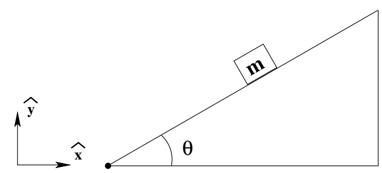
Lagrangian Dynamics with Constraints [1]

Review:



Let's revisit the block on the plane (Example 3). We set the problem up with two degrees of freedom (associated with x and y) and quickly discovered that we needed to incorporate a relevant constraint ($y = x \tan\theta$) to introduce the plane into the problem and reproduce the actual behavior of the system. Introducing the constraint effectively knocked the system down from two degrees of freedom to one. We went from two Euler-Lagrange equations (one for x, one for y) to one; from two equations of motion to one. Knowing how the block moves in x, the equation of constraint determines how it moves in y.

We solved the problem by incorporating the constraints that the system was under into the equations we used to transform the initial set of coordinates into the final set of generalized coordinates. For each constraint we used, the number of degrees of freedom was reduced by one. Fewer degrees of freedom mean fewer equations of motion and this usually means less work. Not surprisingly, less work usually means we get less information out of the system. This is not always desirable.

In this set of notes, we'll explore a modification to Lagrangian dynamics that allows us to, effectively, postpone the use of constraints so that we can pick up that additional information.

[1] Yet another set in the collection of Corbin's notes.

Euler-Lagrange with Explicit Constraints

- Write the Lagrangian in some convenient inertial frame of reference. The more (relevant) degrees of freedom it exhibits, the more information you will have at the end.

- Write out the equations of constraint for the problem. These need to be expressed in a particular form. If the equations were, say:

$$
\begin{aligned}
d &= R\theta \\
x &= u\cos\alpha - v\sin\alpha \\
y &= u\sin\alpha + v\cos\alpha
\end{aligned}
$$

  you'd write:

$$
\begin{aligned}
G_1 &= d - R\theta \\
G_2 &= x - u\cos\alpha + v\sin\alpha \\
G_3 &= y - u\sin\alpha - v\cos\alpha
\end{aligned}
$$

  that is, each function G_i is an equation of constraint re-written so that $G_i \equiv 0$.

- At this point, the number of degrees of freedom in your Lagrangian minus the number of equations of constraint should be equal to the number of actual degrees of freedom in the problem.

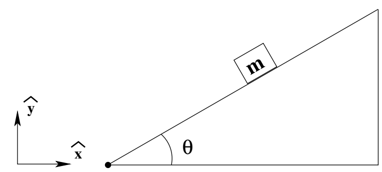
- **The Euler-Lagrange Equation with a side of Explicit Constraints:**

$$
\boxed{\frac{\partial \mathfrak{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_i}\right) + \sum_j \lambda_j(t)\frac{\partial G_j}{\partial q_i} \equiv 0}
$$

- The $\lambda_j(t)$'s are *Lagrange Multipliers* that are solved for in the course of working through the problem.

- The product $\lambda_j(t)\,\frac{\partial G_j}{\partial q_i}$ is equivalent to *the q_i'th component of the force of constraint associated with the constraint G_j* This is the extra bit of information that makes all the trouble worth it!

Example 6: Object on a Plane (Version 3)



We'll set the Lagrangian up with two degrees of freedom:

$$\mathfrak{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

Now, let's set up the constraint:

$$G_1 = y - x\tan\theta$$

The Euler-Lagrange equations look like:

$$\frac{\partial \mathfrak{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right) + \lambda_1(t)\frac{\partial G_1}{\partial x} \equiv 0 \qquad \frac{\partial \mathfrak{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{y}}\right) + \lambda_1(t)\frac{\partial G_1}{\partial y} \equiv 0$$

$$-m\ddot{x} - \lambda_1(t)\tan\theta = 0 \qquad -mg - m\ddot{y} + \lambda_1(t) \equiv 0$$

Admittedly, the presence of $\lambda_1(t)$ in those differential equations worries me a bit. But we have a (all-too-easy-to-forget) proverbial ace up our sleeves. Take a couple time derivatives of our constraint equation $G_1 = 0$. We get:

$$\ddot{y} = \ddot{x}\tan\theta$$

Use this to replace $\ddot{y}$, then eliminate $\ddot{x}$ between the resulting equations - you'll get:

$$\lambda_1(t) = mg\cos^2\theta$$

Plug *this* into the equations of motion, and you should get:

$$\ddot{x} = -g\sin\theta\cos\theta \qquad \ddot{y} = -g\sin^2\theta$$

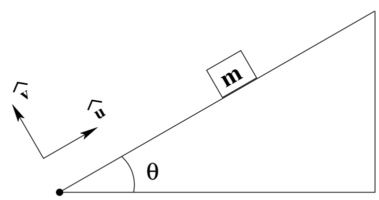
If that's all there is, there's no point in going through all that extra work. But remember, we can use this method to obtain the forces of constraint!

