MATH 131B practice final

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Problem 0.1.

(a) Let E be a subset of a compact metric space (X, d_X). Show that if E is open in X, then $E^c := X \backslash E$ is compact.

(b) Let $f : X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces. Suppose f is bijective and continuous, and (X, d_X) is compact. Show that f is an open map (that is, for any $E \subset X$ open, $f(E) \subset Y$ is open).

(a) Since E is open, its complement is closed. Any closed subset of a compact space is compact, so E^c is compact.

(b) Since E is open, E^c is compact. Because f is continuous, that also means $f(E^c)$ is compact, and since f is bijective, $f(E^c) = f(E)^c$. Every compact space is closed, so $f(E)^c$ is closed, which means $f(E)$ is open.

Problem 0.2. Consider $(\mathbb{R}^2, d_{\ell^2})$.

(a) What is $B_{(\mathbb{R}^2, d_{\ell^2})}((0,0), 1)$ as a set?

(b) Denote $B = B_{(\mathbb{R}^2, d_{\ell^2})}((0,0), 1)$. Let $f : \mathbb{R}^2 \to \mathbb{R}, f(x) = \inf \{d(x,y) | y \in B\}$. Show that f is continuous (with respect to the ℓ^2 metric on $\mathbb{R}^2$ and the standard metric on $\mathbb{R}$).

(a) The set of points distance less than one from the origin can be written as

$$B := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}.$$

(b) f can be written as $g \circ h$, where $h((x,y)) = \|(x,y)\| = \sqrt{x^2 + y^2}$ and $g(x) = \max(0, x - 1)$. Since g and h are both continuous, so is f.

This is not a rigorous proof, a better way to do this would be to use the triangle inequality to show that for any $\varepsilon > 0, (x,y) \in \mathbb{R}^2$, there exists some $\delta > 0$ such that $f(B((x,y), \delta)) \subset B(f((x,y)), \varepsilon)$. But I won't write that whole thing out because it's annoying to do.

Problem 0.3. Let (X, d_X) be a metric space. Suppose for any collection of closed subsets $\{S_\alpha\}_{\alpha \in A}$ (i.e., for every $\alpha \in A$, S_α is a closed subset of X) such that $\cap_{\alpha \in F} S_\alpha \neq \emptyset$ for any finite subset F of A, we have that $\cap_{\alpha \in A} S_\alpha \neq \emptyset$. Show that X is compact.

Suppose X is not compact. Let $\{T_\alpha\}_{\alpha \in A}$ be any open cover of X. There is no finite subcover of that, so for any finite subset $F \subset A$, $\cup_{\alpha \in F} T_\alpha \neq X$, which means $\cap_{\alpha \in F} T_\alpha^c \neq \emptyset$. Then $\cap_{\alpha \in A} T_\alpha^c \neq 0$, which means $\cup_{\alpha \in A} T_\alpha \neq X$, so $\{T_\alpha\}_{\alpha \in A}$ does not cover X. This is a contradiction, so X must be compact.

Problem 0.4. Let $(f_n)_{n=1}^{\infty}$ be a sequence of Riemann-integrable functions from $[a,b]$ to $\mathbb{R}$, and $f : [a,b] \to \mathbb{R}$ another Riemann-integrable function. Suppose $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $[a,b]$.

(a) Show that $\lim_{n\to\infty} \int_{[a,b]} f_n = \int_{[a,b]} f$.

(b) For every n, define $F_n : [a,b] \to \mathbb{R}, F_n(x) = \int_{[} -[a,x] f_n$ and $F : [a,b] \to \mathbb{R}, F(x) = \int_{[a,b]} f$. Show that F_n converges uniformly to F on $[a,b]$.

Problem 0.5.

(a) Let $f : E \to \mathbb{R}$ where $E \subset \mathbb{R}$ and $a \in \operatorname{int}(E)$ (interior of E). State the definition of f being real analytic at $x = a$.

(b) Now let $f(x) = \operatorname{arccot}(x)$ (inverse cotangent, where $\cot(x) := \cos(x)/\sin(x)$). Show that $f'(x)$ is real analytic at $x = 0$ using the geometric series formula

$$\sum_{k=0}^{\infty} cr^k = \frac{c}{1-r}.$$

Find the radius of convergence R for the power series expansion for $f'(x)$ at $x = 0$.

(c) Show that $f(x)$ is real analytic at $x = 0$ (you can use $f(x) = \pi/2 + \int_{[0,x]} f'$ for all $x \in \mathbb{R}$).

Problem 0.6.

(a) State the Fourier theorem for $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.

(b) Let f be a function in $C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. For $n \in \mathbb{Z}_{\geq 0}$, let

$$a_n = 2\int_0^1 f(x)\cos(2\pi nx)\mathrm{d}x,$$
$$b_n = 2\int_0^1 f(x)\sin(2\pi nx)\mathrm{d}x.$$

Use the Fourier series of f to show that the series

$$\frac{a_0}{2 + \sum_{n=1}^{\infty}} \left[a_n \cos(2\pi nx) + b_n \sin(2\pi nx)\right]$$

converges to f in the L^2 metric.

Problem 0.7.

(a) Let $(f_n)_n \subset C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, and $f, g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and $f_n \to f$ uniformly on $\mathbb{R}$. Show that $f_n * g \to f * g$ pointwise and uniformly.

(b) Let $(f_n)_n \subset C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, and $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Suppose for each n, f_n is a periodic $\left(\frac{1}{n}, \frac{1}{2n}\right)$ approximation to the identity. Show that $f_n * f \to f$ uniformly.

There was a typo on this question???

Problem 0.8. Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and g be a trigonometric polynomial. Show that $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$ for all $n \in \mathbb{Z}$.

Problem 0.9. Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. For each $N \in \mathbb{N}$, let $F_N = \sum_{n=-N}^{N} \hat{f}(n) e_n$, and S_n be the collection of trigonometric polynomial $p = \sum_{n=-N}^{N} c_n e_n$ where $\sum_{n=-N}^{N} |c_n|^2 \leq 1$ (recall $e_n(x) := e^{2\pi i n x}$ is the character with frequency n). For each N, show that $|\langle f, p \rangle| \leq \|F_N\|$ for all $p \in S_n$. Find an element $p \in S_N$ such that equality holds.