Physics 245 Homework #9

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Problem 0.1.

(a)

$$\rho = \frac{1}{2}\left(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right) = \frac{1}{2}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}.$$

(b)

$$\begin{aligned} [\sigma_x] &= \operatorname{tr}(\sigma_x\rho) \\ &= \operatorname{tr}\left(\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}\right) \\ &= \frac{1}{2}\operatorname{tr}\left(\begin{bmatrix} r_x + ir_y & 1-r_z \\ 1+r_z & r_x - ir_y \end{bmatrix}\right) \\ &= \frac{1}{2}\left((r_x + ir_y) + (r_x - ir_y)\right) \\ &= r_x. \end{aligned}$$

(c)

$$\begin{aligned} [\sigma_z] &= \operatorname{tr}(\sigma_z\rho) \\ &= \operatorname{tr}\left(\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}\right) \\ &= \frac{1}{2}\operatorname{tr}\left(\begin{bmatrix} 1+r_z & r_x - ir_y \\ -r_x - ir_y & r_z - 1 \end{bmatrix}\right) \\ &= \frac{1}{2}\left((1+r_z) + (r_z - 1)\right) \\ &= r_z. \end{aligned}$$

Problem 0.2.

(a) Using the result from problem 1, we know that $r_x = [\sigma_x] = 1$, $r_y = [\sigma_y] = 0$, and $r_z = [\sigma_z] = 0$, so the density matrix is

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(b) This is a pure state, because if we define

$$|\psi\rangle := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

then $\rho = |\psi\rangle\langle\psi|$.

(c) See the Jupyter notebook. There appears to only be one state/vector, because they overlap perfectly.

(d) By the same reasoning as in part (a),

$$\rho = \frac{1}{2}\left(I + 0.7\sigma_x\right) = \begin{bmatrix} 0.5 & 0.35 \\ 0.35 & 0.5 \end{bmatrix}.$$

(e) A state is pure iff the density matrix squared has a trace of 1.

$$\begin{aligned} \operatorname{tr}(\rho^2) &= \operatorname{tr}\left(\begin{bmatrix} 0.5 & 0.35 \\ 0.35 & 0.5 \end{bmatrix}^2\right) \\ &= \operatorname{tr}\left(\begin{bmatrix} 0.3725 & 0.35 \\ 0.35 & 0.3725 \end{bmatrix}\right) \\ &= 0.745 \\ \operatorname{tr}(\rho^2) &< 1. \end{aligned}$$

Therefore this is not a pure state.

To show it's not a pure state, we just need to show $\rho^2 \neq \rho$, but taking the trace of ρ^2 gives a measure of how mixed the state is (1 means pure, and 0 means completely mixed).

(f) See the Jupyter notebook.

Problem 0.3.

(a) The density matrix is equal to

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left(|1,0,n\rangle\langle 1,0,n| + |0,1,n\rangle\langle 1,0,n| + |1,0,n\rangle\langle 0,1,n| + |0,1,n\rangle\langle 0,1,n|\right)$$

I will not bother to actually write this whole thing out in matrix form, because it would be really messy.

(b) Ignoring the harmonic oscillator, we are left with

$$\begin{aligned} \rho_S &= \frac{1}{2}\left(|1,0\rangle\langle 1,0| + |0,1\rangle\langle 1,0| + |1,0\rangle\langle 0,1| + |0,1\rangle\langle 0,1|\right) \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

This matrix is its own square (that is, $\rho_S^2 = \rho_S$), so it represents a pure state.

(c)
$$\rho = |\psi\rangle\langle\psi| = \left[a\,|1,0,n\rangle + a\,|0,1,n\rangle + b\,|0,0,n+1\rangle\right]\left[a^*\langle 1,0,n| + a^*\langle 0,1,n| + b^*\langle 0,0,n+1|\right]$$

If we expand this out, we get

$$\begin{aligned} \rho = &|a|^2\,|1,0,n\rangle\langle 1,0,n| + |a|^2\,|1,0,n\rangle\langle 0,1,n| + ab^*\,|1,0,n\rangle\langle 0,0,n+1| + \\ &|a|^2\,|0,1,n\rangle\langle 1,0,n| + |a|^2\,|0,1,n\rangle\langle 0,1,n| + ab^*\,|0,1,n\rangle\langle 0,0,n+1| + \\ &a^*b\,|0,0,n+1\rangle\langle 1,0,n| + a^*b\,|0,0,n+1\rangle\langle 0,1,n| + |b|^2\,|0,0,n+1\rangle\langle 0,0,n+1|. \end{aligned}$$

