**Math 115B: Linear Algebra**
**Homework 1**
**Due: Tuesday, January 14 at 11:59 PT**

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- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.

1. ($\frac{-}{20}$) Assume $V$ and $W$ are vector spaces over a field $k$, and let $T : V \to W$ denote a linear transformation between them. Prove that if $T$ has an inverse, then that inverse is a linear function.

2. ($\frac{-}{20}$) Assume that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are vectors in a vector space $V$ over some field $k$. Is it possible that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent but the vectors $\vec{w}_1 = \vec{v}_1 + \vec{v}_2, \vec{w}_2 = \vec{v}_1 + \vec{v}_3, \vec{w}_3 = \vec{v}_2 + \vec{v}_3$ are linearly *dependent*?

3. ($\frac{-}{5+5}$) Assume $V$ is a vector space consisting of 49 vectors over a field $k$.

   (a) Prove that the set of elements in $k$ is finite.

   (b) Prove that $V$ is finite dimensional and that $\dim(V) = 1$ or $\dim(V) = 2$.

4. ($\frac{-}{5+5+5}$) Assume $V$ is a vector space over some field $k$, and let $W_1, W_2$ denote subspaces of $V$. Define
$$W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

   (a) Prove that $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_2 + W_1$.

   (b) Prove that $W_1 + W_2$ is a subspace of $V$.

   (c) Assume that $W \subseteq V$ is a subspace such that $W_1 \subseteq W$ and $W_2 \subseteq W$. Prove $W_1 + W_2 \subseteq W$. (Notice that these results imply that $W_1 + W_2$ is the smallest possible subspace of $V$ which contains both $W_1$ and $W_2$!)

5. ($\frac{-}{5*4}$) Let $k$ denote some field. For each function $f$ below, determine if $f$ is a linear functional and prove your answer is correct. You may assume standard results from calculus.

   (a) $V = \mathbb{R}[x]$, $f(p(x)) := 4p'(0) + p''(1)$, where $q'(x)$ denotes the derivative of $q(x) \in \mathbb{R}[x]$.

   (b) $V = k^2$, $f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 2x \\ 4y \end{pmatrix}$.

   (c) $V = k^{2\times 2}$, $f(A) = \text{tr}(A)$

(d) $V = \mathbb{R}[x]$, $f(p(x)) = \int_0^1 p(x)dx$

(e) $V = \mathbb{Q}^3$, $f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = x^2 + y^2 + z^2$

6. $(\frac{-}{10+5})$ Assume that $m, n$ are positive integers, and fix $x_1, ..., x_m \in \mathbb{R}$ such that $x_i \neq x_j$ if $i, j \in \{1, ..., m\}$ such that $i \neq j$. Let

$$W := \{f \in \mathbb{R}[x]_{\leq n} : f(x_1) = f(x_2) = ... = f(x_m) = 0\}$$

where $\mathbb{R}[x]_{\leq n}$ denotes the degree less than or equal to $n$. (The set $\mathbb{R}[x]_{\leq n}$ is denoted $P_n(\mathbb{R})$ in our textbook.)

(a) Prove that $W$ is a vector space over $\mathbb{R}$. (*Hint:* The set $W$ is, by definition, a sub*set* of $\mathbb{R}[x]_{\leq n}$, which was shown in 115A to be a vector space.)

(b) Compute the dimension of $W$. (*Hint:* It may help to use the rank-nullity theorem.)