Math 180 midterm note sheet

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The addition principle:

$$\#(A \sqcup B) = \#A + \#B.$$

(In this context, "disjoint union" means union of sets which are disjoint.)

The multiplication principle:

$$\#(A \times B) = \#A \times \#B.$$

More generally, the multiplication principle says that if every object in A can be uniquely constructed from a series of k choices, with n_i options for the ith choice, then $\#A = \prod n_i$.

The subtraction principle: If A is a finite set and $B \subset A$, then

$$\#(A \backslash B) = \#A - \#B.$$

Relations: A relation between X and Y is any subset of $X \times Y$, and a (binary) relation on X is any subset of $X \times X$. Sometimes $(x, y) \in R$ is denoted by xRy. A binary relation $R \subset X \times X$ is called

- *Reflexive* iff $(x, x) \in R$ for any $x \in X$
- *Symmetric* iff $(x, y) \in R$ implies $(y, x) \in R$
- *Transitive* iff $(x, y), (y, z) \in R$ implies $(x, z) \in R$
- *(Weakly) antisymmetric* iff $(x, y), (y, x) \in R$ implies $x = y$.
- *Strongly antisymmetric* iff $(x, y) \in R$ implies $(y, x) \notin R$.

Some special types of relations: an **equivalence relation** is reflexive, symmetric, and transitive. A **partial ordering** is reflexive, antisymmetric, and transitive. A partial ordering R on X is also called a total order iff $R \cup R^{-1} = X \times X$, where R^{-1} denotes the swizzled version of R.

A **partition** of a set A is a set of disjoint subsets of A whose union is A (e.g. equivalence classes in A)

The division principle: If $f : A \to B$ is a surjection between finite sets such that $\#f^{-1}(b) = d$ for every $b \in B$, then

$$\#B = \frac{\#A}{d}$$

Falling factorials: The notation for a "falling factorial" is $(n)_k := \frac{n!}{(n-k)!}$.

Pascal's identity states that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

A **composition** of n into k parts is a sequence of k positive integers which sum to n, and a **weak composition** of n into k parts is a sequence of k nonnegative integers which sum to n. There are $\binom{n-1}{k-1}$ compositions and $\binom{n-k-1}{k-1}$ weak compositions (of n into k parts).

Binomial theorem: If n is a nonnegative integer, then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

You can derive a bunch of useful variations of that formula by plugging in values for x or y, or by differentiating both sides with respect to x or y.

A **(finite, undirected, unweighted) graph** $G = (V, E, \varphi)$ consists of a nonempty finite set V of vertices, a finite set E of edges, and a map $\varphi : E \to \{\{u, v\} : u, v \in V\}$ from edges to their endpoints.

A **simple graph** is a graph that contains no loops (edges from a vertex to itself) or multiple edges (meaning φ is injective). Sometimes, we just call these "graphs" and instead call graphs which are not simple "multigraphs".

The **path graph** P_n (where $n \geq 0$) has $n + 1$ vertices connected in a line.

The **cycle graph** C_n (where $n \geq 2$, although C_2 is not simple) has n vertices connected in a circle.

The **complete graph** K_n has n vertices, where every pair of vertices share an edge.

The **complete bipartite graph** $K_{n,m}$ has vertices that can be split into a pair of disjoint subsets of sizes n and m, such that a pair of vertices share an edge if and only if they are not in the same one of those subsets.

A **subgraph** of $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subset V$ and $E' \subset E$. G' is called an **induced subgraph** if and only if every edge in G between vertices in V' is in E'.

A **path** in G is a subgraph of G which is a path graph, and a **cycle** in G is a subgraph of G which is a cycle graph.

A **graph isomorphism** between $G = (V, E, \varphi)$ and $G' = (V', E', \varphi')$ is a bijection $\theta : V \to V'$ such that vertices $u, v \in V$ share an edge if and only if $\theta(u)$ and $\theta(v)$ share an edge.

Two graphs are **isomorphic** iff there exists a graph isomorphism between them. This is an equivalence relation.

An **automorphism** on a graph G is an isomorphism from G to itself.

The number of isomorphic graphs with n vertices is less than or equal to $2^{\binom{n}{2}}$ but greater than or equal to $\frac{2^{\binom{n}{2}}}{n!}$.

A **walk of length** $t \geq 0$ in $G = (V, E)$ is a sequence of $t + 1$ vertices (which are zero-indexed) and t edges such that the ith edge connects the $i - 1$th and ith vertices. A **tour** is a walk whose start and end vertices are the same, and a **trail** is a walk in which no edges are repeated (but vertices can be repeated).

Two vertices are called **connected** in a graph iff there exists a walk between them. This is an equivalence relation, and the equivalence classes are called **connected components**.

