Math 180 midterm note sheet

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The addition principle:

$$\#(A \sqcup B) = \#A + \#B.$$

(In this context, "disjoint union" means union of sets which are disjoint.)

The multiplication principle:

$$\#(A \times B) = \#A \times \#B.$$

More generally, the multiplication principle says that if every object in A can be uniquely constructed from a series of k choices, with n_i options for the ith choice, then $\#A = \prod n_i$.

The subtraction principle: If A is a finite set and $B \subset A$, then

$$\#(A \backslash B) = \#A - \#B.$$

Relations: A relation between X and Y is any subset of $X \times Y$, and a (binary) relation on X is any subset of $X \times X$. Sometimes $(x, y) \in R$ is denoted by xRy. A binary relation $R \subset X \times X$ is called

- *Reflexive* iff $(x, x) \in R$ for any $x \in X$
- *Symmetric* iff $(x, y) \in R$ implies $(y, x) \in R$
- *Transitive* iff $(x, y), (y, z) \in R$ implies $(x, z) \in R$
- *Weakly antisymmetric* iff $(x, y), (y, x) \in R$ implies $x = y$.
- *Strongly antisymmetric* iff $(x, y) \in R$ implies $(y, x) \notin R$.

Some special types of relations: an **equivalence relation** is reflexive, symmetric, and transitive. A **partial ordering** is reflexive, antisymmetric, and transitive. A partial ordering R on X is also called a total order iff $R \cup R^{-1} = X \times X$, where R^{-1} denotes the swizzled version of R.

A **partition** of a set A is a set of disjoint subsets of A whose union is A (e.g. equivalence classes in A)

The division principle: If $f : A \to B$ is a surjection between finite sets such that $\#f^{-1}(b) = d$ for every $b \in B$, then

$$\#B = \frac{\#A}{d}$$

Falling factorials: The notation for a "falling factorial" is $(n)_k := \frac{n!}{(n-k)!}$.
Pascal's identity states that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

A **composition** of n into k parts is a sequence of k positive integers which sum to n, and a **weak composition** of n into k parts is a sequence of k nonnegative integers which sum to n. There are $\binom{n-1}{k-1}$ compositions and $\binom{n_k-1}{k-1}$ weak compositions (of n into k parts).

Binomial theorem: If n is a nonnegative integer, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

You can derive a bunch of useful variations of that formula by plugging in values for x or y, or by differentiating both sides with respect to x or y.

A **finite, undirected, unweighted graph** is $G = (V, E, \varphi)$ consists of a nonempty finite set V of vertices, a finite set E of edges, and a map $\varphi : E \to \{\{u, v\} : u, v \in V\}$ from edges to their endpoints.

A **simple graph** is a graph that contains no loops (edges from a vertex to itself) or multiple edges (meaning φ is injective). Sometimes, we just call these "graphs" and instead call graphs which are not simple "multigraphs".

The **path graph** P_n (where $n \geq 0$) has $n+1$ vertices connected in a line.

The **cycle graph** C_n (where $n \geq 2$, although C_2 is not simple) has n vertices connected in a circle.

The **complete graph** K_n has n vertices, where every pair of vertices share an edge.

The **complete bipartite graph** $K_{n,m}$ has vertices that can be split into a pair of disjoint subsets of sizes n and m, such that a pair of vertices share an edge if and only if they are not in the same one of those subsets.

A **subgraph** of $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subset V$ and $E' \subset E$. G' is called an **induced subgraph** if and only if every edge in G between vertices in V' is in E'.

A **path** in G is a subgraph of G which is a path graph, and a **cycle** in G is a subgraph of G which is a cycle graph.

A **graph isomorphism** between $G = (V, E, \varphi)$ and $G' = (V', E', \varphi')$ is a bijection $\theta : V \to V'$ such that vertices $u, v \in V$ share an edge if and only if $\theta(u)$ and $\theta(v)$ share an edge.

Two graphs are **isomorphic** iff there exists a graph isomorphism between them. This is an equivalence relation.

