Math 110BH Homework 4

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1

Prove that the ideal in $\mathbb{Z}[\sqrt{-5}]$ generated by 2 and $1+\sqrt{-5}$ is not principal.

2

Determine whether the ring $\mathbb{Z}[\sqrt{5}]$ is a PID.

We can prove that it's not a UFD

3

Let $R = \mathbb{Z}[i]$ be the ring of Gauss integers and let p be a prime integer such that $p \equiv 3 \pmod 4$. Prove that p is prime in R.

Suppose there exist $a + bi, c + di \in R$ such that $p = (a + bi)(c + di)$. Then

$$p = (ac - bd) + (bc + ad)i$$

4

Let $R = \mathbb{Z}[i]$ and let p be a prime integer such that $p \equiv 1 \pmod 4$.

- (a) Prove that p is not prime in R. (Hint: use HW 5, Problem 9 in 110AH).
- (b) Prove that there are integers a and b such that $p = a^2 + b^2$.

5

Let R be a PID and let a be a prime element in R. Prove that the ideal pR is maximal.

6

Prove that the product of two Noetherian rings is also Noetherian.

7

An integral domain in which every ideal generated by two elements is principal is called a *Bezout domain*. Prove that a ring R is a PID if and only if R is a Noetherian Bezout domain.

8

Let $R_1 \subset R_2 \subset R_3 \subset \dots$ be a chain of countably many subrings of a ring R such that $R = \cup R_i$. Suppose that all the R_i are UFD and any prime element in R_i is prime in R_{i+1}. Prove that R is a UFD.

9

Prove that the polynomial ring $\mathbb{Z}[x_1, x_2, x_3, \dots]$ in countably many variables is a UFD but not a Noetherian ring.

It's not a Noetherian ring because we have a chain of infinitely many ideals:

$$\mathbb{Z} \subset \mathbb{Z}[x_1] \subset \mathbb{Z}[x_1, x_2] \subset \cdots$$

but it's a UFD because every polynomial in that ring belongs to an ideal of the form

$$\mathbb{Z}[x_1, x_2, \dots, x_n]$$

and we can use induction to prove that all ideals of that form are UFDs.

10

Let $R = \mathbb{Z}[\sqrt{-5}]$. Prove that the product of the two ideals $2R + (1+\sqrt{-5})R$ and $3R + (1+\sqrt{-5})$ in R is the principal ideal $(1+\sqrt{-5})R$.