Math 115B Homework #1

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Problem 0.1.

T^{-1} is linear iff $T^{-1}(w_1 + aw_2) = T^{-1}(w_1) + aT^{-1}(w_2)$ for any $a \in k$ and $w_1, w_2 \in W$. By the definition of an inverse and by the linearity of T,

$$\begin{aligned} T(T^{-1}(w_1 + aw_2)) &= w_1 + aw_2 \\ &= T(T^{-1}(w_1)) + aT(T^{-1}(w_2)) \\ &= T(T^{-1}(w_1) + aT^{-1}(w_2)). \end{aligned}$$

Any invertible function has to be injective, so we can cancel T from both sides, which leaves

$$T^{-1}(w_1 + aw_2) = T^{-1}(w_1) + aT^{-1}(w_2).$$

Therefore T^{-1} is linear.

Problem 0.2.

That is not possible. Suppose v_1, v_2, and v_3 are linearly independent, but $v_1 + v_2$, $v_2 + v_3$, and $v_3 + v_1$ are linearly dependent. Then there exist constants a_1, a_2, a_3 which are not all zero, for which

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_1) = 0.$$

This can be rewritten as

$$(a_3 + a_1)v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 = 0,$$

and since the vectors v_1, v_2, and v_3 are independent, that means $a_3 + a_1$, $a_1 + a_2$, and $a_2 + a_3$ are all zero. However,

$$\begin{aligned} a_1 &= (a_3 + a_1) + (a_1 + a_2) - (a_2 + a_3) = 0 + 0 - 0 = 0 \\ a_2 &= (a_1 + a_2) + (a_2 + a_3) - (a_3 + a_1) = 0 + 0 - 0 = 0 \\ a_3 &= (a_2 + a_3) + (a_3 + a_1) - (a_1 + a_2) = 0 + 0 - 0 = 0. \end{aligned}$$

This contradicts our earlier statement, that a_1, a_2, and a_3 are not all zero. Therefore it is not possible.

Problem 0.3.

V is isomorphic to k^d, for some nonnegative integer d, so $49 = |V| = |k|^d$. This equation is only satisfied if $|k| = 7$ and $d = 2$, or if $|k| = 49$ and $d = 1$. In either case, k is finite and k^d is finite-dimensional (with dimension either 1 or 2), which means V also has dimension 1 or 2.

Problem 0.4.

(a) For any $w_1 \in W_1$, let $w_2 = 0$. Since every subspace contains the zero element, w_2 is in W_2, which means $w_1 + w_2 \in W_1 + W_2$. But $w_1 = w_1 + 0 = w_1 + w_2$, so every $w_1 \in W_1$ is also in $W_1 + W_2$. This means $W_1 \subset W_1 + W_2$, and by the same logic, W_2 is also a subset of $W_2 + W_1$.

(b) $W_1 + W_2$ is clearly a subset of V, since V is closed under addition and scalar multiplication, so I only need to show two things: that $W_1 + W_2$ is closed under addition, and that it's closed under scalar multiplication. For any $u_1 + u_2, v_1 + v_2 \in W_1 + W_2$, we have $(u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1) + (u_2 + v_2)$, which is in $W_1 + W_2$ because $u_1 + v_1$ is in W_1 and $u_2 + v_2$ is in W_2.

$W_1 + W_2$ is closed under scalar multiplication, because for any scalar a and vector $w_1 + w_2 \in W_1 + W_2$, $a(w_1 + w_2) = aw_1 + aw_2$ is in $W_1 + W_2$ because $aw_1 \in W_1$ and $aw_2 \in W_2$.

(c) W is a subspace of V, so W is also a vector space over k. That means you can just take my answer to part (b) and replace V with W.

Problem 0.5.

(a) This is a functional, because if $k := \mathbb{R}$ (or $V = k[x]$), then $f : V \to k$ and we also have

$$\begin{aligned} f(p + cq) &= 4(p + cq)'(0) + (p + cq)''(1) \\ &= (4p'(0) + p''(1)) + c(4q'(0) + q''(1)) \\ &= f(p) + cf(q). \end{aligned}$$

for any $p, q \in \mathbb{R}[x], c \in \mathbb{R}$.

(b) Not a functional, because the output of f is not an element of k.

(c) This is a functional, because tr is a function from V to k, and for any $A, B \in k^{2\times 2}$ and any $c \in k$,

$$\begin{aligned} f(A + cB) &= \operatorname{tr}(A + cB) \\ &= (A + cB)_{1,1} + (A + cB)_{2,2} \\ &= (A_{1,1} + A_{2,2}) + c(B_{1,1} + B_{2,2}) \\ &= \operatorname{tr}(A) + c\operatorname{tr}(B) \\ &= f(A) + cf(B). \end{aligned}$$

(d) This is a functional. Just like in part (a), I will assume $k := \mathbb{R}$(or $V = k[x]$), so $f : V \to k$. f is linear because for any $p, q \in V, c \in k$,

$$\begin{aligned} f(p + cq) &= \int_0^1 (p + cq)(x)\mathrm{d}x \\ &= \left(\int_0^1 p(x)\mathrm{d}x\right) + c\left(\int_0^1 q(x)\mathrm{d}x\right) \\ &= f(p) + cf(q). \end{aligned}$$

(e) Not a functional, because

$$f\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}\right) \neq f\left(\begin{bmatrix}2\\0\\0\end{bmatrix}\right).$$

Problem 0.6.

