

# Matter Waves

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## 1 Cool Integral Trick

Before starting the actual notes, here's a fun integral. It's particularly useful for solving all those problems where you average something over a probability distribution. It assumes  $\text{Re}(a) > 0$  and  $n \in \mathbb{N}$ .

$$\begin{aligned}\int_0^\infty x^n e^{-x/a} dx &= \int_0^\infty (a^n y^n) e^{-y} (a dy) \\ &= a^{n+1} \int_0^\infty y^n e^{-y} dy \\ &= a^{n+1} \Gamma(n+1) \\ &= n! a^{n+1}\end{aligned}$$

## 2 Another Unrelated Topic

We should memorize this definition of the fine-structure constant:

$$\alpha := \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{e^2}{2\epsilon_0\hbar c} \approx \frac{1}{137}$$

It was also recommended that we remember the following approximations:

$$\begin{aligned}\hbar c &\approx 197 \text{ eV nm} \\ \frac{e^2}{4\pi\epsilon_0} &\approx 1.44 \text{ eV nm}\end{aligned}$$

## 3 Dispersion Relations

This is pretty much all we need to know about dispersion relations (for now):

$$\begin{aligned}v_{\text{phase}} &= \omega/k \\ v_{\text{group}} &= \frac{\partial\omega}{\partial k}\end{aligned}$$

For all matter waves, the geometric mean of the phase velocity and the group velocity is  $c$ , the speed of light. For non-dispersive waves, such as light traveling through a vacuum, the phase and group velocities are both  $c$ .

## 4 Uncertainty Principle

Suppose you have some signal, which is a function of time, and you measure  $n$  samples from it, which are evenly spaced time  $\Delta t$  apart. Now you know that that signal exists within a window of time with width  $n\Delta t$ , so that represents your “uncertainty in time”. But suppose you also want to measure the frequency of the signal. According to the Nyquist-Shannon sampling theorem, the highest frequency you can extract from your samples is  $1/(2\Delta t)$ . Trying to measure the frequencies that exist in the signal is equivalent to trying to reconstruct the signal using the sum of several sinusoids, and in this case, you need  $n$  sinusoids whose frequencies are evenly spaced, ranging from zero to  $1/(2\Delta t)$  (including the upper bound, but not including zero). That means the lowest frequency you can measure using the  $n$  samples is  $1/(2n\Delta t)$ , which represents your “uncertainty in frequency”.

THIS SECTION IS INCOMPLETE – STILL NEED TO FIGURE OUT WHEN AND HOW THE FACTORS OF  $2\pi$  COME IN, AND GENERALIZE THE UNCERTAINTY PRINCIPLE TO WORK WITH SPACE AS WELL AS TIME, AND WITH ENERGY AND MOMENTUM.

## 5 Fourier Inversion Theorem

The inverse Fourier transform of the Fourier transform of  $f(x)$  is

$$\begin{aligned}\mathcal{F}^{-1}(\mathcal{F}(f))(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') e^{-ikx'} dx' \right) e^{ikx} dk \\ &= \int_{-\infty}^{\infty} f(x') \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(x-x')} dk \right) dx' \\ &= \int_{-\infty}^{\infty} f(x') \delta(x - x') dx' \\ &= f(x)\end{aligned}$$

This proves that  $\mathcal{F}^{-1}$  and  $\mathcal{F}$ , as we have defined them, are indeed inverses, which is good. That proof above uses the Dirac  $\delta$  “function”, which is more like the limit of a function than an actual function, but it makes everything work out nicely, so we leave out the nitty-gritty details.