1 5/29/2024 lecture

- Casimir element
- Spin groups
- Lenz vector
- Use Frobenius theorem to show that only 0, 1, and 3 dimensional spheres are Lie groups
- Review universal covers
- See Sakurai "Modern QM" chapter 4 and Kirillov Chapter 7
- Show that $\mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{so}(4)$
- Root spaces

1.1 Cartan subalgebras

REVIEW ADJOINT REPRESENTATION

Let G be a real, n-dimensional, compact Lie group, and $\mathfrak{g}$ be it's Lie algebra. For any $x \in \mathfrak{g}$, define the function

$$\mathrm{ad}_x : \mathfrak{g}^{\mathbb{C}} \to g^{\mathbb{C}}, \qquad \mathrm{ad}_x(y) = [x, y].$$

and let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal abelian subalgebra, called the "Cartan subalgebra".

For any linear map $\alpha : \mathfrak{h} \to \mathbb{C}$, which we call a "root", we can define a "root space" $\mathfrak{g}_\alpha$, which is the set of elements $x \in \mathfrak{g}^{\mathbb{C}}$ such that $\mathrm{ad}_h(x) = [h, x] = \alpha(h)x$ for any $h \in \mathfrak{h}$.

If W is the set of weights of the representation $\mathfrak{g}^{\mathbb{C}}$, then

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in W} \mathfrak{g}_\alpha \right)$$

is called the "root space decomposition".

1.2 Killing form

An inner product $\langle \cdot | \cdot \rangle$ is called degenerate iff the statement "$\langle x | y \rangle = 0$ for every y" implies $x = 0$.

In the coordinate system that we talked about, we can define an inner product (x, y) called the "Killing form" as

$$(x, y) = \mathrm{Tr}_{\mathfrak{g}} \left(\mathrm{ad}_x \, \mathrm{ad}_y \right).$$

The Lie algebra $\mathfrak{g}$ is called "semisimple" iff the Killing form is non-degenerate.

Theorem 1.1. $\mathfrak{g}$ is semisimple iff (1) it has no center and (2) it is a direct sum of simple Lie algebras.