Physics 127 Homework #2

Nathan Solomon

April 17, 2025

Problem 0.1.

(a) In the moving reference frame, the photon first moves from the bottom mirror to the top mirror along a line with speed c and horizontal velocity v, so its vertical velocity is $\sqrt{c^2 - v^2} = c/\gamma$. For the return trip, the photon moves from the top mirror to the bottom one with vertical velocity $-c/\gamma$, so the duration of the photon's round trip is $t' = 2L\gamma/c$.

In the lab frame, where the mirrors aren't moving, the horizontal velocity of the photon is also zero, so the duration of the round trip is $t = 2L/c$. This is consistent with the time-dilation formula, $t' = \gamma t$.

(b) Once again, the round trip duration in the lab frame is $t = 2L/c$. But in the moving frame, the right mirror is a distance of L/γ from the left mirror, because of length contraction. The time it takes for the photon to move from the left mirror to the right mirror is $(L/\gamma)/(c-v)$, since the photon is moving to the right with speed c and the right mirror is also moving to the right with speed v. For the same reason, the time it takes the photon to move from the right mirror to the left mirror is $(L/\gamma)/(c+v)$, so the duration of the round trip is

$$t' = \frac{L}{\gamma} \cdot \left(\frac{1}{c+v} + \frac{1}{c-v} \right) = \frac{L}{\gamma} \cdot \frac{2c}{c^2 - v^2} = \frac{2\gamma L}{c},$$

which once again agrees with the formula $t' = \gamma t$.

Problem 0.2. **Coleman problem 1.2:** A spaceship moves with velocity v along its axis of symmetry. A star is at rest; the vector from the spaceship to the star makes an angle θ with this axis. What is the angle at which an observer on the spaceship sees the star?

Suppose the spaceship is at the origin, moving in the $+x$ direction, and the star is a distance L from the spaceship, at $(x = L\cos\theta, y = L\sin\theta)$.

Then in the spaceship's reference frame, the horizontal lengths is contracted by a factor of $\gamma = 1/\sqrt{1 - v^2/c^2}$, but the vertical length is the same. In this frame, the star's coordinates are $(L\cos\theta\sqrt{1 - v^2/c^2}, L\sin\theta)$. Therefore, the angle to the star is

$$\theta' = \arctan\left(\frac{L\sin\theta}{L\cos\theta\sqrt{1 - v^2/c^2}} \right) = \arctan\left(\frac{\tan\theta}{\sqrt{1 - \frac{v^2}{c^2}}} \right).$$

Problem 0.3.

(a) The half-life in the π-meson's frame is $t = 26\,\mathrm{ns}$, so the half-life in the Earth's frame is

$$t = \gamma t = \frac{26\,\mathrm{ns}}{\sqrt{1-0.95^2}} \approx 83.3\,\mathrm{ns}.$$

(b) Since the speed is constant, average distance traveled is proportional to average time before decaying. We know the half-life is $\ln(2)$ times the expected lifetime, so the average distance traveled before decaying is

$$vt/\ln(2) = \frac{(0.95c)(83.3\,\mathrm{ns})}{\ln(2)} = 34.2\,\mathrm{m}.$$

Problem 0.4.

(a) Using the definition of proper time,

$$v^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{u^i}{\gamma},$$

so $u^\mu = \gamma v^\mu$, which means $u^0 = \gamma$ and $u^i = \gamma v^i$ (for $i \in \{1,2,3\}$). Therefore $u \cdot u$ is

$$g_{\mu\nu}u^\mu u^\nu = \gamma^2 g_{\mu\nu} v^\mu v^\nu = \gamma^2\left(1 - v\cdot v\right) = 1.$$

The derivative of the Lorentz factor with respect to proper time is

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}(1-v^2)^{-1/2} = \left(-\frac{1}{2}(1-v^2)^{-3/2}\right)(-2v)\left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right) = \gamma^3 v\left(\frac{\mathrm{d}v}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \gamma^4 v\cdot\dot{v},$$

so the 4-acceleration is

$$\begin{aligned}
a^\mu &= \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\gamma, \gamma v^x, \gamma v^y, \gamma v^z\right) \\
&= \left(\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}, v^x\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} + \gamma\frac{\mathrm{d}v^x}{\mathrm{d}\tau}, v^y\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} + \gamma\frac{\mathrm{d}v^y}{\mathrm{d}\tau}, v^z\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} + \gamma\frac{\mathrm{d}v^z}{\mathrm{d}\tau}\right) \\
&= \left(\gamma^4 v\cdot\dot{v}, v^x\gamma^4 v\cdot\dot{v} + \gamma^2\dot{v}^x, v^y\gamma^4 v\cdot\dot{v} + \gamma^2\dot{v}^y, v^z\gamma^4 v\cdot\dot{v} + \gamma^2\dot{v}^z\right) \\
&= \left(\gamma^4(v\cdot\dot{v}), v\gamma^4(v\cdot\dot{v}) + \gamma^2\dot{v}\right).
\end{aligned}$$

The dot product $a \cdot u$ is

$$\begin{aligned}
a\cdot u &= \left(\gamma^4(v\cdot\dot{v}), v\gamma^4(v\cdot\dot{v}) + \gamma^2\dot{v}\right)\cdot(\gamma, \gamma v) \\
&= \gamma^5(v\cdot\dot{v}) - (\gamma v)\cdot\left(v\gamma^4(v\cdot\dot{v}) + \gamma^2\dot{v}\right) \\
&= 0.
\end{aligned}$$

(b) The magnitude of v^x is always less than 1, so the particle never reaches the speed of light.