The **degree of a vertex** in a simple graph is the number of edges incident to it. In a multigraph, the degree of a vertices is the number of edges incident to it plus the number of loops incident to it (so loops are counted twice).

The **handshaking lemma** says that for a graph $G = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|.$$

The **distance between two vertices**, denoted $d(x, y)$ (where $x, y \in V$), is the minimum of the lengths of all walks between x and y.

The **adjacency matrix** of a graph $G = (V = [n], E)$ is the $n \times n$ matrix A such that $A_{i,j}$ is 1 if vertices i and j share an edge, and 0 otherwise. This is always symmetric, so it has an orthonormal basis of eigenvectors, and all of its eigenvalues are real. The number of walks of length k from i to j is $\left(A^k\right)_{i,j}$.

The **degree sequence**, sometimes called **score**, of a graph $G = (V, E)$ is the multiset

$$\{\deg v_1, \deg v_2, \ldots, \deg v_n\}$$

which is usually written in nondecreasing order.

The **score theorem** states that if $D = (d_1, d_2, \ldots, d_n)$ is a nondecreasing chain of natural numbers, then D is the score of a (simple) graph if and only if $D' := (d'_1, \ldots, d'_{n-1})$ (defined by the following formula) is a (simple) graph score:

$$d'_i = \begin{cases} d_i & i < n - d_n \\ d_i - 1 & i \geq n - d_n \end{cases}$$

The proof of that is equivalent to the Havel-Hakimi algorithm for finding such a graph.

An **Eulerian walk** is a walk that traverses every edge of a graph exactly once (and can use each vertex any number of times). **Hierholzer's theorem** states that a connected graph has a closed Eulerian walk if and only if all of its vertices have even degree. A graph which contains a closed Eulerian walk is called **Eulerian**. More generally, a connected graph has an Eulerian walk if and only if there are either 0 or 2 vertices with odd degree.

A **Hamiltonian cycle** in a graph is a cycle that visits every vertex at least once, and a **Hamiltonian path** is a path which visits every vertex at least once. A **Hamiltonian graph** is a graph which contains a Hamiltonian cycle.

A **Gray code (of degree d)** is a Hamiltonian cycle of the **cube graph** of degree d, which is the skeleton of the d-dimensional cube. Alternatively, the Gray code of degree d is a sequence of all 2^d binary strings with d bits, such that the **Hamming distance** between adjacent strings is 1. For example, the Gray codes of degree 3 are

0	1	2	3	4	5	6	7
000	001	011	010	110	111	101	100

A graph is called **k-connected** iff it has at least $k + 1$ vertices and it remains connected after removing ANY $k - 1$ vertices.

- $G - e$ is the graph obtained by removing edge e
- $G - v$ is the graph obtained by removing vertex v and all edges incident to v
- $G + e$ is the graph obtained by adding a new edge e
- $G\%e$ is the graph obtained by subdividing e and adding a new vertex on e.
- G/e is the graph obtained by contracting e (gluing its 2 vertices and then removing just enough edges to get a simple graph)

A graph is 2-connected if and only if for any two vertices in that graph, there is a cycle containing those two vertices.

A **graph subdivision** of a graph G is a graph that can be obtained by repeatedly subdividing edges of G.

Whitney's theorem says that G is 2-connected if and only if G can be constructed from $K_3 \cong C_3$ by a sequence of subdivisions and edge additions.

The **complement** of a graph $G = (V, E)$ is

$$\overline{G} = \left(V, \binom{V}{2} - E\right)$$

A **tree** is a connected graph with no cycles.

A **leaf** or **end-vertex** is a vertex with degree 1.

The **leaf lemma** says every tree with at least 2 vertices contains at least two leaves.

The **tree-growing lemma** says G is a tree iff for any leaf v of G, $G - v$ is a tree.

The **0th Betti number** b_0 is the number of connected components of a graph, and the **1st Betti number** b_1 is the maximum number of edges that can be removed without changing b_0.

A **forest** is a graph whose connected components are all trees. G is a forest iff $b_1 = 0$. Then $b_0 = |V| - |E|$. In particular, trees have $|V| - |E| = 1$.

For any simple graph, the Euler characteristic (of the 1-skeleton) is $\chi(G) = |V| - |E| = b_0 - b_1$.

A **rooted tree** is a tree in which one vertex is specified as the root. The terms **parent** and **child** are defined as you'd expect in a rooted tree.

An **isomorphism of rooted trees** is a graph isomorphism between rooted trees which preserves the root.

A **planted tree** is a rooted tree with a linear ordering on the children of each vertex. An **isomorphism of planted trees** preserves that ordering.

