Physics 231B Homework #6

Nathan Solomon

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Most of the facts about manifolds we discussed and used, and things about the fundamental group can be found in Nakahara's book "Geometry, Topology, and Physics". The structure theorems on Lie groups and algebras can be found in Kirillov Ch 3 and Hall Ch 5.

Problem 0.1. Suppose G is a closed subgroup of $GL(n, \mathbb{R})$, and $\gamma(t) : (-1, 1) \to G$ is a smooth map with $\gamma(0) = I$. Show that

$$\frac{d}{dt}\gamma(t)|_{t=0} \in \mathfrak{g},$$

that is, that if $A = \frac{d}{dt}\gamma(t)|_{t=0}$, then $\exp(tA) \in G$ for all t. (This gives an identification of the tangent space of G at I with its Lie algebra.)

Problem 0.2. On the last homework you constructed a map $SU(2) \times SU(2) \to SO(4)$. Now show explicitly a Lie algebra isomorphism between $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{so}(4)$.

Problem 0.3.

- (a) Classify the representations of $SO(4)$ in terms of representations of $SU(2) \times SU(2)$.
- (b) What $SU(2) \times SU(2)$ representation corresponds to the vector representation of $SO(4)$?
- (c) (bonus) What $SO(4)$ representation corresponds to the $SU(2) \times SU(2)$ representation $(1, 0)$?

Problem 0.4. There is an invertible $\mathbb{R}$-linear map $F : \mathbb{C}^n \to \mathbb{R}^{2n}$ given by

$$f(x_1 + iy_1, \ldots, x_n + iy_n) = (x_1, \ldots, x_n, y_1, \ldots, y_n).$$

- (a) Prove that $F(iz) = JF(z)$ where J is the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

- (b) Prove that a $2n \times 2n$ real matrix N is the image of an $n \times n$ complex matrix M if and only if $NJ = JN$.
- (c) Given such a pair of matrices (M, N) as above, show that there is a basis of $\mathbb{C}^n$ where M has real entries if and only if there is a $2n \times 2n$ real matrix T (called a real structure) satisfying $T^2 = I$, $TJ = -JT$, and $TN = NT$.

Problem 0.5. A representation (of a group of Lie algebra) on $\mathbb{C}^n$ has a real structure if there is a matrix T as above commuting with each of the representation matrices. In this case, problem 3 implies that these matrices are real in a certain basis, and define also a real representation.

- (a) Show that the spin 1 representation of $SU(2)$ has a real structure but the spin 1/2 representation does not.
- (b) Show that the spin-1/2 representation has a "pseudoreal structure", meaning a matrix T as above satisfying $T^2 = -1$ instead of $T^2 = 1$.

Problem 0.6. Any real Lie algebra $\mathfrak{g}$ has an associated complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ (called its complexification) whose elements are complex linear combinations of elements of $\mathfrak{g}$. We extend the Lie bracket on $\mathfrak{g}$ to $\mathfrak{g}^{\mathbb{C}}$ by linearity: $[iA, B] := i[A, B]$, $[iA, iB] := -[A, B]$ for $A, B \in \mathfrak{g}$.

- (a) Show that for each n, the real Lie algebras $\mathfrak{so}(p, q)$ for $p + q = n$ all have isomorphic complexifications.
- (b) For $n = 4$, use this to construct the Lorentz algebra $\mathfrak{so}(3, 1)$ matrices acting on a 4-vector.