Physics 231B Lecture Notes

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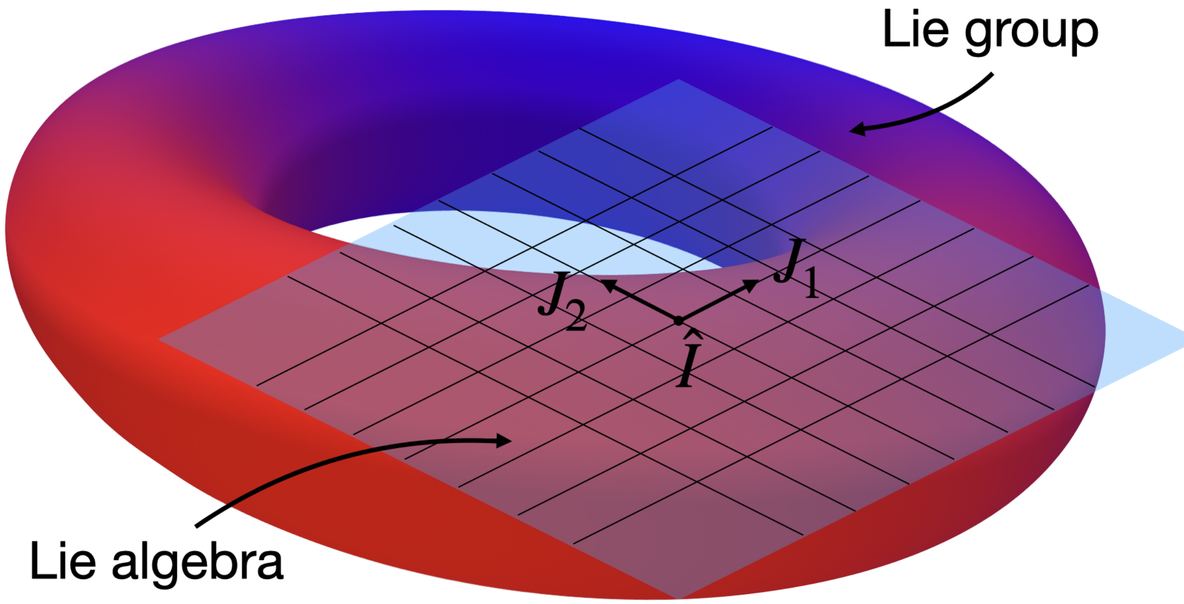

Contents

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1.1 Syllabus

The instructor's email is `ryan.thorngren@physics.ucla.edu` and his office hours will probably be Fridays from 1-2pm. The TA, Yanyan Li, can be reached at `yanyanli@ucla.edu`

Grading is based on weekly problem sets, which will be due Fridays. To get an A, you need to put in a good effort on all of them.

There course topics and textbooks are:

- Group theory and representation theory of finite group (Artin's *Algebra*)
- Crystallographic groups (Sternberg's *Group Theory and Physics*)
- Lie groups (Hall's *Lie Groups, Lie Algebras, and Representations* and Georgi's *Lie Algebras in Particle Physics*)
- If we have spare time, we can cover vector bundles and characteristic classes (Nakahara's *Geometry, Topology, and Physics*)

1.2 Intro to groups

A *group* is a pair $(G, \cdot)$ where G is a set and $\cdot : G \times G \to G$ is a binary operator, such that the following are all true.

- (Unital axiom) There is some $e \in G$, called the *identity element*, such that for any $g \in G$, $eg = g = ge$.
- (Associative axiom) For any 3 elements $a, b, c \in G$, $(ab)c = a(bc)$.
- (Invertible axiom) For any $g \in G$, there exists some element $g^{-1} \in G$ such that $gg^{-1} = e = g^{-1}g$.

A group is called *abelian* iff $ab = ba$ for any two elements a, b in the group. It is called *nonabelian* iff it is not abelian.

Note. Depending on context, the operator $\cdot$ can be called "addition", "multiplication", "composition", or just "the group operation". It is often written as $+$, $\times$, $*$, or $\circ$. Repeated multiplication is denoted by g^n, where $g \in G$ and $n \in \mathbb{N}$, and repeated addition is denoted by $ng = g + g + \cdots + g$ (n times). Additive notation is pretty much only used for abelian groups, but not every abelian group is written additively.

