given by:

$$\eta\nabla^2\vec{v} = \eta\begin{bmatrix} \dfrac{\partial^2 f\left(x,y,z\right)}{\partial x^2} v_x \\ \dfrac{\partial^2 f\left(x,y,z\right)}{\partial y^2} v_y \\ \dfrac{\partial^2 f\left(x,y,z\right)}{\partial z^2} v_z \end{bmatrix} \tag{28}$$

And when dimensionally reduced:

$$\eta\tilde{\nabla}^2\tilde{\vec{v}} = \eta\begin{bmatrix} \dfrac{\partial^2 \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{x}^2} \tilde{v_x} \\ \dfrac{\partial^2 \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{y}^2} \tilde{v_y} \\ \dfrac{\partial^2 \tilde{f}\left(\tilde{x},\tilde{y},\tilde{z}\right)}{\partial \tilde{z}^2} \tilde{v_z} \end{bmatrix} \tag{29}$$

Substituting equation (22) into equation (28) with a critical velocity V_0 gives:

$$\eta \nabla^2 \vec{v} = \eta \begin{bmatrix} \frac{\partial^2 \tilde{f}(\tilde{x},\tilde{y},\tilde{z})}{\partial (L_0 \tilde{x})^2} V_0 \vec{\tilde{v}_x} \\ \frac{\partial^2 \tilde{f}(\tilde{x},\tilde{y},\tilde{z})}{\partial (L_0 \tilde{y})^2} V_0 \vec{\tilde{v}_y} \\ \frac{\partial^2 \tilde{f}(\tilde{x},\tilde{y},\tilde{z})}{\partial (L_0 \tilde{z})^2} V_0 \vec{\tilde{v}_z} \end{bmatrix} \tag{30}$$

Pulling out the critical length and velocity:

$$\eta \nabla^2 \vec{v} = \frac{\eta V_0}{L_0^2} \begin{bmatrix} \frac{\partial^2 \tilde{f}(\tilde{x},\tilde{y},\tilde{z})}{\partial \tilde{x}^2} \vec{\tilde{v}_x} \\ \frac{\partial^2 \tilde{f}(\tilde{x},\tilde{y},\tilde{z})}{\partial \tilde{y}^2} \vec{\tilde{v}_y} \\ \frac{\partial^2 \tilde{f}(\tilde{x},\tilde{y},\tilde{z})}{\partial \tilde{z}^2} \vec{\tilde{v}_z} \end{bmatrix} \tag{31}$$

And finally, substituting equation (29):

$$\eta \nabla^2 \vec{v} = \frac{\eta V_0}{L_0^2} \tilde{\nabla}^2 \vec{\tilde{v}} \ \square \tag{32}$$

vii) The Reynolds Number (Re)

The Reynolds number is a scalar value which modulates the dimensionless velocity gradient:

$$Re = \rho \left(\frac{L_0 V_0}{\eta} \right) \tag{33}$$

In the following section we will see how this number is applied.

viii) Non-Dimensional Navier-Stokes Equations

Substituting equations (15), (16), (20), (27), and (32) into equation (10) gives:

$$\rho\left(\frac{\partial V_0\tilde{\vec{v}}}{\partial t_s\tilde{t}}+\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)\right)=-\frac{1}{L_0}\tilde{\vec{\nabla}}p_s\tilde{p}+\frac{\eta V_0}{L_0^2}\tilde{\nabla}^2\tilde{\vec{v}}+\rho\vec{F} \tag{34}$$

$$\rho\left(\frac{V_0}{t_s}\right)\left(\frac{\partial\tilde{\vec{v}}}{\partial\tilde{t}}\right)+\rho\left(\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)\right)=-\frac{1}{L_0}\tilde{\vec{\nabla}}p_s\tilde{p}+\frac{\eta V_0}{L_0^2}\tilde{\nabla}^2\tilde{\vec{v}}+\rho\vec{F} \tag{35}$$

Dividing by the highest order derivative term and assuming negligible contribution from body forces:

$$\left(\frac{L_0^2}{\eta V_0}\right)\rho\left(\frac{V_0}{t_s}\right)\left(\frac{\partial\tilde{\vec{v}}}{\partial\tilde{t}}\right)+\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)=\left(\frac{L_0^2}{\eta V_0}\right)\left(-\frac{1}{L_0}\tilde{\vec{\nabla}}p_s\tilde{p}\right)+\left(\frac{L_0^2}{\eta V_0}\right)\frac{\eta V_0}{L_0^2}\tilde{\nabla}^2\tilde{\vec{v}} \tag{36}$$

$$\rho\left(\frac{L_0^2}{\eta}\right)\left(\frac{\partial\tilde{\vec{v}}}{\partial\tilde{t}}\right)+\rho\left(\frac{L_0V_0}{\eta}\right)\left(\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)\right)=-\frac{L_0}{\eta V_0}\tilde{\vec{\nabla}}p_s\tilde{p}+\tilde{\nabla}^2\tilde{\vec{v}} \tag{37}$$

