MATH 131B practice final

Nathan Solomon

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Problem 0.1.

(a) Let E be a subset of a compact metric space (X, d_X). Show that if E is open in X, then $E^c := X \backslash E$ is compact.

(b) Let $f : X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces. Suppose f is bijective and continuous, and (X, d_X) is compact. Show that f is an open map (that is, for any $E \subset X$ open, $f(E) \subset Y$ is open).

(a) Since E is open, its complement is closed. Any closed subset of a compact space is compact, so E^c is compact.

(b) Since E is open, E^c is compact. Because f is continuous, that also means $f(E^c)$ is compact, and since f is bijective, $f(E^c) = f(E)^c$. Every compact space is closed, so $f(E)^c$ is closed, which means $f(E)$ is open.

Problem 0.2. Consider $(\mathbb{R}^2, d_{\ell^2})$.

(a) What is $B_{(\mathbb{R}^2, d_{\ell^2})}((0,0), 1)$ as a set?

(b) Denote $B = B_{(\mathbb{R}^2, d_{\ell^2})}((0,0), 1)$. Let $f : \mathbb{R}^2 \to \mathbb{R}, f(x) = \inf \{d(x,y) | y \in B\}$. Show that f is continuous (with respect to the ℓ^2 metric on $\mathbb{R}^2$ and the standard metric on $\mathbb{R}$).

(a) The set of points distance less than one from the origin can be written as

$$B := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}.$$

(b) f can be written as $g \circ h$, where $h((x, y)) = \|(x, y)\| = \sqrt{x^2 + y^2}$ and $g(x) = \max(0, x - 1)$. Since g and h are both continuous, so is f.

This is not a rigorous proof, a better way to do this would be to use the triangle inequality to show that for any $\varepsilon > 0, (x, y) \in \mathbb{R}^2$, there exists some $\delta > 0$ such that $f(B((x, y), \delta)) \subset B(f((x, y)), \varepsilon)$. But I won't write that whole thing out because it's annoying to do.

Problem 0.3. Let (X, d_X) be a metric space. Suppose for any collection of closed subsets $\{S_\alpha\}_{\alpha \in A}$ (i.e., for every $\alpha \in A$, S_α is a closed subset of X) such that $\cap_{\alpha \in F} S_\alpha \neq \emptyset$ for any finite subset F of A, we have that $\cap_{\alpha \in A} S_\alpha \neq \emptyset$. Show that X is compact.

Suppose X is not compact. Let $\{T_\alpha\}_{\alpha \in A}$ be any open cover of X. There is no finite subcover of that, so for any finite subset $F \subset A$, $\cup_{\alpha \in F} T_\alpha \neq X$, which means $\cap_{\alpha \in F} T_\alpha^c \neq \emptyset$. Then $\cap_{\alpha \in A} T_\alpha^c \neq 0$, which means $\cup_{\alpha \in A} T_\alpha \neq X$, so $\{T_\alpha\}_{\alpha \in A}$ does not cover X. This is a contradiction, so X must be compact.

Problem 0.4. Let $(f_n)_{n=1}^{\infty}$ be a sequence of Riemann-integrable functions from $[a,b]$ to $\mathbb{R}$, and $f : [a,b] \to \mathbb{R}$ another Riemann-integrable function. Suppose $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $[a,b]$.

(a) Show that $\lim_{n\to\infty} \int_{[a,b]} f_n = \int_{[a,b]} f$.

(b) For every n, define $F_n : [a,b] \to \mathbb{R}, F_n(x) = \int_{[a,x]} f_n$ and $F : [a,b] \to \mathbb{R}, F(x) = \int_{[a,b]} f$. Show that F_n converges uniformly to F on $[a,b]$.

