Math 110AH Homework 7

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November 22, 2023

Assignment due November 22nd at 11:59 pm.
Problems 1, 3, 5, 7, and 8 are graded.

1

Write out the disjoint cycle decomposition of $\sigma = (1234)(456)(145)$. Determine the order of σ.

$$\sigma = (152346)$$

Since σ is a 6-cycle, it has order 6.

2

Prove that S_7 contains a cyclic subgroup of order 10.

Lemma 2.1. *The order of a permutation is equal to the LCM of the cycle type.*

Proof. For any cycle a, write out the disjoint cycle decomposition, and call those disjoint cycles $a_1, a_2, \ldots, a_n$. For each disjoint cycle a_i, the order of a_i is equal to the length of the cycle, so the cycle type of a is the multiset of the orders of each a_i.

We know that disjoint cycles commute, so for any natural number x,

$$a^x = (a_1 a_2 \cdots a_n)^x = a_1^x a_2^x \cdots a_n^x.$$

If a_i and a_j are disjoint, then a_i^x and a_j^x are also disjoint, which implies a^x is the identity if and only if for every index i, a_i^x is the identity. Therefore the order of a is the least common multiple of the orders of each a_i, or equivalently, the LCM of the cycle type. $\square$

Define $\sigma \in S_7$ as $\sigma = (12)(34567)$. According to lemma 2.1, σ has order 10, so $\langle\sigma\rangle$ is a cyclic subgroup with order 10.

3

Find the largest order of an element in S_5.

For any element $x \in S_5$, the sum of the cycle type is at most five (by "cycle type", I mean the multiset consisting of the length of all the disjoint cycles in x, ignoring 1-cycles). Therefore there are only a few possibilities for the cycle type, and using lemma 2.1, we can calculate the order of x in each of those cases.

Cycle type	order
$\{\}$	1
$\{2\}$	2
$\{3\}$	3
$\{4\}$	4
$\{5\}$	5
$\{2,2\}$	2
$\{2,3\}$	6

From this table, we see that the order of x can be 6, but cannot be larger than 6.

Note: you can generalize this from S_5 to S_n by using the Landau function, which is given by sequence A000793 in the OEIS.

4

Find all elements in S_5 that commute with the cycle (123).

Lemma 4.1. *If x is a cycle of the form $(1, 2, \dots, n)$ and y is another permutation, then $yxy^{-1} = (y(1), y(2), \dots, y(n))$.*

Proof. We proved this in class. $\square$

The cycle (123) commutes with y if and only if $y(123)y^{-1} = (123)$, which, according to lemma 4.1, is true if and only if one of the following is true:

- $y(1) = 1$ and $y(2) = 2$, and $y(3) = 3$
- $y(1) = 2$ and $y(2) = 3$, and $y(3) = 1$
- $y(1) = 3$ and $y(2) = 1$, and $y(3) = 2$

One of those conditions will be satisfied if and only if

$$y \in \{e, (123), (321), (45), (123)(45), (321)(45)\}.$$

5

How many conjugacy classes are there in S_5?

Lemma 5.1. *For any permutation $x \in S_n$, the conjugacy class of x is the set of all permutations $y \in S_n$ which have the same cycle type as x.*

Proof. By applying lemma 4.1 to the disjoint cycles of x, we can easily see that the conjugate of x by any permutation will have the same cycle type as x.

If x and y have the same cycle type, write their disjoint cycle decompositions in order from shortest cycle to longest. Then the written expressions match up perfectly and you can easily use lemma 4.1 to defined a permutation p such that $pxp^{-1} = y$. □

According to the table in problem 3, there are 7 possibilities for the cycle type of an element in S_5, so by lemma 5.1, S_5 has 7 conjugacy classes.

6

- (a) Prove that S_n is generated by $(1,2), (1,3), \dots, (1,n)$. (Hint: Use $(1,j)(1,i)(1,j) = (i,j)$.)
- (b) Prove that S_n is generated by $(1,2), (2,3), \dots, (n-1,n)$.
- (c) Prove that S_n is generated by the two cycles $(1,2)$ and $(1,2,\dots,n)$. (Hint: Use $(1,2,\dots,n)(i-1,i)(1,2,\dots,n)^{-1} = (i,i+1)$.)

Lemma 6.1. *The symmetric group S_n is generated by the set of transpositions (i,j).*

Proof. We know that any permutation can be written as a product of disjoint cycles, so we just need to show that every disjoint cycle is a product of transpositions. For any disjoint cycle of the form $(1,2,\dots,n)$, we can rewrite it as

$$(1,2,\dots,n) = (2,3)(3,4)\cdots(n-1,n)(n,1).$$

So by expanding a permutation into a product of disjoint cycles and then expanding every cycle into a product of transpositions, we see that transpositions generate the symmetric group. □

- (a) Let $x \in S_n$ be any permutation on n elements. Then by lemma 6.1, we can rewrite x as a product of transpositions. Each transposition has the form (i,j), which is equal to $(1,j)(1,i)(1,j)$. Therefore x can be expanded as a product of transpositions which all have the form $(1,i)$ (for some index i).

