Complex Analysis Homework #4

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June 26, 2024

Problem 0.1. prob 1

(a)
$$\sin(i) = \frac{e^{i*i} - e^{-i*i}}{2i} = \frac{1/e - e}{2i} = \frac{ie}{2} - \frac{i}{2e}.$$
Alternatively, we could use the identity $\sin(ix) = i\sinh(x)$ to get
$$\sin(i) = i\sinh(1) = i\left(\frac{e}{2} - \frac{1}{2e}\right).$$

(b) The following equations are equivalent:
$$\begin{aligned} \sin(z) &= 0 \\ \frac{e^{iz} - e^{-iz}}{2i} &= 0 \\ e^{iz} &= e^{-iz} \end{aligned}$$

Let a and b be the real and imaginary components of z, respectively. Then
$$e^{iz} = e^{-b}e^{ia}.$$

This is the polar form of e^{iz} – the magnitude is e^{-b} and the argument is a. Similarly, e^{-iz} has magnitude e^b and argument $-a$. For e^{iz} and e^{-iz} to have the same magnitude, we must have $e^{-b} = e^b$ (which means $b = 0$), and for them to have the same angle, we must have $e^{ia} = e^{-ia}$. That second criterion means $e^{2ia} = 1$, so a is an integer multiple of π.

We have shown that $z = a + bi$ satisfies $\sin(z) = 0$ iff z is a (real) integer multiple of π.

Problem 0.2. prob 2

(a) The coefficients of this power series are
$$\{a_n\}_{n\in\mathbb{N}_0} = \{1, 0, 1, 0, 1, \dots\},$$
so we can use the formula for radius of convergence, which gives
$$R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{1} = 1.$$

That means that within the open disc of radius 1 about the origin, $D_1(0)$, the sum converges absolutely. Because it converges absolutely, we can rearrange it as much as we want.
$$(1 + z^2 + z^4 + \cdots)(1 - z^2) = (1 + z^2 + z^4 + \cdots) - (z^2 + z^4 + z^6 + \cdots) = 1$$
$$1 + z^2 + z^4 + \cdots = \frac{1}{1 - z^2}$$

That sum does not converge anywhere on the boundary of the disc of convergence, because if $|z| = 1$, then each term in that power series has magnitude 1, so the partial sums are not a Cauchy sequence.

(b) The coefficients of this new power series are

$$\{a_n\}_{n\in\mathbb{N}_0} = \left\{\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \cdots\right\}.$$

so the radius of convergence is

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\limsup\left\{\sqrt[n]{4^{-n-1}}\right\}} = \frac{1}{\limsup\left\{\frac{1}{4\sqrt[n]{\frac{1}{4}}}\right\}} = \frac{1}{\frac{1}{4}} = 4.$$

Within that radius of convergence, we have

$$\sum_{n=0}^{\infty} \frac{z^n}{4^{n+1}} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = \frac{1}{4}\cdot\frac{1}{1-\left(\frac{z}{4}\right)} = \frac{1}{4-z}.$$

On the boundary of the disc of convergence, every term of the power series will have magnitude $\frac{1}{4}$, so the partial sums are not a Cauchy sequence, meaning there is no point on that boundary where the power series converges.

Problem 0.3. prob 3

Problem 0.4. prob 4

Problem 0.5. prob 5

Let b_0 be any complex number, and for every positive integer n, let $b_n = \frac{a_{n-1}}{n}$. Then define another power series

$$g(z) := \sum_{n=0}^{\infty} b_n(z-a)^n.$$