Math 115B: Linear Algebra
Homework 3
Due: Thursday, January 31 at 8pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. ($\frac{-}{10}$) In class, we proved the Cayley-Hamilton theorem for matrices. Let A be a 2×2 diagonalizable matrix. Prove the statement of the Cayley-Hamilton theorem directly, using the fact that $A = QDQ^{-1}$ for some invertible $Q \in k^{2\times 2}$ and some diagonal $D \in k^{2\times 2}$.

2. ($\frac{-}{4*3}$) For each linear endomorphism T on the vector space V find an ordered basis for the T-cyclic subspace generated by the vector $\vec{v}$.

   (a) $V = \mathbb{R}^4$, $T(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} w+x \\ x-y \\ w+y \\ w+z \end{pmatrix}, \vec{v} = \vec{e}_1$

   (b) $V = \mathbb{R}[x]_{\leq 3}$, $T(f(x)) = f''(x)$, $\vec{v} = x^2$

   (c) $V = k^{2\times 2}, T(A) = A^T, \vec{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   (d) $V = k^{2\times 2}, T(A) = L_{\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}(A), \vec{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3. ($\frac{-}{4*2}$) For each linear operator T and cyclic subspace W in the previous problem, compute the characteristic polynomial of $T|_W$.

4. ($\frac{-}{2*5}$) Let V and W be non-zero finite dimensional k-vector spaces and let $T : V \to W$ be a linear transformation.

   (a) Prove that T is onto (i.e. surjective) if and only if T^* is one-to-one (i.e. injective).

   (b) Prove that T^* is onto (i.e. surjective) if and only if T is one-to-one (i.e. injective).

5. $(\frac{-}{10})$ Fix some $d \in \mathbb{Z}^{\geq 1}$ and some scalars $a_0, ..., a_{d-1} \in k$. Let A denote the $d \times d$ matrix

$$\begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & ... & 0 & -a_1 \\ 0 & 1 & ... & 0 & -a_2 \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & ... & 0 & -a_{d-2} \\ 0 & 0 & ... & 1 & -a_{d-1} \end{pmatrix}$$

Prove that the characteristic polynomial of A is $(-1)^d(a_0 + a_1 t + ... + a_{d-1}t^{d-1} + t^d)$. (*Hint:* use induction on d, expanding the determinant along the first row.)

6. $(\frac{-}{2*10})$ Let T be a linear endomorphism of a finite dimensional vector space V.

   (a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

   (b) Deduce that if the characteristic polynomial of T splits, then any nonzero T-invariant subspace of V contains an eigenvector of T.

7. $(\frac{-}{2*5})$ Let T be a linear operator on a finite dimensional vector space V, and let W be a T-invariant subspace of V.

   (a) Suppose that $\vec{v}_1, \vec{v}_2, ..., \vec{v}_d$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $\sum_{i=1}^{d} \vec{v}_i$ is in W, then $\vec{v}_i \in W$ for all $i \in \{1, 2, ..., d\}$. (*Hint:* Induct on d.)

   (b) Suppose that $\dim(V) = n$ and T has n distinct eigenvalues. Prove that V itself is a T-cyclic subspace. (*Hint:* Use the previous part to find a vector $\vec{v}$ such that $\{\vec{v}, T(\vec{v}), ..., T^{n-1}(\vec{v})\}$ is linearly independent.)

8. $(\frac{-}{10})$ Prove that the restriction of a diagonalizable linear operator T to any non-trivial T-invariant subspace is also diagonalizable. (*Hint:* Use the first part of the previous problem.)

9. $(\frac{-}{10})$ Let $A \in k^{n \times n}$ for some $n \in \mathbb{Z}^{\geq 0}$. Prove that $\dim(\text{span}\{I_n, A, A^2, A^3, ...\}) \leq n$.