MATH 131B Homework #8

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Problem 0.1. Exercise 5.1.2: Prove lemma 5.1.5.

(a) If $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, define a function $g : [0,1] \to \mathbb{C}$ such that $g(x) = f(x + \mathbb{Z})$ for any $x \in \mathbb{R}$. Since g is a continuous function on the compact interval $[0,1]$, it must be bounded, because continuous functions map compact sets to compact sets, and since $\mathbb{C}$ is isometric to $\mathbb{R}^2$, if the image of g is compact, it is also bounded. Every element of the domain of f can be written as $x + \mathbb{Z}$ for some $x \in [0,1]$, so the image of f is the same as the image of g, meaning f is also bounded.

(b) Ignore the notation I used above – for the rest of this homework, I will treat functions $f, g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ as continuous, $\mathbb{Z}$-periodic functions from $\mathbb{R}$ to $\mathbb{C}$, instead of treating them as continuous functions from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{C}$.

If $f, g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, then f and g are continuous, so $f + g$, $f - g$, and fg are also continuous. Also, for any $x \in \mathbb{R}$, the following are all true:

$$\begin{aligned}
(f+g)(x+1) &= f(x+1) + g(x+1) = f(x) + g(x) = (f+g)(x) \\
(f-g)(x+1) &= f(x+1) - g(x+1) = f(x) - g(x) = (f-g)(x) \\
(fg)(x+1) &= f(x+1)g(x+1) = f(x)g(x) = (fg)(x).
\end{aligned}$$

Therefore $f + x, f - g, fg \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.

For any $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C}), c \in \mathbb{C}$, cf is continuous and for any $x \in \mathbb{R}$,

$$(cf)(x+1) = c \cdot f(x+1) = c \cdot f(x) = (cf)(x),$$

so $cf \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.

(c) We have already shown that if a sequence of continuous functions converges uniformly, that sequence converges to another continuous function, so I only need to show that the sequence converges to another $\mathbb{Z}$-periodic function.

For any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that whenever $n \geq N$, $d(f_n(x), f(x)) < \varepsilon/2$ for every $x \in \mathbb{R}$. Since f_n is $\mathbb{Z}$-periodic, $f_n(x+1) = f_n(x)$ for any $x \in \mathbb{R}$. By the triangle inequality, for any $x \in \mathbb{R}$,

$$\begin{aligned}
d(f(x), f(x+1)) &\leq d(f(x), f_n(x)) + d(f_n(x+1), f(x+1)) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

f is continuous, and the only way $d(f(x), f(x+1))$ can be less than ε for any positive ε is if $f(x) = f(x+1)$, so $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.

Problem 0.2. Exercise 5.2.1: Prove lemma 5.2.5.

(a)

$$\begin{aligned}
\langle g, f\rangle &= \int_{[0,1]} f(x)\overline{g(x)}\mathrm{d}x \\
&= \overline{\int_{[0,1]} g(x)\overline{f(x)}\mathrm{d}x} \\
&= \overline{\langle f, g\rangle}.
\end{aligned}$$

(b) The function $g(x) := \|f(x)\|^2$ is continuous and nonnegative, and

$$\langle f, f\rangle = \int_{[0,1]} g(x)\mathrm{d}x.$$

We can partition the interval $[0, 1]$ into the disjoint subset of Lebesgue-measurable sets A, B, C, where

$$\begin{aligned}
A &:= \{x \in [0,1] : g(x) < 0\} \\
B &:= \{x \in [0,1] : g(x) = 0\} \\
C &:= \{x \in [0,1] : g(x) > 0\}, \\
\therefore \langle f, f\rangle &= \left(\int_A g(x)\mathrm{d}x\right) + \left(\int_B g(x)\mathrm{d}x\right) + \left(\int_C g(x)\mathrm{d}x\right).
\end{aligned}$$

Clearly $A = \emptyset$ though, so I will ignore that first integral. The second integral is also obviously zero, so we only need to consider the integral over C.

If $f \neq 0$, then C is nonempty, so there is some $x_0 \in C$ such that $g(x_0) > 0$, and since g is continuous, that implies there is some interval contained in C for which $g(x) \geq g(x_0)/2 > 0$. The integral of g over that interval is positive, and the integral of g over the remainder of C is nonnegative, so $\langle f, f\rangle > 0$.

