Physics 245 Homework #1

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Problem 0.1.

(a)

$$H = -\mu \cdot B = \frac{g\mu_B}{\hbar} S \cdot B = \frac{g\mu_B}{2} B \cdot \sigma = \frac{g\mu_B}{2} \begin{bmatrix} B_z & B_x \\ B_x & -B_z \end{bmatrix}$$

To find the eigenvalues, we set the characteristic polynomial $\det(H - \lambda I)$ equal to zero.

$$\begin{aligned} 0 &= \left(\frac{g\mu_B B_z}{2} - \lambda \right) \left(-\frac{g\mu_B B_z}{2} - \lambda \right) - \left(\frac{g\mu_B B_x}{2} \right)^2 \\ &= \lambda^2 - \left(\frac{g\mu_B}{2} \right)^2 (B_z^2 + B_x^2) \\ \lambda &= \pm \frac{g\mu_B}{2} \sqrt{B_z^2 + B_x^2} \end{aligned}$$

So the corresponding eigenvectors are vectors in the nullspace of

$$\frac{g\mu_B}{2} \begin{bmatrix} B_z \mp \sqrt{B_z^2 + B_x^2} & B_x \\ B_x & -B_z \mp \sqrt{B_z^2 + B_x^2} \end{bmatrix}.$$

For the positive eigenvalue, the eigenbasis is spanned by the vector

$$\begin{bmatrix} 1 \\ -B_z/B_x + \sqrt{1 + B_z^2/B_x^2} \end{bmatrix}$$

and for the negative eigenvalue, the eigenbasis is spanned by

$$\begin{bmatrix} 1 \\ -B_z/B_x - \sqrt{1 + B_z^2/B_x^2} \end{bmatrix}.$$

I will not bother normalizing the eigenvectors yet, because that would be unnecessarily ugly.

(b) If $B_z >> B_x$, then the positive eigenvalue is $\approx \frac{g\mu_B}{2} B_z$, and its corresponding eigenvector is $\approx \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The negative eigenvalue is $\approx -\frac{g\mu_B}{2} B_z$, and its corresponding eigenvector (up to a constant) is $\approx \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) If $B_x >> B_z$, then the positive eigenvalue is $\approx \frac{g\mu_B}{2} B_x$, and its corresponding eigenvector is $\approx \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The negative eigenvalue is $\approx -\frac{g\mu_B}{2} B_x$, and its corresponding eigenvector is $\approx \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Both of these eigenvectors could be normalized by just dividing by $\sqrt{2}$.

(d) Let $\hat{z'}$ be a unit vector pointing in the direction of the magnetic field B. Then

$$\hat{z'} = \frac{B_x \hat{x} + B_z \hat{z}}{\sqrt{B_x^2 + B_z^2}}.$$

Just like $\sigma_z = \sigma \cdot \hat{z}$ and $\sigma_x = \sigma \cdot \hat{x}$, we can define $\sigma_{\hat{z'}}$ to be

$$\sigma_{\hat{z'}} = \frac{B_x \sigma_x + B_z \sigma_z}{\sqrt{B_x^2 + B_z^2}} = \frac{1}{\sqrt{B_x^2 + B_z^2}} \begin{bmatrix} B_z & B_x \\ B_x & -B_z \end{bmatrix},$$

which can be factored out of the expression I found in part (a) for the Hamiltonian:

$$H = A\sigma_{\hat{z'}} = \frac{g\mu_B \sqrt{B_x^2 + B_z^2}}{2} \sigma_{\hat{z'}}.$$

Here, A is a scalar (or a scalar times the identity matrix, but that wouldn't be interesting), so I will assume the question is asking for the eigenvalues and eigenvectors of H. But those are exactly the same as they were in part (a). The eigenvalues of $\sigma_{\hat{z'}}$ are ± 1 and the eigenvectors of $\sigma_{\hat{z'}}$ are the same as the eigenvectors of H. The direction of the new z' axis is

$$\hat{z'} = \text{unit}(B) = \frac{B_x \hat{x} + B_z \hat{z}}{\sqrt{B_x^2 + B_z^2}}.$$