Physicist generally care about groups because they represent symmetries. For example, if you multiply every electric charge in the universe by a complex number with magnitude 1 (that is, by an element of $U(1)$), the Lagrangian doesn't change.

Example 1.1. The set of transformations of a regular polyhedron forms a group, when you define the group operation to be composition of transformations. Note that no matter which of the 5 regular polyhedra we choose, this is not a commutative group.

Proposition 1.2. The rotational group of a tetrahedron is $T_3 \cong A_4$, the rotational group of a cube or octahedron is S_4, and the rotational group of a dodecahedron or icosahedron is A_5. We will not prove this, for now.

For any integer $n > 0$, we define the *symmetric group* S_n to be the set of permutations of n elements, with the group operation being composition. Alternatively, we can define S_n to be the set of bijections $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Lemma 1.3. Every permutation is a product of disjoint cyclic permutations.

Corollary 1.4. Since every cyclic permutation is a product of transpositions, we can also say that every permutation is a product of transpositions.

The *order* of a finite group G, denoted $|G|$, is the number of elements in G. For example, the order of S_n is $|S_n| = n!$. The *order* of an element $g \in G$ is the smallest integer $n > 0$ such that $g^n = e$.

Note. It's easy to get confused between the terms "symmetry group" and "symmetric group. The symmetry group of something is the set of transformations that preserve some property of it. The symmetric group of a set is the group of permutations of elements in that set.

1.3 Group homomorphisms

Let G, H be groups. A map $\varphi : G \to H$ is called a *(group) homomorphism* iff for every $g_1, g_2 \in G, \varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$. Furthermore, a group homomorphism φ is called a *(group) isomorphism* iff it is also bijective.

Group isomorphism is an equivalence relation, which we denote with $\cong$.

Example 1.5. The map $\exp : \mathbb{R} \to \mathbb{R}_{\neq 0}$ is a group isomorphism (from $(\mathbb{R}, +)$ to $(\mathbb{R}_{\neq 0}, \times)$), because $e^{a+b} = e^a e^b$ for any $a, b \in \mathbb{R}$ and it is bijective.

One helpful trick for proving that a homomorphism is an isomorphism is to note that a group homomorphism $\varphi : G \to H$ is injective iff $\varphi(g) = e_H$ implies $g = e_G$. This is fairly easy to prove.

Theorem 1.6. Let $\widetilde{T_3}$ be the set of symmetries of a regular tetrahedron. Then $\widetilde{T_3} \cong S_4$.

Proof. Label the edges of a regular tetrahedron as $\{1, 2, 3, 4\}$ and let $\varphi : \widetilde{T_3} \to S_4$ be the map which takes a transformation of the tetrahedron to the corresponding permutation of its vertices. This is clearly an injective homomorphism, so we just need to show that it's also surjective.

For any two vertices of a regular tetrahedron, consider the pependicular bisector of the edge that joins them. Reflection across that plane is an element of $\widetilde{T_3}$, and φ takes that element to a transposition in S_4. We can obtain every transposition in S_4 this way, and we know that every element of S_4 is a product of transpositions, so φ is surjective. $\square$

Proposition 1.7. More generally, the symmetry group of an n-simplex is S_{n+1}.

Proof. The proof of this is exactly the same as it is for a 3-simplex (tetrahedron), except the perpendicular bisector is now an $(n - 1)$ dimensional hyperplane instead of a plane, and the number of vertices to be permuted is now $n + 1$ instead of 4. $\square$

1.4 Generators and relations

A subset $S \subset G$ is said to *generate* G iff every element of G can be written as a finite product of the elements of S and the inverses of the elements of S. Elements of the generating set S are called *generators*.

A *word* in a group G is an expression of the form $g_{i_1}^{\pm 1} g_{i_2}^{\pm 1} \cdots g_{i_n}^{\pm 1}$. Two words are considered different if they are written differently, even if the expressions they represent are equal. A word which represents an expression equal to the identity is called a *relation*.

Example 1.8. The symmetric group S_n is generated by the transpositions $\{s_1, s_2, \dots, s_{n-1}\}$, where $s_i = (i, i+1)$. Since every transposition has order 2, we have the relation $s_i^2 = e$. We also have the relation $s_i s_j = s_j s_i$ whenever $|i-j| > 1$, because s_i and s_j are disjoint – this can also be written as $s_i s_j s_i^{-1} s_j^{-1}$. Lastly, we have the relation that $(s_i s_{i+1})^3 = e$ (whenever $i < n-1$), because $s_i s_{i+1}$ is a 3-cycle.