$$F_{1,x} = \lambda_1(t)\frac{\partial G_1}{\partial x} \qquad\qquad F_{1,y} = \lambda_1(t)\frac{\partial G_1}{\partial y}$$

$$F_{1,x} = -mg\cos\theta\sin\theta \qquad\qquad F_{1,y} = mg\cos^2\theta$$

Of course, the 'normal force' ($mg\cos\theta$) projected onto the horizontal and vertical axes yields precisely the same result.

Example 7: Object on a Plane (Version 4)



We're going to do this just because we can. Actually, it's useful to note that if we're going to insist on *two* degrees of freedom, the potential energy depends on both u **and** v!

$$T = \frac{1}{2}m\left(\dot{u}^2 + \dot{v}^2\right)$$
$$V = mg(u\sin\theta + v\cos\theta)$$

$$\mathfrak{L} = \frac{1}{2}m\left(\dot{u}^2 + \dot{v}^2\right) - mg(u\sin\theta + v\cos\theta)$$

$$G_1 = v$$

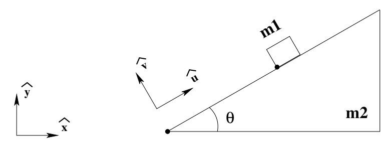
$$\frac{\partial\mathfrak{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial\mathfrak{L}}{\partial\dot{u}}\right) + \lambda_1(t)\frac{\partial G_1}{\partial u} \equiv 0 \qquad\qquad \frac{\partial\mathfrak{L}}{\partial v} - \frac{d}{dt}\left(\frac{\partial\mathfrak{L}}{\partial\dot{v}}\right) + \lambda_1(t)\frac{\partial G_1}{\partial v} \equiv 0$$

$$-mg\sin\theta - m\ddot{u} = 0 \qquad\qquad -mg\cos\theta - m\ddot{v} + \lambda_1(t) = 0$$

From the constraint ($G_1 = 0$), $\ddot{v} = 0$, so $\lambda_1 = mg\cos\theta$.
$F_{1,u} = \lambda_1\frac{\partial G_1}{\partial u}$ $\quad F_{1,v} = \lambda_1\frac{\partial G_1}{\partial v}$ $\quad$ and we get:

$$\ddot{u} = -g\sin\theta \qquad F_{1,v} = mg\cos\theta$$

Example 8: Object on a Sliding Plane (Version 2)



You *knew* it was coming. Imagine you were asked to find the force the table exerts on the wedge as well as the force the wedge exerts on the block. If all you had was Newton's second law, you probably wouldn't be enjoying life much at the moment. Let's see what we can do with Lagrangian dynamics.

$$\mathfrak{L} = \frac{1}{2}\, m_1 \left(\dot{x_1}^2 + \dot{y_1}^2\right) \,+\, \frac{1}{2}\, m_2 \left(\dot{x_2}^2 + \dot{y_2}^2\right) \,-\, m_1 g y_1 \,-\, m_2 g y_2$$

$$\begin{aligned} x_1 &= x_2 + u\cos\theta - v\sin\theta \\ y_1 &= y_2 + u\sin\theta + v\cos\theta \end{aligned}$$

$$\begin{aligned} \mathfrak{L} = \;& \frac{1}{2}\, m_1 \left(\dot{u}^2 + \dot{v}^2\right) \\ +\;& \frac{1}{2}\, (m_1 + m_2) \left(\dot{x_2}^2 + \dot{y_2}^2\right) \\ +\;& m_1 (\dot{u}\dot{x_2}\cos\theta + \dot{v}\dot{y_2}\cos\theta - \dot{v}\dot{x_2}\sin\theta + \dot{u}\dot{y_2}\sin\theta) \\ -\;& m_1 g (u\sin\theta + v\cos\theta) - (m_1 + m_2) g y_2 \end{aligned}$$

$$G_1 = y_2 \qquad\qquad G_2 = v$$

It's at times like these that I start to wonder - *where am I going and why am I doing this to myself?* I think we're going to have to abandon our side-by-side aesthetics, as we're going to have to evaluate four equations of motion.

