Math 151A Homework #2

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Problem 0.1.

Let p be some point such that $f(p) = 0$. Then g has a fixed point at p iff $g(p) = p$.

(a)

$$\begin{aligned} f(p) = 0 &= p^4 + 2p^2 - p - 3 \\ p + 3 - 2p^2 &= p^4 \\ p &= \left(p + 3 - 2p^2\right)^{1/4} \\ p &= g_1(p) \end{aligned}$$

(b)

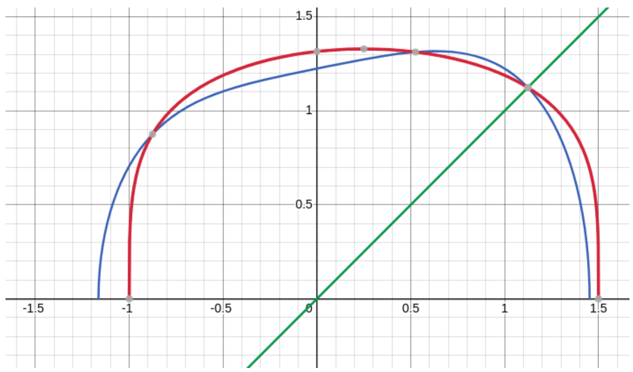
$$\begin{aligned} f(p) = 0 &= p^4 + 2p^2 - p - 3 \\ p + 3 - p^4 &= 2p^2 \\ p &= \left(\frac{x + 3 - x^4}{2}\right)^{1/2} \\ p &= g_2(p) \end{aligned}$$

Problem 0.2.

```
>>> def iterate(f, p_0, n):
...     p = p_0
...     for i in range(n+1):
...         print(f"p_{i} = {p}")
...         p = f(p)
...
>>> def g_1(x): return (3 + x - 2*x**2) ** (1/4)
...
>>> def g_2(x): return ((3 + x - x**4) / 2) ** (1/2)
...
>>> iterate(g_1, 1, 4)
p_0 = 1
p_1 = 1.189207115002721
p_2 = 1.0800577526675623
p_3 = 1.1496714305893827
p_4 = 1.1078205295102599
>>> iterate(g_2, 1, 4)
```

```
p_0 = 1
p_1 = 1.224744871391589
p_2 = 0.9936661590774817
p_3 = 1.228568645274987
p_4 = 0.9875064291508866
```

Here is a graph of g_1 in red and g_2 in blue:



From that graph, we can see that if $p_0 = 1$, then FPI with either g_1 or g_2 will approximate the root $r_1 = 1.124123\ldots$ (and not the root r_2, which is negative). After 4 iterations, FPI with g_1 approximates r_1 to be 1.107821 and FPI with g_2 approximates that same root as 0.987506. Therefore, I think g_1 is better for approximating the root r_1.

Problem 0.3.

There are 4 hypotheses needed to show existence and uniqueness of fixed points. If the following are all true:

- $g \in C([a, b])$
- $g(x) \in [a, b]$ for all $x \in [a, b]$
- $g'(x)$ exists when $x \in (a, b)$
- there exists $k \in (0, 1)$ such that $|g'(x)| \leq k$ for all $x \in (a, b)$

then g has a unique fixed point on $[a, b]$. If the first two criteria are true, then g has a fixed point on $[a, b]$, but that alone would not guarantee it is unique.

Let $a = 0, b = 2\pi, g(x) = \pi + 0.5\sin(x/2)$. Then g is continuous on $[a, b]$. Since the sine of a real number is always in $[-1, 1]$, the image $g([a, b])$ is in $[\pi - 0.5, \pi + 0.5]$. g is differentiably everywhere, and $g'(x) = 0.25\cos(x/2)$. So if $k = 0.25$, then $|g'(x)|$ is always less than or equal to k.

All 4 hypotheses are satisfied, so g has a unique fixed point on $[0, 2\pi]$.

Problem 0.4.

```
>>> from math import sin, cos
>>> def f(x): return -x**3 - cos(x)
...
>>> def f_prime(x): return -3 * x**2 + sin(x)
...
>>> p = -1
```

```
>>> iteration = 0
>>> tolerance = 1e-10
>>> max_iterations = 40
>>> while abs(f(p)) > tolerance and iteration < max_iterations:
...     iteration += 1
...     p -= f(p) / f_prime(p)
...     print(f"{iteration=}\t {p=}\t residual={abs(f(p))}")
...
iteration=1     p=-0.880332899571582      residual=0.04535115463649053
iteration=2     p=-0.8656841631760818     residual=0.0006323133963189731
iteration=3     p=-0.865474075952977      residual=1.289199819121123e-07
iteration=4     p=-0.8654740331016162     residual=5.218048215738236e-15
```

Newton's method converges very quickly. The approximation after two iterations is $p_2 = -0.8656841631760818$ and the residual is $f(p_2) = 0.0006323133963189731$. Newton's method only works because we chose a good value for p_0 though. If p_0 were zero, then $f(p_0)$ would be nonzero and $f'(p_0)$ would be zero, so $p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$ would be undefined.

