Review

Nathan Solomon

December 6, 2022

1 Notecard for midterms & final

This document has all of the most important equations to include on your notecard EXCEPT for the stuff that was in notes from previous weeks (e.g. the Maxwell-Boltzmann speed distribution, the blackbody radiation equation, et cetera).

2 Virial Theorem

For a system in which the potential energy due to each pair of particles is proportional to r^n, where r is the distance between the particles and n is an integer, the total kinetic energy (averaged over a long time) of the system is $n/2$ times the potential energy (averaged over a long time) of the system. Notably, $n = 2$ for a harmonic oscillator, and $n = -1$ for a gravitational system or a hydrogen atom.

We can use this to find that for a giant cloud of N gas particles dominated by gravitational force, the specific heat is

$$C = -\frac{3}{2} N k_B$$

3 Lorentz Invariants

Define the following 4-vectors (with the weird Minkowski metric, unless you prefer to multiply the first component of each by i):

$$\boldsymbol{r} = (ct, x, y, z)$$
$$\boldsymbol{k} = (\omega/c, k_x, k_y, k_z)$$

Then $s := |\boldsymbol{r}|$ and $\phi := \boldsymbol{k} \cdot \boldsymbol{r}$ and $|\boldsymbol{k}|$ are invariants.

4 Partition function Z

$$Z := \sum_i g_i e^{-\beta E_i} = \int g e^{-\beta E} dE$$

$$E_{avg} = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} = \frac{\int E e^{-\beta E} dE}{\int e^{-\beta E} dE} = -\frac{\partial}{\partial \beta} \ln Z$$

5 Spherical coords

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta \\
dx\, dy\, dz &= r^2 \sin\theta \, dr\, d\theta\, d\phi
\end{aligned}
$$

6 Summation identities

The following Taylor series converges on $x \in (-1, 1)$.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Differentiate that and multiply by x to get this formula:

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

7 Binomial coefficients

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

8 Energy-momentum equation

In general, I'll use m to denote the rest mass, and γm to mean the relativistic mass.

$$E^2 = m^2 c^4 + p^2 c^2$$

In that equation, E is the kinetic energy plus the rest energy. The momentum is $p = \gamma m v$, and the rest energy is mc^2, so the kinetic energy must be

$$\begin{aligned}
K &= E - mc^2 \\
&= \sqrt{m^2c^4 + p^2c^2} - mc^2 \\
&= \sqrt{m^2c^4 + \frac{m^2v^2c^2}{1 - \frac{v^2}{c^2}}} - mc^2 \\
&= mc\sqrt{\frac{(c^2 - v^2) + v^2}{1 - \frac{v^2}{c^2}}} - mc^2 \\
&= mc^2\left(\sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} - 1\right) \\
&= (\gamma - 1)mc^2
\end{aligned}$$

This makes sense, because then the total energy is $E = \gamma mc^2$.

9 Maxwell's equations

Let $D := \varepsilon_0 E + P$ where P is the polarization field, and let $H := \frac{B}{\mu_0} - M$ where M is the magnetization field. Then the following version of Maxwell's equations work even if you aren't in a vacuum:

$$\begin{aligned}
\nabla \cdot D &= \rho \\
\nabla \cdot B &= 0 \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \times H &= J + \frac{\partial D}{\partial t}
\end{aligned}$$

If you're in a vacuum, you can use the identity $\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$ to derive the 3D wave equation for electromagnetic waves:

$$\nabla^2 E = \frac{1}{c^2} \cdot \frac{\partial^2 E}{\partial t^2}$$

10 Maxwell-Boltzmann speed distribution

$$\begin{aligned}
\text{probability density} &= 4\pi v^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{mv^2}{2k_BT}} dv \\
&= 2\sqrt{E/\pi}\left(\frac{1}{k_B T}\right)^{3/2} e^{-E/k_BT} dE
\end{aligned}$$

	Speed	Energy
Mean	$\sqrt{\frac{8kT}{\pi m}}$	$\frac{3}{2}kT$
RMS	$\sqrt{\frac{3kT}{m}}$	$\frac{\sqrt{15}}{2}kT$
Mode	$\sqrt{\frac{2kT}{m}}$	$\frac{kT}{2}$

