Math 115B Homework #8

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Problem 0.1.

(a) H is a bilinear form, because $H(f,g) = H(g,f)$ and H is linear in the first argument (therefore also the second). If $a \in \mathbb{R}$ and $f, g, h \in C([0,1])$, then

$$H(af+h,g) = \int_0^1 (af(x)+h(x))g(x)\mathrm{d}x = a\left(\int_0^1 f(x)g(x)\mathrm{d}x\right) + \left(\int_0^1 h(x)g(x)\mathrm{d}x\right) = aH(f,g)+H(h,g).$$

(b) H is not a bilinear form, because if we let v, w be vectors such that $J(v,w) \neq 0$, then $H(v,w)$ is also nonzero, which implies

$$\begin{aligned} H(v+v,w) &= (J(v,w)+J(v,w))^2 \\ &= H(v,w)+H(v,w)+H(v,w)+H(v,w) \\ &= (1+1)(H(v,w)+H(v,w)) \\ &\neq H(v,w)+H(v,w). \end{aligned}$$

(c) H is not a bilinear form, because $H(1,0) = 1$, but linearity in the second argument would imply $H(x,0) = 0$ for any $x \in \mathbb{R}$.

(d) This is a bilinear form, because it gives the determinant of the 2×2 matrix whose columns are the two parameters, and theorem 4.12 states that the determinant is the unique n-linear alternating function from $M_{n\times n}(F)$ to F whish maps I_n to 1. Therefore, the determinant of a 2×2 matrix can be thought of as a bilinear form on its rows, or equivalently, on its columns.

(e) This is a bilinear form. We defined an inner product to be linear in the second argument (first, if you use silly math notation) and conjugate-linear in the first argument (second, if you never bothered to learn quantum mechanics). Since the underlying field of V is $\mathbb{R}$, H is linear in both arguments.

(f) H is not a bilinear form, because if $v, w \neq 0$, then

$$H(iv,w) = \langle iv, w\rangle = -i\langle v, w\rangle = -iH(v,w) \neq iH(v,w).$$

Problem 0.2.

(a) If $\alpha \in k$ and $H_1, H_2 \in \mathbb{B}(V)$, then H_1 and H_2 are both linear in the first argument. Linear combinations of linear functions are linear functions, so $H_1 + H_2$ and αH_1 are also linear in the first argument. By the exact same logic, they're linear in the second argument, so $H_1 + H_2$ and αH_1 are bilinear forms, which implies $\mathbb{B}(V)$ is a vector space (over k).

(b) Theorem 6.32 states that there is an isomorphism from $\mathbb{B}(V)$ and $M_{n\times n}(F)$, where V is an n-dimensional vector space over F. Therefore the dimension of $\mathbb{B}(V)$ is n^2.

Problem 0.3.

Since we're given that H is symmetric, $H(v,w) = H(w,v)$, so

$$\begin{aligned} K(v+w) - K(v) - K(w) &= H(v+w, v+w) - H(v,v) - H(w,w) \\ &= [H(v,v) + H(v,w) + H(w,v) + H(w,w)] - H(v,v) - H(w,w) \\ &= H(v,w) + H(w,v) \\ &= (1+1)H(v,w). \end{aligned}$$

Since k does not have characteristic 2, we can divide both sides by 2 to get

$$H(v,w) = \frac{1}{2}\left(K(v+w) - K(v) - K(w)\right).$$

Problem 0.4.

(a) For any $u, v, w \in V$ and $\alpha \in \mathbb{R}$, we can see that H is linear in the first argument because

$$H(\alpha v + u, w) = \langle \alpha v + u, Tw\rangle = \alpha^* \langle v, Tw\rangle + \langle u, Tw\rangle = \alpha H(v,w) + H(u,w),$$

and it's linear in the second argument because

$$H(v, \alpha w + u) = \langle v, T(\alpha w + u)\rangle = \langle v, \alpha Tw + Tu\rangle = \alpha \langle v, Tw\rangle + \langle v, Tu\rangle = \alpha H(v,w) + H(v,u).$$

(b) T^* is the unique operator such that $\langle v, Tw\rangle = \langle T^* v, w\rangle$ for all $v, w \in V$, so T is Hermitian iff $\langle v, Tw\rangle = \langle Tv, w\rangle$ for all $v, w \in V$. Since V is a real vector space, $\langle Tv, w\rangle = \langle w, Tv\rangle$, so that's equivalent to saying $\langle v, Tw\rangle = \langle w, Tv\rangle$, which is equivalent to saying $H(v,w) = H(w,v)$. Therefore H is symmetric iff T is Hermitian.

Problem 0.5.

Let $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ be an orthonormal basis for V, and let $A = \psi_{\mathcal{B}}(H)$ be the matrix representation of H in that basis.