(e) The (normalized) positive energy eigenstate is

$$\frac{1}{\sqrt{2 + 2Bz^2/B_x^2 - 2B_z/B_x \sqrt{1 + B_z^2/B_x^2}}} \cdot \begin{bmatrix} 1 \\ -B_z/B_x + \sqrt{1 + B_z^2/B_x^2} \end{bmatrix}$$

So the expected value of spin along the z-axis is $\hbar/2$ times the first component squared, plus $-\hbar/2$ times the second component squared:

$$\langle S \rangle = \frac{\hbar}{2} \left(\frac{1}{2 + 2\frac{B_z^2}{B_x^2} - 2\frac{B_z}{B_x}\sqrt{1 + \frac{B_z^2}{B_x^2}}} \right) - \frac{\hbar}{2} \left(\frac{1 + 2\frac{B_z^2}{B_x^2} - 2\frac{B_z}{B_x}\sqrt{1 + \frac{B_z^2}{B_x^2}}}{2 + 2\frac{B_z^2}{B_x^2} - 2\frac{B_z}{B_x}\sqrt{1 + \frac{B_z^2}{B_x^2}}} \right)$$

I didn't want to convert that from LaTeXto whatever format Wolfram Alpha uses, so I used ChatGPT to simplify that expression. The result was

$$\langle S \rangle = \frac{\hbar}{2} \left(\frac{-\frac{B_z^2}{B_x^2} + \frac{B_z}{B_x}\sqrt{1 + \frac{B_z^2}{B_x^2}}}{1 + \frac{B_z^2}{B_x^2} - \frac{B_z}{B_x}\sqrt{1 + \frac{B_z^2}{B_x^2}}} \right).$$

Problem 0.2.

See the jupyter notebook a few pages below this.

Problem 0.3.

I used QuTiP to calculate these, because it would be tedious to do by hand. For each of the Pauli spin matrices, the eigenvalues are ± 1.

(a) The $\lambda = -1$ eigenvector is $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and the $\lambda = +1$ eigenvector is $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) The $\lambda = -1$ eigenvector is $\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ i\end{bmatrix}$ and the $\lambda = +1$ eigenvector is $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}$.

(c) The $\lambda = -1$ eigenvector is $\begin{bmatrix}0\\ 1\end{bmatrix}$ and the $\lambda = +1$ eigenvector is $\begin{bmatrix}1\\ 0\end{bmatrix}$.

Problem 0.4.

(a)

$$\sigma_x^2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}^2 = I$$

(b)

$$\sigma_y^2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}^2 = I$$

(c)

$$\sigma_z^2 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}^2 = I$$

(d)

$$\begin{aligned}\exp(i\theta\sigma_x) &= \left(I + \frac{i\theta}{1!}\sigma_x + \frac{i^2\theta^2}{2!}I + \frac{i^3\theta^3}{3!}\sigma_x + \cdots\right)\\ &= \cos(\theta)\cdot I + i\sin(\theta)\cdot\sigma_x\\ &= \begin{bmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{bmatrix}\end{aligned}$$

(e)

$$\begin{aligned}\exp(i\theta\sigma_y) &= \left(I + \frac{i\theta}{1!}\sigma_y + \frac{i^2\theta^2}{2!}I + \frac{i^3\theta^3}{3!}\sigma_y + \cdots\right)\\ &= \cos(\theta)\cdot I + i\sin(\theta)\cdot\sigma_y\\ &= \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\end{aligned}$$

(f)

$$\exp(i\theta\sigma_z) = \exp\left(\begin{bmatrix}i\theta & 0\\ 0 & -i\theta\end{bmatrix}\right) = \begin{bmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix}$$

Problem 0.5.

