Math 115B Homework #2

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Problem 0.1.

(a) Let

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

be an arbitrary vector in V. Then the dual basis $\mathcal{B}^*$ consists of the following linear functionals:

$$\begin{aligned} u &\mapsto u_1 - 2^{-1}u_2 \\ u &\mapsto 2^{-1}u_2 \\ u &\mapsto u_3 - u_1. \end{aligned}$$

This was obtained by finding the following inverse matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

(b) Given an arbitrary element $u = u_2x^2 + u_1x + u_0 \in V$, the dual basis $\mathcal{B}^*$ consists of the following 3 linear functionals:

$$\begin{aligned} u &\mapsto u_0 = u(0) \\ u &\mapsto u_1 = u'(0) \\ u &\mapsto u_2 = 2^{-1}u''(0). \end{aligned}$$

Problem 0.2.

(a) $T * f$ is the functional which maps $\begin{bmatrix} x \\ y \end{bmatrix}$ to

$$T * f \begin{bmatrix} x \\ y \end{bmatrix} = fT \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} 3x + 2y \\ x \end{bmatrix} = 2(3x + 2y) + (x) = 7x + 4y.$$

(b) For this problem, I will represent all transformations in the standard basis ε and its dual, ε^*.

You can think of f as the row vector $\begin{bmatrix} 2 & 1 \end{bmatrix}$, and T as the matrix

$$T := \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}.$$

T^* can be represented by the transpose of T,

$$T^* = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$$

By looking at the matrix representation I already found for T^* in the dual of the standard basis, we can see $a = 3, b = 1, c = 2, d = 0$.

(c)

$$[T]_\varepsilon = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$
$$([T]_\varepsilon)^T = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = [T^*]_{\varepsilon^*}.$$

Problem 0.3.

(a) Let f, g be arbitrary elements of S^0, and let c be an arbitrary element of k. Then $f + cg \in V^*$, and for all $x \in S$,

$$(f + cg)(x) = f(x) + cg(x) = 0 + 0c = 0,$$

so $f + cg \in S^0$, and S^0 also contains the zero funcitonal, which means S^0 is a subspace of V^*.

(b) Since V is finite dimensional and $x \neq 0$, there exists some basis $\mathcal{B}$ of V which contains x and contains a subset which spans W. Any vector $v \in V$ can be written as a finite sum of scalars in k times basis vectors in $\mathcal{B}$, so there exists a linear functional f which maps v to the coefficient of x in that sum. Since f maps all basis vectors of W to zero, f is in W^0. Lastly, note that $f(x) = 1 \neq 0$.

(c) Let $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a basis for V such that $(x_1, \ldots, x_n)$ is a basis for S. Then $(y_1^*, \ldots, y_m^*)$ is a basis for S^0. By that same reasoning, $(x_1^{**}, \ldots, x_n^{**})$ is a basis for $(S^0)^0$. The span of that last basis is equal to $\text{span}(\psi(S))$.

(d) If $W_1 \neq W_2$, then without loss of generality, there exists $x \in W_1$ such that $x \notin W_2$. That implies there exists $f \in W_2^0$ such that $f(x) \neq 0$, but $f \notin W_1^0$, so $W_1^0 \neq W_2^0$.

If $W_1^0 \neq W_2^0$, then $(W_1^0)^0 \neq (W_2^0)^0$, which means $\text{span}(\psi(W_1)) \neq \text{span}(\psi(W_2))$. Both $\psi(W_1)$ and $\psi(W_2)$ are subspaces, so $\psi(W_1) \neq \psi(W_2)$. ψ is an isomorphism, so $W_1 \neq W_2$.

I have shown that $W_1 \neq W_2$ implies $W_1^0 \neq W_2^0$ and vice versa, so $W_1 = W_2$ iff $W_1^0 = W_2^0$.

(e) A functional $f \in V^*$ is in $(W_1 + W_2)^0$ iff $f(w_1 + w_2) = f(w_1) + f(w_2) = 0$ for any $w_1 \in W_1, w_2 \in W_2$. That is true iff $f(w_1) = 0$ for any $w_1 \in W_1$ and $f(w_2) = 0$ for any $w_2 \in W_2$. This is equivalent to saying $f \in W_1^0$ and $f \in W_2^0$, which is equivalent to saying $f \in W_1^0 \cap W_2^0$.

Problem 0.4.

Dimension is defined as the cardinality of a basis, so in order for that to be defined, I will assume for now that every vector space has a basis.

