UCLA Physics 17: Modern Physics

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Goal

We aim to give an overview the three foundational theories of physics: thermodynamics, quantum mechanics, and relativity. Once we peel back an obscuring layer of culture, we will find that each of these interlinked theories corresponds to a distinct *mathematical* idea, namely statistics, the wave equation, and geometry, respectively.

Lecture notes©

Lecture 1 Introduction

Welcome to 'Physics XVII'. Please complete the pre-course survey if you have not yet. Our goal is to give a modern overview of physics, as described above. After a year of frosh physics, I expect that you are most familiar with:

1. Classical mechanics, developed in the 1600's–1800's

2. Classical E&M, developed in the 1700's–1800's

These theories are obviously very old, and we now know them to be not quite right. They are approximately correct and useful in many or even most practical situations, but they do not tell the whole story. And that's what we'd like to get at. Some examples of where and when classical physics breaks down:

- Newton's laws of motion famously break down when speeds near the speed of light are involved. They also do not explain the diffraction of matter. Double slit experiments with electrons, neutrons, and even moderately large molecules show that matter does not follow deterministic Newtonian trajectories.

- Newton's law of gravity does not explain the precession of the perihelion of Mercury. It does not explain gravitational redshifts, and thus could not be used to construct a functional Global Positioning System (GPS).

- Maxwell's equations do not explain the scattering of light by light (Delbrück scattering). According to Maxwell's equations, which are linear, the electromagnetic field does not interact with itself. They also do not explain the photoelectric effect.

Thus the material that you have learned so far does not capture our best understanding of the physical world.

What constitutes our best understanding? It is encapsulated in three distinct physical theories:

- **Quantum mechanics.** We use this phrase as shorthand for quantum field theory (QFT) and the particle zoo described by the Standard Model: six quarks, six leptons, and the $W^{\pm}$, Z, g, γ, and Higgs bosons.

- **Statistical mechanics.** The second law of thermodynamics cannot be otherwise derived. It provides an obvious arrow of time.

- **General relativity**. Gravity is not part of the Standard Model. You will hear talk of gravitons, the quantum mediator of the gravitational field, but these particles have never been observed.

Everything we understand in physics is rooted in these three theories. Various fields and sub-fields (e.g. atomic physics, plasma physics, classical mechanics, classical electricity and magnetism, condensed mater physics, cosmology, chemistry, etc.) are based on principles that, at least in principle, can be derived from these three theories. Note that the above picture does not include various famous speculations, such as supersymmetry, string theory, supergravity, and M-theory, which are neither predictive nor supported by experiment.[1]

Our three foundational theories are undergirded by distinct (mostly) mathematical principles:

- quantum mechanics ↔ countable waves (a neat mixture of differential equations and the integers)
- statistical mechanics ↔ statistics (obviously)
- general relativity ↔ geometry

Going forward this quarter we will expand on all of these beautiful, physical ideas. And in so doing we will discover that they are often obscured by a thick ooze of human culture.

What is culture in physics? Culture is the part of physics that alien physicists visiting from another solar system would not recognize.

Language is a core and indispensable part of culture. We need language to express any physical idea. But we must not lose sight of the great influence that language can have on our way of thinking. More obviously dispensable are the proper names in physics. For instance, units are often named after dead white men to form a code that, while useful, is certainly jargony.

Even units not named after people have deeply cultural (which is to say quite arbitrary) origins. The meter was originally defined as one-ten-millionth of the distance to the North Pole from the equator along a meridian through Paris. The kilogram was originally defined as the mass of 1000 cm^3 (1 liter) of water at the melting point of ice. The second was originally defined as the time period that results when you divide the Earth's rotational period into 24 hours with 60 minutes per hour and 60 seconds per minute. None of these would be recognizable to our hypothetical alien visitor.

Our definitions of the second, the meter, and the kilogram have improved since the days of the French Revolution. Since 1967 the second has been defined in terms of a hyperfine transition in atomic cesium-133. Since 1983 the meter has been defined in terms of the second by defining the speed of

[1]You might also hear about 'grand unified theories' (GUTs), which aim to unify the electroweak and strong interactions, and 'theories of everything' (TOEs), which unify gravity with a GUT. Presently we do not have a successful GUT, a successful TOE, or even a successful theory of quantum gravity.

light as the fixed constant

$$c = 299,792,458 \text{ m/s}. \tag{1}$$

Amazingly, up until 2019 [1] the kilogram was defined in terms of an actual artifact: a Pt/Ir cylinder stored under nested bell jars in a vault just outside Paris. To get rid of this particular historical legacy, the define-a-constant trick that proved so successful with the meter was employed again. Defining Planck's constant

$$h \equiv 6.62607015 \times 10^{-34} \text{ J}\cdot\text{Hz}^{-1} \tag{2}$$

made the kilogram artifact obsolete. Given this description of the origins and value of Planck's constant, do you think it is fair to consider h to be a nearly arbitrary historical legacy? Or would you join the many authors (including our own Serway) who consider Planck's constant to be 'fundamental'?

Many introductory physics textbooks such as Young and Freedman (used by ~ 40% of the students in this class) begin by discussing units, physical quantities, and vectors. As we have already established, the units we use in physics contain arguably as much culture as they do culture-independent physics. The same can be said for physical quantities such as energy and momentum. But vectors and our standard vector operations (e.g. the dot and cross product) are almost pure math. Surely we can consider these to be relatively free of culture? Alas, no. We will address this topic in more detail next week.

Now it is time to get started with some calculations. You are familiar with Maxwell's equations in their integral form, and perhaps you recall how to derive the wave equation from Maxwell's equations from Physics 1C. For the sake of doing this derivation slightly differently, we will start from Maxwell's equations in their differential form:

$$\nabla \cdot \mathbf{D} = \rho, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad \text{and} \tag{5}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{6}$$

These are easily related to the integral forms via the divergence and Stokes theorems from vector calculus,

$$\int \nabla \cdot \mathbf{F}\, dV = \int \mathbf{F}\, d\mathbf{a} \quad \text{and} \tag{7}$$

$$\int \nabla \times \mathbf{F}\, d\mathbf{a} = \oint \mathbf{F}\, d\ell, \tag{8}$$

where $\mathbf{F}$ is any vector field. In free space (i.e. away from any charge density ρ and current density $\mathbf{J}$), Maxwell's equations simplify to

$$\nabla \cdot \mathbf{E} = 0 \tag{9}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{10}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{and} \tag{11}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{12}$$

Taking the curl of both sides of the last equation, we find

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial t} \nabla \times \mathbf{B} \\ \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla^2 \mathbf{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \end{aligned} \tag{13}$$

which is the 3D wave equation. If you do not know the ∇-identity used to go from the first to the second line, you can derive the 1D wave equation, a less general but simpler case, by assuming that

$$\mathbf{E}(x, y, z, t) = E_x(z, t)\hat{\mathbf{x}}. \tag{14}$$

Doing basically the same operations as before, but writing out the curls on the left hand side explicitly gives the 1D wave equation as promised,

$$\frac{\partial^2 E_x}{\partial z^2} = \epsilon_0 \mu_0 \frac{\partial^2 E_x}{\partial t^2}. \tag{15}$$

Everyone is expected to be able to do this second derivation.

Lecture 2 Units and Dimensions

Our goal today is to arrive at a better understanding of these elementary concepts: energy, momentum, and mass. While I am sure that you are familiar with these concepts from your previous coursework, they are hard to define. In fact, I challenge you to come up with definitions. You might try doing it off the top of your head first, and then try referring to the textbooks that you have learned from. It is remarkable that few books deign to attempt careful definitions of these fundamental and basic concepts.

One source of difficulty is that energy, momentum, and mass are so fundamental that it is hard to come up with terms that are *more* fundamental that can be used to create a definition. And if you use terms that are equally (or even less) fundamental, you run into a circularity or bootstrapping problem. For instance, you can say the force is defined by $F = ma$, but then how is mass defined? If the answer is $m = F/a$ then you have not advanced very far. [2]

Newton had to resort to circular definitions, but we can do better. Surprisingly enough, we now understand energy, momentum, and mass in terms of parameters that describe waves. So let us review the basics of waves.

A generic 1D wave equation, one that might describe light, pressure, or water waves, has the form

$$\frac{\partial^2 F}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}. \tag{16}$$

and its solutions have the form

$$F(z,t) = \tilde{f}(vt - z) + \tilde{g}(vt - z), \tag{17}$$

as we can demonstrate by calculating the derivatives and applying the chain rule. Comparing Eq. 16 with Eq. 15, we see that

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}. \tag{18}$$

Recall that ϵ_0 and μ_0 were constants determined experimentally by making measurements on capacitors, inductors, and the like. In a theoretical triumph Maxwell connected these phenomena to one that previously seemed

[2] If one turtle is standing on the back of another turtle, what is that turtle standing on? Evidently it is turtles all the way down.

unconnected, namely light. Sticking in the numbers, we find $v = 3 \times 10^8$ m/s $= c$, the speed of light, and are drawn to the conclusion that light is an electromagnetic wave.

We can, if we choose, rescale the arguments in Eq. 17 so that the solution reads

$$F = f(\omega t - kz) + g(\omega t + kz), \tag{19}$$

where $\omega/k \equiv v$, the phase velocity. Here ω is the angular temporal frequency and k is the angular spatial frequency [3]. This second form turns out to be much more convenient because its arguments are dimensionless. For instance, we can define the phase ϕ with

$$\phi \equiv \omega t - kz \quad \text{in 1D, or} \tag{20}$$

$$\equiv \omega t - \mathbf{k} \cdot \mathbf{r} \quad \text{in 3D.} \tag{21}$$

Then $f(\phi) = \sin\phi$, $\frac{1}{1+\phi^2}$, and $e^{-\phi^2}$, along with most other functions that you might dream up, are all solutions to the wave equation. Convince yourself that these statements are true. Meanwhile $\sin(vt - z)$ does not make any sense; you cannot take the sine of, say, 10 m.