Apply a time-dependent magnetic field to an electron spin state, so that the hamiltonian is

$$H = \mu_B B_z\sigma_z + \mu_B B_x\cos(\omega t + \phi)\sigma_x.$$

Then we can define $\Omega := \mu_B B_x/\hbar$ to be the Rabi frequency, $\omega_0 := 2\mu_B B_z/\hbar$ to be the qubit frequency, and $\delta := \omega - \omega_0$ to be the detuning. Assume $\phi = 0$. Then

$$\frac{H}{\hbar} = \frac{\omega_0}{2}\sigma_z + \Omega\cos(\omega t)\sigma_x = \begin{bmatrix}\omega_0/2 & \Omega\cos(\omega t)\\ \Omega\cos(\omega t) & -\omega_0/2\end{bmatrix}.$$

We can plug that Hamiltonian into the Schrödinger equation and guess that the solution will have the form

$$|\Psi\rangle = \begin{bmatrix} a\exp\left(-i\omega_0 t/2\right) \\ b\exp\left(i\omega_0 t/2\right) \end{bmatrix},$$

where $a, b \in \mathbb{C}$ may depend on time. The Schrödinger equation for our qubit is

$$\frac{\partial}{\partial t}|\Psi\rangle = i\frac{\partial}{\partial t}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \omega_0/2 & \Omega\cos(\omega t) \\ \Omega\cos(\omega t) & -\omega_0/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = H|\Psi\rangle.$$

From that, we get two coupled differential equations for a and b:

$$\begin{aligned} i\dot{a} &= be^{i\omega_0 t}\Omega\cos(\omega t) \\ i\dot{b} &= ae^{-i\omega_0 t}\Omega\cos(\omega t). \end{aligned}$$

At this point, we will assume $\omega + \omega_0 >> \Omega$, but the driving frequency is probably close to the qubit frequency, so we cannot assume $\delta = \omega - \omega_0 >> \Omega$. This is relevant because we can now make the rotating wave approximation:

$$\begin{aligned} i\dot{a} &= be^{i\omega_0 t}\Omega\frac{e^{i\omega t} + e^{-i\omega t}}{2} \approx \frac{\Omega b}{2}e^{-i\delta t} \\ i\dot{b} &= ae^{-i\omega_0 t}\Omega\frac{e^{i\omega t} + e^{-i\omega t}}{2} \approx \frac{\Omega a}{2}e^{i\delta t}. \end{aligned}$$

Now we can use substitution to uncouple these equations:

$$\begin{aligned} i\ddot{a} &= \frac{\partial}{\partial t}\left(\frac{\Omega b}{2}e^{-i\delta t}\right) \\ &= \frac{\Omega}{2}\dot{b}e^{-i\delta t} - \frac{\Omega}{2}i\delta be^{-i\delta t} \\ &= \frac{\Omega^2}{4i}a - i\delta(i\dot{a}) \\ 0 &= \ddot{a} + i\delta\dot{a} + \frac{\Omega^2}{4}a. \end{aligned}$$

Guess that the fundamental set of solutions is $a \in \text{span}\left\{e^{\lambda_- t}, e^{\lambda_+ t}\right\}$, where $\lambda_\pm$ are roots of the following quadratic equation:

$$\begin{aligned} 0 &= \lambda^2 e^{\lambda t} + i\delta\lambda e^{\lambda t} + \frac{\Omega^2}{4}e^{\lambda t} \\ 0 &= \lambda^2 + i\delta\lambda + \frac{\Omega^2}{4} \\ \lambda &= \frac{-i\delta \pm \sqrt{-\delta^2 - \Omega^2}}{2} \\ \lambda_- &:= \frac{-i\delta - i\Omega'}{2} \\ \lambda_+ &:= \frac{-i\delta + i\Omega'}{2} \end{aligned}$$

Since we are given that $|\Psi\rangle = |0\rangle$ at time $t = 0$, we know $a = 1$ and $b = 0$ at $t = 0$, which means $\dot{a} = 0$ at

$t = 0$.