Given a set S of symbols and a set R of relations in S, we define the *group presentation* $\langle S | R \rangle$ to be the group of equivalence classes of words in S (with the equivalence relation making every element of R equal to the identity, and with the group operation being concatenation).

The *cyclic group (of n elements)* is the group $C_n := \langle s | s^n \rangle$. C_n is an abelian group of order n, in which s has order n.

Proposition 1.9.

$$S_n \cong \langle s_1, \dots, s_{n-1} | \left\{ s_i^2 \right\} \cup \left\{ s_i s_j s_i^{-1} s_j^{-1} : |i-j| > 1 \right\} \cup \left\{ (s_i s_{i+1})^3 \right\} \rangle$$

Proof. The proof of this is a painful exercise. You would need to show that there is a map from the right hand side to S_n which is an injective and surjective homomorphism. □

Theorem 1.10. Good news: every group has a presentation. Bad news: given a finite presentation, there is no algorithm to determine any of the following.

- Whether two words are equal
- Whether the group is finite
- Whether the group is *trivial* (meaning isomorphic to the *trivial group* $\{e\} = 1$)

This is because simplifying a word often requires expanding it before simplifying it, and there is no upper bound on how much you need to expand it before you can start simplifying. Therefore, even a brute force algorithm for any of those three problems could take arbitrarily long to run.

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2.1 Matrix groups

Matrix groups are groups whose elements are matrices. For any field $\mathbb{F}$ and any $n \in \mathbb{N}$, we define the *general linear group* $GL(n, \mathbb{F})$ to be the multiplicative group of invertible (nonsingular) $n \times n$ matrices whose entries are elements of $\mathbb{F}$. $GL(n, \mathbb{F})$ is a subset of $M(n, \mathbb{F})$ (the set of $n \times n$ matrices over $\mathbb{F}$). $M(n, \mathbb{F})$ is an algebra over $\mathbb{F}$ but is not a group.

Note. In this class, we will always let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$.

The determinant is a homomorphism $\det : GL(n, \mathbb{F}) \to \mathbb{F}^\times$. Here, $\mathbb{F}^\times$ is the group of invertible elements in $\mathbb{F}$. More generally, we can define the multiplicative group $R^\times$ of any ring R to be the group of invertible elements (AKA units), but if R is a field, then every element except 0 is invertible, so $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$.

2.2 Subgroup

A *subgroup* H of a group G is a subset of the elements of G which contains the identity, is closed under inversion, and is closed under composition. In other words, H is a subgroup of G if and only if for any $h_1, h_2 \in H$,

- $e_G \in H$
- $h_1^{-1} \in H$
- $h_1 h_2 \in H$

Every subgroup is a group. The notation for "H is a subgroup of G" is $H \leq G$. If H is a *proper subgroup* of G, meaning $H \leq G$ and $H \neq G$, then we write $H < G$.

If H is a group, we can say that $H \leq G$ if and only if there is an injective homomorphism $i : H \to G$.

2.3 Subgroups of the general linear group

We define the *orthogonal group* $O(n)$ to be the set of orthogonal matrices in $GL(n, \mathbb{R})$. Since every orthogonal matrix A satisfies $A^T A = 1$, the determinant of A must be ± 1. Therefore we define the *special orthogonal group* $SO(n)$ to be the set of matrices in $O(n)$ with determinant 1.

$$SO(n) < O(n) < GL(n, \mathbb{R})$$

The *special linear group* $SL(n, \mathbb{F})$ is the group of matrices in $GL(n, \mathbb{F})$ with determinant 1.

Similarly, we define the *unitary group* $U(n)$ to be the subgroup of unitary matrices in $GL(n, \mathbb{C})$, and we define the *special unitary group* $SU(n, \mathbb{C})$ to be $U(n) \cap SL(n, \mathbb{C})$ – that is, the set of $n \times n$ complex unitary matrices with determinant 1.

Example 2.1. Let T_3 be the group of rotational symmetries of a tetrahedron, and let $\widetilde{T_3}$ be the group of all symmetries (rotations plus reflections) of a tetrahedron. Then $T_3 < \widetilde{T_3} < O(3)$, and $T_3 < SO(3)$.

Proposition 2.2. $GL(n, \mathbb{C})$ is a proper subgroup of $GL(2n, \mathbb{R})$.

