MATH 131B Homework #5

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Problem 0.1. Exercise 3.7.1

Since the derivatives f'_n converge uniformly to g, for any $\varepsilon > 0$, there exists some $N' \in \mathbb{N}$ such that $|f'_n(x) - g(x)| < \varepsilon/(3(b-a))$ for any $x \in [a,b]$ and any $n \geq N'$. There also exists $N'' \in \mathbb{N}$ such that $|L - f_n(x_0)| < \varepsilon/3$ whenever $n \geq N''$. Define $N := \max(N', N'')$.

Now I want to show that this will imply $|f(x) - f_n(x)| \leq \varepsilon$.

$$\begin{aligned}
|f(x) - f_n(x)| &= \left| L - \int_{[a,x_0]} g + \int_{[a,x]} g - f_n(x) \right| \\
&= \left| L + \left(\int_{[a,x]} g - \int_{[a,x_0]} g - f_n(x) \right) \right| \\
&= \left| L + \left(\int_{[a,x]} g - \int_{[a,x_0]} g - f_n(x_0) + \int_{[a,x_0]} f'_n - \int_{[a,x]} f'_n \right) \right| \\
&\leq |L - f_n(x_0)| + \left| \int_{[a,x_0]} (g - f'_n) \right| + \left| \int_{[a,x]} (g - f'_n) \right| \\
&\leq \frac{\varepsilon}{3} + \int_{[a,x_0]} \left| \frac{\varepsilon}{3(b-a)} \right| + \int_{[a,x]} \left| \frac{\varepsilon}{3(b-a)} \right| \\
&= \frac{\varepsilon}{3} + (x_0 - a) \left| \frac{\varepsilon}{3(b-a)} \right| + (x - a) \left| \frac{\varepsilon}{3(b-a)} \right| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Therefore f_n converges uniformly to f. The last thing we need to show is that f is differentiable and $f' = g$. Since $f(x)$ is defined as a constant plus $\int_{[a,x]} g$, theorem 11.9.1 from Analysis 1 says that f is differentiable and $f' = g$.

This does not contradict example 1.2.10 from Analysis 1. If we define that function to be

$$f_n(x) = \frac{x^3}{x^2 + \varepsilon^2}, \varepsilon = \frac{1}{n},$$

then the derivatives f'_n do not converge uniformly, so the hypotheses for theorem 3.7.1 are not met.

Problem 0.2. Exercise 3.7.3

Define g_n to be the partial sum $\sum_{k=1}^n f_k$. If n is finite, the derivative g'_n is equal to $\sum_{k=1}^n f'_k$. By the Weierstrass M-test, g'_n converges uniformly to some function h as $n \to \infty$, and we are given that $g_\infty(x_0)$ exists. So by theorem 3.7.1, g_n converges uniformly to a differentiable function g_∞, and $g'_\infty = h$. In other words,

$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x).$$

Problem 0.3. Exercise 4.1.1

(a) If $|x-a| > R$, then
$$\frac{1}{|x-a|} < \frac{1}{R} := \limsup_{n\to\infty} |c_n|^{1/n},$$
meaning there are infinitely many natural numbers n such that $|c_n|^{1/n} > 1/|x-a|$. Therefore there are infinitely many n such that $|c_n(x-a)^n| > 1^n = 1$, so the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is divergent.

(b) If $|x-a| < R$, then
$$\frac{1}{|x-a|} > \frac{1}{R} := \limsup_{n\to\infty} |c_n|^{1/n},$$
so if we define $r = \limsup_{n\to\infty}\left(|c_n|^{1/n}|x-a|\right)$, then $r \in [0,1)$ and there exists some $N \in \mathbb{N}$ such that $|c_n| \cdot |x-a|^n \leq r^n$ whenever $n \geq N$. Therefore
$$\begin{aligned}
\sum_{n=0}^{\infty} c_n(x-a)^n &= \sum_{n=0}^{N-1} c_n(x-a)^n + \sum_{n=N}^{\infty} c_n(x-a)^n \\
&= \text{some finite number } + \sum_{n=N}^{\infty} c_n(x-a)^n \\
&\leq \text{some finite number } + \sum_{n=N}^{\infty} |c_n| \cdot |x-a|^n \\
&\leq \text{some finite number } + \sum_{n=N}^{\infty} r^n,
\end{aligned}$$
which converges to another finite number.

