

Math 110AH Homework 8

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Assignment due November 29th at 11:59 pm.

1

Let G act on a set X . Prove that if $x, y \in X$ satisfy $ax = y$ for some $a \in G$ then $G_y = a \cdot G_x \cdot a^{-1}$.

2

Let G be a group, $a \in G$. Show that the number of elements in the conjugacy class $\{bab^{-1}, b \in G\}$ divides $|G|$.

Let H be the conjugacy class of a . Since H is normal in G , the quotient G/H is a group. By Lagrange's theorem, FIND A SOLUTION THAT WORKS FOR INFINITE GROUPS TOO

3

An action of a group G on a set X is called *double transitive* if for any two pairs (x_1, x_2) and (y_1, y_2) of elements of X such that $x_1 \neq x_2$ and $y_1 \neq y_2$ there is $a \in G$ such that $ax_1 = y_1$ and $ax_2 = y_2$. Prove that $|G| \geq |X|^2 - |X|$. (Hint: Consider an action of G on $X \times X$.)

4

- (a) Let H be a subgroup of a finite group G . Prove that the number of different conjugate subgroups $xHx^{-1}, x \in G$, is at most $[G : H]$.
- (b) Let H be a subgroup of a finite group G . Prove that if G is the union of xHx^{-1} over all $x \in G$, then $H = G$.
- (c) Let a finite group G act transitively on a set X consisting of at least two elements. Prove that there exists a $g \in G$ fixing no element of X .

5

Determine all Sylow p -subgroups of A_5 .

6

Find the number of all Sylow p -subgroups of S_p (p is prime).

7

Prove the following Useful Counting Result. Let H be a *proper* subgroup of a finite group G . Suppose that $|G|$ does not divide $[G : H]!$. Then G contains a nontrivial normal subgroup N such that N is a subgroup of H . In particular, G is not simple.

8

Prove that all groups of order $2p^n$ (p is prime, $n \geq 1$) are not simple.

9

- (a) Let $H \subset G$ be a subgroup. Prove that if H is contained in the center of G and the quotient group G/H is cyclic, then G is abelian.
- (b) Prove that any group of order p^2 is abelian (p is prime).

10

Prove that $\{e\}$, S_n , and A_n are the only normal subgroups in S_n if $n \geq 5$.