Proof. The map which takes every entry $a+bi$ to the 2×2 block $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is an isomorphism between $GL(n, \mathbb{C})$ and a proper subgroup of $GL(2n, \mathbb{R})$. □

2.4 Representations

A *representation* of G is a homomorphism from G to $GL(n, \mathbb{F})$. This allows us to make groups slightly more intuitive, since we can see them as linear transformations on $\mathbb{F}^n$. A representation is called *faithful* iff it is injective.

2.5 Cosets and quotient groups

If $H \leq G$, then for any $g \in G$, define the *left coset* gH to be $\{gh : h \in H\}$, and let the *right coset* Hg be $\{hg : h \in H\}$. Note that cosets are not necessariy subgroups – for example, the set of odd integers is not a subgroup of $\mathbb{Z}^+$ (the additive group of integers).

Example 2.3. There are only two distinct cosets of $SO(n)$ in $O(n)$: those with determinant 1, and those with determinant -1.

If H is a subgroup of G, we define the *quotient* G/H to be the set of left H-cosets.

The *index of H in G*, denoted $[G : H]$, is defined as $|G/H|$ (the number of left cosets of H in G).

Theorem 2.4. **Lagrange's theorem:** If H is a subgroup of a finite group G, then

$$|G/H| = |G|/|H|.$$

Proof. Every element of G is in at least one coset, because $g \in gH$. For any g, $|gH| = |H|$. Therefore the number of cosets ($|G/H|$) times the number of elements per coset ($|H|$) is equal to the total number of elements in the group G. □

Corollary 2.5. The order of a finite group is divisible by the order of any element.

A subgroup $H \leq G$ is called a *normal subgroup*, and denoted by $H \trianglelefteq G$ iff $g^{-1}hg \in H$ for any $g \in G, h \in H$ – that is, iff H is invariant under *conjugation* by any $g \in G$.

Theorem 2.6. For any cycle $(x_1 \;\; x_2 \;\; \ldots \;\; x_k) \in S_n$, the conjugate of that cycle by some permutation $\sigma \in S_n$ is

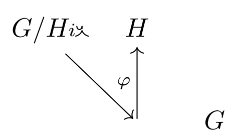
$$\sigma^{-1}(x_1 \;\; x_2 \;\; \ldots \;\; x_k)\sigma = (\sigma(x_1) \;\; \sigma(x_2) \;\; \ldots \;\; \sigma(x_k)).$$

Theorem 2.7. G/H is a group iff $H \trianglelefteq G$. In this case, G/H is called the *quotient group*.

Proof. If $H \trianglelefteq G$, then we can define the product of two cosets $g_1H, g_2H \in G/H$ to be g_1g_2H. If H were not a normal subgroup of H, than G/H would not be closed under multiplication. □

2.6 First isomorphism theorem

If $\varphi : G \to K$ is a homomorphism, then define the *kernel of* φ, denote $\ker \varphi$, to be the set of elements in G which φ maps to e_K. The kernel of any homomorphism forms a normal subgroup of the codomain. REMEMBE TO PROVE THIS TOO. ALSO DEFINE THE IMAGE AND USE A COMMUTATIVE DIAGRAM TO PROVE THE FIRST ISOMORPHISM THEOREM



2.7 Product of groups

If G, H are groups, then the group $G \times H$ is the group of pairs (g, h) where composition is given by componentwise composition. If G and H are finite, then $|G \times H| = |G| \times |H|$. ISNT THIS CALLED THE DIRECT SUM??

ALSO INCLUDE NOTES ON SHORT EXACT SEQUENCES, SIMPLE GROUPS (note that integers modulo a prime are a simple group), AND EXTENSIONS

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3.1 Quotient groups

Example 3.1. $SO(n)$ is a normal subgroup of $O(n)$, so we can define the quotient group $O(n)/SO(n)$, which is isomorphic to $C_2 := \langle x | \, x^2 = e \rangle \cong \{\pm 1\}^\times$.

Let $n\mathbb{Z}$ be the subgroup $\{nm : m \in \mathbb{Z}\}$. Since $\mathbb{Z}$ is an additive group, be sure not to confuse $n\mathbb{Z}$ with the coset $n + \mathbb{Z}$. We know that $n\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$ – it's easy to prove that every subgroup of an abelian group is normal.

