Math 246A HW 2

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Notes 1, Exercises 5, 9, 14, 17; Stein-Shakarchi Chapter 1, Exercises 13, 18. Due Friday, October 13th

1 Exercises from Notes 1

5 (Gauss-Lucas Theorem). Let $P(z)$ be a complex polynomial that is factored as

$$P(z) = c(z - z_1) \dots (z - z_n)$$

for some non-zero constant $c \in \mathbb{C}$ and roots $z_1, \dots, z_n \in \mathbb{C}$ (not necessarily distinct) with $n \geq 1$.

- (i) Suppose that $z_1, \dots, z_n$ all lie in the upper half-plane $\{z \in \mathbb{C} : \text{Im}(z) \geq 0\}$. Show that any root of the derivative $P'(z)$ also lies in the upper half-plane. (*Hint:* use the product rule to decompose the *log-derivative* $\frac{P'(z)}{P(z)}$ into partial fractions, and then investigate the sign of the imaginary part of this log-derivative for z outside the upper half-plane.)
- (ii) Show that all the roots of P' lie in the convex hull of the set $z_1, \dots, z_n$ of roots of P, that is to say the smallest convex polygon that contains $z_1, \dots, z_n$.

- (i) According to the product rule, the derivative of $P(z)$ is

$$P'(z) = c \sum_{i=1}^{n} \left(\prod_{j \in \{1,2,\dots n\} \setminus \{i\}} (z - z_j) \right)$$

Therefore the log-derivative is

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \dots + \frac{1}{z - z_n}$$

Outside of the upper half-plane, the imaginary component of z is negative, and since $\text{Im}(z_i) \geq 0$ that means $\text{Im}(z - z_i) < 0$. The reciprocal of $z - z_i$ is a positive multiple of the conjugate of $z - z_i$, so we can conclude that

$$\mathrm{Im}\left(\frac{1}{z - z_i}\right) > 0$$

The log-derivative of $P(z)$ is the sum of finitely many terms which all have a positive imaginary component, so the log-derivative of $P(z)$ has a positive imaginary component whenever z is outside the upper half-plane.

Now suppose z is a root of P' and z is outside the upper half-plane. This cannot be true, since we just proved that if z is outside of the upper half-plane then $P'(z)/P(z)$ has a positive imaginary component, but if z is a root of P', the log-derivative must be zero. Since that's a contradiction, any root of P' must be in the upper half-plane.

- (ii) According to a lemma shared in Reid Johnson's office hours, the complex hull Ω of a finite set of complex numbers is closed and convex, and therefore satisfies the property the following property:

  For any $z \in \mathbb{C}\backslash\Omega$, there exists a complex number c and a real number θ such that

  $$z \in \{c + e^{i\theta}t : \mathrm{Im}(t) < 0\}$$

  and

  $$\Omega \subset \{c + e^{i\theta}t : \mathrm{Im}(t) \geq 0.\}$$

  Suppose Ω is the complex hull of the roots of P and z_0 is a root of P' which lies outside Ω. Let c and θ be constants which satisfy the property above.

  For each root z_i of P, $e^{-i\theta}(z_i - c)$ must be in the upper half-plane, but $e^{-i\theta}(z_0 - c)$ is not in the upper half-plane. However, this implies the polynomial $P(c + e^{i\theta}z)$ has roots only in the upper half plane. By the chain rule, the derivative of that polynomial is $e^{i\theta}P'(c + e^{i\theta}z)$, which has at least one root (namely, $e^{-i\theta}(z_0 - c)$) outside the upper half-plane. That contradicts the result from part (i) though, so we have proven that every root of P' lies in the convex hull of the roots of P.

9 (Ratio test). If $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is a formal power series with the a_n non-zero for all sufficiently large n, show that the radius of convergence R of the series obeys the lower bound

$$\limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \geq R \geq \liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \tag{1}$$

In particular, if the limit $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ exists, then it is equal to R. Give examples to show that strict inequality can hold in either of the bounds in (1). (For an extra challenge, provide an example where *both* bounds are simultaneously strict.)