The components of the 4-velocity are

$$v^\mu = \begin{bmatrix} 1 \\ gt/\sqrt{1+g^2t^2} \\ 0 \\ 0 \end{bmatrix}.$$

The Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1-(v^x)^2}} = 1 + g^2 t^2,$$

therefore the proper time τ as a function of t is

$$\tau = \int_{s=0}^{s=t} \frac{\mathrm{d}s}{\gamma(s)} = \int_{s=0}^{s=t} \frac{1}{1+g^2 s^2} \mathrm{d}s = \frac{\arctan(gt)}{g}.$$

We can also write t as a function of τ now:

$$t = \frac{\tan(g\tau)}{g},$$

which allows us to also write x as a function of τ:

$$x = \frac{gt}{\sqrt{1+g^2t^2}} = \frac{\tan^2(g\tau)}{1+\tan^2(g\tau)} = \frac{\tan^2(g\tau)}{\sec^2(g\tau)} = \sin^2(g\tau).$$

Problem 0.5. Optional problem

In the pole's reference frame, the doors are never simultaneously closed.

Relativity Physics 127 Homework 2

Due Wednesday April 16th 2025, 11:59pm on gradescope.

1. Time Dilation and Light Clocks.

   One way to understand time dilation is to consider a light clock, which uses light rays bouncing between mirrors placed at a fixed distance from each other. Each tick of the clock amounts to the time it takes the light to make one round trip between the mirrors. [This setup is described in detail in the Feynman lectures cahpter 15 (see extra material module on Canvas)]

   a) Consider two parallel mirrors extended along the x-direction, and a distance L apart in the y-direction. In the reference frame in which the mirrors are at rest, consider a light beam making a round trip from $y = 0$ to $y = L$ and back to $y = 0$, traveling only in the y-direction. Compute the time t it takes the light to make this trip. Now analyze the same process in a frame in which mirrors are moving with speed v to the right along the x-axis, so that the mirrors remain parallel to their direction of motion. Compute the light round-trip time t' in this frame and show that t is related to t' by the usual time-dilation formula.

   b) Consider the same setup as in the previous problem, but with the mirrors extended along the y-direction, at $x = 0$ and $x = L$ in their rest frame. Now the mirrors are perpendicular to their direction of motion.

2. Abberation: Coleman Problem 1.2

3. Time dilation and lifetime of unstable particles.

   The average lifetime ("half-life") of a π-meson in its own frame is 26 nanoseconds.

   a) Assume that a pion in produced in a cosmic ray collision in the atmosphere moves with a velocity $v = 0.95c$ with respect to the stationary earth frame. what is its lifetime as measured by an observer at rest on Earth?

   b) What is the average distance it travels before decaying as measured by an observer at rest on Earth?

4. Relation of 4-velocity and 4-acceleration to ordinary velocity and acceleration.

a) Consider a particle moving along a trajectory $x^\mu(\tau)$ parametrized by its proper time. Recall that the 4-velocity of the particle is

$$u^\mu = \frac{dx^\mu}{d\tau} \ , \qquad u \cdot u = g_{\mu\nu} u^\mu u^\nu = 1 \ .$$

Express the components of u^μ in terms of the particle's ordinary 3-velocity $v^i = \frac{dx^i}{dt}$ and check that the resulting expression always satisfies $u \cdot u = 1$.
Similarly, express the components of the 4-acceleration $a^\mu = du^\mu/d\tau$ in terms of the 3-velocity v^i and the 3-acceleration $a^i = d^2x^i/dt^2$ and check that the resulting expression satisfies $a \cdot u = 0$ (which we derived in class). Use your expression to show that, in a frame where $v^i = 0$, we have $a^\mu = (0, a^i)$. This justifies some of the statements I made in class about the accelerating rocket.

b.) Consider a particle moving along the x-axis with ordinary velocity

$$v^x = \frac{dx}{dt} = \frac{gt}{\sqrt{1 + (gt)^2}} \ , \qquad g = \text{constant} \ .$$

Does the particle ever reach the speed of light? Calculate the components of the particle's 4-velocity. Express x and t as functions of the proper time τ along the trajectory.

5. [**Extra credit will not be relevant for quiz 2**]. The Pole-In-Barn Paradox: a pole of proper length L_* (i.e. the length as measured in its rest frame) is carried so fast along the direction of its length that it appears to have much shorter length $L < L_*$ in the lab frame. The runner carries the pole into a barn of length exactly L, so that for just an instant both the front and the back door of the barn can be shut, fully enclosing the pole, before they are opened again as the runner exits. From the runner and pole's point of view, however, the pole has length L_* greater than the length of the barn, which appears contracted to a length less than L. Thus the pole can never be enclosed in the barn! Explain, using a spacetime diagram depicting the barn doors and the pole ends, how this apparent paradox is resolved.