Physics 127 Homework #4

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Problem 0.1.

Since Λ is a linear operator, it is equal to the Jacobian. So the new measure after the Lorentz transformation $y^\mu \to \Lambda^\mu_\nu x^\nu$ is given by

$$\mathrm{d}^4x \to \mathrm{d}^4y = J\mathrm{d}^4x = \det\left(\frac{\partial y^i}{\partial x^j}\right)\mathrm{d}^4x = \det\left(\frac{\partial(\Lambda x^i)}{\partial x^j}\right) = \det(\Lambda)\mathrm{d}^4x.$$

By the definition of a Lorentz transformation, $\Lambda^T g \Lambda = g$. Taking the determinant of both sides gives $\det(\Lambda)\det(g)\det(\Lambda) = \det(g)$, so $\det(\Lambda)^2 = 1$, which means $\det(\Lambda) = \pm 1$. If we assume that Λ is a proper Lorentz transformation, then by definition, it has determinant 1.

For a 4-dimensional region V of spacetime, since the Lorentz transformation maps d^4x to $\det(\Lambda)\mathrm{d}^4x = \mathrm{d}^4x$, and the Lagrangian density $\mathcal{L}$ is Lorentz invariant, the action

$$I = \int_V \mathcal{L}\mathrm{d}^4x$$

is Lorentz invariant. However, if we do not assume that Λ is proper, then only $|I|$ is invariant.

Problem 0.2.

If V_4 is a 4-dimensional region of spacetime with boundary $\partial V_4 = \Sigma_3$, and V^μ is a conserved current with gradient $\partial_\mu V^\mu = 0$, then the following integral does not depend on what Σ_3 is, because it is always zero:

$$\int_{\Sigma_3} V^\mu \mathrm{d}\sigma_\mu = \int_{V_4} \partial_\mu V^\mu \mathrm{d}^4x = \int_{V_4} 0 \cdot \mathrm{d}^4x = 0.$$

However, since we defined Σ_3 to be the boundary of V_4, $\Sigma_3 := \partial V_4$ must be a closed surface. Therefore, I'm not sure how to generalize this to work when Σ_3 is a timeslice, because a timeslice cannot be a closed surface.

Problem 0.3.

Problem 0.4.

Problem 0.5.

Relativity Physics 127 Homework 4

Due Wednesday April 30th 2025, 11:59pm on gradescope.
There is a 24hr grace period this time.

1. Recall from multivariable calculus that a change of variables $x^i \to y^i(x)$ transforms the measure via the determinant of the Jacobian matrix associated with the change of variables:

$$d^3x \to d^3y = Jd^3x \ , \qquad J = \det\left(\frac{\partial y^i}{\partial x^j}\right) \ .$$

Argue that the corresponding formula for the spacetime volume element d^4x under a Lorentz transformation $y^\mu = \Lambda^\mu{}_\nu x^\nu$ is given by

$$d^4x \to \det\Lambda \, d^4x \ .$$

Use the defining properties of Lorentz transformations to show that $\det\Lambda = 1$, so that the measure d^4x is Lorentz invariant. Conclude that the action

$$I = \int d^4x \mathscr{L}$$

is Lorentz invariant if the Lagrangian density $\mathscr{L}$ is a Lorentz scalar as well.

2. Gauss' theorem applied to a region V_4 in spacetime with boundary $\partial V_4 = \Sigma_3$ takes the form

$$\int_{V_4} d^4x \partial_\mu V^\mu = \int_{\Sigma_3} d\sigma_\mu V^\mu \ .$$

Here $d\sigma_\mu$ is the outward pointing area element of the boundary Σ_3. By choosing a suitable V_4 (see section 2.4.1 in Coleman's book), show that for a conserved current satisfying $\partial_\mu V^\mu = 0$, we have

$$\int_{\Sigma_3} d\sigma_\mu V^\mu = \int_{\Sigma_3'} d\sigma_\mu V^\mu$$

for any two spacelike hypersurfaces (i.e. timeslices) Σ_3 and Σ_3'.

Use this to conclude that the charge obtained by integrating the 0-component V^0 of the current over a constant x^0 slice,

$$Q = \int d^3x \, V^0 \ ,$$

is in fact a Lorentz scalar.

3. In ordinary particle mechanics the canonical momentum p of q is given by

$$p = \frac{\partial L}{\partial \dot{q}} \ ,$$

and the Hamiltonian or total energy is

$$H = p\dot{q} - L \ .$$

Consider the free scalar field $\phi(x)$ with Lagrangian density

$$\mathscr{L} = \frac{1}{2}\left(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2\right) \ .$$

Find the canonical momentum density

$$\pi^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \ ,$$

and the Hamiltonian density

$$\mathscr{H} = \pi^0 \partial_0 \phi - \mathscr{L} \ ,$$

and show that $\mathscr{H} = T^{00}$ is the energy density obtained from the stress tensor of the scalar field defined in class.

4. Use the conservation and symmetry of the stress tensor $T^{\mu\nu}$ to show that

$$\partial^\mu (T_{\mu\sigma} x_\lambda - T_{\mu\lambda} x_\sigma) = 0 \ .$$

Conclude that

$$J^{\mu\nu} = \int_{\Sigma_3} d\sigma_\rho \left(T^{\rho\mu} x^\nu - T^{\rho\nu} x^\mu\right)$$

is independent of the choice of spacelike hypersurface Σ_3, e.g. we could take it to be any constant time slice. This shows that $J^{\mu\nu}$ is the conserved angular momentum.

5. Coleman problem 2.6.