Math 115B Homework #6

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Problem 0.1.

Problem 0.2.

Suppose V has an orthonormal basis of eigenvectors of T with corresponding eigenvalues which all have magnitude 1. Then V can be decomposed into orthogonal eigenspaces $V_1 \oplus V_2 \oplus \cdots \oplus V_n$, and if we let $\lambda_i \in \mathbb{C}$ be the eigenvalue corresponding to the eigenspace V_i and let T_i be the orthogonal projection onto V_i, then

$$T = \sum_{i=1}^{n} \lambda_i T_i.$$

We know want to check that $TT^* = I$. Since we have an orthonormal basis of eigenvectors, whenver v_i and v_j are eigenvectors of T with different eigenvalues, $\langle v_i, v_j \rangle = 0$. That means if $i \neq j$, then every element of V_i is orthogonal to every element of V_j, which implies $T_i T_j = 0$. Therefore

$$\begin{aligned}
TT^* &= \left(\sum_{i=1}^{n} \lambda_i T_i \right) \left(\sum_{i=1}^{n} \lambda_i T_i \right)^* \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j^* T_i T_j^* \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \delta(i=j) \lambda_i \lambda_j^* T_i T_j \\
&= \sum_{i=1}^{n} \lambda_i \lambda_i^* T_i^2 \\
&= \sum_{i=1}^{n} T_i \\
&= T,
\end{aligned}$$

so T is unitary.

Now suppose T is unitary, and we want to show that V has an orthonormal basis of eigenvectors of T which all have magnitude 1.

Problem 0.3.

- **Reflexivity:** $A \sim A$ because I is unitary and $I^{-1} A I$.

- **Symmetry:** If $A \sim B$, then there exists a unitary matrix P such that $A = P^*BP$. P^* is also unitary, so
$$(P^*)^*AP^* = PAP^* = PP^*BPP^* = IBI = B,$$
which means $B \sim A$.
- **Transitivity:** If $A \sim B$ and $B \sim C$, then there exist unitary matrices P and Q such that $A = P^*BP$ and $B = Q^*CQ$. If we let $R = QP$, then R is also unitary, and
$$A = P^*(Q^*CQ)P = R^*CR,$$
which means that $A \sim C$.

If you replace $\mathbb{C}$ with $\mathbb{R}$, and "unitary" with "orthogonal", then this exact same method proves that orthogonal equivalence is an equivalence relation.

Problem 0.4.

(a) We know that $T_iT_j = \delta(i == j)$, so by induction,
$$T^n = \lambda_i^n T_i + \cdots + \lambda_m^n T_m = \sum_{i=1}^m \lambda_i^n T_i$$
for any nonnegative integer n. $g(T)$ is a linear combination of such powers of T, so if $g(t) = a_0 + a_1t + a^2t^2 + \cdots + a^dt^d$ then
$$g(T) = \sum_{i=0}^d a^iT^i = \sum_{i=0}^d a^i \sum_{j=1}^m \lambda_j^i T_j^i = \sum_{j=1}^m \sum_{i=0}^d a^i\lambda_j^i T_j = \sum_{j=1}^m g(\lambda_j)T_j.$$

(b) I just showed that
$$T^n = \lambda_i^n T_i + \cdots + \lambda_m^n T_m = \sum_{i=1}^m \lambda_i^n T_i,$$
so if $T^n = 0$, then each λ_i is zero (because none of the T_is are zero). That means T is zero.

(c)

(d) For every complex number λ_i, there exists at least one complex number c_i such that $c_i^2 = \lambda_i$. Let $U = \sum_i c_iT_i$. Then by the logic I used in part (a), $U^2 = T$. Since U is a diagonal matrix (in the basis spanned by eigenvectors v_i of each T_i), U is normal.

(e) If there is some i such that $\lambda_i = 0$, then T is not injective, which means T is not invertible. However, if every $\lambda_i = 0$, then let $U = \sum_i (1/\lambda_i)T_i$. This satisfies $UT = I = TU$, so T is invertible.

(f) In part (a), I showed that
$$T^n = \lambda_i^n T_i + \cdots + \lambda_m^n T_m = \sum_{i=1}^m \lambda_i^n T_i.$$
Plugging in $n = 2$ and recalling that the T_is are linearly independent, we see that $T^2 = T$ if and only if $\lambda_i^2 = \lambda_i$ for every index i. That means every λ_i is either zero or one.

(g) Orthogonal projections satisfy $T_j^* = T_j$, so if $T = -T^*$, then
$$\sum_j \lambda_jT_j = -\left(\sum_j \lambda_jT_j\right)^* = \sum_j \left(-\lambda_j^*\right)T_j,$$

which means $\lambda_j = -\lambda_j^*$ for each j. Each λ_j can be written as $a + bi$ where $a, b \in \mathbb{R}$, so we have

$$a + bi = -(a + bi)^* = -(a - bi) = -a + bi,$$

so $a = 0$, meaning λ_j is purely imaginary.

Problem 0.5.

Since T is normal, there exists an orthonormal basis of eigenvectors $\{v_1, \ldots, v_n\}$ and corresponding eigenvectors λ_i such that T can be written as

$$T = \sum_{i=1}^{n} \lambda_i P_i,$$

where P_i is the orthogonal projection onto the span of v_i (so $P_i^* = P_i$). Then

$$\begin{aligned} UT^* &= U\left(\sum_{i=1}^{n} \lambda_i P_i\right)^* \\ &= U\left(\sum_{i=1}^{n} \lambda_i^* P_i^*\right) \\ &= \sum_{i=1}^{n} \lambda_i^* U P_i^* \\ &= \sum_{i=1}^{n} \lambda_i^* (P_i U^*)^* \end{aligned}$$

Problem 0.6.