(d) Taking the partial trace over the QHO leaves us with

$$\rho_S = |a|^2 (|10\rangle + |01\rangle)(\langle 10| + \langle 01|) + |b|^2 |00\rangle \langle 00| .$$

The square of the reduced density matrix is

$$\rho_S^2 = 2 |a|^4 (|10\rangle + |01\rangle)(\langle 10| + \langle 01|) + |b|^4 |00\rangle \langle 00| ,$$

which is equal to ρ_S iff $|a|^2 \in \left\{0, \frac{1}{2}\right\}$ and $|b|^2 \in \{0, 1\}$. Since the normalization of $|\psi\rangle$ implies $2|a|^2 + |b|^2 = 1$, the only way this can be a pure state is if one of the following is true:

- $b = 0$, and a is some phase factor times $1/\sqrt{2}$
- b is some phase factor times 1, and $a = 0$.

Problem 0.4.

(a)

$$\begin{aligned} K_0 &= \langle e_0| U_{SE} |e_0\rangle \\ &= \sqrt{1-p} I_S \\ K_1 &= \langle e_1| U_S E |e_0\rangle \\ &= \sqrt{p} X_S, \end{aligned}$$

and for any $i \notin \{0, 1\}$, $K_i = 0$.

(b)

$$\begin{aligned} \rho(t) &= \sum_{i=0}^{1} K_i \rho(0) K_i^\dagger \\ &= \sqrt{1-p} I_S \rho(0) I_S^\dagger + \sqrt{p} X_S \rho(0) X_S^\dagger \\ &= \sqrt{1-p} |0\rangle \langle 0| + \sqrt{p} (X_S |0\rangle)(\langle 0| X_S^\dagger) \\ &= \sqrt{1-p} |0\rangle \langle 0| + \sqrt{p} |1\rangle \langle 1| . \end{aligned}$$

(c)

$$\begin{aligned} \rho(0) &= |\psi\rangle \langle \psi| \\ \rho(t) &= \sum_{i=0}^{1} (K_i |\psi\rangle)(K_i |\psi\rangle)^\dagger \\ &= \left(\sqrt{1-p} I |\psi\rangle\right) \left(\sqrt{1-p} I |\psi\rangle\right)^\dagger + (\sqrt{p} X |\psi\rangle) (\sqrt{p} X |\psi\rangle)^\dagger \\ &= (1-p) |\psi\rangle \langle \psi| + p |\psi\rangle \langle \psi| \\ &= |\psi\rangle \langle \psi| . \end{aligned}$$

Problem 0.5.

See Jupyter notebook.

notebook

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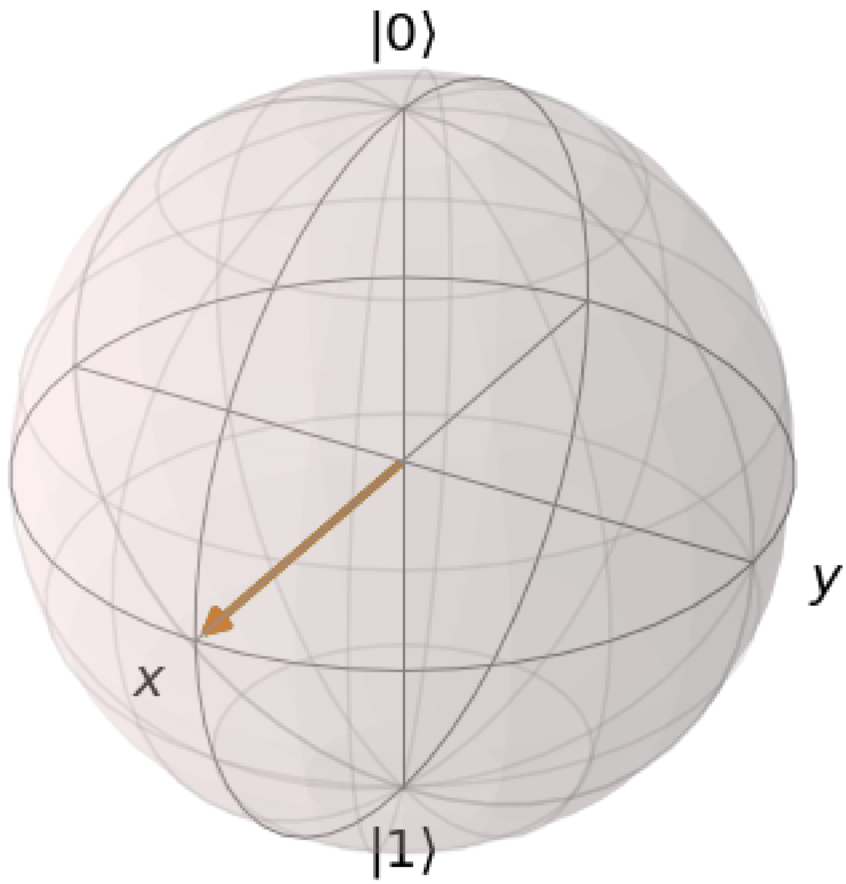
[1]:
```
import qutip as qt
import matplotlib.pyplot as plt
import numpy as np
```

