Conversion from SI to natural units

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The dimensions for any quantity can be written as a column vector in $\mathbb{Z}^3$ by listing the powers of mass, length, and time (in that order) as a column vector.

$$[M] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [L] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad [T] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then energy, the reduced Planck constant, and the speed of light have dimensions represented by the following vectors:

$$[E] = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \qquad [\hbar] = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad [c] = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

However, we could also write the dimensions for any quantity in another basis. When using natural units, it's convenient to work in the basis where

$$[E] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [\hbar] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad [c] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To convert from the $([E], [\hbar], [c])$ basis to the $([M], [L], [T])$ basis, all we need to do is left multiply by a change-of-basis matrix, which I'll call A.

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ -2 & -1 & -1 \end{bmatrix}$$

The inverse of that is

$$A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

so to convert from the $([M], [L], [T])$ basis to the $([E], [\hbar], [c])$ basis, multiply by A^{-1}.

By looking at the entries of A^{-1}, we can read off the following conversions:

$$\begin{aligned} [M] &= [E/c^2] \\ [L] &= [\hbar c/E] \\ [T] &= [\hbar/E] \end{aligned}$$