And finally applying equation (33) to equation (37):

$$\rho\left(\frac{L_0^2}{\eta}\right)\left(\frac{\partial\tilde{\vec{v}}}{\partial\tilde{t}}\right)+Re\left[\tilde{\vec{v}}\left(\tilde{\vec{\nabla}}\cdot\tilde{\vec{v}}\right)\right]=-\frac{L_0}{\eta V_0}\tilde{\vec{\nabla}}p_s\tilde{p}+\tilde{\nabla}^2\tilde{\vec{v}}\ \blacksquare \tag{38}$$

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Per equation (33) Re is defined as:

$$Re = \rho \left(\frac{L_0 V_0}{\eta} \right)$$

Where L_0 and V_0 are critical length (hydrodynamic diameter) and critical velocity respectively, and ρ and η are the density and dynamic viscosity of the fluid respectively If pressure and temperature are assumed to be constant and body forces are assumed to be negligible, ρ and η can be assumed to be constants. Re is therefore dependent on two forces; viscosity forces dependent on ρ and η and inertial forces dependent on L_0 and V_0. We can construct two simplified mental models of the physical effect of Re using our intuition and own experience. We know that viscous fluids like honey have a greater tendency to 'stick' and less tendency to 'flow'. When honey does flow it is slow and smooth with little bubbles or turbulence. However, upon heating honey it becomes much easier to pour which can be modeled as increasing the fluid's velocity. The flow of hot honey is noticeably more chaotic with bubbles and turbulence. Another way to increase this observed chaos would be to increase the diameter of the flowing liquid; honey flowing from the tip of a squeeze bottle is much more smooth than honey flowing from an upturned jar.

In the first model where fluid flow is 'smooth', the system is dominated by viscosity forces ($Re < 1$) and the fluid flow is said to be laminar:

$$\frac{\rho}{\eta} > L_0 V_0 \tag{39}$$

The second model with 'chaotic' flow is dominated by inertial forces (Re>>1) and the flow is said to be turbulent:

$$\frac{\rho}{\eta} << L_0 V_0 \tag{40}$$

Intermediate values are neither fully turbulent nor fully laminar and have contributions by both forces.

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Hagen-Poiseuille Flow between Parallel Plates

Given a cross section perpendicular to the flow of the fluid, $\partial\mathcal{C}$, the flow is said to be under Hagen-Poiseuille conditions if:

1. *The system is translationally invariant in the direction of the flow*
2. *Pressure is applied in one direction*
3. *No body forces act upon the system*
4. *For a given cross-section of a pipe, $\mathcal{C}$, the velocity at any point along the wall, $\partial\mathcal{C}$, is zero (no-slip condition)*
5. *Pressure boundary conditions hold:* $p(t_i) = p_0 + \Delta p$, $p(t_f) = p_0$

i) Navier-Stokes Equations for Incompressible Liquids in Cartesian Coordinates

The convective term of the Navier Stokes equation (eq. 10) is:

$$\left(\vec{v} \cdot \vec{\nabla}\right) \vec{v}$$

If we choose the z-axis as our translationally invariant axis, we can ignore the x and y equations of the Navier-Stokes equations since they will not contribute to the flow of the system. Equation (10) reduces to:

$$\left(\vec{v}_z \cdot \vec{\nabla}_z\right) \vec{v}_z = v_x \left(\frac{\partial}{\partial x} v_z \vec{i}\right) + v_y \left(\frac{\partial}{\partial y} v_z \vec{j}\right) + v_z \left(\frac{\partial}{\partial z} v_z \vec{k}\right) \tag{41}$$

Since the velocity vector is dependent only on pressure, it will only point along the z-axis and be zero everywhere else. Thus equation (41) reduces to:

$$\left(\vec{v}_z \cdot \vec{\nabla}_z\right) \vec{v}_z = (0) \left(\frac{\partial}{\partial x} v_z \vec{i}\right) + (0) \left(\frac{\partial}{\partial y} v_z \vec{j}\right) + v_z \left(\frac{\partial}{\partial z} v_z \vec{k}\right) = v_z \left(\frac{\partial}{\partial z} v_z \vec{k}\right) \tag{42}$$

And finally, since the z-axis is translationally invariant, the change in velocity is zero:

$$\left(\vec{v}_z \cdot \vec{\nabla}_z\right) \vec{v}_z = (0) \left(\frac{\partial}{\partial z} v_z \vec{k}\right) = 0 \ \blacksquare \tag{43}$$

By equation (43), a system under Hagen-Poiseuille flow has no convection and is therefore laminar.

