MATH 131B Homework #3

Nathan Solomon

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Problem 0.1. 3.1.2. Prove proposition 3.1.5.

$\boxed{(a) \Leftrightarrow (b)}$ By the definitions of limits and of convergence, both (a) and (b) are equivalent to the statement that for any $\varepsilon/ > 0$, there exists $\delta > 0$ such that whenever $d(x, x_0) < \delta, x \in E$, $d(f(x), L) < \epsilon$.

$\boxed{(b) \Rightarrow (c)}$

Problem 0.2. 3.1.5

This question is phrased weirdly, so I will assume that E is a subset of X, and that the statement we want to prove is $\lim_{x \to x_0, x \in E} g \circ f(x) = z_0$ (instead of $\lim_{x \in x_0, x \in E} g \circ f(x) = z_0$).

For any $\varepsilon > 0$, there exists $\delta > 0$ such that $B(y_0, \delta) \subset g^{-1}(B(z_0, \varepsilon))$, and there also exists $\gamma > 0$ such that $B(x_0, \gamma) \subset f^{-1}(B(y_0, \delta))$. We have found a positive γ such that $B(x_0, \gamma) \subset (g \circ f)^{-1}(B(z_0, \varepsilon))$, so $\lim_{x \to x_0, x \in E} g \circ f(x) = z_0$.

Problem 0.3. 3.2.1

Part (a). If f is continuous, then at each $x_0 \in \mathbb{R}$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $d(x, x_0) < \delta$, $d(f(x), f(x_0)) < \varepsilon$. If a_n converges to zero, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $d(a_n, 0) < \delta$. So for any $n \geq N$, $f_{a_n}(x_0) = f(x_0 - a_n)$ which is equal to $f(x)$ for some x such that $d(x, x_0) < \delta$. That implies $d(f(x), f(x_0)) < \varepsilon$, so $d(f_{a_n}(x_0), f(x_0)) < \varepsilon$. Since this works at any $x_0 \in \mathbb{R}$, the shifted functions f_{a_n} converge pointwise to f.

If the shifted functions converge pointwise to f (for ANY sequence a_n which converges to 0), then suppose for the sake of contradiction that f is not continuous. That would mean there is $x_0 \in X, \varepsilon > 0$ such that there is no $\delta > 0$ for which $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$. Then for each $n \in \mathbb{N}$, let a_n be a number in $B(x_0, \delta)$ such that $d(f(x_0), f(x_0 - a)) \geq \varepsilon$. a_n is a sequence which converges to zero, but for which $f_{a_n}(x_0)$ does not converge to zero.

Part (b). If f is continuous, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that at any x_0, whenever $d(x, x_0) < \delta$, $d(f(x), f(x_0)) < \varepsilon$. If a_n converges to zero, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $d(a_n, 0) < \delta$. So for any $n \geq N$, $f_{a_n}(x_0) = f(x_0 - a_n)$ which is equal to $f(x)$ for some x such that $d(x, x_0) < \delta$. That implies $d(f(x), f(x_0)) < \varepsilon$, so $d(f_{a_n}(x_0), f(x_0)) < \varepsilon$. Therefore the shifted functions f_{a_n} converge uniformly to f.

If the shifted functions f_{a_n} converge uniformly to f (for any sequence a_n which converges to 0), then suppose for the sake of contradiction that f is not uniformly continuous. That would mean there exists some $\varepsilon > 0$ such that there is no $\delta > 0$ such that whenever $d(x_1, x_2) < \delta$, $d(f(x_1), f(x_2)) < \varepsilon$. That means there exists some $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, there exist $x_1^{(n)}, x_2^{(n)} \in X$ satisfying $d(x_1^{(n)}, x_2^{(n)}) < \frac{1}{n}$ and $d(f(x_1^{(n)}), f(x_2^{(n)})) \geq \varepsilon$. Let $a_n = x_1^{(n)} - x_2^{(n)}$. Then for any n, $d(f(x_1^{(n)}), f_{a_n}(x_1^{(n)})) \geq \varepsilon$, so the shifted functions do not converge uniformly to f, which contradicts our assumption.

