MATH 131B Homework #1

Nathan Solomon

December 13, 2024

Problem 0.1. Exercises 1.1.13, 1.2.2, 1.2.3, 1.4.7, and 1.5.2 from the textbook

Exercise 1.1.13. Prove Proposition 1.1.19.

If there exists $N \geq m$ such that $x^{(n)} = x$ for all $n \geq N$, then the distance from $x^{(n)}$ to x in the discrete metric is zero for all $n \geq N$, which means for any $\varepsilon > 0$ and any $n \geq N$, $d(x^{(n)}, x) \leq \varepsilon$, so the sequence converges to x.

If the sequence converges to x in the discrete metric, then let ε be a positive number less than 1. By definition 1.1.14, there exists some $N \geq m$ such that for any $n \geq N$, $d(x^{(n)}, x) \leq \varepsilon < 1$. But in the discrete metric, $d(x^{(n)}, x)$ can only be less than one if $x^{(n)} = x$, so for all $n \geq N$, $x^{(n)} = x$.

Exercise 1.2.2. Prove Proposition 1.2.10. (Hint: for some of the implications one will need the axiom of choice, as in Lemma 8.4.5.)

$\boxed{(a \Rightarrow b)}$ If x_0 is an adherent point of E, then for every radius $r > 0$, the intersection $B(x_0, r) \cap E \neq 0$, which means x_0 is not an exterior point of E. Since a boundary point is defined as any point that is not an interior point or an exterior point, x_0 must be an interior point or a boundary point.

$\boxed{(b \Rightarrow c)}$ If x_0 is an interior point of E, let $x_n = x_0$ for every n. This sequence clearly converges to x_0.

If x_0 is a boundary point of E, and we assume the axiom of choice, then for any $n \in \mathbb{N}$, we can define x_n to be some point in $B(x_0, \frac{1}{n}) \cap E$. We know that $B(x_0, \frac{1}{n}) \cap E$ is nonempty because x_0 is not an exterior point of E, and we know this sequence converges to x_0 because $d(x_n, x_0) < \frac{1}{n}$.

$\boxed{(c \Rightarrow a)}$ Let x_n be a sequence in E which converges to x_0. For any $r > 0$, there is some N such that for any $n \geq N$, $d(x_n, x_0) < r$. This means $B(x_0, r)$ contains $\{x_N, x_{N+1}, \dots\}$, which are all points in E, so $B(x_0, r) \cap E$ is nonempty. Therefore x_0 is an adherent point of E.

Exercise 1.2.3. Prove Proposition 1.2.15. (Hint: you can use earlier parts of the proposition to prove later ones.)

(a) If E is open, then $\partial E \cap E = \emptyset$. Let x be any point in E. Then $x \notin \partial E$. Since x is not in the exterior of E or in the boundary of E, it must be in the interior of E. Therefore E is a subset of the interior of E. But of course, if x is in the interior of E, there is some $r > 0$ such that $B(x, r) \subset E$, which means $x \in E$. Therefore the interior of E is also a subset of E, so E is equal to its own interior.

    If E is equal to its own interior, then $E \cap \partial E = \text{int}(E) \cap \partial E$. But since the boundary is defined as the points which aren't in the interior or exterior, the boundary and the interior are disjoint, so that intersection is empty, which means E is open.

(b) If E is closed, then $\partial E \subset E$, which means every $x \in E$ is an interior point or a boundary point. By proposition 1.2.10, which we just proved, this means x is an adherent point of E.

    Every boundary point of E is an adherent point (because adherent points are defined as any point not in the exterior, and boundary points are defined as any point not in the exterior or interior). Therefore if E contains all of its adherent points, it must also contain all its boundary points, so E is closed.

(c) Let x be any element of $B(x_0, r)$, and let $r' = (r - d(x_0, x))/2$. Then $B(x, r')$ must be fully contained in $B(x_0, r)$, because for any $x' \in B(x, r')$,

$$d(x', x_0) \leq d(x', x) + d(x, x_0) \leq r' + d(x_0, x) = \frac{r + d(x_0, x)}{2} \leq \frac{r + r}{2} = r.$$

By proposition 1.2.15 part (a), this means $B(x_0, r)$ is open.