Since convection is zero under Hagen-Poiseuille flow, pressure is the only force which will be acting on our system. Additionally, pressure will only be acting along the z-axis. Applying equation (43) to equation (10):

$$0 = -\vec{\nabla} p + \eta \nabla^2 \vec{v_z} \tag{44}$$

$$\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} = \eta \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \tag{45}$$

$$\frac{\partial p}{\partial z} = \eta \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{46}$$

$$\frac{\partial p}{\partial z} = \eta \left(\frac{\partial^2 v_z}{\partial y^2} \right) \quad \blacksquare \tag{47}$$

ii) Derivation of Velocity

Given Hagen-Poiseuille flow, the velocity on $\partial \mathcal{C}$ is zero. Moving away from $\partial \mathcal{C}$ towards the center of the cross section $\mathcal{C}$ will increase the velocity with a maxima at the center of $\mathcal{C}$. We will define this center point as $y_0 = 0$. Physically, the decrease in velocity as we approach the wall of the pipe is due to sheer stress along the pipe walls increasing drag and therefore decreasing velocity. Since velocity is symmetrical about y_0 the change in velocity with respect to position will be zero, $\frac{\partial v_z}{\partial y} = 0$. Applying to equation (47) gives a simple second-order ordinary differential equation:

$$\int \eta \frac{\partial^2 v_z}{\partial y^2} dy = \int \frac{\partial p}{\partial z} dy \tag{48}$$

$$\eta \frac{\partial v_z}{\partial y} + C_0 = \frac{\partial p}{\partial z} y \tag{49}$$

$$C_0 = \left(\frac{\partial p}{\partial z} \right) \left(\frac{y_0}{\eta} \right) - \frac{\partial v_{z_0}}{\partial y} \tag{50}$$

$$C_0 = \left(\frac{\partial p}{\partial z} \right) \left(\frac{0}{\eta} \right) - 0 \tag{51}$$

$$C_0 = 0 \tag{52}$$

$$\therefore \frac{\partial p}{\partial z} y = \eta \frac{\partial v_z}{\partial y} \quad \square \tag{53}$$

The no-slip boundary condition states that the velocity on $\partial\mathcal{C}$ is zero. Let us define the vertical position at the boundary as y_1 and the velocity at the boundary as $v_{z_1} = 0$. Applying to equation (53) leaves a first-order ordinary differential equation:

$$\int \eta \frac{\partial v_z}{\partial y} dy = \int \frac{\partial p}{\partial z} y dy \tag{54}$$

$$v_z = \left(\frac{y^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) + C_1 \tag{55}$$

$$C_1 = v_{z_1} - \left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{56}$$

$$C_1 = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{57}$$

$$v_z = \left(\frac{y^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) - \left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{58}$$

$$\therefore v_z = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)\left(1 - \frac{y^2}{y_1^2}\right) \ \blacksquare \tag{59}$$

iii) Compliance with No-Slip Conditions

If the velocity is non-zero on the boundary of $\mathcal{C}$, then the velocity will not be symmetrical. Physically this is due to the fluid experiencing shear stress and drag across the wall which causes non-zero velocity at the wall ('slip') and non-symmetrical velocity. No-slip conditions hold that the velocity at the wall is zero. This is trivial to prove as $y^2 = y_1^2$ on $\partial\mathcal{C}$. Substituting into equation (59) gives:

$$v_z = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)\left(1 - \frac{y_1^2}{y_1^2}\right) \tag{60}$$

$$v_z = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)(1 - 1) \tag{61}$$

$$v_z = -\left(\frac{y_1^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)(0) \tag{62}$$

$$\therefore v_z = 0 \ \blacksquare \tag{63}$$

Hagen-Poiseuille Flow in a Circular Channel

i) Navier-Stokes Equation for Incompressible Liquids in Cylindrical Coordinates

Mapping the cartesian z-component from equation (10) to cylindrical coordinates gives:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\left(\frac{v_\theta}{r}\right)\left(\frac{\partial v_z}{\partial \theta}\right)+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta\nabla^2 v_z+\rho g_z \tag{64}$$

Given Hagen-Poiseuille flow this reduces to:

$$\frac{\partial p}{\partial z}=\eta\nabla^2 v_z \tag{65}$$

$$\frac{\partial p}{\partial z}=\eta\left[\frac{1}{r}\left(\frac{\partial}{\partial r}\right)\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2 v_z}{\partial \theta^2}\right)+\frac{\partial^2 v_z}{\partial z^2}\right] \tag{66}$$

$$\frac{\partial p}{\partial z}=\frac{\eta}{r}\left[\left(\frac{\partial}{\partial r}\right)\left(r\frac{\partial v_z}{\partial r}\right)\right] \tag{67}$$

$$\left(\frac{r}{\eta}\right)\left(\frac{\partial p}{\partial z}\right)dr=\partial\left(r\frac{\partial v_z}{\partial r}\right)\quad\square \tag{68}$$

ii) Derivation of Velocity

Equations (47) & (68) are analogous, so the method used to derive equation (53) may be used here as well. As with equation (47) let the boundary conditions be $r_0=0$, $v_{z_0}=0$:

$$\int\left(\frac{r}{\eta}\right)\left(\frac{\partial p}{\partial z}\right)dr=\int\partial\left(r\frac{\partial v_z}{\partial r}\right) \tag{69}$$

$$\left(\frac{r^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)=r\left(\frac{\partial v_z}{\partial r}\right)+C_0 \tag{70}$$

$$\because C_0=\left(\frac{r_0^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)-r_0\left(\frac{\partial v_z}{\partial r_0}\right)=0 \tag{71}$$

$$\therefore\left(\frac{r^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right)=r\left(\frac{\partial v_z}{\partial r}\right)\quad\square \tag{72}$$