To understand why we call v the 'phase velocity', we pick some value of ϕ, call it ϕ_0, solve for z, and evaluate dz/dt to find the speed at which that particular phase value (i.e. a point on the wave with amplitude $f(\phi_0)$) moves through space. This exercise takes two lines and is well worth doing.

The phase ϕ has an interesting form that reminds us of special relativity (which you know from Physics 1C). The Lorentz transformations are hyperbolic rotations (that might be new to you). Compare a rotation:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{22}$$

with a Lorentz transformation:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \tag{23}$$

[3] By 'angular' we mean that ω has units of radians per second and k has units of radians per meter, as opposed to cycles per second and cycles per meter, respectively.

Here we have the definitions $\beta \equiv v/c$, $\gamma \equiv 1/\sqrt{1-\beta^2} \equiv \cosh\theta$, and $\sinh\theta = \gamma\beta$. In the general case these transformations change the coördinates x, y, z, and t. But they preserve something too. All of these matrices have determinants that are equal to 1, so they preserve the 'length' of the vector. The exact definition of 'length' will vary, depending on the context. Rotations preserve the radius-squared,

$$r^2 = x^2 + y^2 + z^2, \tag{24}$$

because the determinant of the Eq. 22 matrix is $\cos^2\theta + \sin^2\theta = 1$. Equation 24 is just the Pythagorean theorem in 3D, and it says that, regardless of how you set up your coördinate system, the distance between the tail and the head of the vector is always the same.

Similarly, Lorentz transformations preserve the 'interval',

$$s^2 = (ct)^2 - x^2 - y^2 - z^2, \tag{25}$$

because the determinant of the matrix Eq. 23 is $\cosh^2\theta - \sinh^2\theta = 1$. We say that vectors ($\mathbf{r}, \mathbf{F}, \mathbf{p}$, etc.) transform under rotations, but scalars ($\mathbf{r}^2 = \mathbf{r}\cdot\mathbf{r}, \mathbf{r}\cdot\mathbf{F}$, etc.) do not; they are invariant. This is pure geometry, but in the case of the Lorentz transformation the space is hyperbolic, not Euclidean, and thus less familiar.

In 3D Euclidean space we know how to write $\mathbf{r}^2 = \mathbf{r}\cdot\mathbf{r}$ so that we get Eq. 24. If we want, we can generalize our 'dot' product to accommodate more complicated geometries by introducing a 'metric'. In Euclidean space the metric g is just a unit matrix, which is why we can (and usually do) ignore it. But we could write out $r^2 = \mathbf{r}\cdot\mathbf{r}$ as

$$r^2 = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2, \tag{26}$$

where the Euclidean metric is

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} \tag{27}$$

For flat (hyperbolic) spacetime the metric is, according to our 'particle physics' convention,

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{28}$$

so we write the interval as

$$s^2 = \begin{pmatrix} ct & x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = (ct)^2 - (x^2 + y^2 + z^2). \tag{29}$$

In general relativity the metric becomes even more complicated, with off-diagonal terms that describe the curvature of spacetime.

Now here is the crux. In special relativity we have these '4-vectors' (ct, x, y, z) that describe positions in spacetime that we call 'events'. The coördinates of an event might change depending on the motion of the observer (Eq. 23) in the same way that the coördinates of a 3D position can change depending on how an observer sets up their coördinate axes (Eq. 22). These transformations are just geometry: hyperbolic geometry for Lorentz boosts and Euclidean geometry for rotations. But looking at our definition of ϕ (Eq. 21), we see that we can write it as

$$\phi = \begin{pmatrix} \omega/c & k_x & k_y & k_z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \omega t - \mathbf{k} \cdot \mathbf{r} \tag{30}$$

if we introduce a new 4-vector $k = (\omega/c, k_x, k_y, k_z)$.

Now we have two ways to understand that all observers agree on the value of ϕ, just as all observers agree on the value of the interval s^2. First, ϕ is a dimensionless number, like the number of eggs in a carton. Such numbers do not change depending on the relative speed of the observer. Second, the ϕ is the result of a 'dot' product between two 4-vectors, and as such must be a Lorentz invariant. This invariance is exactly analogous to the way that the dot product between two 3-vectors is a rotational invariant. It does not matter how two different observers set up their coördinate systems; while they might report vectors that have different individual coördinate values, they will report the same value for the dot product between two vectors.

With two 4-vectors we can write down three Lorentz invariants:

$$s^2 = r \cdot r = (ct)^2 - \mathbf{r}^2, \tag{31}$$

$$\phi = k \cdot r = \omega t - \mathbf{k} \cdot \mathbf{r}, \quad \text{and} \tag{32}$$

$$k_C^2 = k \cdot k = (\omega/c)^2 - \mathbf{k}^2. \tag{33}$$

All observers agree on the values of s^2, ϕ, and our new invariant, k_C^2.

To recapitulate, we are talking about waves (with temporal frequencies f and spatial frequencies $\mathbf{k}$) in hyperbolic spacetime, and we are emphasizing geometric relationships. We notice, for instance, that c is appearing because we need it to put time t in the same units as space $\mathbf{r}$. From the physicist's point of view, the equations would be simpler if we measured time in meters, not seconds. Equation 31 is basically the Pythagorean theorem for hyperbolic geometry, and we are measuring different sides of the same triangle in different units. Because we learned to measure time and space separately, before we knew that they were both aspects of a unified spacetime, we got in the habit of expressing them in different units. As discussed in the last lecture, we figured this out and, in 1983, we made a partial fix to this historical accident by defining the speed of light. I say partial because a complete solution would have been deciding to junk one of the units entirely, meters or seconds, and deciding to henceforth to either measure distance in seconds or time in meters (depending on which unit we decided to keep).

I want to emphasize that both Eq. 31 and Eq. 33 are the Pythagorean theorem. So fine, ok, to be more pedantic it's a pseudo-Pythagorean theorem adapted for hyperbolic space, because of the special minus signs, but all the same — it is really just geometry. The geometry looks more complicated than it otherwise would because of culture and custom. We use γ and β more often than cosh and sinh (which hides the geometry), and we measure time and space in different units (which also hides the geometry).

These ideas are presented carefully in the excellent textbook, *Spacetime Physics*, by Taylor and Wheeler. To illustrate the point, they tell the "Parable of the Surveyors", which is about two land surveyors working in one town. Both measure north-south distances in miles and east-west distances in meters (just as we measure time in seconds and distance in meters). That's bad, but one aligns their coördinate system with magnetic north, while the other aligns their coördinate system with true north (just as observers might use different inertial reference frames). What a mess! Eventually these surveyors need to talk to each other about their measurements and hilarity ensues. The main point here is that, if you make poor decisions about the units you use, conceptually simple coordinate transformations can look rather complicated.

Our final pseudo-Pythagorean theorem (Eq. 33) might look unfamiliar. We can put this simple, geometric relationship into its traditional and more familiar form by multiplying on both sides by $(\hbar c)^2$, where $\hbar = h/2\pi$ is the reduced Planck's constant. To do so we need to define some 'new' concepts:

energy $E = \hbar\omega$, momentum $\mathbf{p} = \hbar\mathbf{k}$, and mass $m = \hbar k_C/c$. Then Eq. 33 becomes

$$m^2c^4 = E^2 - \mathbf{p}^2c^2, \tag{34}$$

the famous Einstein energy-momentum relation, which is no doubt familiar to you. This version of our pseudo-Pythagorean theorem says that different observers might report different energy and momenta, but they will all report the same mass. But admire how deeply the geometry is now hidden. All three sides of the triangle are in different dimensions — not just feet vs. miles, we have kilograms vs. Joules! Traditional, Newtonian language (energy, momentum, etc.) couldn't do more to hide the underlying geometry. Planck's constant allows us to go back to that old language, if we need to. But to better see the underlying physics, it is better to think in more "natural" units, i.e. frequencies. Forget about these abstract quantities E, $\mathbf{p}$, and m; use temporal frequency f, spatial frequency $\mathbf{k}$, and the invariant frequency k_C (the Compton wavenumber) instead.

Note the advantages of the frequency picture. The frequencies f and $\mathbf{k}$ can be defined mathematically. Energy and momentum are difficult to define. The frequencies f and $\mathbf{k}$ can be visualized or drawn explicitly. Energy and momentum are difficult to visualize or draw.

To review one more time: In the 1790's some random French people made some nearly random decisions defining the second, the meter, the liter, and the kilogram. These definition led to several Pt/Ir kilogram prototypes. Human physicists have since decided to define h and c, choosing fixed values to achieve the best possible continuity with the historical definitions embodied by the physical prototypes. Like the prototypes themselves, the specific values of h and c are artifacts of decisions made by human beings long ago. Despite what you have been and will be told, nothing about these values can properly be described as "fundamental". In fact, the existence of c, and especially of h, obscures the underlying physics, which is based on the geometry of waves in hyperbolic spacetime. Frequencies are simple mathematical concepts. Energy and momentum are not. The architects of Newtonian physics had no idea about the wave nature of matter when they were developing their system. We have learned a lot — namely relativity and quantum mechanics — since the Newtonian era. We see connections that the pioneers of classical physics did not, and, when we peel off the historical vocabulary, we find that the physics underneath is simpler and more geometrical.

Lecture 3 Complex Numbers

Today we step away from physics proper to discuss the main language of physics: mathematics. You might think that mathematics, unconstrained by units or experiments, would be free of human culture. If so, I am sorry to disappoint you. Mathematics, like any other human activity, is not immune to custom and fashion. In many instances the choices mathematicians make can be considered matters of taste, about which there should be no dispute ("de gustibus non disputandum est"). However, sometimes there is a right and a wrong. [4]

Vectors are a critical tool in the physicists' toolkit, and the vector-algebra system that we use to manipulate vectors has some glaring deficiencies in comparison to another system, geometric algebra. We continue to use our Roman-numeral-like vector algebra system because of historical inertia. I did not learn of geometric algebra until a few years ago, and I regret that I did not learn of it earlier. Geometric algebra is not standard. You might not see it again. But I am going to introduce it here. Knowing that it exists, you have the option to incorporate it into your way of thinking while your mind is still young, plastic, and adept at absorbing new ideas.

