MATH 131B Homework #6

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Problem 0.1. 4.5.1. Prove proposition 4.5.2.

(a) For any $x \in \mathbb{R}$, let $a_n = x^n/n!$. We want to show that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent. One way to do this is with the ratio test:

$$L := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1} n!}{x^n (n+1)!} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0.$$

Since the limit L exists and $L < 1$, the ratio test says that $\sum a_n$ converges absolutely.

This implies that $\exp(x)$ exists and is real for any $x \in \mathbb{R}$, the power series has an infinite radius of convergence, and that exp is a real analytic function on $\mathbb{R} = (-\infty, \infty)$.

(b) Since we have radius of convergence $R = \infty$, theorem 4.1.6(d) says that exp is differentiable on $(-\infty, \infty)$. For any $x \in \mathbb{R}$,

$$\exp'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} n \frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x).$$

(c) By theorem 4.1.6(c), exp is continuous on $\mathbb{R}$, and by 4.1.6(e),

$$\int_{[a,b]} \exp(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} \cdot \frac{b^{n+1} - a^{n+1}}{n+1} \right) = \sum_{n=1}^{\infty} \left(\frac{b^n}{n!} - \frac{a^n}{n!} \right) = \exp(b) - \exp(a).$$

(d) Despite what the hint says, theorem 4.4.1 doesn't really help here.

The following steps are allowed, because $\exp(x)\exp(y)$ is absolutely convergent for any $x, y \in \mathbb{R}$. All I am doing is reindexing the terms, so that $l = n + m$.

$$\begin{aligned}
\exp(x)\exp(y) &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{y^m}{m!} \right) \\
&= \sum_{l=0}^{\infty} \sum_{n=0}^{l} \frac{x^n y^{l-n}}{n!(l-n)!} \\
&= \sum_{l=0}^{\infty} \sum_{n=0}^{l} \binom{l}{n} \frac{x^n y^{l-n}}{l!} \\
&= \sum_{l=0}^{\infty} \frac{(x+y)^l}{l!} \\
&= \exp(x+y).
\end{aligned}$$

(e)

$$\exp(0) = \frac{0^0}{0!} + \frac{0^1}{1!} + \cdots = 0^0 = 1.$$

Because of the result from part (d), we have

$$\exp(x)\exp(-x) = \exp(0) = 1$$

for any $x \in \mathbb{R}$. That means $\exp(x)$ (and $\exp(-x)$) can never be zero. Since exp is a continuous function, and $\exp(0) = 1$ is positive, $\exp(x)$ can never be negative, because if it were, the intermediate value theorem would imply there is a point where $\exp(x) = 0$. Therefore $\exp(x)$ is always positive, and

$$\exp(-x) = \frac{1}{\exp(x)}.$$

(f) exp is real analytic everywhere, and its first derivative is exp, which we just showed is always positive. Therefore exp is strictly monotone increasing.

A more rigorous way to do this problem is to observe that whenever $x > 0$, $\exp(x) = 1 + \sum_{n=1}^{\infty} x^n/n! > 1$. So if $y > x$, then $\exp(y - x) > 1$, which means $\exp(y) = \exp(x)\exp(y - x) > \exp(x)$. Alternatively, if $y < x$, then you can swap the symbols x and y to get that $\exp(x) > \exp(y)$.

Problem 0.2. 4.5.3. Prove proposition 4.5.4.

$\boxed{\text{If } x \text{ is a natural number (including zero),}}$ we can prove this with induction. If $x = 0$, then $\exp(x)$ and e^x are both one. If $\exp(x) = e^x$ for some $x \in \mathbb{N}$, then $\exp(x+1) = \exp(x)\exp(1) = e^x \exp(1) = e \cdot e^x = e^{x+1}$. So by induction, the statement $\exp(x) = e^x$ is true for any $x \in \mathbb{N}$.

$\boxed{\text{If } x \text{ is a negative integer,}}$ then $-x$ is a natural number (or zero), so $\exp(x) = 1/\exp(-x) = 1/e^{-x} = e^x$. So the statement $\exp(x) = e^x$ works for any $x \in \mathbb{Z}$.

$\boxed{\text{If } x \text{ is a rational number,}}$ then $x = p/q$ for some $p, q \in \mathbb{Z}$ such that $q \neq 0$. $e^{p/q}$ can be defined as $\sqrt[q]{e^p}$ – the unique positive number such that $\sqrt[q]{e^p}^q = e^p$. Also, $\exp(p/q)$ is the unique positive number such that

$$\prod_{n=1}^{q} \exp\left(\frac{p}{q}\right) = \exp\left(\sum_{n=1}^{q} \frac{p}{q}\right) = \exp(p) = e^p.$$

Therefore, $\exp(x) = e^x$ when $x \in \mathbb{Q}$.