Equations (72) & (53) are analogous, so the same method applies:

$$\int \left(\frac{r^2}{2\eta}\right)\left(\frac{\partial p}{\partial z}\right) dr = \int r \left(\frac{\partial v_z}{\partial r}\right) \tag{73}$$

$$v_z = \left(\frac{r^2}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right) + C_1 \tag{74}$$

$$C_1 = v_{z_1} - \left(\frac{r_1^2}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{75}$$

$$C_1 = -\left(\frac{r_1^2}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right) \tag{76}$$

$$\therefore v_z\left(r,\theta\right) = -\left(\frac{1}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right)\left(r_1^2 - r^2\right) \quad \blacksquare \tag{77}$$

iii) Compliance with No-Slip Conditions

By setting the radius $\left(r^2\right)$ to the boundary, from equation we get (77):

$$v_z\left(r,\theta\right) = -\left(\frac{1}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right)\left(r_1^2 - r_1^2\right) \tag{78}$$

$$v_z\left(r,\theta\right) = -\left(\frac{1}{4\eta}\right)\left(\frac{\partial p}{\partial z}\right)(0) \tag{79}$$

$$\therefore v_z\left(r,\theta\right) = 0 \quad \blacksquare \tag{80}$$

We can prove no-slip conditions in a circular pipe in cartesian coordinates as well. To convert back we set $\dfrac{\partial p}{\partial z} = \dfrac{\Delta p}{L_0}$, and $r^2 = a^2$. Since $\mathcal{C}$ is a circle, any point on $\partial\mathcal{C}$ can be used to derive the radius: $r = a = \sqrt{x_{boundary}^2 + y_{boundary}^2}$. If we make these substitutions into equation (77) it is trivial to prove compliance with no-slip boundary conditions:

$$\vec{v_z}\left(x,y\right) = -\left(\frac{\Delta p}{4\eta L_0}\right)\left(a^2 - \left(\sqrt{x_{boundary}^2 + y_{boundary}^2}\right)^2\right) \tag{81}$$

$$\vec{v_z}\left(x,y\right) = -\left(\frac{\Delta p}{4\eta L_0}\right)\left(a^2 - a^2\right) \tag{82}$$

$$\therefore \vec{v_z}\left(x,y\right) = 0 \quad \blacksquare \tag{83}$$

Hagen-Poiseuille Flow in a Rectangular Channel

Unlike a channel between parallel plates with theoretically 'zero' height and circular channels which are symmetrical in length and width, rectangular channels introduce asymmetry between the x and y axis. Therefore the velocity gradient will now have x and y components. As we have done previously, we will assume pressure is only applied in the z-direction and the z-axis is translationally invariant. Starting from equation (46), our velocity function becomes:

$$\nabla^2 \vec{v_z}\left(x,y\right) = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} \tag{84}$$

Unfortunately there is no known analytical solution to this differential equation and the best we can do is a Fourier approximation. Per the Fourier transform, any periodic function function $f\left(u\right),\ u \in [-D, +D]$ can be represented as a series of cosines and sines:

$$f\left(u\right) = \sum_{n=odd}^{\infty} b_n\left(u\right)\sin\left(\frac{n\pi u}{D}\right) + \sum_{n=even}^{\infty} a_n\left(u\right)\cos\left(\frac{n\pi u}{D}\right) \tag{85}$$

Let the y-axis be the height of the channel with bounds $0 \le y \le h$, let the x-axis be the width of the channel with bounds $-w_{1/2} \le x \le +w_{1/2}$ and finally let the z-axis be the direction of the flow. The sign of the width terms denotes which side of the midpoint $(x = 0)$ they are on. As we did in the last two sections, this will give boundary conditions $x \in \left[-w_{1/2}, +w_{1/2}\right]$, $y \in [0, h]$. As with the previous examples, velocity is constant across the y-axis (the height of the channel) but variable along the x-axis (the width of the channel). Using equation (85), the general form of the solution for equation (84) will be:

$$\vec{v_z}\left(x,y\right) = \sum_{n=odd}^{\infty} b_n\left(x\right)\sin\left(\frac{n\pi x}{h}\right) + \sum_{n=even}^{\infty} a_n\left(x\right)\cos\left(\frac{n\pi x}{h}\right) \tag{86}$$