(a) For any $\varepsilon > 0$, there exists an N such that $|f(x) - f_n(x)| < \varepsilon/(b-a)$ for any $n \geq N, x \in [a,b]$. Then

$$\begin{aligned} d\left(\int_{[a,b]} f_n, \int_{[a,b]} f\right) &\leq \left|\int_{[a,b]} (f_n - f)(x)\mathrm{d}x\right| \\ &= \int_{[a,b]} |f_n(x) - f(x)|\, \mathrm{d}x \\ &\leq \int_{[a,b]} \frac{\varepsilon}{b-a} \mathrm{d}x \\ &= \varepsilon. \end{aligned}$$

Therefore $\int_{[a,b]} f_n$ converges to $\int_{[a,b]} f$ as n goes to ∞.

(b) Define N to be the same as in part (a). For any $x \in [a,b]$,

$$\begin{aligned} d(F_N(x), F(x)) &\leq \left|\int_{[a,x]} (f_n(x) - f(x))\, \mathrm{d}x\right| \\ &\leq \int_{[a,x]} |f_n(x) - f(x)|\, \mathrm{d}x \\ &\leq \int_{[a,x]} \frac{\varepsilon}{b-a} \mathrm{d}x \\ &= \varepsilon \cdot \frac{x-a}{b-a} \\ &\leq \varepsilon. \end{aligned}$$

Problem 0.5.

(a) Let $f : E \to \mathbb{R}$ where $E \subset \mathbb{R}$ and $a \in \mathrm{int}(E)$ (interior of E). State the definition of f being real analytic at $x = a$.

(b) Now let $f(x) = \mathrm{arccot}(x)$ (inverse cotangent, where $\cot(x) := \cos(x)/\sin(x)$). Show that $f'(x)$ is real analytic at $x = 0$ using the geometric series formula

$$\sum_{k=0}^{\infty} cr^k = \frac{c}{1-r}.$$

Find the radius of convergence R for the power series expansion for $f'(x)$ at $x = 0$.

(c) Show that $f(x)$ is real analytic at $x = 0$ (you can use $f(x) = \pi/2 + \int_{[0,x]} f'$ for all $x \in \mathbb{R}$).

(a) f is real analytic at $x = a$ iff there exists some $\delta > 0$ such that f is equal to its power series expansion in $(a - \delta, a + \delta)$.

(b) First, we want to find the derivative of f using implicit differentiation. Let $y = f(x)$.

$$\begin{aligned} x &= \cot(y) \\ \frac{\partial x}{\partial y} &= \frac{1}{\sin^2(y)} = 1 + \cot^2(y) = 1 + x^2 \\ f'(x) &= \left(\frac{\partial x}{\partial y}\right)^{-1} = \frac{1}{1 + x^2} = \sum_{k=0}^{\infty} (-x^2)^k. \end{aligned}$$

Therefore the radius of convergence for f' is

$$R := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}},$$

where $c_n = 0, 0, -1, 0, 1, 0, -1, 0, 1, \ldots$, so the radius of convergence is 1.

(c) The function

$$x \mapsto \int_{[0,x]} f'(y)\mathrm{d}y$$

is real analytic with the same radius of convergence, so $f(x)$ is also real analytic at $x = 0$.

Problem 0.6.

(a) State the Fourier theorem for $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.

(b) Let f be a function in $C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. For $n \in \mathbb{Z}_{\geq 0}$, let

$$\begin{aligned} a_n &= 2\int_0^1 f(x)\cos(2\pi n x)\mathrm{d}x, \\ b_n &= 2\int_0^1 f(x)\sin(2\pi n x)\mathrm{d}x. \end{aligned}$$

Use the Fourier series of f to show that the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi n x) + b_n \sin(2\pi n x)]$$

converges to f in the L^2 metric.

(a) For any $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, $\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ converges to f in the L^2 metric.

(b)

$$
\begin{aligned}
f &= \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n \\
&= \sum_{n=0}^{\infty} \left(\hat{f}(n) e_n + \hat{f}(-n) e_{-n} \right) \\
&= \sum_{n=0}^{\infty} \left(\left(\int_{[0,1]} f(x) e_{-n}(x) \mathrm{d}x \right) e_n + \left(\int_{[0,1]} f(x) e_n(x) \mathrm{d}x \right) e_{-n} \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{a_n - ib_n}{2} \cdot e_n + \frac{a_n + ib_n}{2} \cdot e_{-n} \right) \\
&= \sum_{n=0}^{\infty} \left(a_n \cdot \frac{e_n + e_{-n}}{2} + b_n \cdot \frac{e_n - e_{-n}}{2i} \right) \\
f(x) &= \sum_{n=0}^{\infty} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \right).
\end{aligned}
$$

Problem 0.7.