$\boxed{\text{If } x \text{ is a real number,}}$ then e^x has no intuitive definition other than being the continuous function uniquely defined by its value when x is rational. Since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, any continuous function on $\mathbb{R}$ can be uniquely defined by its value on $\mathbb{Q}$. Therefore, if e^x and $\exp(x)$ are equal for any rational x, and since exp is also continuous, they must be equal on all of $\mathbb{R}$.

Problem 0.3. 4.5.4

The nth derivative of f exists at $x = 0$ iff $\lim_{x\to 0^-} f^{(n)}(x) = \lim_{x\to 0^+} f^{(n)}(x)$. The left-hand side of that equation will always be zero. The right-hand side, for $n = 1, 2, 3, \ldots$ is

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \left(-\frac{1}{x^2}\right) \exp\left(-\frac{1}{x}\right) = \lim_{x\to 0^+} = 0$$

$$\lim_{x\to 0^+} f^{(n)}(x) = \lim_{x\to 0^+} (\text{some rational function}) \exp\left(-\frac{1}{x}\right) = 0.$$

You can use induction to show that you will always get some rational function there. Specifically, you will get a polynomial function of $1/x$ times $\exp(-1/x)$. As $x \to 0$ from the positive side, $1/x \to +\infty$, which means a polynomial function of $1/x$ times $\exp(-1/x)$ will go to zero.

Therefore f is infinitely differentiable, and $f^{(k)}(0) = 0$ for any $k \in \mathbb{N}$. However, f is not real analytic at $x = 0$, because if it were, there would be some ε neighborhood of zero in which f is equal to its own power series expansion (around zero). But since all derivatives of f are zero, its power series expansion is the zero function. But for any $x > 0$, $f(x) = \exp(-1/x) > 0$.

Problem 0.4. 4.5.5. Prove theorem 4.5.6.

(a) The inverse function theorem says that if f is differentiable at x, $f(x) = y$, $f'(x) \neq 0$, and f is invertible, then

$$(f^{-1})'(y) = \frac{1}{f'(x)}.$$

If $f = \exp$, then $f^{-1} = \ln$, so $\ln'(y) = 1/\exp'(x) = 1/y$, which could also be written as $\ln'(x) = 1/x$ for some x in the image of exp, which is $(0, \infty)$.

The fundamental theorem of calculus says that

$$\int_a^b \frac{1}{x} dx = \ln(b) - \ln(a).$$

(b) If $x, y \in (0, \infty)$, then there exists some $a, b \in (0, \infty)$ such that $\exp(a) = x$ and $\exp(b) = y$. Thus,

$$\ln(xy) = \ln\left(\exp(a)\exp(b)\right) = \ln\left(\exp(a+b)\right) = a + b = \ln(x) + \ln(y).$$

(c) $\exp : \mathbb{R} \to (0, \infty)$ is a bijective map, so if we apply exp to both sides, and the equation is true, then it must have been true originally as well.

$$\begin{aligned}
\exp(\ln(1)) &= \exp(0) \\
1 &= \exp(0) \\
\exp\left(\ln\left(\frac{1}{x}\right)\right) &= \exp\left(-\ln(x)\right) \\
\frac{1}{x} &= \frac{1}{x}.
\end{aligned}$$

Therefore, both of the equations were true to begin with.

(d) Let $z = \ln(x^y)$, so that $e^z = x^y$. Then $e^{z/y} = \sqrt[y]{e^z} = x$, and taking the log of both sides of that, $z/y = \ln(x)$, so $z = y\ln(x)$. Therefore

$$\ln(x^y) = y\ln(x).$$

(e) The nth derivative of $\ln(x)$ (for $x > 0$ and $n > 0$) is

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n},$$

which you can prove by induction of by just noticing the pattern. Therefore

$$f^{(n)}(1) = -(-1)^n(n-1)!,$$

and $f(1) = 0$, so the Taylor series expansion of ln around $a = 1$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \left(-(-1)^n(n-1)!\right)\frac{(x-1)^n}{n!} = \sum_{n=1}^{\infty} -\frac{(-1)^n x^n}{n},$$

which can also be written as

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

That series converges iff $|x| < 1$, so that equation is true iff $x \in (-1, 1)$. Substituting in $y = 1 - x$ to that equation, we get

$$\ln(y) = -\sum_{n=1}^{\infty} \frac{(1-y)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(y-1)^n$$

for any $y \in (0, 2)$.

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(1) Exercise: 4.5.1, 4.5.3, 4.5.4, 4.5.5.