(a) Since we already know $\mathbb{R}[x]_{\leq n}$ is a vector space, we just need to show that W is a subspace. If $f, g \in W$ and $c \in \mathbb{R}$, then for any $i \in \{1, 2, \ldots, m\}$,

$$(f + cg)(x_i) = f(x_i) + cg(x_i) = 0 + 0c = 0.$$

By the definition of W and the fact that $\mathbb{R}[x]_{\leq n}$ is a vector space, that means $f + cg \in W$. Since W is a subspace of a vector space over $\mathbb{R}$, W is also a vector space over $\mathbb{R}$.

(b) Let $A : \mathbb{R}[x]_{\leq n} \to \mathbb{R}^m$ be the function defined by

$$A(f) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \end{bmatrix}.$$

This is clearly a linear map. Since $W := \ker(A)$, the rank-nullity theorem says

$$\dim(W) = \dim(V) - \dim(\operatorname{im}(A)) = (n + 1) - \dim(\operatorname{im}(A)).$$

The last step for this problem is to show when A is surjective. For any basis element e_i of $\mathbb{R}^m$, let

$$f_i(x) = \prod_{j \in \{1,2,\ldots,m\} \setminus \{i\}} (x - x_j).$$

f_i is a degree $m - 1$ polynomial, so $f_i \in W$ iff $m - 1 \leq n$. If $m > n + 1$, then by the fundamental theorem of algebra, $W = \{0\}$, which means $\operatorname{im}(A)$ is also the zero vector space. If $m \leq n + 1$, then $f_i \in W$, and $A(f_i)$ is a nonzero scalar multiple of the basis vector e_i. In this case, all basis vectors of $\mathbb{R}^m$ are in $\operatorname{im}(A)$, so $\operatorname{im}(A) \cong \mathbb{R}^m$.

The image of A is zero-dimensional if $m > n + 1$ and m-dimensional if $m \leq n + 1$, so

$$\dim(W) = n + 1 - \dim(\operatorname{im}(A)) = \max(0, n + 1 - m).$$

Math 115B: Linear Algebra
Homework 1
Due: Tuesday, January 14 at 11:59 PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.

1. ($\frac{-}{20}$) Assume V and W are vector spaces over a field k, and let $T : V \to W$ denote a linear transformation between them. Prove that if T has an inverse, then that inverse is a linear function.

2. ($\frac{-}{20}$) Assume that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are vectors in a vector space V over some field k. Is it possible that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent but the vectors $\vec{w}_1 = \vec{v}_1 + \vec{v}_2, \vec{w}_2 = \vec{v}_1 + \vec{v}_3, \vec{w}_3 = \vec{v}_2 + \vec{v}_3$ are linearly *dependent*?

3. ($\frac{-}{5+5}$) Assume V is a vector space consisting of 49 vectors over a field k.

   (a) Prove that the set of elements in k is finite.

   (b) Prove that V is finite dimensional and that $\dim(V) = 1$ or $\dim(V) = 2$.

4. ($\frac{-}{5+5+5}$) Assume V is a vector space over some field k, and let W_1, W_2 denote subspaces of V. Define
$$W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

   (a) Prove that $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_2 + W_1$.

   (b) Prove that $W_1 + W_2$ is a subspace of V.

   (c) Assume that $W \subseteq V$ is a subspace such that $W_1 \subseteq W$ and $W_2 \subseteq W$. Prove $W_1 + W_2 \subseteq W$. (Notice that these results imply that $W_1 + W_2$ is the smallest possible subspace of V which contains both W_1 and W_2!)

5. ($\frac{-}{5*4}$) Let k denote some field. For each function f below, determine if f is a linear functional and prove your answer is correct. You may assume standard results from calculus.

   (a) $V = \mathbb{R}[x]$, $f(p(x)) := 4p'(0) + p''(1)$, where $q'(x)$ denotes the derivative of $q(x) \in \mathbb{R}[x]$.

   (b) $V = k^2$, $f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 2x \\ 4y \end{pmatrix}$.

   (c) $V = k^{2\times 2}$, $f(A) = \text{tr}(A)$

(d) $V = \mathbb{R}[x]$, $f(p(x)) = \int_0^1 p(x)dx$

(e) $V = \mathbb{Q}^3$, $f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = x^2 + y^2 + z^2$

6. ($\frac{-}{10+5}$) Assume that m, n are positive integers, and fix $x_1, ..., x_m \in \mathbb{R}$ such that $x_i \neq x_j$ if $i, j \in \{1, ..., m\}$ such that $i \neq j$. Let

$$W := \{f \in \mathbb{R}[x]_{\leq n} : f(x_1) = f(x_2) = ... = f(x_m) = 0\}$$

where $\mathbb{R}[x]_{\leq n}$ denotes the degree less than or equal to n. (The set $\mathbb{R}[x]_{\leq n}$ is denoted $P_n(\mathbb{R})$ in our textbook.)

(a) Prove that W is a vector space over $\mathbb{R}$. (*Hint:* The set W is, by definition, a sub*set* of $\mathbb{R}[x]_{\leq n}$, which was shown in 115A to be a vector space.)

(b) Compute the dimension of W. (*Hint:* It may help to use the rank-nullity theorem.)