Microstates

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1 Macrostates vs. microstates

Suppose you have N particles which each have a nonnegative integer amount of energy, and the total energy of those particles is E, where $E \geq N$. In this example, suppose N=6 and E=8. Each way of distributing the energy between N unordered particles is a macrostate, and each way of distributing the energy between N ordered particles is a microstate. Counting microstates is useful because every microstate is equally likely, so we can figure out the probability a that particle will be at a given energy level. In this example, there are 20 macrostates and 1287 microstates.

Each microstate can be represented by a "stars-and-bars" diagram, where each star is a unit of energy. For example, if one particles has 4 energy, 2 particles have 2 energy, and the rest have zero energy, the diagram could look like

But by convention, each partition in a stars-and-bars diagram must have at least one star, so we add one star to each partition by default, and instead write

This diagram has the same distribution of energies, but in a different order, meaning it represents the same macrostate, but a different microstate.

$$\star\star\star|\star\star\star\star\star|\star\star\star|\star|\star|\star|\star$$

That microstate can be permuted $\frac{6!}{2!3!} = 60$ different ways, meaning that that macrostate contains 60 microstates.

2 Nasty combinatorics stuff

$$\Omega(E,N) := \# \text{ of microstates} = \sum_{\text{macrostates with } N \text{ particles \& energy } E} (\# \text{ of permutations of that macrostate})$$

Going back to the first diagram (the one where some partitions have no stars), we have a total of N + E - 1 characters: E stars and N - 1 bars. So the number of ways to arrange those characters is

$$\Omega(E, N) = \binom{N+E-1}{E} = \binom{N+E-1}{N-1} = \frac{(N+E-1)!}{E!(N-1)!}$$

In the example where E=8 and N=6, that tells us that there are $\Omega(8,6)=1287$ microstates.

3 Deriving the discrete probability distribution

Define the function

$$f(e,E,N) := \sum_{\substack{\text{microstates with N particles \& total energy E}} (\# \text{ of particles with energy e})$$

$$= \sum_{n=1}^{\lfloor E/e \rfloor} n \cdot (\# \text{ of microstates where exactly n particles have energy e})$$

Since each microstate has N particles, we know that

$$\sum_{e=0}^{E} f(e, E, N) = N \cdot \Omega(E, N)$$

We also know that

$$f(E, E, N) = N$$

and that ????

$$f(e, E, N) = N {E + N - e - 2 \choose N - 2} = N {E + N - e - 2 \choose E - e}$$

I HAVE ABSOLUTELY NO IDEA HOW TO GET THAT FORMULA BUT I THINK IT MIGHT WORK???

Now we can calculate the probability p(e, E, N) of finding a particle with energy e (given that there are N particles whose total energy is E). Note that the average number of particles in each microstate with energy e is equal to N times p(e, E, N).

$$p(e, E, N) := \frac{f(e, E, N)}{N\Omega(E, N)} = \frac{(E + N - e - 2)!}{(E - e)!(N - 2)!} \cdot \frac{E!(N - 1)!}{(N + E - 1)!} = \frac{N - 1}{E + N - 1 - e} \cdot \prod_{i=1}^{e} \frac{E + 1 - i}{E + N - i}$$