**Math 115B: Linear Algebra**
**Homework 6**
**Due: Saturday, February 22nd at 11:59pm PT**

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- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated $k$ denotes an arbitrary field and all vector spaces are over $k$. All inner product spaces are defined over a field $F$ which is either $\mathbb{R}$ or $\mathbb{C}$.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. ($\frac{-}{10}$) Let $T$ be a normal operator on a finite-dimensional real inner product space $V$ whose characteristic polynomial splits. Prove that $V$ has an orthonormal basis of eigenvectors of $T$, and use this to prove $T$ is self adjoint.
2. ($\frac{-}{10}$) Prove Corollary 2 to Theorem 6.18. That is, prove that if $T$ is a linear operator on a finite-dimensional complex inner product space $V$, then: $V$ has an orthonormal basis of eigenvectors of $T$ with corresponding eigenvalues of absolute value 1 if and only if $T$ is unitary.
3. ($\frac{-}{10}$) We say that matrix $A \in \mathbb{C}^{n\times n}$ is *unitarily equivalent* to $B \in \mathbb{C}^{n\times n}$ if there exists a unitary matrix $P \in \mathbb{C}^{n\times n}$ such that $A = P^{-1}BP$. Prove that this is an equivalence relation on $\mathbb{C}^{n\times n}$. (Replacing the field $\mathbb{C}$ with the field $\mathbb{R}$ and the word 'unitary' with 'orthogonal,' the same definition gives when two matrices are *orthogonally equivalent.*)
4. ($\frac{-}{6*5}$) Let $T$ be a normal operator on a finite-dimensional complex inner product space $V$. Use the spectral decomposition $\lambda_1 T_1 + ... + \lambda_m T_m$ of $T$ (where $\lambda_i \in \mathbb{C}$) to prove the following:
   (a) If $g(t) \in \mathbb{C}[t]$ then $g(T) = \sum_{i=1}^{m} g(\lambda_i)T_i$.
   (b) If some positive power of $T$ is the zero transformation, then $T$ itself is the zero transformation.
   (c) Let $U$ be a linear operator on $V$. Then $U$ commutes with $T$ if and only if $U$ commutes with each $T_i$ for all $i \in \{1, 2, ..., m\}$.
   (d) There exists a normal operator $U$ on $V$ such that $U^2 = T$.
   (e) $T$ is invertible if and only if $\lambda_i \neq 0$ for all $i \in \{1, 2, ..., m\}$.
   (f) $T$ is a projection if and only if every eigenvalue of T is 1 or 0.

(g) $T = -T^*$ if and only if every $\lambda_i$ is *purely imaginary*, that is, it lies in the set $i\mathbb{R} := \{ix : x \in \mathbb{R}\}$.

5. $(\frac{-}{10})$ Show that if $T$ is a normal operator on a complex finite-dimensional inner product space and $U$ is a linear operator that commutes with $T$, then $U$ also commutes with $T^*$.

6. $(\frac{-}{10})$ Let $T$ be a normal operator on a finite-dimensional inner product space. Prove that if $T$ is a projection, then it is also an orthogonal projection.

7. $(\frac{-}{2*5})$ Let $U$ be a unitary operator on an inner product space $V$, and let $W$ be a finite-dimensional $U$-invariant subspace of $V$. Prove that:

   (a) $U(W) = W$

   (b) $W^\perp$ is $U$-invariant.

8. $(\frac{-}{10})$ Prove part $(c)$ of the spectral theorem. In other words: assume that for a linear operator $T$ on a finite dimensional inner product space $V$ which is normal if $F = \mathbb{C}$ and self adjoint if $F = \mathbb{R}$, and let $W_1, \ldots, W_m$ denote the eigenspaces corresponding to distinct eigenvalues of $T$ and let $T_i$ be the orthogonal projection of $V$ onto $W_i$. Then $T_iT_j = \delta_{ij}T_i$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.