MATH 131B Homework #9

Nathan Solomon

December 8, 2024

Problem 0.1. Exercise 5.3.3: prove corollary 5.3.6.

If $-N \leq n \leq N$, then by the linearity (in the first argument) of the inner product,

$$\langle f, e_n \rangle = \sum_{m=-N}^{N} c_m \langle e_m, e_n \rangle .$$

Lemma 5.3.5 tells us that

$$\langle e_m, e_n \rangle = \delta_{m,n} := \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases},$$

so all of the terms except the one where $m = n$ are zero, and we are left with

$$\langle f, e_n \rangle = c_n.$$

If $n < -N$ or $n > N$, then

$$\langle f, e_n \rangle = \sum_{m=-N}^{N} c_m \langle e_m, e_n \rangle = 0,$$

because m will never be equal to n.

Lastly, we have the identity

$$\begin{aligned} \|f\|^2 &= \langle f, f \rangle \\ &= \left\langle \sum_{n=-N}^{N} c_n e_n, \sum_{m=-N}^{N} c_m e_m \right\rangle \\ &= \sum_{n=-N}^{N} c_n \sum_{m=-N}^{N} \overline{c_m} \langle e_n, e_m \rangle \\ &= \sum_{n=-N}^{N} c_n \sum_{m=-N}^{N} \overline{c_m} \delta_{n,m} \\ &= \sum_{n=-N}^{N} \|c_n\|^2 . \end{aligned}$$

Problem 0.2. Exercise 5.4.2: prove lemma 5.4.4.

(a) Suppose $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

$$\begin{aligned}
(f * g)(x+n) &= \int_{[0,1]} f(y)g(x+n-y)\mathrm{d}y \\
&= \int_{[0,1]} f(y)g(x-y)\mathrm{d}y \\
&= (f * g)(x),
\end{aligned}$$

so $(f * g)$ is $\mathbb{Z}$-periodic.

Next, I want to show that $f * g$ is continuous. Since f is continuous on the closed interval $[0,1]$, f is also bounded on $[0,1]$, so there exists some $M \in \mathbb{R}$ such that $f(x) < M$ for any x. Similarly, g is continuous on $[0,1]$, so g is uniformly continuous on $[0,1]$, and since g is $\mathbb{Z}$-periodic, that means g in uniformly continuous everywhere.

Therefore, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x, y \in \mathbb{R}$, $d(x,y) < \delta$ implies $|g(x) - g(y)| < \varepsilon/M$. Now consider the difference between $(f * g)(x)$ and $(f * g)(x')$ for some $x' \in (x - \delta, x + \delta)$:

$$\begin{aligned}
|(f * g)(x') - (f * g)(x)| &= \left| \int_{[0,1]} \left(f(y)g(x'-y) - f(y)g(x-y) \right) \mathrm{d}y \right| \\
&\leq M \left| \int_{[0,1]} \left(g(x'-y) - g(x-y) \right) \mathrm{d}y \right|.
\end{aligned}$$

But $|g(x'-y) - g(x-y)| < \varepsilon/M$ because g is uniformly continuous everywhere, so the value of that integral is less than ε/M everywhere, so that entire expression is less than ε, meaning $f*g$ is (uniformly) continuous.

(b) The following steps substitute $u = x - y$, then use the facts that f and g are both $\mathbb{Z}$-periodic (so we can shift the interval we're integrating over by any integer amount) and that $x - 1 < \lfloor x \rfloor \leq x$ for any $x \in \mathbb{R}$.

$$\begin{aligned}
(f * g)(x) &= \int_{y=0}^{1} f(y)g(x-y)\mathrm{d}y \\
&= \int_{u=x}^{x-1} f(x-u)g(u)(-\mathrm{d}u) \\
&= \int_{[x-1,x]} g(u)f(x-u)\mathrm{d}u \\
&= \left(\int_{[x-1,\lfloor x \rfloor]} g(u)f(x-u)\mathrm{d}u \right) + \left(\int_{[\lfloor x \rfloor, x]} g(u)f(x-u)\mathrm{d}u \right) \\
&= \left(\int_{[x-1,\lfloor x \rfloor]} g(u)f(x-u)\mathrm{d}u \right) + \left(\int_{[\lfloor x \rfloor - 1, x-1]} g(u)f(x-u)\mathrm{d}u \right) \\
&= \int_{[\lfloor x \rfloor - 1, \lfloor x \rfloor]} g(u)f(x-u)\mathrm{d}u \\
&= \int_{[0,1]} g(u)f(x-u)\mathrm{d}u \\
&= (g * f)(x).
\end{aligned}$$

