Physics 231B Homework #2

Nathan Solomon

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Problem 0.1. Artin Chapter 6 Problem 4.2, page 188.

- (a) List all subgroups of the dihedral group D_4, and decide which ones are normal.
- (b) List all the proper subgroups N of the dihedral group D_{15}, and identify the quotient groups D_{15}/N.
- (c) List the subgroups of D_6 that do not contain x^3.

- (a) Using the notation from Artin,

  $$D_4 = \left\langle x, y \middle| \, x^4 = y^2 = xyxy = e \right\rangle .$$

  Since this group has order 8, by Lagrange's theorem, every subgroup must have order 1, 2, 4, or 8. Knowing that, we can methodically list all subgroups. The trivial group $\{e\}$ and the entire group D_4 are both normal subgroups of D_4, so we only need to think about the proper subgroups, which have 2 or 8 elements.

  If a subgroup of D_4 has 2 elements, it must be generated by one element of order two; therefore it must be either $\left\{e, x^2\right\}$ or $\{e, y\}$. $\left\{e, x^2\right\}$ is normal, but $\{e, y\}$ is not, because $x^{-1}yx = yx^2 \notin \{e, y\}$.

  If a subgroup of D_4 has 4 elements, it is isomorphic to either C_4 or K_4. If it is isomorphic to C_4, it is generated by an element of order 4, so it must be $\left\{e, x, x^2, x^3\right\}$, which is normal in D_4. If the subgroup is isomorphic to K_4, it is generated by the only two elements in D_4 which have order 2, so it is $\left\{e, x^2, y, x^2y\right\}$, which is also normal in D_4

- (b) Since D_{15} has order 30, every proper subgroup of it must have order 2, 3, 5, 6, 10, or 15. From here, we can guess all of those subgroups:

  - $\langle y \rangle$
  - $\left\langle x^5 \right\rangle$
  - $\left\langle x^3 \right\rangle$
  - $\left\langle y, x^5 \right\rangle$
  - $\left\langle y, x^3 \right\rangle$
  - $\left\langle x^3, x^5 \right\rangle = \langle x \rangle$

  To find out the quotient by each of those subgroups, consider some homomorphism from D_{15} to N which takes an element y, x^5, or x^3 to e iff that element is in N. Then the image of that homomorphism is the quotient group, and is generated by the intersection of N with $\left\{y, x^5, x^3\right\}$.

  - $D_{15}/\langle y \rangle = \langle x \rangle \cong C_{15}$
  - $D_{15}/\left\langle x^5 \right\rangle = \left\langle y, x^3 \right\rangle \cong D_5$

- $D_{15}/\langle x^3\rangle = \langle y, x^5\rangle \cong D_3$
- $D_{15}/\langle y, x^5\rangle = \langle x^3\rangle \cong C_5$
- $D_{15}/\langle y, x^3\rangle = \langle x^5\rangle \cong C_3$
- $D_{15}/\langle x\rangle = \langle y\rangle \cong C_2$

- (c) Since D_6 is generated by y, x^3, and x^2, the subgroups of D_6 which do not contain x^3 are
  - $\{e\}$
  - $\langle y\rangle = \{e, y\} \cong C_2$
  - $\langle x^2\rangle = \{e, x^2, x^4\} \cong C_3$
  - $\langle y, x^2\rangle = \{e, x^2, x^4, y, yx^2, yx^4\} \cong D_3$

Problem 0.2. Artin Chapter 6 Problem 4.3, page 188.

- (a) Compute the left cosets of the subgroup $H = \{1, x^5\}$ in the dihedral group D_{10}.
- (b) Prove that H is normal and that D_{10}/H is isomorphic to D_5.
- (c) Is D_{10} isomorphic to $D_5 \times H$?

