1 5/3/2024 lecture

Talk about orthogonality theorem for characters. Note that an "irrep" is an irreducible representation, and also talk about how the word "representation" can be used in a few different ways. Define what it means to take the direct sum and tensor product of representations.

1.1 Schur's lemma

There are two theorems which we call "Schur's lemma".

Lemma 1.1. If R_g is an n-dimensional irrep and R'_g is an m-dimensional irrep and A is an $m \times n$ matrix such that

$$R'_g A = A R_g,$$

then either $A = 0$ or FINISH COPYING DOWN THE STATEMENT

Proof. Suppose $v \in \ker A$. Then $\ker A$ is a G-invariant subspace of $\mathbb{F}^n$, so ???
Suppose $w \in \operatorname{im} A$. □

Here's the other version of Schur's lemma:

Lemma 1.2. Suppose A is an $n \times n$ matrix over $\mathbb{C}$ (or some other algebraically-closed field) and let R be an irrep which commutes with A. Then A is a scalar multiple of the identity matrix.

Proof. Since we're working in $\mathbb{C}$, A must have at least one eigenvector, which we will call v. Then $Av = \lambda v$ for some $v \in \mathbb{C}$. The λ-eigenspace of A is a non-trivial G-invariant subspace, and since R is irreducible, that means it must be all of $\mathbb{C}^n$. □

Consider a quantum mechanical system with n states and some symmetry. Then there is a hamiltonian H acting on $\mathbb{C}^n$ and a representation R (which is probably unitary, but doesn't need to be) of a group G on $\mathbb{C}^n$ such that R commutes with H. Then all states have the same energy. For example, in a spin-$\frac{1}{2}$ system, with $G = SU(2)$ symmetry, there is double degeneracy.

1.2 Sum of two irreps

Suppose $R = R_1 \oplus R_2$, where R_1 and R_2 are irreducible representations. Then any matrix A which commutes with R is a block-diagonal matrix where the top left block is $\lambda_1 I$ and the bottom right block is $\lambda_2 I$.