Physics 231B Homework #7

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Consider the n-fold tensor product

$$\mathbb{C}^{2^n} = \bigotimes_{i=1}^{n} \mathbb{C}^2$$

and define the Pauli matrices X_i, Y_i, Z_i as acting on the ith factor via

$$\begin{aligned}
X_1 &= X \otimes I \otimes \cdots \otimes I, \\
X_2 &= I \otimes X \otimes I \cdots \otimes I, \\
&\cdots \\
X_n &= I \otimes I \otimes \cdots X,
\end{aligned}$$

and so on (using the shorthand $\sigma^x = X, \sigma^y = Y, \sigma^z = Z$). With these, we define $2n$ "Majorana" operators via $\gamma_1 = X_1, \gamma_2 = Y_1$, and for $1 \leq k < n$:

$$\begin{aligned}
\gamma_{2k+1} &= \left(\prod_{i=1}^{k} Z_i \right) X_i \\
\gamma_{2k+2} &= \left(\prod_{i=1}^{k} Z_i \right) Y_i.
\end{aligned}$$

This is called the Jordan-Wigner transformation. The next few problems will develop spinor representations from these variables.

Problem 0.1. Prove that these satisfy the Clifford algebra relations:

$$\begin{cases} \gamma_i^2 = 1 & \text{for any } i \\ \gamma_i \gamma_j = -\gamma_j \gamma_i & \text{whenever } i \neq j \end{cases}$$

Problem 0.2. Recall $\mathfrak{so}(2)$ is the Lie algebra of anti-symmetric real matrices. Prove that

$$A_{\mu\nu} \mapsto \frac{1}{4} A_{\mu\nu} \gamma_\mu \gamma_\nu$$

defines a Lie algebra representation of $\mathfrak{so}(2n)$ on $\mathbb{C}^{2^n}$. In other words, prove

$$\frac{1}{16} A_{ij} B_{kl} [\gamma_i \gamma_j, \gamma_k, \gamma_l] = \frac{1}{4} [A, B]_{ab} \gamma_a \gamma_b.$$

Problem 0.3. We define the group $Spin(2n)$ by exponentiating these operators inside $U(2^n)$:

$$\exp\left(\frac{1}{2}A_{\mu\nu}\gamma_\mu\gamma_\nu\right).$$

Show that the conjugation action of these operators on the Clifford algebra acts on γ_i according to the $2n$-dimensional representation on the index i, thus giving a surjective map $Spin(2n) \to SO(2n)$.

Problem 0.4. Show that the representation of $Spin(2n)$ on $\mathbb{C}^{2^n}$ we have obtained from this construction does not define an $SO(2n)$ representation. Decompose this representations into irreducibles of $Spin(4) = SU(2) \times SU(2)$ in the case $n = 2$.

Problem 0.5. The action of $Spin(2n)$ on $\mathbb{C}^{2^n}$ is reducible because of the commuting element $\gamma_c := \prod_{i=1}^n Z_i$. Prove that this element satisfies

$$\gamma_c^2 = 1$$
$$\gamma_i\gamma_c = -\gamma_c\gamma_i.$$

Thus we get a 2^n dimensional representation of $\mathfrak{so}(2n+1)$ and a corresponding double cover group $Spin(2n+1)$. Verify for $n = 1$ we get the identification $Spin(3) = SU(2)$.