Let $\mathcal{B}_W$ be a basis of W, and extend that to a basis $\mathcal{B}_V$ for V – in other words, when choosing a basis, make sure that $\mathcal{B}_W \subset \mathcal{B}_V$. Then each element of $\mathcal{B}_W^*$ will map the corresponding basis vector in $\mathcal{B}_W$ to a nonzero value, so none of the dual vectors in $\mathcal{B}_W^*$ are in W^0. However, every dual basis vector in $\mathcal{B}_V^* \backslash \mathcal{B}_W^*$ will map all elements of $\mathcal{B}_W$ to zero. If $\dim(W^0)$ is finite, then $\mathcal{B}_V^* \backslash \mathcal{B}_W^*$ is a basis for W^0. We now have

$$\begin{aligned} \dim(W) + \dim(W^0) &= |\mathcal{B}_W| + |\mathcal{B}_V^* \backslash \mathcal{B}_W^*| \\ &= |\mathcal{B}_V| \\ &= \dim(V). \end{aligned}$$

If $\dim(W^0)$ is infinite, I'm not sure how to solve this problem.

Problem 0.5.

If $g \in \ker(T^*)$, then for any $v \in V$, we have $T^*gv = 0$, but we also know $T^*(g) = g \circ T$, so $0 = (g \circ T)(v) = g(Tv)$. Every element of $R(T)$ can be written as Tv for some $v \in V$, so g maps every element of $R(T)$ to zero, which means $\ker(T^*) \subset R(T)^0$.

This also works in reverse – if $g \in R(T)^0$, then for any $v \in W$, we can let $u = Tv$, and since $u \in R(T)$, $g(u) = 0$. Equivalently, $(g \circ T)(v) = 0$, which means $(T^*(g))(v) = 0$, so $g \in \ker(T^*)$. That means $R(T)^0 \subset \ker(T^*)$, so $R(T)^0 = \ker(T^*)$.

Problem 0.6.

The characteristic polynomial is

$$\begin{aligned}
\det(R - \lambda I) &= \det\left(\begin{bmatrix} -3-\lambda & -3 & -4 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{bmatrix}\right) \\
&= (-3-\lambda)(2-\lambda)(-1-\lambda) - (-3)(2)(-1-\lambda) \\
&= (6 + 5\lambda - 2\lambda^2 - \lambda^3) - (6 + 6\lambda) \\
&= -\lambda^3 - 2\lambda^2 - \lambda \\
&= -\lambda(\lambda+1)^2.
\end{aligned}$$

The roots of that are $\lambda = 0$ (with multiplicity 1, so the corresponding eigenspace is 1D) and $\lambda = -1$ (with multiplicity 2, so the corresponding eigenspace is 2D).

The $\lambda = 0$ eigenspace is the nullspace of R, which is

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$$

and the $\lambda = -1$ eigenspace is the nullspace of

$$R - (-1)I = R + I = \begin{bmatrix} -2 & -3 & -4 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

which is

$$\operatorname{span}\left(\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right).$$

Problem 0.7.

In the standard basis ε,

$$[T]_\varepsilon = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix},$$

which has eigenvectors $v_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (with eigenvalue 3) and $v_2 := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (with eigenvector 3). This means in the basis $\mathcal{B} := (v_1, v_2)$, T is diagonal:

$$[T]_\mathcal{B} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

This works because $Tv_1 = 3v_1$ and $Tv_2 = 2v_2$.

Problem 0.8.

(a) This is T-invariant, because the derivative of any polynomial with degree less than or equal to 2 will be another polynomial with degree less than or equal to 2.

(b) This is not T-invariant, because $x^2 \in W$, but $T(x^2) = x^3 \notin W$.

(c) This is T-invariant, because for any vector in the image of T, all 3 components of that vector are the same, which means the vector is in W.

(d) This is T-invariant, because if $f \in W$ is the function which maps any $t \in [0,1]$ to $at + b$, then Tf is the function which maps $t \in [0,1]$ to

$$\left(\int_0^1 f(x)\mathrm{d}x\right) t = \left[\frac{a}{2}x^2 + bx\right]_{x=0}^1 t = (b + a/2)t,$$

which is a linear function of t. Since T maps any affine function $f(t) = at + b$ to another affine function $(Tf)(t) = (b + a/2)t$, W is a T-invariant subspace.

(e) This is not T-invariant, because any symmetric 2×2 matrix A can be written as

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

which means

$$TA = \begin{bmatrix} b & c \\ a & b \end{bmatrix},$$

which is not symmetric when $a = 0$ and $c = 1$.

Math 115B: Linear Algebra
Homework 2
Due: Thursday, January 23 at 8pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. $(\frac{-}{5+5})$ For each of the following vector spaces V and each (ordered) basis $\mathcal{B}$, find an explicit formula for each vector in the dual basis $\mathcal{B}^*$.

   (a) $V = k^3, \mathcal{B} = \{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$.

   (b) $V = k[x]_{\leq 2}, \mathcal{B} = \{1, x, x^2\}$.

