Math 110BH homework 1

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Due Tuesday, January 16th

1

Show that if $1 = 0$ in a ring R, then R is the zero ring.

2

Find an example of a subring of $\mathbb{Q}$ different from $\mathbb{Z}$ and $\mathbb{Q}$.

set of rational numbers where the denominator is a power of 2???

3

Find all zero divisors in $\mathbb{Z}/m\mathbb{Z}$.

the set of integers which are not coprime to m???

4

Prove that the ring $\mathrm{End}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

Every endomorphism in Z is multiplication by an integer Z

5

Show that a subring of an integral domain is an integral domain. Is it true that a subring of a field is a field?

6

Prove that a finite integral domain is a field.

For any nonzero element b of a finite integral domain R, let $m_b : R \to R$ be the function defined by $m_b(a) = ab$. If a and a' are nonzero elements of R for which $m_b(a) = m_b(a')$, then $0 = m_b(a) - m_b(a') = ab - a'b = (a - a')b$. Since b is nonzero, $a - a'$ is also nonzero. We have shown that $m_b(a) = m_b(a')$ implies $a = a'$, meaning that m_b is injective.

Since m_b is an injective function from a finite set to itself, it must also be a bijection. HOW DOES THIS PROVE EVERY ELEMENT IS INVERTIBLE??? PIGEONHOLE SOMETHING??? WHAT IS A FIELD???

7

- (a) Find a ring A such that for any ring R there is exactly one ring homomorphism $A \to R$.
- (b) Find a ring B such that for any ring R there is exactly one ring homomorphism $R \to B$.

8

By "an ideal", in this problem, we mean left (respectively, right or two-sided) ideal. Let $f : R \to S$ be a ring homomorphism.

- (a) Let J be an ideal of S. Show that $f^{-1}(J)$ is an ideal of R that contains $\mathrm{Ker}(f)$.
- Prove that if f is surjective and I is an ideal of R, then $f(I)$ is an ideal of S. Show that the correspondence $I \mapsto f(I)$ yields a bijection between the set of all ideals of R that contain $\mathrm{Ker}(f)$ and the set of all ideals of S. Determine the inverse bijection.

9

- (a) An element a of a ring R is called *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$. Show that if R is a commutative ring, then the set $\mathrm{Nil}(R)$ of all nilpotent elements in R is an ideal (called the *nilradical* of R).
- (b) Prove that a polynomial $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ over a commutative ring R is nilpotent if and only if all a_i are nilpotent in R.

10

- (a) Prove that if a is a nilpotent element of a ring R, then the element $1+a$ is invertible. (Hint: Use the identity $1 - X^n = (1 - X)(1 + X + \cdots X^{n-1})$.)
- (b) Prove that a polynomial $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$ over a commutative ring R is invertible in $R[X]$ if and only if a_0 is invertible in R and all a_i are nilpotent in R for $i \geq 1$. (Hint: Let $g(X) = b_0 + b_1 X + \cdots + b_m X^m \in R[X]$ be the inverse of $f(X)$. Prove first that $a_n^{m+1} = 0$. Then use induction.)