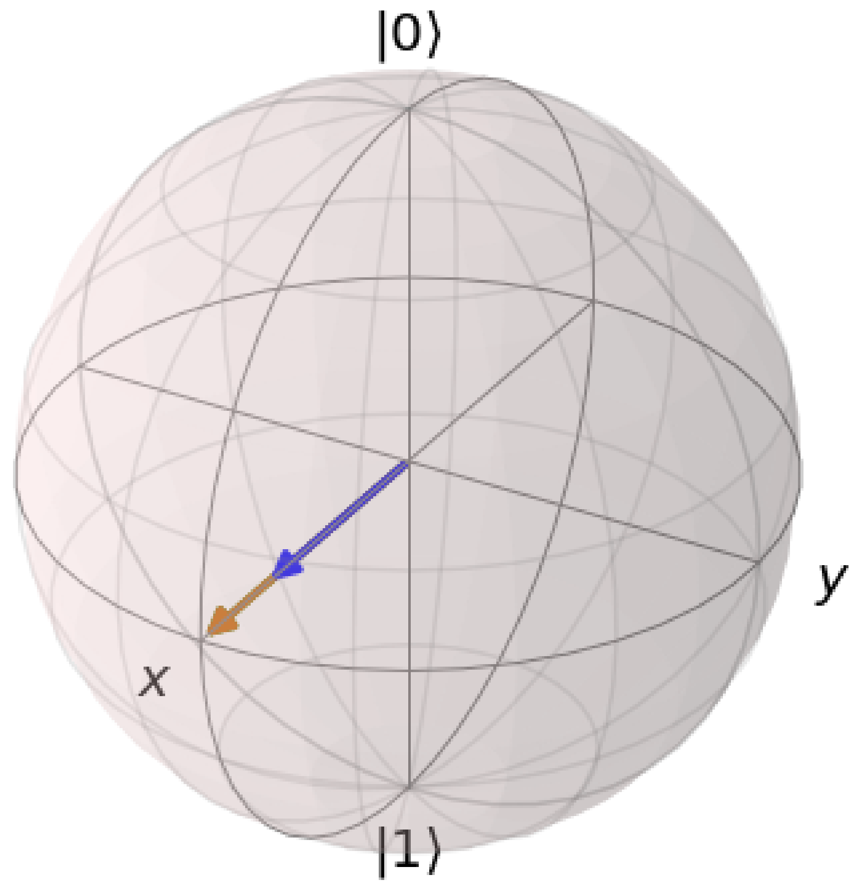
[2]:
```
# Problem 2
Psi = (qt.ket("0") + qt.ket("1")).unit()
r = (1, 0, 0)

b = qt.Bloch()
b.add_states(Psi)
b.add_vectors(r)
b.show()

r = (0.7, 0, 0)
b.add_vectors(r)
b.show()
```



