1 5/22/2024 lecture

A Lie group representation is a smooth group homomorphism $\rho : G \to GL(n, \mathbb{F})$. This induces a map $\rho_* : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{F})$. Lets define a *Lie algebra representation* as a Lie algebra homomorphism $R : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{F})$.

Note that in order for R to be a Lie algebra homomorphism, it must satisfy the criterion that $R([A, B]) = R(A)R(B) - R(B)R(A)$.

Proposition 1.1. If G is connected and simply connected, then a Lie group representation of G is equivalent to a Lie algebra representation of $\mathfrak{g}$.

Proof. Use the structure theorems. □

In general, if G is a finite-dimensional matrix Lie group, we can consider G as a subgroup of $GL(n, \mathbb{F})$ (where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$) and define its Lie group $\mathfrak{g}$ as

$$\mathfrak{g} = \left\{ A \in \mathfrak{gl}(n, \mathbb{F}) : \forall t \in \mathbb{F}, e^{tA} \in G \right\}.$$

Example 1.2. $\mathfrak{sl}(n, \mathbb{F})$ is the set of $n \times n$ matrices over $\mathbb{F}$ with zero trace.

Example 1.3. $\mathfrak{sl}(2, \mathbb{C})$ is the set of matrices of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, which has dimension 6 over $\mathbb{R}$. We can define the following basis vectors:

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

so that $[h, e] = 2e$ and $[e, f] = h$ and $[f, h] = 2f$. You can think of this as a two state system where the lower state has h-eigenvalue -1 and the higher state has h-eigenvalue 1. Then e behaves as a raising operator and f behaves as a lowering operator.

In general, a Lie algebra can be defined by basis vectors $\{A_i\}$ and "structure constants" t_{ij} satisfying $t_{ij} = -t_{ji}$ and $[A_i, A_j] = \sum_k t_{ij}^k A_k$

Theorem 1.4. In general, a Lie group and its corresponding Lie algebra will have the same dimension. PROVE THIS STATEMENT, ALSO FIND ANY EXCEPTIONS

Example 1.5. $\mathfrak{u}(n)$ is the set of complex-valued matrices A such that $(\exp(sA))^\dagger (\exp(sA)) = I_n$ for all $s \in \mathbb{R}$ (COULD WE HAVE SAID $s \in \mathbb{C}$ INSTEAD??? WHY NOT, WOULD IT IMPLY THAT $\mathfrak{u}(n)$ IS A COMPLEX MANIFORLD, WHICH IT ISNT??) Then we can write this as a formal power series in s and use that to find that $A^\dagger = -A$ (A is "anti-Hermitian"). In other words, A is i times some Hermitian matrix. Counting the degrees of freedom for a Hermitian matrix, we see that $\dim_{\mathbb{R}} \mathfrak{u}(n) = \dim_{\mathbb{R}} (U(n)) = n^2$.
$\mathfrak{su}(n)$ is the Lie algebra of traceless anti-Hermitian matrices, so it has dimension $n^2 - 1$ over $\mathbb{R}$. Every $A \in \mathfrak{u}(n)$ can be written as $B + \frac{\mathrm{Tr}(A)}{n} \cdot I$ for some $B \in \mathfrak{su}(n)$, which implies $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathbb{R}$.

A Lie subalgebra is a subalgebra of a Lie algebra which is closed under the Lie bracket (e.g. $\mathfrak{su}(n) \subset \mathfrak{u}(n)$).

The set of 2×2 Hermitian matrices is a real vector space with basis vectors $\{I, \sigma_x, \sigma_y, \sigma_z\}$, where the Pauli spin matrices are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then the set of anti-Hermitian matrices is $\mathfrak{u}(2) = \text{span}_{\mathbb{R}} \{iI, i\sigma_x, i\sigma_y, i\sigma_z\}$ and the set of traceless anti-Hermitian matrices is $\mathfrak{su}(2) = \text{span}_{\mathbb{R}} \{i\sigma_x, i\sigma_y, i\sigma_z\}$.

Define the Levi-Cevita symbol ε^{ijk} to be $+1$ if ijk is an even permutation of 123 (or xyz, or abc), -1 if ijk is an odd permutation of 123, and 0 if any indices are repeated (e.g. if $i = j$). It is implicitly summed over all i, j, k. Then

$$\sigma_a \sigma_b = i\varepsilon^{abc}\sigma_c$$

and

$$[i\sigma_a, i\sigma_b] = -2i\varepsilon^{abc}\sigma_c.$$

1.1 Complexification

The *complexification* of a real Lie algebra $\mathfrak{g}$ is a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of elements of the form $A + iB$, where $A, B \in \mathfrak{g}$. The Lie bracket for $\mathfrak{g}^{\mathbb{C}}$ is then defined such that

$$[A + iB, C + iD] = [A, C] + i[A, D] + i[B, C] - [B, D].$$

Proposition 1.6. $\mathfrak{su}(n)^{\mathbb{C}} \cong \mathfrak{sl}(n, \mathbb{C})$

Proof. Define $\varphi : \mathfrak{su}(n)^{\mathbb{C}} \to \mathfrak{sl}(n, \mathbb{C})$ to be the Lie algebra homomorphism which takes $A + iB$ to FINISH THIS PROOF, AND USE THE SAME TRICK TO SHOW THAT $\mathfrak{u}(n)^{\mathbb{C}} \cong \mathfrak{gl}(n, \mathbb{R})$ AND THAT $\mathfrak{gl}(n, \mathbb{R})^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. $\square$

GET NOTES FROM RYAN ABOUT RELATING COMPLEX REPRESENTATIONS OF SU2 TO COMPLEX REPRESENTATIONS OF SL(2,C).

1.2 Highest weight vectors