Math 180 Homework 7

Nathan Solomon

March 8, 2024

1

> Prove or disprove: if T is a minimal spanning tree of a weighted graph (G, wt) and u, v are two vertices, then the u, v-path in T is a minimal weight u, v-path in G.

False. Consider the cycle graph C_{100}, where u, v are two adjacent vertices, the edge between them has weight 2, and every other edge has weight 1. Then the only minimal spanning tree T is the path graph P_{99} in which u and v are leaves, so the path between them in T has weight 99. But in G, there is a path between u and v with weight 2, so the path in T (which has weight 99) is not minimal in G.

Note: this was the first example I though of, but a simpler example would be $G = C_3$, where all edges in G have weight 1. For any two distinct vertices u, v in G, there is an MST T of G such that u, v are not adjacent in T. Then the path between them in T has weight 2, but there is a path between them in G of weight 1.

2

> Let T be a minimal spanning tree in a connected weighted graph (G, wt). Prove that T omits a heaviest edge from every cycle in G.

Let e be the heaviest edge in some cycle C contained in G, and let e' be any other edge in C. Then removing e from T and adding in e' will decrease the weight of T by $\mathrm{wt}(e) - \mathrm{wt}(e') > 0$. Replacing e with e' will not change the fact that all vertices of G are connected in T, because the vertices in the cycle are still connected to each other, and every connected component of $G - C$ is still connected in G to at least one of the vertices in C. We also will not change the fact that T is a tree, because a connected component is a tree iff $|V| - |E| = 1$, and we didn't change either $|V|$ or $|E|$. This new tree is a spanning tree with less weight than the original.

Therefore if T were a minimal tree, it could not contain a heaviest edge from any cycle in G.

3

> **Section 5.4, Exercise 4.** Let G be a connected graph with a weight function w on the edges, and assume that w is injective. Prove that the minimum spanning tree of G is determined uniquely.

Since there is a unique way to order the edges of G from lowest to highest weight, there is only one possible output of Kruskal's algorithm. Let T be that output, and let T' be a different MST of G. Let e be an edge which T contains but T' does not. Then $T'+e$ contains a cycle C, and there is some edge e' which is in C but not in T. Now $T+e-e'$ is a spanning tree of G. Because T has minimal weight of all spanning trees, $w(T+e-e') > w(T)$, which implies $w(e) > w(e')$. But by the exact same logic, $T'+e'-e$ must be a spanning tree, and since T' is an MST, $w(e') > w(e)$. This is a contradiction, so there cannot be any MST of G which is different from T.

4

> **Section 8.1, Exercise 3.** *Hint: Count the number of spanning trees along with an edge to remove.*
> Put $T_n = T(K_n)$. Prove the recurrent formula
>
> $$(n-1)T_n = \sum_{k=1}^{n-1} k(n-k)\binom{n-1}{k-1} T_k T_{n-k}.$$
>
> *Remark.* Theorem 8.1.1 (Cayley's Formula) can be derived from this recurrence too, but it's not so easy.

Every spanning tree of K_n vertices can be constructed by the following process:

1. Pick some integer $k \in [1, n-1]$.
2. Split the vertices of K_n into two groups: G_1, which contains k vertices, and G_2, which contains the other $n-k$ vertices. WLOG, assume G_1 contains the vertex labeled 1. There are $\binom{n-1}{k-1}$ ways to make this choice.
3. Pick a spanning tree for the vertices in G_1, and a spanning tree for the vertices in G_2. There are $T_k T n - k$ ways to make these two choices.
4. Pick any vertex in G_1 and any vertex in G_2. There are $k(n-k)$ ways to make this choice. Connect those two vertices by an edge to obtain a spanning tree of K_n.

This construction will always give a spanning tree, but we need to worry now about overcounting. Given a spanning tree of K_n, we can choose any of the $n-1$ edges to be the bridge constructed between G_1 and G_2 in step 4. Then it becomes easy to go through the 4 steps in reverse and identify the choices made at each step. In fact, once we choose which edge in our spanning tree is the bridge between G_1 and G_2, all the other decisions are uniquely determined. Therefore, the number of spanning trees of K_n with one edge labeled as the bridge is $(n-1)T_n$, but it's also equal to

$$\sum_{k=1}^{n-1} k(n-k)\binom{n-1}{k-1} T_k T_{n-k}.$$

5

Find the spanning tree of K_9 with Prüfer code $(2,8,9,4,7,7,2)$.