The **code** of a planted tree P with n vertices (drawn with the root at the top and children ordered left-to-right) is a sequence of $2(n - 1$ characters on the alphabet $\{\pm 1\}$ (or $\{-, 1\}$), defined by tightly looping counterclockwise around the edges, starting and ending at the root, writing "1" every time you go down and "-" every time you go up. A string that is the code of a planted tree is called a **ballot sequence**. All partial sums of a ballot sequence are nonnegative.

The n**th Catalan number** C_n is the number of ballot sequences of length $2n$ (equivalently, the number of unlabeled planted trees):

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}.$$

The **eccentricity** $\mathrm{ex}_G(v)$ of a vertex v in a connected graph is the maximum distance to any other vertex. The **center** $C(G)$ is the set of vertices in a graph with the minimum eccentricity. The center of any tree is either one vertex or two, and if it's two vertices, those two share an edge.

A graph is **planar** iff it can be embedded in a plane or sphere so that edges don't intersect. Embeddings are called **distinct** iff they are not isotopic. The connected components of the complement of an embedding of a graph in $\mathbb{R}^2$ are called **faces**. Every planar graph has at least one unbounded face, called an **outer face**.

The number of faces in an embedded planar graph is $|F| = b_1 + 1$, so for any planar graph, $|V| - |E| + |F| = b_0 + 1$, and for any connected planar graph, $|V| - |E| + |F| = 2$.

The **degree of a face** is the degree of the dual vertex. The sum of the degrees of all faces is equal to $2|E|$.

IF G has a subgraph that is a subdivision of a nonplanar graph, then G is nonplanar.

Kuratowski's theorem says that a graph is planar iff it does not have any subgraph isomorphic to either K_5 or $K_{3,3}$.

Any connected planar graph with $|V| \geq 3$ has $|E| \leq 3|V| - 6$.

If every vertex has degree d and every face has degree k in a connected graph, it must be one of the following:

d	k	**Convex polyhedron name**	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert F\rvert$	$\chi = \lvert V\rvert - \lvert E\rvert + \lvert F\rvert$	Name of dual
3	3	**Tetrahedron**	4	6	4	2	Tetrahedron
3	4	**Cube**	8	12	6	2	Octahedron
4	3	**Octahedron**	6	12	8	2	Cube
3	5	**Dodecahdron**	20	30	12	2	Icosahdron
5	3	**Icosahedron**	12	30	20	2	Dodecahdron

A **proper vertex coloring** is a vertex coloring (assignment of a color to each vertex) which is proper (meaning adjacent vertices are not the same color). A graph is k**-colorable** iff it has a proper vertex coloring with $\leq k$ colors. A graph is **bipartite** iff it is 2-colorable. The **4 color theorem** says that every planar graph is 4-colorable.

The **chromatic number** $\chi(G)$ of a graph G is the smallest k such that G is k-colorable.

A **clique** is a subgraph isomorphic to K_n, and the **clique number** $\omega(G)$ is the number of vertices in the largest clique of G. The clique number is always les than or equal to the chromatic number.

An **independent set** is a subset of the vertices in the graph such that no two vertices share an edge (equivalently, it is a clique in the complement of the graph). The **independence number** $\alpha(G)$ is the size of the largest independent set in G.

For any graph $G = (V, E)$, $|V| \leq \chi(G)\alpha(G)$. Also, $\chi(G) \leq \max_{v \in V}(\deg(v)) + 1$.

The **chromatic polynomial** $p_G(k)$ is the polynomial function $p_G : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ such that $p_G(k)$ is the number of proper vertex colorings of G in k colors. Then $\chi(G)$ is the smallest k such that $p_G(k) \neq 0$.

If $G = K_n$ then $p_G(k) = k(k-1)\cdots(k-n+1)$. If G is a tree with n vertices, then $p_G(k) = k(k-1)^{n-1}$. If G is a forest with c connected components, $p_G(k) = k^c(k-1)^{n-c}$.

The **deletion-contraction formula** says that

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k)$$

The chromatic polynomial is always monic and always has degree equal to the number of vertices. This is because for k much larger than $|V|$, "almost all" colorings are proper, meaning $\lim_{k\to\infty} \frac{p_G(k)}{k^{|V|}} = 1$.

An **acyclic orientation** of G is a **directed graph** obtained by orienting edges of G without creating a cycle. The number of acyclic orientations of G is denoted $\mathrm{ao}(G)$.

If G is a tree, then $\mathrm{ao}(G) = 2^{|E|}$. If $G = C_n$, then $\mathrm{ao}(G) = 2^n - 2$. If $G = K_n$, then $\mathrm{ao}(G) = n!$.