(a) Let $(f_n)_n \subset C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, and $f, g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and $f_n \to f$ uniformly on $\mathbb{R}$. Show that $f_n * g \to f * g$ pointwise and uniformly.

(b) Let $(f_n)_n \subset C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, and $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Suppose for each n, f_n is a periodic $\left(\frac{1}{n}, \frac{1}{2n}\right)$ approximation to the identity. Show that $f_n * f \to f$ uniformly.

(a) g is bounded by $\|g\|_\infty$, and for any $\varepsilon > 0$, there exists N such that for any $x \in \mathbb{R}$ and any $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{\|g\|_\infty}$. Then

$$
\begin{aligned}
\|f_n * g - f * g\| &= \left| \int_{[0,1]} f_n(y) - f(y)) g(x-y) \mathrm{d}y \right| \\
&\leq \int_{[0,1]} |f_n(y) - f(y)| \, g(x-y) \mathrm{d}y \\
&\leq \frac{\varepsilon}{\|g\|_\infty} \int_{[0,1]} g(x-y) \mathrm{d}y \\
&\leq \varepsilon.
\end{aligned}
$$

Therefore $f_n * g$ converges uniformly (and therefore, also pointwise) to $f * g$.

(b)

$$
\begin{aligned}
(f_n * f)(x) &= \int_{[-1/2,1/2]} f_n(y) f(x-y) \mathrm{d}y \\
&= \left(\int_{[-1/2,-1/2n] \cup [1/2n,1/2]} f_n(y) f(x-y) \mathrm{d}y \right) + \left(\int_{[-1/2n,1/2n]} f_n(y) f(x-y) \mathrm{d}y \right).
\end{aligned}
$$

Now, use the facts that $0 \leq f_n(y) < \frac{1}{n}$ and that f is bounded by $\|f\|_\infty$ (the magnitude of f in the supremum norm), so for any $\varepsilon > 0$, there exists N_1 such that $f_n(y) < \frac{\varepsilon}{3\|f\|_\infty}$ whenever $n \geq N_1$. Therefore that first term can be bounded as follows:

$$\begin{aligned}\int_{[-1/2,-1/2n]\cup[1/2n,1/2]} f_n(y)f(x-y)\mathrm{d}y &\leq \int_{[-1/2,1/2]} f_n(y)f(x-y)\mathrm{d}y \\ &\leq \int_{[-1/2,1/2]} \left(\frac{\varepsilon}{3\|f\|_\infty}\right)(\|f\|_\infty)\,\mathrm{d}y \\ &\leq \frac{\varepsilon}{3}.\end{aligned}$$

Also, f is uniformly continuous, so there exists $N_2 \in \mathbb{N}$ such that $|f(x-y) - f(x)| < \varepsilon/3$ for any $y \in (-\frac{1}{2n}, \frac{1}{2n})$. Therefore

$$\begin{aligned}\int_{[-1/2n,1/2n]} f_n(y)\,|f(x-y)-f(x)|\,\mathrm{d}y &\leq \frac{\varepsilon}{3}\int_{[-1/2n,1/2n]} f_n(y)\mathrm{d}y \\ &\leq \frac{\varepsilon}{3}.\end{aligned}$$

Lastly, there exists $N_3 \in \mathbb{N}$ such that whenever $n \geq N_3$,

$$\left|1 - \int_{[-1/2n,1/2n]} f_n(y)\mathrm{d}y\right| < \frac{\varepsilon}{3\|f\|_\infty}.$$

