Math 151A
HW #5, due on Friday, November 15, 2024 at 11:59pm PST.

[1]
Let $f(x) = \sin(2\pi x)$ and let $[a, b] = [0, 1]$.

(a) Construct a piecewise-linear polynomial that interpolates f at $\{x_0, x_1, x_2\} = \{0, 1/2, 1\}$. Let's call this object $P_{1,2}$.

(b) Construct a piecewise linear polynomial that interpolates f at $\{x_0, x_1, x_2, x_3, x_4\} = \{0, 1/4, 1/2, 3/4, 1\}$. Let's call this object $P_{1,4}$.

(c) Draw a graph (by hand, or if you'd like, with MATLAB) for $x \in [0, 1]$ of:

i) $f(x)$

ii) the answer to part (a)

iii) the answer to part (b)

iv) a piecewise-linear polynomial that interpolates f at $x = \{0, 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 1\}$ (no need to derive a formula). Let's call this last object $P_{1,8}$.

(d) What is the value of $\lim_{n\to+\infty} |f(x) - P_{1,n}|$ and why?

[2] Suppose that $f(x)$ is a polynomial of degree 3. Show that $f(x)$ is it's own clamped cubic spline, but that it cannot be its own natural cubic spline.

[3] In this problem we will work with the data:

x	$f(x)$
0.1	-0.29004996
0.2	-0.56079734
0.3	-0.81401972

These values correspond to the function $f(x) = x^2 \cos(x) - 3x$.

(a) Construct the natural cubic spline $s(x)$ for the data above.
Recall that we need:

i) $s(x)$ to interpolate $f(x)$

ii) $s(x)$, $s'(x)$, $s''(x)$ continuous

iii) $s''(0.1) = s''(0.3) = 0$

Remember also that on each interval, there are 4 coefficients to determine. You should be able to immediately know three of them. For the other five, write down 5 equations in matrix form; you can then invert the matrix equation in Matlab. If $A\vec{x} = \vec{b}$ is the matrix equation, then, to solve for $\vec{x}$ the MATLAB command is `A\b`.
To get full credit you should report

(i) The conditions you are imposing to determine the coefficients.

(ii) The system of equations in matrix form that you need to solve to determine the coefficients.

(iii) The result you obtain for the 8 coefficients.

(b) Approximate $f(0.18)$ and $f'(0.18)$ using $s(x)$ and $s'(x)$, respectively, and list the relative errors.

(c) Approximate $f'(0.2)$ using $s'(x)$. Do the values $f'(0.2)$ and $s'(0.2)$ agree? Based on the definition of a cubic spline, should they agree? What other method could have we used instead of Splines if we wanted those derivatives to agree?

(d) Find an approximation of

$$\int_{0.1}^{0.3} x^2 \cos(x) - 3x \, dx$$

by evaluating

$$\int_{0.1}^{0.3} S(x) \, dx$$

what is the relative error? Please attach your code. (*Hint:* You may need to rewrite the integral of S as a sum of two integrals)
To compute integrals in MATLAB you can refer to integEx.m as an example.

[4] Computational Exercise
In the previous problem you computed by hand the spline interpolant. In this problem we will use the built in function in MATLAB to compute a spline.
Consider the function $f(x) = \cos(ax) * x^2 + 10x$ for $a = 1, 3, 5$.
Define a vector `xvals` made of 15 equispaced points between 0 and 10 (you can use `linspace` to do this).
Set $a = 1$ and define a vector `fvals` containing the values of f at each point of `xvals`.
Use the `spline` function in MATLAB to generate the spline for `xvals` and `fvals` (use `help spline` in the command window in MATLAB to look up how to use this function).
Make a plot of the true function vs. the spline you computed. Make sure to also add to your plot the points that you used to compute the spline (i.e. `xvals`, `fvals`).
Repeat the same problem for $a = 3, 5$ and report the plot you obtain in these cases. What do you observe? Why does this happen?