In physics you have encountered or will encounter:

- scalars, vectors, and tensors
- dot and cross products
- div, grad, and curl (vector derivatives ∇)
- rotations and other (e.g. Lorentz) coördinate transformations
- normed division algebras: complex numbers, quaternions, octonians
- linear algebra
- spinors, Pauli, and Dirac matrices

[4] Compare, for instance, the Roman system (Physics XVII) to the Arabic system (Physics 17) for writing numerals. The Roman system is comparatively awkward. The Arabic system is just better, and everyone uses it now. Given the difficulties that the United States has experienced with adopting the metric system, you have to wonder, would we still be using Roman numerals if the Roman Empire had never fallen?

In the usual presentation, some of these ideas come across as (and in fact are) ad hoc. The Pauli matrices, for instance, were a one-off invented to produce a certain multiplication table that would help address a particular physics problem. Geometric algebra puts all of these ideas under a single umbrella, so that they can all be discussed using a uniform and axiomatic vocabulary.

First we discuss the deficiencies of the system we know. The elementary operations of arithmetic ($+, -, \times, \div$) can all be applied to scalars. For example, given $a = 3$ and $b = 6$ we can calculate $ab = 18$ and $b/a = 2$. With vectors it is not so simple. Given $\mathbf{a} = (1, 0, 2)$ and $\mathbf{b} = (-3, 1, 2)$ we can perform two multiplication operations, the 'dot' product

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &\equiv a_x b_x + a_y b_y + a_z b_z \qquad (35)\\ &= 1\cdot(-3) + 0\cdot 1 + 2\cdot 2\\ &= 1, \qquad (36)\end{aligned}$$

and the 'cross' product

$$\mathbf{a}\times\mathbf{b} \equiv \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} = \hat{\mathbf{x}}\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix} - \hat{\mathbf{y}}\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix} \qquad (37)$$

$$= -2\hat{\mathbf{x}} - 8\hat{\mathbf{y}} + \hat{\mathbf{z}} = (-2, -8, 1) \qquad (38)$$

Fantastic, we have two kinds of vector multiplication. What about vector division? Well, $\mathbf{a}/\mathbf{b}$ (a vector divided by another vector) does not make sense, so we need to ask the question, is there an inversion operation? By definition, an operator a can be inverted if we can find a^{-1} such that $aa^{-1} = 1$, i.e. $aa^{-1} = a/a = 1$.

Immediately we see that the dot product (Eq. 35) cannot be inverted. It is only one equation, and the unknown vector has three unknown components. Thus the system is under-determined. The cross product (Eq. 37) is less obvious. While in this case there are three equations (one for each component of the resultant vector), still the system is under-determined (see this week's homework). For the problem of finding $\mathbf{b}$ given $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{a}\times\mathbf{b} = \mathbf{c}$, any solution $\mathbf{b}$ allows an infinitude of equally valid solutions $\mathbf{b}' = \alpha\mathbf{a} + \mathbf{b}$, where α can be any number. So there is no uniquely determined solution. We have two kinds of vector multiplication, and zero kinds of vector division.

Thus given the work W and the distance $\boldsymbol{\ell}$, you cannot find the force $\mathbf{F}$ from $W = \mathbf{F}\cdot\boldsymbol{\ell}$. Likewise, given the torque $\boldsymbol{\tau}$ and the lever arm $\mathbf{r}$, you cannot find the force $\mathbf{F}$ from $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Does this deficiency of standard vector algebra not seem lamentable?

The cross product in particular is worthy of our dissatisfaction for the following reasons:

1. The cross product is not invertible and it seems like it should be, as just described.

2. The recipe (Eq. 37) for calculating the cross product is arcane. The use of a determinant is unmotivated [5] and looks bizarre from a dimensional standpoint, as the first row is populated with vectors while the other two rows contain numbers.

3. The cross product only works in three dimensions. Real physicists avoid 3D whenever they can. Think 2DEGs (two dimensional electron gasses), string theory (prefers 10, 11, or 26 dimensions), and plain ol' $3+1 = 4$D spacetime. So we do not even have this vector product, weak as it is, in instances where we might want it.

If we are to be fully realized as physicists, we should be equipped with the proper and most powerful tools. We want, need, and should demand a division algebra.

Well, it turns out that we have one already, and it was mentioned at the beginning of the lecture. We only need to generalize it. So let us review the complex numbers.

We start by defining the unit imaginary

$$i \equiv \sqrt{-1} \tag{39}$$

because we are (for the purposes of this lecture) crazy mathematicians and we like to make up stuff. This particular invention is handy because it allows us to take the square root of negative numbers and to solve otherwise insoluble polynomial equations. Heartened by these successes, we define complex numbers,

$$Z \equiv X + iY, \tag{40}$$

[5] Wikipedia puts it very nicely, writing: "The cross product can also be expressed as a formal determinant" and defining "a formal calculation is a calculation that is systematic but without rigorous justification".

with $\mathrm{Re}[Z] = X$ and $\mathrm{Im}[Z] = Y$. To see what happens, we calculate the square of Z,

$$Z^2 = (X + iY)^2 = X^2 + 2iXY - Y^2, \tag{41}$$

but we find that this expression does not have an obvious geometric interpretation. Undeterred, we define complex conjugation [6], which we designate with an asterisk and write

$$Z^* = X - iY, \tag{42}$$

which is to say that we make the replacement $i \to -i$ when we take the complex conjugate. We can then define a new kind of multiplication

$$Z^* Z = |Z|^2 = (X - iY)(X + iY) = X^2 + Y^2, \tag{43}$$

which we recognize as the Pythagorean theorem, and which comes with a lovely geometric interpretation! We now define/invent (lots of inventions here) the complex plane, and demonstrate the polar representation of the complex numbers,

$$Z \equiv X + iY = |Z|e^{i\theta}, \tag{44}$$

with $X = |Z|\cos\theta$ and $Y = |Z|\sin\theta$. With the complex plane we see that we can interpret any complex number as a vector in 2D.

In Lecture Lecture 2 you might have felt that I was somehow unjustified in adjusting the definition of the dot product to accommodate a hyperbolic geometry. Today's lecture and Lecture Lecture 2 demonstrate that such 'adjustments' (e.g. making up completely new rules that you call 'multiplication') are common. They can be useful, and, if they are, that is justification enough. So now I am going to make another 'adjustment', defining a new vector product where Eq. 43 is the special case arising when multiplying the vector by itself. Instead of changing the multiplication operation by introducing a metric, here we change it by incorporating complex conjugation. For $Z = X + iY$ and $W = U + iV$ we define the product

$$\begin{aligned} Z^* W &= (X - iY)(U + iV) \\ &= (XU + YV) + i(XV - YU) = \underbrace{XU + YV}_{\text{like 'dot'}} + \underbrace{i \det \begin{vmatrix} X & Y \\ U & V \end{vmatrix}}_{\text{like 'cross'}}, \end{aligned} \tag{45}$$

[6] Near as I can tell, good math often consists of just making stuff up and seeing if it is cool in some way.

where obvious similarities with traditional vector algebra have been labeled. These same similarities are recognizable when we do the calculation in polar coordinates:

$$\begin{aligned} Z^* W &= |Z| e^{-i\theta} |W| e^{i\phi} \\ &= |Z||W| e^{i(\phi-\theta)} = \underbrace{|Z||W|\cos(\phi-\theta)}_{\text{like `dot'}} + i \underbrace{|Z||W|\sin(\phi-\theta)}_{\text{like `cross'}}, \end{aligned} \tag{46}$$

Dot (cross) products are proportional to the magnitudes of the multiplied vectors, and exhibit the cosine (sine) of the angle between the vectors. Thus this complex 'vector' multiplication is showing the *exact* behavior that we expect for dot and cross products, but it includes division! For instance, if we have

$$Z^* W = Q, \quad \text{then} \tag{47}$$

$$Z Z^* W = Z Q \quad \text{and} \tag{48}$$

$$W = \frac{ZQ}{|Z|^2}. \tag{49}$$

In other words, any 'vector' Z has an inverse

$$Z^{-1} = \frac{Z^*}{|Z|^2} \tag{50}$$

because

$$Z^{-1} Z = \frac{Z^* Z}{|Z|^2} = \frac{|Z|^2}{|Z|^2} = 1, \tag{51}$$

which is the definition of an inverse. We understand now why the normal vector dot and cross products do not allow an inverse: the inverse operation requires knowledge of both. While normal vector algebra separates the two products, the complex numbers carry both together, which makes the inverse possible.

Representing complex numbers in a plane has one other advantage. When we draw the two complex 'vector's in the plane, we see that the 'dot' and 'cross' (i.e. real and imaginary) pieces of the product (Eq. 45 or Eq. 46) each has a nice geometric interpretion. The former gives the projection of one vector onto the other, while the latter gives the area of the parallelogram circumscribed by the vectors. There is a lot to like here. The question now arises, is there a way to generalize the division algebra of the complex numbers to dimensions $D > 2$?

Lecture 4 Geometric Algebra

As discussed in Lecture Lecture 3, our usual dot (Eq. 35) and cross (Eq. 37) products are not invertible. However, complex numbers can be interpreted as vectors, and with complex numbers we can construct a 'vector' product (Eqs. 45–46) that is invertible. Since vectors are common in physics, it would be nice to have a division algebra for them, and it would be even better if this algebra worked in an arbitrary number of dimensions. The example of the complex numbers gives us hope that such an algebra might exist, and, amazingly[7], it does.

An enumeration of the axioms of **geometric algebra** and associated notation follows:

(i) The algebra is associative: $(ab)c = a(bc) = abc$.

(ii) The algebra is distributive over addition: $a(b + c) = ab + ac$.