(c) For any $f, g, h \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and any $c \in \mathbb{C}$,

$$\begin{aligned}
(f * (g+h))(x) &= \int_{[0,1]} f(y)(g+h)(x-y)\mathrm{d}y \\
&= \int_{[0,1]} f(y)\left(g(x-y)+h(x-y)\right)\mathrm{d}y \\
&= \left(\int_{[0,1]} f(y)g(x-y)\mathrm{d}y\right)\left(\int_{[0,1]} f(y)h(x-y)\mathrm{d}y\right) \\
&= (f*g+f*h)(x). \\
((f+g)*h))(x) &= \int_{[0,1]} (f+g)(y)h(x-y)\mathrm{d}y \\
&= \int_{[0,1]} \left(f(y)+g(y)\right)h(x-y)\mathrm{d}y \\
&= \left(\int_{[0,1]} f(y)h(x-y)\mathrm{d}y\right)\left(\int_{[0,1]} g(y)h(x-y)\mathrm{d}y\right) \\
&= (f*h+g*h)(x). \\
(cf)*g &= \int_{[0,1]} (cf)(y)g(x-y)\mathrm{d}y \\
&= c\int [0,1]f(y)g(x-y)\mathrm{d}y \\
&= c(f*g). \\
f*(cg) &= \int_{[0,1]} f(y)(cg)(x-y)\mathrm{d}y \\
&= c\int [0,1]f(y)g(x-y)\mathrm{d}y \\
&= c(f*g).
\end{aligned}$$

Problem 0.3. Exercise 5.5.2

(a) First, I will compute the a_n and b_n described in exercise 5.5.1:

$$\begin{aligned}
b_n &= 2\int_{[0,1]} (1-2x)^2\sin(2\pi nx)\mathrm{d}x \\
&= \left(\int_{[0,1/2]} (1-2x)^2\sin(2\pi nx)\mathrm{d}x\right) + \left(\int_{x=1/2}^{1} (1-2x)^2\sin(2\pi nx)\mathrm{d}x\right) \\
&= \left(\int_{[0,1/2]} (1-2x)^2\sin(2\pi nx)\mathrm{d}x\right) + \left(\int_{u=1/2}^{0} (1-2(1-u))^2\sin(2\pi n(1-u))(-\mathrm{d}u)\right) \\
&= \left(\int_{[0,1/2]} (1-2x)^2\sin(2\pi nx)\mathrm{d}x\right) + \left(\int_{[0,1/2]} (2u-1)^2\sin(2\pi n(1-u))\mathrm{d}u\right) \\
&= \left(\int_{[0,1/2]} (1-2x)^2\sin(2\pi nx)\mathrm{d}x\right) + \left(\int_{[0,1/2]} (1-2u)^2\sin(2\pi nu)\mathrm{d}u\right) \\
&= 0.
\end{aligned}$$

To get that, I substituted $u = 1 - x$, then used the fact that $x \mapsto \sin(2\pi n x)$ is odd and $\mathbb{Z}$-periodic. Similarly, to find a_n, I will use the fact that $x \mapsto \cos(2\pi n x)$ is even and $\mathbb{Z}$-periodic. If $n > 0$, then

$$\begin{aligned}
a_n &= 2\int_{[0,1]} (1-2x)^2 \cos(2\pi n x)\mathrm{d}x \\
&= \int_{[0,1]} \left(2 - 8x + 8x^2\right) \cos\left(2\pi n x\right) \mathrm{d}x \\
&= \left(\int_{[0,1]} 2\cos\left(2\pi n x\right)\mathrm{d}x\right) + \left(\int_{[0,1]} 8x\cos\left(2\pi n x\right)\mathrm{d}x\right) + \left(\int_{[0,1]} 8x^2\cos\left(2\pi n x\right)\mathrm{d}x\right) \\
&= \left(\left[\frac{1}{\pi n}\sin\left(2\pi n x\right)\right]_{x=0}^{1}\right) + \left(\int_{[0,1]} 8x\cos\left(2\pi n x\right)\mathrm{d}x\right) + \left(\int_{[0,1]} 8x^2\cos\left(2\pi n x\right)\mathrm{d}x\right) \\
&= \left(\int_{[0,1]} 8x\cos\left(2\pi n x\right)\mathrm{d}x\right) + \left(\int_{[0,1]} 8x^2\cos\left(2\pi n x\right)\mathrm{d}x\right) \\
&= \left(\left[(8x)\left(\frac{1}{2\pi n}\sin(2\pi n x)\right)\right]_{x=0}^{1} - \int_{[0,1]} (8)\left(\frac{1}{2\pi n}\sin(2\pi n x)\right)\mathrm{d}x\right) + \left(\int_{[0,1]} 8x^2\cos\left(2\pi n x\right)\mathrm{d}x\right) \\
&= \left(0 - \int_{[0,1]} \frac{4}{\pi n}\sin(2\pi n x)\mathrm{d}x\right) + \left(\int_{[0,1]} 8x^2\cos\left(2\pi n x\right)\mathrm{d}x\right) \\
&= 0 + \int_{[0,1]} 8x^2\cos\left(2\pi n x\right)\mathrm{d}x \\
&= \left[(8x^2)\left(\frac{\sin(2\pi n x)}{2\pi n}\right)\right]_{x=0}^{1} - \int_{[0,1]} (16x)\frac{\sin(2\pi n x)}{2\pi n}\mathrm{d}x \\
&= 0 - \int_{[0,1]} \frac{8x}{\pi n}\sin(2\pi n x)\mathrm{d}x \\
&= -\left[\left(\frac{8x}{\pi n}\right)\left(\frac{-\cos(2\pi n x)}{2\pi n}\right)\right]_{x=0}^{1} - \int_{[0,1]} \frac{8}{\pi n}\left(\frac{-\cos(2\pi n x)}{2\pi n}\right)\mathrm{d}x \\
&= \left[\frac{4x\cos(2\pi n x)}{\pi^2 n^2}\right]_{x=0}^{1} + \int_{[0,1]} \frac{4}{\pi^2 n^2}\cos(2\pi n x)\mathrm{d}x \\
&= \frac{4}{\pi^2 n^2} + \left[\left(\frac{4}{\pi^2 n^2}\right)\left(\frac{\sin(2\pi n x)}{2\pi n}\right)\right]_{x=0}^{1} \\
&= \frac{4}{\pi^2 n^2}.
\end{aligned}$$

