Math 180 Homework 3

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March 4, 2024

1

> Prove or disprove: if u and v are the only vertices in a graph with odd degree, then there is a path from u to v.

Assume this question is talking about simple graph, and not multigraphs. If multigraphs are allowed, there is an obvious counterexample: the graph with two vertices which each have one edge to themselves, but are not connected to each other.

Suppose there exists a (simple) graph G in which u and v are the only vertices with odd degree, and there is no path from u to v. Then let U be the connected component of G which contains u but not v. The only vertex in U with odd degree is u, so the sum of the degrees of each vertex in U is odd. But because U is a simple graph, it must follow the handshaking lemma, so this is a contradiction. Therefore there must be a path in G from u to v.

2

> Let G be the complete bipartite graph $K_{m,n}$, where $m, n > 0$. Characterize the m, n such that:
>
> - (1) G has a closed Eulerian walk.
> - (2) G has an Eulerian walk.

- (1) m and n must both be even, since every vertex in an Eulerian graph has even degree.
- (2) Exactly 0 or 2 vertices can have odd degree. Therefore (m, n) must be either $(1, 1)$, or $(2, \text{any odd number}$, or $(\text{any odd number}, 2)$, or both m and n can be even.

3

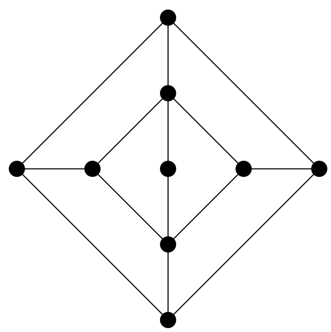
How many Hamiltonian cycles does K_n have? Two Hamiltonian cycles are considered distinct if their set of edges are different.

Given a Hamiltonian cycle of K_n, we can write the cycle as a permutation of $[n]$ by first choosing which of the n vertices to start at, then choosing which direction to go in (if $n > 2$, there are two choices of which direction to go in, otherwise there are only one). If n is either 1 or 2, then there is exactly one Hamiltonian cycle in K_n. Otherwise, there are exactly $2n$ ways to write each Hamiltonian cycle of K_n as a permutation of $[n]$.

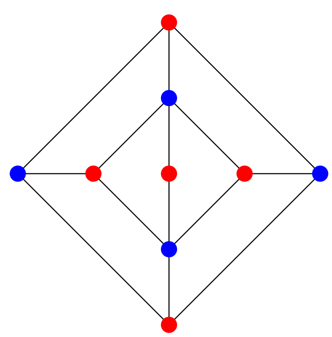
For any $\sigma \in \mathrm{Aut}([n])$, the sequence of vertices $(\sigma(1), \sigma(2), \ldots, \sigma(n), \sigma(1))$ represents a Hamiltonian cycle in K_n. Therefore every permutation of $[n]$ corresponds to a Hamiltonian cycle of K_n, so if $n > 2$, there are $n!/(2n) = (n-1)!/2$ Hamiltonian cycles of K_n.

4

Prove that the graph below is *not* Hamiltonian.



Every edge in this graph connects a red and a blue node. Since a Hamiltonian cycle is a subgraph, it would also have that property, so it would visit the same number of red and blue vertices. However, to be a Hamiltonian cycle, it must visit each of the 4 blue vertices and the 5 red vertices exactly once. That's a contradiction, so the graph is not Hamiltonian.



5

Prove that if an edge e appears an odd number of times in a closed walk W, then W contains the edges of a cycle through e. *Hint: Use induction on the length of* W.

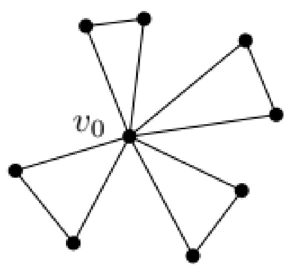
If W doesn't repeat any vertices, this is clearly true. Otherwise, we can split it into to shorter walks, one of which uses e an odd number of times. If $|W| = 1$, this statement is true, so by induction, it's always true.