Suppose $\limsup_{n\to\infty} |a_n|/|a_{n+1}| < R$. Then there exists a natural number N such that for any $n \geq N$, the ratio $|a_n|/|a_{n+1}|$ is strictly less than R. That implies that for any $n \geq N$,

$$|a_n| > R^{N-n}|a_N|.$$

Since both sides of that inequality are positive, we also know that

$$\frac{1}{\sqrt[n]{|a_n|}} < \frac{1}{\sqrt[n]{R^{N-n}|a_N|}}.$$

This is useful because the radius of convergence R has been defined in Notes 1, equation (3), as

$$R := \liminf_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}.$$

The bound we found above is true for any $n \geq N$, but for the sake of comparing limit inferiors, we can ignore a finite number of terms at the beginning, so we have the inequality:

$$R < \liminf_{n\to\infty} \frac{1}{\sqrt[n]{R^{N-n}|a_N|}}.$$

The right hand side of that inequality can be rewritten as

$$\liminf_{n\to\infty} \left(|a_N|^{-1/n} R^{1-N/n}\right) = \liminf_{n\to\infty} \frac{R}{\sqrt[n]{|a_N|R^N}} = R.$$

Plugging that back in to the last inequality, we get $R < R$, which cannot be true. Therefore we conclude that

$$\limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \geq R.$$

The proof to show that

$$R \geq \liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

is exactly the same, except with all inequalities reversed. Of course, if the limit of $|a_n|/|a_{n+1}|$ exists, then it is equal to the limit supremum and the limit inferior, so it must be equal to R.

One example (inspired by `https://math.stackexchange.com/a/1708611`) of a case where both bounds are simultaneously strict is when

$$a_n = 2 + (-1)^n$$

because in that case, we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} &= 3 \\ R = \liminf_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} &= 1 \\ \liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} &= \frac{1}{3} \end{aligned}$$

14.

- (i) (Summation by parts formula) Let $a_0, a_1, a_2, \dots, a_N$ be a finite sequence of complex numbers, and let $A_n := a_0 + \cdots + a_n$ be the partial sums for $n = 0, \dots, N$. Show that for any complex numbers $b_0, \dots, b_N$ that

$$\sum_{n=0}^{N} a_n b_n = \sum_{n=0}^{N-1} A_n(b_n - b_{n+1}) + b_N A_N$$

- (ii) Let $a_0, a_1, \dots$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} a_n$ is convergent (not necessarily absolutely) to zero. Show that for any $0 < r < 1$, the series $\sum_{n=0}^{\infty} a_n r^n$ is absolutely convergnt, and

$$\lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n r^n = 0$$

(*Hint:* use summation by parts and a limiting argument to express $\sum_{n=0}^{\infty} a_n r^n$ in terms of the partial sums $A_n = a_0 + \cdots + a_n$.)

- (iii) (Abel's theorem) Let $F(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series with a finite positive radius of convergence R, and let $z_1 := z_0 + R \cdot e^{i\theta}$ be a point on the boundary of the disk of convergence at which the series $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges (not necessarily absolutely). Show that $\lim_{r \to R^-} F(z_0 + re^{i\theta}) = F(z_1)$. (*Hint:* use various translations and rotations to reduce to the case considered in (ii).)

- (i) First, consider the base case ($N = 1$):

$$\begin{aligned} \sum_{n=0}^{N-1} A_n(b_n - b_{n+1}) + b_N A_N &= \\ \sum_{n=0}^{0} A_n(b_n - b_{n+1}) + b_1 A_1 &= \\ a_0(b_0 - b_1) + b_1(a_0 + a_1) &= \\ a_0 b_0 + a_1 b_1 &= \sum_{n=0}^{N} a_n b_n \end{aligned}$$

Therefore the statement is true when $N = 1$. If it is true for some natural number N, then

$$\sum_{n=0}^{(N+1)-1} A_n(b_n - b_{n+1}) + b_{N+1}A_{N+1} =$$

$$\sum_{n=0}^{N} A_n(b_n - b_{n+1}) + b_{N+1}(A_N + a_{N+1}) =$$

$$\sum_{n=0}^{N} A_n b_n + b_{N+1}a_{N+1} =$$

$$\left(\sum_{n=0}^{N-1} A_n b_n + b_N a_N\right) + b_{N+1}a_{N+1} =$$

$$\sum_{n=0}^{N} a_n b_n + b_{N+1}a_{N+1} = \sum_{n=0}^{N+1} a_n b_n$$

meaning the statement is true for $N+1$ as well. By induction, the statement must be true for any natural number N.

- (ii)
- (iii)

17 (Taylor expansion and uniqueness of power series). Let $F(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with a positive radius of convergence. Show that $a_n = \frac{1}{n!}F^{(n)}(z_0)$, where $F^{(n)}$ denotes the n^{th} complex derivative of F. In particular, if $G(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ is another power series around z_0 with a positive radius of convergence which agrees with F on some neighborhood U of z_0 (thus, $F(z) = G(z)$ for all $z \in U$), show that the coefficients of F and G are identical, that is to say that $a_n = b_n$ for all $n \geq 0$.

