MATH 131B Homework #4

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Problem 0.1. 3.3.2

Let $y_n = \lim_{x \to x_0, x \in E} f^{(n)}(x)$. For any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that $d(f^{(n)}(x), y_n) < \frac{\varepsilon}{4}$ whenever $d(x, x_0) < \delta_n$.

Because the sequence of functions converges uniformly to f, there also exists $N \in \mathbb{N}$ such that $d(f(x), f^{(n)}(x)) < \frac{\varepsilon}{4}$ for any $n \geq N$ and any $x \in E$. By the triangle inequality, for this particular N,

$$d(y_n, y_m) < d(y_n, f^{(n)}(x)) + d(f^{(n)}(x), f(x)) + d(f(x), f^{(m)}(x_0)) + d(f^{(m)}(x), y_m) < \varepsilon$$

whenever $d(x, x_0) < \min(\delta_n, \delta_m)$. That shows that y_n is a Cauchy sequence, and since Y is complete, that sequence converges to $y \in Y$.

We want to show that $\lim_{x \to x_0, x \in E} f(x) = y$. This is fairly straighforward – there exists $N \in \mathbb{N}$ such that $d(f(x), f^{(N)}(x)) < \frac{\varepsilon}{3}$ for any $x \in X$, N can be further increased to ensure that $d(y_N, y) < \frac{\varepsilon}{3}$, and there exists $\delta_N > 0$ such that $d(f^{(N)}(x), y_N) < \frac{\varepsilon}{3}$ whenever $d(x, x_0) < \delta_N$. Then

$$d(f(x), y) \leq d(f(x), f^{(N)}(x)) + d(f^{(N)}(x), y_N) + d(y_N, y) < \varepsilon,$$

meaning $f(x)$ converges to y. So in the equation we want to prove is true, both sides are equal to y.

Problem 0.2. 3.3.4

Problem 0.3. 3.3.5

Let $f^{(n)} : [0, 1] \to [0, 1]$ be the function which maps x to x^n, and let $x^{(n)} = \sqrt[n]{\varepsilon} = \varepsilon^{1/n}$ for some $\varepsilon \in (0, 1)$. Then $f^{(n)}$ converges **pointwise** (but not uniformly) to f, where

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & x < 1, \end{cases}$$

so $f(x^{(n)}) = 0$ for any $n \in \mathbb{N}$. However, $x^{(n)}$ converges to 1, and $f(1) = 1 \neq 0$.

Problem 0.4. 3.3.6

Let y_0 be any point in Y. For each $n \in \mathbb{N}$, $f^{(n)}$ is bounded, meaning there exists $R_n \in \mathbb{R}$ such that $f^{(n)}(X) \subset B(y_0, R_n)$. Since the sequence of functions converges uniformly to f, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(f(x), f^{(n)}(x)) < \varepsilon$ whenever $n \geq N$ and $x \in X$. Therefore,

$$d(f(x), y_0) \leq d(f(x), f^{(N)}(x)) + d(f^{(n)}(x), y_0) < \varepsilon + R_N,$$

so $f(X) \subset B(y_0, R_N + \varepsilon)$, meaning f is bounded.

Problem 0.5. 3.5.1

For each $i \in \{1, \ldots, N\}$, define

$$\left\| f^{(i)} \right\|_\infty = \sup_{x \in X} \left| f^{(i)}(x) \right|$$

Then for any $x \in X$,

$$\left(\sum_{i=1}^{N} f^{(i)} \right)(x) = \sum_{i=1}^{N} \left(f^{(i)}(x) \right) \leq \sum_{i=1}^{N} \left\| f^{(i)} \right\|_\infty ,$$

which is the sum of finitely many real numbers, so the sum is bounded.

If each $f^{(i)}$ is continuous at some $x_0 \in X$, then for any $\varepsilon > 0$, there exists $\delta_i > 0$ for every $i \in \{1, \ldots, N\}$ such that

$$d_X(f^{(i)}(x), f^{(i)}(x_0)) < \frac{\varepsilon}{N}$$

whenever $d(x, x_0) < \delta_i$. This implies

$$d_X \left(\sum_{i=1}^{N} f^{(i)}(x), \sum_{i=1}^{N} f^{(i)}(x_0) \right) < \varepsilon,$$

so $\sum f^{(i)}$ is continuous at x_0. And of course, if each $f^{(i)}$ is continuous at every $x_0 \in X$, then so is $\sum f^{(i)}$.

This proof would work just as easily if we had said "uniformly continuous" instead of "continuous". The only difference is that δ_i would need to ensure $d_X(f^{(i)}(x), f^{(i)}(x_0)) < \frac{\varepsilon}{N}$ for any $x, x_0 \in X$ which are distance less than δ_i apart, instead of for one specific $x_0 \in X$ and any x withint a distance δ_i of x_0.

Problem 0.6. 3.5.2

24F-MATH-131B HOMEWORK 4
DUE SUNDAY, NOVEMBER 3

(1) Exercise: 3.3.2, 3.3.4, 3.3.5, 3.3.6, 3.5.1, 3.5.2.