Physics 231B Homework #4

Nathan Solomon

June 14, 2024

Problem 0.1. Artin Chapter 10 Problem 2.1, page 314. Prove that the standard three-dimensional representation of the tetrahedral group T is irreducible as a complex representation.

The best way to approach this problem is to imagine a regular tetrahedron whose vertices are in a unit sphere, and think of T is the set of rotations preserving that tetrahedron.

Suppose T is reducible, meaning there is a nontrivial subspace of $\mathbb{R}^3$ which is invariant under the action of T. Such a subspace must be either a line through the origin or a plane through the origin. Since a plane is the orthogonal complement of a line in $\mathbb{R}^3$, finding a T-invariant plane and finding a T-invariant line are the same problem. Therefore I will only consider a T-invariant line.

A T-invariant line could not exist because the group of rotations which fixes that line is $O(2)$. Every finite subgroup of $O(2)$ is the trivial group, a cyclic group, or a dihedral group. T is none of those, so the line is not T-invariant.

This is a contradiction, so T is irreducible.

Problem 0.2. Artin Chapter 10 Problem 3.1, page 315. Let G be a cyclic group of order 3. The matrix $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ has order 3, so it defines a matrix representation of G. Use the averaging process to produce a G-invariant form from the standard Hermitian product $X * Y$ on $\mathbb{C}^2$.

Note. "$*$" denotes elementwise matrix multiplication, just like in Python.

The elements of G are

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\}.$$

The average of the Hermitian norm squared of each element of G is

$$\begin{aligned}
\frac{1}{3} \sum_{g \in G} g^\dagger g &= \frac{1}{3} \left(I_2^\dagger I_2 + A^\dagger A + (A^2)^\dagger (A^2) \right) \\
&= \frac{1}{3} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right) \\
&= \frac{1}{3} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right) \\
&= \frac{2}{3} \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right)
\end{aligned}$$

Therefore we can define an inner product form which maps $v \in \mathbb{C}^2$ to $v^\dagger \left(\frac{2}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) v$. This is G invariant because

$$
\begin{aligned}
\langle Au|\, Av\rangle &= (Au)^\dagger \left(\frac{2}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)(Av)\\
&= u^\dagger A^\dagger \left(\frac{2}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right) Av\\
&= u^\dagger \left(\frac{2}{3}\begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}\right) v\\
&= u^\dagger \left(\frac{2}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right) v\\
&= \langle u|\, v\rangle\,,
\end{aligned}
$$

meaning the action of I_2, A, or A^2 on any vectors $u, v \in \mathbb{C}^2$ will not change the inner product $\langle u|\, v\rangle$.

Problem 0.3. Artin Chapter 10 Problem 3.2, page 315. Let $\rho : G \to GL(V)$ be a representation of a finite group on a real vector space V. Prove the following:

- (a) There exists a G-invariant, positive-definite symmetric form $\langle , \rangle$ on V.
- (b) ρ is a direct sum of irreducible representations.
- (c) Every finite subgroup of $GL_n(\mathbb{R})$ is conjugate to a subgroup of $O(n)$.

- (a) We can use the same averaging process as in the above problem. Define a matrix M to be

  $$M := \sum_{g\in G} \rho(g)^T \rho(g)$$

  and define a symmetric form $\langle x|\, y\rangle = x^T M y$. This is bilinear, symmetric, positive-definite, and G-invariant.
- (b) Assume V is finite dimensional. If there is some nontrivial subspace U of V, then $V = U \oplus U^\perp$. Every action $\rho(g)$ will map elements of U to elements of U and elements of $U^\perp$ to elements of $U^\perp$. Therefore the image of ρ is contained in $GL(U) \oplus GL(U^\perp)$.

  This shows that any reducible finite-dimensional representation of G can be rewritten as a direct sum of representations with strictly smaller dimensions. Every one-dimensional representation is trivially irreducible, so by induction, ρ is a direct sum of irreducible representations.
- (c)

Problem 0.4. Given a representation $R = \bigoplus_i n_i R_i$ decomposed into irreps R_i, prove that the operator

$$P_i = \frac{\dim(R_i)}{|G|}\sum_{g\in G} \chi_i(g)^* R_g$$

is a projector onto the subspace $n_i R_i$ of R.

