Hamiltonian Dynamics

For a full treatment of Hamiltonian dynamics, you'll want to refer to a good textbook. What follows is more of a test-drive :)

Generalized Momenta

$$\boxed{P_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}}$$

The generalized Momentum associated with the generalized Coordinate q_i

The Hamiltonian

... is obtained in two steps...

(1) $H = \sum_i P_i \dot{q}_i - \mathcal{L}$

I've been known to call this the "pre-Hamiltonian" :)

(2) $H = H(P_i, q_i, t)$

it's not really the Hamiltonian until all the $\dot{q}_i$'s have been re-written in terms of P_i's $\Rightarrow$ (get rid of the $\dot{q}_i$'s)

⇒ If...

- the potential energy of a system is independent of the velocities in the system

<u>and</u>

- the transformations from the original coordinates (in some inertial frame) into generalized coordinates do not explicitly involve time

⇒ then...

The Hamiltonian is equal to the total energy of the system (expressed in terms of position and momentum): ($H = E = T + V$)

* If, in addition, the Hamiltonian is not an explicit function of time, the Hamiltonian is constant over time (and thus, energy is conserved).

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Taken together, the observations on this page mean you can often 'eyeball' the Hamiltonian ~ that is, write it directly from the statement of the problem

Hamiltonian ("Canonical") Equations of Motion

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ -\dot{p}_i &= \frac{\partial H}{\partial q_i} \end{aligned}}$$

Considered to be quite attractive for the fact that they're first-order and symmetric

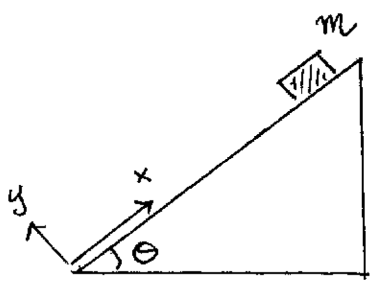
Procedure:

* By eyeball... 1) write $H = E$ (in terms of p's and q's)
2) obtain the canonical equations of motion

* More general... 1) write the Lagrangian for the system
2) obtain the generalized momentum for each generalized coordinate
3) Write $H = \sum p_i \dot{q}_i - \mathcal{L}$ (in terms of p's & q's)
4) obtain the canonical equations of motion

Let's take it out for a spin...

Example Block on a fixed plane (by eyeball)



$T = \frac{1}{2}m\dot{x}^2$

$V = mgx\sin\theta$

$E = T + V$

$E = \frac{1}{2}m\dot{x}^2 + mgx\sin\theta$

Here, we encounter our first problem... We need to replace velocity with generalized momentum, and -technically- we don't have the generalized momenta yet.

However - when the coordinates you are using for each degree of freedom tie back directly to a common inertial frame of reference, the generalized momentum is usually the actual momentum (or angular momentum) component associated with the coordinate. Keep in mind that you are probably making an assumption here - when in doubt, use the general approach (via the Lagrangian)!

→ with all that in mind, I feel pretty comfortable betting that $P_x = m\dot{x}$ so ↘

$$H = \frac{P_x^2}{2m} + mgx\sin\theta$$

→ The Canonical Equations of Motion?

$\dot{x} = \frac{\partial H}{\partial P_x}$

$\dot{x} = P_x/m$

$\dot{P}_x = m\ddot{x}$

$-\dot{P}_x = \frac{\partial H}{\partial x}$

$-\dot{P}_x = mg\sin\theta$

$m\ddot{x} = -mg\sin\theta$

$\ddot{x} = -g\sin\theta$

↓

Easy enough! ¿

Example: Block on a fixed plane (formal)

$$T = \tfrac{1}{2} m\dot{x}^2$$
$$V = mgx\sin\theta$$
$$\mathcal{L} = \tfrac{1}{2} m\dot{x}^2 - mgx\sin\theta$$

(from previous example)

