Math 115B: Linear Algebra
Homework 7
Due: *Wednesday*, March 5 at 11:59pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k. All inner product spaces are defined over a field F which is either $\mathbb{R}$ or $\mathbb{C}$.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. ($\frac{-}{10}$) Prove all orthogonal projections are self adjoint.

2. ($\frac{-}{2+9+9}$) Let T be an orthogonal (unitary) operator on a finite-dimensional real (respectively, complex) inner product space V. If W is a T-invariant subspace of V, prove the following:

   (a) $T|_W$ is an orthogonal (respectively, unitary) operator on W.

   (b) $W^\perp$ is a T-invariant subspace of V. (Hint: use the fact that $T|_W$ is one-to-one and onto to conclude that for any $\vec{w} \in W, T^*(\vec{w}) = T^{-1}(\vec{w}) \in W$.)

   (c) $T|_{W^\perp}$ is an orthogonal (respectively, unitary) operator.

3. ($\frac{-}{15}$) Let V be a real inner product space of dimension two. Prove that rotations, reflections and compositions of rotations and reflections are orthogonal operators.

4. ($\frac{-}{5+5}$) For any real number $\theta \in \mathbb{R}$, let $A_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$.

   (a) Prove that L_{A_θ} is a reflection.

   (b) Find the subspace of $\mathbb{R}^2$ about which L_{A_θ} reflects.

5. ($\frac{-}{5+5+5}$) For any real number $\theta \in \mathbb{R}$, define $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation given by left multiplication by the matrix $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

   (a) Prove that any rotation on $\mathbb{R}^2$ is of the form R_θ for some $\theta \in \mathbb{R}$.

   (b) Prove that $R_\theta R_{\theta'} = R_{\theta+\theta'}$ for any $\theta, \theta' \in \mathbb{R}$.

   (c) Show that any two rotations on $\mathbb{R}^2$ commute.

6. ($\frac{-}{10}$) Prove that no orthogonal operator on a two dimensional real inner product space can be both a rotation and a reflection.

7. ($\frac{-}{10}$) Let V be a finite-dimensional real inner product space. Define $T : V \to V$ via the formula $T(\vec{v}) = -\vec{v}$. Prove that T is a direct sum of rotations[1] if and only if the dimension of V is even.

8. ($\frac{-}{10}$) Let V be a real inner product space of dimension 2. For any $\vec{v}, \vec{w} \in V$ such that $||\vec{v}|| = ||\vec{w}|| = 1$, show that there exists a unique rotation R on V such that $R(\vec{v}) = \vec{w}$.

9. ($\frac{-}{\text{No points but it's a pretty fun exercise so you should still try it}}$) For a given positive integer n, define the *special unitary group* SU_n to be the set of $n \times n$ unitary complex matrices which have determinant one. Construct a bijection of sets between SU_2 and the 3-sphere $S^3 := \{x \in \mathbb{R}^4 : ||x|| = 1\}$.

[1] In other words, there exists some T-invariant subspaces $W_1, ..., W_m$ such that $V = W_1 \oplus ... \oplus W_m$ and such that $T|_{W_i} : W_i \to W_i$ is a rotation for all $i \in \{1, 2, ..., m\}$.