MATH 131B Homework #7

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November 29, 2024

Problem 0.1. Exercise 4.7.1: prove theorem 4.7.2.

(a)

$$\begin{aligned}
\sin(x)^2 + \cos(x)^2 &= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 + \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 \\
&= \left(\frac{e^{2ix} - 2e^0 + e^{-2ix}}{-4}\right) + \left(\frac{e^{2ix} + 2e^0 + e^{-2ix}}{4}\right) \\
&= \frac{-(e^{2ix} - 2 + e^{-2ix}) + (e^{2ix} + 2 + e^{-2ix})}{4} \\
&= 1.
\end{aligned}$$

Since $x \in \mathbb{R}$, $\sin(x)$ and $\cos(x)$ are also real numbers, so $\sin(x)^2$ and $\cos(x)^2$ are both positive. That means $\sin(x)^2$ and $\cos(x)^2$ are both less than or equal to one, so $\sin(x), \cos(x) \in [-1, 1]$.

(b)

$$\begin{aligned}
\sin'(x) &= \frac{d}{dx}\left(\frac{e^{ix} - e^{-ix}}{2i}\right) \\
&= \frac{ie^{ix} + ie^{-ix}}{2i} \\
&= \cos(x). \\
\cos'(x) &= \frac{d}{dx}\left(\frac{e^{ix} + e^{-ix}}{2}\right) \\
&= \frac{ie^{ix} - ie^{-ix}}{2} \\
&= -\sin(x).
\end{aligned}$$

(c)

$$\begin{aligned}
\sin(-x) &= \frac{e^{i(-x)} - e^{-i(-x)}}{2i} \\
&= \frac{e^{-ix} - e^{ix}}{2i} \\
&= -\sin(x). \\
\cos(-x) &= \frac{e^{i(-x)} + e^{-i(-x)}}{2} \\
&= \cos(x).
\end{aligned}$$

(d)

$$\begin{aligned}
\cos(x+y) &= \frac{e^{i(x+y)} + e^{-i(x+y)}}{2} \\
&= \frac{(e^{ix} + e^{-ix})(e^{iy} + e^{-iy})}{4} + \frac{(e^{ix} - e^{-ix})(e^{iy} - e^{-iy})}{4} \\
&= \cos(x)\cos(y) - \sin(x)\sin(y). \\
\sin(x+y) &= \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} \\
&= \frac{(e^{ix} + e^{-ix})(e^{iy} - e^{-iy})}{4i} + \frac{(e^{ix} - e^{-ix})(e^{iy} + e^{-iy})}{4i} \\
&= \cos(x)\sin(y) + \sin(x)\cos(y).
\end{aligned}$$

(e)

$$\begin{aligned}
\sin(0) &= \frac{e^{0i} - e^{-0i}}{2i} \\
&= \frac{1-1}{2i} \\
&= 0. \\
\cos(0) &= \frac{e^{0i} + e^{-0i}}{2} \\
&= \frac{1+1}{2} \\
&= 1.
\end{aligned}$$

(f)

$$\begin{aligned}
\cos(x) + i\sin(x) &= \frac{e^{ix} + e^{-ix}}{2} + i \cdot \frac{e^{ix} - e^{-ix}}{2i} \\
&= \frac{(e^{ix} + e^{-ix}) + (e^{ix} - e^{-ix})}{2} \\
&= e^{ix}. \\
e^{-ix} &= \overline{e^{ix}} \\
&= \overline{\cos(x) + i\sin(x)} \\
&= \cos(x) - i\sin(x). \\
\cos(x) &= \Re(e^{ix}). \\
\sin(x) &= \Im(e^{ix}).
\end{aligned}$$

Problem 0.2. Exercise 4.7.3: prove theorem 4.7.5.

(a) From the problem above, we have

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

and

$$\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y).$$

Plugging in $y = \pi$, this becomes

$$\begin{aligned} \cos(x+\pi) &= \cos(x)(-1) - \sin(x)(0) && = -\cos(x) \\ \sin(x+\pi) &= \cos(x)(0) + \sin(x)(-1) && = -\sin(x). \end{aligned}$$

Applying that first formula twice will give you $\cos(x + 2\pi) = \cos(x)$, and applying that second formula twice will give you $\sin(x + 2\pi) = \sin(x)$.

(b) I am going to use the lemma that if $n \in \mathbb{Z}$, then $\sin(n\pi) = 0$.

