Math 151A

HW #1, due on Friday, October 11, 2024 at 11:59pm PST.

You can use a calculator for the following problems. (If you don't have a hand calculator, Matlab or even Google should suffice.)
Please write your answers clearly. To get full credit you need to show all your work. If you are required to write code, please attach a screenshot of your code and all the outputs and plots to your homework.
Homework should be submitted on Gradescope.

[**1**]

(a) Describe intuitively (i.e., in "plain English") what the Intermediate Value Theorem (IVT) says.

(b) Use the IVT to show that there exists a solution to the nonlinear equation

$$x\cos(x) - 2x^2 + 3x - 1 = 0$$

on the interval $[0.2, 0.3]$.

[**2**]

Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x+1}$ about $x_0 = 0$. Approximate $\sqrt{0.5}$, $\sqrt{0.75}$, $\sqrt{1.25}$, and $\sqrt{1.5}$ using $P_3(x)$ and compute the absolute errors.

[**3**] Perform the following computations (i) $\frac{4}{5} + \frac{1}{3}$ (ii) $\left(\frac{1}{3} + \frac{3}{11}\right) - \frac{3}{20}$

(a) exactly

(b) using three-digit chopping arithmetic

(c) using three-digit rounding arithmetic.

(d) Compute the relative errors in part (b) and (c)

[**4**]

Find an approximation to $\sqrt{3}$ correct to within 10^{-4} using the Bisection Method on [0,2].
You can either do this with a calculator or you can use the code provided in class with the necessary adjustments. (*Hint*: consider $f(x) = x^2 - 3$)
To get full credit you need to show:

1. The list of midpoints given by the bisection method.
2. The number of iterations needed to converge.
3. The approximation you obtain of $\sqrt{3}$.

[**5**]

(a) Show that the sequence $p_n = (\frac{1}{10})^n$ converges linearly (i.e. with order $\alpha = 1$) to $p = 0$.

(b) Show that the sequence $p_n = 10^{-2^n}$ converges quadratically (i.e. with order $\alpha = 2$) to $p = 0$.

[**6**][*Modified notion of order of convergence*]

In this problem we explore an alternative notion of convergence that is more well suited for the Bisection Method. First, recall the definition given in lecture:

Suppose that $(p_n)_n$ is a sequence converging to p with $p_n \neq p$ for all n. If there exists positive constants α and λ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

then we say that the sequence $(p_n)_n$ converges with order α *to p.*

Suppose now that (i) there exists some sequence $(\epsilon_n)_n$ such that

$$|p_n - p| \leq \epsilon_n \qquad \forall n$$

and (ii) $(\epsilon_n)_n$ itself converges to $\epsilon = 0$ with order $\alpha > 0$. Then we say that $(p_n)_n$ converges to p with *at least* order α.

Show that the Bisection Method converges with at least order $\alpha = 1$.

[**7**]

Suppose a numerical method at the k-th step produces an approximation x_k for the root of some continuous function f in the interval $[a_k, b_k]$.
Find the root (the zero) of the line that passes through the points $(a_k, f(a_k))$ and $(b_k, f(b_k))$. (*Hint*: the equation of the line passing through the two given points can be found with 'point-slope' form from precalculus.)

(*Note*: the solution to this problem will be useful in problem [8].)

[8] Computational exercise A common activity of a numerical analyst is to be responsible for learning about new computational techniques and implementing them for comparison purposes. Typically, this involves understanding the concept behind the new technique and then taking an existing program and modifying it to incorporate the new technique.

The purpose of this exercise is to have you follow this process and implement an "improved" variant of the bisection method called the Reguli-Falsi ("False Position") method. Rather than taking the approximate root p_k to be the midpoint of the interval $[a_k, b_k]$ as the Bisection Method does, the Reguli-Falsi method takes p_k to be the zero of the line passing through the points $(a_k, f(a_k))$ and $(b_k, f(b_k)$, as found in exercise [7].

(a) Create a copy of the m-file 'bisect.m' (found on Canvas) and rename it 'falsep.m'. Implement the Reguli-Falsi method by appropriately modifying the file falsep.m.

(b) Compare the Reguli-Falsi method to the bisection method by using both to find the root of $(x-1)(x-2)(x-3)$ with a starting interval of $[1.75, 2.95]$. Stop the iteration when the residual is less than 10^{-6}.

1. How many iterations does each take?
2. What is the approximate root for each method?

(c) Repeat the comparison for the function $x^6 - x - 1$ using an initial interval of both $[1.0, 1.2]$ and $[1.0, 2.0]$. Stop the iteration when the residual is less than 10^{-6}.

1. How many iterations does each take?
2. What is the approximate root for each method?
3. Is the Reguli-Falsi method always better than the bisection method?