$$\begin{aligned}
a &= C_1 e^{\lambda_- t} + C_2 e^{\lambda_+ t} \\
a(t=0) &= C_1 + C_2 \\
\dot{a}(t=0) &= 0 = \lambda_- C_1 + \lambda_+ C_2 \\
\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ \frac{i}{2}(-\delta - \Omega') & \frac{i}{2}(-\delta + \Omega') \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1 - \frac{\delta}{\Omega'}\right) \\ \frac{1}{2}\left(1 + \frac{\delta}{\Omega'}\right) \end{bmatrix} \\
a &= \left(\frac{1}{2} - \frac{\delta}{2\Omega'}\right) \exp\left(\frac{-i\delta - i\Omega'}{2} t\right) + \left(\frac{1}{2} + \frac{\delta}{2\Omega'}\right) \exp\left(\frac{-i\delta + i\Omega'}{2} t\right) \\
&= \exp(-i\delta t/2) \left(\cos\left(\frac{\Omega' t}{2}\right) + \frac{i\delta}{\Omega'} \sin\left(\frac{\Omega' t}{2}\right)\right)
\end{aligned}$$

The probability of being in state $|0\rangle$ is

$$\begin{aligned}
P_0 = a^* a &= \cos^2\left(\frac{\Omega' t}{2}\right) + \frac{\delta^2}{\Omega'^2} \sin^2\left(\frac{\Omega' t}{2}\right) \\
&= 1 - \left(1 - \frac{\delta^2}{\Omega'^2}\right) \sin^2\left(\frac{\Omega' t}{2}\right) \\
&= 1 - \frac{\Omega^2}{\Omega'^2} \sin^2\left(\frac{\Omega' t}{2}\right).
\end{aligned}$$

If $\Omega t = \pi$, then $P_1 = 1 - P_0$ can be written as a function of only δ (and Ω, which is a constant):

$$\begin{aligned}
P_1 &= \frac{\Omega^2}{\Omega^2 + \delta^2} \sin^2(\pi/2) \\
&= \frac{\Omega^2}{\Omega^2 + \delta^2}.
\end{aligned}$$

See the jupyter notebook below for a plot of this function.

notebook

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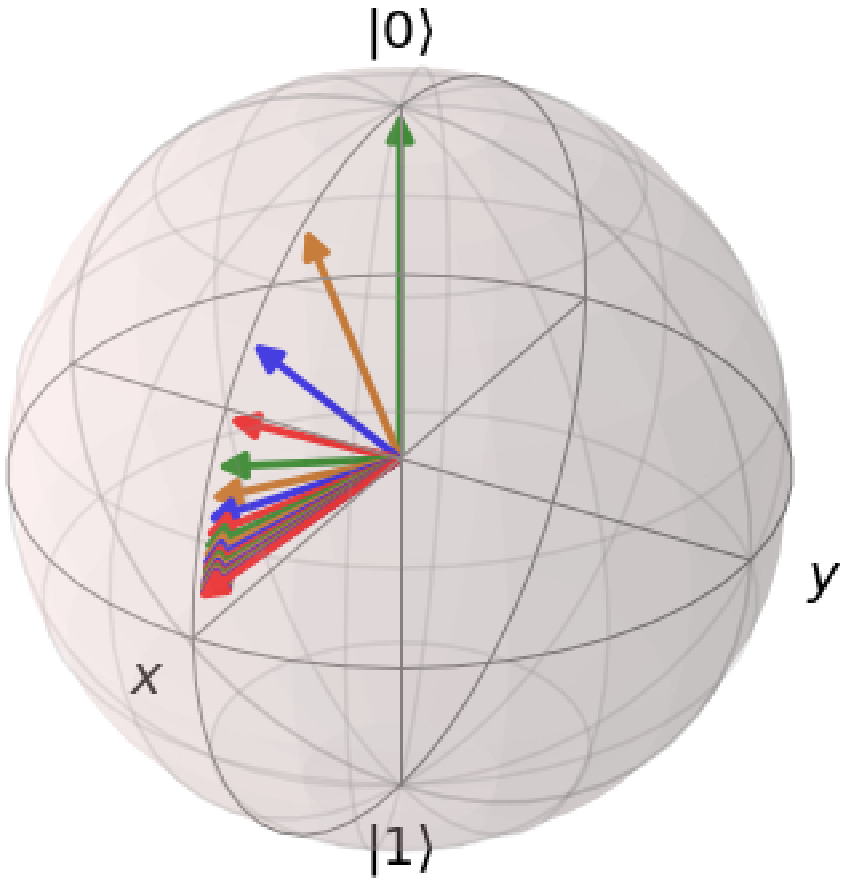
[1]:
```
import qutip as qt
import matplotlib.pyplot as plt
import numpy as np
```