Suppose $H(v,w) = \langle v, Tw\rangle$ for all $v, w \in V$. Then we can let $v = e_i$ and $w = e_j$ be basis vectors, so that $H(v,w) = A_{i,j}$, which implies $A_{i,j} = \langle e_i, Te_j\rangle$. But $\langle e_i, Te_j\rangle$ is just the (i,j)th entry of $[T]_{\mathcal{B}}$. Therefore T is the linear operator such that $[T]_{\mathcal{B}} = A$, which means T exists and is unique.

Problem 0.6.

Suppose for the sake of contradiction that p is not prime, so it is composite. Then there exist smaller positive integers a, b such that $ab = p$. By the distributive rule,

$$\left(\sum_{i=1}^{a} 1\right)\left(\sum_{i=1}^{b} 1\right) = \sum_{i=1}^{p} 1 = 0.$$

Since a field has no nonzero divisors of zero, either the sum of a ones is zero, or the sum of b ones is zero. But either of those cases would contradict the assumption that p is the smallest positive integer for which

$$\sum_{i=1}^{p} 1 = 0.$$

Therefore p must be prime.

Math 115B: Linear Algebra
Homework 8
Due: *Friday*, March 14th at 11:59pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- *In this homework assignment* k *always denotes a field for which* $1 + 1 \neq 0$.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. $(\frac{-}{7*6})$ Determine which of the following mappings given below are bilinear forms. Justify your answers.
   (a) Let $C[0, 1]$ be the set of continuous real valued functions with domain $[0, 1]$. For $f, g \in C[0, 1]$, define $H(f, g) := \int_0^1 f(x)g(x)dx$.
   (b) Let V be a vector space over k, and let $J \in \mathbb{B}(V)$ be nonzero. Define $H : V \times V \to k$ by the formula $H(\vec{v}, \vec{w}) = J(\vec{v}, \vec{w})^2$ for all $\vec{v}, \vec{w} \in V$.
   (c) The function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by the formula $H(t_1, t_2) := t_1 + 2t_2$.
   (d) The function $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by the formula $D(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}) := ad - bc$.
   (e) Let V be a real inner product space, and let $H : V \times V \to \mathbb{R}$ be the function $H(\vec{v}, \vec{w}) := \langle \vec{v}, \vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$.
   (f) Let V be a *complex* inner product space, and let $H : V \times V \to \mathbb{C}$ be the function $H(\vec{v}, \vec{w}) := \langle \vec{v}, \vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$.

2. $(\frac{-}{10+5})$ Assume V is a vector space and $\mathbb{B}(V)$ is the set of bilinear forms on V.
   (a) Prove Theorem 6.31. That is, prove that if $H_1, H_2 \in \mathbb{B}(V)$ and $\alpha \in k$ implies $H_1 + H_2 \in \mathbb{B}(V)$ and $\alpha H_1 \in \mathbb{B}(V)$ and that $\mathbb{B}(V)$ is a vector space over k with respect to these operations.
   (b) Assume the dimension of V is $n \in \mathbb{Z}^{\geq 0}$. Compute the dimension of $\mathbb{B}(V)$.

3. $(\frac{-}{15})$ Let V be a vector space over a field k (whose characteristic we have assumed is not two!) and let H denote a symmetric bilinear form on V. Prove if we define the function $K : V \times V \to k$ by the formula $K(\vec{v}) := H(\vec{v}, \vec{v})$ for all $\vec{v} \in V$, then

$$H(\vec{v}, \vec{w}) = \frac{1}{2}(K(\vec{v} + \vec{w}) - K(\vec{v}) - K(\vec{w}))$$

   for all $\vec{v}, \vec{w} \in V$.

4. $(\frac{-}{2+8})$ Assume T is a linear operator (endomorphism) on a finite dimensional real inner product space V, and define the function $H : V \times V \to \mathbb{R}$ by the formula $H(\vec{v}, \vec{w}) = \langle \vec{v}, T\vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$.

   (a) Prove that H is a bilinear form.

   (b) Prove that H is symmetric if and only if T is self adjoint.

5. $(\frac{-}{13})$ Prove that if V is a finite dimensional real inner product space and H is a bilinear form on V, then there exists a unique linear operator $T : V \to V$ such that $H(\vec{v}, \vec{w}) = \langle \vec{v}, T\vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$. (*Hint*: Choose an orthonormal basis $\mathcal{B}$ for V, and let A be the matrix representation of H for this basis. Let $T : V \to V$ be the linear transformation for which $[T(\vec{v})]_{\mathcal{B}} = A[\vec{v}]_{\mathcal{B}}$.)

6. $(\frac{-}{5})$ Assume k is a field such that, for some positive integer m, $\sum_{i=1}^{m} 1 = 1 + 1 + \ldots + 1 = 0$. Prove the smallest positive integer p for which $\sum_{i=1}^{p} 1 = 0$ is prime. (This prime number is called the *characteristic* of the field k, and if $\sum_{i=1}^{m} 1 = 1 + 1 + \ldots + 1 \neq 0$ for all positive integers m, we say that k has *characteristic zero.*