This one ended up being really hard for them and way too long. I guess they hadn't all heard of perturbation theory etc. Shorten this one up.

Phys 245 Quantum Computation
Homework 6

1. *[31] Neutral atom tweezers as a QHO*. Neutral atom tweezers function by shining a very red detuned laser onto an atom. In class, we derived the Stark shift due to such a laser as $E = -\frac{\hbar|\Omega|^2}{4\delta}$, where δ is the detuning from resonance. For a laser focused on an atom the Rabi frequency can be written as $\Omega = dE/\hbar$, where d is the atomic dipole moment (just a constant and a property of the atom) and E is the electric field of the laser. The intensity of the laser can be found from the electric field as $I = \frac{1}{2}c\epsilon_o|E|^2$. Assuming at the focus of the laser beam the electric field of the laser is described by a Gaussian beam of the form $E = E_o e^{-\frac{x^2+y^2}{w_o}} e^{\iota(kz-\omega t+\phi)}\hat{x}$. Do the following:
   a. [10] Find the Stark shift on the atom
   b. [10] Approximate the result from part (a) to describe the confining potential as a harmonic oscillator and identify the trap frequency ω.
   c. [10] Assume that $d = e\, a_o$ (e is the electron charge and a_o the Bohr radius), the laser power is 100 mW, the spot size is $w_o = 10\ \mu$m, and the detuning is $\delta = 10$ THz, what is ω?
   d. [1] What is $\hbar\omega$ in temperature units?

2. *[40] An anharmonic oscillator.* Suppose a harmonic oscillator with frequency ω is modified such that its potential goes from $V = \frac{1}{2}m\omega^2x^2 \rightarrow$ V = $\frac{1}{2}m\omega^2x^2 + \lambda x^4$. For this problem let $\omega = 1$ and $\lambda = 0.01$.
   a. [10] Use perturbation theory to calculate the first-order correction to the states $|0\rangle$ and $|1\rangle$.
   b. [15] Use perturbation theory to calculate the second-order correction to the energy. For this problem you'll have to deal with the problem of having an infinite basis. You'll need to choose a maximum n at which to truncate the basis. I recommend doing this step numerically and then exploring the effect of changing n_{max} on your final energy.
   c. [15] Instead of using perturbation theory, write the Hamiltonian as an $n_{max} \times n_{max}$ matrix and diagonalize it to find the eigenenergies – tip: QuTip has a built-in function for finding eigenvalues. For the same n_{max} do the methods of part (b) and part (c) agree or disagree?

3. [80] *Composite quantum systems.* In this exercise, we'll try out our new tools for dealing with composite quantum systems.

a. [20] For a system composed of two qubits with qubit one in the state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and qubit two in the state $|\psi_2\rangle = |0\rangle$, do the following:
   i. [5] Write down the total state vector.
   ii. [5] From the composite state vector, find the expectation value of the $\hat{\sigma}_z^{(1)}$, i.e. the expectation value of $\hat{\sigma}_z$ for qubit 1.
   iii. [5] From the composite state vector, find the expectation value of the $\hat{\sigma}_z^{(2)}$, i.e. the expectation value of $\hat{\sigma}_z$ for qubit 2.
   iv. [5] From the composite state vector, find the expectation value of $\hat{\sigma}_z^{(1)} \otimes \hat{\sigma}_z^{(2)}$.

b. [15] For a system composed of two qubits with qubit one in the state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and qubit two in the state $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, do the following:
   i. [5] Write down the total state vector.
   ii. [5] From the composite state vector, find the expectation value of $\hat{\sigma}_z^{(1)} \otimes \hat{\sigma}_z^{(2)}$.
   iii. [5] From the composite state vector, find the expectation value of $\hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)}$.

c. [10] Assume that the total state vector is $|\psi_{Tot}\rangle = \frac{1}{\sqrt{2}}(\,|00\rangle + |11\rangle)$ and do the following:
   i. [5] From the composite state vector, find the expectation value of $\hat{\sigma}_z^{(1)} \otimes \hat{\sigma}_z^{(2)}$.
   ii. [5] From the composite state vector, find the expectation value of $\hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)}$.

d. [15] Write down the total statevector for a general three qubit system. That is, starting with $|\psi_1\rangle = (a_1|0\rangle + b_1|1\rangle)$, $|\psi_2\rangle = (a_2|0\rangle + b_2|1\rangle)$, and $|\psi_3\rangle = (a_3|0\rangle + b_3|1\rangle)$ find the total statevector in the composite Hilbert space.

e. [20] For a system composed of a qubit in the state $|\psi\rangle = (a_1|0\rangle + b_1|1\rangle)$ and a quantum harmonic oscillator in a superposition of Fock states $|\phi\rangle = \frac{1}{\sqrt{3}}(\,|0\rangle + |1\rangle + |3\rangle)$, do the following:
   i. [5] Write down the total state vector.
   ii. [5] From the composite statevector, find the expectation value of $\hat{\sigma}_z$ for the qubit.
   iii. [5] From the composite statevector, find the expectation value of $\hat{N}$ for the harmonic oscillator.
   iv. [5] From the composite statevector, what is the probability of finding the system with qubit in state $|1\rangle$ and the harmonic oscillator in state $|3\rangle$?

4. [70] *Jayne says!* The Jaynes-Cummings Hamiltonian is:

$$H/\hbar = \frac{\omega_o}{2}\hat{\sigma}_z + \omega\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right) + \frac{g}{2}(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger)$$

Implement the Jaynes-Cummings Hamiltonian in QuTip and use it to find the time dynamics of the following situations. For simplicity assume $\omega_o = 2\pi$, and $g = \omega_o/100$. Since you cannot use an infinite dimensioned Hilbert space in QuTip you'll need to truncate the basis at some maximum $|n\rangle$. Make sure and choose that maximum n large enough to not affect your answer.

a. [50] For $\omega = \omega_o$ find the time evolution of the population in the initial state for the following initial states over one period of oscillation:
   i. [10] $|\psi(\mathrm{t}=0)\rangle = |00\rangle$ -- that is the qubit in the excited state and the harmonic oscillator in its ground state
   ii. [10] $|\psi(\mathrm{t}=0)\rangle = |10\rangle$ -- that is the qubit in the ground state and the harmonic oscillator in its ground state
   iii. [10] $|\psi(\mathrm{t}=0)\rangle = |01\rangle$ -- that is the qubit in the excited state and the harmonic oscillator in its first excited state
   iv. [10] $|\psi(\mathrm{t}=0)\rangle = |05\rangle$ -- that is the qubit in the excited state and the harmonic oscillator in its fifth excited state
   v. [10] $|\psi(\mathrm{t}=0)\rangle = |16\rangle$ -- that is the qubit in the excited state and the harmonic oscillator in its fifth excited state

b. [20] For $\omega = 1.03\,\omega_o$ find the time evolution of the population in the initial state for the following initial states over one period of oscillation:
   i. [10] $|\psi(\mathrm{t}=0)\rangle = |05\rangle$ -- that is the qubit in the excited state and the harmonic oscillator in its fifth excited state
   ii. [10] $|\psi(\mathrm{t}=0)\rangle = |16\rangle$ -- that is the qubit in the excited state and the harmonic oscillator in its fifth excited state