Microstates

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1 Macrostates vs. microstates

Suppose you have N particles which each have a nonnegative integer amount of energy, and the total energy of those particles is E, where $E \geq N$. In this example, suppose $N = 6$ and $E = 8$. Each way of distributing the energy between N unordered particles is a macrostate, and each way of distributing the energy between N ordered particles is a microstate. Counting microstates is useful because every microstate is equally likely, so we can figure out the probability a that particle will be at a given energy level. In this example, there are 20 macrostates and 1287 microstates.

Each microstate can be represented by a "stars-and-bars" diagram, where each star is a unit of energy. For example, if one particles has 4 energy, 2 particles have 2 energy, and the rest have zero energy, the diagram could look like

$$\star\star\star\star\,|\,\star\star|\,\star\star|||$$

But by convention, each partition in a stars-and-bars diagram must have at least one star, so we add one star to each partition by default, and instead write

$$\star\star\star\star\star|\,\star\star\star\,|\,\star\star\star\,|\,\star\,|\,\star\,|\star$$

This diagram has the same distribution of energies, but in a different order, meaning it represents the same macrostate, but a different microstate.

$$\star\star\star|\,\star\star\star\star\star\,|\,\star\star\star\,|\,\star\,|\,\star\,|\star$$

That microstate can be permuted $\frac{6!}{2!3!} = 60$ different ways, meaning that that macrostate contains 60 microstates.

2 Nasty combinatorics stuff

$$\Omega(E,N) := \#\text{ of microstates} = \sum_{\text{macrostates with N particles \& energy E}} (\#\text{ of permutations of that macrostate})$$

Going back to the first diagram (the one where some partitions have no stars), we have a total of $N+E-1$ characters: E stars and $N-1$ bars. So the number of ways to arrange those characters is

$$\Omega(E,N) = \binom{N+E-1}{E} = \binom{N+E-1}{N-1} = \frac{(N+E-1)!}{E!(N-1)!}$$

In the example where $E=8$ and $N=6$, that tells us that there are $\Omega(8,6)=1287$ microstates.

3 Deriving the discrete probability distribution

Define the function

$$\begin{aligned} f(e,E,N) :&= \sum_{\text{microstates with N particles \& total energy E}} (\#\text{ of particles with energy } e) \\ &= \sum_{n=1}^{\lfloor E/e \rfloor} n \cdot (\#\text{ of microstates where exactly n particles have energy e}) \end{aligned}$$

Since each microstate has N particles, we know that

$$\sum_{e=0}^{E} f(e,E,N) = N \cdot \Omega(E,N)$$

We also know that

$$f(E,E,N) = N$$

and that

???

$$f(e,E,N) = N\binom{E+N-e-2}{N-2} = N\binom{E+N-e-2}{E-e}$$

I HAVE ABSOLUTELY NO IDEA HOW TO GET THAT FORMULA BUT I THINK IT MIGHT WORK???

Now we can calculate the probability $p(e,E,N)$ of finding a particle with energy e (given that there are N particles whose total energy is E). Note that the average number of particles in each microstate with energy e is equal to N times $p(e,E,N)$.

$$p(e,E,N) := \frac{f(e,E,N)}{N\Omega(E,N)} = \frac{(E+N-e-2)!}{(E-e)!(N-2)!} \cdot \frac{E!(N-1)!}{(N+E-1)!} = \frac{N-1}{E+N-1-e} \cdot \prod_{i=1}^{e} \frac{E+1-i}{E+N-i}$$