(iii) Squaring a vector gives a real number (i.e. a scalar): $aa = a^2 \in \mathbb{R}$.

(iv) The dot product reduces the grade: $A_r \cdot B_s = \langle AB\rangle_{|r-s|}$.

(v) The wedge product increases the grade: $A_r \wedge B_s = \langle AB\rangle_{r+s}$.

Here the notation $\langle A\rangle_r$ says "the r^{th}-grade element of the multivector A". In other words, we can decompose a multivector A into a sum of pure grade terms:

$$A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \cdots = \sum_r \langle A\rangle_r \tag{52}$$

Multivectors are generally represented by capital letters $A,\ B,\ C\ldots$, vectors by lowercase letters $a,\ b,\ c,\ \ \ldots$, and scalars by Greek letters $\alpha,\ \beta, \gamma\ldots$ or sub-scripted lowercase letters $a_1, a_2, b_1\ldots$ A 'homogeneous' multivector contains only one grade and is written without the angle brackets as A_r.

Note one familiar axiom is missing from our list (i)–(v). We are missing any statement about commutativity. We are *not* postulating that $ab = ba$, for instance. The fact that this axiom is missing turns out to be a key enabling feature in geometric algebra.

Armed with these axioms, we first split the geometric product:

[7]The amazing thing is that everyone is not using it.

$$\begin{aligned} ab &= \overbrace{\frac{ab+ba}{2}}^{\text{symmetric}} + \overbrace{\frac{ab-ba}{2}}^{\text{antisymmetric}} && \text{(53a)} \\ &= \underbrace{a \cdot b}_{\text{inner, dot}} + \underbrace{a \wedge b}_{\text{outer, wedge}} . && \text{(53b)} \end{aligned}$$

Equation 53a is a trivial rearrangement of the geometric product ab into pieces that are symmetric and antisymmetric under the exchange $a \leftrightarrow b$. Equation 53b is just introducing shorthand notation for those arrangements,

$$\left.\begin{array}{r}\text{the symmetric, inner,}\\ \text{or dot product}\end{array}\right\} \quad a \cdot b \equiv \frac{ab+ba}{2} \quad \text{and} \tag{54}$$

$$\left.\begin{array}{r}\text{the antisymmetric, outer,}\\ \text{exterior, or wedge product}\end{array}\right\} \quad a \wedge b \equiv \frac{ab-ba}{2}. \tag{55}$$

Of the synonyms given for the two products, symmetric and antisymmetric are the most descriptive, but we tend to prefer dot and wedge because they are easier to say and write.

From the definitions (Eqs. 54–55) two important identities immediately follow:

$$a \cdot b = b \cdot a \quad \text{and} \tag{56}$$

$$a \wedge b = -b \wedge a. \tag{57}$$

Thus the arguments of a dot product commute, while the arguments of a wedge product anticommute (which should remind you of the cross product). Unless a is either parallel to or perpendicular to b, the full geometric product ab has no particular symmetry under the exchange $a \leftrightarrow b$.

We prove that the dot product must produce a scalar by defining $c = a + b$ and calculating

$$(a+b)^2 = a^2 + ab + ba + b^2. \tag{58}$$

Rearranging and dividing by 2, we find

$$\frac{ab+ba}{2} = \frac{(a+b)^2 - a^2 - b^2}{2}. \tag{59}$$

Everything on the right side is a scalar by the contraction axiom (iii) above. If we take $a = \alpha b$ (i.e. a parallel to b, or $a||b$), then we find $ab = \alpha a \cdot a + 0$, so:

- The symmetric (dot) product captures all of the parallel ($\|$) part.
- The antisymmetric (wedge) product captures the part that is not $\|$, so it must be perpendicular ($\perp$).[8]

Now we define perpendicularity using the Pythagorean theorem [9]. Say $c = a + b$. Then

$$c^2 = a^2 + b^2 \tag{60}$$

if and only if (iff) a is perpendicular to b (i.e. $a \perp b$). But

$$c^2 = (a+b)^2 = a^2 + b^2 + ab + ba. \tag{61}$$

Thus $ab = -ba$ iff $a \perp b$. Moreover, since $a \cdot b = (ab + ba)/2$, $a \cdot b = 0$ iff $a \perp b$. So we also have:

- The symmetric (dot) product captures the part that is not $\perp$.
- The antisymmetric (wedge) product captures all of the $\perp$ part.

Evidently the properties of parallelism and perpendicularity in vectors a and b are *intrinsic* to their symmetry and anti-symmetry, respectively, under the interchange $a \leftrightarrow b$ in the geometric product ab. I find these relationships to very beautiful, and you should too.[10]

We demonstrate that we have a division algebra by setting $a = b$ in the geometric product ab. The geometric product $aa = a^2 = a \cdot a + a \wedge a = a \cdot a = |a|^2$ is a scalar by the contraction axiom (iii). Thus

$$\frac{aa}{|a|^2} = 1. \tag{62}$$

Since the inverse is defined by

$$aa^{-1} = 1 \tag{63}$$

[8] Exercise: prove that $a \wedge b$ does not contain a scalar component by noting that any scalar commutes with vectors a and b.

[9] It is interesting to note how central this theorem has been to this lecture and the two previous.

[10] If you don't, something might be wrong with you — see me if you have a question about this.

we can identify the inverse of any vector a as

$$a^{-1} = \frac{a}{|a|^2}. \tag{64}$$

Knowing the inverse allows us to 'divide' by any vector.

We can now establish the connection between the geometric product (Eq. 53) and the complex product (Eqs. 45–46). Given two vectors a and b, they live in, at most, a plane. ('At most' because they only define a line if they are parallel.) We can then express these two vectors in terms of orthonormal unit vectors $\{e_1, e_2\}$ defining the plane:

$$a = a_1 e_1 + a_2 e_2, \tag{65a}$$
$$b = b_1 e_1 + b_2 e_2, \tag{65b}$$

with $e_1 e_1 = e_2 e_2 = 1$, and $e_1 e_2 = e_1 \wedge e_2 = -e_2 \wedge e_1 = -e_2 e_1$. We summarize these relations with

$$e_i e_j + e_j e_i = 2\delta_{ij}, \tag{66}$$

where the Kronecker-δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{else.} \end{cases} \tag{67}$$

We interpret the bivector $e_1 \wedge e_2$ as an oriented unit area in the plane established by $\{e_1, e_2\}$, where the orientation is defined by the sense of the circumnavigation about the $\{e_1, e_2\}$ square that starts along e_1, then proceeds along e_2, then along $-e_1$, and finally along $-e_2$ to return to the starting position. Expanding the geometric product

$$\begin{aligned} ab &= (a_1 e_1 + a_2 e_2)(b_1 e_1 + b_2 e_2) \\ &= a_1 b_1 e_1 e_1 + a_1 b_2 e_1 e_2 + a_2 b_1 e_2 e_1 + a_2 b_2 e_2 e_2 \\ &= a_1 b_1 + a_2 b_2 + e_1 e_2 (a_1 b_2 - a_2 b_1), \end{aligned} \tag{68}$$

we find an expression exactly analogous to the complex product in Eq. 45, but with the replacement $i \to e_1 e_2$. The unit scalar imaginary i is defined by the property $i^2 = -1$. Squaring $I \equiv e_1 e_2$ we find

$$\begin{aligned} I^2 &= (e_1 e_2)^2 = e_1 e_2 e_1 e_2 \\ &= e_1 (e_2 \wedge e_1) e_2 = -e_1 (e_1 \wedge e_2) e_2 = -e_1 e_1 e_2 e_2 = -1. \end{aligned} \tag{69}$$

In other words, e_1e_2 has the defining characteristic of the unit scalar imaginary *and* a simple geometric interpretation as a directed unit area. According to one viewpoint [2], the familiar i, whenever it appears in physics, can be replaced with advantage by a pseudoscalar I from geometric algebra. Seeing that we can give a geometric interpretation to a quantity that squares to −1, it seems clear to me that our old language of 'imaginary' was crippling, in that it placed this concept in a category also occupied by fairies and dragons and thus discouraged further investigation.

We use the capital 'I', as opposed to the lowercase 'i', to remind ourselves that these two quantities do not necessarily have all of the same properties. In particular, depending on the dimensionality of the problem, I may or may not commute with all of the other elements of the algebra, in contrast to the scalar i, which always commutes with everything. In our 2D example, for instance, we find

$$\begin{aligned} aI &= (a_1e_1 + a_2e_2)e_1e_2 \\ &= a_1e_1e_1e_2 + a_2e_2e_1e_2 \\ &= -a_1e_1e_2e_1 - a_2e_1e_2e_2 \\ &= -Ia. \end{aligned} \tag{70}$$

Thus the pseudoscalar in 2D, I_2, anticommutes with any vector a. Note also that

$$\begin{aligned} aI &= (a_1e_1 + a_2e_2)e_1e_2 \\ &= a_1e_1e_1e_2 + a_2e_2e_1e_2 \\ &= a_1e_2 - a_2e_1. \end{aligned} \tag{71}$$

Thus right-multiplying a vector a by I_2 gives a new vector rotated counterclockwise from a by 90°, just as we see in complex numbers: $e^{i\theta}i = e^{i\theta}e^{i\frac{\pi}{2}} = e^{i(\theta+\frac{\pi}{2})}$. Oh, and look what happens when we right-multiply a vector a by I twice: we get two 90° rotations (i.e. one 180° rotation), which is to say that the rotated vector is now pointing in the direction opposite the original direction (i.e. $a' = aII = a\,(-1) = -a$). Isn't that lovely?

