For the following discussions, u and v are continuous and differentiable functions of the generalized coordinates and momenta associated with some system...

Poisson Brackets

$$[u,v] \equiv \sum_i \left(\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} - \frac{\partial v}{\partial p_i}\frac{\partial u}{\partial q_i} \right)$$

Properties of the Poisson Bracket

$$[u,v] = -[v,u]$$

$$[q_j,q_k] = [q_k,q_j] = 0$$

$$[p_j,p_k] = [p_k,p_j] = 0$$

$$[p_j,q_k] = [q_k,p_j] = 0 \qquad \text{if } j \neq k$$

$$[p_j,q_j] = -[q_j,p_j] = 1$$

Vocabulary

* quantities u and v that satisfy

$$[u, v] = 0$$

are said to "Commute", as one may change their order within the Poisson bracket without changing the result

* quantities u and v that satisfy

$$[u_i, u_j] = 0$$

$$[v_i, v_j] = 0$$

$$[u_i, v_j] = \delta_{ij}$$

are said to be "Canonically Conjugate"
(thus the quantities p & q are canonically conjugate)

Algebra

$$[u, c] = 0 \qquad (c = \text{Constant})$$

$$[u, v+w] = [u, v] + [u, w]$$

$$[u+v, w] = [u, w] + [v, w]$$

$$[uv, w] = u[v, w] + v[u, w]$$

$$[u, vw] = w[u, v] + v[u, w]$$

Fun and Games

Suppose $u = u(q_1, q_2 \ldots, P_1, P_2, \ldots)$

$$\dot{u} = \Sigma \left(\frac{\partial u}{\partial q_i} \dot{q}_i + \frac{\partial u}{\partial p_i} \dot{P}_i \right)$$ by the chain rule

Now $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $-\dot{P}_i = \frac{\partial H}{\partial q_i}$ Canonical equations of motion

So...

$$\dot{u} = \Sigma \left(\frac{\partial H}{\partial p_i} \frac{\partial u}{\partial q_i} - \frac{\partial u}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

which is to say...

$$\dot{u} = [H, u]$$

* NOTE THAT QUANTITIES THAT COMMUTE WITH THE HAMILTONIAN ARE CONSTANT IN TIME (Conserved)

Now, replace u with q_i & P_i ... The canonical equations of motion can be written

$$\dot{q}_i = [H, q_i]$$
$$\dot{P}_i = [H, P_i]$$

Transformations

This could get ugly - stick close :'

$$u_i = u_i(q_1, q_2, \ldots p_1, p_2 \ldots)$$
$$V_i = V_i(q_1, q_2, \ldots p_1, p_2 \ldots)$$

So $\dot{u}_i = [H, u_i]$ and $\dot{V}_i = [H, V_i]$

Given the manifest symmetry, let's focus on $\dot{u}_i$

$$\dot{u}_i = \sum_j \left(\frac{\partial H}{\partial p_j} \frac{\partial u_i}{\partial q_j} - \frac{\partial u_i}{\partial p_j} \frac{\partial H}{\partial q_j} \right)$$

We can write (Chain-rule)...

$$\frac{\partial H}{\partial p_j} = \sum_K \left(\frac{\partial H}{\partial u_K} \frac{\partial u_K}{\partial p_j} + \frac{\partial H}{\partial V_K} \frac{\partial V_K}{\partial p_j} \right)$$

$$\frac{\partial H}{\partial q_j} = \sum_K \left(\frac{\partial H}{\partial u_K} \frac{\partial u_K}{\partial q_j} + \frac{\partial H}{\partial V_K} \frac{\partial V_K}{\partial q_j} \right)$$

and ...(I'm sorry)... stick this back into $\dot{u}_i$

$$\dot{u}_i = \sum_K \sum_j \left\{ \left(\frac{\partial H}{\partial u_K} \frac{\partial u_K}{\partial p_j} + \frac{\partial H}{\partial v_K} \frac{\partial v_K}{\partial p_j} \right) \frac{\partial u_i}{\partial q_j} - \frac{\partial u_i}{\partial p_j} \left(\frac{\partial H}{\partial u_K} \frac{\partial u_K}{\partial q_j} + \frac{\partial H}{\partial v_K} \frac{\partial v_K}{\partial q_j} \right) \right\}$$

$$\dot{u}_i = \sum_K \frac{\partial H}{\partial u_K} \sum_j \left(\frac{\partial u_K}{\partial p_j} \frac{\partial u_i}{\partial q_j} - \frac{\partial u_i}{\partial p_j} \frac{\partial u_K}{\partial q_j} \right) + \sum_K \frac{\partial H}{\partial v_K} \sum_j \left(\frac{\partial v_K}{\partial p_j} \frac{\partial u_i}{\partial q_j} - \frac{\partial u_i}{\partial p_j} \frac{\partial v_K}{\partial q_j} \right)$$

Check out the quantities in parentheses...

$$\dot{u}_i = \sum_K \frac{\partial H}{\partial u_K} \left[u_K, u_i \right] - \sum_K \frac{\partial H}{\partial v_K} \left[u_i, v_K \right]$$

and by symmetry...

$$\dot{v}_i = \sum_K \frac{\partial H}{\partial u_K} \left[u_K, v_i \right] - \sum_K \frac{\partial H}{\partial v_K} \left[v_i, v_K \right]$$

** If u and v are Canonically-Conjugate,

$$[u_i, u_j] = 0$$
$$[v_i, v_j] = 0$$
$$[u_i, v_j] = \delta_{ij}$$

and...

$$\dot{u}_i = \sum_K \frac{\partial H}{\partial u_K} (0) - \sum_K \frac{\partial H}{\partial v_K} \delta_{iK}$$

$$\dot{v}_i = \sum_K \frac{\partial H}{\partial u_K} \delta_{K,i} - \sum_K \frac{\partial H}{\partial v_K} (0)$$

That is...

$$\dot{u}_i = -\frac{\partial H}{\partial v_i} \qquad\qquad v_i = \frac{\partial H}{\partial u_i}$$

Compare these to

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \qquad\qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

* If u and v are functions of q and p <u>and</u> <u>if they are</u> <u>canonically-conjugate</u>, then they satisfy the canonical equations of motion!

(This sort of transformation into an alternate set of canonically-conjugate variables is known as a "point transformation" or "contact transformation")

* It should probably be pointed out that the canonical equations in commutator form...

$$\dot{u}_i = [H, u_i] \qquad \dot{v}_i = [H, v_i]$$

hold regardless of whether u and v are canonically-conjugate