The even expansion satisfies $f\left(-t\right) = f\left(t\right)$ whereas the odd expansion satisfies $f\left(-t\right) = -f\left(t\right)$. Only the odd terms will be used as they satisfy the boundary conditions:

$$\vec{v_z}\left(x,y\right) = \sum_{n=odd}^{\infty} b_n\left(x\right)\sin\left(\frac{n\pi x}{h}\right) \tag{87}$$

Subbing equation (87) into equation (84) gives:

$$\nabla^2 \vec{v_z}\,(x,y) = \sum_{n=odd}^{\infty} \left(\frac{\partial^2 v_z}{\partial x^2} \left[b_n\,(x) \sin(\frac{n\pi x}{h}) \right] + \left[\frac{\partial^2 v_z}{\partial y^2} b_n\,(x) \sin(\frac{n\pi x}{h}) \right] \right) \tag{88}$$

$$\nabla^2 \vec{v_z}\,(x,y) = \sum_{n=odd}^{\infty} \frac{\partial v_z}{\partial x} \left[b_n^{'}\,(x) \sin(\frac{n\pi x}{h}) + b_n^{'}\,(x)\, \frac{n\pi x}{h} \cos(\frac{n\pi x}{h}) \right] \tag{89}$$

$$\nabla^2 \vec{v_z}\,(x,y) = \sum_{n=odd}^{\infty} \left[b_n^{''}\,(x) \sin(\frac{n\pi x}{h}) - b_n\,(x)\, \frac{n^2\pi^2x^2}{h^2} \sin(\frac{n\pi}{h}) \right] + \sum_{n=even}^{\infty} 2 \left[b^{'}\,(x)\, \frac{n\pi x}{h} \cos(\frac{n\pi x}{h}) \right] \tag{90}$$

Since we are only concerned with the odd terms, the equation reduces to:

$$\nabla^2 \vec{v_z}\,(x,y) = \sum_{n=odd}^{\infty} \left[b_n^{''}\,(x) - b_n\,(x)\, \frac{n^2\pi^2x^2}{h^2} \right] \sin(\frac{n\pi x}{h}) \tag{91}$$

To make our solution satisfy the boundary conditions, we must do a Fourier expansion of the pressure term, which is a constant. The general form for a sine approximation of a constant is:

$$U = \sum_{n=odd}^{\infty} \frac{4U}{n\pi}\,(U) \sin(\frac{n\pi u}{D}) \tag{92}$$

Now let us sub in the bounds described above, the L.H.S of equation (46), and using the relationship for the pressure differential described in equation (83).

$$\frac{\Delta p}{4\eta L_0} = \sum_{n=odd}^{\infty} \left(\frac{\Delta p}{\eta L_0} \right) \left(\frac{4}{n\pi} \right) \sin(\frac{n\pi x}{h}) \tag{93}$$

The velocity any point not at the wall will be bound by $-w_{1/2} < x < -w_{1/2}$ and $0 < y < h$. Subbing equations (92) and (94) into equation (46):

$$\sum_{n=odd}^{\infty} \left[b_n^{''}\,(x) - b_n\,(x)\, \frac{n^2\pi^2x^2}{h^2} \right] \sin(\frac{n\pi x}{h}) = \sum_{n=odd}^{\infty} \left(\frac{\Delta p}{\eta L_0} \right) \left(\frac{4}{n\pi} \right) \sin(\frac{n\pi x}{h}) \tag{94}$$

$$\left[b_n^{''}\,(x) - b_n\,(x)\, \frac{n^2\pi^2x^2}{h^2} \right] \sin(\frac{n\pi x}{h}) = \left(\frac{\Delta p}{\eta L_0} \right) \left(\frac{4}{n\pi} \right) \sin(\frac{n\pi x}{h}) \tag{95}$$

$$b_n^{''}\,(x) - b_n\,(x)\, \frac{n^2\pi^2x^2}{h^2} = \left(\frac{\Delta p}{\eta L_0} \right) \left(\frac{4}{n\pi} \right) \quad \square \tag{96}$$

Which per Bruss gives the following solution for velocity:

$$\vec{v_z}\,(x,y) = \sum_{n=odd}^{\infty} \left[1 - \frac{\cosh(\frac{n\pi x}{h})}{\cosh(\frac{n\pi w}{2h})} \right] \frac{4h^2 \Delta p \sin(n\pi \frac{y}{h})}{\pi^3 \eta L_0 n^3} \quad \blacksquare \tag{97}$$

