# Physics 127 Homework #5

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**Problem 0.1.**

The Faraday tensor is defined as

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu.$$

So after the gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$, that becomes

$$F_{\mu\nu} := \partial_\mu (A_\nu + \partial_\nu \lambda) - \partial_\nu (A_\mu + \partial_\mu \lambda),$$

which simplifies to

$$F_{\mu\nu} = \partial_\mu A_\nu + \partial_\mu \partial_\nu \lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \lambda = F_{\mu\nu} + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\lambda = F_{\mu\nu},$$

so $F_{\mu\nu}$ is invariant.

Suppose we want to choose $\lambda(x)$ such that $A'_\mu = A_\mu + \partial_\mu \lambda$ satisfies $\partial_\mu A'^\mu = 0$. Then

$$\partial_\mu (A^\mu + \partial^\mu \lambda) = 0.$$

Such a $\lambda(x)$ can always be constructed, so there is always a gauge in which $\partial_\mu A^\mu = 0$. In that gauge, the equation $\partial_\mu F^{\mu\nu} = 0$ becomes

$$\begin{aligned}
0 &= \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu \\
&= \Box A^\nu - \partial^\nu 0 \\
&= \Box A^\mu.
\end{aligned}$$

**Problem 0.2.**

The Lagrangian density is

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\
&= -\frac{1}{4}\left(F^{00}F_{00} + F^{i0}F_{i0} + F^{0j}F_{0j} + F^{ij}F_{ij}\right) \\
&= -\frac{1}{4}\left(0 \cdot 0 + F^{i0}F_{i0} + F^{0j}F_{0j} + F^{ij}F_{ij}\right) \\
&= -\frac{1}{4}\left((E^i)(E_i) + (-E^j)(-E_j) + (-\varepsilon^{ijk}B_k)(-\varepsilon_{ijk}B^k)\right) \\
&= -\frac{1}{4}\left(-2E \cdot E + (\varepsilon^{ijk}\varepsilon_{ijk})(B_k B^k)\right) \\
&= -\frac{1}{4}\left(-2E^2 + 2B_k B^k\right) \\
&= \frac{1}{2}\left(E^2 - B^2\right).
\end{aligned}$$

If $A^\mu$ changes by a small amount $\delta A^\mu$ (meaning terms containing $\delta^2$ can be ignored), then the action changes by

$$\begin{aligned}\delta S &= \left(-\frac{1}{4}\int (\partial^\mu(A^\nu+\delta A^\nu)-\partial^\nu(A^\mu+\delta A^\mu))(\partial_\mu(A_\nu+\delta A_\nu)-\partial_\nu(A_\mu+\delta A_\mu))\mathrm{d}^4x\right)-\left(-\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}\mathrm{d}^4x\right)\\ &= -\frac{1}{4}\int \left(\partial^\mu\delta A^\nu F_{\mu\nu}-\partial^\nu\delta A^\mu F_{\mu\nu}+F^{\mu\nu}\partial_\mu\delta A_\nu-F^{\mu\nu}\partial_\nu\delta A_\mu\right)\mathrm{d}^4x\\ &= \int \delta A_\nu\partial_\mu F^{\mu\nu}\mathrm{d}^4x.\end{aligned}$$

The variational principle states that $\delta S=0$ for any path we integrate along, which means $\delta A_\nu\partial_\mu F^{\mu\nu}=0$, so we get Maxwell's equation $\partial_\mu F^{\mu\nu}=0$.

**Problem 0.3.**

(a) This is symmetric because

$$\begin{aligned}T^{\mu\nu} &= F^{\mu\lambda}F^\nu_\lambda+\frac{1}{4}g^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\\ &= (F^\nu_\lambda)(F^{\mu\lambda})+\frac{1}{4}g^{\nu\mu}F_{\lambda\sigma}F^{\lambda\sigma}\\ &= (-F^\lambda_\nu)(-F^{\lambda\mu})+\frac{1}{4}g^{\nu\mu}F_{\lambda\sigma}F^{\lambda\sigma}\\ &= F^{\lambda\mu}F^\lambda_\nu+\frac{1}{4}g^{\nu\mu}F_{\lambda\sigma}F^{\lambda\sigma}\end{aligned}$$

# Relativity Physics 127 Homework 5

Due Wednesday May 7th 2025, 11:59pm on gradescope.
No grace period since HW is relevant for quiz 3.

