Math 132 Homework #1

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Problem 0.1. Chapter I, section 1, exercise 3

Using the facts that $|s|^2 = s\overline{s}$ and $2\Re(s) = s + \overline{s}$ for any $s \in \mathbb{C}$, the equation can be simplified as follows:

$$\begin{aligned}
|z|^2 - 2\Re(\overline{a}z) + |a|^2 &= \rho^2 \\
z\overline{z} - (\overline{a}z + a\overline{z}) + a\overline{a} &= \rho^2 \\
(z-a)(\overline{z} - \overline{a}) &= \rho^2 \\
|z-a|^2 &= \rho^2 \\
d(z,a) &= \rho.
\end{aligned}$$

That last line is equivalent to saying z lies in the circle of radius $\rho \geq 0$ centered at $a \in \mathbb{C}$.

Problem 0.2. Chapter I, section 1, exercise 4

Let $a, b \in \mathbb{R}$ be the unique numbers such that $z = a + bi$. Then the inequality $|z| \leq |\Re z| + |\Im z|$ can be rewritten as:

$$\begin{aligned}
|a + bi| &\leq |a| + |b| \\
\sqrt{a^2 + b^2} &\leq |a| + |b| \\
a^2 + b^2 &\leq a^2 + b^2 + 2|a||b| \\
0 &\leq |2ab|.
\end{aligned}$$

That inequality is clearly true, and both sides are equal iff $a = 0$ or $b = 0$. If you sketch the set of points $z \in \mathbb{C}$ for which equality holds, it will look like a plus-sign centered at the origin (that is, the union of the real and imaginary axes).

Problem 0.3. Chapter I, section 2, exercise 8

Those theorems are true for any $\theta \in \mathbb{C}$, but my proofs below only work when $\theta \in \mathbb{R}$.

$$\begin{aligned}
\cos(2\theta) &= \Re(\exp(2i\theta)) \\
&= \Re(\exp(2i\theta)^2) \\
&= \Re\left((\cos\theta + i\sin\theta)^2\right) \\
&= \Re(\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta) \\
&= \cos^2\theta - \sin^2\theta. \\
\sin(2\theta) &= \Im(\exp(2i\theta)) \\
&= \Im(\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta) \\
&= 2\cos\theta\sin\theta. \\
\cos(4\theta) &= \Re\left(\exp(4i\theta)\right) \\
&= \Re\left((\cos\theta + i\sin\theta)^4\right) \\
&= \Re\left(\cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta\right) \\
&= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta. \\
\sin(4\theta) &= \Im\left(\exp(4i\theta)\right) \\
&= \Im\left((\cos\theta + i\sin\theta)^4\right) \\
&= \Im\left(\cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta\right) \\
&= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.
\end{aligned}$$

In general, the identities $\cos(x) = \Re(\exp(ix))$ and $\sin(x) = \Im(\exp(ix))$ only work when $x \in \mathbb{R}$. If you wanted these proofs to also work when $\theta \in \mathbb{C}$, you'd have to instead use the identities $\cos(x) = (e^{ix} + e^{-ix})/2$ and $\sin(x) = (e^{ix} - e^{-ix})/(2i)$.

Problem 0.4. Chapter I, section 3, exercise 4

If you take a point $a + bi$ on the complex plane, then the corresponding point on the Riemann sphere is (X, Y, Z), where

$$\begin{aligned}
X &= \frac{2a}{|a+bi|^2 + 1} \\
Y &= \frac{2b}{|a+bi|^2 + 1} \\
Z &= \frac{|a+bi|^2 - 1}{|a+bi|^2 + 1}.
\end{aligned}$$

Rotating that sphere by 180 degrees about the X axis maps it to $(X', Y', Z') = (X, -Y, -Z)$:

$$\begin{aligned}
X' &= \frac{2a}{|a+bi|^2 + 1} \\
Y' &= \frac{-2b}{|a+bi|^2 + 1} \\
Z' &= \frac{1 - |a+bi|^2}{|a+bi|^2 + 1}.
\end{aligned}$$