Problem 0.5.

```
>>> from math import cos
>>> def f(x): return -x**3 - cos(x)
...
>>> p = [-1, 0]
>>> while len(p) <= 3:
...     p.append((p[-2] * f(p[-1]) - p[-1] * f(p[-2])) / (f(p[-1]) - f(p[-2])))
...
>>> for i in range(len(p)):
...     print(f"p_{i} = {p[i]} residual={abs(f(p[i]))}")
...
p_0 = -1 residual=0.45969769413186023
p_1 = 0 residual=1.0
p_2 = -0.6850733573260451 residual=0.45285023447500383
p_3 = -1.252076488909229 residual=1.649523592018598
```

After 2 iterations, our approximation (p_3) is pretty bad, but if we kept going, it would eventually converge:

```
p_4 = -0.8072055385060928        residual=0.16556011526065983
p_5 = -0.8477837694325692        residual=0.05231270017050571
p_6 = -0.8665281869207425        residual=0.0031747088825289094
p_7 = -0.8654557261640932        residual=5.5076148661736823e-05
p_8 = -0.8654740143806453        residual=5.632275656974883e-08
p_9 = -0.8654740331019471        residual=1.0009770790020411e-12
p_10 = -0.8654740331016145       residual=2.220446049250313e-16
```

Problem 0.6.

Assume $f'(p_0) \neq 0$, and define $g(x) = x - f(x)/f'(p_0)$. Let $p_1 = g(p_0), p_2 = g(p_1)$, and so on. f is continuous on $[a, b]$ and $f'(p_0)$ is assumed to be nonzero, so g is also continuous on $[a, b]$

Suppose p_n converges to p as n goes to infinity. By Taylor's theorem,

$$g(x) = g(x_0) + g'(\xi_x) \cdot (x - x_0)$$

for some ξ_x between x_0 and x. If we let $x = p_{n-1}$ and $x_0 = p_0$, that equation becomes

$$g(p_{n-1}) = g(p) + g'(\xi_n) \cdot (p_{n-1} - p)$$

(for some ξ_n between p_{n-1} and p). That simplifies to

$$p_n = p + g'(\xi_n) \cdot (p_{n-1} - p),$$

which can be rewritten as

$$g'(\xi_n) = \frac{p_n - p}{(p_{n-1} - p)}.$$

Since ξ_n is between p_{n-1} and p, and p_n converges to p, ξ_n must also converge to p. The function g' is continuous (because $g'(x) = 1 - f'(x)/f'(p_0)$, and f is coninuously differentiable), so that means $g'(\xi_n)$ converges to $g'(p) = 1 - f'(p)/f'(p_0)$. We are given that $f'(p) \neq f'(p_0)$, so

$$\lim_{n\to\infty} \frac{|p_n - p|}{|p_{n-1} - p|} = |g'(p)| > 0.$$

We do not have enough information to show that $L < 1$, where $L := |g'(p)|$, but if we assume $L < 1$, then we have shown p_n converges linearly to p.

Problem 0.7.

Note thar for this question, there is some ambiguity about whether to start with p_0 or p_1 for Newton's method. I chose to ignore the given $p_1 = 5$ for Newton's method, and instead start with p_0.