According to theorem 15 from Notes 1, the derivative of $F(z)$ is another power series with the same radius of convergence:

$$F'(z) = \sum_{k=0}^{\infty}(k+1)a_{k+1}(z-z_0)^k$$

Repeating that step n times, we get

$$F^{(n)}(z) = \sum_{k=0}^{\infty}\left[(k+1)(k+2)\cdots(k+n)\right]a_{k+n}(z-z_0)^k$$

on the same disc of convergence. To evaluate $F^{(n)}(z_0)$, we ignore all terms except $k=0$, since they are multiplied by $(z-z_0)^k$ and are therefore zero. The result is

$$F^{(n)}(z_0) = n!a_n \to a_n = \frac{F^{(n)}(z_0)}{n!}$$

Assume U is open, or remove the limit points so it becomes open, and suppose $F(z) = G(z)$ whenever $z \in U$. The restriction of F to U is exactly the same as the restriction of G to U. Also, U is an open set containing z_0, and U is fully contained in the radius of convergence of F. Since the derivative of $F|_U(z) = G|_U(z)$ is well defined, the following statements are all true:

$$
\begin{aligned}
F(z_0) &= G(z_0) \\
F'(z_0) &= G'(z_0) \\
\cdots &= \cdots \\
\forall n \in \mathbb{N}, F^{(n)}(z_0) &= G^{(n)}(z_0)
\end{aligned}
$$

Using the result from the first half of this problem (and the fact that factorials are nonzero), that implies $a_n = b_n$ for any nonnegative integer n.

2 Exercises from Stein & Shakarchi Chapter 1

13. Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

- (a) $\mathrm{Re}(f)$ is constant;
- (b) $\mathrm{Im}(f)$ is constant;
- (c) $|f|$ is constant;

one can conclude that f is constant.

For convenience, instead of working with $f : \mathbb{C} \to \mathbb{C}$, I'll use $u(x, y)$ to mean $\mathrm{Re}(f(x+iy))$ and $v(x, y)$ to mean $\mathrm{Im}(f(x + iy))$.

According to the Cauchy-Reimann equations (Notes 1, equations (10) and (11)), f cannot be holomorphic unless both of the following are true:

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}
\end{aligned}
$$

- (a) If the real component of f is constant, then u is constant with respect to x and with respect to y, so both sides of both of the above equations are zero. That implies v is also constant with respect to both x and y, and since u and v are both constant, f is too.
- (b) f is constant for the same reason as in part (a), except instead of using the fact that u is constant to prove v is constant, we use the fact that v is constant to prove u is constant.

- (c) If $|f|$ is constant, then so is u^2+v^2, meaning

$$\begin{aligned}
0 &= \Delta(u^2+v^2)\\
&= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)\\
&= \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}\right)\\
&= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+2u\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+2v\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]\\
&= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]
\end{aligned}$$

  That last step is possible because of Laplace's equation (Notes 1, equation (13)), which we can apply to u and v because they're harmonic. From here, it's clear that all those derivatives must be zero, since otherwise, the sum of their squares would be positive. That means both u and v are constant with respect to both x and y, therefore f is constant.

18. Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.
[Hint: Write $z = z_0 + (z - z_0)$ and use the binomial expansion for z^n.]

Let a_n be the coefficient of z^n in $f(z)$, so

$$f(z)=\sum_{n=0}^{\infty}a_n z^n$$

Also let R be the radius of convergence of f and let z_0 be any complex number with magnitude less than R – that is, z_0 is in the disc of convergence of f.

Now we can rewrite $f(z)$ in terms of a binomial expansion, and guess that swapping the order of summations is valid (we will prove this shortly):

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty}a_n(z_0+(z-z_0))^n\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}a_n\binom{n}{k}z_0^{n-k}(z-z_0)^k\\
&= \sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty}a_n\binom{n}{k}z_0^{n-k}\right](z-z_0)^k.
\end{aligned}$$

According to `https://qr.ae/pKtSXO`, swapping the order of the summations is allowed if it converges absolutely – that is, if

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left|a_n\binom{n}{k}z_0^{n-k}(z-z_0)^k\right|<\infty.$$

Alas, I can't figure out how to prove this.