- (b) Using the result of part (a), any permutation $x \in S_n$ can be expanded as a product of transpositions of the form $(1, i)$. Then each of those can be rewritten as

  $$(1, i) = (1, 2)(2, 3)(3, 4) \cdots (i-2, i-1)(i-1, i)(i-2, i-1)(i-3, i-2) \cdots (2, 3)(1, 2).$$

  Therefore x can be expanded as a product of adjacent swaps (that is, transpositions of the form $(i-1, i)$).

- (c) Using the result from part (b), any permutation $x \in S_n$ can be written as a product of "adjacent swaps" $(i, i+1)$. Each of those can be written as

  $$(1, 2, \ldots, n)^{i-1}(1, 2)(1, 2, \ldots, n)^{1-i}.$$

  Therefore x can be written as a product of $(1, 2)$ and $(1, 2, \ldots, n)$, meaning S_n is generated by those two elements alone.

7

Show that the alternating groups A_n $(n \geq 4)$ have trivial center.

Let p be any permutation in A_n other than the identity.

- If p contains only 2-cycles, we can choose symbols a, b, c, d such that $p(a) = b$ and $p(b) = a$. In this case, p does not commute with the 3-cycle (abc), because $p \circ (abc)$ maps a to a and $(abc) \circ p$ maps a to c.

- If p does not contain only 2-cycles, then since $p \neq e$, p must contain a cycle whose length is at least 3, meaning we can choose symbols a, b, c, d such that $p(a) = b$ and $p(b) = c$. Then p does not commute with $(ab)(cd)$, because $(ab)(cd) \circ p$ maps a to a, and $p \circ (ab)(cd)$ maps a to c.

In either case, we have found an even permutation that p does not commute with. Therefore the center of A_n is trivial (when $n \geq 4$).

8

Show that $Aut(S_3)$ is isomorphic to S_3.

Any permutation $p \in S_3$ can be written as a product of $r := (123)$ and $s := (12)$. Specifically, p can be written as $r^a s^b$ where $a \in \{0, 1, 2\}$ and $b \in \{0, 1\}$. This comes from the result from problem 6 part (c), which states r and s generate S_3, as well as the relations $srs = r^{-1}$ and $s^2 = e$.

Define $\varphi : S_3 \to Aut(S_3)$ by $\varphi(p) = f$, where $f : S_3 \to S_3$ is the isomorphism which maps r to r^a and s to s^b. One can easily check that f and φ are both well-defined, they're both homomorphisms, and they're both bijective, so φ is an isomorphism from S_3 to $Aut(S_3)$.

9

Let $N = \{e, (12)(34), (13)(24), (14)(23)\} \subset S_4$. Show that S_4/N is isomorphic to S_3.

Let $f : S_4 \to S_4$ be a homomorphism defined by the following process: for any permutation $x \in S_4$, let y be the permutation consisting of two disjoint transpositions, such that $x(1) = y(1)$. Given x, there is a unique y that satisfies that property. Then we can define f by $f(x) = xy$, and from that definition, we see that f is indeed a well-defined homomorphism.

A permutation x is in the kernel of f if an only the y that corresponds to x is the inverse of x. Since y is made of 2-cycles, it's equal to its own inverse, so $Ker(f) = N$.

Applying the first isomorphism theorem, S_4/N is isomorphic to $Im(f)$. By Lagrange's theorem, $Im(f)$ has $24/4 = 6$ elements, so if we find 6 distinct elements of $Im(f)$, those are the only elements of $Im(f)$. For any permutation x which acts only on the elements 2, 3, and 4, the corresponding y will be the identity, so $f(x) = x$, meaning S_3 is a subset of $Im(f)$, and S_3 also has 6 elements, so $S_3 \cong S_4/N$.

10

Prove that the alternating group A_4 does not have a subgroup of order 6.

Every group of order 6 is either C_6 (which is isomorphic to $C_2 \times C_3$) or D_6 (which is isomorphic to S_3). Repeating the process that was used in problem 3, we see that no element of A_4 can have order greater than 4, so A_4 cannot have a subgroup isomorphic to C_6.

This implies that if A_4 has a subgroup of order 6, then that subgroup is isomorphic to D_6, which means there exist elements $r, s \in A_4$ such that $srs = r^{-1}$ and $s^2 = e$ and $r^3 = e$. Every element of A_4 is either the identity, a 3-cycle, or the product of two disjoint transpositions. Since the identity commutes with everything $r \neq e \neq s$, so we infer that r is a 3-cycle and s is either a transposition or two disjoint transpositions.

Without loss of generality, we can call one of the transpositions in s (ab), and call r either (abc) or (bcd). In the first case, srs maps a to either c or d (depending on whether s is one transposition or two), so $srs \neq r^{-1}$. In the second case, srs maps b to b, which also implies $srs \neq r^{-1}$. We have reached a contradiction in either case, so no subgroup of A_4 can be isomorphic to D_6.

But since every group of order 6 is isomorphic to either C_6 or D_6, and we have shown no subgroup of A_4 can be isomorphic to either of those, there cannot be any subgroup of A_4 which has order 6.