Now we can define the *group of integers modulo n* to be $\mathbb{Z}/n\mathbb{Z}$. Some people write this as $\mathbb{Z}_n$, because that's shorter.

Theorem 3.2. For any $N \in \mathbb{N}$, the cyclic group $C_n := \langle x | \, x^n = e \rangle$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Therefore, we can use C_n and $\mathbb{Z}_n$ interchangeably.

Proof. Let $\varphi : C_n \to \mathbb{Z}/n\mathbb{Z}$ be the homomorphism which maps x to the coset $1 + n\mathbb{Z}$ (and thus, also maps x^m to $m + n\mathbb{Z}$). You can easily shows that φ is an injective and surjective homomorphism. □

3.2 Exact sequences and extensions

A path in a commutative diagram is called an *exact sequence* iff the kernel of each morphism (except the first one) is equal to the image of the previous one. Right now, we only care about the category **Grp**, in which morphisms are group homomorphism. For example, if $H \trianglelefteq G$, then

$$0 \hookrightarrow H \xhookrightarrow{i} G \xrightarrow{\pi} G/H \twoheadrightarrow 0$$

is an exact sequence because $\ker \pi = \operatorname{im} i$.

A group G is called an *extension of Q by K* iff there is an exact sequence

$$0 \hookrightarrow K \xhookrightarrow{i} G \xrightarrow{\pi} Q \twoheadrightarrow 0.$$

Example 3.3. The *Klein 4-group* $K_4 := \mathbb{Z}_2 \times \mathbb{Z}_2$ and the group $\mathbb{Z}_4$ are distinct extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_2$.

3.3 Conjugacy classes

Two elements $g_1, g_2 \in G$ are called *conjugate* iff there exists some $h \in G$ such that $hg_1h^{-1} = g_2$. Conjugacy is an equaivalence relation, and the equivalence classes of that relation are called the *conjugacy classes*. The conjugacy class of $g \in G$ is written as

$$C(g) := \{h \in G : \; h \text{ and } g \text{ are conjugate}\}.$$

For matrices, conjugacy is the same as similarity, meaning two matrices are conjugate iff they represent the same linear transformation in different bases.

Since every permutation $\sigma \in S_n$, σ can be written as the product of disjoint cycles (by lemma 1.2 WHY IS THIS NUMBER OFF?), we can define the *cycle type* of a permutation to be the multiset of the lengths of those cycles (CHECK THAT THIS IS UNIQUELY DEFINED).

Theorem 3.4. The conjugacy class of some permutation $\sigma \in S_n$ is the set of permutations in S_n with the same cycle type as σ.

Proof. By 2.5. FINISH THIS PROOF. $\square$

Problem 3.5. How many conjugacy classes does S_n have? If this is too hard, just consider the $n = 4$ case.

Any permutation which is conjugate to $\sigma \in S_n$ must have the same number of 1-cycles as σ, the same number of 2-cycles, etc. Therefore each conjugacy class of S_n can be uniquely determined by a partition of n of the form $n = a_1 + a_2 + \cdots + a_m$, where $a_1 > a_2 > \cdots > a_m$. So if $n = 4$, there are 5 conjugacy classes of S_n:

- $4 = 4$
- $4 = 3 + 1$
- $4 = 2 + 2$
- $4 = 2 + 1 + 1$
- $4 = 1 + 1 + 1 + 1$

IS THERE A GENERAL FORMULA FOR THE NUMBER OF CONJUGACY CLASSES OF THE SYMMETRIC GROUP

3.4 The alternating group

The *sign of a permutation* is 1 if it can be written as a product of an even number of permutations, and -1 otherwise.

Let *the permutation matrix* $P(\sigma)$ of some permutation $\sigma \in S_n$ be the orthogonal matrix which permutes the basis vectors $e_i \in \mathbb{R}^n$. Then we can define the sign of σ to be $\det(P(\sigma))$. Note that the sign of any transposition is -1.

$$S_n \overset{P}{\hookrightarrow} O(n) \overset{\det}{\twoheadrightarrow} \mathbb{Z}_2.$$

Now we can define *the alternating group* A_n to be the kernel of $\det \circ P$. By Lagrange's theorem (CITE THAT), $|A_n| = n!/2$.

Proposition 3.6. For $n \geq 5$, A_5 is simple. In fact, every group of order less than 60 is *solvable*. This is not really relevant to us, but in Galois theory, this is used to prove the Abel-Ruffini theorem.

