Math 151A
HW #3, due on Friday, October 25, 2024 at 11:59pm PST.

You can use a calculator for the following problems. (If you don't have a hand calculator, Matlab or even Google should suffice.)
Please write your answers clearly. To get full credit you need to show all your work. If you are required to write code, please attach your code and all the outputs and plots to your homework.
Homework should be submitted on Gradescope.

[**1**]

(a) Let $a > 0$. Verify that $\sqrt{a}$ is a fixed point of the function

$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

(b) Assume that, for some p_0, the fixed point iteration $p_{n+1} = g(p_n)$ converges to $p = \sqrt{a}$ as $n \to \infty$. Determine the order of convergence.

[**2**] **Computational exercise**
Consider the function $f(x) = e^x - 1 - x - x^2/2$.

(a) Show that $x = 0$ is a zero of f. What is the multiplicity?

(b) Implement the modified Newton's method to calculate the root with tolerance 10^{-6} (i.e. stop the iteration when $|x_n - x| \le 10^{-6}$. Then, use the usual Newton's method and compute the root with same tolerance. Use as initial point $x_0 = 1$ and set the maximum number of iterations to 1000. For each method, report the number of iterations, the approximated root and the residual (i.e. $|f(x_n)|$). Describe your results. Which method converges faster and why?

(c) Repeat the experiment but now set the tolerance to 10^{-10}. Do the methods converge? Why?

[**3**] The sequence defined by $p_n = 1 + 1/n$ is linearly convergent to $p = 1$. Compute the first five terms (by hand, or using a computer) of the sequence $\hat{p}_n$ using Aitken's Δ^2 method. Report both $\hat{p}_n$ and the elements p_n that you needed to compute $\hat{p}_n$. What do you notice?

[4]

Let $f(x) = \ln x$.

(a) Construct the Lagrange polynomial for f passing through the points $(1, \ln 1)$, $(2, \ln 2)$, and $(3, \ln 3)$.

(b) Using the answer in part (a), approximate the values $\ln(1.5)$ and $\ln(2.4)$ and report the absolute error when comparing with the true values $\ln(1.5) = 0.405465108$ and $\ln(2.4) = 0.875468737$.

(c) What is the maximum error when the polynomial is used to approximate $f(x)$ on the interval $[1, 3]$?

[5] Computational exercise

This problem will make use of the data in Table 1.

Year	1960	1970	1980	1990	2000	2010
Population (in thousands)	179,323	203,302	226,542	249,633	281,422	308,746

Table 1: Population in the United States, in thousands, for the last six censuses.

(a) Use MATLAB to construct the interpolating polynomial $P(x)$ for this data (since we have $n = 6$ data points, it should be a degree 5 polynomial). It is convenient let x equal the number of years since 1960, so that 1960 corresponds to $x = 0$, 1970 corresponds to $x = 10$, and so on.

(b) Use $P(x)$ to 'predict' the population of the United States in 2020 (in thousands). How does the result compare with the true US population in 2020 of 329 500 thousand? What is the relative error?