|0⟩
y
x
|1⟩

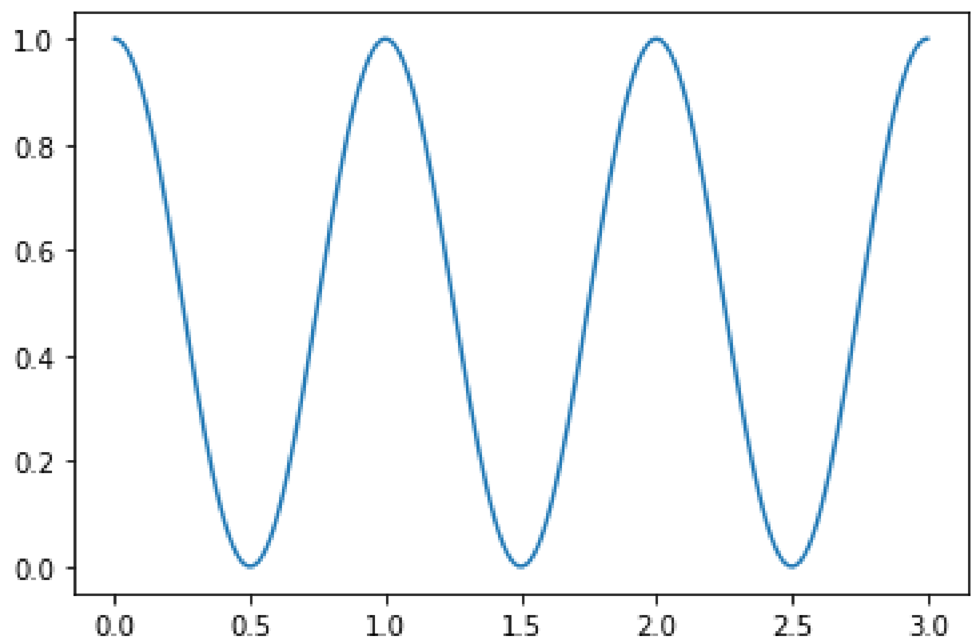


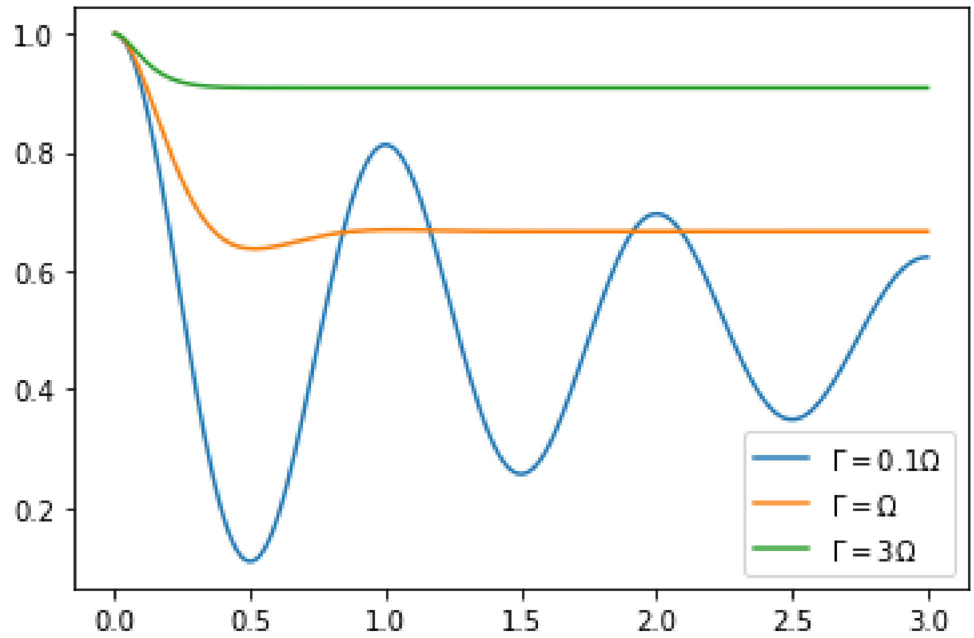
[3]:
```
# Problem 5
delta = 0
Omega = 2 * np.pi
H = - delta / 2 * qt.sigmaz() + Omega / 2 * qt.sigmax()
Psi = qt.ket("1")
rho_0 = Psi * Psi.dag()
times = np.linspace(0, 3, 200)
rho_t = qt.mesolve(H, rho_0, times).states
plt.plot(times, [qt.expect(rho, Psi) for rho in rho_t])
plt.show()

for Gamma, label in [(0.1 * Omega, "$\Gamma=0.1\Omega$"),
                     (Omega, "$\Gamma=\Omega$"),
                     (3 * Omega, "$\Gamma=3\Omega$")]:
    c_ops = np.sqrt(Gamma) * qt.sigmam()
    rho_t = qt.mesolve(H, rho_0, times, c_ops).states
```

```python
    plt.plot(times, [qt.expect(rho, Psi) for rho in rho_t], label=label)
plt.legend()
plt.show()

for ntraj in [1, 10, 100]:
    for Gamma, label in [(0.1 * Omega, "$\Gamma=0.1\Omega$"),
                         (Omega, "$\Gamma=\Omega$"),
                         (3 * Omega, "$\Gamma=3\Omega$")]:
        c_ops = np.sqrt(Gamma) * qt.sigmam()
        rho_t = qt.mcsolve(H, Psi, times, c_ops, ntraj=ntraj).states
        plt.plot(times, [qt.expect(rho, Psi) for rho in rho_t], label=label)
    plt.legend()
    plt.show()
```



