Math 115B Homework #1

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Problem 0.1.

Problem 0.2.

That is not possible. If $v_1 + v_2$, $v_2 + v_3$, and $v_3 + v_1$ are linearly dependent, then there exist constants a_1, a_2, a_3 which are not all zero, for which

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_1) = 0.$$

Problem 0.3.

V is isomorphic to k^d, for some nonnegative integer d, so $49 = |V| = |k|^d$. This equation is only satisfied if $|k| = 7$ and $d = 2$, or if $|k| = 49$ and $d = 1$. In either case, k is finite and k^d is finite-dimensional (with dimension either 1 or 2), which means V also has dimension 1 or 2.

Problem 0.4.

(a) For any $w_1 \in W_1$, let $w_2 = 0$. Since every subspace contains the zero element, w_2 is in W_2, which means $w_1 + w_2 \in W_1 + W_2$. But $w_1 = w_1 + 0 = w_1 + w_2$, so every $w_1 \in W_1$ is also in $W_1 + W_2$. This means $W_1 \subset W_1 + W_2$, and by the same logic, W_2 is also a subset of $W_2 + W_1$.

(b) $W_1 + W_2$ is clearly a subset of V, since V is closed under addition and scalar multiplication, so I only need to show two things: that $W_1 + W_2$ is closed under addition, and that it's closed under scalar multiplication. For any $u_1 + u_2, v_1 + v_2 \in W_1 + W_2$, we have $(u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1) + (u_2 + v_2)$, which is in $W_1 + W_2$ because $u_1 + v_1$ is in W_1 and $u_2 + v_2$ is in W_2.

$W_1 + W_2$ is closed under scalar multiplication, because for any scalar a and vector $w_1 + w_2 \in W_1 + W_2$, $a(w_1 + w_2) = aw_1 + aw_2$ is in $W_1 + W_2$ because $aw_1 \in W_1$ and $aw_2 \in W_2$.

(c)

Problem 0.5.

(a)

(b) Not a functional, because the output of f is not an element of k.

(c)

(d)

(e)

Problem 0.6.

(a)

(b)

Math 115B: Linear Algebra
Homework 1
Due: Tuesday, January 14 at 11:59 PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.

1. $(\frac{-}{20})$ Assume V and W are vector spaces over a field k, and let $T : V \to W$ denote a linear transformation between them. Prove that if T has an inverse, then that inverse is a linear function.

2. $(\frac{-}{20})$ Assume that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are vectors in a vector space V over some field k. Is it possible that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent but the vectors $\vec{w}_1 = \vec{v}_1 + \vec{v}_2, \vec{w}_2 = \vec{v}_1 + \vec{v}_3, \vec{w}_3 = \vec{v}_2 + \vec{v}_3$ are linearly *dependent*?

3. $(\frac{-}{5+5})$ Assume V is a vector space consisting of 49 vectors over a field k.

   (a) Prove that the set of elements in k is finite.

   (b) Prove that V is finite dimensional and that $\dim(V) = 1$ or $\dim(V) = 2$.

4. $(\frac{-}{5+5+5})$ Assume V is a vector space over some field k, and let W_1, W_2 denote subspaces of V. Define
$$W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

   (a) Prove that $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_2 + W_1$.

   (b) Prove that $W_1 + W_2$ is a subspace of V.

   (c) Assume that $W \subseteq V$ is a subspace such that $W_1 \subseteq W$ and $W_2 \subseteq W$. Prove $W_1 + W_2 \subseteq W$. (Notice that these results imply that $W_1 + W_2$ is the smallest possible subspace of V which contains both W_1 and W_2!)

5. $(\frac{-}{5*4})$ Let k denote some field. For each function f below, determine if f is a linear functional and prove your answer is correct. You may assume standard results from calculus.

   (a) $V = \mathbb{R}[x]$, $f(p(x)) := 4p'(0) + p''(1)$, where $q'(x)$ denotes the derivative of $q(x) \in \mathbb{R}[x]$.

   (b) $V = k^2$, $f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 2x \\ 4y \end{pmatrix}$.

   (c) $V = k^{2\times 2}$, $f(A) = \text{tr}(A)$

(d) $V = \mathbb{R}[x]$, $f(p(x)) = \int_0^1 p(x)dx$

(e) $V = \mathbb{Q}^3$, $f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = x^2 + y^2 + z^2$

6. ($\frac{-}{10+5}$) Assume that m, n are positive integers, and fix $x_1, ..., x_m \in \mathbb{R}$ such that $x_i \neq x_j$ if $i, j \in \{1, ..., m\}$ such that $i \neq j$. Let

$$W := \{f \in \mathbb{R}[x]_{\leq n} : f(x_1) = f(x_2) = ... = f(x_m) = 0\}$$

where $\mathbb{R}[x]_{\leq n}$ denotes the degree less than or equal to n. (The set $\mathbb{R}[x]_{\leq n}$ is denoted $P_n(\mathbb{R})$ in our textbook.)

(a) Prove that W is a vector space over $\mathbb{R}$. (*Hint:* The set W is, by definition, a sub*set* of $\mathbb{R}[x]_{\leq n}$, which was shown in 115A to be a vector space.)

(b) Compute the dimension of W. (*Hint:* It may help to use the rank-nullity theorem.)