Math 115B: Linear Algebra
Homework 8
Due: *Friday*, March 14th at 11:59pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- *In this homework assignment k always denotes a field for which* $1 + 1 \neq 0$.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. ($\frac{-}{7*6}$) Determine which of the following mappings given below are bilinear forms. Justify your answers.
   (a) Let $C[0,1]$ be the set of continuous real valued functions with domain $[0,1]$. For $f, g \in C[0,1]$, define $H(f,g) := \int_0^1 f(x)g(x)dx$.
   (b) Let V be a vector space over k, and let $J \in \mathbb{B}(V)$ be nonzero. Define $H : V \times V \to k$ by the formula $H(\vec{v}, \vec{w}) = J(\vec{v}, \vec{w})^2$ for all $\vec{v}, \vec{w} \in V$.
   (c) The function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by the formula $H(t_1, t_2) := t_1 + 2t_2$.
   (d) The function $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by the formula $D(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}) := ad - bc$.
   (e) Let V be a real inner product space, and let $H : V \times V \to \mathbb{R}$ be the function $H(\vec{v}, \vec{w}) := \langle \vec{v}, \vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$.
   (f) Let V be a *complex* inner product space, and let $H : V \times V \to \mathbb{C}$ be the function $H(\vec{v}, \vec{w}) := \langle \vec{v}, \vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$.

2. ($\frac{-}{10+5}$) Assume V is a vector space and $\mathbb{B}(V)$ is the set of bilinear forms on V.
   (a) Prove Theorem 6.31. That is, prove that if $H_1, H_2 \in \mathbb{B}(V)$ and $\alpha \in k$ implies $H_1 + H_2 \in \mathbb{B}(V)$ and $\alpha H_1 \in \mathbb{B}(V)$ and that $\mathbb{B}(V)$ is a vector space over k with respect to these operations.
   (b) Assume the dimension of V is $n \in \mathbb{Z}^{\geq 0}$. Compute the dimension of $\mathbb{B}(V)$.

3. ($\frac{-}{15}$) Let V be a vector space over a field k (whose characteristic we have assumed is not two!) and let H denote a symmetric bilinear form on V. Prove if we define the function $K : V \times V \to k$ by the formula $K(\vec{v}) := H(\vec{v}, \vec{v})$ for all $\vec{v} \in V$, then

$$H(\vec{v}, \vec{w}) = \frac{1}{2}(K(\vec{v} + \vec{w}) - K(\vec{v}) - K(\vec{w}))$$

for all $\vec{v}, \vec{w} \in V$.

4. $(\frac{-}{2+8})$ Assume T is a linear operator (endomorphism) on a finite dimensional real inner product space V, and define the function $H : V \times V \to \mathbb{R}$ by the formula $H(\vec{v}, \vec{w}) = \langle \vec{v}, T\vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$.

   (a) Prove that H is a bilinear form.

   (b) Prove that H is symmetric if and only if T is self adjoint.

5. $(\frac{-}{13})$ Prove that if V is a finite dimensional real inner product space and H is a bilinear form on V, then there exists a unique linear operator $T : V \to V$ such that $H(\vec{v}, \vec{w}) = \langle \vec{v}, T\vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$. (*Hint*: Choose an orthonormal basis $\mathcal{B}$ for V, and let A be the matrix representation of H for this basis. Let $T : V \to V$ be the linear transformation for which $[T(\vec{v})]_{\mathcal{B}} = A[\vec{v}]_{\mathcal{B}}$.)

6. $(\frac{-}{5})$ Assume k is a field such that, for some positive integer m, $\sum_{i=1}^{m} 1 = 1 + 1 + \ldots + 1 = 0$. Prove the smallest positive integer p for which $\sum_{i=1}^{p} 1 = 0$ is prime. (This prime number is called the *characteristic* of the field k, and if $\sum_{i=1}^{m} 1 = 1 + 1 + \ldots + 1 \neq 0$ for all positive integers m, we say that k has *characteristic zero*.