Math 115B Homework #3

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Problem 0.1.

We know similar matrices have the same eigenvalues (with the same multiplicities), and since a characteristic polynomial can be uniquely determined from the eigenvalues and their multiplicities, A and D must have the same characteristic polynomial. The matrix D can be written as

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

so its characteristic polynomial is $p_D(x) = (\lambda_1 - x)(\lambda_2 - x) = p_A(x)$. When we plug A into that polynomial, we get

$$\begin{aligned} p_A(A) &= (\lambda_1 - A)(\lambda_2 - A) \\ &= \lambda_1\lambda_2 - \lambda_1 A - \lambda_2 A + A^2 \\ &= \lambda_1\lambda_2 - \lambda_1 QDQ^{-1} - \lambda_2 QDQ^{-1} + QD^2Q^{-1} \\ &= Q\left(\lambda_1\lambda_2 I - \lambda_1 D - \lambda_2 D + D^2\right)Q^{-1} \\ &= Q\begin{bmatrix} \lambda_1\lambda_2 - \lambda_1^2 - \lambda_2\lambda_1 + \lambda_1^2 & 0 \\ 0 & \lambda_1\lambda_2 - \lambda_1\lambda_2 - \lambda_2^2 + \lambda_2^2 \end{bmatrix}Q^{-1} \\ &= 0. \end{aligned}$$

Problem 0.2.

(a) The first few powers of T acting on v are

$$T^0 v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, T^1 v = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, T^2 v = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}, T^3 v = \begin{bmatrix} 0 \\ -3 \\ 3 \\ 3 \end{bmatrix}, \ldots$$

Those first 3 vectors are clearly linearly independent, but by induction, we know that the last two components of $T^n(v)$ (that is, $e_3^* T^n v$ and $e_4^* T^n v$) will always be the same, so the dimension of the T-cyclic subspace generated by v cannot be more than 3. Therefore, $\left(v, Tv, T^2 v\right)$ is a basis for that subspace.

(b) $v = x^2$ and $Tv = 2$, and for any $n > 1, T^n v = 0$. Therefore (v, Tv) is a basis for the T-cyclic subspace generated by v.

(c) $Tv = v$, so $T^n v = v$ for any n, which means the singleton (v) is a basis for the T-cyclic subspace generated by v.

(d)

$$v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Tv = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, T^2 v = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}, \ldots$$

We can see that v and Tv are linearly independent, but $T^2v = 3Tv$, which means $T^n v \propto Tv$ whenever $n > 0$. Therefore the T-cyclic subspace generated by v is 2-dimensional, so (v, Tv) is a basis for it.

Problem 0.3.

(a) In the basis I chose in the previous problem, $T|_W$ is described by the matrix

$$T|_W = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$$

because $T^3 v = 3T^2 v - 3Tv$. Therefore the characteristic polynomial is

$$p(\lambda) = (-\lambda)(-\lambda)(3 - \lambda) - (\lambda)(-3)(1) = -\lambda^3 + 3\lambda^2 + 3\lambda.$$

(b) In the basis I chose in the previous problem, $T|_W$ is described by the matrix

$$T|_W = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

so the characteristic polynomial is $p(\lambda) = \lambda^2$.

(c) $T|_W$ is just the identity matrix, so its characteristic polynomial is $p(\lambda) = 1 - \lambda$.

(d) In the basis I chose in the previous problem, $T|_W$ is described by the matrix

$$T|_W = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix},$$

so the characteristic polynomial is $p(\lambda) = (-\lambda)(3 - \lambda) = \lambda^2 - 3\lambda$.

Problem 0.4.