On the other hand, if $f = 0$, then $\langle f, f\rangle = 0$.

(c)

$$\begin{aligned}
\langle f + g, h\rangle &= \int_{[0,1]} (f+g)(x)\overline{h(x)}\mathrm{d}x \\
&= \left(\int_{[0,1]} f(x)\overline{h(x)}\mathrm{d}x\right) + \left(\int_{[0,1]} g(x)\overline{h(x)}\mathrm{d}x\right) \\
&= \langle f, h\rangle + \langle g, h\rangle. \\
\langle cf, g\rangle &= \int_{[0,1]} (cf)(x)\overline{h(x)}\mathrm{d}x \\
&= c \cdot \int_{[0,1]} f(x)\overline{h(x)}\mathrm{d}x \\
&= c\,\langle f, g\rangle.
\end{aligned}$$

(d)

$$\begin{aligned}
\langle f, g + h\rangle &= \int_{[0,1]} f(x)\overline{(g+h)(x)}\mathrm{d}x \\
&= \left(\int_{[0,1]} f(x)\overline{g(x)}\mathrm{d}x\right) + \left(\int_{[0,1]} f(x)\overline{h(x)}\mathrm{d}x\right) \\
&= \langle f, g\rangle + \langle f, h\rangle. \\
\langle f, cg\rangle &= \overline{\langle cg, f\rangle} = \overline{c\,\langle g, f\rangle} = \overline{c} \cdot \overline{\langle g, f\rangle} = \overline{c}\,\langle f, g\rangle.
\end{aligned}$$

Problem 0.3. Exercise 5.2.2

$\boxed{d(f,f) = 0 \text{ for any } f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C}),}$ because

$$d(f,f) = \sqrt{\int_{[0,1]} (f-f)(x)\mathrm{d}x} = \sqrt{\int_{[0,1]} 0\mathrm{d}x} = 0.$$

$\boxed{d(f,g) = d(g,f) \text{ for any } f,g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C}),}$ because

$$\begin{aligned} d(f,g) &= \sqrt{\int_{[0,1]} |f(x) - g(x)|^2 \, \mathrm{d}x} \\ &= \sqrt{\int_{[0,1]} |g(x) - f(x)|^2 \, \mathrm{d}x} \\ &= d(g,f). \end{aligned}$$

$\boxed{d(f,g) > 0 \text{ whenever } f \neq g,}$ because

$$d(f,g) = \sqrt{\int_{[0,1]} |f(x) - g(x)|^2 \, \mathrm{d}x},$$

and the function $h(x) = |f(x) - g(x)|^2$ is continuous, positive somewhere, and nonnegative everywhere. I showed in the previous problem that the integral of such a function must be positive, so $d(f,g) > 0$.

$\boxed{d(f,g) + d(g,h) \geq d(f,h) \text{ for any } f,g,h \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C}),}$ because

$$\begin{aligned} d(f,h)^2 &= \langle f-h, f-h \rangle \\ &= \langle (f-g) + (g-h), (f-g) + (g-h) \rangle \\ &= \langle f-g, f-g \rangle + \langle g-h, g-h \rangle + \langle f-g, g-h \rangle + \overline{\langle f-g, g-h \rangle} \\ &= \|f-g\|^2 + \|g-h\|^2 + 2\Re\left(\langle f-g, g-h \rangle\right) \\ &\leq \|f-g\|^2 + \|g-h\|^2 + 2\left|\langle f-g, g-h \rangle\right| \\ &\leq \|f-g\|^2 + \|g-h\|^2 + 2 \cdot \|f-g\| \cdot \|g-h\| \\ &= (\|f-g\| + \|g-h\|)^2 \\ &= (d(f,g) + d(g,h))^2, \end{aligned}$$

which implies $d(f,h) \leq d(f,g) + d(g,h)$. Note that I used the Cauchy-Schwarz inequality, which says that

$$|\langle u, v \rangle|^2 \leq \|u\| \cdot \|v\|$$

for any u, v in some inner product space. In this case, $u = f - g$ and $v = g - h$.

