Geometric Algebra

Nathan Solomon

October 10, 2022

1 General rules

These should work no matter what the degree of a and b are. Note that the degree of a multivector is only defined if all its terms have the same degree.

- Associativity: $(ab)c = a(bc)$
- Left distributivity: $a(b+c) = ab + ac$
- Right distributivity: $(a+b)c = ac + bc$
- The dot product and wedge product are also both left and right distributive
- Scalar multiplication (let λ be a scalar): $\lambda a = a\lambda = \lambda \cdot a = a \cdot \lambda = \lambda \wedge a = a \wedge \lambda$
- Dot product commutativity: $a \cdot b = b \cdot a$
- Wedge product anticommutativity: $a \wedge b = (-1)^{\deg(a)\deg(b)} b \wedge a$
- $\deg(a \cdot b) = |\deg(a) - \deg(b)|$
- $\deg(a \wedge b) = \deg(a) + \deg(b)$ (if the degree of a multivector is greater than the dimension of the vector space it lives in, it must be zero. For example, any trivector in 2D will be written in terms of $e_1 \wedge e_1$ and $e_2 \wedge e_2$, which are both zero)

2 Rules that work for blades (but not all multivectors)

Terminology: a k-blade is a multivector that can be written as the wedge product of k vectors. A multivector is any combination of blades – for example, $e_1e_2 + e_3e_4 \in G(4,0)$ and $1 + e_1 \in G(1,0)$ are multivectors but not blades.

- Euclidean metric: $a^2 = ||a||^2$ where $||a||$ is a scalar
- Factorability: if $\deg(a) = x + y$, then a can be written as the geometric product of an x-blade and a y-blade.

3 Rules that might not work unless a and b are vectors

- $ab = a \cdot b + a \wedge b$
- $a \wedge b = -b \wedge a$
- $a \cdot b = \frac{ab+ba}{2}$
- $a \wedge b = \frac{ab-ba}{2}$

4 Unit pseudoscalar I

Let $I \in G(n, 0) = e_1 \wedge e_2 \wedge \cdots \wedge e_n$. Then I commutes with everything. If $n \in (4\mathbb{Z} + \{0, 1\})$, we have the very nice property that $I^{-1} = I$, meaning that taking the dual of something twice has no net effect. If $n \in (4\mathbb{Z} + \{2, 3\})$, we have the elegant-but-not-as-nice property that $I^{-1} = -I$, meaning that the unit pseudoscalar I behaves like the imaginary unit $i = \sqrt{-1}$. Using geometric algebras in mischievous ways, we can coerce them into representing complex numbers, or even quaternions.

Reversing a list of n unique elements by swapping adjacent elements requires $n(n-1)/2$ swaps (you can prove that with induction). That number of swaps is even if $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$, and odd otherwise. Thus, the permutation that reverses $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ is even if $n \in (4\mathbb{Z} + \{0, 1\})$ and odd otherwise, which is why that property above works.

4.1 Cross product & wedge product

The cross product of vectors exists only in $\mathbb{R}^3$, so the following properties work only in $G(3, 0)$.

$$I(a \times b) = a \wedge b$$

$$a \times b = -I(a \wedge b) = I(b \wedge a)$$

5 Cool application of combinations

The set of multivectors of degree p in $\mathbb{F}^n$ has dimension $\binom{n}{p} = \frac{n!}{p!(n-p)!}$ over $\mathbb{F}$. Thus, the set of all multivectors in $\mathbb{F}^n$ has dimension 2^n. For example, the set of multivectors in $\mathbb{R}^3$ has the basis $\{1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, e_1e_2e_3\}$ over $\mathbb{R}$, which matches the 1-3-3-1 row from Pascal's triangle.