To see how the algebra expands as we go to higher dimension D, we draw Pascal's triangle:

D							
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1
	0	1	2	3	4	5	6
				r			

The values in the columns indicate how many elements of a given grade r the algebra has in that dimension D, where $r = 0, 1, 2, 3$, etc.. The index r refers to scalars, vectors, bivectors, trivectors, etc., respectively. Thus in dimension $D = 3$ there is one scalar (1), three vectors $\{e_1, e_2, e_3\}$, three bivectors $\{e_1e_2, e_2e_3, e_3e_1\}$, and one trivector $e_1e_2e_3 = e_1 \wedge e_2 \wedge e_3$. The highest grade element in a D-dimensional algebra, of which there is always only one, is an oriented D−dimensional 'volume', or hyper-volume. For $D > 2$ we also identify this highest grade element as the pseudoscalar. In the cases of $D = 2$ and $D = 3$ we have $I_2^2 = -1$ and $I_3^2 = -1$, but it is not always the case that $I_D^2 = -1$. In general the sign of I_D^2 depends on both the dimension and the metric of the space. [11]

The Pascal's triangle construction shows vividly that 3D has a unique coincidence: only in $D = 3$ are the number of vectors and the number of bivectors equal. This coincidence is the reason that the cross product 'works' in 3D and only in 3D. As you will see in this week's homework, in 3D you can assign a unique vector to every bivector, and vice-versa, using the relation

$$a \wedge b = I\, a \times b. \tag{72}$$

Finally, we explain one more bit of vocabulary. In geometric algebra any multivector that can be written purely as the outer product of a set of vectors is called a 'blade'. Note that not every homogeneous multivector is necessarily a blade. (A homogeneous multivector contains terms belonging to just one grade.) [12]

[11] We have been assuming a Euclidean space where $e_ie_j = 1$ for $i = j$, but in the $3+1$D hyperbolic space appropriate for special relativity we might choose $e_0e_0 = 1$ and $e_1e_1 = e_2e_2 = e_3e_3 = -1$. (See the Minkowski/hyperbolic metric Eq. 28.). Thus in arguably the three most important cases for physics — 2D, 3D, and spacetime — we have $I^2 = -1$.

[12] Exercise: $e_1 \wedge e_2 + e_3 \wedge e_4$ cannot be written as a single outer product and thus is not

Lecture 5 Counting States: Ω

Today we start statistical mechanics, which will allow us to finally make good contact with our textbook. Serway begins with:

- *Chapter 1: Relativity I*, which treats special relativity in real space $(ct, \mathbf{r})$.
- *Chapter 2: Relativity II*, which treats special relativity in reciprocal space $(\omega/c, \mathbf{k})$, also known as energy-momentum space $(E/c, \mathbf{p})$.
- *Chapter 3: The quantum theory of light*, which starts with Maxwell's equations and then begins discussing blackbody radiation.

Other than some obvious unit problems, the first two chapters are a fantastic beginning to a modern physics class. However, Serway loses his audience when he jumps to blackbody radiation. The formulas he presents are difficult to understand without some statistical mechanics preliminaries, and those do not appear in the text until *Chapter 10: Statistical physics.* So we will skip ahead to *Chapter 10* and then come back.

More than any other field in physics, thermo/stat mech makes for good quotes. Goodstein's *States of Matter* begins with,

> Ludwig Boltzmann, who spent much of his life studying statistical mechanics, died in 1906, by his own hand. Paul Ehrenfest, carrying on the work, died similarly in 1933. Now it is our turn to study statistical mechanics. Perhaps it will be wise to approach the subject cautiously.

Taking this sound advice to heart, we begin by enumerating the laws of thermodynamics:

0^{th}:	If two systems are in thermal equilibrium with a third, then they are in thermal equilibrium with each other.
1^{st}:	Energy is conserved.
2^{nd}:	In any process the total entropy change $\Delta S \geq 0$.
3^{rd}:	The entropy $S \to \{\text{a constant}\}$, usually zero, as the temperature $T \to 0$.

a blade. Contrast $e_1 \wedge e_2 + e_2 \wedge e_3$, which is a blade — can you demonstrate this fact?

These laws[13] include a lot of vocabulary that we have yet to define. We will develop these definitions fully in the next lecture, but in brief the four key, specifically-thermodynamic concepts are:

- $\Omega(E)$, the number of states accessible to a system A with energy E in the range $[E, E + dE]$,
- the entropy S, given by
$$S \equiv k_B \ln \Omega, \tag{73}$$
- the temperature T, defined by
$$\frac{1}{T} \equiv \frac{\partial S}{\partial E}, \quad \text{and} \tag{74}$$
- thermal equilibrium, the condition where two systems A and A' have equal temperature
$$T = T'. \tag{75}$$

Boltzmann's constant $k_B \equiv 1.380649 \times 10^{-23}$ J/K appears explicitly in the definition of entropy. It is another stupid human conversion constant like Planck's h. Note that, like h, Boltzmann's constant k_B has its numerical value defined exactly in the SI[1]. Boltzmann's constant arises because entropy and temperature were studied and measured before they were understood. If we were starting from scratch and aiming to keep things simple, k_B would not exist, entropy would be dimensionless, and temperature would be measured in energy (or better, frequency) units.

Despite their similar origin stories and current status in the SI, however, not everyone views k_B and h the same way. If asked, nearly all physicists will quickly agree that k_B is a relic conversion factor with value "obtained from historical specifications of the temperature scale"[1]. The same cannot be said about h — even experts state that h is "properly described as fundamental"[1].

To get a feeling for the meaning of the number of accessible states $\Omega(E)$, we start by working an example from Serway that appears in the homework. Imagine you have 6 distinguishable particles, each of which can have energy $E_i = 0, \epsilon, 2\epsilon, 3\epsilon, \ldots$ (or, if you like, frequency $f = 0, f, 2f, 3f, \ldots$) In other

[13] A cute parody of these laws is: (0) There is a game; (1) You can't win; (2) You can't break even; and (3) You have to play.

words, the particles have non-negative energy that is 'quantized' in units of ϵ. The justification for this hypothesis comes from quantum mechanics, and we will come back to it later. It also happens to be the hypothesis that Planck made (without justification) to arrive at his successful blackbody formula. For now just accept it.

Given that our system has total energy $E = \sum_i E_i$, where the particle index i runs from 1 to 6, what is $\Omega(E)$?

We calculate Ω for $E \in [0, 5\epsilon]$:

$$\Omega(0) = 1 \quad \text{(all the particles are in the ground state)} \tag{76}$$

$$\begin{aligned}\Omega(\epsilon) &= \{1 \text{ particle with } 1\epsilon\} \\ &= \{\text{'six choose 1'}\} = \binom{6}{1} = \frac{6!}{1!5!} = 6\end{aligned} \tag{77}$$

$$\begin{aligned}\Omega(2\epsilon) &= \{1 \text{ particle with } 2\epsilon\} + \{2 \text{ particles with } 1\epsilon\} \\ &= \{\text{'six choose 1'}\} + \{\text{'six choose 2'}\} \\ &= \binom{6}{1} + \binom{6}{2} = \frac{6!}{1!5!} + \frac{6!}{2!4!} = 6 + 15 = 21\end{aligned} \tag{78}$$

$$\begin{aligned}\Omega(3\epsilon) &= \{1 \text{ with } 3\epsilon\} + \{1 \text{ with } 2\epsilon, 1 \text{ with } 1\epsilon\} + \{3 \text{ with } 1\epsilon\} \\ &= \frac{6!}{1!5!} + \frac{6!}{1!1!4!} + \frac{6!}{3!3!} = 6 + 30 + 20 = 56\end{aligned} \tag{79}$$

$$\begin{aligned}\Omega(4\epsilon) &= \{1\text{w}4\epsilon\} + \{1\text{w}3\epsilon, 1\text{w}1\epsilon\} + \{2\text{w}2\epsilon\} + \{1\text{w}2\epsilon, 2\text{w}1\epsilon\} + \{4\text{w}1\epsilon\} \\ &= \frac{6!}{1!5!} + \frac{6!}{1!1!4!} + \frac{6!}{2!4!} + \frac{6!}{1!2!3!} + \frac{6!}{4!2!} \\ &= 6 + 30 + 15 + 60 + 15 = 126\end{aligned} \tag{80}$$

$$\begin{aligned}\Omega(5\epsilon) &= \{1\text{w}5\epsilon\} + \{1\text{w}4\epsilon, 1\text{w}1\epsilon\} + \{1\text{w}3\epsilon, 1\text{w}2\epsilon\} + \{1\text{w}3\epsilon, 2\text{w}1\epsilon\} \\ &\qquad + \{2\text{w}2\epsilon, 1\text{w}1\epsilon\} + \{1\text{w}2\epsilon, 3\text{w}1\epsilon\} + \{5\text{w}1\epsilon\} \\ &= \frac{6!}{1!5!} + \frac{6!}{1!1!4!} + \frac{6!}{1!1!4!} + \frac{6!}{1!2!3!} \\ &\qquad + \frac{6!}{2!1!3!} + \frac{6!}{1!3!2!} + \frac{6!}{5!1!} \\ &= (6 + 30 + 30 + 60) + (60 + 60 + 6) = 252\end{aligned} \tag{81}$$

For the first three calculations we use the binomial coefficients "n choose k", written

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{82}$$

However, once we reach $\Omega(3\epsilon)$ the binomial coefficients are no longer sufficient, because we are sorting particles into more than two energy bins. We must employ the more general Maxwell-Boltzmann combinatoric formula, of which the binomial coefficients are a special case:

$$N_{MB} = \frac{N!}{n_1!n_2!n_3!\dots}, \tag{83}$$

where $\sum n_i = N$. The motivation for this expression is as follows. Say we are putting 6 unique balls into 3 bins. Or, as we are here with $\{1$ with 2ϵ, 1 with $1\epsilon\}$, assigning 3 energies (2ϵ, 1ϵ, and 0ϵ) to 6 particles. We can order the N particles in $N!$ different ways. We are going to put the first particle in the 2ϵ bin, and the second in the ϵ bin. The last four go in the 0ϵ bin, but we don't care what order they go in — we only care which ones are in that bin when we are done. Thus, for any choice of the first two particles, there are 4! orderings that give the same final state. The 4! factorial in the denominator of $\frac{6!}{1!1!4!}$ fixes this over-counting.

