Relativity Physics 127 Homework 4

Due Wednesday April 30th 2025, 11:59pm on gradescope.
There is a 24hr grace period this time.

1. Recall from multivariable calculus that a change of variables $x^i \to y^i(x)$ transforms the measure via the determinant of the Jacobian matrix associated with the change of variables:
$$d^3x \to d^3y = J d^3x \ , \qquad J = \det\left(\frac{\partial y^i}{\partial x^j}\right) \ .$$
Argue that the corresponding formula for the spacetime volume element d^4x under a Lorentz transformation $y^\mu = \Lambda^\mu{}_\nu x^\nu$ is given by
$$d^4x \to \det\Lambda \, d^4x \ .$$
Use the defining properties of Lorentz transformations to show that $\det\Lambda = 1$, so that the measure d^4x is Lorentz invariant. Conclude that the action
$$I = \int d^4x \mathscr{L}$$
is Lorentz invariant if the Lagrangian density $\mathscr{L}$ is a Lorentz scalar as well.

2. Gauss' theorem applied to a region V_4 in spacetime with boundary $\partial V_4 = \Sigma_3$ takes the form
$$\int_{V_4} d^4x \partial_\mu V^\mu = \int_{\Sigma_3} d\sigma_\mu V^\mu \ .$$
Here $d\sigma_\mu$ is the outward pointing area element of the boundary Σ_3. By choosing a suitable V_4 (see section 2.4.1 in Coleman's book), show that for a conserved current satisfying $\partial_\mu V^\mu = 0$, we have
$$\int_{\Sigma_3} d\sigma_\mu V^\mu = \int_{\Sigma_3'} d\sigma_\mu V^\mu$$
for any two spacelike hypersurfaces (i.e. timeslices) Σ_3 and Σ_3'.

Use this to conclude that the charge obtained by integrating the 0-component V^0 of the current over a constant x^0 slice,
$$Q = \int d^3x \, V^0 \ ,$$

is in fact a Lorentz scalar.

3. In ordinary particle mechanics the canonical momentum p of q is given by

$$p = \frac{\partial L}{\partial \dot{q}} \ ,$$

and the Hamiltonian or total energy is

$$H = p\dot{q} - L \ .$$

Consider the free scalar field $\phi(x)$ with Lagrangian density

$$\mathscr{L} = \frac{1}{2}\left(\partial^\mu \phi \partial_\mu \phi - m^2\phi^2\right) \ .$$

Find the canonical momentum density

$$\pi^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \ ,$$

and the Hamiltonian density

$$\mathscr{H} = \pi^0 \partial_0 \phi - \mathscr{L} \ ,$$

and show that $\mathscr{H} = T^{00}$ is the energy density obtained from the stress tensor of the scalar field defined in class.

4. Use the conservation and symmetry of the stress tensor $T^{\mu\nu}$ to show that

$$\partial^\mu (T_{\mu\sigma} x_\lambda - T_{\mu\lambda} x_\sigma) = 0 \ .$$

Conclude that

$$J^{\mu\nu} = \int_{\Sigma_3} d\sigma_\rho \left(T^{\rho\mu} x^\nu - T^{\rho\nu} x^\mu\right)$$

is independent of the choice of spacelike hypersurface Σ_3, e.g. we could take it to be any constant time slice. This shows that $J^{\mu\nu}$ is the conserved angular momentum.

5. Coleman problem 2.6.