(c) Let $m = \limsup_{n\to\infty}\left(|c_n|^{1/n} r^n\right) \in [0,1)$. Then by the difinition of a limit supremum, there exists $N' \in \mathbb{N}$ such that $m \geq |c_n|^{1/n} r$ (and therefore, $m^n \leq |c_n| r^n$) for any $n \geq N$. For any $\varepsilon > 0$, let $N'' = \lceil \log_m(1-m) \rceil$, and let $N = \max(N', N'')$. Now, for any $n \geq N$ and any $x \in [a-r, a+r]$,
$$\begin{aligned}
\left| f(x) - \sum_{k=0}^{n} c_k(x-a)^k \right| &\leq \sum_{k=n+1}^{\infty} |c_k| \cdot |x-a|^k \\
&\leq \sum_{k=n+1}^{\infty} m^k \\
&= \frac{m^{n+1}}{1-m} \\
&\leq \varepsilon.
\end{aligned}$$
Therefore $\sum_{k=0}^{n} c_k(x-a)^k$ converges to $f(x)$ as n goes to infinity, and since the choice of N did not depend on x, this convergence is uniform. The metric space of bounded continuous functions with the supremum norm is complete (and the partial sums are all bounded, continuous functions), so f is also continuous on $[a-r, a+r]$. Since this works for any $r \in (0,R)$, f is continuous on $(a-R, a+R)$.

(d) By the formula for the radius of convergence, $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ also has radius of convergence R.

For any $r \in (0,R)$, and any $n \in \{0\} \cup \mathbb{N}$, let $f_n(x) = c_n(x-a)^n$. Then f_n is continuously differentiable, $f_n'(x) = nc_n(x-a)^{n-1}$, and the partial sums of $f_n'(x)$ converge uniformly. By theorem 3.7.1, the partial sums of f_n converge to a differentiable function (which of course, is f), and the derivative of f is $\sum_{n=0}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

Since this method works for any $r \in (0, R)$, f is also differentiable on $(a-R, a+R)$.

(e) Once again, let $f_n(x) = c_n(x-a)^n$. For any $[y,z] \subset (a-R, a+R)$ (assume $y < z$), we have shown $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent. Corollary 3.6.2 says that

$$\begin{aligned}
\int_{[y,z]} f = \int_{[y,z]} \sum_{n=0}^{\infty} f_n(x) &= \sum_{n=0}^{\infty} \int_{[y,z]} f_n(x) \\
&= \sum_{n=0}^{\infty} \int_{[y,z]} c_n(x-a)^n \\
&= \sum_{n=0}^{\infty} \left[c_n \frac{(x-a)^{n+1}}{n+1} \right]_{x=y}^{x=z} \\
&= \sum_{n=0}^{\infty} \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}.
\end{aligned}$$

Problem 0.4. Exercise 4.2.3

Base case: If $k = 0$, this is true, because anything can be differentiated zero times, and the 0th derivative of any function of itself. Proposition 4.2.6 is true whenever $k = 0$, because

$$f^{(0)}(x) = \sum_{n=0}^{\infty} c_{n+0} \frac{(n+0)!}{n!} (x-a)^n = f(x).$$

Inductive step: Suppose proposition 4.2.6 is true when $k = k'$. Let

$$g(x) = f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!} (x-a)^n$$

for all $x \in (a-r, a+r)$. Then by theorem 4.1.6.d,

$$\begin{aligned}
g'(x) &= \frac{d}{dx} \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!} (x-a)^n \\
&= \sum_{n=1}^{\infty} nc_{n+k} \frac{(n+k)!}{n!} (x-a)^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1) c_{n+k+1} \frac{(n+k+1)!}{(n+1)!} (x-a)^n \\
&= \sum_{n=0}^{\infty} c_{n+k+1} \frac{(n+k+1)!}{n!} (x-a)^n \\
&= f^{(k+1)}(x)
\end{aligned}$$

for any $x \in (a-r, a+r)$. Therefore proposition 4.2.6 is also true when $k = k'+1$.

Conclusion: By induction, proposition 4.2.6 is true for any integer $k \geq 0$.

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(1) Exercise: 3.7.1, 3.7.3, 4.1.1, 4.2.3.