We summarize the results of the $\Omega(E)$ calculation with a table.

energy E (in units of ϵ)	# of arrangements (i.e. number of {} terms)	$\Omega(E)$
0	1	1
1	1	6
2	2	21
3	3	56
4	5	126
5	7	252
$\vdots$	$\vdots$	$\vdots$
8	20	1287

Table 1: Number of states accessible to 6 distinguishable particles with energy quantized in units of ϵ.

The takeaway lesson from the progression shown in Table 1 is that both the number of arrangements and Ω increase rapidly with E when E becomes large, although this progression is admittedly less obvious in the former case. We can write

$$\Omega(E) = \sum_{\text{arrangements}} N_{\text{MB}}. \tag{84}$$

It is important that the number of arrangements becomes large on its own, independent of the Maxwell-Boltzmann factor N_{MB}, because that factor arises due the distinguishability of the particles. Later we will encounter situations where the particles are indistinguishable, and in those cases the usually-enormous combinatoric Maxwell-Boltzmann factor N_{MB} will be replaced with a '1'. (There are six ways to put ϵ into six particles if you can distinguish the particles, but only one way if you cannot.) But, because the number of arrangements is large, $\Omega(E)$ is large, and our statistical arguments that depend on large numbers will still be valid.

For distinguishable particles there is a faster and more elegant method for calculating $\Omega(E)$ called 'sticks and stones'. Generally n stones can be sorted into k bins in $\binom{n+k-1}{k-1}$ ways. The bottom number is the number of sticks required to define k bins, assuming the outside edges are pre-defined. The top number is the total number of objects (sticks and stones) being ordered. Thus the binomial coefficient gives the number of ways that you can choose to place the sticks among the stones, i.e. bin them. We have $n+k-1$ different positions, and we are counting how many unique final distributions there are of the sticks among those positions (those positions not filled by sticks get filled by the stones).

Compared to our laborious approach, sticks and stones is quick. For example, to reproduce the calculation of $\Omega(5\epsilon)$ (Eq. 81), we identify the number of energy units ϵ with the number of stones n and the number of particles with the number of bins k. In other words, we are putting n energy units into k particles. To figure out how many different ways there are to accomplish this sorting task, we then write:

$$\Omega(5\epsilon) = \binom{5+6-1}{6-1} = \frac{10!}{5!5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 252. \tag{85}$$

The disadvantage of this method is that it only works in the classical, distinguishable-particle case.

Lecture 6 Deriving Thermodynamics: S and T

Today we use a statistical mechanics perspective to develop definitions of our thermodynamic quantities. We imagine a system A^0 that is isolated from the rest of the universe. We divide, perhaps only conceptually, the system A^0 into two pieces, A and A', and we allow them to exchange energy (later we will allow them to exchange particles). Eventually we will come think of A as the system of interest and A' as a 'heat bath' or thermal reservoir.

We make two postulates:

(I) Energy is conserved (the 1st law).

(II) The fundamental statistical postulate (FSP): the probability of finding an isolated, in-equilibrium system with energy E is proportional to the number of accessible states Ω.

Observe how far these postulates get us. The first postulate gives

$$E^0 = E + E', \tag{86}$$

which we can re-write as

$$E' = E^0 - E, \tag{87}$$

Deep, right? This equation says that, given constant E^0, specifying E specifies E'. Thus we can write that the number of states accessible to A^0 is a function of E only: $\Omega^0(E)$.

Now using the second (FSP) postulate, we say that the probability that the system A has energy in $[E, E + \delta E]$ is

$$P(E) = C\Omega^0(E). \tag{88}$$

The constant C is readily determined by enforcing the requirement that A has to have *some* value of energy, i.e. $\sum_E P(E) = 1$. Choosing

$$C = \frac{1}{\sum_E \Omega^0(E)} \tag{89}$$

gives

$$\sum_E P(E) = \sum_E \left(\frac{\Omega^0(E)}{\sum_{E''} \Omega^0(E'')} \right) = 1, \tag{90}$$

and thus satisfies the requirement. The E'' in the denominator is a dummy variable — the two primes remind us that the sum over E'' is independent of the sum over E.

According to our definition of Ω:

- When system A has energy in $[E, E + \delta E]$, it can be in one of $\Omega(E)$ states.

- When system A' has energy in $[E', E'+\delta E]$, it can be in one of $\Omega'(E') = \Omega'(E^0 - E)$ states.

Every accessible state of A can combine with every accessible state of A' to give a distinct, accessible state of A^0. Thus

$$\Omega^0(E) = \Omega(E)\,\Omega'(E^0 - E) \tag{91}$$

and

$$\begin{aligned} P(E) &= C\Omega^0(E) && (88)\\ &= C\,\Omega(E)\,\Omega'(E^0 - E). && (92) \end{aligned}$$

Both $\Omega(E)$ and $\Omega'(E')$ are rapidly increasing functions of their arguments, as we saw last lecture. But, because $E' = E^0 - E$, Ω' is a rapidly decreasing function of E while Ω is a rapidly increasing function of E. Their product thus has a sharp maximum at some particular value of E, say $\tilde{E}$.

The energy $\tilde{E}$ that maximizes $P(E)$ is the energy that the subsystem A is most likely to have (and it determines the energy E' that the heat bath A' is most likely to have). Now if $P(E)$ is maximized, so is $\ln P(E)$, because the logarithm is a monotonically increasing function. It turns out that $\ln P \sim \ln\Omega$ is a more convenient measure than $P \propto \Omega$ itself. I can't pretend that that's not strange. Three factors seem to figure in:

- We work with the number of accessible states Ω more with than we do with the probabilities P. Since $P \propto \Omega$, you can think of Ω as unnormalized probabilities.
- Ω in a macroscopic system is absurdly large, whereas $\ln\Omega$ is merely ginormous.
- We would rather add than multiply.

This last advantage is already familiar to you in the context of digital data storage. Hard drives are specified according to the number of bits b that they have, not by the number of different datasets/messages m that they might store. These numbers are related by

$$\begin{aligned} b &= \log_2 m \quad \text{and} && (93)\\ 2^b &= m. && (94) \end{aligned}$$

Every additional bit doubles the number of data sets you might store, but it does not double your storage. You need to buy a whole second hard drive to do that. Note that hard drives have a ginormous number of bits; a 1 TB hard drive has $\sim 8 \times 10^{12} \sim 2^{43}$ bits. But the number of datasets that it can store, $2^{2^{43}} \sim 10^{10^{10}}$, is exponentially larger[14]. We will develop this analogy more thoroughly at the end of the lecture (Table 2).

To find the energy $\tilde{E}$ maximizing $\ln P$ we set $\partial \ln P/\partial E = 0$. The logarithm of a product is the sum of the logarithms, so by Eq. 92

$$\ln P = \ln C + \ln \Omega + \ln \Omega'. \tag{95}$$

Performing the derivative gives

$$\begin{aligned} 0 \overset{\text{set}}{=} \quad \frac{\partial \ln P}{\partial E} &= 0 + \frac{\partial \ln \Omega(E)}{\partial E} + \frac{\partial \ln \Omega'(E^0 - E)}{\partial E} & (96)\\ &= 0 + \frac{\partial \ln \Omega(E)}{\partial E} + \frac{\partial \ln \Omega'(E')}{\partial E'}\frac{\partial E'}{\partial E} & (97)\\ &= 0 + \frac{\partial \ln \Omega(E)}{\partial E} - \frac{\partial \ln \Omega'(E')}{\partial E'} & (98)\end{aligned}$$

because $\partial E'/\partial E = -1$. Thus the maximum in the probability P occurs when

$$\begin{aligned} \ln\Omega + \ln\Omega' = \text{maximum} \quad &\text{by Eq. 95} \quad \text{and} & (99)\\ \frac{\partial \ln \Omega(E)}{\partial E} = \frac{\partial \ln \Omega'(E')}{\partial E'} \quad &\text{by Eq. 98} & (100)\end{aligned}$$

These relationships are a mouthful, so we define some new words to help us use and discuss them. These words make contact with the historical, thermodynamic concepts that were (amazingly) developed before the statistical underpinnings were understood: entropy (Eq. 73), temperature (Eq. 74), and thermal equilibrium (Eq. 75). You have doubtless heard these terms previously, but now you understand their precise meaning in terms of state-counting with Ω.

Thus the probability maximum conditions (Eqs.99–100) can be re-written in the new vocabulary as

$$\begin{aligned} S + S' = \text{maximum} \quad \text{and} & \qquad (101)\\ T = T', & \qquad (102)\end{aligned}$$

[14] Note that number of degrees-of-freedom $\leftrightarrow$ bits in a macroscopic thermodynamic system makes the number of bits in a hard drive look very, very small. For instance, compare Avogadro's number, 6×10^{23} atoms/mole, to 8×10^{12} bits/TB.

In the old/new words, the **most probable** condition for the combined system is **thermal equilibrium**, in which the combined **entropy is maximized** and the **temperatures are equal**.[15]

Now we ask, what happens if we have two separate, isolated systems A and A', and we bring them into thermal contact? In general, the initial condition is exceedingly improbable. Energy will flow from one system to the other until the conditions Eqs. 101–102 are achieved.

