Math 115B Homework #5

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Problem 0.1.

Problem 0.2.

(a)

(b)

Problem 0.3.

Problem 0.4.

(a)

(b)

(c)

(d)

Problem 0.5.

Problem 0.6.

(a)

(b)

(c)

Problem 0.7.

(a)

(b)

(c)

(d)

Problem 0.8.

Math 115B: Linear Algebra
Homework 5
Due: Thursday, February 13th at 11:59pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k. All inner product spaces are defined over a field F which is either $\mathbb{R}$ or $\mathbb{C}$.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. $(\frac{-}{5})$ Give an example of an inner product space V and a linear operator (endomorphism) $T : V \to V$ such that the kernel of T and the kernel of T^* are not equal.
2. $(\frac{-}{10+10})$ Let V be a finite dimensional inner product space, and let W be a subspace.
   (a) Prove $V = W \oplus W^\perp$.
   (b) Show that if T is a projection on W along $W^\perp$, then $T = T^*$.
3. $(\frac{-}{5})$ Let T be a linear operator on an inner product space V. Prove that $||T(\vec{v})|| = ||\vec{v}||$ for all $\vec{v} \in V$ if and only if $\langle \vec{v}, \vec{w} \rangle = \langle T(\vec{v}), T(\vec{w}) \rangle$ for all $\vec{v}, \vec{w} \in V$.
4. $(\frac{-}{4*5})$ For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither.
   (a) $V = \mathbb{R}^2$ with the standard inner product, $T(\begin{pmatrix} x \\ y \end{pmatrix}) = (\begin{pmatrix} 2x - 2y \\ -2x + 5y \end{pmatrix})$
   (b) $V = \mathbb{C}^2$ with the standard inner product, $T(\begin{pmatrix} x \\ y \end{pmatrix}) = (\begin{pmatrix} 2x + iy \\ x + 2y \end{pmatrix})$
   (c) $V = \mathbb{R}[x]_{\leq 2}, T(f) = f'$, where $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$.
   (d) $V = \mathbb{R}^{2\times 2}$ and $T(M) = M^t$, where $\langle A, B \rangle = \text{trace}(B^*A)$.
5. $(\frac{-}{10})$ Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.
6. $(\frac{-}{3*5})$ Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}(T + T^*) \text{ and } T_2 = \frac{1}{2i}(T - T^*).$$

(a) Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.

(b) Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.

(c) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.

7. $(\frac{-}{5*4})$ Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

   (a) If T is self-adjoint, then $T|_W$ is self-adjoint.

   (b) $W^\perp$ is T^*-invariant.

   (c) If W is both T- and T^*-invariant, then $(T|_W)^* = (T^*)|_W$.

   (d) If W is both T- and T^*-invariant and T is normal then $T|_W$ is normal.

8. $(\frac{-}{5})$ Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T^*-invariant. (This problem is a bit harder so the low point total is designed to allow you to skip it without hurting your grade too much.)