Complex Analysis Homework #5

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Problem 0.1. Calculate
$$\int_{|z|=R} \overline{z}^{\,n}\, dz$$
for $R > 0$ and all $n \geq 1$ integers.
(Hint: Orient the circle counterclockwise.)

Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed curve defined by $\gamma(t) = Re^{it}$. Then the integral we want to find is

$$\int_\gamma \overline{z}^n \mathrm{d}z = \int_{t=0}^{2\pi} \left(e^{-it}\right)^n \gamma'(t) \mathrm{d}t = \int_{t=0}^{2\pi} i e^{(1-n)t} \mathrm{d}t$$

Problem 0.2. Let γ be the unit circle with positive orientation: $\{z : |z| = 1\}$. Find

1. $$\int_\gamma \frac{e^z}{z^4} dz$$

2. $$\int_\gamma \frac{\sin z}{z^2} dz$$

(Hint: Find their antiderivatives.)

Problem 0.3. Show that
$$\left| \int_\gamma e^{\sin z}\, dz \right| \leq 1$$
where γ is the straight line segment from $z = 0$ to $z = i$.

Problem 0.4. Suppose f is continuously complex differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem (and the CR equations) to show that $\int_T f = 0$. (This provides a proof of Goursat's theorem under the additional assumption that f' is continuous.)