- **Normal** means that $TT^* = T^*T$
- **Projection** means that $T^2 = T$
- **Orthogonal projection** means that $T^2 = T = T^*$

Since T is normal, there exists an orthonormal basis of eigenvectors $\{v_1, \ldots, v_n\}$ and corresponding eigenvectors λ_i such that T can be written as

$$T = \sum_{i=1}^{n} \lambda_i P_i,$$

where P_i is the orthogonal projection onto the span of v_i (so $P_i^* = P_i$). Since $P_i P_j = \delta(i = j)$,

$$T^2 = \sum_{i=1}^{n} \lambda_i^2 P_i,$$

which means every λ_i is either 0 or 1. Then

$$T^* = \left(\sum_{i=1}^{n} \lambda_i P_i\right)^* = \sum_{i=1}^{n} \lambda_i^* P_i^* = \sum_{i=1}^{n} \lambda_i * P_i = \sum_{i=1}^{n} \lambda_i P_i = T,$$

so T is an orthogonal projection.

Problem 0.7.

(a) By the definition of being U-invariant, UW is a subspace of W. If the dimension of UW is less than the dimension of W, then that would imply the determinant of U is zero, but a unitary matrix always has a determinant with magnitude 1, so $\dim(UW)$ is not less than $\dim(W)$ (and both of those are finite). Combined with the fact that $UW \subset W$, that means $UW = W$.

(b) Suppose for the sake of contradiction that $a \in W^{\perp}$ but $Ua \notin W^{\perp}$. That is, $P_W a = 0$ but $P_W U a \neq 0$, where P_W is the orthogonal projection onto W. Applying U to both sides of the first equation, $UP_W a = 0$

Problem 0.8.

Let v be any vector in V. Then $T_j v$ is in W_j. If $i \neq j$, then since W_i is orthogonal to W_j, $T_i T_j v = 0$. If $i = j$, then elements of W_j are unaffected by T_i, so $T_i T_j v = T_j v$. In conclusion, $T_i T_j = \delta(i == j) T_i$.

Math 115B: Linear Algebra
Homework 6
Due: Saturday, February 22nd at 11:59pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k. All inner product spaces are defined over a field F which is either $\mathbb{R}$ or $\mathbb{C}$.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. ($\frac{-}{10}$) Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T, and use this to prove T is self adjoint.
2. ($\frac{-}{10}$) Prove Corollary 2 to Theorem 6.18. That is, prove that if T is a linear operator on a finite-dimensional complex inner product space V, then: V has an orthonormal basis of eigenvectors of T with corresponding eigenvalues of absolute value 1 if and only if T is unitary.
3. ($\frac{-}{10}$) We say that matrix $A \in \mathbb{C}^{n\times n}$ is *unitarily equivalent* to $B \in \mathbb{C}^{n\times n}$ if there exists a unitary matrix $P \in \mathbb{C}^{n\times n}$ such that $A = P^{-1}BP$. Prove that this is an equivalence relation on $\mathbb{C}^{n\times n}$. (Replacing the field $\mathbb{C}$ with the field $\mathbb{R}$ and the word 'unitary' with 'orthogonal,' the same definition gives when two matrices are *orthogonally equivalent.*)
4. ($\frac{-}{6*5}$) Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1 T_1 + \ldots + \lambda_m T_m$ of T (where $\lambda_i \in \mathbb{C}$) to prove the following:
   (a) If $g(t) \in \mathbb{C}[t]$ then $g(T) = \sum_{i=1}^{m} g(\lambda_i)T_i$.
   (b) If some positive power of T is the zero transformation, then T itself is the zero transformation.
   (c) Let U be a linear operator on V. Then U commutes with T if and only if U commutes with each T_i for all $i \in \{1, 2, \ldots, m\}$.
   (d) There exists a normal operator U on V such that $U^2 = T$.
   (e) T is invertible if and only if $\lambda_i \neq 0$ for all $i \in \{1, 2, \ldots, m\}$.
   (f) T is a projection if and only if every eigenvalue of T is 1 or 0.

(g) $T = -T^*$ if and only if every λ_i is *purely imaginary*, that is, it lies in the set $i\mathbb{R} := \{ix : x \in \mathbb{R}\}$.

5. $(\frac{-}{10})$ Show that if T is a normal operator on a complex finite-dimensional inner product space and U is a linear operator that commutes with T, then U also commutes with T^*.

6. $(\frac{-}{10})$ Let T be a normal operator on a finite-dimensional inner product space. Prove that if T is a projection, then it is also an orthogonal projection.

7. $(\frac{-}{2*5})$ Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that:

   (a) $U(W) = W$

   (b) $W^\perp$ is U-invariant.

8. $(\frac{-}{10})$ Prove part (c) of the spectral theorem. In other words: assume that for a linear operator T on a finite dimensional inner product space V which is normal if $F = \mathbb{C}$ and self adjoint if $F = \mathbb{R}$, and let $W_1, ..., W_m$ denote the eigenspaces corresponding to distinct eigenvalues of T and let T_i be the orthogonal projection of V onto W_i. Then $T_iT_j = \delta_{ij}T_i$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.