Midterm Notes

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1 Notecard for midterms & final

This document has all of the most important equations to include on your notecard EXCEPT for the stuff that was in notes from previous weeks (e.g. the Maxwell-Boltzmann speed distribution, the blackbody radiation equation, et cetera).

2 Partition function Z

$$Z := \sum_i g_i e^{-\beta E_i} = \int g e^{-\beta E} dE$$

$$E_{avg} = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} = \frac{\int E e^{-\beta E} dE}{\int e^{-\beta E} dE} = -\frac{\partial}{\partial \beta} \ln Z$$

3 Spherical coords

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \\ dx\,dy\,dz &= r^2 \sin\theta\, dr\, d\theta\, d\phi \end{aligned}$$

4 Summation identities

The following Taylor series converges on $x \in (-1, 1)$.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Differentiate that and multiply by x to get this formula:

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

5 Binomial coefficients

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

6 Energy-momentum equation

In general, I'll use m to denote the rest mass, and γm to mean the relativistic mass.

$$E^2 = m^2c^4 + p^2c^2$$

In that equation, E is the kinetic energy plus the rest energy. The momentum is $p = \gamma m v$, and the rest energy is mc^2, so the kinetic energy must be

$$\begin{aligned}
K &= E - mc^2 \\
&= \sqrt{m^2c^4 + p^2c^2} - mc^2 \\
&= \sqrt{m^2c^4 + \frac{m^2v^2c^2}{1 - \frac{v^2}{c^2}}} - mc^2 \\
&= mc\sqrt{\frac{(c^2 - v^2) + v^2}{1 - \frac{v^2}{c^2}}} - mc^2 \\
&= mc^2\left(\sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} - 1\right) \\
&= (\gamma - 1)mc^2
\end{aligned}$$

This makes sense, because then the total energy is $E = \gamma mc^2$.

7 Maxwell's equations

Let $D := \varepsilon_0 E + P$ where P is the polarization field, and let $H := \frac{B}{\mu_0} - M$ where M is the magnetization field. Then the following version of Maxwell's equations work even if you aren't in a vacuum:

$$\begin{aligned}
\nabla \cdot D &= \rho \\
\nabla \cdot B &= 0 \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \times H &= J + \frac{\partial D}{\partial t}
\end{aligned}$$

If you're in a vacuum, you can use the identity $\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$ to derive the 3D wave equation for electromagnetic waves:

$$\nabla^2 E = \frac{1}{c^2} \cdot \frac{\partial^2 E}{\partial t^2}$$

8 Maxwell-Boltzmann speed distribution

$$\begin{aligned}\text{probability density} &= 4\pi v^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{mv^2}{2k_B T}} dv \\ &= 4\sqrt{E/\pi}\left(\frac{1}{k_B T}\right)^{3/2} e^{-E/k_B T} dE\end{aligned}$$

mean speed: $\sqrt{\frac{8kT}{\pi m}}$
RMS speed: $\sqrt{\frac{3kT}{m}}$
mode speed: $\sqrt{\frac{2kT}{m}}$
TODO: ADD MEAN, RMS, AND MODE ENERGIES TOO

9 Wien's law

$$\lambda_{\max} T(\lambda_{\max}) = 2.898 \times 10^{-3} mK$$

10 Angular frequency version of Fourier transform

If f is a function of x:

$$(\mathcal{F}f)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

If f is a function of k:

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(k) dk$$

11 Bohr model

$$\begin{aligned} r &= \frac{n^2 h^2 \varepsilon_0}{\pi m Z e^2} \approx & \frac{n^2}{Z}(5.292 \times 10^{-11} \text{ m}) \\ v &= \frac{Z e^2}{2 n \varepsilon_0 h} \approx & \frac{Z}{n}(2.188 \times 10^{6} \text{ m/s}) \end{aligned}$$

12 Constants

$$
\begin{aligned}
k_B &\approx 1.381 \times 10^{-23}\,\frac{\mathrm{J}}{\mathrm{K}} \approx 8.617 \times 10^{-5}\,\frac{\mathrm{eV}}{\mathrm{K}} \\
h &\approx 6.626 \times 10^{-34}\ \mathrm{J\ s} \approx 4.136 \times 10^{-15}\ \mathrm{eV\ s} \\
\hbar := \frac{h}{2\pi} &\approx 1.055 \times 10^{-34}\ \mathrm{J\ s} \approx 6.582 \times 10^{-16}\ \mathrm{eV\ s} \\
c &\approx 2.998 \times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}^2} \\
k := \frac{1}{4\pi\varepsilon_0} &\approx 8.988 \times 10^{9}\,\frac{\mathrm{N\ m}^2}{\mathrm{C}^2} \\
\varepsilon_0 &\approx 8.854 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}} \\
\mu_0 &\approx 1.257 \times 10^{-6}\,\frac{\mathrm{N}}{\mathrm{A}^2} \\
1\ \mathrm{mol} &\approx 6.022 \times 10^{23}\ \text{molecules} \\
\alpha := \frac{e^2}{2\varepsilon_0 hc} &\approx 0.007297 \approx \frac{1}{137} \\
a_0 = \frac{\varepsilon_0 h^2}{\pi e^2 m_e} &\approx 5.292 \times 10^{-11}\ \mathrm{m} \\
G &\approx 6.674 \times 10^{-11}\,\frac{\mathrm{N\ m}^2}{\mathrm{kg}^2} \\
e &\approx 1.602 \times 10^{-19}\ \mathrm{C} \\
m_e &\approx 9.109 \times 10^{-31}\ \mathrm{kg} \approx 0.511\ \mathrm{MeV/c}^2 \\
m_p &\approx 1.673 \times 10^{-27}\ \mathrm{kg} \approx 1836\, m_e \approx 938.3\ \mathrm{MeV/c}^2 \\
\sigma := \frac{2\pi^5 k^4}{15 h^3 c^2} &\approx 5.670 \times 10^{-8}\,\frac{W}{m^2 K^4} \\
R &\approx \frac{25}{3}\,\frac{J}{mol \cdot K}
\end{aligned}
$$

13 Other random stuff

- Average energy at 0 Kelvin is 3/5 of the Fermi energy (prove by integrating over spherical shells in k-space)
- Bragg scattering: $n\lambda = 2d \sin\theta$
- $p = \hbar k = h/\lambda_{\text{de Broglie}}$
- $E = hf = \hbar\omega = \frac{hc}{\lambda} = \frac{\hbar c}{k}$
- $\lambda k = 2\pi$

- If the work function ϕ is $e \cdot V_{\text{stopping}}$, then $KE_e = \frac{hc}{\lambda} - \phi$
- $v_{\text{phase}} = c\sqrt{1 + \left(\frac{mc}{\hbar k}\right)^2}$
- $dN = (4\pi k^2 dk)\left(\frac{L}{2\pi}\right)^3$ (# of polarization states)