4

Hagen-Poiseuille's Law

i) Derivation of Hagen-Poiseuille's Law in a Circular Channel

For the sake of simplicity, we will derive Hagen-Poisseuille's Law in a cirular channel. First we must derive the flow rate, 'Q', which is obtained by integrating the velocity over the area of our cross-section $\mathcal{C}$. We must define some terms before performing the integration. First, let us define the bounds of integration on $\mathcal{C}$. Given a circular cross-section, the bounds will be from the the minimum radius in the center of $\mathcal{C}$, $r_{min} = 0$, to the maximum radius on $\partial\mathcal{C}$, $r_{max} = R$. Next we will make the substitution $\frac{\partial p}{\partial z} = \frac{\Delta p}{L_0}$. Finally, we must define our area of integration and the Jacobian (scaling factor). The area of integration, dA, will be 2π and the Jacobian for cylindrical coordinates is simply $J(r, \theta, z) = r$. Now we are ready to derive Q:

$$Q = -\int_{r_{min}}^{r_{max}} v_z J(r, \theta, z)\, dA \tag{98}$$

$$= -\int_{r=0}^{r=R} -\left(\frac{1}{4\eta}\right)\left(\frac{\Delta p}{L_0}\right)\left(R^2 - r^2\right) r 2\pi dr \tag{99}$$

$$= \left(\frac{\pi}{2\eta}\right)\left(\frac{\Delta p}{L_0}\right)\int_{r=0}^{r=R}\left(R^2 r - r^3\right) dr \tag{100}$$

$$= \left(\frac{\pi}{2\eta}\right)\left(\frac{\Delta p}{L_0}\right)\left(\frac{R^2 r^2}{2} - \frac{r^4}{4}\right)\Big|_{r=0}^{r=R} \tag{101}$$

$$= \left(\frac{R^4 \pi}{8\eta}\right)\left(\frac{\Delta p}{L_0}\right) \quad \square \tag{102}$$

It is easy to see how Δp of a system is proportional to the rate of fluid flow and how easily the fluid can pass through the medium of $\partial\mathcal{C}$. This ability of a fluid to pass through a medium is called 'hydraulic resistance', $R_{hyd} = \frac{8\eta L_0}{R^4\pi}$, multiplied by the flow rate, Δp, is the generalized Hagen-Poiseuille Law. Starting from equation (89):

$$\frac{QL_0}{\Delta p} = \frac{R^4\pi}{8\eta} \tag{103}$$

$$\Delta p = \frac{8\eta L_0}{R^4\pi} Q \tag{104}$$

$$\Delta p = R_{hyd} Q \quad \blacksquare \tag{105}$$

Though a circular channel was used to derive Hagen-Poiseuille's law, it still holds irrespective of the shape of the channel. That being said, since the velocity vector is dependent on the shape of the channel, so will the exact derivation of hydraulic resistance and flow rate.

ii) Experimental Verification of Hagen-Poiseuille's Law

$\boldsymbol{\Delta p\,(mbar)}$	$\boldsymbol{H_2O_{mass}\,(g)}$
50	0.0348
250	0.2327
500	0.484
750	0.775
950	0.9061

Table 1: *Experimental Data*

The data above was collected by applying pressure across a microfluidic device with a rectangular channel and collecting the mass of water at the outlet for 300s. The width of the channel is $650\mu m$ and its height is $30\mu m$, and the temperature of the water was $20°\mathrm{C}$, $\eta = 1.0(mPa)s$. Plugging in the first row of values from table 1, we can use equation (105) to calculate the volumetric flow rate, 'Q', and the hydraulic resistance, 'R_{hyd}' :

$$Q = \frac{(H_2O_g)\left(m^3\right)}{(seconds)\,(10^5g)} \tag{106}$$

$$Q = \frac{(0.0348g)\left(m^3\right)}{(300s)\,(10^5g)} = 1.16\times10^{-9}\left({}^{m^3}\!/_{s}\right) \quad \square \tag{107}$$

$$R_{hyd} = \frac{(\Delta p)\left(\frac{100Pa}{mbar}\right)}{(seconds)\,(10^5g)} \tag{108}$$