Suppose x is some point a distance greater than r from x_0. Then by the triangle inequality, $B(x, d(x, x_0) - r)$ does not intersect with the closed ball of radius r centered at x_0. Therefore x is an exterior point of that closed ball. Every point a distance less than or equal to r away from x_0 must then be either an interior point or a boundary point of $\{x \in X | d(x, x_0) \leq r\}$. Since that set contains all of its own boundary points, it is closed.

(d) Take "any" infinite sequence of points in $\{x_0\}$ (there is only one such sequence). Every element in that sequence is the same point, so the sequence converges to that point, which is x_0. By proposition 1.2.15 part (b), this means $\{x_0\}$ is closed.

(e) If E is open, then $E \cap \partial E = \emptyset$, which means $\partial E \subset (X \backslash E)$. By the definition of a boundary, the boundary of E is the same as the boundary of $X \backslash E$, so $X \backslash E$ contains all its boundary points, so it is closed.

If $X \backslash E$ is closed, then it contains all of its boundary points, so it contains ∂E, so E contains none of its boundary points, so E is open.

(f) Let x be any element of x. Then there exists some $r_1, \ldots, r_n > 0$ such that $B(x, r_1) \subset E_1, \ldots, B(x, r_n) \subset E_n$. Let r be the minimum of $\{r_1, \ldots, r_n\}$. Then $r > 0$ and $B(x, r) \subset E_1 \cap E_2 \cap \cdots \cap E_n$, so that union is open.

If $F_1, \ldots, F_n$ is a finite collection of closed sets in X, then the complement of their union is the intersection of their complements (by De Morgan's law). Their complements are all open, so the intersection of their complements is open, so the complement of their union is open, so their union is closed.

(g) For any $x \in \cup_{\alpha \in I} E_\alpha$, there exists some $\alpha \in I$ such that $x \in E_\alpha$. Since E_α is open, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset E_\alpha$, and that same ball is also contained in the union $\cup_{\alpha \in I} E_\alpha$, so that union is open.

If F_α is a collection of closed sets in X, then $\cup_{\alpha \in I}(X \backslash F_\alpha)$ is the union of a collection of open sets, so it is open. That means its complement, $\cap_{\alpha \in I} F_\alpha$, is closed.

(h) For any $x \in V$, there exists a ball around x which is contained in V, and therefore also contained in E, so x is in the interior of E. That means V is contained in the interior of E.

If E is a subset of a closed set K, and $x \in E$ is an adherent point, then x is not in the exterior of E, which means its not in the exterior of K, so it's in K. Therefore the closure of E is a subset of K.

Exercise 1.4.7. Prove Proposition 1.4.12.

(a) Suppose Y is complete, and let y be any boundary point of Y. For any n, let y_n be any point in $B(y, \frac{1}{n}) \cap Y$. That intersection is nonempty by the definition of a boundary point, and I am assuming the axiom of choice in order to choose a point from each. Then y_n is a Cauchy sequence which clearly converges to y, and since Y is complete, that means $y \in Y$. Therefore Y contains all its boundary points, so Y is closed.

(b) Let y_n be a Cauchy sequence in Y. Since X is complete, y_n must converge to some $x \in X$. Suppose $x \notin Y$. Since Y is closed, that means x is not an interior or boundary point of Y, so it's an exterior point. The definition of an exterior point is that there exists some r such that $B(x, r) \cap Y = \emptyset$, but since y_n converges to x, there must be infinitly many n such that $d(y_n, x) < r$ (for any positive r). This is a contradiction, so x (the point that y_n converges to) must be in Y, which means Y is complete.

Exercise 1.5.2. Prove Proposition 1.5.5. (Hint: prove the completeness and boundedness separately. For both claims, use proof by contradiction. You will need the axiom of choice, as in Lemma 8.4.5.)