TO PROVE THAT A_5 IS SIMPLE, FIND THE SIZES OF ALL CONJUGACY CLASSES SINCE EVERY SUBGROUP OF A_5 CONTAINS EITHER AN ENTIRE CONJUGACY CLASS OF ITS ELEMENTS, THE SIZE OF ANY SUBGROUP OF A_5 IS THE SUM OF SOME SUBSET OF $(1, 15, 20, 12, 12)$ BUT THAT SUM CAN ONLY DIVIDE 60 IF IT IS EITHER 1 OR 60.

ALSO TALK ABOUT THE SYLOW THEOREMS

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4.1 Group actions

A *(left) group action* (of a group G on a set M) is a map $\cdot : G \times M \to M$ which is

1. Unital, meaning $e \cdot m = m$ for any $M \in M$, and
2. Associative, meaning $(gh) \cdot m = g \cdot (h \cdot m)$ for any $g \in G, m \in M$.

For any group G, there is an obvious (left) group action of G on itself, given by (left) multiplication.

GET NOTES ON LEFT/RIGHT GROUP ACTION, CAYLEY'S THEOREM, ORBITS (define conjugacy classes as orbits of the conjugation action), STABILIZER SUBGROUP, CENTER, DIHEDRAL GROUP, ORBIT-STABILIZER THEOREM, etc.

4.2 Dihedral group

We can define the action of this group on the elements of the xy-plane in terms of reflections and rotations. ALSO GIVE GROUP PRESENTATION

Note. W use D_n to mean the set of symmetries of a regular n-gon, which is a group with $2n$ elements, but some people use D_{2n} to mean that same group.

4.3 Orbit-stabilizer theorem

Theorem 4.1. If G acts on M, then

$$|G \cdot m| \times |G_m| = |G|$$

for any $m \in M$.
If G acts on M via a right group action instead of a left group action, then replace $G \cdot m$ with $m \cdot G$.

Proof. Let $f : G/G_m \to G \cdot m$ be a homomorphism defined by $f(gG_m) = g \cdot m$. This is well-defined because for any two representative elements gG_m and ghG_m of G/G_m,

$$f(gG_m) = g \cdot m = g \cdot (h \cdot m) = (gh) \cdot m = f(ghG_m).$$

Also, f is surjective because every element of the orbit has the form $g \cdot m$. It's injective because if $g \cdot m = g' \cdot m$, then

$$(g')^{-1} \cdot (g \cdot m) = (g')^{-1} \cdot (g' \cdot m) = m,$$

which implies $(g')^{-1} g \in G_m$, so $gG_m = g'G_m$. $\square$

Proposition 4.2. Let $C(g)$ be the conjugacy class of some element g of a group. If that group is S_n, then $C(g)$ can also be denoted as $C(a_1, \ldots, a_n)$, where a_i is the number of i-cycles in g. Then

$$|C(a_1, \ldots, a_n)| = \frac{n!}{\prod_{j=1}^{n} j^{a_j} (a_j!)}.$$

INCLUDE PROOF OF THIS

4.4 Fixed point theorem

Let G be a finite group acting on a set M such that for any $g \in G, g \neq e$, only finitely many m are fixed by g.

Let $Y = \{(m, g) : g \neq e, g \cdot m = m\}$. Then

$$\sum_{g \neq e} |\{m \in M : g \cdot m = m\}| = |Y| = \sum_{m \in M} (|G_m| - 1).$$

This is also equal to

$$\sum_{\text{orbits}} \left| \frac{G}{G_m} \right| (|G_m| - 1) \,.$$

where m is any element of the orbit.

Corollary 4.3. If G is a finite subgroup of $SO(3)$ which acts on the unit sphere $M = S^2$, then

$$2\left(1 - \frac{1}{n}\right) = r - \sum i = 1^r \frac{1}{n_i},$$

where r is the number of group orbits of ??? FINISH THIS PROOF AND EXPLAIN INTUITION FOR IT

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5.1 Finite subgroups of $SO(3)$

Let G be any finite subgroup of $SO(3)$. Let G act on the unit sphere S^2 and let P be the set of points with a nontrivial stabilizer – that is, the set of points on S^2 which are fixed by some nontrivial element of G. Then by the fixed-point theorem (SEE WHAT ARTIN SAYS ABOUT THIS THEOREM), the number of G-orbits in P is $r \leq 3$.