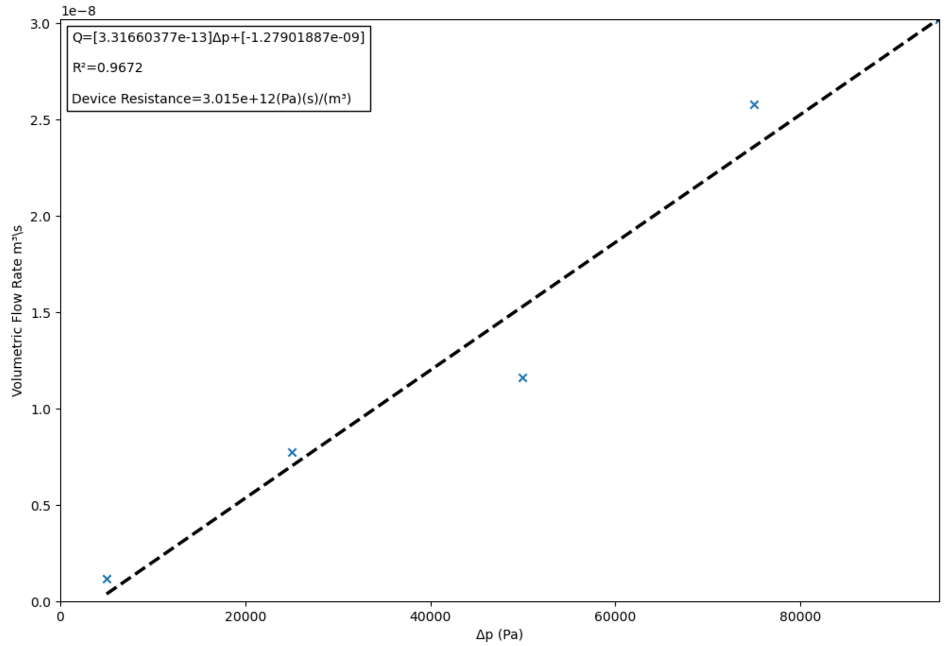
$$R_{hyd} = \frac{(50mbar)\left(\frac{100Pa}{mbar}\right)}{\left(\frac{0.0348g}{300s}\right)\left(\frac{m^3}{10^5g}\right)} = 4.31\times10^{12}\left({}^{Pa\ s}\!/_{m^3}\right) \quad \square \tag{109}$$

$\boldsymbol{\Delta p\,(Pa)}$	$\boldsymbol{Q\,(m^3s^{-1})}$	$\boldsymbol{R_{hyd}\,(Pa)\,(s)\,(m^{-3})}$
5000	1.16×10^{-9}	4.31×10^{12}
25000	7.76×10^{-9}	3.22×10^{12}
50000	1.61×10^{-8}	3.10×10^{12}
75000	2.58×10^{-8}	2.90×10^{12}
95000	3.02×10^{-8}	3.14×10^{10}

Table 2: *Pressure, Volumetric Flow Rate, and Hydraulic Resistance*

By plotting the volumetric flow rate against pressure and performing a linear regression, we get:

$$Q = \frac{1}{R_{device}} \times \Delta p + b \tag{110}$$



Hydraulic resistance is analogous to electrical resistance. As such, total hydraulic resistance in series is the sum of each 'resistor'and in parallel the resistance is the sum of the reciprocal of each resistor. If we were to add a tube of a specific resistance to the above device, the total resistance would increase by the tube's hydraulic resistance R_{tubes}. However, the resistance of the tubes, $R_{tubes} = 1 \times 10^{8}\,(Pa)\,(s)\,m^{-3}$, is so much lower than the device's resistance it would relate to a negligible increase in resistance.

Given a pressure of 150mbar, we can use equation (110) to estimate Q:

$$Q = \frac{\Delta p(Pa)}{3.31660377 \times 10^{13 \, {}^{(Pa)(s)}/_{m^3}}} - 1.27901887 \times 10^{-9}\left({}^{m^3}/_{s}\right) \tag{111}$$

$$Q = \frac{(150mbar)\left(\dfrac{100Pa}{mbar}\right)}{3.31660377 \times 10^{13 \, {}^{(Pa)(s)}/_{m^3}}} - 1.27901887 \times 10^{-9}\left({}^{m^3}/_{s}\right) \tag{112}$$

$$Q = 3.70 \times 10^{-09}\left({}^{m^3}/_{s}\right) \quad \square \tag{113}$$

We may also convert volumetric flow rate $Q = \left({}^{m^3}\!/_{s}\right)$ to average linear flow velocity $V_0 = ({}^{m}\!/_{s})$. Doing this will allow us to calculate the Reynold's number. To perform this conversion, we must divide Q by the cross-sectional area of the channel:

$$V_0 = \frac{Q}{\textit{channel width} \times \textit{channel height}} \tag{114}$$

$$V_0 = \frac{3.70 \times 10^{-09} \left({}^{m^3}\!/_{s}\right)}{\dfrac{(650\mu m)\,(m)}{10^6 \mu m} \times \dfrac{(30\mu m)\,(m)}{10^6 \mu m}} \tag{115}$$

$$V_0 = 0.190\,({}^{m}\!/_{s}) \quad \square \tag{116}$$

By equation (33) the Reynolds number is:

$$Re = \rho \left(\frac{L_0 V_0}{\eta}\right)$$

For non-circular channels, oftentimes hydraulic diameter (D_H) is used in place of L_0:

$$L_0 = D_H = \frac{2wh}{w+h} \tag{117}$$

Substituting equations (116) & (117) into equation (33) gives us:

$$Re = \rho \left(\frac{D_H V_0}{\eta}\right) \tag{118}$$

$$Re = 2 \times \left(\frac{\dfrac{(650\mu m)\,(m)}{10^6 \mu m} \times \dfrac{(30\mu m)\,(m)}{10^6 \mu m}}{\dfrac{(650\mu m)\,(m)}{10^6 \mu m} + \dfrac{(30\mu m)\,(m)}{10^6 \mu m}} \right) \times \frac{0.190\,({}^{m}\!/_{s})}{\left(\dfrac{1.0(mPa)(s) \times (Pa)(s)}{10^3 (Pa)(s)}\right)\left(\dfrac{kg}{(m)(s)}\right)} \times \left(\frac{998.2kg}{m^3}\right) \tag{119}$$

$$Re \approx 10.9 \quad \square \tag{120}$$

In microfluidics, laminar flow occurs when $Re < 2000$. Therefore the experimental microfluidic device was experiencing laminar flow.