Canonical Momenta:

$$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$P_x = m\dot{x}$$

{Cool! our previous assumption was right ☺}

Hamiltonian:

$$H = P_x \dot{x} - \mathcal{L}$$
$$H = m\dot{x}^2 - \tfrac{1}{2} m\dot{x}^2 + mgx\sin\theta$$
$$H = \tfrac{1}{2} m\dot{x}^2 + mgx\sin\theta$$

{The total energy - Good! Don't forget to swap out $\dot{x}$}

$P_x = m\dot{x} \longrightarrow$

$$H = \frac{P_x^2}{2m} + mgx\sin\theta$$

Canonical Equations of Motion:

$$\dot{x} = \frac{\partial H}{\partial P_x} \qquad -\dot{P}_x = \frac{\partial H}{\partial x}$$

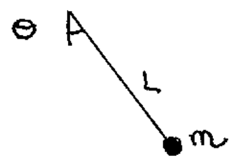
$$\dot{x} = \frac{P_x}{m} \qquad -\dot{P}_x = mg\sin\theta$$

$$m\ddot{x} = -mg\sin\theta$$

$$\ddot{x} = -g\sin\theta$$

{a bit redundant when we already have P_x}

Example: Simple Pendulum



$$T = \tfrac{1}{2} mL^2 \dot{\theta}^2$$
$$V = -mgL \cos\theta$$
$$\mathcal{L} = \tfrac{1}{2} mL^2 \dot{\theta}^2 + mgL \cos\theta$$

Canonical Momentum:

$$P_\theta = -\frac{\partial \mathcal{L}}{\partial \dot{\theta}}$$
$$P_\theta = mL^2 \dot{\theta}$$

The actual angular momentum of the pendulum about the pivot!

Hamiltonian:

$$H = P_\theta \dot{\theta} - \mathcal{L}$$
$$H = mL^2 \dot{\theta}^2 - \tfrac{1}{2} mL^2 \dot{\theta}^2 - mgL \cos\theta$$
$$H = \tfrac{1}{2} mL^2 \dot{\theta}^2 - mgL \cos\theta$$

TOTAL ENERGY - WE Could have eyeballed this

... swap out the $\dot{\theta}$'s ...

$$H = \frac{P_\theta^2}{2mL^2} - mgL \cos\theta$$

Canonical Equations of motion:

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} \qquad -\dot{P}_\theta = \frac{\partial H}{\partial \theta}$$

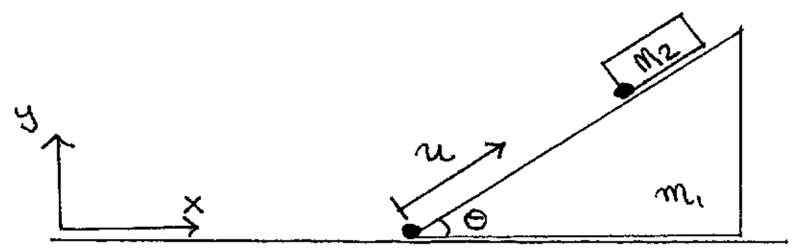
$$\dot{\theta} = \frac{P_\theta}{mL^2} \qquad -\dot{P}_\theta = mgL \sin\theta$$

$$-mL^2 \ddot{\theta} = mgL \sin\theta$$
$$\ddot{\theta} + \frac{g}{L} \sin\theta = 0$$

should look familiar :)

⇒ Enough of the easy stuff :)

Example: Block on a sliding ramp



In the inertial frame:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$V = m_2 g y_2$$

$$\mathcal{L} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) - m_2 g y_2$$

Generalized Coordinates:

$$x_2 = x_1 + u \cos\theta$$

$$y_2 = u \sin\theta$$

$$\mathcal{L} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1^2 + \dot{u}^2 + 2\dot{u}\dot{x}_1 \cos\theta) - m_2 g u \sin\theta$$