We proved in class that "compact is equivalent to sequentially compact", meaning in a compact metric space, every sequence has a convergent subsequence. Let x_n be any Cauchy sequence in X. Then there exists some subsequence $x_{(i_n)}$ which converges to a point we will call x. By the definition of convergence, that means that for any $\varepsilon > 0$, there exists N such that for any $i_n \geq N$, $d(x_{(i_n)}, x) < \frac{\varepsilon}{2}$. By the definition of a Cauchy sequence, for any $\varepsilon > 0$, there exists M such that for any $j, k \geq M$, $d(x_j, x_k) < \frac{\varepsilon}{2}$. Now let $N' = \max(N, M)$. Then for any $m, i_n \geq N'$,

$$d(x_m, x) \leq d(x_m, x_{(i_n)}) + d(x_{(i_n)}, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so x_m converges to x, which means $\boxed{X \text{ is complete}}$.

Let x be any point in X, and cover X with infinitely many open balls centered at x:

$$\{B(x,1), B(x,2), B(x,3), \dots\}.$$

Since X is compact, there exists a finite subset of that cover which still covers X, and for any such finite subcover, we can define r to be the radius of the largest ball in the subcover. By the triangle inequality, since no point in X can be a distance greater than r from x, no two points in X can be a distance greater than $2r$ apart, so $\boxed{X \text{ is bounded}}$.

Problem 0.2. Let (X, d) be a metric space and $E \subset X$ a subset.

(a) Show that if E is a finite set, then E is compact.

(b) Suppose in addition that $d = d_{disc}$ is the discrete metric. Show that if E is compact, then E is a finite set.

(a) Let $\{a_n\}_{n\in\mathbb{N}}$ be any sequence of elements of E. By the pigeonhole principle, since E is finite and $\{a_n\}$ is infinite, there must be some $e \in E$ such that e occurs infinitely many times in $\{a_n\}$. That forms a convergent subsequence of $\{a_n\}$, so E is sequentially compact, which means E is also compact (we proved in class that for metric spaces, sequentially compact is equivalent to compact).

(b) Cover E with singleton sets, which are open in the discrete metric. If E is compact, then it can be covered by finitely many of those singleton sets, which means E has finitely many elements.

Problem 0.3. Give the following examples.

(a) Find a metric space (X, d) and an infinite collection of open subsets $\{U_n\}_{n\in\mathbb{N}}$ of X such that $\cap_{n\in\mathbb{N}} U_n$ is not open.

(b) Find a metric space (X, d) and an infinite collection of closed subsets $\{K_n\}_{n\in\mathbb{N}}$ of X such that $\cup_{n\in\mathbb{N}} K_n$ is not closed.

For both (a) and (b), let (X, d) be $\mathbb{R}$ with the Euclidean metric.

(a) Let $U_n = \left(0, 1 + \frac{1}{n}\right)$. Then $\cap_{n\in\mathbb{N}} U_n = (0, 1]$. Every U_n is open, but $(0, 1]$ is not open.

(b) Let $K_n = \left[0, 1 - \frac{1}{n}\right]$. Then $\cup_{n\in\mathbb{N}} = [0, 1)$. Every K_n is closed, but $[0, 1)$ is not closed.

24F-MATH-131B HOMEWORK 1
DUE SUNDAY, OCTOBER 13

(1) Exercise: 1.1.13, 1.2.2, 1.2.3, 1.4.7, 1.5.2

(2) Let (X, d) be a metric space and $E \subset X$ a subset.
   (a) Show that if E is a finite set then E is compact.
   (b) Suppose in addition that $d = d_{\text{disc}}$ is the discrete metric. Show that if E is compact then E is a finite set.

(3) Give the following examples.
   (a) Find a metric space (X, d) and an infinite collection of open subsets $\{U_n\}_{n \in \mathbb{N}}$ of X such that $\cap_{n \in \mathbb{N}} U_n$ is not open.
   (b) Find a metric space (X, d) and an infinite collection of closed subsets $\{K_n\}_{n \in \mathbb{N}}$ of X such that $\cup_{n \in \mathbb{N}} K_n$ is not closed.