Hamiltonian Dynamics (TL;DR)[1]

The Set-Up:

- 1) Start in an inertial frame of reference. Write out the Lagrangian for the system ($\mathfrak{L}$).

- 2) Transform coordinates specified in the inertial frame ("*inertial coordinates*") to more convenient "*generalized coordinates*" (q_i).

- 3) Find the "*generalized momentum*" associated with each generalized coordinate:
$$p_i \equiv \frac{\partial \mathfrak{L}}{\partial \dot{q}_i}$$

- 4) Write out the "pre-Hamiltonian":[2]
$$H = \sum_i p_i \dot{q}_i - \mathfrak{L}$$

- 5) Obtain the *actual* Hamiltonian from the pre-Hamiltonian by replacing the $\dot{q}_i$'s with appropriate expressions in p_i. [3]

- 6) The "*canonical equations of motion*" are obtained from: [4]
$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$-\dot{p}_i = \frac{\partial H}{\partial q_i}$$

[1] Another installment in the ever-growing collection of Corbin's notes.

[2] The actual Hamiltonian cannot have any $\dot{q}_i$'s in it.

[3] If *i*) the equations that transform the inertial coordinates into generalized coordinates are independent of time **and** *ii*) the potential energy is independent of velocity, **then** H will be equivalent to the total mechanical energy of the system expressed in terms of q_i and p_i (but not $\dot{q}_i$).

[4] p_i and q_i, so related through H by the canonical equations of motion, are said to be "*canonically conjugate*".

Poisson Brackets

- Defined: If u and v are functions of the q_i and p_i,

$$[u, v] \equiv \sum_i \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} - \frac{\partial v}{\partial p_i}\frac{\partial u}{\partial q_i}$$

- Algebra: If u, v and w are functions of the q_i and p_i and c is some constant:

$$[u, c] = 0$$

$$\begin{aligned}
[u, v] &= -[v, u] \\
[u, v + w] &= [u, v] + [u, w] \\
[u + v, w] &= [u, w] + [v, w] \\
[uv, w] &= u[v, w] + v[u, w] \\
[u, vw] &= v[u, w] + w[u, v]
\end{aligned}$$

- Evolution in Time: If u is a function of the q_i and p_i:[5]

$$\dot{u} = [H, u]$$

- Conserved Quantities: Quantities that commute with the Hamiltonian[6] are constant in time.

[5] This is true for any and all u that are functions of q_i and p_i.
[6] $[H, u] = [u, H] = 0$

Canonical Conjugates:

- When applied to generalized coordinates and momenta:

$$[q_i, q_j] = 0$$
$$[p_i, p_j] = 0$$
$$[p_i, q_j] = \delta_{i,j}$$

- Any set of u_i and v_i that satisfy:

$$[u_i, u_j] = 0$$
$$[v_i, v_j] = 0$$
$$[v_i, u_j] = \delta_{i,j}$$

  are said to be "*canonically-conjugate*".[7]

- Canonically-conjugate u_i and v_i satisfy the canonical equations of motion:

$$\dot{u}_i = \frac{\partial H}{\partial v_i}$$
$$-\dot{v}_i = \frac{\partial H}{\partial u_i}$$

- Transformation into an alternate set of canonically-conjugate variables is known as a "*point transformation*" or "*contact transformation*".

[7]...thus the generalized coordinates and generalized momenta, which - not surprisingly, satisfy the canonical equations of motion - are canonically-conjugate.