Practice Midterm 1

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1

There is a subgraph of $K_{4,4}$ that is isomorphic to K_4. True or False?

False, because K_4 contains an 3-cycle, and $K_{4,4}$ doesn't.

2

There are exactly two (nonempty) nonisomorphic regular trees. True or False? (A graph is *regular* if every vertex has the same degree.)

True, they'e the isolated vertex P_0 and the graph P_1. Since every tree has at least one leaf, this implies all vertices are leaves.

A more elegant method is to use the handshaking lemma, which gives $2(n-1) = dn$, where d is the degree of each vertex and n is the number of vertices. Then the only valid solutions are $(n, d) = (2, 1)$ and $(n, d) = (1, 0)$.

3

The relation $x \preceq y$ if $|x| \leq |y|$ is an ordering on the set

$$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}.$$

True or False?

False. $1 \preceq -1$ and $-1 \preceq 1$, but $1 \neq -1$. Since the relation is not antisymmetric, it's not an ordering.

4

> Suppose that there exist two connected graphs G and H and a bijection from $f : V(G) \to V(H)$ such that $d_G(u, v) = d_H(f(u), f(v))$ for every two vertices u and v of G. Then, G and H are isomorphic. True or False?

True. This means u and v are distance 1 apart iff $f(u)$ and $f(v)$ are distance 1 apart. Being distance 1 apart is the same as sharing an edge, so this is equivalent to the definition of a graph isomorphism.

5

> Let $G = (V, E)$ be a graph with $|V| = n$. If for every two nonadjacent vertices u and v of G,
>
> $$\deg_G(u) + \deg_G(v) \geq n - 1,$$
>
> then show that any two vertices are connected by a path of length ≤ 2.

If u and v are adjacent, we're done. Otherwise, consider the set of vertices adjacent to u and the set of vertices adjacent to v. The sum of the size of those two sets is at least $n - 1$, but there are only $n - 2$ vertices in G which aren't either u or v. By pigeonhole, those sets have a nonempty union, and any of the vertices in that union will share an edge with both u and v, so there is a path of length 2 from u to v.

6

> The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \binom{V}{2} - E)$. Find all trees T such that $\overline{T}$ is also a tree. *Hint:* How many vertices can T have?

T and $\overline{T}$ are both trees with $|V|$ vertices, so they must both have $|V| - 1$ edges, so $\binom{V}{2} = 2|V| - 2$. Therefore $|V|$ is either 1 or 4. Now we just gotta go through all cases, which I'm not gonna do here.

7

> How many compositions of n into k parts of size 1 and 2 are there?

There must be $n - k$ parts of size 2 and $2k - n$ parts of size 1, so the answer is $\binom{n}{n-k}$.

8

Let $G = (V, E)$ be a connected graph. An edge $e \in E$ is a *cut-edge* if $G - e$ is disconnected. Show that if G is Eulerian, then there exist no cut-edges in G.

Every vertex in G has even degree, so every vertex in $G - e$ except for exactly two have even degree. That means G has an Eulerian walk, so $G - e$ is connected, meaning no edge e is a cut-edge.