[2]:
```
# Problem 2
b = qt.Bloch()

for ratio in np.linspace(0, 10, 20):
    # Rewrite formula so that you don't have to divide by (B_x / B_z), in case
 ↪it's zero
    positive_eigenvector = qt.basis(2, 0) + qt.basis(2, 1) * ratio / (1 + np.
 ↪sqrt(1 + ratio ** 2))
    b.add_states(positive_eigenvector.unit())

b.show()
```



[3]:
```
# Problem 3
print(qt.sigmax().eigenstates(), "\n\n")
print(qt.sigmay().eigenstates(), "\n\n")
print(qt.sigmaz().eigenstates())
```

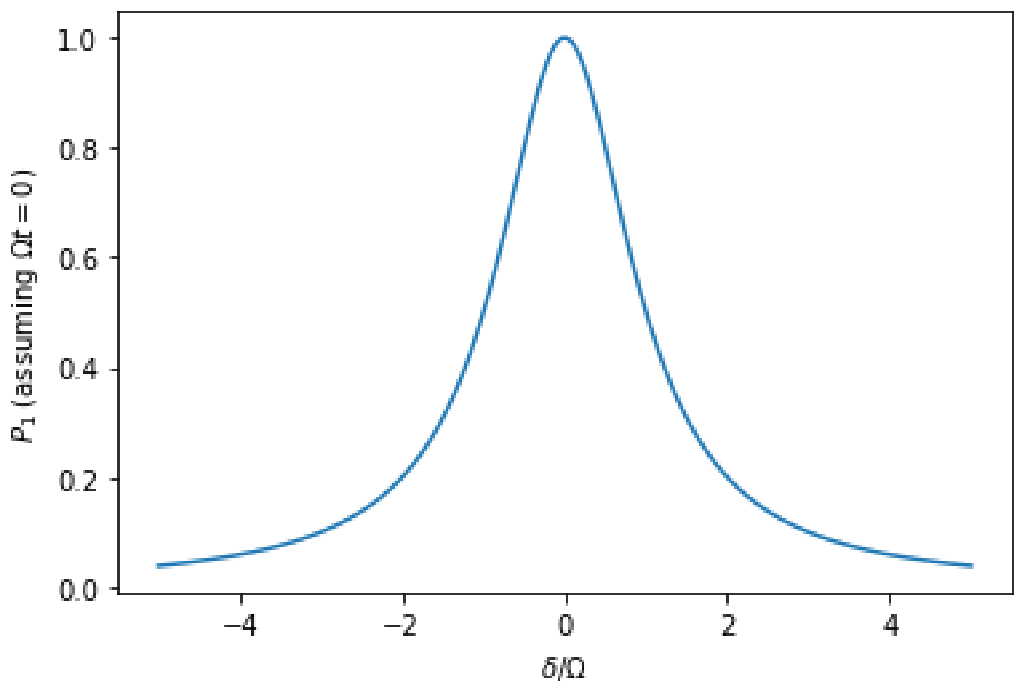
```
(array([-1.,  1.]), array([Quantum object: dims=[[2], [1]], shape=(2, 1),
type='ket', dtype=Dense
       Qobj data =
       [[-0.70710678]
        [ 0.70710678]]                                                      ,
       Quantum object: dims=[[2], [1]], shape=(2, 1), type='ket', dtype=Dense
       Qobj data =
       [[0.70710678]
        [0.70710678]]                                                       ],
      dtype=object))
```

```
(array([-1.,  1.]), array([Quantum object: dims=[[2], [1]], shape=(2, 1),
type='ket', dtype=Dense
       Qobj data =
       [[-0.70710678+0.j        ]
        [ 0.        +0.70710678j]]                                           ,
       Quantum object: dims=[[2], [1]], shape=(2, 1), type='ket', dtype=Dense
       Qobj data =
       [[-0.70710678+0.j        ]
        [ 0.        -0.70710678j]]                                           ],
      dtype=object))
```

```
(array([-1.,  1.]), array([Quantum object: dims=[[2], [1]], shape=(2, 1),
type='ket', dtype=Dense
       Qobj data =
       [[ 0.]
        [-1.]]                                                               ,
       Quantum object: dims=[[2], [1]], shape=(2, 1), type='ket', dtype=Dense
       Qobj data =
       [[-1.]
        [-0.]]                                                               ],
      dtype=object))
```

