Math 110AH Homework 9

Nathan Solomon

December 9, 2023

Assignment due December 6th at 11:59 pm.

1

Let $H \subset G = \mathbb{Z} \times \mathbb{Z}$ be the cyclic group generated by $(2, 4)$. Is the quotient group G/H isomorphic to $\mathbb{Z}$? (Hint: Consider elements of finite order of G/H.)

No. Let $x = (1, 2)$ and consider the coset $x + H$ in G/H. Since x is not in H, xH is not the identity in G/H. However, $2x$ is in H, so $2x + H$ is the identity in G/H. Therefore the coset $x + H$ has order 2 in G/H. We know that $\mathbb{Z}$ has no elements of finite order, so it is not isomorphic to G/H.

2

Determine all subgroups of the alternating group A_4.

By going through all elements methodically and seeing what subgroups they generate, we get the following list of subgroups:

Order	Subgroup of A_4
1	$\{e\}$
2	$\langle(12)(34)\rangle$
2	$\langle(13)(24)\rangle$
2	$\langle(14)(23)\rangle$
3	$\langle(123)\rangle$
3	$\langle(124)\rangle$
3	$\langle(134)\rangle$
3	$\langle(234)\rangle$
4	$\{e, (12)(34), (13)(24), (14)(23)\}$
12	A_4

3

> Let $g \in S_n$ be an odd element.
>
> - (a) Show that the map $f_n : A_n \to A_n$ given by $f_n(x) = gxg^{-1}$, is an automorphism of A_n.
> - (b) Prove that the automorphism f_n of A_n is not inner for $n \geq 3$.

In order to define the alternating group, we used the sign operator sgn from the symmetric group to the multiplicative group $\{-1, 1\}$. The sign operator is uniquely defined (and well defined) by the fact that it is a homomorphism and the fact that it maps every transposition to -1.

- (a) We already know that conjugation by g is an automorphism on S_n, so we just need to show that it maps elements of A_n to elements of A_n. Using the fact that sgn is a homomorphism and the fact that g is odd,

  $$\operatorname{sgn}(f_n(x)) = \operatorname{sgn}(g)\operatorname{sgn}(x)\operatorname{sgn}(g^{-1}) = (-1)\operatorname{sgn}(x)(-1) = \operatorname{sgn}(x).$$

  This shows that if x is an even permutation, then $f_n(x)$ is too, so f_n is an automorphism on A_n.

- (b)

4

> Show that the group $\mathbb{Q}/\mathbb{Z}$ cannot be generated by a finite set of elements.

Suppose $\mathbb{Q}/\mathbb{Z}$ is generated by a finite set of elements. Call those elements $a_1, a_2, \ldots a_n$. Each a_i is rational, so it can be written as a fraction with denominator $b_i \in \mathbb{N}$. Then $b_i \cdot a_i$ is an integer, so every a_i has finite order.

For convenience, assume b_i is the smallest natural number such that $b_i \cdot a_i$ is an integer. In other words, the fractions have been reduced so that the numerator and denominator are coprime, and the denominator b_i is the order of a_i in $\mathbb{Q}/\mathbb{Z}$.

Since $\mathbb{Q}/\mathbb{Z}$ is an abelian group generated by $a_1, a_2 \ldots a_n$, every element can be written as $a_1^{c_1} a_2^{c_2} \cdots a_n^{c_n}$ for some set of integers $c_1, c_2, \ldots, c_n$. But because every a_i has finite order b_i, we can write that element in a unique way, by assuming $0 \leq c_i < b_i$. Writing each element this way makes it clear that there are only a finite number of elements in $\mathbb{Q}/\mathbb{Z}$.

However, $\mathbb{Q}/\mathbb{Z}$ has infinitely many elements, so this is a contradiction. Therefore $\mathbb{Q}/\mathbb{Z}$ cannot be generated by finitely many elements.

5

Let G be an (additively written) abelian group. An element $a \in G$ is called *torsion* if $na = 0$ for some integer $n > 0$.

- (a) Prove that the set G_{tors} of all torsion elements in G is a subgroup of G.
- (b) Determine $(\mathbb{R}/\mathbb{Z})_{tors}$.
- (c) Determine $(\mathbb{Q}^{\times})_{tors}$.

- (a) For any elements $a, b \in G_{tors}$, let n_a and n_b be positive integers such that $n_a a = 0 = n_b b$. Then $\mathrm{lcm}(n_a n_b)(a + b) = 0$, so $a + b$ is in the torsion group. If $n_a = 1$ then $a = 0$, so $-a = 0$ is also in the torsion group. Otherwise, $n_a(n_a - 1)(-a) = n_a a = 0$, so $-a$ is in the torsion group. Because G_{tors} is closed under multiplication and inversion, it is a subgroup of G.
- (b) If x is a rational number in that group, let a, b be integers such that $b > 0$ and $x = a/b$. Then $bx = 0$, so x is in the torsion group. If x is an irrational number, there is integer n such that nx is an integer, and therefore $nx \neq 0$, so x is not in the torsion group. Therefore
$$(\mathbb{R}/\mathbb{Z})_{tors} = (\mathbb{Q}/\mathbb{Z})$$
- (c) Suppose a/b is an element of that group, where $0 \leq a < b$ and $b > 0$. Then there exists a natural number n such that a^n/b^n is an integer. This is only possible if b^n divides a^n, meaning every number appears in the prime factorization of a^n at least as many times as it appears in the prime factorization of b^n. Of course, that can only be true if every number appears in the prime factorization of a at least as many times as it appears in the prime factorization of b. This is equivalent to saying b divides a, so $a/b =$ an integer $= 0$. Therefore $(\mathbb{Q}^{\times})_{tors}$ is the trivial subgroup.

6

Prove that A_n is generated by all n-cycles if n is odd.

7

Describe all conjugacy classes in A_4 and S_4.

8

A *commutator* of G is an element of the form $xyx^{-1}y^{-1}$ where $x, y \in G$. Let G' be the subgroup of G generated by all commutators. We call G' the *commutator subgroup* of G. Show all the following are true.

- (a) G' is normal in G.
- (b) G/G' is abelian.
- (c) If N is a normal subgroup of G and G/N is abelian then $G' \subset N$.

9

A group G is called *perfect* if the commutator subgroup G' coincides with G. Find all n such that the alternating group A_n is perfect.

10

Let N be a normal subgroup of G and let K be a subgroup of G such that the restriction $K \to G/N$ of the canonical homomorphism $G \to G/N$ is an isomorphism. Prove that G is a semidirect product of N and K.