11 Wien's law

$$\lambda_{\max}T(\lambda_{\max}) = 2.898 \times 10^{-3} mK$$

12 Angular frequency version of Fourier transform

If f is a function of x:

$$(\mathcal{F}f)(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx$$

If f is a function of k:

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(k)dk$$

13 Bohr model

$$\begin{aligned}
r &= \frac{n^2h^2\varepsilon_0}{\pi m_e Z e^2} \approx \frac{n^2}{Z}(5.292 \times 10^{-11} \text{ m}) \\
v &= \frac{Ze^2}{2n\varepsilon_0 h} \approx \frac{Z}{n}(2.188 \times 10^{6} \text{ m/s}) \\
E &= -\frac{Z^2 m_e e^4}{8\varepsilon_0^2 h^2 n^2} \approx \frac{Z^2}{n^2}(-13.6 \text{ eV})
\end{aligned}$$

14 Constants

$$
\begin{aligned}
k_B &\approx 1.381 \times 10^{-23}\,\frac{\mathrm{J}}{\mathrm{K}} \approx 8.617 \times 10^{-5}\,\frac{\mathrm{eV}}{\mathrm{K}} \\
h &\approx 6.626 \times 10^{-34}\ \mathrm{J\ s} \approx 4.136 \times 10^{-15}\ \mathrm{eV\ s} \\
\hbar := \frac{h}{2\pi} &\approx 1.055 \times 10^{-34}\ \mathrm{J\ s} \approx 6.582 \times 10^{-16}\ \mathrm{eV\ s} \\
c &\approx 2.998 \times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}} \\
k := \frac{1}{4\pi\varepsilon_0} &\approx 8.988 \times 10^{9}\,\frac{\mathrm{N\ m^2}}{\mathrm{C^2}} \\
\varepsilon_0 &\approx 8.854 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}} \\
\mu_0 &\approx 1.257 \times 10^{-6}\,\frac{\mathrm{N}}{\mathrm{A^2}} \\
1\ \mathrm{mol} &\approx 6.022 \times 10^{23}\ \mathrm{molecules} \\
\alpha := \frac{e^2}{2\varepsilon_0 hc} &\approx 0.007297 \approx \frac{1}{137} \\
a_0 = \frac{\varepsilon_0 h^2}{\pi e^2 m_e} &\approx 5.292 \times 10^{-11}\ \mathrm{m} \\
G &\approx 6.674 \times 10^{-11}\,\frac{\mathrm{N\ m^2}}{\mathrm{kg^2}} \\
e &\approx 1.602 \times 10^{-19}\ \mathrm{C} \\
m_e &\approx 9.109 \times 10^{-31}\ \mathrm{kg} \approx 0.511\ \mathrm{MeV/c^2} \\
m_p &\approx 1.673 \times 10^{-27}\ \mathrm{kg} \approx 1836\, m_e \approx 938.3\ \mathrm{MeV/c^2} \\
\sigma := \frac{2\pi^5 k^4}{15 h^3 c^2} &\approx 5.670 \times 10^{-8}\,\frac{W}{m^2 K^4} \\
R &\approx \frac{25}{3}\,\frac{J}{mol \cdot K}
\end{aligned}
$$

15 Other random stuff

- Average energy at 0 Kelvin is 3/5 of the Fermi energy (prove by integrating over spherical shells in k-space)
- Bragg scattering: $n\lambda = 2d\sin\theta$
- $p = \hbar k = h/\lambda_{\text{de Broglie}}$
- $E = hf = \hbar\omega = \frac{hc}{\lambda} = \hbar ck$
- $\lambda k = 2\pi$

- If the work function ϕ is $e \cdot V_{\text{stopping}}$, then $KE_e = \frac{hc}{\lambda} - \phi$
- $v_{\text{phase}} = c\sqrt{1 + \left(\frac{mc}{\hbar k}\right)^2}$
- $dN = (4\pi k^2 dk)\left(\frac{L}{2\pi}\right)^3$ (# of polarization states)
- For a blackbody, a "typical" photon has $\lambda = hc/5kT$
- Schwarzschild radius: $r_s = 2GM/c^2$
- $\lambda_{mfp} = 1/n\sigma$ where λ_{mfp} is the mean free path, n is the number of scatterers per volume, and σ (the "cross-section") is the scattered flux divided by the incident area per flux