Problem 0.4. 3.2.2.ab

Part (a). If $f^{(n)}$ converges uniformly to f, then for any $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for any $n \geq N(\varepsilon)$ and any $x \in X$, $d(f_n(x), f(x)) < \varepsilon$. So for any $\varepsilon > 0$ and any $x_0 \in X$, let $N(\varepsilon, x_0)$ be equal to that same $N(\varepsilon)$. Then for any $n \geq N(\varepsilon, x_0)$, $d(f_n(x_0), f(x_0)) < \varepsilon$, so f_n converges pointwise to f.

Part (b). For any $x_0 \in (-1, 1)$ and any $\varepsilon > 0$, let $N = \lceil \log \varepsilon / \log |x_0| \rceil$. Then for any $n \geq N$,

$$\begin{aligned}
\left| f^{(n)}(x_0) \right| &= |x_0|^n \\
&= |x_0|^{\lceil \log \varepsilon / \log |x_0| \rceil} \\
&\leq |x_0|^{\log \varepsilon / \log |x_0|} \\
&= \exp\left(\frac{\log |x_0| \log \varepsilon}{\log |x_0|} \right) \\
&= \varepsilon \\
d(f^{(n)}(x_0), 0) &\leq \varepsilon.
\end{aligned}$$

Therefore $f^{(n)}$ converges pointwise to the zero function.

Now define $f^{(n)'}$ so that the domain is $(-1, 1]$ (and it also maps x to x^n). These converge to a function $f' : (-1, 1] \to \mathbb{R}$ which maps 1 to 1 and every other point to zero. However, that convergence is not uniform, because a sequence of continuous functions cannot converge uniformly to a discontinuous function. Since the convergence is not uniform, there exists some $\varepsilon > 0$ such that for any $\delta > 0$, for arbitrarily large n, you can find $x_1, x_2 \in (-1, 1]$ such that $x_1 < x_2$, $d(x_1, x_2) < \delta$, and $d(f^{(n)'}(x_1), f^{(n)'}(x_2)) > 2\varepsilon$. Since $f^{(n)'}$ is continuous, you could also find x_3 (for arbitrarily large n) such that $x_1 < x_3 < x_2$ and $d(f^{(n)'}(x_2), f^{(n)'}(x_3)) < \varepsilon$, so by the triangle inequality $d(f^{(n)'}(x_1), f^{(n)'}(x_3)) > \varepsilon$. x_1 and x_3 are both in $(-1, 1)$, so even if we restricted the domain to $(-1, 1)$, this would still prove that the sequence of functions doesn't converge uniformly to f.

Problem 0.5. 3.2.4

We are given that f is bounded, meaning there exists $r \in \mathbb{R}$ such that for any $x \in X$, $d(f(x), y_0) < r$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$ and any $x \in X$, $d(f_n(x), f(x)) < \varepsilon$. The triangle inequality says that

$$\begin{aligned}
d(f_n(x), y_0) &\leq d(f_n(x), f(x)) + d(f(x), y_0) \\
&\leq \varepsilon + r.
\end{aligned}$$

So if we define $R := r + \varepsilon$, then $f_n(x) \in B(y_0, R)$ whenever $x \in X$ and $n \in \mathbb{N}$. We know that finitely many balls of finite radius can be enclosed in another ball of finite radius, and we are given that every f_n is bounded, so there exists a ball which encloses $B(y_0, R)$ as well as $f_1(X), f_2(X), \ldots, f_{n-1}(X)$.

Problem 0.6. 3.3.1. Prove theorem 3.3.1 (optional: use prove and use proposition 3.3.3).

For any $\varepsilon > 0$, since the funcitons converge uniformly to f, there exists $N \in \mathbb{N}$ such that for any $n \geq N$ and any $x \in X$, $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$. Since f_N is continuous at x_0, there exists $\delta > 0$ such that for any x a distance less than δ away from x_0, $d(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3}$. The triangle inequality says

$$\begin{aligned}
d(f(x), f(x_0)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This works for any x a distance less that δ away from x_0, so f is continuous at x_0.

If the convergence was pointwise instead of uniform, this reasoning would not work, because then the choice of N would have to depend on what x is. But x is determined after δ, and δ is determined after N.

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(1) Exercise: 3.1.2, 3.1.5, 3.2.1, (a) and (b) of 3.2.2, 3.2.4, 3.3.1.