Math 110BH Homework 7

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1

Prove that the intersection of two principal ideals in a UFD is a principal ideal.

2

Find an example of a non-free submodule $N \subset M$ of a free module M over some domain R.

3

Show that a submodule N of a module M generated by n elements over a PID also can be generated by n elements.

4

Prove that the group $\mathbb{Z}^n$ cannot be generated by $n-1$ elements.

5

Find two non-free modules M and N over $\mathbb{Z}/6\mathbb{Z}$ such that $M \oplus N$ is free.

Let $M = \mathbb{Z}/2\mathbb{Z}$ and let $N = \mathbb{Z}/3\mathbb{Z}$.

6

Let R be a PID and let M be a torsion finitely generated R-module with the invariant factors $d_1|d_2|\cdots|d_k$. Set

$$I = \{a \in R \text{ such that } aM = 0\}.$$

Prove that $I = d_k R$.

7

Classify all abelian groups of order 300.

Any abelian group of order 300 is isomorphic to one of the four following groups:

$$\begin{aligned}
&(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z}) \\
&\qquad (\mathbb{Z}/2^2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z}) \\
&(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/5^2\mathbb{Z}) \\
&\qquad (\mathbb{Z}/2^2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/5^5\mathbb{Z}).
\end{aligned}$$

8

Find the rank of the subgroup in $\mathbb{Z}^3$ generated by $(2, -2, 0)$, $(0, 4, -4)$, and $(5, 0, -5)$.

9

Determine the invariant factors of the factor group $\mathbb{Z}^3/N$, where N is generated by $(3, -3, 3)$, $(0, 6, 12)$, and $(9, 0, -9)$.

10

Let M be a finitely generated torsion module over a PID R. Prove that M is cyclic if and only if every two elementary divisors of M are relatively prime.