Conversion from SI to natural units

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The dimensions for any quantity can be written as a column vector in $\mathbb{Z}^3$ by listing the powers of mass, length, and time (in that order) as a column vector.

$$[M] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [L] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad [T] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then energy, the reduced Planck constant, and the speed of light have dimensions represented by the following vectors:

$$[E] = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \qquad [\hbar] = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad [c] = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

However, we could also write the dimensions for any quantity in another basis. When using natural units, it's convenient to work in the basis where

$$[E] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [\hbar] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad [c] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To convert from the $([E], [\hbar], [c])$ basis to the $([M], [L], [T])$ basis, all we need to do is left-multiply by a change-of-basis matrix, which I'll call A.

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ -2 & -1 & -1 \end{bmatrix}$$

The inverse of that is

$$A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

so to convert from the $([M], [L], [T])$ basis to the $([E], [\hbar], [c])$ basis, multiply by A^{-1}.

By looking at the entries of A^{-1}, we can read off the following conversions:

$$\begin{aligned} [M] &= [E/c^2] \\ [L] &= [\hbar c/E] \\ [T] &= [\hbar/E] \end{aligned}$$

This is super cool because it immediately tells us how many times we need to multiply or divide by $\hbar$ and c to get a quantity that's a power of energy. For example, viscosity (μ) has units of

$$[\mu] = [M] \cdot [L]^{-1} \cdot [T]^{-1} = [E/c^2] \cdot [\hbar c/E]^{-1} \cdot [\hbar/E]^{-1} = [E]^3 \cdot [\hbar]^{-2} \cdot [c]^{-3}$$

so to get μ from SI units to natural units, we need to multiply by $\hbar^2 c^3$.

Suppose the quantity we want to convert to natural units is $\mu = 3\,\mathrm{kg\,m^{-1}\,s^{-1}}$. Multiplying by $\hbar^2 c^3$ is annoying because we would still need to do a lot of math, but there is another way to do this.

First, take the logarithm of the quantity we want to convert. *Note that we're using the base 10 log here instead of the natural log, since this will make it easier to convert between eV, keV, MeV, and GeV later on.*

$$\log(\mu) = \log(3) + \log(\mathrm{kg}) - \log(\mathrm{m}) - \log(\mathrm{s}) \in \mathrm{span}_{\mathbb{R}}\left(\{1, \log(\mathrm{kg}), \log(\mathrm{m}), \log(\mathrm{s})\}\right)$$

can be written in the $(1, \log(\mathrm{kg}), \log(\mathrm{m}), \log(\mathrm{s}))$ basis as

$$\begin{bmatrix} \log 3 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.477121 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

and we can convert that to the $(1, \log(\mathrm{eV}), \log(\hbar), \log(\mathrm{c}))$ basis by left-multiplying by the inverse of some matrix B, where B is the matrix we left-multiply by to convert from the $(1, \log(\mathrm{eV}), \log(\hbar), \log(c))$ basis to the $(1, \log(\mathrm{kg}), \log(\mathrm{m}), \log(\mathrm{s}))$ basis.

$$B = \begin{bmatrix} 1 & -18.795290 & -33.976924 & 8.476821 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & -2 & -1 & -1 \end{bmatrix}$$
$$B^{-1} = \begin{bmatrix} 1 & 35.748931 & 6.704814 & 15.181634 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

The matrix B was derived by writing out 1, eV, $\hbar$, and c in SI units and taking the base 10 log of both sides to obtain a system of equations. For example,

$$c = 299792458\,\mathrm{m/s}$$
$$\log(c) = 8.476821 + 0 \cdot \log(\mathrm{kg}) + 1 \cdot \log(\mathrm{m}) + (-1) \cdot \log(\mathrm{s})$$

Note that if we leave off the "real component" (in that case, the 8.476821 term), this process is equivalent to deriving the matrix A that we used earlier, which is why the bottom-right 3×3 submatrix of B is A, and the bottom-right 3×3 submatrix of B^{-1} is A^{-1}.

If we left-multiply the vector we wrote for $\log(\mu)$ (in the $(1, \log(\mathrm{kg}), \log(\mathrm{m}), \log(\mathrm{s}))$ basis) by B^{-1}, we get

$$\begin{bmatrix} 1 & 35.748931 & 6.704814 & 15.181634 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.477121 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 14.339605 \\ 3 \\ -2 \\ -3 \end{bmatrix}$$

which means that

$$\mu = 3\,\mathrm{kg}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1} = 10^{14.339605}\,\mathrm{eV}^3\,\hbar^{-2}\,c^{-3}$$

so in natural units (the unit system where $\hbar = 1 = c$), μ is

$$10^{14.339605}\,\mathrm{eV}^3 = 2.185770247 \times 10^{14}\,\mathrm{eV}^3.$$