Surjective is equivalent to having a right inverse, and injective is equivalent to having a left inverse

(a) T is surjective iff it has a right inverse, meaning there exists a linear map $T^{-1} : W \to V$ such that $T \circ T^{-1} = I_W$. The dual of that is $(T \circ T^{-1})^* = I_{w^*}$, and by the definition of a dual, the dual of $(T \circ T^{-1}) : W \to W$ is the linear functional which maps $w^* \in W^*$ to $w^* \circ T \circ T^{-1}$, which can also be written as

$$W^* \circ T \circ T^{-1} = (T^{-1})^* \circ (w^* \circ T) = (T^{-1})^* \circ (T^* \circ w^*) = ((T^{-1})^* \circ T^*)w^*.$$

Therefore, the dual of $(T \circ T^{-1})$ is $(T^{-1})^* \circ T^*$, and vice versa. $(T^{-1})^*$ is a left inverse of T^*, and if we did not already know T was surjective, we could have instead defined T^{-1} such that $(T^{-1})^*$ is a left inverse of T^*. Thus, a right inverse of T exists iff a left inverse of T^* exists. That's equivalent to saying T is surjective iff T^* is injective.

(b) This can be derived from the statement I proved in part (a) by replacing the symbols T^* with T, V^* with W, and W^* with V. Then the statement becomes "T^* is surjective iff T^{**} is injective", but we know T^{**} is indistinguishable from T.

Problem 0.5.

The characteristic polynomial of A is equal to

$$p_A(\lambda) = \det\left(\begin{bmatrix} -\lambda & 0 & \cdots & 0 & -a_0 \\ 1 & -\lambda & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & -a_{d-2} \\ 0 & 0 & \cdots & 1 & -a_{d-1} - \lambda \end{bmatrix}\right)$$

If $d = 1$, that matrix has only one entry, which is $-a_0 - \lambda$, therefore the characteristic polynomial is $p_A(\lambda) = (-1)^1(a_0 + \lambda)$, so the statement we want to prove is true.

If the statement is true for $d - 1$, then consider the submatrix of $A - \lambda I$ with the first row and column removed. The determinant of that matrix is

$$(-1)^{d-1}(a_1 + a_2\lambda + \cdots + a_{d-1}\lambda^{d-2} + \lambda^{d-1}).$$

Now we want to calculate the determinant of the entire matrix $A - \lambda I$, whose first row is all zeros, except for the leftmost entry, which is $-\lambda$, and the rightmost entry, which is a_0. Using the general, explicit formula for the determinant of a matrix, the determinant of $A - \lambda I$ is then

$$(-\lambda) \cdot (-1)^{d-1}(a_1 + a_2\lambda + \cdots + a_{d-1}\lambda^{d-2} + \lambda^{d-1}) + (-a_0)(1)(-1)^{d-1} = (-1)^d(a_0 + a_1\lambda + \cdots + a_{d-1}\lambda + \lambda^d).$$

That means the statement is also true for d, so by induction, it is true whenever d is a positive integer.

Problem 0.6.

Let W be any T-invariant subspace of V.

(a) Let p_W be the characteristic polynomial of T restricted to W, and let p_V be the characteristic polynomial of T. We know that p_W divides p_V, meaning there exists a polynomial q such that $p_V = q \cdot p_W$. If p_W does not split, then neither does p_V. Conversely, if p_V splits, then so does p_W.

(b) If p_V splits, then so does p_W, which means $T|_W$ has at least one eigenvalue. Since we assumed W is nonzero, that implies W has at least one eigenvector.

Problem 0.7.

(a) **Base case:** if $d = 1$, then $\sum_{i=1}^{d} v_i \in W$ implies $v_1 \in W$.

Inductive step: if this statement is true for $d - 1$, then consider whether it's true for d. Suppose $v_1, v_2, \ldots, v_d$ are eigenvectors of T with distinct eigenvalues, and $\sum_{i=1}^{d} v_i \in W$. By my assumption, $u := \sum_{i=1}^{d-1} v_i$ is in W, and $u + v_d$ is in W, which means $(u + v_d) - u = v_d$ is in W.

By induction, the statement is true for any positive integer d.

(b)

Problem 0.8.

Problem 0.9.

