Math 115B: Linear Algebra
Homework 4
Due: Thursday, February 6 at 11:59 PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. $(\frac{-}{15})$ Assume V is a finite dimensional vector space and $W_1 \subseteq V$ and $W_2 \subseteq V$ are subspaces of V. Prove directly (i.e. without citing Theorem 5.9 in our textbook) that $V = W_1 \oplus W_2$ (which in class we defined to mean that any $\vec{v} \in V$ can be written as $\vec{v} = \vec{w}_1 + \vec{w}_2$ for some $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$ in one and only one way) if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

2. $(\frac{-}{10})$ Let V be a finite-dimensional vector space with a basis $\mathcal{B}$, and let $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_m$ be a partition of $\mathcal{B}$ (that is, $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_m$ are subsets of $\mathcal{B}$ such that $\mathcal{B} = \cup_{i=1}^m \mathcal{B}_i$ and $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ if i and j are distinct elements in $\{1, 2, ..., m\}$.) Prove that $V = \oplus_{i=1}^m \text{span}(\mathcal{B}_i)$. (This is the correction to the equivalence of (e) in Theorem 5.9 in our textbook. You are welcome to cite the equivalence of (a)-(c) as part of your argument.)

3. $(\frac{-}{10})$ Let T be a linear operator on a finite-dimensional vector space V. Prove that T is diagonalizable if and only if V is the direct sum of *one-dimensional* T-invariant subspaces. (Hint: Use the fact that T is diagonalizable if and only if V is the direct sum of its eigenspaces.)

4. $(\frac{-}{5+10})$ Let V be a finite dimensional vector space.

   Definition: If W_1, W_2 be subspaces of V such that $V = W_1 \oplus W_2$, then we say that a linear endomorphism T of V is a *projection onto* W_1 *along* W_2 if, whenever $\vec{v} = \vec{v}_1 + \vec{v}_2$ for some $\vec{v}_1 \in W_1$ and $\vec{v}_2 \in W_2$, then $T(\vec{v}) = \vec{v}_1$. We also say that T is a *projection* if it a projection onto a subspace W_1 of V along some subspace W_2 of V such that $V = W_1 \oplus W_2$.

   (a) Assume T is a projection onto W_1 along W_2. Show that the range of T is W_1 and the kernel of T is W_2.

   (b) Prove that a linear endomorphism $T : V \to V$ is a projection if and only if $T = T^2$.

5. $(\frac{}{15})$ Let $n \in \mathbb{Z}^{>0}$ and let $A \in k^{n\times n}$ be the matrix

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{pmatrix}.$$

Compute the characteristic polynomial of A. (Hint: first show that A has rank 2 and that $\text{span}\{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}\}$ is L_A-invariant.)

6. $(\frac{}{10})$ Assume $T : V \to V$ is some invertible linear endomorphism of an inner product space. Show that T^* is invertible and moreover $(T^*)^{-1} = (T^{-1})^*$. (Here, T^* is the *adjoint* or *conjugatge transpose*, not the transpose.)

7. $(\frac{}{5*3})$ For each of the following inner product spaces V and linear operators T on V, evaluate the adjoint of T at the given vector in V.

   (a) $V = \mathbb{R}^2, T(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} 2a+2b \\ a-3b \end{pmatrix}, \vec{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

   (b) $V = \mathbb{C}^2, T(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \begin{pmatrix} 2z_1 + iz_2 \\ (1-i)z_1 \end{pmatrix}, \vec{v} = \begin{pmatrix} 3-i \\ 1+2i \end{pmatrix}$

   (c) $V = \mathbb{R}[x]_{\leq 1}$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx, T(f) = f' + 3f, \vec{v} = 4 - 2x$

8. $(\frac{}{5+5})$ Let T be a linear endomorphism (operator) on an inner product space V. We say any linear operator $U : V \to V$ is *self adjoint* if V is its own conjugate transpose.

   (a) Let $U_1 = T + T^*$ and $U_2 = TT^*$. Prove that U_1 and U_2 are self adjoint, that is, U_1 is its own adjoint and that U_2 is its own adjoint.

   (b) Assume T is a linear endomorphism of an inner product space V. Prove that T preserves lengths of all vectors if and only if it preserves all inner products. More precisely, prove that $||T(\vec{v})|| = ||\vec{v}||$ for all $\vec{v} \in V$ if and only if $\langle T(\vec{v}), T(\vec{w})\rangle = \langle \vec{v}, \vec{w}\rangle$ for all $\vec{v}, \vec{w} \in V$.