Applications of Variational Calculus to Physics [1]

Overview:

- **Hamilton's Principle:** "Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies."
- The '**Lagrangian**' is defined to be the difference between the kinetic and potential energies of the system:

$$\mathfrak{L} \equiv T - V$$

- '**Action**' is defined to be the time integral of the Lagrangian:

$$S \equiv \int \mathfrak{L}\, dt$$
$$S = \int (T - V)\, dt$$

- ... and Hamilton's Principle is often shortened to 'Nature follows the path of least action'.

- Minimizing the action integral is precisely the sort of problem the *calculus of variations* was derived to address. If the Lagrangian is written in terms of the coordinates $< q_i >$ and their time-derivatives $< \dot{q}_i >$, then the action is minimized when the corresponding Euler equations (called 'Euler-Lagrange' equations in the context of dynamics) are satisfied:

$$\frac{\partial \mathfrak{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_i}\right) \equiv 0$$

[1] Yet another set in the collection of Corbin's notes.

Advantage:

- Kinetic and potential energies are fundamentally Newtonian concepts and therefore the Lagrangian really needs to be evaluated in an inertial frame of reference.
- The path from the Lagrangian in the action integral to the Euler-Lagrange equations is purely mathematical. It neither contains nor requires any dynamical content.
- The *huge* advantage one gains from using 'Lagrangian Dynamics' (that is, from using the Euler-Lagrange equation to generate equations of motion) is that, having obtained the Lagrangian in some inertial frame of reference, you may transform any (or all) of the initial set of coordinates into *any* other set of convenient coordinates, in any frame (or frames) of reference, inertial or not. The Euler-Lagrange equation will give you valid equations of motion for each of the new *generalized* coordinates.

  To lay the best foundation for understanding what we're doing, let's first use Lagrangian dynamics to explore a handful of well-known examples.

Example 1: A Very Simple 1-D System

$$T = \frac{1}{2}\, m\, \dot{x}^2$$
$$V = V(x)$$

The Lagrangian:

$$\mathfrak{L} = \frac{1}{2}\, m\, \dot{x}^2 - V(x)$$

inserted into the Euler-Lagrange equation:

$$\frac{\partial \mathfrak{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right) \;\equiv\; 0$$

results in:

$$-\frac{\partial V(x)}{\partial x} \;-\; m\, \ddot{x} \;=\; 0$$

Since $\vec{F} = -\vec{\nabla} V$, we can write:

$$\vec{F} \;=\; m\, \ddot{x}$$

Using Lagrangian dynamics, we have recovered Newton's second law.

Example 2: Free Fall

Fumble-fingers, here, has managed to drop some small furry critter from from a great height. Taking x and y in the usual directions:

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$
$$V = mgy$$

$$\mathfrak{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

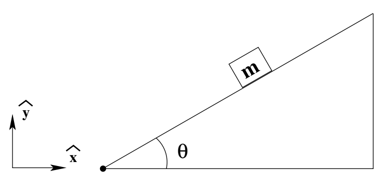
$$\frac{\partial \mathfrak{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right) \equiv 0 \qquad \frac{\partial \mathfrak{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{y}}\right) \equiv 0$$

$$-m\ddot{x} = 0 \qquad -mg - m\ddot{y} = 0$$

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

No surprises here (I hope!).

Example 3: Object on a Plane (Version 1)



Let's see how the classic inclined-plane problem from introductory mechanics holds up Lagrangian dynamics.

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$
$$V = mgy$$

$$\mathfrak{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$$\frac{\partial \mathfrak{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right) \equiv 0 \qquad \frac{\partial \mathfrak{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{y}}\right) \equiv 0$$

$$-m\ddot{x} = 0 \qquad - mg - m\ddot{y} = 0$$

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

If I didn't know any better, I'd swear the block was in free-fall! What happened?

The answer is sort of subtle. The Lagrangian for the block implies that the block is free to move, without constraints, in two dimensions (we say there are two '*degrees of freedom*' to the system). The presence of the plane says otherwise! Since the block is constrained to move along the line $y = x \tan\theta$, there is really only *one* degree of freedom. The existence of the constraint means that the value of x, once known, determines the value of y. There is only one coordinate in which the system can move freely, without constraint; the other coordinate is forced to a value by the constraint.