Example 5.1. Consider the subgroup of $SO(3)$ generated by a rotation by $2\pi/n$ around the z-axis and a rotation by π about the x-axis, where n is some odd number. This group turns out to be isomorphic to the dihedral group D_n. INCLUDE DRAWING OF THE AXES OF ROTATION FOR THIS GROUP.

Now consider other finite groups of $SO(3)$. These must satisfy the equation

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{n_3} = 1 + \frac{2}{n}.$$

HOW DID WE GET THIS EQUATION, WHAT ARE n and n_3??? The only solutions are $n \in \{12, 24, 60\}$, which corresponds to the rotational symmetry groups of the 5 platonic solids.

IS EVERY FINITE SUBGROUP OF $SO(3)$ ISOMORPHIC TO EITHER THE DIHEDRAL GROUP OR A POLYHEDRAL GROUP???

5.2 Lattice groups

A *d-dimensional lattice group* is a discrete subgroup L of $\mathbb{R}^d$ which is isomorphic to $\mathbb{Z}^d$. Equivalently, if we have a basis $\{v_1, v_2, \ldots, v_d\}$ of $\mathbb{R}^d$, then L is the additive group generated by (translations by) those vectors.

Suppose the rotation $R \in SO(3)$ preserves the lattice $L := \langle v_1, \ldots, v_d \rangle$. Let M be the change-of-basis matrix which maps e_i to v_i. Then $n_{ij} := M^{-1}RM$ is a matrix with integer entries which satisfies

$$R \cdot v_i = \sum_{j=1}^{d} n_{ij} v_j.$$

That means the trace of n_{ij} is an integer, but since $\operatorname{tr} R = \operatorname{tr} n$, the trace of R also has to be an integer. For example, if R is the rotation in the xy-plane

$$R = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then we have $\operatorname{tr} R = 2\cos\varphi + 1 \in \mathbb{Z}$, which means φ is an integer multiple of 2π divided by 1, 2, 3, 4, or 6.

5.3 Finite subgroups of $O(3)$

The center of $O(3)$ is $\{\pm I\}^{\times}$.

Let $\widetilde{O}$ be the set of symmetries of the cube in $O(3)$. This just happens to be isomorphic to S_4. HOW DO WE PROVE THIS???

More generally, consider a finite subgroup G of $O(3)$ which is not a subgroup of $SO(3)$. Then there are two cases of interest:

1. If $-I \in G$, then $G = G_+ \cup (-I) \cdot G_+$, where $G_+ := G \cap SO(3)$ is the subgroup of G with positive determinant and $G_- = G \backslash G_+ = (-I) \cdot G_+$ is the subset of G with negative determinant. This is the simpler, easier case.

2. If $-I \notin G$, then $G_- = G \backslash G_+ = \{g \in G : \det(g) = -1\}$. Define a new set $G' = G_+ \cup (-I) \cdot G_-$, which we can verify is a closed, finite subgroup of $SO(3)$. Also, we know that G_+ is a normal subgroup of G' (with index 2 in G'), because the conjugate of any element $R \in G_+$ by some $-S \in G' \backslash G_+$ is $(-S)^{-1}R(-S) = S^{-1}RS$ which is in G and has determinant 1, so it's in G_+.

Theorem 5.2. The set of symmetries of a cube is S_4.

Proof. The set of rotational symmetries of a tetrahedron is A_4. We can embed the vertices of a tetrahedron in the vertices of a cube in two different ways, and a reflection of the cube will convert between those two tetrahedra. □

Corollary 5.3. Given a cube, choose 4 vertices such that none of the vertices are adjacent to each other (there are two ways to do this) and label those vertices. Then any symmetry of the cube is uniquely determined by a permutation of those 4 vertices.

Problem 5.4. SHow that there are no subgroups of A_4 of size 6 (i.e. of index 2).

5.4 Crystallographic groups

A group G is called a *crystallographic group* iff it is a discrete subgroup of the group of affine transformations of $\mathbb{R}^d$ (that is, the group generated by translations, rotations, and reflections) and it contains a d-dimensional lattice subgroup L.

It follows that L is normal, and G/L is a finite subgroup of $O(3)$.

6 4/19/2024 lecture

6.1 Affine transformations of $\mathbb{R}^d$

Recall that there are 32 finite subgroups of $O(3)$ which act on a lattice. Since a lattice in 3 dimensions can be defined by 3 basis vectors, the image of any representation in that basis will have only integer entries.