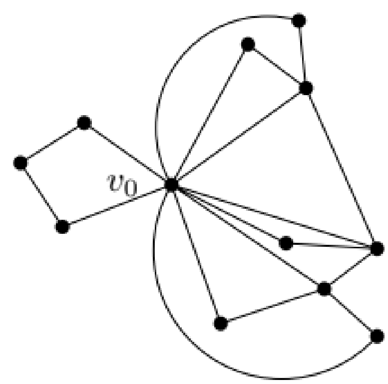
6

Section 4.4, Exercise 10

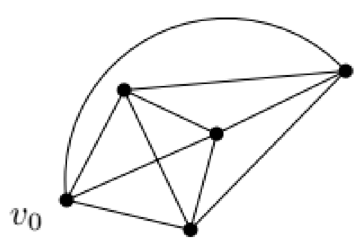
10. We say that a graph $G = (V, E)$ is *randomly Eulerian* from a vertex v_0 if every maximal tour starting at v_0 is already a closed Eulerian tour in G. That is, if we start at v_0 and draw edges one by one, choosing a continuation arbitrarily among the yet unused edges, we can never get stuck. (It would be nice if art galleries or zoos were randomly Eulerian, but unfortunately they seldom are. The result in part (c) below indicates why.)

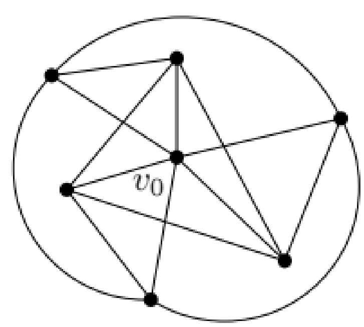
(a) Prove that the following graphs are randomly Eulerian:





(b) Show that the graphs below are not randomly Eulerian:

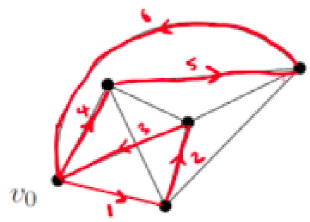


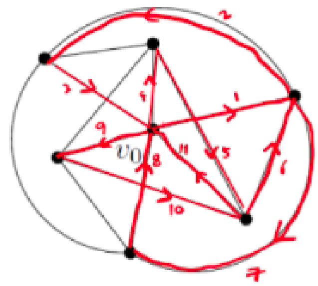


(c) *Prove the following characterization of randomly Eulerian graphs. A connected graph $G = (V, E)$ all of whose vertices have even degree is randomly Eulerian from a vertex v_0 if and only if the graph $(V \setminus \{v_0\}, \{e \in E\colon\ v_0 \notin e\})$ contains no cycle.

Problem 0.6. Section 4.4, Exercise 10.

Solution. (a) Use part (c).





(b)

(c) $\Longrightarrow$: Suppose G is randomly Eulerian. Assume for contradiction that $G - v_0$ contains a cycle C. Let G' denote the graph obtained from G by removing all edges in C. Then G' has all vertices of even degree, so G' has a closed Eulerian walk T. Then T is a tour in G from v_0 to v_0 that uses all the edges incident to v_0, so it is maximal, but is not a closed Eulerian walk in G, a contradiction.

$\Longleftarrow$: Suppose $G - v_0$ contains no cycle. Let $T = (v_0, e_1, v_1, \ldots, e_m, v_m)$ be a maximal tour in G starting at v_0. Since v_m has even degree and T is maximal, we must have $v_m = v_0$. Therefore, T is a tour starting and ending at v_0 and using all the edges incident to v_0. Let G' denote the graph obtained by removing all the edges in T and also the vertex v_0. Then G' has all vertices of even degree as well. Since $G - v_0$ has no cycle, neither does G'. By the lemma proved in class, a connected graph in which every vertex has even degree is either a single vertex or has a cycle. Therefore, every connected component of G' is an isolated