We continue our development by asking, what is the probability of finding A in a particular microstate with energy E_r, assuming that the system A is small in comparison the the bath A'? A sub-system microstate, for instance, in the problem of lecture Lecture 5, might be the condition where particle #2 ("A") has energy $E_r = 2\epsilon$. We will commonly take the microstate to be a single energy level with energy E_r in, say, an atom, and use the following argument to determine the probability that that energy level is occupied. Recalling Eq. 92, we write

$$P(E_r) = C\,\Omega(E_r)\,\Omega'(E^0 - E_r). \tag{92}$$
$$= C \cdot 1 \cdot \Omega'(E^0 - E_r) \tag{103}$$

since there is only one way to be in a particular microstate by the definition of 'microstate'. Now we use $E_r \ll E^0$ and Taylor series expand the logarithm of Ω':

$$\ln\Omega'(E^0 - E_r) \simeq \ln\Omega'(E^0) + E_r \frac{\partial \ln\Omega'}{\partial E'}\frac{\partial E'}{\partial E} \tag{104}$$
$$\simeq \ln\Omega'(E^0) - E_r \frac{\partial \ln\Omega'}{\partial E'} \tag{105}$$
$$\simeq \ln\Omega'(E^0) - \frac{E_r}{k_B T'} \tag{106}$$
$$\simeq \ln\Omega'(E^0) - \frac{E_r}{k_B T}, \tag{107}$$

where we have invoked thermal equilibrium to make the last step. Exponen-

[15]Without the historical language, we have Eqs. 99–100 instead of Eqs. 101–102. In words, the most probable condition for the combined system maximizes the sum of the natural logarithms of the number of states accessible to the subsystems, which occurs when these logarithms have equal rates of change with respect to the subsystems' respective energies/frequencies. Although jargony and dimensionally impaired, the historical vocabulary is clearly more economical.

tiating both sides and applying the FSP gives

$$P(E_r) = C' e^{-E_r/k_B T}. \tag{108}$$

The contant of proportionality C' is again determined by the requirement that $\sum P = 1$. Thus

$$P(E_r) = \frac{e^{-E_r/k_B T}}{Z}, \tag{109}$$

where we have defined the 'partition function' Z with

$$Z \equiv \sum_E e^{-E/k_B T}. \tag{110}$$

Finally, we can (and often do) calculate probability *ratios* without troubling to evaluate the partition function, since in its fully evaluated form Z is only a normalization constant. For instance,

$$\frac{P(E)}{P(0)} = \frac{e^{-E/k_B T}}{1} = e^{-E/k_B T}, \text{ 'the Boltzmann factor'}, \tag{111}$$

gives the probability for a state with energy E to be occupied relative to the occupation probability of the ground state, which is defined to have zero energy. Relative probabilities are found via pairwise comparison to the ground state.

Lecture 7 Z and the Ideal Gas Law

Entropy S is defined (Eq. 73) as proportional to the logarithm of the number of accessible states Ω. As probabilities associated with independent systems multiply, so entropies add. This property makes entropy an "extensive" quantity, like energy or volume: scaling the system size will scale the entropy. In contrast "intensive" properties such as temperature T and pressure P do not scale. We can understand why T is intensive by noting that it is defined (Eq. 74) as the (differential) ratio of two extensive quantities, S and E.

Preview of comming attractions: On the first day of class we drew a Venn diagram depicting the three pillars of known physics: quantum mechanics, general relativity, and statistical mechanics. We further advertised that, by the end of this quarter, we will be able to understand

statistical mechanics		**analogy**	
isolated system	A^0		your life
subsystem	A		your hard drive
# of states accessible to A with E	$\Omega(E) = e^{S/k_B}$	$m = 2^b$	# of states accessible to hard drive
entropy of A	$S = k_B \ln \Omega$	$b = \log_2 m$	size of drive in bits
energy of A	E		price of hard drive
total energy	E^0		your total budget
# of states accessible to total system	Ω^0		# of things that you and your hard drive can do

Table 2: How your life (in the absence of outside income) is like an isolated thermodynamic system. In both cases you maximize the last line. If you spend more money on your hard drive, your hard drive is more capable but you have less money for other things. Spend too little on your hard drive, and you can't do your homework, install software, and store cat pictures. Spend too much on your hard drive, and you can't eat, buy clothes, pay rent, and go to the movies. You decide how much to spend on your hard drive with an eye toward maximizing the total capability of you and your hard drive together.

> some non-obvious aspects of cosmology, a subfield of physics that lives in the Venn diagram's intersection region and is normally outside the purview of an undergraduate education.
>
> In Lecture 6 we developed a key concept necessary for that discussion, namely thermal equilibrium. We are in the process of developing another, blackbody radiation, but I want to skip ahead briefly to set up the cosmology problem.
>
> The cosmic microwave background (CMB) is relic blackbody radiation that formed 379,000 years after the Big Bang. At that time, the expanding universe was cooling through 3000K, allowing atoms to form. Electrons and protons, which had previously existed separately as a plasma, combined to form neutral (and thus transparent) hydrogen gas. Radiation decoupled from matter and has been propagating freely ever since. With the expansion of the universe continuing over subsequent billions of years, the CMB has cooled from its initial

temperature of 3000 K to its current value of 2.7 K. Moreover, the radiation is exceedingly isotropic, showing directional variations only at the one-part-in-10^5 level.

Given this uniformity, it seems that the universe was very nearly in thermal equilibrium a few hundred thousand years after the Big Bang. Now, however, we look around and see a universe that is far from thermal equilibrium. Deep space is characterized by 2.7 K, but — to pick purely local examples — the Earth's surface is at 300 K, the surface of the sun is at 5800 K, and its core is at 15.7×10^6 K. Such orders-of-magnitude temperature variations were not present in the early universe when the CMB was being created.

In Lecture 6 we learned that when two systems at different temperatures come into thermal contact, energy flows from one to the other until the temperatures equalize. The universe, which was near equilibrium early in its life, is now 13.8 billion years old. It currently shows huge place-to-place temperature variations and thus seems much *farther* from equilibrium. We ourselves represent systems that are not in thermal equilibrium. Putting gratitude off to the side, we have to ask, "how is it possible that we are here? Why, after all of this time, hasn't the universe come to thermal equilibrium?"

We now develop some applications of the Boltzmann factor (Eq. 111) and the partition function (Eq. 110). We first calculate the average energy $\langle E \rangle$ of a two-level system,

$$\langle E \rangle = \frac{\sum_E E\, P(E)}{\sum_E P(E)} = \frac{E\, e^{-E/kT} + 0 \cdot 1}{e^{-E/kT} + 1} = \frac{E}{1 + e^{E/kT}}, \tag{112}$$

where we have defined the energy of the lower state to be the energy origin ($E = 0$) and the energy of the upper state to be E. This expression has interesting limits. As $T \to 0$ the average energy $\langle E \rangle \to 0$, which is to say that the system is most likely to be in its ground state. As $T \to \infty$ the average energy $\langle E \rangle \to E/2$, which is to say that the excited level has only a 50%/50% chance of being populated. The condition where $\langle E \rangle > E/2$ is unusual but not impossible. Called a "population inversion", this condition exists in a laser and corresponds to $T < 0$, as is evident from Eq. 112.

We now generalize from two levels to many levels.

$$\langle E\rangle = \frac{\sum_E E\, e^{-E/kT}}{\sum e^{-E/kT}} = \frac{\sum E e^{-\beta E}}{\sum_E e^{-\beta E}} \tag{113}$$

$$= \frac{\sum(-\frac{\partial}{\partial\beta})e^{-\beta E}}{\sum e^{-\beta E}} = -\frac{\frac{\partial}{\partial\beta}\sum e^{-\beta E}}{\sum e^{-\beta E}} \tag{114}$$

$$= \frac{\sum(-\frac{\partial}{\partial\beta})e^{-\beta E}}{\sum e^{-\beta E}} = -\frac{\frac{\partial}{\partial\beta}Z}{Z} = -\frac{\partial}{\partial\beta}\ln Z, \tag{115}$$

where

$$\beta \equiv \frac{1}{k_B T} \tag{116}$$

If we can compute the partition function Z of a system in thermodynamic equilibrium, this Eq. 115 gives us a quick way to find the system's mean energy. It does not require a large number of states and can perfectly well reproduce our two-state result (Eq. 112), as you might check yourself.

In fact, the partition function is a powerful tool for many thermodynamic calculations. We demonstrate one application: deriving the ideal gas law (classically). Consider N non-interacting particles in a box of volume $V = L_x L_y L_z$. Classically a particle's state is completely determined by its position and its momentum/wavevector[16]. As usual, when we say a particle is at position $\mathbf{r}$ we mean that its position is in $[\mathbf{r}, \mathbf{r} + d^3\mathbf{r}]$, and similarly for $\mathbf{p}$. In terms of the components we have

$$\mathbf{r} \in [(x,y,z);(x+dx, y+dy, z+dz)] \tag{117}$$

$$\mathbf{p} \in [(p_x,p_y,p_z);(p_x+dp_x, p_y+dp_y, p_z+dp_z)] \tag{118}$$

Thus every particle lives in a 6D "phase" space: three real-space dimensions and three momentum- (or reciprocal-) space dimension. We take both the position and the momentum coördinates to be continuous, replace the sums with integrals, and note that the only energy in the problem is kinetic. Thus

[16] I want to remind you to try to think in terms of wavevector $\mathbf{k}$ as opposed to momentum $\mathbf{p}$, but for the purpose of agreeing with our textbooks I will use the traditional language for the remainder of this discussion.

the partition function

$$Z = \sum_{\text{states}} e^{-\beta E} = \iint d^3\mathbf{r}\, d^3\mathbf{p}\, e^{-\beta E} \tag{119}$$
$$= V \int d\Omega\, p^2\, dp\, e^{-\beta p^2/2m} \tag{120}$$
$$= 4\pi V \left(\frac{2m}{\beta}\right)^{3/2} \int_0^\infty \theta^2\, d\theta\, e^{-\theta^2}. \tag{121}$$

The remaining integral is very doable, but we don't need to for this calculation, since it's just a constant number. Collecting all the constants together, we write

$$\ln Z = \ln C + \ln V - \frac{3}{2}\ln\beta. \tag{122}$$

Using Eq. 115 we can immediately write down the average energy per atom:

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z = \frac{3}{2\beta} = \frac{3}{2}k_B T, \tag{123}$$

a nice result showing that the average kinetic energy is of-order the thermal energy scale $k_B T$. This result is a special case of the equipartition theorem, which we will derive next week.

