Math 132 Homework #5

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May 11, 2025

Problem 0.1. Chapter IV section 4 exercise 1(a)

Let $f(z) = z^n$ (where $n \geq 0$) and let D be the positively oriented closed disk of radius 2 centered at the origin. Then f is analytic on D, so

$$\begin{aligned}\oint_{|z|=2} \frac{z^n}{z-1} \mathrm{d}z &= \oint_{\partial D} \frac{f(z)}{z-1} \mathrm{d}z \\ &= 2\pi i f(1) \\ &= 2\pi i.\end{aligned}$$

Problem 0.2. Chapter IV section 4 exercise 1(c)

Let $f(z) = \sin(z)$ and let D be the positively oriented closed disk of radius 1 centered at the origin. Then f is analytic on D, so

$$\begin{aligned}\oint_{|z|=1} \frac{\sin z}{z} \mathrm{d}z &= \oint_{\partial D} \frac{f(z)}{z-0} \mathrm{d}z \\ &= 2\pi i f(0) \\ &= 0.\end{aligned}$$

Problem 0.3. Chapter IV section 4 exercise 1(h)

Let D be the positively oriented closed disk of radius 3 centered at 1. Then you can use a partial fraction decomposition along with the Cauchy integral formula to obtain

$$\begin{aligned}
\frac{1}{z(z+2)(z-2)} &= \frac{A}{z} + \frac{B}{z+2} + \frac{C}{z-2} \\
1 &= A(z+2)(z-2) + B(z)(z-2) + C(z)(z+2) \\
&= (A+B+C)z^2 + (2C-2B)z + (-4A) \\
\begin{bmatrix} A \\ B \\ C \end{bmatrix} &= \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1/8 \\ 1/8 \end{bmatrix} \\
\oint_{|z-1|=3} \frac{\mathrm{d}z}{z(z^2-4)e^z} &= \oint_{\partial D} \frac{e^{-z}\mathrm{d}z}{z(z+2)(z-2)} \\
&= \oint_{\partial D} e^{-z}\left(\frac{A}{z} + \frac{B}{z+2} + \frac{C}{z-2}\right)\mathrm{d}z \\
&= \oint_{\partial D} \left(\frac{-e^{-z}/4}{z} + \frac{e^{-z}/8}{z+2} + \frac{e^{-z}/8}{z-2}\right)\mathrm{d}z \\
&= 2\pi i\left(\frac{-e^{-0}}{4} + \frac{e^{-(-2)}}{8} + \frac{e^{-2}}{8}\right) \\
&= \frac{\pi i}{2}\left(\cosh(2) - 1\right).
\end{aligned}$$

Problem 0.4. Chapter IV section 5 exercise 2

Let f be an entire function, and suppose there is some disk $D \subset \mathbb{C}$ such that for any $z \in \mathbb{C}$, $f(z) \notin D$. Then let z_0 be the center of D, let r be the radius of D, and let $g(z) = 1/(f(z) - z_0)$. The magnitude of $g(z)$ can never be more than $1/r$ on that domain, since $|f(z) - z_0| \leq r$. Also, g is an entire function, which means you can apply the Louiville theorem to prove that g is constant. Since $g(z) = 1/(f(z) - z_0)$ is constant, $f(z)$ must also be constant.

Problem 0.5. Chapter V section 1 exercise 3

If $p > 1$, you can find an upper bound for the difference between $S = \zeta(p)$ and its partial sums using the integral test:

$$\begin{aligned}
\left| S - \sum k = 1^n \frac{1}{k^p} \right| &= \sum_{k=n+1}^{\infty} \frac{1}{k^p} \\
&= \int_{k=n}^{\infty} \frac{1}{\lceil k \rceil^p} \mathrm{d}k \\
&< \int_{k=n}^{\infty} \frac{1}{k^p} \mathrm{d}k \\
&= \left[\frac{1}{(p-1)k^{p-1}} \right]_{k=n}^{\infty} \\
&= \frac{1}{(p-1)n^{p-1}}.
\end{aligned}$$

Problem 0.6. Chapter V section 2 exercise 2

For any $x \in [0, \infty)$, the sequence $g_k(x)$ will converge pointwise to the function g, defined as follows:

$$g(x) := \begin{cases} 0 & x < 1 \\ \frac{1}{2} & x = 1 \\ 1 & x > 1 \end{cases}.$$

However, the sequence of functions does not converge uniformly. For any k and any $a \in (0, \frac{1}{2})$, the function g_k is a bijection from $[0, \infty)$ to $[0, 1)$, so you can define $x = g_k^{-1}(a)$, and that will guarantee $x \in (0, 1)$. Therefore $|g_k(x) - g(x)| = |a - 0| = a$, so g_k does not converge uniformly to g (but since g_k converges pointwise to g, that means g does not converge uniformly at all).

The sequence of functions g_k does converge uniformly to g if we restrict the domain of all these functions to be only a subset of $[0, \infty)$ whose boundary does not include 1.

Problem 0.7. Chapter V section 2 exercise 8

Assume $|z| < 1$. Let $f_n(z) := \sum_{n=1}^{k} z^k/k^2$ be the sequence of partial sums. This converges pointwise, because for any such z, the series whose terms are z^k/k^2 converges by the ratio test (the ratio from one term to the next always has absolute value less than $|z|$, which is a fixed number less than 1). Therefore we can define $f(z)$ to be the pointwise limit of $f_n(z)$, so if f_n converges uniformly, it must converge to f.

For any $\varepsilon > 0$, let $N \in \mathbb{N}$ be any integer high enough that $\pi^2/6 - \sum_{k=1}^{N} k^{-2} < \varepsilon$. This is always possible, because the series $\sum_{k=1}^{\infty} k^{-2}$ converges to $\zeta(2) = \pi^2/6$. Then for any $n \geq N$,

$$\begin{aligned}
|f_n(z) - f(z)| &= \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k^2} \right| \\
&\leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k^2} \\
&< \sum_{k=n+1} \frac{1}{k^2} \\
&< \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \\
&< \frac{\pi^2}{6} - \sum_{k=1}^{N} \frac{1}{k^2} \\
&< \varepsilon,
\end{aligned}$$

so f_n converges uniformly to f (when $|z| < 1$).

Homework Assignment 5

MATH 132 LEC 1&2

Due May 11th, Sunday 11:59 PM

Please submit your work to Gradescope!

- IV.4 Exercises: #1(a), #1(c), #1(h),
- IV.5 Exercises: #2,
- V.1 Exercises: #3,
- V.2 Exercises: #2, #8.