1. Show that $F_{\mu\nu}$ is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \lambda \ ,$$

with $\lambda(x)$ a scalar function of spacetime. Show that we can always choose $\lambda(x)$ in such a way that $A_\mu$ satisfies the (Lorentz-invariant) Lorenz gauge condition,

$$\partial_\mu A^\mu = 0 \ .$$

(Hint: Take a general $A_\mu$ and study the chance of $\partial_\mu A^\mu$ under a gauge transformation.) Finally show that in this gauge the Maxwell equation $\partial_\mu F^{\mu\nu} = 0$ reduces to the relativistic wave equation

$$\Box A_\mu = 0 \ , \qquad \Box \equiv \partial^\mu \partial_\mu \ .$$

2. In class we discussed the Lorentz-invariant action for the electromagnetic field is

$$I = -\frac{1}{4} \int d^4x \, F^{\mu\nu} F_{\mu\nu} \ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

Show that the Lagrangian density that appears in this action is

$$-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \ .$$

Vary the action $I$ with respect to $A_\mu(x)$ and deduce that

$$\delta S = \int d^4x \, \delta A_\nu \partial_\mu F^{\mu\nu} \ ,$$

and hence that the variational principle implies Maxwell's equation $\partial_\mu F^{\mu\nu} = 0$.

3. The electromagnetic stress tensor is given by the following expression

$$T^{\mu\nu} = F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \ ,$$

**a)** Check that $T^{\mu\nu}$ is conserved and symmetric. (Hint: you need to use all Maxwell equations.) Show that the energy and momentum densities are given by the following familiar formulas from electromagnetism.

$$T^{00} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) \ , \qquad T^{0i} = (\vec{E} \times \vec{B})_i \ .$$

Note that the momentum density is the Poynting vector.

**b)** Check that $T^{\mu\nu}$ is traceless, i.e. $T^{\mu}{}_{\mu} = 0$

**c)** Show that for a traceless, conserved and symmetric stress tensor the current

$$j^\mu = x^\nu T_\nu{}^\mu$$

is conserved, i.e. $\partial_\mu j^\mu = 0$. (The symmetry associated with this conserved quantity are dilations $x^\mu \to a x^\mu$ )

4.) Consider a change of variables $x^\mu \to x'^\mu(x)$ with inverse function $x^\mu(x')$.

**a)** Show that the Jacobian matrices of these two functions are inverse matrices:

$$\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\rho} = \delta^\mu_\rho \ , \qquad \frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\rho} = \delta^\mu_\rho \ .$$

**b)** Show that the gradient $b_\mu = \partial_\mu \phi(x)$ of a scalar function $\phi(x)$ transforms like any covariant vector,

$$b'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} b_\nu \ .$$

**c)** Assume that the electromagnetic vector potential transforms like a covariant vector,

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu \ .$$

Show that $\partial_\mu A_\nu$ does not transform like a rank-2 tensor, but that the anti-symmetric field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ ,$$

does transform like a rank-2 tensor.

5. **Extra credit - not relevant for quiz 3**] Consider a point particle with trajectory $y^\mu(s)$ and mass $m$ coupled to a relativistic scalar field $\phi(x)$ via the following

action,

$$I = -m \int ds \, \exp\left(\phi(y(s))\right) \sqrt{\dot{y}^\mu \dot{y}_\mu} \ .$$

Vary this action with respect to the particle trajectory $y^\mu(s)$. Consider the parametrization $s = y^0$ via ordinary laboratory-frame time. Show that if the particle moves non-relativistically in that frame,

$$\left| \frac{d\vec{y}}{dy^0} \right| \ll 1 \ ,$$

then the equation of motion you found above reduces to Newton's second law in a gravitational field $\vec{g} = -\vec{\nabla}\phi$. Thus $\phi(x)$ can be viewed as a fully relativistic version of the Newtonian gravitational potential. Explain why this relativistic version of Newtonian gravity does not obey the strong equivalence principle. It can be shown that light rays are not affected by gravity in this theory, which is both inconsistent with the equivalence principle and with experiment.