For that new point on the Riemann sphere, the corresponding value of t' (that is, the t defined in I.3 of the textbook) is $t' = 1/(1 - Z')$:

$$t' = \frac{1}{1 - Z'} = \frac{1}{\frac{|a+bi|^2+1}{|a+bi|^2+1} - \frac{1-|a+bi|^2}{|a+bi|^2+1}} = \frac{|a+bi|^2 + 1}{2\,|a+bi|^2}.$$

So after rotating the Riemann sphere, the number on the complex plane which corresponds to the new point is $a' + b'i$, where

$$\begin{aligned} a' = t'X' &= \frac{a}{|a+bi|^2} \\ b' = t'Y' &= \frac{-b}{|a+bi|^2} \\ a' + b'i &= \frac{\overline{a+bi}}{|a+bi|^2} = \frac{1}{a+bi}. \end{aligned}$$

Therefore, taking the multiplicative inverse of a point on the complex plane is equivalent to mapping it onto the Riemann sphere, rotating 180 degrees around the X-axis, then mapping back to the complex plane.

Problem 0.5. Chapter I, section 4, exercise 3

The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = w = z^3$ can be visualized as the function which takes a complex number, cubes its magnitude, and triples its argument. As a point z rotates about the origin, $f(z)$ rotates about the origin in the same direction at 3 times the speed.

Define A_1 to be the subset of C containing 0 and all points with principal argument in $(-\pi/3, \pi/3]$. Similarly, let A_2 be the region with zero and all points whose principal argument is in $(\pi/3, \pi]$, and let A_3 be the region with zero and all numbers whose principal argument is in $(-\pi, -\pi/3]$. Then there are three branch cuts: $f_i : \mathbb{C} \to A_i$, for $i \in \{1, 2, 3\}$. They can be defined as

$$\begin{aligned} f_1(w) &= |w|^{1/3} \exp\left(i \operatorname{Arg}(w)/3\right) \\ f_2(w) &= |w|^{1/3} \exp\left(i \operatorname{Arg}(w)/3 - 2\pi i/3\right) \\ f_3(w) &= |w|^{1/3} \exp\left(i \operatorname{Arg}(w)/3 + 2\pi i/3\right) \\ f_1(0) = f_2(0) = f_3(0) = 0. \end{aligned}$$

The Riemann surface is all of $\mathbb{C}$.

Problem 0.6. Chapter I, section 5, exercise 3

Let a and b be the real and imaginary components of z, respectively, so

$$\begin{aligned} e^{\overline{z}} &= e^{\overline{a+bi}} \\ &= e^{a-bi} \\ &= e^a(\cos(-b) + i\sin(-b)) \\ &= e^a(\cos b - i \sin b) \\ &= e^a \cdot \overline{(\cos b + i \sin b)} \\ &= \overline{e^a} \cdot \overline{e^{ib}} \\ &= \overline{e^a e^{ib}} \\ &= \overline{e^z}. \end{aligned}$$

Problem 0.7. Chapter I, section 5, exercise 4

Let $a = \Re(\lambda)$ and $b = \Im(\lambda)$, so $\lambda = a + bi$. Multiplying both sides by e^{-z} gives

$$1 = e^z e^{-z} = e^{z+\lambda} e^{-z} = e^\lambda = e^a e^{ib}.$$

The magnitude of that equation is

$$1 = |e^a| \left|e^{ib}\right| = |e^a| = e^a,$$

so dividing both sides by $e^a = 1$ gives

$$1 + 0i = e^{ib} = \cos b + i \sin b,$$

which implies $\cos(b) = 1$, so b is an integer multiple of 2π. The condition $e^a = 1$ is equivalent to $a = 0$, so $\lambda = a + bi$ is an integer multiple of $2\pi i$.

Homework Assignment 1

MATH 132 LEC 1&2

Due April 6th, Sunday 11:59 PM

Please submit your work to Gradescope!

- I.1 Exercises: #3, #4,
- I.2 Exercises: #8,
- I.3 Exercises: #4,
- I.4 Exercises: #3,
- I.5 Exercises: #3, #4.