- (a) By Lagrange's theorem, there are 10 left cosets of H. Those are $\{1, x^5\}$, $\{x, x^6\}$, $\{x^2, x^7\}$, $\{x^3, x^8\}$, $\{x^4, x^9\}$, $\{y, yx^5\}$, $\{yx, yx^6\}$, $\{yx^2, yx^7\}$, $\{yx^3, yx^8\}$, and $\{yx^4, yx^9\}$.
- (b)
$$x^{-1}Hx = \{x^{-1}1x, x^{-1}x^5x\} = H$$
$$y^{-1}Hy = \{yy, yx^5y\} = H$$
Since H is closed under conjugation by x and by y, and since $D_{10} = \langle x, y\rangle$, $H \trianglelefteq D_{10}$.
If you label the coset $\{y, yx^5\}$ as y' and $\{x, x^6\}$ as x', then we can observe that $\langle x', y'\rangle$ has 10 elements, and that $x'^5 = y'^2 = x'y'x'y' = H$, so $D_{10}/H \cong D_5$.
- (c) Yes, because D_{10} is generated by x^2, x^5, and y. Since x^5 commutes with both x^2 and y, we can say
$$D_{10} = \langle x^2, x^5, y\rangle \cong \langle x^2, y\rangle \times \langle x^5\rangle = D_5 \times H.$$

Problem 0.3. Artin Chapter 6 Problem 5.1, page 188. Let ℓ_1 and ℓ_2 be lines through the origin in $\mathbb{R}^2$ that intersect in an angle π/n, and let r_i be the reflection about ℓ_i. Prove that r_1 and r_2 generate a dihedral group D_n.

Let $q = r_1r_2$. Since q is the product of 2 reflections, it must be a rotation, but we want to know by how much. By considering a point (other than the origin) on ℓ_1, we can see that q is a rotation about the origin by $2\pi/n$. Therefore q has order n, so r_1 and q together generate all $2n$ elements of D_n.

Problem 0.4. Artin Chapter 6 Problem 5.10, page 189. Let f and g be rotations of the plane about distinct points, with arbitrary nonzero angles of rotation θ and ϕ. Prove that the group generated by f and g contains a translation.

There is a homomorphism from the group of rotation and reflections of the plane to $U(1)$, defined by mapping translations to 1 and mapping a rotation by an angle x to e^{ix}. The kernel of this map is the group of translations of the plane. Using the definition of a homomorphism, $f^{-1}g^{-1}fg$ is in that kernel of that map, since it gets mapped to $\exp(i \cdot (-\theta - \phi + \theta + \phi)) = 1$. In other words, the total angle of rotation is zero.

This means that to prove $f^{-1}g^{-1}fg$ is a translation, we just need to show that it is not the identity map. f and g do not commute because if x is the fixed point of f, then $gfx = gx$, and since $gx \neq x$, $fgx \neq gx$. Since $gf \neq fg$, $f^{-1}g^{-1}fg$ is a non-identity element of the group of translations of the plane.

Problem 0.5. Artin Chapter 6 Problem 7.7, page 191. Let $G = GL_n(\mathbb{R})$ operate on the set $V = \mathbb{R}^n$ by left multiplication.

- (a) Describe the decomposition of V into orbits for this operation.
- (b) What is the stabilizer of e_1?

- (a) There are only two orbits: one contains the zero vector, and the other contains everything else. That's because every matrix maps the zero vector to itself, and for any nonzero vectors $x, y \in V$, there is an invertible matrix which maps x to y.
- (b) The stabilizer of e_1 is the set of matrices A such that $Ae_1 = e_1$. That is true iff the top left entry of A is 1 and every other entry in the leftmost column of A is 0.

Problem 0.6. Artin Chapter 6 Problem 9.1, page 191. Use the counting formula to determine the orders of the groups of rotational symmetries of a cube and of a tetrahedron.

For both the cube and the tetrahedron, consider a point x in the center of a face. In the case of a cube, the stabilizer of x has order 4, because the face it's on can be rotated 4 ways, and the orbit of x has 6 distinct points (because there are 6 faces, and a rotation could map x to the center of any face. By the counting formula, the group of rotational symmetries of a cube has order $4 \cdot 6 = 24$.

Similarly, if x is in the center of a face of a tetrahedron, then its orbit has 4 points (the centers of each face), and the stabilizer of x has order 3 (there are 3 rotational symmetries of the face which x lies on). Therefore the group of rotational symmetries of a tetrahedron has order $3 \cdot 4 = 12$.