This implies

$$\begin{aligned}\left|f(x) - \int_{[-1/2n,1/2n]} f_n(y)f(x-y)\mathrm{d}y\right| &\leq \left|f(x)\left(1 - \int_{[-1/2n,1/2n]} f_n(y)\mathrm{d}y\right)\right| + \left|\int_{[-1/2n,1/2n]} f_n(y)(f(x-y)-f(x))\mathrm{d}y\right| \\ &\leq \|f\|_\infty \cdot \frac{\varepsilon}{3\|f\|_\infty} + \int_{[-1/2n,1/2n]} f_n(y)\,|f(x-y)-f(x)|\,\mathrm{d}y \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.\end{aligned}$$

If we let

$$N := \max(N_1, N_2, N_3),$$

then by the triangle inequality,

$$|f(x) - (f_n * f)(x)| \leq \varepsilon,$$

so $f_n * f$ converges to f, and since the choice of N did not depend on x, that convergence is uniform.

Problem 0.8. Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and g be a trigonometric polynomial. Show that $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$ for all $n \in \mathbb{Z}$.

$$
\begin{aligned}
\widehat{f*g}(n) &= \langle f*g, e_n\rangle \\
&= \int_{[0,1]} (f*g)(x)e_n^*(x)\mathrm{d}x \\
&= \int_{[0,1]} \left(\int_{[0,1]} f(y)g(x-y)\mathrm{d}y\right) e_n^*(x)\mathrm{d}x \\
&= \int_{[0,1]} \left(\int_{[0,1]} f(y)\sum_{k=-\infty}^{\infty}\hat{g}(k)e_k(x-y)\mathrm{d}y\right) e_n^*(x)\mathrm{d}x \\
&= \int_{[0,1]} \sum_{k=-\infty}^{\infty}\hat{g}(k)e_k(x)\left(\int_{[0,1]} f(y)e_k^*(y)\mathrm{d}y\right) e_n^*(x)\mathrm{d}x \\
&= \int_{[0,1]} \sum_{k=-\infty}^{\infty}\hat{g}(k)e_k(x)\hat{f}(k)e_n^*(x)\mathrm{d}x \\
&= \sum_{k=-\infty}^{\infty}\hat{g}(k)e_k(x)\hat{f}(k)\delta_{k,n} \\
&= \hat{f}(n)\hat{g}(n).
\end{aligned}
$$

Problem 0.9.
Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. For each $N \in \mathbb{N}$, let $F_N = \sum_{n=-N}^{N}\hat{f}(n)e_n$, and S_N be the collection of trigonometric polynomial $p = \sum_{n=-N}^{N} c_n e_n$ where $\sum_{n=-N}^{N}|c_n|^2 \leq 1$ (recall $e_n(x) := e^{2\pi i n x}$ is the character with frequency n). For each N, show that $|\langle f, p\rangle| \leq \|F_N\|$ for all $p \in S_N$. Find an element $p \in S_N$ such that equality holds.

$$
\begin{aligned}
\langle f, p\rangle &= \int_{[0,1]} f(x)p^*(x)\mathrm{d}x \\
&= \int_{[0,1]} \left(f(x)\sum_{n=-N}^{N} c_n^* e_n^*(x)\right)\mathrm{d}x \\
&= \sum_{n=-N}^{N} \langle f, c_n e_n\rangle \\
&= \sum_{n=-N}^{N} c_n^*\hat{f}(n) \\
&= \sum_{n=-N}^{N} \hat{f}(n)\langle e_n, c_n e_n\rangle \\
&= \langle F_N, p\rangle .
\end{aligned}
$$

Now that we know $\langle f, p\rangle = \langle F_N, p\rangle$, we can use the Cauchy-Shwarz inequality and the fact that $\|p\| \leq 1$:

$$
\begin{aligned}
|\langle f, p\rangle| &= |\langle F_N, p\rangle| \\
&\leq \|F_N\| \cdot \|p\| \\
&\leq \|F_N\| .
\end{aligned}
$$

Now we want to find some p such that $|\langle f, p\rangle| = \|F_N\|$. If $F_N = 0$, then p can be anything, otherwise, let

$$p = \frac{F_N}{\|F_N\|}.$$