1 5/24/2024 lecture

1.1 Haar measure

1.2 Peter-Weyl theorem

1.3 Highest weight representation of $\mathfrak{su}(2)$

Spin is one half of the highest weight. WHAT DOES THIS MEAN? ALSO, WHAT IS THE MEANING OF CHARGE IN THIS CONTEXT?

1.4 Adjoint representations

Relate adjoint representation of a Lie algebra to conjugation in the corresponding Lie group.

Also, note that the fundamental representation can also be called the defining representation or the vector representation

1.5 Simple Lie algebras

Prove that $\mathfrak{su}(2)$ is simple

1.6 $SU(2)$ is a double cover of $SO(3)$

Discuss the adjoint spin-1 representation of $SU(2)$.

Consequence of that: $\pi_1(SO(3), I) = \mathbb{Z}_2$

1.7 Universal covering maps

1.8 Lie algebra $\mathfrak{so}(n)$

Note. Since an anti-Hermitian real matrix must have zeros on the diagonal, $SO(n)$ and $O(n)$ have the same Lie algebra. We call it $\mathfrak{so}(n)$ instead of $\mathfrak{o}(n)$, because that way the exponential of $\mathfrak{so}(n)$ is $SO(n)$.

$\mathfrak{so}(3)$ is the algebra spanned (over $\mathbb{R}$) by $\{i, j, k\}$ with the Lie bracket being the cross product.

Define the following matrices as a basis for $\mathfrak{so}(3)$:

$$J_z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, J_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, J_y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Then there is a Lie algebra iisomorphism from $\mathfrak{so}(3)$ to $\mathfrak{su}(2)$ which maps J_i to $\frac{i\sigma_i}{2}$.

Prove that $SO(3)$ is not simply connected, but $SU(2)$ is.

1.9 The Lie group $U(2)$

Problem 1.1. Use Schur's lemma to show that irreps of $U(1) \times SU(2)$ can be characterized by a spin $s \in \frac{1}{2}\mathbb{Z}$ and a charge $q \in \mathbb{Z}$.

Theorem 1.2. $U(2)$ representations are those with $\frac{q}{2} + s \in \mathbb{Z}$.

Corollary 1.3. $U(2) = U(1) \times SU(2)/\mathbb{Z}_2$

TALK ABOUT HOW TO VISUALIZE $SO(3)$ AS $\mathbb{RP}^3$.