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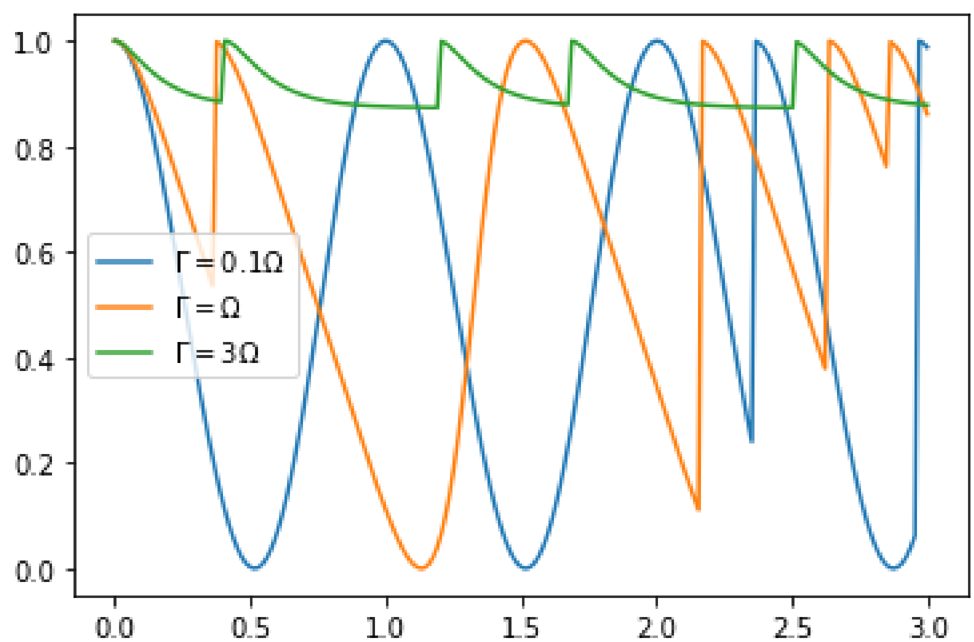
100.0%. Run time:   0.00s. Est. time left: 00:00:00:00
Total run time:   0.01s
100.0%. Run time:   0.00s. Est. time left: 00:00:00:00
Total run time:   0.02s
```



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70.0%. Run time:   0.08s. Est. time left: 00:00:00:00
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90.0%. Run time:   0.10s. Est. time left: 00:00:00:00
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Total run time:   0.13s
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80.0%. Run time:   0.10s. Est. time left: 00:00:00:00
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60.0%. Run time:   0.08s. Est. time left: 00:00:00:00
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80.0%. Run time:   0.11s. Est. time left: 00:00:00:00
90.0%. Run time:   0.13s. Est. time left: 00:00:00:00
100.0%. Run time:   0.14s. Est. time left: 00:00:00:00
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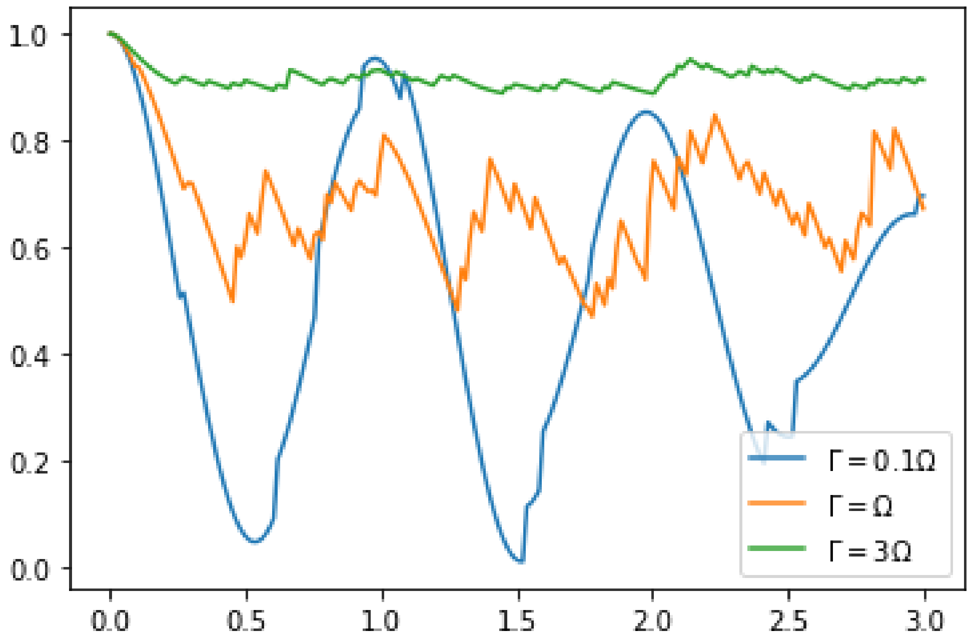
```