[4]:

```
# Problem 5
plt.xlabel("$\delta / \Omega$")
plt.ylabel("$P_1$ (assuming $\Omega t=0$)")
x = np.linspace(-5, 5, 200)
y = 1 / (1 + x**2)
plt.plot(x, y)
plt.show()
```


P_1 (assuming $\Omega t = 0$)
1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
2
4
δ/Ω

Phys 245 Quantum Computation
Homework 2

1. [25+5] *The Zeeman effect*. Suppose we have an electron in an uniform magnetic field described by $\vec{B} = B_x \hat{x} + B_z \hat{z}$. The electron has a magnetic moment $\vec{\mu} = -\frac{g\mu_B}{\hbar}(\hat{S}_x\hat{x} + \hat{S}_y\hat{y} + \hat{S}_z\hat{z})$ and it interacts with this magnetic field according to the Hamiltonian: $\hat{H} = -\vec{\mu} \cdot \vec{B}$. Calculate the following:
a.) [10] In the given coordinate system, calculating the eigenvalues and eigenvectors of the electron spin for arbitrary values of B_x and B_z.
b.) [5] Find the eigenvectors and eigenvalues in the limit of $B_z \gg B_x$.
c.) [5] Find the eigenvectors and eigenvalues in the limit of $B_x \gg B_z$.
d.) [5] Show how the Hamiltonian can be written in a form $\hat{H} = A\hat{\sigma}_z$, determine A, write down the eigenvectors and eigenvalues, and determine the direction of this new z-axis.
e.) [5] Bonus: Suppose your system is in the positive energy eigenstate of the Hamiltonian in part d and you make a measurement of the spin along the original z-direction (i.e. the z-direction in parts (a)-(c). What is the expectation value of that measurement?

2. [20] The Zeeman Effect and the Bloch Sphere. Take your answer from 1(a) and plot on the Bloch Sphere, using e.g. QuTIP. The positive eigenvector for the several points ranging from B_x/B_z = 0 and $B_x/B_z = 10$.

3. [30] Calculate eigenvalues and eigenvectors of the three Pauli matrices:
   a. [10] $\hat{\sigma}_x$
   b. [10] $\hat{\sigma}_y$
   c. [10] $\hat{\sigma}_z$.

4. [30] Calculate the following (remember those are matrix exponentials):
   a. [5] $\hat{\sigma}_x^2$
   b. [5] $\hat{\sigma}_y^2$
   c. [5] $\hat{\sigma}_z^2$
   d. [5] $\exp(\imath\,\theta\,\hat{\sigma}_x)$
   e. [5] $\exp(\imath\,\theta\,\hat{\sigma}_y)$
   f. [5] $\exp(\imath\,\theta\,\hat{\sigma}_z)$

5. [30] Rabi flopping with detuning. In class, we found the Rabi flopping evolution for a qubit with resonant drive. Now, redo that derivation but with $\delta \neq 0$. You may assume $\phi = 0$. Find:
   a. [25] Show that if the system initially starts in $|\psi\rangle = |0\rangle$, the probability to find it in $|0\rangle$ at a later time is: $P_0 = 1 - \frac{\Omega^2}{\Omega'^2}\sin^2\left(\frac{\Omega' t}{2}\right)$, where the generalized Rabi frequency is $\Omega' = \sqrt{\Omega^2 + \delta^2}$.
   b. [5] Suppose $\Omega t = \pi$. Plot the probability of being in $|1\rangle$ as a function of δ.