110AH Section Worksheet 5

Warm-up. Describe the correspondence between subgroups of G/N and subgroups of G that contain N. Use this to classify the subgroups of $\mathbb{Z}/n\mathbb{Z}$ and in particular the group $\mathbb{Z}/p^n\mathbb{Z}$ from last week's warm-up.

Universal property of quotients. The following diagram illustrates the universal property for group quotients. State it precisely, and prove it.

$$\begin{tikzcd}
N \arrow[r] \arrow[drr, "1"'] & G \arrow[r, "q"] \arrow[dr, "f"] & G/N \arrow[d, dotted, "\exists!"] \\
& & X
\end{tikzcd}$$

Group presentations. Last week we defined the dihedral group of order $2n$ using the *group presentation*

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle.$$

In general, given a set of symbols S and a set of relations R which are words in these symbols, the group $\langle S \mid R \rangle$ is the quotient of the free group generated by S by the normal subgroup generated by R. Find a presentation of the groups $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z}$. Complete the recipe (from two weeks ago) for the pushout of $f\colon A \to C$ and $g\colon B \to C$.

Quotients of D_8. Recall that the subgroups of D_8 are as follows:

$$\begin{tikzcd}
& & D_8 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
& K = \langle r^2, s\rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & C_4 = \langle r\rangle \arrow[d, dash] & K' = \langle r^2, rs\rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
L_0 = \langle s\rangle \arrow[drr, dash] & L_1 = \langle r^2 s\rangle \arrow[dr, dash] & C_2 = \langle r^2\rangle \arrow[d, dash] & L_0' = \langle rs\rangle \arrow[dl, dash] & L_1' = \langle r^3 s\rangle \arrow[dll, dash] \\
& & 1 & &
\end{tikzcd}$$

For each normal subgroup $N \trianglelefteq D_8$, compute D_8/N.

Group operation on cosets.

- Review the argument that for $N \trianglelefteq G$, the operation $(aN)(bN) = (ab)N$ on G/N is well-defined.
- For each nonnormal subgroup $H \leq D_8$, explain why the operation on D_8/H is not well-defined. *Hint*: Recall that K fixes L_0 whereas rK swaps L_0 and L_1 (ditto with primes).

Solution. Letting $q\colon G \to G/N$ denote the quotient map, the correspondence is $\overline{H} \mapsto q^{-1}(\overline{H})$ and $q(H) \mapsfrom H$ The subgroups of $\mathbb{Z}/n\mathbb{Z}$ are therefore the image under q of the subgroups containing $n\mathbb{Z}$, which are just $d\mathbb{Z}$ for $d \mid n$, i.e. they are the subgroups that are generated by a divisor of n. □

Solution. Let $N \trianglelefteq G$ be a normal subgroup of a group G, and denote by $q\colon G \to G/N$ the quotient map. The universal property of G/N is that for every homomorphism $f\colon G \to X$ that is trivial on N, there exists a unique homomorphism $\overline{f}\colon G/N \to X$ such that $\overline{f}q = f$. Indeed, the definition $\overline{f}(gN) = f(g)$ is forced, and this is well-defined because if $gN = g'N$, then $f(g) = f(gg^{-1}g') = f(g')$. □

Solution. Certainly $\mathbb{Z} = \langle a \mid \varnothing \rangle$ via the isomorphism $1 \mapsto a$. Thus

$$\mathbb{Z}/n\mathbb{Z} = \langle a \mid \varnothing \rangle / \langle a^n \rangle = \langle a \mid a^n = 1 \rangle.$$

Finally, noting that $\mathbb{Z} \times \mathbb{Z}$ is generated by $(1,0)$ and $(0,1)$ but that $(1,0)$ and $(0,1)$ commute, the best guess for a presentation is $\langle a, b \mid aba^{-1}b^{-1} \rangle$. Indeed

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} & \rightleftarrows & \langle a, b \mid aba^{-1}b^{-1} \rangle \\ (n, m) & \mapstofrom & a^n b^m \end{array}$$

exhibit inverse homomorphisms, where the reverse map exists by the universal property of quotients. □

Solution. Obviously D_8/K, D_k/C_4, and D_8/K' are all C_2 since they have order two. Now D_8/C_2 has order four, hence is either Klein-four or C_4. It is the former because the elements sC_2, rC_2, srC_2 all have order two. □

Solution. The operation $(aN)(bN) = (ab)N$ is well-defined by the following argument. Let $n, m \in N$. By normality $b^{-1}Nb = N$, so $nb = bn'$ for some $n' \in N$. Therefore

$$((an)N)((bm)N) = (anbm)N = (abn'm)N = (ab)N.$$

Concretely, if $b^{-1}Hb \neq H$, then picking $b^{-1}hb \notin H$ shows that

$$((ah)H)(bH) = (ahb)H = ab(b^{-1}hb)H \neq (ab)H = (aH)(bH).$$

Thus for example for L_0, we have $r^{-1}sr \notin L_0$, so

$$(sL_0)(rL_0) = (sr)L_0 \neq rL_0 = (L_0)(rL_0).$$

□