Math 110BH Homework 9

Nathan Solomon

March 20, 2024

1

Find the invariant factors of the quotient group $\mathbb{Z}^3/N$, where N is generated by $(-4, 4, 2)$, $(16, -4, -8)$, $(12, 0, -6)$, and $(8, 4, 2)$.

The element $(12, 0, -6)$ is generated by the other elements, since $(-4, 4, 2)+(16, -4, -8) = (12, 0, -6)$. That means we can ignore the generator $(12, 0, -6)$, and the other 3 elements will still generate N.

The coefficient matrix is then

$$A = \begin{pmatrix} -4 & 4 & 2 \\ 16 & -4 & -8 \\ 8 & 4 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 4 & -4 \\ -8 & -4 & 16 \\ 2 & 4 & 8 \end{pmatrix} \to \begin{pmatrix} 2 & -4 & 4 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

so the invariant factors are $\{2\mathbb{Z}, 12\mathbb{Z}, 12\mathbb{Z}\}$.

2

Find the rational canonical form over $\mathbb{Q}$ of the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{pmatrix}$$

Row reducing (xI minus that matrix) gives

$$\begin{bmatrix} x+2 & 0 & 0 \\ -1 & x+4 & -1 \\ 2 & 4 & x \end{bmatrix} \to \begin{bmatrix} 1 & -x-4 & 1 \\ x+2 & 0 & 0 \\ 2 & 4 & x \end{bmatrix} \to \begin{bmatrix} 1 & -x-4 & 1 \\ 0 & x^2+6x+8 & -x-2 \\ 0 & 2x+12 & x-2 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2x+12 & x-2 \\ 0 & x^2+6x+8 & -x-2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & x-2 \\ 0 & x^2+8x+12 & -x-2 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & x-2 \\ 0 & x^2+8x+12 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & x-2 & 16 \\ 0 & -4 & x^2+8x+12 \end{bmatrix}$$

Since the only invariant factor is $(x+2)^3 = x^3+6x^2+12x+8$, the RCF is

$$\begin{bmatrix} 0 & 0 & -8 \\ 1 & 0 & -12 \\ 0 & 1 & -6 \end{bmatrix}$$

NOTE: THIS IS WRONG, THERE SHOULD BE TWO INVARIANT FACTORS

3

Find the rational canonical form over $\mathbb{Z}/2\mathbb{Z}$ of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Call that matrix A. Then by row-reducing $xI - A$, we get

$$xI - A = \begin{bmatrix} x-1 & 1 & 0 \\ 0 & x-1 & 1 \\ 0 & 0 & x-1 \end{bmatrix} \to \begin{bmatrix} 1 & x-1 & 0 \\ x-1 & 0 & 1 \\ 0 & 0 & x-1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(x-1)^2 & 1 \\ 0 & 0 & x-1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2+1 & 1 \\ 0 & 0 & x+1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -(x+1)(x^2+1) \end{bmatrix}.$$

Therefore the only invariant factor is x^3+x^2+x+1, so the RCF is

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

but in $\mathbb{Z}/2\mathbb{Z}$, that's the same as

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

4

Let $V \subset \mathbb{R}[x, y]$ be the subspace of all polynomials of the form $ax + by + c$, where $a, b, c \in \mathbb{R}$. Let $\mathcal{A}$ be a linear operator in V defined by

$$\mathcal{A}(ax + by + c) = a(x + 1) + b(y - 1) + c.$$

Find the elementary divisors and the canonical form of $\mathcal{A}$.

5

Find the Jordan canonical form over $\mathbb{C}$ of the matrix

$$\begin{pmatrix} 2i & 1 \\ 1 & 0 \end{pmatrix}$$

6

Prove that two 2×2 matrices over a field that are not scalar matrices are similar if and only if they have the same characteristic polynomials.

7

Prove that two 3×3 matrices are similar if and only if they have the same characteristic and the same minimal polynomials.

8

Show that the minimal polynomial of an $n \times n$ matrix A has the same irreducible divisors as the characteristic polynomial of A.

9

Let A be a nilpotent $n \times n$ matrix (that is, $A^N = 0$ for some $N > 0$). Show that the invariant factors of A are powers of X. Prove that $A^n = 0$.

10

Prove that any $n \times n$ matrix A is similar to its transpose A^t.