Practice midterm 2 answers

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1

> There is a planar graph with 5 faces such that for any pair of faces F_1, F_2, there is an edge incident to both F_1 and F_2. True or False?

False. This would imply the dual graph contains K_5 as a subgraph, but that would force it to not be planar (WHY?), so the original graph also couldn't be planar.

2

> Suppose G is a simple connected graph and e is an edge of G. There is a spanning tree of G containing e. True or False?

True, it can be constructed using Kruskal's algorithm if you give e a lower weight than all other edges.

3

> Let G be a connected simple graph. An edge e is called a *bridge* if $G - e$ is disconnected. Suppose every spanning tree of G contains an edge e. Then e is a bridge. True or False?

True. Suppose e is not a bridge. Then $G - e$ is connected. Let T be a spanning tree in $G - e$. It is also a spanning tree of G but does not contain e, a contradiction.

4

> If a simple connected graph G on 7 vertices has degree sequence $(4, 2, 2, 1, 1, 1, 1)$, then G is a tree. True or False?

True. By the degree score theorem, if you remove the degree 4 vertex and all its adjacent edges, you would get two isolated vertices and 2 copies of P_2. For the original graph to have been connected, the degree 4 vertex must be connected to each of those components, so it must have been a tree.

Alternate solution: $|E| = 6 = |V| - 1$. By the characterization of trees, G is a tree.

5

A graph is *bipartite* if it can be colored using two colors. There is a simple planar bipartite graph with 8 vertices and 13 edges. True or False?

Suppose there is such a graph. It must be obtainable by removing either 3 of the 16 edges from $K_{4,4}$ or by removing 2 of the 15 edges from $K_{3,5}$. In either case, the result is a connected planar graph, so

$$|V| - |E| + |F| = 2$$

which implies there are 7 faces. But in bipartite graphs, every face has degree at least 4 (that is, there are no triangles), so $2|E| \geq 4|F|$. 26 is not less than 28, so the statement is false.

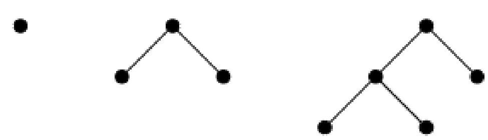
6

Suppose $G = (V, E)$ is a simple graph with $|V| \geq 3$. If $|E| \leq 3|V| - 6$, then G is planar. True or False?

False. G could be K_5 plus a million isolated vertices. We know K_5 isn't planar, so then G isn't either.

7

(10 points) A rooted tree (T, r) in which every vertex has either 0 or 2 children is called a *binary tree*. The following are some examples of binary trees.



Show that the number of leaves in a binary tree with n vertices is $\frac{n+1}{2}$.

Hint: Induction on n.

If $n > 1$, then you can remove any leaf and along with the other leaf that shared the same parent. Doing so will reduce the number of leaves by 1 (since the parent becomes a leaf) and reduce n by 2, leaving a new binary tree. If $n = 1$ then there is 1 leaf, so by induction, the number of leaves is $(n+1)/2$.

8

(10 points) A graph is said to be *unicylic* if it contains exactly one cycle. Show that for a graph $G = (V, E)$, any two of the following conditions implies the third.

1. G is unicyclic.
2. G is connected.
3. $|V| = |E|$.

If G is unicyclic, then $|F| = 2$ and G is planar. If it's also connected, then $|V| - |E| + |F| = 2$. So (1) and (2) implies (3). Similarly, if (2) and (3) are true, then $|F| = 2$ which means it is unicyclic. If (1) and (3) are true, then $|V| - |E| + |F| = 2$, which implies it is connected (since it is homeomorphic to circle, which has Euler characteristic 0, and we know that $b_2 = 0$ and $b_1 = 1$, so $b_0 = 1$).