

Physics 245 Homework #3

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Problem 0.1.

Every qubit can be written as some global phase factor times

$$|\Psi\rangle = \cos(\varphi/2) |0\rangle + e^{i\phi} \sin(\varphi/2) |1\rangle,$$

which is uniquely represented in the Bloch sphere by the coordinates

$$x = \sin \varphi \cos \phi$$

$$y = \sin \varphi \sin \phi$$

$$z = \cos \varphi.$$

Normally, we would use θ in place of φ to represent the polar angle, but I want to reserve θ to represent the angle of the rotation matrices.

(a) This is going to map our original state to

$$\begin{aligned} \hat{R}_x(\theta) |\Psi\rangle &= \begin{bmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos \frac{\varphi}{2} \\ e^{i\phi} \sin \frac{\varphi}{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - i e^{i\phi} \sin \frac{\theta}{2} \sin \frac{\varphi}{2} \\ -i \sin \frac{\theta}{2} \cos \frac{\varphi}{2} + e^{i\phi} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \end{bmatrix} \end{aligned}$$

(b)

(c)

(d)

(e)

Problem 0.2.

(a)

(b)

(c)

Problem 0.3.

(a)

(b)

(c)

(d)

Phys 245 Quantum Computation
Homework 3

1. [50] *Single Qubit Gates! We got 'em!* In math, $SU(2)$ is the group of unitary 2×2 matrices with determinant equal to 1. This group is important in quantum mechanics because, for a qubit, all possible quantum operations can be written as an $SU(2)$ matrix. This means that any time evolution operator you find for a qubit will always be in $SU(2)$. Now, $SU(2)$ is closely related to the 3D rotation group, $SO(3)$. $SO(3)$ is just the group of rotations about an origin in Euclidean space. So, we can expect just like in $SO(3)$, where we can define matrices that rotate a vector, we should be able to find matrices in $SU(2)$ that rotate our qubit around the Bloch sphere. Show, using a conveniently chosen initial state, that:

- a. [10] $\hat{R}_x(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$ rotates a qubit state vector about the x axis on the Bloch sphere by an angle θ .

- b. [10] $\hat{R}_y(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$ rotates a qubit state vector about the y axis on the Bloch sphere by an angle θ .

- c. [10] $\hat{R}_z(\theta) = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$ rotates a qubit state vector about the z axis on the Bloch sphere by an angle θ .

- d. [10] Show how each of these rotation matrices can be realized from the time-evolution operator we found in class:

$$\hat{U}_R = \cos\left(\frac{\Omega' t}{2}\right) \hat{I} - i \sin\left(\frac{\Omega' t}{2}\right) \left(-\frac{\delta}{\Omega'} \hat{\sigma}_z + \frac{\Omega}{\Omega'} (\cos \phi \hat{\sigma}_x + \sin \phi \hat{\sigma}_y) \right)$$

- e. [10] Show how starting from state $|0\rangle$, \hat{U}_R can be used to create any point on the Bloch sphere – that is any single qubit state can be realized.

2. [50] *Using QuTiP to find the evolution of a real system.* In a real quantum computer, to effect single qubit gates the pulses of radiation are pulsed on and off when an algorithm calls for a certain gate. Due to hardware limitations the pulses do not instantly turn on and off. Therefore, a gate evolution would follow from a Hamiltonian similar to:

$$\hat{H} = \frac{\omega_0}{2} \hat{\sigma}_z + \Omega f(t) \cos \omega t \hat{\sigma}_x$$

where $f(t)$ is an 'envelope function' that turns on and off smoothly as dictated by the hardware.

- a.) [10] Transform this Hamiltonian to the interaction picture with respect to the qubit energies and make the RWA.
- b.) [20] Use QuTiP to numerically find the evolution of a system initially in $|0\rangle$ under the Hamiltonian you found in part (a) and with $f(t) = 1$. Here you should use `sesolve`, `mcsolve`, or `mesolve`. Choose the parameters as you wish, but make sure they are appropriate to accomplish a π -pulse.

c.) [20] Use QuTip to numerically solve the problem with $f(t) = A \operatorname{sech}^2 \frac{t}{\Delta t}$. Choose the parameters to accomplish a π -pulse. For the parameters you chose, calculate the integral $\int_{-\infty}^{\infty} A \operatorname{sech}^2 \frac{t}{\Delta t} dt$.

3. [40] *Composite Pulse Sequences*. Often a quantum computing gate is composed of more pulses than you might expect. For example, as you now know it is possible rotate a qubit from $|0\rangle$ to $|1\rangle$ with a single π -pulse. However, sometimes this is done with more than one pulse. One such composite Pulse sequence is the Knill sequence. In this sequence instead of one π pulse being applied to the qubit there are five π -pulses applied! Each of those pulses has a different phase. The phases of the pulses are: $\phi_1 = \frac{\pi}{6}$, $\phi_2 = 0$, $\phi_3 = \frac{\pi}{2}$, $\phi_4 = 0$, $\phi_5 = \frac{\pi}{6}$.
- [10] Find the time evolution operator that propagates from before the application of the Knill sequence to after it – i.e find the time evolution operator for the Knill sequence.
 - [10] Now assume that the Rabi frequency can vary away from the π -pulse condition. Assuming the qubit started in $|0\rangle$, plot the population in $|1\rangle$ as a function of Ω for Ω greater and less than the π pulse condition.
 - [10] Now assume that the drive frequency can vary away from the resonance condition. Assuming the qubit started in $|0\rangle$, plot the population in $|1\rangle$ as a function of δ for δ greater and less than the resonance condition.
 - [10] Based on (b) and (c) why do you think, despite its extra complexity, that the Knill sequence is used in real devices?

notebook

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```
[1]: import qutip as qt  
import matplotlib.pyplot as plt  
import numpy as np
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