Math 115B Homework #7

Nathan Solomon

March 9, 2025

Problem 0.1.

By theorem 6.24, an operator T is an orthogonal projection iff $T^2 = T = T^*$. So if T is an orthogonal projection, then $T = T^*$.

Problem 0.2.

(a) For any $v \in V$, $\|v\| = \|T(v)\|$. For any $w \in W$, w is also in V, so $\|w\| = \|T|_W(w)\|$.

(b) Since $T|_W^* = T|_W^{-1}$, we know $T|_W$ is invertible, so it's a bijection from W to W. That means $T^{-1}(w) \in W$ for any $w \in W$, so the preimage of W under T. Conversely, T cannot map any element of $W^\perp$ to a nonzero element of W, so $W^\perp$ is T-invariant.

(c) For any $v \in V$, $\|v\| = \|T(v)\|$. For any $w \in W^\perp$, w is also in V, so $\|w\| = \|T|_{W^\perp}(w)\|$, meaning $T|_{W^\perp}$ is unitary.

Problem 0.3.

If T is either a rotation or reflection on V, then there exists $\theta \in \mathbb{R}$ and a basis $\varepsilon = \{e_1, e_2\}$ of V such that

$$[T]_\varepsilon \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}.$$

In the first case, T is a reflection, and $T^* = T$. In the second case, T is a rotation, and $T^* = T$. Either way, T is unitary (which is equivalent to orthogonal, since V is a real inner product space).

The composition of any unitary operators A, B is also unitary because for any $v \in V$,

$$\|v\| = \|Bv\| = \|ABv\|.$$

Problem 0.4.

Define

$$x_1 = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}, x_2 = \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix}.$$

Then $Lx_1 = x_1$ and $Lx_2 = -x_2$. Therefore, $W = \operatorname{span}(x_2)$ is a one-dimensional subspace such that $Lw = w$ for any $w \in W$ and $Lv = -v$ for any $v \in W^\perp$, meaning L is a reflection about $W^\perp = \operatorname{span}(x_1)$.

Problem 0.5.

(a) Let T be a rotation. Then T is orthogonal, so $\|Te_1\| = \|e_1\| = 1$. Therefore Te_1 is on the unit circle, so there exists $\theta \in \mathbb{R}$ such that
$$Te_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$
We also know $0 = \langle e_1, e_2 \rangle = \langle Te_1, Te_2 \rangle$. Since Te_2 is perpendicular to Te_1, there exists a constant $a \in \mathbb{R}$ such that
$$Te_2 = \begin{bmatrix} -a \cdot \sin\theta \\ a \cdot \cos\theta \end{bmatrix}.$$
In matrix form, T can be written as
$$[T]_\varepsilon = R_\theta = \begin{bmatrix} \cos\theta & -a \cdot \sin\theta \\ \sin\theta & a \cdot \cos\theta \end{bmatrix}.$$
The determinant of that is $a\cos^2\theta - a(-\sin^2\theta) = a$, but the determinant of an orthogonal matrix is always one, so $a = 1$.

(b)
$$\begin{aligned} R_\theta R_\varphi &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\cos\varphi \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & \sin\theta\sin\varphi + \cos\theta\cos\varphi \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{bmatrix} \\ &= R_{\theta+\varphi}. \end{aligned}$$

(c)
$$R_\theta R_\varphi = R_{\theta+\varphi} = R_{\varphi+\theta} = R_\varphi R_\theta.$$

Problem 0.6.

From the previous few problems, it is obvious that the determinant of a rotation is one.

If T is a reflection, then there exists a one dimensional subspace W such that $Tx = -x$ for any $x \in W$ and $Tx = x$ for any $x \in W^\perp$. Therefore T is diagonaizable, one of its eigenvalues is -1, and the rest are 1. That means $\det T = -1$, so T cannot also be a rotation.

Problem 0.7.

T is a direct sum of rotations iff it can be written as the composition of rotation operators. If $\dim(V)$ is odd, then $\det(T) = \det(-I_V) = (-1)^{\dim(V)} = -1$. Rotation operators always have determinant one, so their composition also does, so T cannot be a direct sum of rotations.

If $\dim(V)$ is even, then let $v_1, v_2, \ldots, v_{2n}$ be an orthonormal basis of V, let $W_i = \operatorname{span}\{v_{2i-1}, v_{2i}\}$, and let R_i be the rotation of W_i by π radians. Then $T = R_1 \oplus R_2 \oplus \cdots \oplus R_n$.

