Math 110BH homework 2

Nathan Solomon

January 22, 2024

Due January 23rd

1

Prove that every (left) ideal of the product $R \times S$ of two rings is a product $I \times J$, where $I \subset R$ and $J \subset S$ are (left) ideals.

2

- (a) Find all idempotents in $\mathbb{Z}/105\mathbb{Z}$.
- (b) Prove that $\mathbb{Z}/p^n\mathbb{Z}$, p a prime, has no nontrivial idempotents.

- (a)
$$\{0, 1, 15, 21, 36, 70, 85, 91\}$$
- (b)

3

Suppose a commutative ring has finitely many idempotents. Prove that the number of idempotents is a power of 2.

4

Show that the ring $M_2(\mathbb{R})$ has infinitely many idempotents.

Consider projection matrices

5

Describe all homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. In each case determine the kernel and the image.

6

Prove that an element a of a commutative ring R is invertible if and only if a does not belong to any maximal ideal of R.

7

Determine all maximal and prime ideals of $\mathbb{Z}/n\mathbb{Z}$.

8

Let R be a commutative ring. The *radical* $\mathrm{Rad}(R)$ of R is the intersection of all maximal ideals in R.

- (a) Determine $\mathrm{Rad}(\mathbb{Z})$ and $\mathrm{Rad}(\mathbb{Z}/12\mathbb{Z})$.
- (b) Prove that $\mathrm{Rad}(R)$ consists of all elments $a \in R$ such that $1 + ab$ is invertible for all $b \in R$.

9

- (a) Prove that every nilradical $\mathrm{Nil}(R)$ of a commutative ring R is contained in every prime ideal of R.
- (b) Prove that $\mathrm{Nil}(R) \subset \mathrm{Rad}(R)$.

10

Let A be an abelian group (written additively). Define a product on the (additive) group $R = \mathbb{Z} \oplus A$ by $(n, a) \cdot (m, b) = (nm, nb_m a)$.

- (a) Prove that R is a ring.
- (b) Determine all prime and maximal ideals of R.