There are a few ways to show that P_i is a projection operator – one method is to show that both $P_i^2 = P_i$ and $\mathrm{Tr}(P_i) = \dim(P_i)$.

$$
\begin{aligned}
\mathrm{Tr}(P_i) &= \frac{\dim(P_i)}{|G|}\sum g \in G\chi_i^*(g)\chi_i(g)\\
&= \dim(R_i).
\end{aligned}
$$

Showing that $P_i^2 = P_i$ is a bit harder because it requires re-indexing:

$$\begin{aligned}
P_i^2 &= \frac{\dim(R_i)^2}{|G|^2} \sum_{g_1 \in G} \sum_{g_2 \in G} \chi_i^*(g_1)\chi_i^*(g_2) R_{g_1} R_{g_2} \\
&= \frac{\dim(R_i)}{|G|^2} \sum_{g_1 \in G} \sum_{g_1 g_2 \in G} \chi_i^*(g_1)\chi_i^*(g_1 g_2) R_{g_1 g_2} \\
&= \frac{\dim(R_i)}{|G|^2} \sum_{g_1 \in G} \chi_i^{reg}(g_1) \sum_{g_1 g_2 \in G} \chi_i^*(g_1 g_2) R_{g_1 g_2} \\
&= \frac{\dim(R_i)}{|G|} \sum_{(g_1 g_2) \in G} \chi_i^*(g_1 g_2) R_{(g_1 g_2)} \\
&= P_i.
\end{aligned}$$

Problem 0.5.

- (a) Given a representation R, prove that the following defines a representation:

$$R_g^\vee = \left(R_g^{-1}\right)^T.$$

  ($R^\vee$ is called the dual representation.)

- (b) Show there is a G-equivariant linear map $R \otimes R^\vee \to \mathbb{C}$.
- (c) Prove that if M is the vector space of matrices on R, conjugation of these matrices endows M with a G representation isomorphic to $R \otimes R^\vee$.

- (a) Let R be a representation which maps $g \in G$ to $R_g \in GL(V)$. Then let φ be the function which maps R_g to $R_g^\vee := (R_g^{-1})^T$. Then $\varphi \circ R$ is also a representation, because

$$(\varphi \circ R)(g)(\varphi \circ R)(g') = (R_g^{-1})^T (R_{g'}^{-1})^T = (R_{g'}^{-1} R_g^{-1})^T = ((R_g R_{g'})^{-1})^T = (\varphi \circ R)(gg').$$

- (b) Let R' be shorthand for $R \otimes R^\vee$. We want to find some linear map $A : R' \to \mathbb{C}$ such that
- (c)

Problem 0.6. For each pair R_i, R_j of irreps of S_3, compute the decomposition of $R_i \otimes R_j$ into irreps. (Together with direct sum, this gives irreps of G the structure of a ring called the "representation ring".)

The 3 irreps of S_3 are the trivial representation R_1, the sign representation R_2, and the faithful representation R_3.

Conjugacy class	Trivial permutation	2-cycle	3-cycle
Character in R_1	1	1	1
Character in R_2	1	-1	1
Character in R_3	2	0	-1
# of elements	1	3	2

Now that we have the character table, we can simply use the rules $\chi(R_i \otimes R_j) = \chi(R_i)\chi(R_j)$ and $\chi(R_i \oplus R_j) = \chi(R_i) + \chi(R_j)$ to decompose all the pairs $R_i \otimes R_j$ into irreps.

$$
\begin{aligned}
R_1 \otimes R_1 &= R_1 \\
R_1 \otimes R_2 &= R_2 \\
R_1 \otimes R_3 &= R_3 \\
R_2 \otimes R_2 &= R_1 \\
R_2 \otimes R_3 &= R_3 \\
R_3 \otimes R_3 &= R_1 \oplus R_2 \oplus R_3
\end{aligned}
$$