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)\dot{x}_1^2 + \frac{1}{2} m_2 \dot{u}_2^2 + m_2 \dot{u}\dot{x}_1 \cos\theta - m_2 g u \sin\theta$$

Canonical Momenta:

$$P_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$P_{x_1} = (m_1+m_2)\dot{x}_1 + m_2 \dot{u} \cos\theta$$

Cool! $P_{x_1} = m_1\dot{x}_1 + m_2\dot{x}_2$ ~ the horizontal component of the total momentum. But careful- had you tried to guess this you would probably have been wrong... The transformations into generalized coordinates mix things up...

$$P_u = m_2 \dot{u} + m_2 \dot{x}_1 \cos\theta$$

while it's tempting to try to interpret this as the total momentum projected onto the u-axis... keep in mind that the u-frame of reference is non-inertial and this may not be meaningful...

Personally, I take those comments to mean that it's great when you can confidently interpret a Canonical Momentum... Don't expect that to happen all the time :'

Hamiltonian:

$$H = P_{x_1}\dot{x}_1 + P_u\dot{u} - \mathcal{L}$$

$$H = \tfrac{1}{2}(m_1+m_2)\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{u}^2 + m_2\dot{u}\dot{x}_1\cos\theta + m_2 g u \sin\theta$$

OK, but this is really a "pre-Hamiltonian" - we need to replace the $\dot{x}_1$'s and $\dot{u}$'s with functions of P_{x_1} and P_u. How do we do this? Well... return to the equations for Canonical Momenta:

$$P_{x_1} = (m_1+m_2)\dot{x}_1 + m_2\cos\theta\,\dot{u}$$

$$P_u = m_2\cos\theta\,\dot{x}_1 + m_2\dot{u}$$

$\Rightarrow$ We can solve these simultaneously for $\dot{u}$ & $\dot{x}_1$

$$\dot{x}_1 = \frac{P_{x_1} - P_u\cos\theta}{m_1+m_2-m_2\cos^2\theta}$$

(yes, I used Mathematica :')

$$\dot{u} = \frac{(m_1+m_2)P_u - m_2 P_{x_1}\cos\theta}{m_2(m_1+m_2-m_2\cos^2\theta)}$$

So now, this goes back into the pre-Hamiltonian:

$$H = \frac{(m_1+m_2)P_u^2 - 2m_2\cos\theta P_u P_{x_1} + m_2\left(g m_2(2m_1+m_2-m_2\cos(2\theta))\sin\theta\, u + P_{x_1}^2\right)}{m_2(2m_1+m_2-m_2\cos(2\theta))}$$

(again, Mathematica: typos are mine - I just want to get the sense of how the solution looks...)

Finally...

Canonical Equations of Motion: $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\quad -\dot{p}_i = \frac{\partial H}{\partial q_i}$

Now before we get started, it may be convenient to point out that we already have the equations for $\dot{q}_i$. We solved these from the canonical momenta equations so we could write the Hamiltonian in its final form

$$\dot{x}_1 = \frac{P_{x_1} - P_u \cos\theta}{m_1 + m_2 - m_2 \cos^2\theta}$$

$$\dot{u} = \frac{(m_1+m_2) P_u - m_2 P_{x_1} \cos\theta}{m_2 (m_1 + m_2 - m_2 \cos^2\theta)}$$

So, to these, we'll add...

$$\dot{P}_{x_1} = 0$$

recall P_{x_1} is the total horizontal momentum... and it is evidently conserved :)

$$\dot{P}_u = -m_2 g \sin\theta$$

while P_u is changing at a constant rate :)

To finish... I would take the equations for canonical momenta

$$P_{x_1} = (m_1+m_2)\dot{x}_1 + m_2 \cos\theta \, \dot{u}$$

$$P_u = m_2 \cos\theta \, \dot{x}_1 + m_2 \dot{u}$$

substitute them in for $\dot{P}_{x_1}$ & $\dot{P}_u$ (after taking derivatives!) and solve the resulting second-order differential equations...