Math 151A
HW #6, due on Friday, November 22, 2024 at 11:59pm PST.

[1] [*Centered difference approximation of second derivative*]
Let $f \in C^4([a,b])$.

(a) Use Taylor's theorem to write f as a third order (cubic) Taylor polynomial plus a fourth order (quartic) remainder term. Expand about the point x_0.

(b) Use the result from (a) to evaluate $f(x)$ at the points $x = x_0 + h$ and $x = x_0 - h$. Add the two results together to derive the centered difference approximation to the second derivative:

$$\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0) + \frac{h^2}{4!}(f^{(4)}(\xi_1) + f^{(4)}(\xi_2))$$

(c) What is the order or this method in terms of big-O notation?

[2] In Lecture 18 we derived a formula for the error in the $O(h^2)$ approximation of f'. In particular we showed that if $\varepsilon_1, \varepsilon_2$ are the roundoff errors respectively in $f(x_0+h), f(x_0-h)$ and $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, then

$$\left| f'(x_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0-h)}{2h} \right| \leq \left| -\frac{h^2 M}{6} \right| + \left| \frac{\varepsilon}{h} \right|$$

Assume $h > 0$, $\varepsilon > 0$. Find a formula for the optimal h.

[3] [*Richardson extrapolation for a 2nd order accurate approximation*]
Using Taylor's theorem, it can be shown that if $f \in C^5([a,b])$, then the centered difference approximation formula for the first derivative is given by

$$\underbrace{f'(x_0)}_{\text{true value}} = \underbrace{\frac{f(x_0+h) - f(x_0-h)}{2h}}_{\text{approximation}} - \left(\underbrace{\frac{h^2}{6} f'''(x_0) + \frac{h^4}{120} f^{(5)}(\xi)}_{\text{error}} \right) \qquad (*)$$

(a) Re-write this formula using step size $h/2$ instead of h.

(b) Multiply your answer from (a) by 4, subtract $(*)$ from the result, and then divide everything by 3. We now should have an approximation to $f'(x_0)$ based on $f(x_0 + h,\ f(x_0 - h),\ f(x_0 + h/2)$ and $f(x_0 - h/2)$. What is the error in this new approximation $f'(x_0)$?

[4]
Let $f(x) = \sin(x)$. Use the backwards difference formula to approximate $f'(x = \pi/3)$ using $h = 0.1$, $h = 0.01$, and $h = 0.001$ and record the absolute error. By how much does the error decrease each time?

[5]
Let $f(x) = 3xe^x - \cos(x)$. Use the data below and your answer to exercise [1] to approximate $f''(1.3)$ with $h = 0.1$ and $h = 0.01$. Compare your results to the true $f''(1.3)$ using the relative error.

x	1.20	1.29	1.30	1.31	1.40
$f(x)$	11.59006	13.78176	14.04276	14.30741	16.86187

[6] Computational Problem
Consider the function $f(x) = x^2 \ln(x)$. We want to numerically approximate the derivative of the function at the point $x = 2$.

(a) Use the first order forward difference scheme

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

to approximate $f'(2)$.

(b) Write a code to find the optimal h (the h for which the error is smallest) by numerical experimentation. Report your estimate of the optimal h.

(c) Create a graph that shows how the error decreases and then starts to increase as h continues to decrease. Why isn't the error always decreasing?

Hint: It might be helpful to use a `log-log` plot to visualize the error. Additionally when thinking about values of h to try you can try different increments, but a good starting place is incrementing by $1/10^n$ for $n = 1, \ldots$

Note: You can identify the "optimal" h as the value that gives the lowest error when using different h vectors. You can calculate the true derivative of f by hand.