Math 151A
HW #2, due on Friday, October 18, 2024 at 11:59pm PST.

You can use a calculator for the following problems. (If you don't have a hand calculator, Matlab or even Google should suffice.)
Please write your answers clearly. To get full credit you need to show all your work. If you are required to write code, please attach your code and all the outputs and plots to your homework.
Homework should be submitted on Gradescope.

[**1**] Use algebraic manipulations to show that the functions

$$g_1(x) = (3 + x - 2x^2)^{1/4}$$

$$g_2(x) = \left(\frac{x + 3 - x^4}{2}\right)^{1/2}$$

have a fixed point at p precisely when $f(p) = 0$, where

$$f(x) = x^4 + 2x^2 - x - 3.$$

(i.e. you need to show that $f(p) = 0 \iff g_i(p) = p$ for $i = 1, 2$)

[**2**]

(a) Perform 4 iterations, if possible, on the functions g_1 and g_2 defined in exercise [1] using fixed point iteration ($p_{n+1} = g(p_n)$) with $p_0 = 1$. Report the result of each ieration.
You can either do it by hand or by implementing the fixed point iteration in MATLAB (or any other programming language).

(b) Which function do you think gives the best approximation to the solution? The real roots of f are given by

$$r_1 = 1.124123..., \quad r_2 = -0.876053...$$

[**3**] Use the Theorem on Existence and Uniqueness of Fixed Points from the class notes to show that there *exists* a *unique* fixed point to the function $g(x) = \pi + 0.5\sin(x/2)$ on the interval $[0, 2\pi]$.

Note: Problems [4] and [5] can be either solved by hand or by implementing the methods in MATLAB (or any other programming language).

[**4**] Let $f(x) = -x^3 - \cos(x)$ and $p_0 = -1$. Use Newton's method to find p_2. Report the result of each iteration and the residual. Could $p_0 = 0$ be used?

[**5**] Let $f(x) = -x^3 - \cos(x)$, $p_0 = -1$ and $p_1 = 0$. Use the secant method to find p_3. Report the result of each iteration and the residual.

[**6**] Let $p \in [a, b]$ be the root of $f \in C^1([a, b])$, and assume $f'(p) \neq f'(p_0)$ for some $p_0 \in [a, b]$. Consider an iteration scheme that is similar to, but different from Newton's method: given p_0, define

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_0)}, \qquad n \geq 0.$$

Assuming that the iterative scheme converges, i.e. that $p_n \to p$ as $n \to \infty$, show that this method has order of convergence $\alpha = 1$.
(*Hint:* Let $g(x) = x - \frac{f(x)}{f'(p_0)}$ and use similar steps as in the proof or the order of convergence of FPI in Lecture 5.)

[7] **Computational exercise**
The function $f(x) = x + \cos(x)$ has a zero $p \approx -0.7390851332$.

(a) Let $p_0 = -5$ and $p_1 = 5$. Compute an approximated solution using (1) Newton's method, (2) Secant method and (3) the method defined in Problem [6]. Stop the iteration when $|f(x)| < 10^{-10}$. You can use as a starting point for your code the file Newt.m available on BruinLearn. Please attach your code. Report:

   (a) The number of iterations needed by each method to achieve the desired tolerance.

   (b) The approximated solution.

   (c) The residual $|f(x)|$.

   Comment on your results. Which method converges faster? Why does this happen?

(b) Repeat the experiment but now use $p_0 = -0.9$ and $p_1 = 5$.
Do you see any changes? Why does this happen?