Note. Any repeating pattern repeats according to translations which define a lattice group.

Let G be the group of all translations and orthogonal transformations in $\mathbb{R}^d$, with the group operation being composition. The subgroup of translations (the Lie group $\mathbb{R}^d$) is normal in G.

Let $\mathbb{R}^d \rtimes O(d) = \{(u, M) \in \mathbb{R}^d \times O(d)\}$ be the set with the group operation $(u_1, M_1) \circ (u_2, M_2) = (u_1 + M_1 u_2, M_1 M_2)$.

Problem 6.1. Verify that $\mathbb{R}^d \rtimes O(d)$ is a group.

Proposition 6.2. There is an isomorphism $\varphi : \mathbb{R}^d \times O(d) \to G$.

An *automorphism* of G is an isomorphism $\varphi : G \to G$. The *automorphism group* $\mathrm{Aut}(G)$ is the set of automorphisms of G with the group operation being composition.

An *action* of a group H on a group G is a homomorphism from H to $\mathrm{Aut}(G)$.

Given two groups G, H and an action $\alpha : H \to \mathrm{Aut}(G)$, define the *semi-direct product* $\rtimes_\alpha$ so that $G \rtimes_\alpha H$ is the set $G \times H$ with the group operation $(g_1, h_1) \circ (g_2, h_2) = (g_1 \alpha(h_1)(g_2), h_1 h_2)$.

Problem 6.3. Show that for any two groups G and H and any action $\alpha : H \to \mathrm{Aut}(G)$, the semi-direct product $G \rtimes_\alpha H$ is a group. Also show that $G \trianglelefteq G \rtimes_\alpha H$.

Problem 6.4. Show that $\mathbb{Z}_3 \rtimes \mathbb{Z}_2 \cong S_3$.

6.2 Crystallographic groups

A group G is called a *d-dimensional crystallographic group* iff $G < (\mathbb{R}^d \rtimes O(d))$ is discrete and $G \cap (\mathbb{R}^d \times \{I\})$ is a d-dimensional lattice group $L = \langle v_1, v_2, \ldots, v_d \rangle < \mathbb{R}^d$.

Proposition 6.5. $L \trianglelefteq G$. The proof of this is an exercise.

Problem 6.6. Show that G/L is a finite subgroup of $O(d)$, and if $d = 3$, then G/L is one of the 32 groups we classified.

We sometimes call G/L the "point group", and denote it by G_{pt}. There is an exact sequence

$$L \hookrightarrow G \twoheadrightarrow G/L.$$

Alternatively, given G_{pt} and the lattice group L on which it acts, we can define G as $L \rtimes G_{pt}$. In G, the stabilizer of the origin is G_{pt}, and the stabilizer of any other point in L is conjugate to G_{pt}.

If $d = 2$, these groups are called "wallpaper groups".

TALK ABOUT NON-SYMMORPHIC GROUPS AND GLIDE/SCREW GROUPS, ALSO EXPLAIN WHAT THE 32 POINT GROUPS (IN 3D) ARE

6.3 Recognizing semi-direct products

Given an extension

$$0 \hookrightarrow L \hookrightarrow G \xrightarrow{\pi} G_{pt} \twoheadrightarrow 0,$$

a *section* is a map $s : G_{pt} \to G$ (not necessarily a homomorphism) such that $s(e_{G_{pt}}) = e_G$ and $\pi \circ s = \mathrm{id}_G$.

Theorem 6.7. Thee exists a section s which is a homomorphism iff $G = L \rtimes G_{pt}$.

Proof. REDO THIS PROOF ∎

Proposition 6.8. A d-dimensional space group is symmorphic iff it is a semi-direct product. JUSTIFY WHY THIS IS TRUE AND DEFINE WHAT IT MEANS TO BE SYMMORPHIC

6.4 Group cohomology

SEE PAGE 91 OF BROWN'S "COHOMOLOGY OF GROUPS"

There is a technique that can be used to measure how far a group is from being a semidirect product. GET NOTES FROM CLASSMATES ABOUT THIS, AND INCLUDE THE 2-COCYCLE EQUATION. THE LINEARITY OF THAT EQUATION CAN BE USED TO CLASSIFY AND ENUMERATE THE 230 SPACE GROUPS.

WHAT IS THE DIFFERENCE BETWEEN A CRYSTALLOGRAPHIC GROUP AND A SPACE GROUP?