Math 180 Homework 8

Nathan Solomon

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1

Consider a random graph G on $3n$ vertices where each edge is chosen independently with probability $\frac{1}{2}$. Show that the probability that G contains a triangle (i.e. K_3) goes to 1 as $n \to \infty$.

Let a be any vertex of G. Since there are $3n-1$ other vertices, the probability that the $\deg(a) = y$ is $\binom{3n-1}{y} 2^{1-3n}$. If a is connected to y vertices, the chance that there is at least one edge in the subgraph made of those y vertices is $1 - (\frac{1}{2})^{\binom{y}{2}}$. Therefore, the chance that G contains a triangle which contains a is

$$\sum_{y=2}^{3n-1} \left(\binom{3n-1}{y} 2^{1-3n} \left(1 - \frac{1}{2^{y(y-1)/2}} \right) \right),$$

and the chance that G contains a triangle is even higher. We can evaluate the limit of that sum as $n \to \infty$ by noting that if we ignore the terms for which $y << n$, then the sum becomes

$$\sum_{y > \varepsilon n}^{3n-1} \left(2^{1-3n} \binom{3n-1}{y} \cdot 1 \right) \approx \sum_{y=0}^{3n-1} \left(2^{1-3n} \binom{3n-1}{y} \right) = 1.$$

Alternate method: show that if G has $n > 2$ vertices and at least $n^2/4$ edges, then it must contain a triangle, and the probability it has at least $n^2/4$ edges approaches 1 as $n \to \infty$.

2

Section 10.3, Problem 7. (Markov inequality) Let X be a random variable on some probability space attaining nonnegative values only. Let $\mu = \mathbb{E}[X]$ be its expectation, and let $t \geq 1$ be a real number. Prove that the probability that X attains a value $\geq t\mu$ is at most $\frac{1}{t}$; in symbols,

$$P\left(\{\omega \in \Omega : X(\omega) \geq t\mu\}\right) \leq \frac{1}{t}.$$

(This is a simple but quite important inequality. It is often used if we want to show that the probability of some quantity getting too big is small.)

Assume that (Ω, P) is a finite probability space. If it's not, we can just replace the sum here with an integral.

$$\begin{aligned}
\mu = \mathbb{E}[X] &= \sum_{\omega \in \Omega} X(\omega) P(\omega) \\
&= \left(\sum_{\omega \in \Omega, X(\omega) \geq t\mu} X(\omega) P(\omega) \right) + \left(\sum_{\omega \in \Omega, X(\omega) < t\mu} X(\omega) P(\omega) \right)
\end{aligned}$$

The first term in parentheses must be at least $t\mu P(X(\omega) \geq t\mu)$ and the second term in parentheses is nonnegative, so

$$\mu \leq t\mu P\left(\{\omega \in \Omega : X(\omega) \geq t\mu\}\right) \leq \frac{1}{t}$$

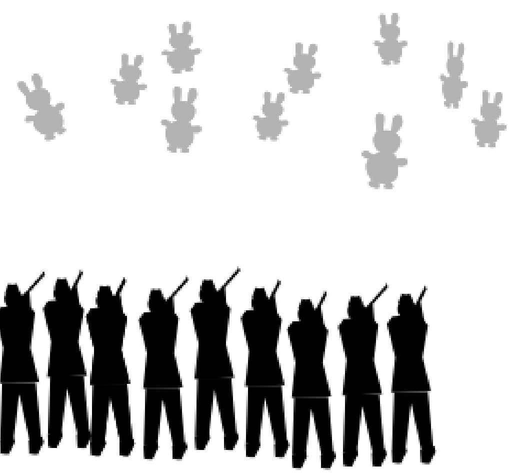
which means

$$P\left(\{\omega \in \Omega : X(\omega) \geq t\mu\}\right) \leq \frac{1}{t}.$$

3

Section 10.3, Problem 8(a). What is the expected number of surviving rabbits in Example 10.3.3 if there are m rabbits and n hunters?

10.3.3 Example (Number of surviving rabbits). Each of n hunters selects a rabbit at random from a group of n rabbits, aims a gun at it, and then all the hunters shoot at once. (We feel sorry for the rabbits but this is what really happens sometimes.) A random variable f_2 is the number of rabbits that survive (assuming that no hunter misses). Formally, the probability space here is the set of all mappings $\alpha\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$, each of them having the probability n^{-n}, and $f_2(\alpha) = |\{1,2,\ldots,n\} \setminus \alpha(\{1,2,\ldots,n\})|$.



Each rabbit has a $\frac{1}{m}$ chance of being aimed at by each of the n numters. Since every hunter aims independently, the chance of a rabbit being aimed at by no hunter is

$$\left(1 - \frac{1}{m}\right)^n$$

so the expected number of rabbits that survives is approximately

$$m \cdot \left(1 - \frac{1}{m}\right)^n.$$

4

Given a graph $G = (V, E)$, the MAXCUT problem is the problem of finding a partition of the vertices into disjoint subsets A, B such that the number of edges with one endpoint in A and other endpoint in B is maximized. Show that there is a choice of A, B with at least half the edges of G going between them as follows.

1. Choose a random subset A of V. You can do this in many ways, but it is easier if you do it in a way that maximizes independence.
2. For each edge e, consider the indicator random variable for the event that e has an endpoint in each part. Use linearity of expectation to compute the expected number of edges between A and B.
3. Use the pigeonhole principle for expectation and the probabilistic method.

1. Let A be a random subset $A \subset V$ such that $|A| = |V|/2$ (or $|A| = (|V| - 1)/2$, if $|V|$ is odd). Let $B = V \backslash A$.
2. Let $f : E \to \{0, 1\}$ be the function such that $f(e) = 1$ if and only if e connects a vertex in A to a vertex in B. The probability of f being 1 is $\mathbb{E}(f) = \frac{1}{2}$. *This is not quite true if $|V|$ is odd, but we can dumb this down by pretending $|V|$ is even, so the expected value of f is one half. In reality, that probability is $\frac{2|A| \cdot |B|}{|V|^2}$, which could be a tiny bit less than one half. Part (3) of this proof would still work, but it would be a lot more painful.* By linearity of expectation, the expected number of edges between A and B is $\frac{|E|}{2}$.
3. Either every choice of A gives exactly $\frac{|E|}{2}$ edges between A and B, or there are some choices of A for which the number of edges between A and B is less than $\frac{|E|}{2}$ and some choices of A for which it is more.