Math 110BH homework 5

Nathan Solomon

February 6, 2024

1

Show that over any field the exist infinitely many non-associate irreducible polynomials.

2

Prove that the factor ring $\mathbb{Z}[i]/(1+i)\mathbb{Z}[i]$ is a field of two elements.

3

Let $f, g \in \mathbb{Q}[X]$ with $fg \in \mathbb{Z}[X]$. Prove that there is $a \in \mathbb{Q}^\times$ such that $af \in \mathbb{Z}[X]$ and $a^{-1}g \in \mathbb{Z}[X]$.

4

Let F be a field. Prove that the set R of all polynomials in $F[X]$ whose X-coefficient is equal to 0 is a subring of $F[X]$ and that R is not a UFD. (Hint: Use $X^6 = (X^2)^3 = (X^3)^2$.)

5

Find all irreducible polynomials of degree ≤ 4 in $(\mathbb{Z}/2\mathbb{Z})[X]$.

6

Let $f \in \mathbb{Z}[X], a, b \in \mathbb{Z}, a \neq b$. Prove that $a - b$ divides $f(a) - f(b)$. (Hint: $a - b$ divides $a^n - b^n$.)

7

Prove that $X^n + Y^n - 1$ is irreducible in $\mathbb{Z}[X, Y]$ for every $n > 0$. (Hint: Use Eisenstein's Criterion.)

8

Let f be a monic polynomial in $\mathbb{Z}[X]$. Prove that if $a \in \mathbb{Q}$ is a root of f then $a \in \mathbb{Z}$.

9

Find all roots of $f = X^p - X$ in $(\mathbb{Z}/p\mathbb{Z})[X]$ (p prime) and factor f into a product of irreducible polynomials. (Hint: Use Fermat's Little Theorem.)

10

Determine whether $X^4 + 4$ is irreducible in $\mathbb{Z}[X]$.