MATH 131B Homework #2

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Problem 0.1. Exercises 1.5.6, 1.5.7, 1.5.8, 2.1.1, 2.1.2, 2.3.3, 2.3.4, and 2.4.4 from the textbook.

Exercise 1.5.6: prove corollary 1.5.9.

For every $n \in \mathbb{N}$, let $V_n = K_1 - K_n$, so that V_n is open in K_1. Suppose for the sake of contradiction that $\cap_{n=1}^{\infty} K_n = \emptyset$. That is equivalent to saying $\cup_{n=1}^{\infty} V_n = K_1$, so by theorem 1.5.8, there is a finite subset $\{V_i\}_{i \in I}$ which covers K_1 – that is, $\cup_{i \in I} V_i = K_1$. Applying De Morgan's law again, $\cap_{i \in I} K_i = \emptyset$. Since I is finite, we can let $j = \max(I)$. K_j is nonempty, and K_j is a subset of K_i for any $i \in I$. Therefore the intersection $\cap_{i \in I} K_i$ must contain K_j, which contradicts our earlier statement that that intersection is empty. Since we have reached a contradiction, $\cap_{n \in \mathbb{N}} K_n$ is nonempty.

Exercise 1.5.7: prove theorem 1.5.10.

(a) If Z is closed, then let $\{U_i\}_{i \in I}$ be an open cover of Z. The complement of Z is open, so $\{X - Z\} \cup \{U_i\}_{i \in I}$ is an open cover of Y, which we know is compact. Therefore there exists some finite $I' \subset I$ such that $\{X - Z\} \cup \{U_i\}_{i \in I'}$ is an open cover of Y, which means $\{U_i\}_{i \in I'}$ is a finite subset of $\{U_i\}_{i \in I}$ which still covers Z. That implies that if Z is closed, it is also compact.

If Z is not closed, then there exists some $x \in \partial Z - Z$. Define $U_n = Z - B(x, 1/n)$ for any $n \in \mathbb{N}$, so that $\{U_n\}$ is an open cover of Z. Suppose Z is compact, so there exists a finite set $I \subset \mathbb{N}$ such that $\cup_{i \in I} U_i = Z$. Take $j := \max(I)$, so then $B(x, 1/j) \cap Z = \emptyset$. Since x is an adherent point of Z, for any $\varepsilon > 0$, the intersection $Z \cap B(x, \varepsilon)$ is nonempty, so this is a contradiction. Therefore if Z is closed, Z is also compact.

(b) Let $\{U_i\}_{i \in I}$ be an open cover of $Y_1 \cup \cdots \cup Y_n$. Then there is a finite subset of $\{U_i\}$ which covers Y_1, a finite subset of $\{U_i\}$ which covers Y_2, and so on. Now take the union of all the finite subcovers, and call that new set $\{V_i\}_{i \in I'}$. That new set clearly covers the union of each Y_j, and it's finite, because it's defined as the union of n finite sets. We have found a finite subcover for the union of each Y_j, so that union is compact.

(c) The empty set is vacuously compact, so assume X is finite but nonempty. Let $x_1, x_2, x_3, \ldots$ be a sequence in X. By the pigeonhole principle, there must be some $x \in X$ which occurs infinitely many times in that sequence, so we can take an infinite subsequence just by taking the terms which are exactly x. That subsequence converges to $x \in X$, so X is compact.

Exercise 1.5.8.

Let $e^{(n)}, e^{(m)}$ be any two sequences in $E := \left\{e^{(n)} : n \in \mathbb{N}\right\}$. If $n \neq m$, then those sequences differ by one at exactly two indices, and are equal at every other index, so their distance apart in X is 2, meaning E is bounded.

Also, for any $f \in X - E$, since all elements of E are distance 2 apart, the triangle inequality guarantees there is at most one element of E which is distance $r < 1$ from f. Thus, $B(f, r/2) \cap E = \emptyset$, so $X - E$ is open, so E is closed.

However, E is not compact. Since every element of E is a distance of 2 apart, no sequence in E can be Cauchy, therefore no sequence in E can have a convergent subsequence (because the convergent subsequence would have to be Cauchy).

Exercise 2.1.1: prove theorem 2.1.4.

$\boxed{(a) \Rightarrow (c)}$ For any $V \subset Y$ which contains $f(x_0)$, there exists some $\varepsilon > 0$ such that $B(f(x_0), \varepsilon) \subset V$. Since f is continuous at x_0, there exists δ such that whenever $d(x, x_0) < \delta$, $d(f(x), f(x_0)) < \varepsilon$. Define $U := B(x_0, \delta)$. Then $f(U) \subset B(f(x_0), \varepsilon) \subset V$.