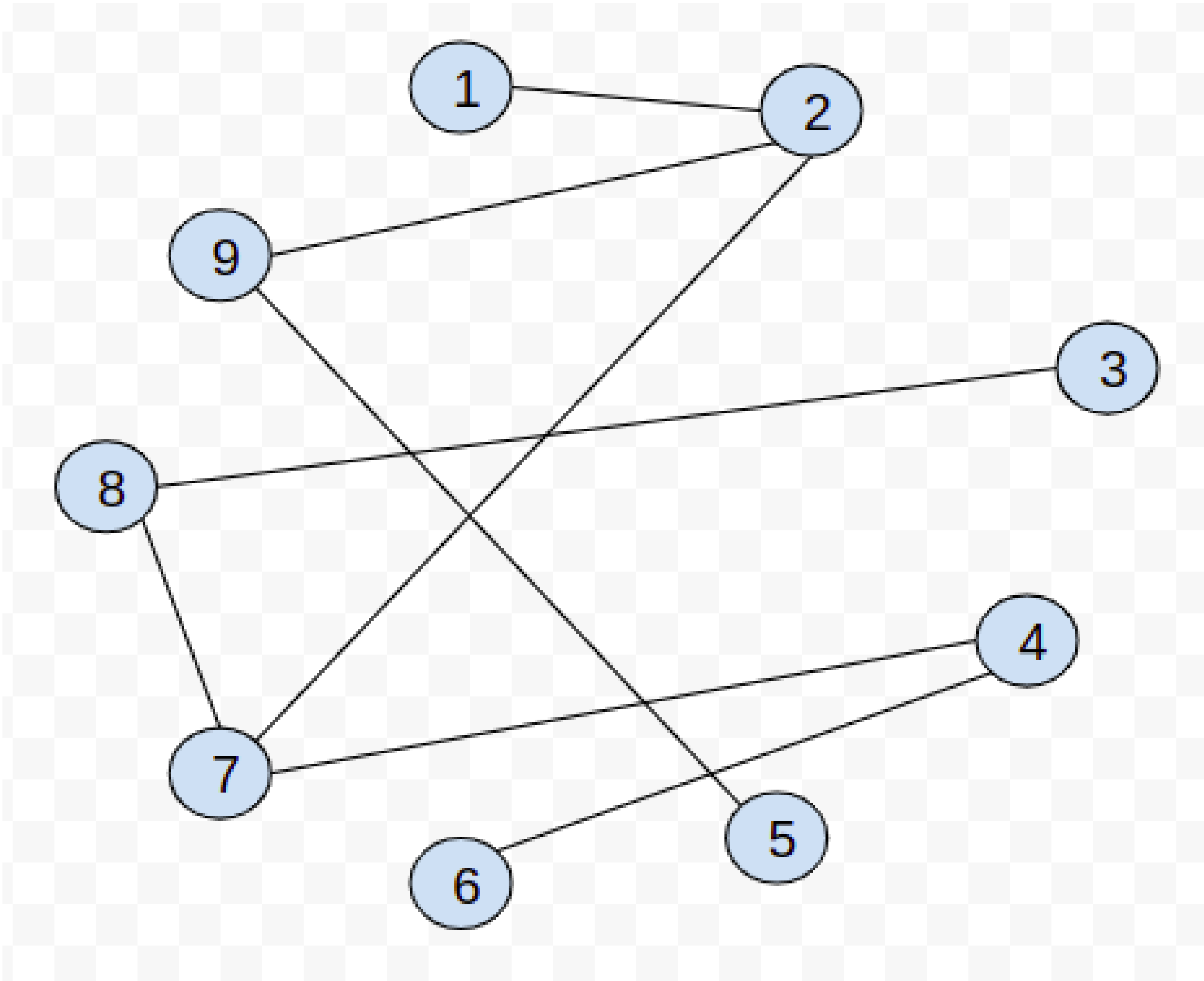
Using the algorithm from the lecture notes, we make a list of vertices which have not yet been connected to the spanning tree, called A, and we repeatedly add an edge to our spanning tree which connects the first element (a_j) from the Prüfer code C to the smallest-numbered vertex (a_i) which is in A but not in C. After each step, we remove a_j from C and a_i from A. Once C is empty, we use the last 2 elements of A to make one last edge.