$$\frac{\partial \mathfrak{L}}{\partial u} \;-\; \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{u}}\right) \;+\; \lambda_1(t)\frac{\partial G_1}{\partial u} \;+\; \lambda_2(t)\frac{\partial G_2}{\partial u} \;\equiv\; 0$$

$$-m_1 g \sin\theta - m_1 \ddot{u} - m_1 \ddot{x}_2 \cos\theta - m_1 \ddot{y}_2 \sin\theta = 0$$

$$g \sin\theta + \ddot{u} + \ddot{x}_2 \cos\theta + \ddot{y}_2 \sin\theta = 0 \tag{1}$$

$$\frac{\partial \mathfrak{L}}{\partial v} \;-\; \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{v}}\right) \;+\; \lambda_1(t)\frac{\partial G_1}{\partial v} \;+\; \lambda_2(t)\frac{\partial G_2}{\partial v} \;\equiv\; 0$$

$$-m_1 g \cos\theta - m_1 \ddot{y}_2 \cos\theta + m_1 \ddot{x}_2 \sin\theta + \lambda_2(t) = 0 \tag{2}$$

$$\frac{\partial \mathfrak{L}}{\partial x_2} \;-\; \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}_2}\right) \;+\; \lambda_1(t)\frac{\partial G_1}{\partial x_2} \;+\; \lambda_2(t)\frac{\partial G_2}{\partial x_2} \;\equiv\; 0$$

$$-(m_1 + m_2)\ddot{x}_2 - m_1 \ddot{u} \cos\theta + m_1 \ddot{v} \sin\theta = 0 \tag{3}$$

$$\frac{\partial \mathfrak{L}}{\partial y_2} \;-\; \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{y}_2}\right) \;+\; \lambda_1(t)\frac{\partial G_1}{\partial y_2} \;+\; \lambda_2(t)\frac{\partial G_2}{\partial y_2} \;\equiv\; 0$$

$$-(m_1 + m_2)g - (m_1 + m_2)\ddot{y}_2 - m_1 \ddot{v} \cos\theta - m_1 \ddot{u} \sin\theta + \lambda_1(t) = 0 \tag{4}$$

The constraint equations tell us: $\ddot{v} = 0$ and $\ddot{y}_2 = 0$. The equations of motion become:

$$g \sin\theta + \ddot{u} + \ddot{x}_2 \cos\theta = 0$$

$$-m_1 g \cos\theta + m_1 \ddot{x}_2 \sin\theta + \lambda_2(t) = 0$$

$$-(m_1 + m_2)\ddot{x}_2 - m_1 \ddot{u} \cos\theta = 0$$

$$-(m_1 + m_2)g - m_1 \ddot{u} \sin\theta + \lambda_1(t) = 0$$

Four equations, four unknowns. I trust you can solve that system in your head to obtain the following:

$$\ddot{x}_2 = \frac{m_1 g \sin\theta \cos\theta}{m_1 \sin^2\theta + m_2}$$

$$\ddot{u} = \frac{-(m_1 + m_2)\, g \sin\theta}{m_1 \sin^2\theta + m_2}$$

$$\lambda_1(t) = \frac{m_2(m_1 + m_2)\, g}{m_1 \sin^2\theta + m_2}$$

$$\lambda_2(t) = \frac{m_1 m_2 g \cos\theta}{m_1 \sin^2\theta + m_2}$$

The forces of constraint:

$$F_v = \lambda_1(t)\frac{\partial G_1}{\partial v} + \lambda_2(t)\frac{\partial G_2}{\partial v} \qquad F_{y_2} = \lambda_1(t)\frac{\partial G_1}{\partial y_2} + \lambda_2(t)\frac{\partial G_2}{\partial y_2}$$

$$F_v = \frac{m_1 m_2 g \cos\theta}{m_1 \sin^2\theta + m_2} \qquad F_{y_2} = \frac{m_2(m_1 + m_2)\, g}{m_1 \sin^2\theta + m_2}$$

In the end, we probably could have solved this using Newton's second law (I dare you!) and it *might* not have been all that much more work. How would you approach the problem if the flat ramp were replaced by a curved ramp? I chose the examples I did because they were easy and familiar and seemed like a real good way both to understand how the steps you need to take are applied to problems and to build intuition. As you begin to do more interesting and complicated problems, take a little time to ponder why Lagrangian dynamics might give you an advantage over trying to tackle the problem with Newtonian dynamics. And most importantly, *have fun!*