

110AH Section Worksheet 5

Warm-up. Describe the correspondence between subgroups of G/N and subgroups of G that contain N . Use this to classify the subgroups of $\mathbb{Z}/n\mathbb{Z}$ and in particular the group $\mathbb{Z}/p^n\mathbb{Z}$ from last week's warm-up.

Universal property of quotients. The following diagram illustrates the universal property for group quotients. State it precisely, and prove it.

$$\begin{array}{ccccc}
 N & \longrightarrow & G & \xrightarrow{q} & G/N \\
 & \searrow & \downarrow f & & \downarrow \exists! \\
 & & & & X
 \end{array}$$

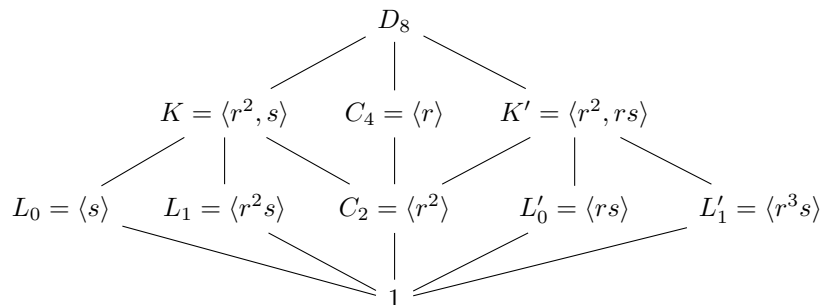
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Group presentations. Last week we defined the dihedral group of order $2n$ using the *group presentation*

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle.$$

In general, given a set of symbols S and a set of relations R which are words in these symbols, the group $\langle S \mid R \rangle$ is the quotient of the free group generated by S by the normal subgroup generated by R . Find a presentation of the groups \mathbb{Z} , $\mathbb{Z}/n\mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z}$. Complete the recipe (from two weeks ago) for the pushout of $f: A \rightarrow C$ and $g: B \rightarrow C$.

Quotients of D_8 . Recall that the subgroups of D_8 are as follows:



For each normal subgroup $N \trianglelefteq D_8$, compute D_8/N .

Group operation on cosets.

- Review the argument that for $N \trianglelefteq G$, the operation $(aN)(bN) = (ab)N$ on G/N is well-defined.
- For each nonnormal subgroup $H \leq D_8$, explain why the operation on D_8/H is not well-defined. *Hint:* Recall that K fixes L_0 whereas rK swaps L_0 and L_1 (ditto with primes).

Solution. Letting $q: G \rightarrow G/N$ denote the quotient map, the correspondence is $\overline{H} \mapsto q^{-1}(\overline{H})$ and $q(H) \hookrightarrow \overline{H}$. The subgroups of $\mathbb{Z}/n\mathbb{Z}$ are therefore the image under q of the subgroups containing $n\mathbb{Z}$, which are just $d\mathbb{Z}$ for $d \mid n$, i.e. they are the subgroups that are generated by a divisor of n . \square

Solution. Let $N \trianglelefteq G$ be a normal subgroup of a group G , and denote by $q: G \rightarrow G/N$ the quotient map. The universal property of G/N is that for every homomorphism $f: G \rightarrow X$ that is trivial on N , there exists a unique homomorphism $\bar{f}: G/N \rightarrow X$ such that $\bar{f}q = f$. Indeed, the definition $\bar{f}(gN) = f(g)$ is forced, and this is well-defined because if $gN = g'N$, then $f(g) = f(gg^{-1}g') = f(g')$. \square

Solution. Certainly $\mathbb{Z} = \langle a \mid \emptyset \rangle$ via the isomorphism $1 \mapsto a$. Thus

$$\mathbb{Z}/n\mathbb{Z} = \langle a \mid \emptyset \rangle / \langle a^n \rangle = \langle a \mid a^n = 1 \rangle.$$

Finally, noting that $\mathbb{Z} \times \mathbb{Z}$ is generated by $(1, 0)$ and $(0, 1)$ but that $(1, 0)$ and $(0, 1)$ commute, the best guess for a presentation is $\langle a, b \mid aba^{-1}b^{-1} \rangle$. Indeed

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z} &\xrightarrow{\quad} \langle a, b \mid aba^{-1}b^{-1} \rangle \\ (n, m) &\xrightarrow{\quad} a^n b^m \end{aligned}$$

exhibit inverse homomorphisms, where the reverse map exists by the universal property of quotients. \square

Solution. Obviously D_8/K , D_8/C_4 , and D_8/K' are all C_2 since they have order two. Now D_8/C_2 has order four, hence is either Klein-four or C_4 . It is the former because the elements sC_2 , rC_2 , srC_2 all have order two. \square

Solution. The operation $(aN)(bN) = (ab)N$ is well-defined by the following argument. Let $n, m \in N$. By normality $b^{-1}Nb = N$, so $nb = bn'$ for some $n' \in N$. Therefore

$$((an)N)((bm)N) = (anbm)N = (abn'm)N = (ab)N.$$

Concretely, if $b^{-1}Hb \neq H$, then picking $b^{-1}hb \notin H$ shows that

$$((ah)H)(bH) = (ahb)H = ab(b^{-1}hb)H \neq (ab)H = (aH)(bH).$$

Thus for example for L_0 , we have $r^{-1}sr \notin L_0$, so

$$(sL_0)(rL_0) = (sr)L_0 \neq rL_0 = (L_0)(rL_0).$$

\square