Matter Waves

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1 Cool Integral Trick

Before starting the actual notes, here's a fun integral. It's particularly useful for solving all those problems where you average something over a probability distribution. It assumes $\mathrm{Re}(a) > 0$ and $n \in \mathbb{N}$.

$$\begin{aligned}\int_0^\infty x^n e^{-x/a} dx &= \int_0^\infty (a^n y^n) e^{-y} (a\, dy) \\ &= a^{n+1} \int_0^\infty y^n e^{-y} dy \\ &= a^{n+1} \Gamma(n+1) \\ &= n!\, a^{n+1}\end{aligned}$$

2 Another Unrelated Topic

We should memorize this definition of the fine-structure constant:

$$\alpha := \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{e^2}{2\varepsilon_0 h c} \approx \frac{1}{137}$$

It was also recommended that we remember the following approximations:

$$\hbar c \approx 197 \text{ eV nm}$$

$$\frac{e^2}{4\pi\varepsilon_0} \approx 1.44 \text{ eV nm}$$

3 Dispersion Relations

This is pretty much all we need to know about dispersion relations (for now):

$$v_{\text{phase}} = \omega / k$$

$$v_{\text{group}} = \frac{\partial \omega}{\partial k}$$

For all matter waves, the geometric mean of the phase velocity and the group velocity is c, the speed of light. For non-dispersive waves, such as light traveling through a vacuum, the phase and group velocities are both c.

4 Uncertainty Principle

Suppose you have some signal, which is a function of time, and you measure n samples from it, which are evenly spaced time Δt apart. Now you know that that signal exists within a window of time with width $\sigma_t := n\Delta t$, so that represents your "uncertainty in time". But suppose you also want to measure the frequency of the signal. According to the Nyquist-Shannon sampling theorem, the highest frequency you can extract from your samples is $1/(2\Delta t)$. Trying to measure the frequencies that exist in the signal is equivalent to trying to reconstruct the signal using the sum of several sinusoids, and in this case, you need n sinusoids whose frequencies are evenly spaced, ranging from zero to $1/(2\Delta t)$ (including the upper bound, but not including zero). That means the lowest frequency you can measure using the n samples is $\sigma_f := 1/(2n\Delta t)$, which represents your "uncertainty in frequency". This allows you to write the uncertainty principle, which is true no matter what you choose n and Δt to be.

$$\sigma_t \sigma_f \geq \frac{1}{2}$$

However, if you define σ to be the standard deviation of a function, instead of the width of the window where the function is non-zero, then you can get an even smaller bound. In next week's notes, we'll see how to minimize that bound and do a whole bunch of other cool stuff with that.

5 Fourier Inversion Theorem

The inverse Fourier transform of the Fourier transform of $f(x)$ is

$$\begin{aligned}
\mathcal{F}^{-1}(\mathcal{F}(f))(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') e^{-ikx} dx' \right) e^{ikx} dk \\
&= \int_{-\infty}^{\infty} f(x') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(x-x')} dk \right) dx' \\
&= \int_{-\infty}^{\infty} f(x') \delta(x - x') dx' \\
&= f(x)
\end{aligned}$$

This proves that $\mathcal{F}^{-1}$ and $\mathcal{F}$, as we have defined them, are indeed inverses, which is good. That proof above uses the Dirac δ "function", which is more like the limit of a function than an actual function, but it makes everything work out nicely, so we leave out the nitty-gritty details.