C	A	a_j	a_i
2894772	123456789	2	1
894772	23456789	8	3
94772	2456789	9	5
4772	246789	4	6
772	24789	7	4
72	2789	7	8
2	279	2	7
	29		

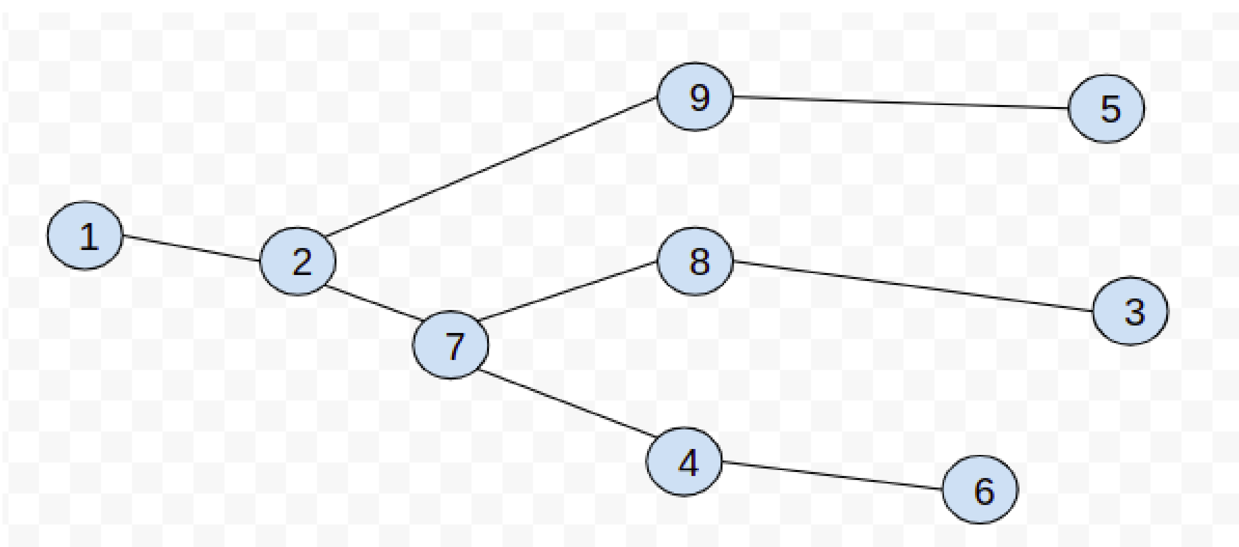
The list of edges (a_j, a_i) we obtained is

$$\{(2,1),(8,3),(9,5),(4,6),(7,4),(7,8),(2,7),(2,9)\}$$

This gives us the following graph:



which can also be drawn as



6

> Determine the number of spanning trees T of K_9 in each of the following scenarios:
>
> 1. T has 5 and 8 as leaves.
>
> 2. T does not have 5 as a leaf.

A Prüfer code for a spanning tree T on n vertices is a sequence of $n-2$ numbers from the alphabet $[n]$, and the number of times a number $i \in [n]$ appears in that code is one less than the degree of the vertex labeled i in T. Therefore, the vertex labeled i is a leaf of T iff it never appears in the Prüfer code of T.

1. This is equivalent to counting how many 7-letter words there are from the alphabet [9] which do not contain 5 or 8, but that's equivalent to counting how many 7-letter words there are from the alphabet [7], which is

$$7^7 = 823543.$$

2. This is equivalent to counting how many 7-letter words there are from the alphabet [9] which contain 5 at least once. That count is equal to the number of 7-letter words from the alphabet [9], minus the number of 7-letter words from the alphabet [8]. That is equal to

$$9^7 - 8^7 = 2685817.$$

7

> Determine which trees have Prüfer codes that have distinct values in all positions.

We know that the number of times a vertex v appears in a Prüfer code is one less than the degree of v. The $n-2$ vertices which occur in the Prüfer code exactly once must have degree 2, and the 2 vertices which never appear in the Prüfer code have degree 1. Therefore, if the Prüfer code for a tree T doesn't repeat any numbers, then T is a path graph.