Math 115B: Linear Algebra
Homework 5
Due: Thursday, February 13th at 11:59pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k. All inner product spaces are defined over a field F which is either $\mathbb{R}$ or $\mathbb{C}$.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. ($\frac{-}{5}$) Give an example of an inner product space V and a linear operator (endomorphism) $T : V \to V$ such that the kernel of T and the kernel of T^* are not equal.
2. ($\frac{-}{10+10}$) Let V be a finite dimensional inner product space, and let W be a subspace.
   (a) Prove $V = W \oplus W^\perp$.
   (b) Show that if T is a projection on W along $W^\perp$, then $T = T^*$.
3. ($\frac{-}{5}$) Let T be a linear operator on an inner product space V. Prove that $||T(\vec{v})|| = ||\vec{v}||$ for all $\vec{v} \in V$ if and only if $\langle \vec{v}, \vec{w} \rangle = \langle T(\vec{v}), T(\vec{w}) \rangle$ for all $\vec{v}, \vec{w} \in V$.
4. ($\frac{-}{4*5}$) For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither.
   (a) $V = \mathbb{R}^2$ with the standard inner product, $T(\begin{pmatrix} x \\ y \end{pmatrix}) = (\begin{pmatrix} 2x - 2y \\ -2x + 5y \end{pmatrix})$
   (b) $V = \mathbb{C}^2$ with the standard inner product, $T(\begin{pmatrix} x \\ y \end{pmatrix}) = (\begin{pmatrix} 2x + iy \\ x + 2y \end{pmatrix})$
   (c) $V = \mathbb{R}[x]_{\leq 2}, T(f) = f'$, where $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$.
   (d) $V = \mathbb{R}^{2\times 2}$ and $T(M) = M^t$, where $\langle A, B \rangle = \text{trace}(B^* A)$.
5. ($\frac{-}{10}$) Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.
6. ($\frac{-}{3*5}$) Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}(T + T^*) \text{ and } T_2 = \frac{1}{2i}(T - T^*).$$

(a) Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.

(b) Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.

(c) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.

7. ($\frac{-}{5*4}$) Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

   (a) If T is self-adjoint, then $T|_W$ is self-adjoint.

   (b) $W^\perp$ is T^*-invariant.

   (c) If W is both T- and T^*-invariant, then $(T|_W)^* = (T^*)|_W$.

   (d) If W is both T- and T^*-invariant and T is normal then $T|_W$ is normal.

8. ($\frac{-}{5}$) Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T^*-invariant. (This problem is a bit harder so the low point total is designed to allow you to skip it without hurting your grade too much.)