

Geometric Algebra

Nathan Solomon

October 16, 2022

1 General rules

Terminology: a k -blade is a multivector that can be written as the wedge product of k vectors. A degree k homogeneous multivector can be written as the sum of a bunch of k -blades. The rules below should work for any multivectors $a, b, c \in G(n, 0)$, although of course the rules that use the deg operator on a multivector also require that multivector to be homogeneous.

- Associativity: $(ab)c = a(bc)$
- Left distributivity: $a(b + c) = ab + ac$
- Right distributivity: $(a + b)c = ac + bc$
- The dot product and wedge product are also both left and right distributive
- Scalar multiplication (let λ be a scalar): $\lambda a = a\lambda = \lambda \cdot a = a \cdot \lambda = \lambda \wedge a = a \wedge \lambda$
- Dot product commutativity: $a \cdot b = b \cdot a$
- Wedge product anticommutativity: $a \wedge b = (-1)^{\deg(a)\deg(b)} b \wedge a$
- $\deg(a \cdot b) = |\deg(a) - \deg(b)|$
- $\deg(a \wedge b) = \deg(a) + \deg(b)$ if $\deg(a) + \deg(b) \leq n$ otherwise $a \wedge b = 0$

2 Rules that might not work unless a and b are vectors

- Euclidean metric: $a^2 = ||a||^2$ where $||a||$ is a scalar
- $ab = a \cdot b + a \wedge b$
- $a \wedge b = -b \wedge a$
- $a \cdot b = \frac{1}{2}(ab + ba)$
- $a \wedge b = \frac{1}{2}(ab - ba)$

3 Unit pseudoscalar I

Let $I \in G(n, 0) = e_1 \wedge e_2 \wedge \cdots \wedge e_n$. Then I commutes with everything. If $n \in (4\mathbb{Z} + \{0, 1\})$, we have the very nice property that $I^{-1} = I$, meaning that taking the dual of something twice has no net effect. If $n \in (4\mathbb{Z} + \{2, 3\})$, we have the elegant-but-not-as-nice property that $I^{-1} = -I$, meaning that the unit pseudoscalar I behaves like the imaginary unit $i = \sqrt{-1}$. Using geometric algebras in mischievous ways, we can coerce them into representing complex numbers, or even quaternions.

Reversing a list of n unique elements by swapping adjacent elements requires $n(n-1)/2$ swaps (you can prove that with induction). That number of swaps is even if $n \equiv 0 \pmod{4}$ or $n \equiv 1 \pmod{4}$, and odd otherwise. Thus, the permutation that reverses $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ is even if $n \in (4\mathbb{Z} + \{0, 1\})$ and odd otherwise, which is why that property above works.

3.1 Cross product & wedge product

The cross product of vectors exists only in \mathbb{R}^3 , so the following properties work only in $G(3, 0)$.

$$\begin{aligned} I(a \times b) &= a \wedge b \\ a \times b &= -I(a \wedge b) = I(b \wedge a) \end{aligned}$$

4 Cool application of combinations

The set of multivectors of degree p in \mathbb{F}^n has dimension $\binom{n}{p} = \frac{n!}{p!(n-p)!}$ over \mathbb{F} . Thus, the set of all multivectors in \mathbb{F}^n has dimension 2^n . For example, $G(3, 0)$ (the set of multivectors in \mathbb{R}^3) has the basis $\{1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, e_1e_2e_3\}$ over \mathbb{R} , which matches the 1-3-3-1 row from Pascal's triangle.