2. $(\frac{-}{5+10+5})$ Define some $f \in (\mathbb{R}^2)^*$ $f \begin{pmatrix} x \\ y \end{pmatrix} = 2x + y$ and a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ via the formula $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ x \end{pmatrix}$

   (a) Compute $T^*(f)$. (The book uses the term T^t for what we call T^*.)

   (b) Compute $[T^*]_{\mathcal{E}^*}$, where $\mathcal{E}$ is the standard ordered basis for $\mathbb{R}^2$ and $\mathcal{E}^* = \{\vec{e}_1^*, \vec{e}_2^*\}$ is the dual basis, explicitly by finding scalars a, b, c, d such that $T^*(\vec{e}_1^*) = a\vec{e}_1^* + c\vec{e}_2^*$ and $T^*(\vec{e}_2^*) = b\vec{e}_1^* + d\vec{e}_2^*$

   (c) Compute $[T]_{\mathcal{E}}$ and $([T]_{\mathcal{E}})^t$ and compare your result with your answer to the last question (you don't need to write anything about this comparison).

3. $(\frac{-}{5+5+5+5+5})$ Let V denote a finite dimensional k-vector space. For any subset $S \subseteq V$, define the *annihilator* S^0 of S as

   $$S^0 := \{f \in V^* : f(x) = 0 \text{ for all } x \in S\}.$$

   (a) Prove that S^0 is a subspace of V^*. (Your proof will likely not use the fact that V is finite dimensional.)

   (b) If W is a subspace of V and $x \notin W$, prove that there exists some $f \in W^0$ such that $f(x) \neq 0$.

(c) In class, we constructed an isomorphism $\psi : V \to V^{**}$. Prove that $(S^0)^0 = \mathrm{span}(\psi(S))$, where $\psi(S) := \{\psi(s) : s \in S\}$.

(d) For subspaces W_1 and W_2 of V, prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

(e) For subspaces W_1 and W_2, prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

4. $(\frac{-}{10})$ Prove that if W is a subspace of V, then $\dim(W) + \dim(W^0) = \dim(V)$. (For one point less: you may assume that $\dim(V) < \infty$. *Hint:* Extend an ordered basis $\{\vec{w}_1, ..., \vec{w}_k\}$ of W to an ordered basis $\mathcal{B} = \{\vec{w}_1, ..., \vec{w}_k, ..., \vec{w}_n\}$ of V. Let $\mathcal{B}^* = \{\vec{w}_1^*, ..., \vec{w}_k^*, ..., \vec{w}_n^*\}$. Prove that $\{\vec{w}_{k+1}^*, ..., \vec{w}_n^*\}$ is a basis for W^0.)

5. $(\frac{-}{15})$ Suppose that W is a finite dimensional vector space and $T : V \to W$ is a linear transformation. Prove that $\ker(T^*) = R(T)^0$.

   Here, the *kernel* of a linear transformation $U : X \to Y$ is $\{\vec{x} \in X : U(\vec{x}) = \vec{0}\}$ which is denoted as $N(U)$ in the textbook and is also referred to as the *null space* of U. Similarly, the *range* of U, written $R(U)$, is defined as $\{U(\vec{x}) : \vec{x} \in X\}$.

6. $(\frac{-}{5})$ Let R denote the 3×3 real matrix $\begin{pmatrix} -3 & -3 & -4 \\ 2 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$. Find all eigenvalues of R. For each eigenvalue, compute the corresponding eigen*space*.

7. $(\frac{-}{5})$ For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by the formula $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix}$, find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_{\mathcal{B}}$ is diagonal (and prove your answer is correct).[1]

8. $(\frac{-}{2+2+2+2+2})$ Given some vector space V and a linear *endomorphism* $T : V \to V$ (i.e. a linear transformation with the same domain and codomain, often also called a linear *operator*), we define a T*-invariant subspace* of V to be a subspace $W \subseteq V$ such that $T(W) \subseteq W$. For each of the following linear endomorphisms $T : V \to V$ determine whether the given subspace W is a T-invariant subspace of V.

   (a) $V = \mathbb{R}[x]$, $T(f(x)) = f'(x)$, $W = \mathbb{R}[x]_{\leq 2}$

   (b) $V = \mathbb{R}[x]$, $T(f(x)) = xf(x)$, $W = \mathbb{R}[x]_{\leq 2}$

   (c) $V = k^3$, $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}$, $W = \{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = x_2 = x_3\}$.

   (d) V is the set of all continuous functions $[0, 1] \to \mathbb{R}$, $T(f(t)) = (\int_0^1 f(x)dx)t$, $W = \{f \in V : f(t) = at + b \text{ for some } a, b \in \mathbb{R}\}$.

   (e) $V = k^{2\times 2}$, $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, W is the subspace of *symmetric* 2×2 matrices, i.e. those 2×2 matrices satisfying $A^t = A$.

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[1] Note you don't have to 'show your work' as to how you got the answer, but make sure you are able to derive the answer on your own!