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70.0%. Run time:   1.03s. Est. time left: 00:00:00:00
80.0%. Run time:   1.19s. Est. time left: 00:00:00:00
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40.0%. Run time:   0.68s. Est. time left: 00:00:00:01
```

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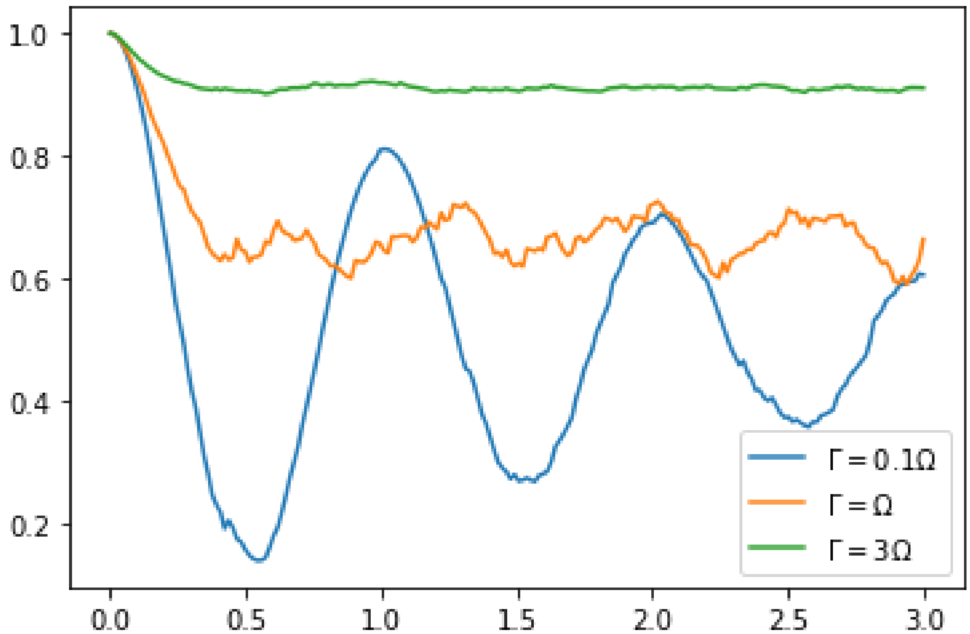