The Cayley-Hamilton theorem says that A^n is a linear combination of $I, A, A^2, \dots, A^{n-1}$. By induction, we also know that A^m is a linear combination of $I, A, A^2, \dots, A^{n-1}$ whenever $m \geq n$, so

$$\operatorname{span}\left(\left\{I, A, A^2, \dots\right\}\right) = \operatorname{span}\left(\left\{I, A, A^2, \dots, A^{n-1}\right\}\right)$$
$$\dim\left(\operatorname{span}\left(\left\{I, A, A^2, \dots\right\}\right)\right) = \dim\left(\operatorname{span}\left(\left\{I, A, A^2, \dots, A^{n-1}\right\}\right)\right) \leq n.$$

Math 115B: Linear Algebra
Homework 3
Due: Thursday, January 31 at 8pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. $(\frac{-}{10})$ In class, we proved the Cayley-Hamilton theorem for matrices. Let A be a 2×2 diagonalizable matrix. Prove the statement of the Cayley-Hamilton theorem directly, using the fact that $A = QDQ^{-1}$ for some invertible $Q \in k^{2\times 2}$ and some diagonal $D \in k^{2\times 2}$.

2. $(\frac{-}{4*3})$ For each linear endomorphism T on the vector space V find an ordered basis for the T-cyclic subspace generated by the vector $\vec{v}$.

   (a) $V = \mathbb{R}^4$, $T(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} w+x \\ x-y \\ w+y \\ w+z \end{pmatrix}$, $\vec{v} = \vec{e}_1$

   (b) $V = \mathbb{R}[x]_{\leq 3}$, $T(f(x)) = f''(x)$, $\vec{v} = x^2$

   (c) $V = k^{2\times 2}, T(A) = A^T, \vec{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   (d) $V = k^{2\times 2}, T(A) = L_{\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}(A), \vec{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3. $(\frac{-}{4*2})$ For each linear operator T and cyclic subspace W in the previous problem, compute the characteristic polynomial of $T|_W$.

4. $(\frac{-}{2*5})$ Let V and W be non-zero finite dimensional k-vector spaces and let $T : V \to W$ be a linear transformation.

   (a) Prove that T is onto (i.e. surjective) if and only if T^* is one-to-one (i.e. injective).

   (b) Prove that T^* is onto (i.e. surjective) if and only if T is one-to-one (i.e. injective).

5. $(\frac{-}{10})$ Fix some $d \in \mathbb{Z}^{\geq 1}$ and some scalars $a_0, ..., a_{d-1} \in k$. Let A denote the $d \times d$ matrix

$$\begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & ... & 0 & -a_1 \\ 0 & 1 & ... & 0 & -a_2 \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & ... & 0 & -a_{d-2} \\ 0 & 0 & ... & 1 & -a_{d-1} \end{pmatrix}$$

Prove that the characteristic polynomial of A is $(-1)^d(a_0 + a_1 t + ... + a_{d-1}t^{d-1} + t^d)$. (*Hint:* use induction on d, expanding the determinant along the first row.)

6. $(\frac{-}{2*10})$ Let T be a linear endomorphism of a finite dimensional vector space V.

   (a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

   (b) Deduce that if the characteristic polynomial of T splits, then any nonzero T-invariant subspace of V contains an eigenvector of T.

7. $(\frac{-}{2*5})$ Let T be a linear operator on a finite dimensional vector space V, and let W be a T-invariant subspace of V.

   (a) Suppose that $\vec{v}_1, \vec{v}_2, ..., \vec{v}_d$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $\sum_{i=1}^{d} \vec{v}_i$ is in W, then $\vec{v}_i \in W$ for all $i \in \{1, 2, ..., d\}$. (*Hint:* Induct on d.)

   (b) Suppose that $\dim(V) = n$ and T has n distinct eigenvalues. Prove that V itself is a T-cyclic subspace. (*Hint:* Use the previous part to find a vector $\vec{v}$ such that $\{\vec{v}, T(\vec{v}), ..., T^{n-1}(\vec{v})\}$ is linearly independent.)

8. $(\frac{-}{10})$ Prove that the restriction of a diagonalizable linear operator T to any non-trivial T-invariant subspace is also diagonalizable. (*Hint:* Use the first part of the previous problem.)

9. $(\frac{-}{10})$ Let $A \in k^{n \times n}$ for some $n \in \mathbb{Z}^{\geq 0}$. Prove that $\dim(\text{span}\{I_n, A, A^2, A^3, ...\}) \leq n$.