> *Proof.* **Base case:** If $n = 0$, then $\sin(n\pi) = \sin(0) = 0$.
> **Inductive step $P(n) \Rightarrow P(n+1)$:** If $\sin(n\pi) = 0$, then $\sin((n+1)\pi) = \sin(n\pi + \pi) = -\sin(n\pi) = -0 = 0$.
>
> So by induction, this statement is true for any nonnegative integer n. If n is a negative integer, then $\sin(n\pi) = \sin(-(-n)\pi) = -\sin((-n)\pi) = -0 = 0$, so this statement is actually true for any $n \in \mathbb{Z}$. □

Let x be any real number. Then we can write $x/\pi = n + a$, for some $n \in \mathbb{Z}, a \in [0, 1)$. If $a = 0$, then $x = n\pi$, so $\sin(x) = \sin(n\pi) = 0$. If $a \neq 0$, then $\sin(x) = \sin(n\pi + a\pi) = \pm\sin(a\pi)$ (also by induction), which is nonzero because $a \in (0, \pi)$. Therefore $\sin(x) = 0$ iff x/π is an integer.

(c) Using those same angle-addition identities with $x = y = \pi/2$, we get

$$\begin{aligned} -1 = \cos(\pi) &= \cos(\pi/2)^2 - \sin(\pi/2)^2 \\ 0 = \sin(\pi) &= 2\cos(\pi/2)\sin(\pi/2). \end{aligned}$$

We also know that for any real number $\sin(x)^2$ and $\cos(x)^2$ are both in $[0, 1]$, so that first equation can only be true if $\cos(\pi/2) = 0$ and $\sin(\pi/2) = \pm 1$. But we already proved that $\sin(x)$ is positive when $x \in (0, \pi)$, so $\sin(\pi/2) = 1$. Now we can use the angle-addition identity again with $y = \pi/2$ and any $x \in \mathbb{R}$:

$$\sin(x + \pi/2) = \cos(x)\sin(\pi/2) - \sin(x)\cos(\pi/2) = \cos(x).$$

So the cosine of x is zero iff $\sin(x + \pi/2) = 0$ which we just showed occurs iff $(x + \pi/2)/\pi$ is an integer. Therefore $\cos(x) = 0$ iff x/π is an integer plus $1/2$.

Problem 0.3. Exercise 4.7.5

If $re^{i\theta} = se^{i\alpha}$, then $\|re^{i\theta}\| = \|se^{i\alpha}\|$, which simplifies to $r = s$.

Now that we know $r = s$, we can divide both sides by r to get $e^{i\theta} = e^{i\alpha}$, which is equivalent to $e^{i(\theta-\alpha)} = 1$. That becomes

$$\cos(\theta - \alpha) + i\sin(\theta - \alpha) = 1,$$

which is true iff $\sin(\theta - \alpha) = 0$ and $\cos(\theta - \alpha) = 1$. We showed that $\sin(x) = 0$ iff x is an integer multiple of π, so $\theta - \alpha$ is an integer multiple of π. However, if $\theta - \alpha$ is an odd multiple of π, then $\cos(\theta - \alpha) = \cos(2\pi n + \pi)$ for some $n \in \mathbb{Z}$, and $\cos(2\pi n + \pi) = \cos(\pi) = -1 \neq 1$. Therefore $\theta - \alpha$ can only be an even multiple of 2π. That is, $\theta = \alpha + 2\pi k$ for some $k \in \mathbb{Z}$.

Problem 0.4. Exercise 4.7.6

First, I will find one value of (r, θ) which works, then I will prove it is unique.

Let $r = \|z\|$, so then $\Re(z/r)^2 + \Im(z/r)^2 = 1$. By the result from exercise 4.7.4, there is a unique $\theta \in (-\pi, \pi]$ such that $\sin(\theta) = \Im(z/r)$ and $\cos(\theta) = \Re(z/r)$. Then

$$z = r\left(\cos(\theta) + i\sin(\theta)\right) = re^{i\theta}.$$

If there was some other (r', θ') such that $z = r'e^{i\theta'}$, then $\|z\| = \|r'\| \cdot \left\|e^{i\theta'}\right\| = \|r'\| = r'$, so $r = r'$. Also, by the result from exercise 4.7.5, θ and θ' must differ by an integer multiple of 2π, but since they are both in $(-\pi, \pi]$, $\theta = \theta'$. Therefore (r, θ) is the only pair that works.

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(1) Exercise: 4.7.1, 4.7.3, 4.7.5, 4.7.6.