$\boxed{(a) \Rightarrow (b)}$ Suppose $\left(x^{(n)}\right)_{n=1}^{\infty}$ converges to x_0 in X, and $f : X \to Y$ is continuous at x_0. For any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $d(x, x_0) \leq \delta$, $d(f(x), f(x_0)) \leq \varepsilon$. There also exists some $N \in \mathbb{N}$ such that for any $m \geq N$, $d(x^{(m)}, 0) \leq \delta$. We have found some $N \in \mathbb{N}$ for every $\varepsilon > 0$ such that whenever $m \geq N$, $d(f(x^{(m)}), f(x_0)) \leq \varepsilon$, so $f(x^{(m)})$ converges to $f(x_0)$.

$\boxed{(b) \Rightarrow (c)}$

$\boxed{(c) \Rightarrow (a)}$ For any $\varepsilon > 0$, let $V = B(f(x_0), \varepsilon)$. Then there exists some open set $U \subset X$ containing x_0 such that $f(U) \subset V$. Since U is open, there exists $\delta > 0$ such that $B(x_0, \delta) \subset U$. Whenever $d(x, x_0) < \delta$, $d(f(x), f(x_0)) < \varepsilon$, so f is continuous at x_0.

Exercise 2.1.2: prove theorem 2.1.5.

$\boxed{(a) \Rightarrow (b)}$ This is proven in the previous problem.

$\boxed{(b) \Rightarrow (c)}$

$\boxed{(c) \Rightarrow (d)}$

$\boxed{(d) \Rightarrow (a)}$

Exercise 2.3.3. Let f be a uniformly continuous function. Then f is continuous.

Suppose f is uniformly continuous, meaning for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever x, x' are in the domain of f and are distance less than δ apart, $d(f(x), f(x')) < \varepsilon$. For any x_0 in the domain of f, we could let $x' = x_0$. Then whenever $d(x, x_0) < \delta$, $d(f(x), f(x_0)) < \varepsilon$).

Let f be the function on $(0, 1)$ which maps x to $1/x$. The domain and codomain of f are both $\mathbb{R}$ with the Euclidean metric. f is continuous but not uniformly continuous.

Exercise 2.3.4.

For any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $d(y, y') < \delta$, $d(g(y), g(y')) < \varepsilon$. Also, there exists $\gamma > 0$ such that whenever $d(x, x') < \gamma$, $d(f(x), f(x')) < \delta$. Thus, whenever $d(x, x') < \gamma$, $d(g(f(x)), g(f(x'))) < \varepsilon$, so $g \circ f$ is uniformly continuous.

Exercise 2.4.4.

Let Y_1 be one connected component of $f(E)$ which contains $f(x)$, let $Y_2 = f(E) - Y_1$, and assume for the sake of contradiction that Y_2 is nonempty. Then Y_1 and Y_2 are both open in $f(E)$, they are both nonempty, their intersection is empty, and their union is $f(E)$. The preimages $f^{-1}(Y_1)$ and $f^{-1}(Y_2)$ are both open and nonempty, their union is E, and their intersection is empty. This would mean E is disconnected, which is a contradiction, so $f(E)$ must be connected.

Problem 0.2. Complete the proof of case 3 from Theorem 1.5.8.

Case 3 is when $r_0 = \infty$, but that can only occur if $r(y) = \infty$ for every $y \in Y$. That would imply $V_\alpha = X$ for every $\alpha \in A$, in which case for any $a \in A$, the singleton set $\{V_a\}$ covers Y.

Problem 0.3.

(a) Let (X, d_X) and (Y, d_Y) both be $\mathbb{R}$ with the Euclidean metric, and let $U = (0, 1)$. Then let $f(x) = 0$ for all x, so U is open but $f(U) = \{0\}$ is not open.

(b) Once again, let X and Y be $\mathbb{R}$ with the Euclidean metric. Let $K = X$ and let $f(x) = \tanh(x)$, so K is closed but $f(K) = (-1, 1)$ is not closed.

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(1) Exercise: 1.5.6, 1.5.7, 1.5.8, 2.1.1, 2.1.2, 2.3.3, 2.3.4, 2.4.4.
(2) Complete the proof of case 3 of Theorem 1.5.8.
(3) Given following examples.
  (a) A continuous function $f : (X, d_X) \to (Y, d_Y)$ and an open set $U \subset X$ such that $f(U) \subset Y$ is not open.
  (b) A continuous function $f : (X, d_X) \to (Y, d_Y)$ and a closed set $K \subset X$ such that $f(K) \subset Y$ is not closed.