Homework 2

Math 182, Winter 2025

For some questions, you may wish to use the algorithms discussed in class (eg BFS or DFS). If you use them directly, you can simply call them in your algorithm. If you modify them, you must provide details of the modified version.

When designing an algorithm, you must include the following:

1. A short (1-2 sentence) sketch of the main idea of the algorithm.
2. The algorithm itself, written in pseudo-code.
3. The runtime of the algorithm.
4. A proof that the algorithm is correct.
5. A proof that the runtime is correct.

Question 1:

A binary tree is a tree in which every node has at most two children. Prove that the number of nodes with exactly two children is one less than the number of leaves (nodes with no children).

Question 2:

Prove that if G is a connected undirected graph, there is a vertex v such that G remains connected when v, and all edges containing v, are removed.

Hint: Consider the DFS search tree.

Question 3:

Given a connected undirected graph $G = (V, E)$ and an edge $(x, y) \in E$, design an efficient algorithm to detect whether there is a cycle containing the edge (x, y). For full credit, this algorithm should run in $O(|V| + |E|)$ time.

Question 4:

Scientists have collected n specimens and are attempting to sort them into two groups. They do this by examining pairs of specimens, and determining if they belong in the same groups or different groups. If it is unclear, they will simply not make a decision. They worry that they may have made inconsistent judgments: maybe it is impossible to sort the specimens into two groups.

Given n specimens and m judgments of "same" or "different" for pairs of specimens, design an efficient algorithm to determine whether the judgments are consistent. That is, determine whether or not the specimens can be labeled A or B, such that for each pair (i, j) labeled "same" i and j will be labeled the same, and for each pair (i, j) labeled "different", i and j will be labeled differently. For full points, the algorithm should run in $O(n + m)$ time.

Question 5:

Extra Practice: This problem will not be graded. Suppose the breadth-first tree and depth-first tree produced from a graph G are the same, and include all vertices of G. Prove that $G = T$. (That is, prove that G cannot contain any edges that do not belong to T.)

Question 6:

Extra Practice: This problem will not be graded. Prove that if G has n vertices, all of which have degree at least $n/2$, then G is connected.