And in the case where $n = 0$, we have

$$\begin{aligned}
b_0 &= 2\int_{[0,1]} (1-2x)^2 \sin(0)\mathrm{d}x \\
&= 0. \\
a_0 &= 2\int_{[0,1]} (1-2x)^2 \cos(0)\mathrm{d}x \\
&= \int_{[0,1]} (2 - 8x + 8x^2)\mathrm{d}x \\
&= \left[2x - 4x^2 + \frac{8x^3}{3}\right]_{x=0}^{1} \\
&= 2 - 4 + \frac{8}{3} \\
&= \frac{2}{3}.
\end{aligned}$$

By the result from exercise 5.5.1,

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(2\pi n x) + b_n \sin(2\pi n x)) \\
&= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x).
\end{aligned}$$

(b) At $x = 0$, we have

$$\begin{aligned}
1 &= (1 - 2(0)^2 \\
&= f(0) \\
&= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(0) \\
\frac{2}{3} &= \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \\
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}.
\end{aligned}$$

(c)

$$
\begin{aligned}
\|f\|^2 &= \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 \\
\langle f, f\rangle &= \left(\sum_{n=-\infty}^{-1} \left|\hat{f}(n)\right|^2\right) + \left|\hat{f}(0)\right|^2 + \left(\sum_{n=1}^{\infty} \left|\hat{f}(n)\right|^2\right) \\
\int_{[0,1]} (1-2x)^4 \mathrm{d}x &= \left(\sum_{n=-\infty}^{-1} \left|\hat{f}(n)\right|^2\right) + \frac{1}{9} + \left(\sum_{n=1}^{\infty} \left|\hat{f}(n)\right|^2\right) \\
\left[\frac{(1-2x)^5}{-10}\right]_{x=0}^{1} &= \left(\sum_{n=-\infty}^{-1} \left(\frac{a_{-n}}{2}\right)^2\right) + \frac{1}{9} + \left(\sum_{n=1}^{\infty} \left(\frac{a_n}{2}\right)^2\right) \\
\frac{1}{10} - \left(-\frac{1}{10}\right) &= \frac{1}{9} + 2\sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2}\right)^2 \\
\frac{2}{10} - \frac{1}{9} &= \sum_{n=1}^{\infty} \frac{8}{\pi^4 n^4} \\
\frac{8}{90} &= \frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \\
\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}.
\end{aligned}
$$

Problem 0.4. Exercise 5.5.4

We are given that f' is continuous, and it is $\mathbb{Z}$-periodic because for any $k \in \mathbb{Z}, x \in \mathbb{R}$,

$$
\begin{aligned}
f'(x+k) &= \lim_{\varepsilon \to 0} \frac{f(x+k+\varepsilon) - f(x+k)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \\
&= f'(x).
\end{aligned}
$$

This means that $f' \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.

$$
\begin{aligned}
\hat{f}'(n) &= \langle f', e_n \rangle \\
&= \int_{[0,1]} \left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right) e^{-2i\pi nx} \mathrm{d}x \\
&= \left[f(x) e^{-2i\pi nx}\right]_{x=0}^{1} - \int_{[0,1]} f(x)(-2i\pi n e^{-2i\pi nx}) \mathrm{d}x \\
&= 0 + 2i\pi n \int_{[0,1]} f(x) e^{-2i\pi nx} \mathrm{d}x \\
&= 2i\pi n \hat{f}(n).
\end{aligned}
$$

24F-MATH-131B HOMEWORK 9
DUE SUNDAY, DECEMBER 8

(1) Exercise: 5.3.3, 5.4.2, 5.5.2, 5.5.4.