Here is the python program and output I used for this problem. I took out the line that attempted to use the variation of Newton's method from problem 6 with $p_0 = -5$ to find the root, because that did not converge.

```
#!/usr/bin/env python3
from math import cos, sin

def f(x): return x + cos(x)

def f_prime(x): return 1 - sin(x)

def newtons_method(f, f_prime, p_0, tolerance, weird_version=False):
    if weird_version:
        print(f"\nVariation of Newton's method from problem 6")
    else:
        print(f"\nNewton's method")
    print(f"{p_0=}\t{tolerance=}\n{'='*53}")
    i = 0
    p = p_0
    while abs(f(p)) > tolerance:
        i += 1
        if weird_version:
            p -= f(p) / f_prime(p_0)
        else:
            p -= f(p) / f_prime(p)
        print(f"p_{i:02} = {p:>16.10f},  residual = {abs(f(p)):>16.10f}")

def secant_method(f, p_0, p_1, tolerance):
    print(f"\nSecant method")
```

```
    print(f"{p_0=}\t{p_1=}\t{tolerance=}\n{'='*53}")
    p = [p_0, p_1]
    while abs(f(p[-1])) > tolerance:
        p.append((p[-2]*f(p[-1])-p[-1]*f(p[-2])) / (f(p[-1])-f(p[-2])))
    for i in range(len(p)):
        print(f"p_{i:02} = {p[i]:>16.10f},  residual = {abs(f(p[i])):>16.10f}")

newtons_method(f, f_prime, -5, 1e-10)
secant_method(f, -5, 5, 1e-10)
newtons_method(f, f_prime, -.9, 1e-10)
newtons_method(f, f_prime, -.9, 1e-10, weird_version=True)
secant_method(f, -.9, 5, 1e-10)
```

```
Newton's method
p_0=-5  tolerance=1e-10
=====================================================
p_01 =   109.8205607048,  residual =   108.8296839085
p_02 =   -15.9607775611,  residual =    16.9289898794
p_03 =     6.6151154617,  residual =     7.5605305903
p_04 =    -4.6000997047,  residual =     4.7121531554
p_05 =   743.6197251930,  residual =   743.0281085215
p_06 = -3090.7583906372,  residual =  3089.9158322043
p_07 =  3606.1400727092,  residual =  3607.0578723786
p_08 =  1024.2182226354,  residual =  1025.2164816044
p_09 =   -65.2592470776,  residual =    66.0148654568
p_10 =   -25.3714027324,  residual =    24.3997474637
p_11 =    -5.6369297160,  residual =     4.8385854382
p_12 =     6.5264723344,  residual =     7.4970237282
p_13 =    -3.3496519727,  residual =     4.3280855996
p_14 =     2.1051947425,  residual =     1.5958713194
p_15 =    -9.3409121107,  residual =    10.3373974311
p_16 =     0.1974775656,  residual =     1.1780421553
p_17 =    -1.2681072761,  residual =     0.9700191926
p_18 =    -0.7718165872,  residual =     0.0551716918
p_19 =    -0.7393136716,  residual =     0.0003825038
p_20 =    -0.7390851447,  residual =     0.0000000193
p_21 =    -0.7390851332,  residual =     0.0000000000

Secant method
p_0=-5  p_1=5   tolerance=1e-10
=====================================================
p_00 =    -5.0000000000,  residual =     4.7163378145
p_01 =     5.0000000000,  residual =     5.2836621855
p_02 =    -0.2836621855,  residual =     0.6763747448
p_03 =    -1.0593324054,  residual =     0.5698780527
p_04 =    -0.7046391719,  residual =     0.0572061591
p_05 =    -0.7369962894,  residual =     0.0034943006
p_06 =    -0.7391013273,  residual =     0.0000271027
p_07 =    -0.7390851257,  residual =     0.0000000125
p_08 =    -0.7390851332,  residual =     0.0000000000

Newton's method
p_0=-0.9        tolerance=1e-10
```

```
================================================================
p_01 =       -0.7438928778,   residual =       0.0080548285
p_02 =       -0.7390902113,   residual =       0.0000084988
p_03 =       -0.7390851332,   residual =       0.0000000000

Variation  of  Newton's method from problem 6
p_0=-0.9         tolerance=1e-10
================================================================
p_01 =     -0.7438928778,  residual =     0.0080548285
p_02 =     -0.7393761353,  residual =     0.0004870559
p_03 =     -0.7391030189,  residual =     0.0000299338
p_04 =     -0.7390862335,  residual =     0.0000018415
p_05 =     -0.7390852009,  residual =     0.0000001133
p_06 =     -0.7390851374,  residual =     0.0000000070
p_07 =     -0.7390851335,  residual =     0.0000000004
p_08 =     -0.7390851332,  residual =     0.0000000000

Secant method
p_0=-0.9         p_1=5    tolerance=1e-10
================================================================
p_00 =     -0.9000000000,  residual =     0.2783900317
p_01 =      5.0000000000,  residual =     5.2836621855
p_02 =     -0.6046951156,  residual =     0.2179803505
p_03 =     -0.8458696397,  residual =     0.1827890735
p_04 =     -0.7358710590,  residual =     0.0053752920
p_05 =     -0.7390133888,  residual =     0.0001200704
p_06 =     -0.7390851842,  residual =     0.0000000854
p_07 =     -0.7390851332,  residual =     0.0000000000
```

(a) With $p_0 = -5$, Newton's method took 21 iterations to reach the desired tolerance. With $p_0 = -5$ and $p_1 = 5$, the secant method took only 7 iterations. With $p_0 = -5$, the variation of Newton's method from problem 6 diverged, which I believe is related to the fact that we couldn't prove $L < 1$ in problem 6. When $L > 1$, the method will "diverge linearly" instead of converging linearly.

With $p_0 = -0.9$, Newton's method took 3 iterations to reach the desired tolerance. With $p_0 = -0.9$ and $p_1 = 5$, the secant method took 6 iterations. With $p_0 = -0.9$, the variation of Newton's method (from problem 6) took 7 iterations.

(b) When p_0 was -5 and p_1 was 5, the approximations given by Newton's method and the secant method were -0.739085133215161 (residual 0.000000000000000) and -0.739085133215134 (residual 0.000000000000045), respectively.

To obtain these results, I modified the code above to print with more precision – otherwise the approximations from each method would be the same.

(c) See the program output above.

Newton's method converges quadratically, which makes it very good, but only if p_0 is chosen well. When p_0 was 5, it took a while to get close to the answer, but once it did get close, it converged super fast. The variation of Newton's method didn't converge at all, because $f'(p_0)$ was almost exactly zero. On the other hand, when we chose $p_0 = -0.9$, Newton's method converged super fast (only 3 iterations), and the variation of Newton's method converged linearly. The secant method, on the other hand, converged in 7 iterations when p_0 was -5, and in 6 iterations when p_0 was -0.9. The secant method is more consistent because it doesn't depend as much on the initial guess, but it doesn't have the advantage of converging quadratically.