Problem 0.8.

Since v and w both lie on the unit circle, there exists $\theta, \varphi \in \mathbb{R}$ such that

$$v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, w = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}.$$

Then $R_\theta e_1 = v$ and $R_\varphi e_1 = w$, so

$$\begin{aligned} R_{\varphi-\theta} v &= R_\varphi R_{-\theta} v \\ &= R_\varphi R_\theta^{-1} v \\ &= R_\varphi e_1 \\ &= w. \end{aligned}$$

Suppose there is another rotation, R_ϕ, such that $R_\phi v = w$. Then $v = R_\phi^{-1} w = R_\phi^{-1} R_{\varphi-\theta} v = R_{\varphi-\theta-\phi} v$. The only 2D rotation which maps a nonzero vector v to itself is the identity, R_0, so $\varphi - \theta - \phi \in 2\pi\mathbb{Z}$. That would mean $R_\phi = R_{\varphi-\theta}$, so the rotation is unique.

Math 115B: Linear Algebra
Homework 7
Due: *Wednesday*, March 5 at 11:59pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k. All inner product spaces are defined over a field F which is either $\mathbb{R}$ or $\mathbb{C}$.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. ($\frac{-}{10}$) Prove all orthogonal projections are self adjoint.

2. ($\frac{-}{2+9+9}$) Let T be an orthogonal (unitary) operator on a finite-dimensional real (respectively, complex) inner product space V. If W is a T-invariant subspace of V, prove the following:

   (a) $T|_W$ is an orthogonal (respectively, unitary) operator on W.

   (b) $W^\perp$ is a T-invariant subspace of V. (Hint: use the fact that $T|_W$ is one-to-one and onto to conclude that for any $\vec{w} \in W, T^*(\vec{w}) = T^{-1}(\vec{w}) \in W$.)

   (c) $T|_{W^\perp}$ is an orthogonal (respectively, unitary) operator.

3. ($\frac{-}{15}$) Let V be a real inner product space of dimension two. Prove that rotations, reflections and compositions of rotations and reflections are orthogonal operators.

4. ($\frac{-}{5+5}$) For any real number $\theta \in \mathbb{R}$, let $A_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$.

   (a) Prove that L_{A_θ} is a reflection.

   (b) Find the subspace of $\mathbb{R}^2$ about which L_{A_θ} reflects.

5. ($\frac{-}{5+5+5}$) For any real number $\theta \in \mathbb{R}$, define $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation given by left multiplication by the matrix $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

   (a) Prove that any rotation on $\mathbb{R}^2$ is of the form R_θ for some $\theta \in \mathbb{R}$.

   (b) Prove that $R_\theta R_{\theta'} = R_{\theta+\theta'}$ for any $\theta, \theta' \in \mathbb{R}$.

   (c) Show that any two rotations on $\mathbb{R}^2$ commute.

6. $(\frac{-}{10})$ Prove that no orthogonal operator on a two dimensional real inner product space can be both a rotation and a reflection.

7. $(\frac{-}{10})$ Let V be a finite-dimensional real inner product space. Define $T : V \to V$ via the formula $T(\vec{v}) = -\vec{v}$. Prove that T is a direct sum of rotations[1] if and only if the dimension of V is even.

8. $(\frac{-}{10})$ Let V be a real inner product space of dimension 2. For any $\vec{v}, \vec{w} \in V$ such that $||\vec{v}|| = ||\vec{w}|| = 1$, show that there exists a unique rotation R on V such that $R(\vec{v}) = \vec{w}$.

9. $(\frac{-}{\text{No points but it's a pretty fun exercise so you should still try it}})$ For a given positive integer n, define the *special unitary group* SU_n to be the set of $n \times n$ unitary complex matrices which have determinant one. Construct a bijection of sets between SU_2 and the 3-sphere $S^3 := \{x \in \mathbb{R}^4 : ||x|| = 1\}$.

[1]In other words, there exists some T-invariant subspaces $W_1, ..., W_m$ such that $V = W_1 \oplus ... \oplus W_m$ and such that $T|_{W_i} : W_i \to W_i$ is a rotation for all $i \in \{1, 2, ..., m\}$.