5

Theoretical Application of an H-Filter

The time it takes a solute with diffusion coefficient D to diffuse across the width of a T-mixer is:

$$T \approx \frac{w^2}{D} \tag{121}$$

And the distance travelled in that time is:

$$Z = V_0 T = \frac{V_0 w^2}{D} \tag{122}$$

Combining equations (121) and (122) gives us the Peclet number, or the number of channel widths required for complete mixing of a solute:

$$Pe = \frac{Z}{w} = \frac{V_0}{D} \tag{123}$$

In principle, T-mixers and H-filters are quite similar. However, unlike T-mixers which only have one outlet, the goal of an H-filter is to separate solutes of interest from a matrix. To understand how this works, two critical times must be taken into account: the time it takes for convective forces to induce advection of a solute compared to the time it takes for diffusive forces to induce diffusion of the solute across the width of the channel:

$$\tau_{\text{conv}} = \frac{L}{V_0} \tag{124}$$

$$\tau_{\text{diff}} = \frac{\left({}^{w}\!/_{2}\right)^2}{D} = \frac{w^2}{4D} \tag{125}$$

If $\tau_{\text{conv}} \ll \tau_{\text{diff}}, \;\; (Pe < 1)$, the solutes will largely be carried with the original buffer into the waste stream (ie advection of solute dominates). On the other hand, if $\tau_{\text{conv}} \geq \tau_{\text{diff}} \;\; (Pe \geq 1)$, the solutes will be mixed into the extraction stream (ie diffusion of solute dominates). By setting $\tau_{\text{conv}} = \tau_{\text{diff}}$ and subbing equations (124) & (125) we can derive the minimum length required for extraction of a solute:

$$\tau_{\text{conv}} = \tau_{\text{diff}} \tag{126}$$

$$\frac{L}{V_0} = \frac{w^2}{4D} \tag{127}$$

$$L = \frac{w^2 V_0}{4D} \;\; \blacksquare \tag{128}$$

Given a blood sample, three fractions are of interest: dissolved ions and solutes $\left(D_1 = 2 \times 10^{-9} m^2 s^- 1\right)$, proteins $\left(D_2 = 6 \times 10^{-11} m^2 s^- 1\right)$ and red, white, and platelet cells $\left(D_3 = 2 \times 10^{-14} m^2 s^{-1}\right)$. Let our theoretical H-filter have an average linear flow velocity of $V_0 = 1\,({}^{mm}\!/_{s})$, a height of $h = 20\mu m$ a width of $w = 100\mu m$, and a length ≤ 50mm.

To determine the length of the H-filter needed which would only extract the dissolved ions, solutes, we would substitute D_1 into equation (128):

$$L = \frac{\left(100\mu m \frac{m}{10^6 \mu m}\right)^2 \left(\frac{1mm}{s}\right)\left(\frac{m}{10^3 mm}\right)}{4\left(2 \times 10^{-9} m^2 s{-1}\right)} \tag{129}$$

$$L = 1.25mm \ \square \tag{130}$$

If we wished to additionally extract the protein fraction, we would set D to D_2, as $D_1 \ll D_2 \ll D_3$:

$$L = \frac{\left(100\mu m \frac{m}{10^6 \mu m}\right)^2 \left(\frac{1mm}{s}\right)\left(\frac{m}{10^3 mm}\right)}{4\left(6 \times 10^{-11} m^2 s{-1}\right)} \tag{131}$$

$$L \approx 41.7mm \ \square \tag{132}$$

Finally, if we wished to extract all three fractions:

$$L = \frac{\left(100\mu m \frac{m}{10^6 \mu m}\right)^2 \left(\frac{1mm}{s}\right)\left(\frac{m}{10^3 mm}\right)}{4\left(2 \times 10^{-14} m^2 s^{-1}\right)} \tag{133}$$

$$L = 125m \ \square \tag{134}$$

Given these critical lengths and the maximum length allowed, an H-filter with a length of 41.7mm would be able to extract the dissolved ions, solutes, and proteins fractions. Since diffusion is a stochastic process, 100% recovery of the solute(s) of interest impossible. Though the H-filter can be optimized to attain near ideal extraction, the process will never recover all of the solutes. Additionally, some waste will make it through into the extraction stream and reduce the purity of the extracted solutes.