```
[ ]:
```

Phys 245 Quantum Computation
Homework 9

1. *[25] Quantum State Tomography Prep.* In class we realized that any qubit density matrix could be written as:
$$\hat{\rho} = \frac{1}{2}(\hat{I} + \vec{r}\cdot\vec{\sigma})$$
where
$$\vec{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$$
$$\vec{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$$
   a.) [5] Show that the density matrix can be written as:
$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}$$
   b.) [10] Find $[\sigma_x]$ in terms of r_x, r_y, and r_z
   c.) [10] Find $[\sigma_z]$ in terms of r_x, r_y, and r_z

2. *[45] Quantum State Tomography.* Suppose that a qubit on one of the IBM machines is repeatedly prepared and measured and it's found that $[\hat{\sigma}_x] = 1$, $[\hat{\sigma}_y] = 0$, and $[\hat{\sigma}_z] = 0$.
   a. [10] Find the density matrix describing this qubit.
   b. [10] Is this a pure state? Prove it. And if so, what is the state vector of the qubit?
   c. [10] Plot the state vector you found in part (b) on the Bloch sphere. Also plot on that Bloch sphere the vector $\vec{r}$ you found in part(a).

   Now suppose that a few hours later the measurement were repeated and it's found that $[\hat{\sigma}_x] = 0.7$, $[\hat{\sigma}_y] = 0$, and $[\hat{\sigma}_z] = 0$.
   d. [5] Find the density matrix describing this qubit.
   e. [5] Is this a pure state? Prove it.
   f. [5] Plot on the Bloch sphere from part (c) the vector $\vec{r}$ you found in part (d).

3. *[40] Reduced Density Matrix Test Drive.* Suppose we have two qubits coupled to one quantum harmonic oscillator. Labeling the states as $|qubit\ 1, qubit\ 2, QHO\rangle$, suppose the state of the system is $|\psi\rangle = \frac{1}{\sqrt{2}}(\,|1,0,n\rangle + |0,1,n\rangle)$.
   a. [5] Write down the full density matrix.
   b. [15] Find the reduced density matrix for just the qubits – that is stop keeping track of the harmonic oscillator. Is this a pure state?

   Now suppose the density matrix changes to $|\psi\rangle = a(\,|1,0,n\rangle + |0,1,n\rangle) + b|0,0,(n+1)\rangle$.
   c. [5] Write down the full density matrix.
   d. [15] Find the reduced density matrix for just the qubits – that is stop keeping track of the harmonic oscillator. Is this a pure state if $b \neq 0$?

4. [45] *Bit flip quantum channel.* In many cases, we have a qubit that we're operating on and the effect of the uncontrolled environment is that it flips the qubit with some probability,

p. We don't really need to model the whole microscopic dynamics to know what's going to happen. The evolution is just given by the unitary:

$$\hat{U}_{SE} = \sqrt{1-p}\,\hat{I}_S \otimes \hat{I}_E + \sqrt{p}\,\hat{X}_S \otimes \hat{G}_E$$

Where $\hat{G}_E$ is some operator that describes what happens to the environment during the interaction that flips the qubit. You may assume $\langle e_i|\hat{G}|e_o\rangle = \delta_{i,G}$ -- that is, there is only one state with $i = G$, where the matrix element is non-zero.

(Some hopefully helpful remarks: Now, in reality, there could be a whole bunch of different terms like the second one, but let's assume for simplicity there is only one that occurs and it happens with probability p. You may also be wondering why that second term couldn't have just been $\sqrt{p}\hat{X}_S \otimes \hat{I}_E$? The point is that in order for the environment to flip the qubit something had to change in the environment due to the interaction between the system and environment.)

From the given time evolution operator:

a.) [15] Find the Kraus operators.

b.) [15] Assuming the system initially started in the state $|0\rangle$ find the density matrix after the operation is complete.

c.) [15] Assuming the system initially started in the state $|X\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, find the density matrix after the operation is complete.

5. *[BONUS 45] Decoherence in QuTip.* Assume we have a qubit being driven by near resonant radiation, such that the Hamiltonian in the rotating frame after the RWA is the usual:

$$\hat{H} = -\frac{\delta}{2}\hat{\sigma}_z + \frac{\Omega}{2}\hat{\sigma}_x$$

a.) [5] Use QuTip's built-in master equation solver mesolve to find the evolution of the density matrix and plot ρ_{22} as a function of time. Pick $\delta = 0$ and $\Omega = 2\pi$.

b.) [20] Redo part (a) but add a collapse operator (c_ops) for spontaneous emission. The appropriate operator is $\sqrt{\Gamma}\hat{\sigma}_-$, where Γ is the spontaneous emission rate. Plot the evolution you find for ρ_{22} as a function of time for $\Gamma = 0.1\Omega$, $\Gamma = \Omega$, and $\Gamma = 3\Omega$.

c.) [20] Redo part (b) but use mcsolve instead of mesolve. For this step plot the result for using 1, 10, 100 trajectories. For more information, on mcsolve and changing the trajectory number see: https://qutip.org/docs/latest/guide/dynamics/dynamics-monte.html