Re-write the Lagrangian to reflect the fact that there is only one degree of freedom.

$$\begin{aligned} T &= \frac{1}{2}\, m\, (\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}\, m\, \dot{x}^2\, (1 + \tan^2\theta) \\ &= \frac{1}{2}\, m\, \dot{x}^2 \sec^2\theta \end{aligned}$$

$$\begin{aligned} V &= mgy \\ &= mgx\tan\theta \end{aligned}$$

$$\mathfrak{L} = \frac{1}{2}\, m\, \dot{x}^2 \sec^2\theta - mgx\tan\theta$$

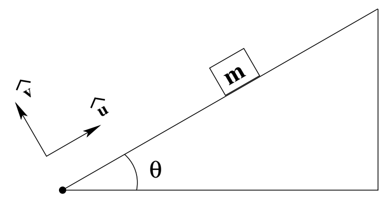
$$\frac{\partial \mathfrak{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right) \equiv 0$$

$$-mg\tan\theta - m\ddot{x}\sec^2\theta = 0$$

$$\ddot{x} = -g\sin\theta\cos\theta$$

Stare at this for a second or two, and it's obvious that this is the familiar solution projected onto less-familiar coordinates.

Example 4: Object on a Plane (Version 2)



We could have been smarter with our setup. Orient the u-axis up the plane and the v-axis perpendicular to it. Take advantage of the constraint $v = 0$.

$$T = \frac{1}{2}\, m\, \dot{u}^2$$
$$V = mgu \sin\theta$$

$$\mathfrak{L} = \frac{1}{2}\, m\, \dot{u}^2 - mgu \sin\theta$$

$$\frac{\partial \mathfrak{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{u}}\right) \equiv 0$$

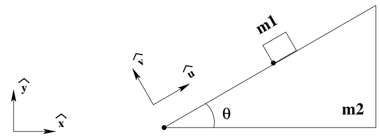
$$-mg \sin\theta - m\ddot{u} = 0$$

$$\ddot{u} = -g \sin\theta$$

That's the boring result you probably expected to see.

Let's finish this set with an example that would probably be a little intimidating in Newtonian dynamics.

Example 5: Object on a Sliding Plane



Now, a small box of mass m_1 is free to slide down a wedge of mass m_2 that is allowed to slide freely over a horizontal table. We'll track their locations using the base of the ramp and the lowest corner of the block.

The Lagrangian needs to be set up in an inertial frame. The u, v-frame (attached to the base of the ramp) is most assuredly *not* inertial (once the wedge starts to accelerate) so ignore it for now.

$$\begin{aligned} T &= \frac{1}{2}\, m_1 \,({\dot{x}_1}^2 + {\dot{y}_1}^2) \;+\; \frac{1}{2}\, m_2 \, {\dot{x}_2}^2 \\ V &= m_1 g y_1 \end{aligned}$$

$$\mathfrak{L} = \frac{1}{2}\, m_1 \,({\dot{x}_1}^2 + {\dot{y}_1}^2) \;+\; \frac{1}{2}\, m_2 \, {\dot{x}_2}^2 \;-\; m_1 g y_1$$

With the Lagrangian (derived in an inertial frame) in hand, we're free, now, to shift to more convenient (generalized) coordinates.

$$\begin{aligned} x_1 &= x_2 + u\cos\theta \\ y_1 &= u\sin\theta \end{aligned}$$

and the Lagrangian becomes...

$$\mathfrak{L} = \frac{1}{2} m_1 (\dot{x}_2^2 + \dot{u}^2 + 2\,\dot{x}_2 \dot{u}\cos\theta) + \frac{1}{2} m_2 \dot{x}_2^2 - m_1 g u \sin\theta$$

Note there are *two* degrees of freedom (one associated with x_2 and another associated with u).

$$\frac{\partial \mathfrak{L}}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x_2}}\right) \equiv 0 \qquad \frac{\partial \mathfrak{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{u}}\right) \equiv 0$$

$$0 - \frac{d}{dt}(m_1\dot{x}_2 + m_1\dot{u}\cos\theta + m_2\dot{x}_2) = 0 \qquad -m_1 g\sin\theta - \frac{d}{dt}(m_1\dot{u} + m_1\dot{x}_2\cos\theta) = 0$$

$$m_1\ddot{x}_2 + m_1\ddot{u}\cos\theta + m_2\ddot{x}_2 = 0 \qquad -m_1 g\sin\theta - m_1\ddot{u} - m_1\ddot{x}_2\cos\theta = 0$$

A quick spell of algebra and we obtain...

$$\begin{aligned}
\ddot{x}_2 &= \frac{m_1 g\sin\theta\cos\theta}{m_1\sin^2\theta + m_2} \\
\ddot{u} &= \frac{-(m_1 + m_2)\, g\sin\theta}{m_1\sin^2\theta + m_2}
\end{aligned}$$

which is easy enough to verify in the limits $m_2 \gg m_1$ and $m_1 \gg m_2$.