To complete the derivation of the idea gas law we consider one atom bouncing off of the wall of the box (assumed to be a rectangular parallelepiped oriented with our coördinate axes) in the yz plane. The atom exerts force F_x, moves the wall dL_x, and loses energy dE such that

$$dE = -F_x dL_x. \tag{124}$$

With probability P_r of being in the state r, the mean force on the wall is then

$$\langle F_x\rangle = \sum_{\text{states } r} F_{xr} P_r = \frac{\sum(-\frac{\partial E}{\partial L_x})e^{-\beta E}}{\sum e^{-\beta E}} \tag{125}$$
$$= \frac{\sum \frac{1}{\beta}(\frac{\partial}{\partial L_x})e^{-\beta E}}{\sum e^{-\beta E}} = \frac{1}{\beta}\frac{(\frac{\partial}{\partial L_x})\sum e^{-\beta E}}{\sum e^{-\beta E}} \tag{126}$$
$$= \frac{1}{\beta}\frac{\partial}{\partial L_x}\ln Z. \tag{127}$$

Equation 127 illustrates a general method: wiggle some external parameter (here the dimension of the box in the x-direction, L_x), and this expression gives the opposite of the force resisting that wiggle.

Using Eq. 122, we then have

$$\langle F_x \rangle = \frac{1}{\beta}\frac{\partial}{\partial L_x}\ln Z = \frac{1}{\beta}\frac{\partial}{\partial L_x}\ln V = \frac{k_B T}{L_x}. \tag{128}$$

Dividing on both sides by the area of the side of the box, we have

$$P_1 = \frac{\langle F_x \rangle}{L_y L_z} = \frac{k_B T}{L_x L_y L_z} = \frac{k_B T}{V} \tag{129}$$

for the pressure from one atom P_1. Since the atoms are non-interacting by hypothesis, the total pressure is N times greater, giving $P = NP_1$ and

$$PV = Nk_B T, \tag{130}$$

which we recognize as our desired result, the ideal gas law. Note that since

$$\langle F_{\text{total}} \rangle = N\langle F_{\text{one}} \rangle = \frac{N}{\beta}\frac{\partial}{\partial L_x}\ln Z = \frac{1}{\beta}\frac{\partial}{\partial L_x}\ln Z^N, \tag{131}$$

we must write

$$Z_{\text{total}} = (Z_{\text{one}})^N. \tag{132}$$

For independent systems (here non-interacting atoms), partition functions, like probabilities, multiply. We expect this behavior because we think of the partition function as a sum of un-normalized probabilities.

Lecture 8 Kinetic Theory of Gases

Last lecture we derived the ideal gas law (Eq. 130) assuming nothing more than basic Newtonian mechanics (energy conservation and the definition of work) and the FSP. Today we will derive the Maxwell-Boltzmann velocity and speed distributions to develop a more detailed picture of the ideal gas. These ideas were worked out in the late 1800's, before the existence of atoms was well established. Resistance to these ideas likely contributed to Boltzmann's depression and eventual suicide in 1906[17]

[17]Remarkably, it was in 1905 that Einstein first gave the explanation of Brownian motion that ended up convincing the scientific community at large that the atomic hypothesis was correct.

Our first problem is to find the number of atoms N per unit volume V that have velocity in $[\mathbf{v}, \mathbf{v}+\delta\mathbf{v}]$. In other words, we want to find $f(\mathbf{v})$ such that

$$n \equiv \int d^3\mathbf{v} f(\mathbf{v}). \tag{133}$$

We proceed by writing down the atomic number density n and then apply a few identities:

$$n = \frac{N}{V} = \frac{N}{V}\cdot 1 = \frac{N}{V}\sum P = \frac{N}{V}\frac{\sum e^{-\beta E}}{Z} \tag{134}$$

$$= \frac{N}{V}\frac{1}{Z}\iint d^3\mathbf{r}\, d^3\mathbf{p}\, e^{-\beta \mathbf{p}^2/2m} = \frac{N}{V}\frac{Vm^3}{Z}\int d^3\mathbf{v}\, e^{-\beta m\mathbf{v}^2/2} \tag{135}$$

$$= C\int d^3\mathbf{v}\, e^{-\beta m\mathbf{v}^2/2} \tag{136}$$

$$= C\int dv_x\, e^{-\beta m v_x^2/2}\int dv_y\, e^{-\beta m v_y^2/2}\int dv_z\, e^{-\beta m v_z^2/2}. \tag{137}$$

The determine the constant of proportionality C we need to evaluate the definite integral

$$I = \int_{-\infty}^{+\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{138}$$

which we can do by squaring the integral and switching from Cartesian to polar coördinates. Using this identity we find $C = n(m/2\pi k_B T)^{3/2}$, which, comparing with Eq. 133 and Eq. 136, gives the Maxwellian velocity distribution

$$f(\mathbf{v})\, d^3\mathbf{v} = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\beta m\mathbf{v}^2/2}\, d^3\mathbf{v}. \tag{139}$$

Because the argument of the velocity distribution function f is a vector, it is difficult to plot f. But we can integrate out the directional part to find the Maxwellian speed distribution $F(v)$,

$$F(v)dv = \int_\Omega f(\mathbf{v})\, d^3\mathbf{v} = \int d\Omega\, v^2\, dv\, f(\mathbf{v}) = 4\pi v^2 f(v)dv \tag{140}$$

$$= 4\pi n\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2\, e^{-\beta m v^2/2}\, dv, \tag{141}$$

which can be plotted.

For a distribution $G(x)$ we can consider the following parameters:

- **Mode:** The most probable value, found by setting $dG/dx = 0$.

- **Mean:** The average value $\langle x \rangle$, found with $\langle x \rangle = \int x\, G(x) dx / \int G(x) dx$ for a continuous distribution. Think of it as the answer to the question, "where would you place a fulcrum so that the distribution balances?" The factor in the denominator is necessary if $G(x)$ is not a normalized probability distribution. For instance, to find $\langle v \rangle$ from the Maxwell-Boltzmann speed distribution (Eq. 141) the denominator is necessary, otherwise the result would have units of [velocity]/[number density], not [velocity] as desired.[18]

- **Median:** The value x_{med} for which half of the distribution is above and half the distribution is below, found from $\int_{-\infty}^{x_{\text{med}}} G(x) dx = \frac{1}{2} \int_{-\infty}^{+\infty} G(x) dx$. Think of cutting the distribution in half such that the two halves balance on the Scales of Justice.

An atomic beam is made by constructing an oven (basically a stainless steel box in vacuum), filling the oven with the desired beam material, and allowing gas to escape the oven through a small hole. Atomic beams have formed the basis of many famous experiments, e.g. the Stern-Gerlach experiment (Ag beam) and your instructor's PhD experiment (Tl and Na beams). Say the hole has diameter D. By 'small' we mean that $D \ll L, \ell_{\text{mfp}}$, where L is the dimension of the box and ℓ_{mfp} is the mean free path of the atoms. In the case of the ideal gas we neglect the interactions between atoms, so $\ell_{\text{mfp}} \simeq L$ (i.e. an atom travels from one side of the oven to the other before it undergoes a collision), but in a dense, non-ideal gas ℓ_{mfp} might be smaller than L. Regardless, D needs to be small for the subsequent arguments to hold; we are going to assume that the existence of the hole does not directly affect the vapor pressure in the oven.

First we ask, what is the "flux" out of the oven through the hole? The units of flux are 'number per time'. We consider a hole in the top of an oven. Given that there are six cardinal directions (e.g. up, down, east, west, north, and south), roughly $1/6^{\text{th}}$ of the atoms are moving in the right direction to escape through the hole (i.e. 'up'). For atoms with speed v, a volume of $\sim (v/6) A dt$ escapes in time dt, which corresponds to a number $\sim n(v/6)A$ in time dt, ignoring the distribution over the speeds. The factor of v here reflects the fact that faster atoms are more likely to escape. Putting the

[18] Exercise: what are the units of $F(v)/f(\mathbf{v})$, where $F(v)$ and $f(v)$ are the Maxwell-Boltzmann speed and velocity distributions, respectively?

distribution in, we expect the flux's speed distribution $\Phi(v)$ to be roughly

$$\Phi(v)\,dv \sim \frac{Av}{6}F(v)\,dv \propto v^3 e^{-mv^2/2k_BT}, \tag{142}$$

which is the famous 'v-cubed Maxwellian'. Because faster atoms are more likely to escape, the most probable, mean, and median speeds are all larger in the beam than they are in the oven. For instance, it takes only a couple of lines to show that

$$v_{\text{mode oven}} = \sqrt{\frac{2k_BT}{m}} < \sqrt{\frac{3k_BT}{m}} = v_{\text{mode beam}}. \tag{143}$$

The most probable (and in fact all the characteristic) speeds increase like the square-root of the temperature.

The argument that gave the $1/6^{\text{th}}$ is fast but crude. We can do better. In fact, the differential volume escaping through the hole is not $dV = \frac{v}{6}\,A\,dt$, but $dV = v_z\,A\,dt = v\cos\theta\,A\,dt$. Adding the cosine factor means that the angular integral that we did to get from the velocity to the speed distributions (i.e. from Eq. 139 to Eq. 141) is no longer trivial. Redoing that integral and this time integrating over the upper half sphere Ω' (i.e. over all directions that are at least partly 'up'), we find that

$$\int_{\Omega'} \cos\theta d\Omega = \int \cos\theta\sin\theta d\theta d\phi = 2\pi \int \cos\theta\, d(cos(\theta)) = \pi, \tag{144}$$

not $\int d\Omega = 4\pi$ as we found before. Thus our guess of 1/6 was a little small; the correct answer is $\int_{\Omega'}\cos\theta d\Omega / \int d\Omega = \pi/4\pi = 1/4$, and the full expression for the flux distribution is

$$\Phi(v)\,dv = \frac{1}{4}AvF(v)\,dv = \pi nA\left(\frac{m}{2\pi k_BT}\right)^{3/2} v^3\, e^{-\beta mv^2/2}\,dv. \tag{145}$$

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