Stanley's theorem says that

$$\mathrm{ao}(G) = (-1)^{|V|} \cdot p_G(-1).$$

The **deletion-contraction formula for edges** says that

$$\mathrm{ao}(G) = \mathrm{ao}(G-e) + \mathrm{ao}(G/e).$$

A **spanning tree** of G is a subgraph which is a tree and which contains all vertices of G.

A **weight** is a function wt from the edges of a graph G to $\mathbb{R}_{\geq 0}$. A **weighted graph** is a pair (G, wt). The weight of a graph or subgraph is the sum of the weights of all the edges.

A **minimal spanning tree (MST)** of a connected weighted graph is a spanning tree T such that $\mathrm{wt}(T)$ is minimized.

Kruskal's algorithm for finding an MST requires iterating through the edges in order by increasing weight, and appending edges to the list iff doing so doesn't create a cycle.

Cayley's formula says the number of spanning trees in K_n is n^{n-2}. **Joyal's proof** counts the number of **vertebrates**, which are trees with a specified head vertex and a specified tail vertex. **Prüfer's proof** instead counts the number of **Prüfer codes**, which are sequences in $[n]^{n-2}$. A Prüfer code for an MST is constructed by repeatedly removing the smallest-numbered leaf from the tree and then appending the number of its neighbor to the sequence, stopping when exactly two vertices remain.

The number of times a vertex appears in a Prüfer sequence is one less than the degree of that vertex in corresponding MST.

A **finite probability space** (Ω, p) is a finite set Ω (called the **sample space**) and a function $p : \Omega \to [0,1]$ such that $\sum_{\omega\in\Omega} p(\omega) = 1$. The values $p(\omega)$ are called **elementary probabilities**, a subset $A \subset \Omega$ is called an **event**, $\mathbb{P}[A] := \sum_{\omega\in A} p(\omega)$ is called the **probability of** A, a function $X : \omega \to \mathbb{R}$ is called a **random variable**, and the **expected value** or **mean** of X is $\mathbb{E}[X] := \sum \omega \in \Omega X(\omega)p(\omega)$.

A **uniform probability distribution** is a probability space (Ω, p) for which all elementary events have probability $\frac{1}{|\Omega|}$.

The probability space of **random graphs** is (Ω, p) where Ω is the set of (labeled) graphs with vertex set $[n]$ and the probability of each graph is $2^{-\binom{n}{2}}$.

The **union bound** says that for any two events A, B, $\mathbb{P}[A \cup B] \leq \mathbb{P}[A] + \mathbb{P}[B]$.

Events $A_1, \ldots, A_k$ are **mutually independent** iff for any $I \subset [k]$, $\mathbb{P}[\cap_{i\in I} A_i] = \prod_{i\in I} \mathbb{P}[A_i]$.

The probability space $\mathcal{G}(n,p)$ (where $n \in \mathbb{N}, p \in [0,1]$) of **Erdős-Renyi random graphs** is the set of (simple, labeled) graphs $G = ([n], E)$ for which the elementary probability of each G is $p^{|E|}$. Intuitively, this means every edge has probability p of existing.

Linearity of expectation says that if X, Y are random variables, and $a, b \in \mathbb{R}$, then $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$.

If $A \subset \Omega$ is an event, then the **indicator random variable** $\mathbf{1}_A : \Omega \to \{0,1\}$ is a function such that $\mathbf{1}_A(\omega)$ is 1 if $\omega \in A$ and 0 otherwise.

The **pigeonhole principle for expectation** says that if $\mathbb{E}[X] > a$, then $\mathbb{P}[X > a] > 0$.

A **tournament** T is an orientation of the edges of K_n. For any $n \in \mathbb{N}$, there is a tournament with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

The **probabilistic method** is to "show that a combinatorial object with certain properties exists by showing that if we choose a random object from an appropriate probability space of objects, there is a nonzero probability that we get an object with the desired properties".

The **Ramsey number** $R(k,l)$ is the largest n such that every graph on n vertices has clique number $\omega(G) \geq k$ or independence number $\alpha(G) \geq l$. Erdős showed in 1947 that $R(k,k) > 2^{\frac{k}{2}-1}$.

A k**-uniform hypergraph** is a pair $H = (V, E)$ where V is a set of vvertices and $E \subset \binom{V}{k}$ is a set of **(hyper)edges**.

A **proper** c**-coloring** of H is a function from V to $[c]$ such that there is no hyperedge in H whose vertices are all the same color. $m(k)$ is the function defined as

$$m(k) := \min\left(\{m' \in \mathbb{N} : \text{there exists a } k\text{-uniform hypergraph with } m' \text{ edges which is not 2-colorable}\}\right).$$

By the probabilistic method, $m(k) \geq 2^{k-1}$.