Here is a proof of the Cauchy-Schwarz inequality, which I copied from Wikipedia:
If $v = 0$, then we have $0 \leq \langle u, u \rangle + 0$, which we already know is true, so we only need to consider the case where $v \neq 0$. Let

$$z := u - \frac{\langle u, v \rangle}{\langle v, v \rangle} v.$$

Note that v is orthogonal to z, so

$$\|u\|^2 = \left| \frac{\langle u, v \rangle}{\langle v, v \rangle} \right|^2 + \|z\|^2 = \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \|z\|^2 \geq \frac{|\langle u, v \rangle|^2}{\|v\|^2}.$$

Therefore $\|v\|^2 \cdot \|u\|^2 \geq |\langle u, v \rangle|^2$, so

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|.$$

Problem 0.4. Exercise 5.2.6

(a) If f_n converges uniformly to f, then for every $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for any $n \geq N, x \in \mathbb{R}$, $|f_n(x) - f(x)| < \sqrt{\varepsilon}$. The L^2 distance between f_n and f can be written as

$$d(f_n, f) = \int_{[0,1]} g(x) \mathrm{d}x,$$

where $g(x) := |f_n(x) - f(x)|^2$.

For any $x \in \mathbb{R}$, $0 \leq g(x) < \varepsilon$, so

$$0 \leq d(f_n, f) < \varepsilon.$$

Therefore f_n converges to f in the L^2 metric.

(b) Let $f = 0$, and let $f_n = \cos(\pi x)^n$. Then

$$\begin{aligned}
d(f_n, f) &= \sqrt{\int_{[0,1]} \cos(\pi x)^{2n} \mathrm{d}x} \\
&= \sqrt{\int_{[0,1]} \left(\frac{e^{i\pi x} + e^{-i\pi x}}{2}\right)^{2n} \mathrm{d}x} \\
&= \sqrt{\int_{[0,1]} \left(\frac{1}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} e^{2i\pi(k-n)x}\right) \mathrm{d}x} \\
&= \sqrt{\int_{[0,1]} \frac{1}{4^n} \binom{2n}{n} \mathrm{d}x} \\
&= \sqrt{\frac{(2n)!}{4^n (n!)^2}} \\
&= \frac{\sqrt{(2n)!}}{2^n n!} \\
&= \prod_{k=1}^{n} \frac{\sqrt{(2k-1)(2k)}}{2k} \\
&= \prod_{k=1}^{n} \sqrt{\frac{2k-1}{2k}} \\
&= \exp\left(\frac{1}{2} \sum_{k=1}^{n} (\log(2k-1) - \log(2k))\right).
\end{aligned}$$

This is useful because that summation goes to $-\infty$ (which you can prove by bounding the sum with an integral, evaluating the integral, and then taking the limit as $n \to \infty$), so as $n \to \infty$, $d(f_n, f) \to 0$. Therefore f_n converges to f in the L^2 metric, but it does ot converge uniformly, because $f_n(0)$ is always $\cos(0)^n = 1$, and $f(0)$ is always zero.

(c) My example from part (b) works here too.

(d) Define f_n by

$$f_n(x) = \begin{cases} n \sin(n\pi x) & 0 \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} \le x \le 1 \\ f(x - \lfloor x \rfloor) & x \notin [0,1] \end{cases}$$

and let $f = 0$. In the L^2 norm,

$$\begin{aligned}
d(f_n, f) &= \sqrt{\int_{[0,1]} |f_n(x) - f(x)|^2 \, \mathrm{d}x} \\
&= \sqrt{\int_{[0,1/n]} n^2 \sin(n\pi x)^2 \mathrm{d}x} \\
&= \sqrt{\int_{u=0}^{1} n \sin(\pi u)^2 \mathrm{d}u} \\
&= \sqrt{\frac{n}{2}},
\end{aligned}$$

so f_n does not converge to f. But f_n does converge pointwise to f, because for any $x \in [0,1]$, either $x = 0$, in which case $f_n(x)$ is always zero, or $f_n(x) = 0$ for any $n \ge 1/x$.

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(1) Exercise: 5.1.2, 5.2.1, 5.2.2, 5.2.6.