An **automorphism** on a graph G is an isomorphism from G to itself.

The number of isomorphic graphs with n vertices is less than or equal to $2^{\binom{n}{2}}$ but greater than or equal to $\frac{2^{\binom{n}{2}}}{n!}$.

A **walk of length** $t \geq 0$ in $G = (V, E)$ is a sequence of $t+1$ vertices (which are zero-indexed) and t edges such that the ith edge connects the $i-1$th and ith vertices.

Two vertices are called **connected** in a graph iff there exists a walk between them. This is an equivalence relation, and the equivalence classes are called **connected components**.

The **degree of a vertex** in a simple graph is the number of edges incident to it. In a multigraph, the degree of a vertices is the number of edges incident to it plus the number of loops incident to it (so loops are counted twice).

The **handshaking lemma** says that for a graph $G = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|.$$

The **distance between two vertices**, denoted $d(x, y)$ (where $x, y \in V$), is the infimum of the lengths of all walks between x and y.

The **adjacency matrix** of a graph $G = (V = [n], E)$ is the $n \times n$ matrix A such that $A_{i,j}$ is 1 if vertices i and j share an edge, and 0 otherwise. This is always symmetric, so it has an orthonormal basis of eigenvectors, and all of its eigenvalues are real. The number of walks of length k from i to j is $\left(A^k\right)_{i,j}$.

The **degree sequence**, sometimes called **score**, of a graph $G = (V, E)$ is the multiset

$$\{\deg v_1, \deg v_2, \ldots, \deg v_n\}$$

which is usually written in nondecreasing order.

The **score theorem** states that if $D = (d_1, d_2, \ldots, d_n)$ is a chain of natural numbers, then D is the score of a (simple) graph if and only if $D' := (d_1', \ldots, d_{n-1}')$ (defined by the following formula) is a (simple) graph score:

$$d_i' = \begin{cases} d_i & i < n - d_n \\ d_i - 1 & i \geq n - d_n \end{cases}$$

The proof of that is equivalent to the Havel-Hakimi algorithm for finding such a graph.

An **Eulerian walk** is a walk that traverses every edge of a graph exactly once (and can use each vertex any number of times). **Hierholzer's theorem** states that a connected graph has a closed Eulerian walk if and only if all of its vertices have even degree. A graph which contains a closed Eulerian walk is called **Eulerian**. More generally, a connected graph has an Eulerian walk if and only if there are either 0 or 2 vertices with odd degree.

A **Hamiltonian cycle** in a graph is a cycle that visits every vertex at least once, and a **Hamiltonian path** is a path which visits every vertex at least once. A **Hamiltonian graph** is a graph which contains a Hamiltonian cycle.

A **Gray code (of degree d)** is a hamiltonian cycle of the **cube graph** of degree d, which is the skeleton of the d-dimensional cube. Alternatively, the Gray code of degree d is a sequence of all 2^d binary strings with d bits, such that the **Hamming distance** between adjacent strings is 1. For example, the Gray codes of degree 3 are

0	000
1	001
2	011
3	010
4	110
5	111
6	101
7	100

A graph is called k**-connected** iff it has at least $k+1$ vertices and it remains connected after removing ANY $k-1$ vertices. In general,

- $G - e$ is the graph obtained by removing edge e
- $G - v$ is the graph obtained by removing vertex v and all edges incident to v
- $G + e$ is the graph obtained by adding a new edge e
- $G\%e$ is the graph obtained by subdividing e and adding a new vertex on e.

A graph is 2-connected if and only if for any two vertices in that graph, there is a cycle containing those two vertices.

A **graph subdivision** of a graph G is a graph that can be obtained by repeatedly subdividing edges of G.

Whitney's theorem says that G is 2-connected if and only if G can be constructed from $K_3 \cong C_3$ by a sequence of subdivisions and edge additions.

The **complement** of a graph $G = (V, E)$ is

$$\overline{G} = \left(V, \binom{V}{2} - E\right)$$