Math 180 Homework 5

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1

Section 5.2, Problem 2. Find two nonisomorphic trees with the same score.

Consider the tree formed by taking a copy of P_2 and two copies of P_3, then gluing together one leaf of each to form a tree. This has degree sequence $(1, 1, 1, 2, 2, 3)$. If we instead used 2 copies of P_2 and one copy of P_4, we would get the same degree sequence.

2

Section 5.2, Problem 6. Prove that there exist at most 4^n pairwise nonisomorphic trees on n vertices.

- There is an injection from the set of unlabeled trees with n vertices to the set of unlabeled planted trees with n vertices, defined by first choosing one of the leaves to be the root, so we obtain an unlabeled rooted tree, and then choosing an order of the branches (such as the "canonical planting" defined by a lexicographical ordering).
- There is an injection from the set of unlabeled planted trees to the set of strings of length $2n$ whose characters are all either -1 or +1, defined by taking the code of the tree.

Composing those two functions, we get an injection from the set of trees on n vertices to a set which we know has 2^{2n} elements. This implies the number of (nonisomorphic) tree with n vertices is at most 4^n.

3

Let P be a convex polyhedron whose faces are all a-gons or b-gons, and whose vertices are each adjacent to three edges. Let p_a, p_b, and n denote the number of a-gonal faces, b-gonal faces, and vertices of P, respectively. Express the number of edges of P in two different ways, and use this to prove that

$$p_a \cdot (6-a) + p_b \cdot (6-b) = 12.$$

By the handshaking lemma, the sum of the degrees of each vertex is $2|E|$, and since every vertex has degree 3, $2 \cdot |E| = 3 \cdot |V|$. If we take the dual graph, then we get a polyhedron with $|E|$ edges and $p_a + p_b$ vertices. Specifically, there are p_a vertices of degree a, and p_b vertices of degree b, so applying the same trick as earlier, we get $2 \cdot |E| = a \cdot p_a + b \cdot p_b$. Now we have the following equations:

$$\begin{aligned} |V| - |E| + |F| &= 2 \\ 2 \cdot |E| &= 3 \cdot |V| \\ 2 \cdot |E| &= a \cdot p_a + b \cdot p_b \end{aligned}$$

Now rearranging those, we get

$$\begin{aligned} 2 &= |V| - |E| + (p_a + p_b) \\ &= \left(\frac{2}{3} \cdot |E|\right) - |E| + (p_a + p_b) \\ &= \left(\frac{-1}{3} \cdot |E|\right) + (p_a + p_b) \\ &= -\frac{1}{6}(a \cdot p_a + b \cdot p_b) + (p_a + p_b) \\ \therefore 12 &= (6-a) \cdot p_a + (6-b) \cdot p_b \end{aligned}$$

4

Show that if a G is a planar graph with no vertices of degree less than 3 or cycles with length less than 4, then G must have at least 8 vertices and at least 12 edges. Give an example to show that these bounds cannot be improved.

Since there's no cycles with length less than 4, there's no triangles in the graph, meaning all of the faces have at least 4 edges. Applying the handshaking lemma to the dual graph, that implies $4 \cdot |F| \geq 2 \cdot |E|$ (every vertex in the dual graph has degree at least 4). Applying the handshaking lemma to the original graph, we see that $3 \cdot |V| \geq 2 \cdot |E|$ (since very vertex in the original graph has degree at least 3). Putting those inequalities together, we have

$$2 = |V| - |E| + |F| \geq \frac{|E|}{2} - |E| + \frac{2}{3} \cdot |E| = \frac{|E|}{6}.$$

Therefore there are at least 12 edges, which implies there are at least 8 vertices.

The cube graph (stereographically projected onto a plane) has 8 vertices and 12 edges, and if you modify it to have fewer vertices or fewer edges, you will either create a cycle of length less than 4 or make a vertex of degree less than 3.

5

> Prove than any simple planar graph has *two* vertices of degree at most 5.

Suppose every vertex but one of a planar graph G has degree at least 6. Then $2 \cdot |E| = \sum_{v \in V} \deg(v) \geq 6(|V|-1)+1 = 6 \cdot |V| - 5$. For any planar graph, $|V|-|E|+|F| = b_0+1 \geq 2$, so

$$\begin{aligned} 2 &\leq |V| - |E| + |F| \\ &\leq |V| - \left(3 \cdot |V| - \frac{5}{2}\right) + |F| \\ \implies 2 \cdot |V| &\leq \frac{1}{2} + |F| \end{aligned}$$

But for planar graphs, every face has at least 3 edges, meaning in the dual graph, every vertex has degree at least 3, so $3 \cdot |F| \leq 2 \cdot |E|$, which implies

$$2 \cdot |V| \leq \frac{1}{2} + \frac{2 \cdot |E|}{3}$$

which implies

$$2 \cdot |V| \leq |E| = \frac{1}{2} \cdot \sum_{v \in V} \deg(v)$$

and by the handshaking lemma, that means there is a vertex of degree less than or equal to 4. This is a contradiction. Therefore every simple planar graph has at least 2 vertices of degree ≤ 5.

6

> **Section 6.3, Exercise 2.**
>
> - (a) Show that a topological planar graph with $n \geq 3$ vertices has at most $2n-4$ faces.
> - (b) Show that a topological planar graph without triangles has at most $n-2$ faces.

- (a) Since $|E| \leq 3|V| - 6$, $|F| = 2 - |V| + |E| \leq 2 - |V| + 3|V| - 6 = 2|V| - 4$.
- (b) There are no triangles, so $2|E| \geq 4|F|$, which means $|F| \leq n - 2$.

7

Section 6.3, Exercise 3. Prove that a planar graph in which each vertex has degree at least 5 must have at least 12 vertices.

Since there is at least one connected component with at least 5 vertices, we can use the formula $|E| \leq 3|V| - 6$. We also know by the handshaking lemma that $5|V| \leq 2|E|$, so we have $5|V| \leq 2|E| \leq 6|V| - 12$, which implies $|V| \geq 12$.