Relativity Physics 127 Homework 1

Due Wednesday April 9th 2025, 11:59pm on gradescope.

1.) The Einstein summation convention is defined as follows: encountering a repeated spacetime index μ (one raised and one lowered) in an expression, we sum over that index from 0 to 3, e.g. $A^\mu B_\mu = \sum_{\mu=0}^{3} A^\mu B_\mu$.

a.) Consider four-vectors v^μ and w^μ in Mikowski space, and the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Think of v^μ, w^μ as column vectors v, w, and of $g_{\mu\nu}$ as a square matrix g.

The Mikowski inner (or dot) product $v \cdot w$ between v and w is defined by the index-free expression

$$v \cdot w = v^T g w \ .$$

Here T denotes the transpose. Express this in terms of the components $v^\mu, w^\mu, g_{\mu\nu}$ using Einstein summation convention. Then show that the inner product is symmetric, $v \cdot w = w \cdot v$. Recall that you are free to rename pairs of dummy indices that are summed over.

Set $v = w = x$, where x^μ is a spacetime point and show that $x \cdot x$ reduces to the distance-squared in Minkowski space we discussed in class.

b.) Consider a Lorentz transformation $\Lambda^\mu{}_\nu$, which you can think of as a 4×4 matrix Λ with entries $\Lambda^\mu{}_\nu$. Here the left index μ always denotes the row and the right index ν the column.

Argue that the transpose matrices Λ^T should have components

$$(\Lambda^T)_\mu{}^\nu = \Lambda^\nu{}_\mu \ .$$

The condition that Λ is a Lorentz transformation, expressed in components, takes the form

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma} \ .$$

Show that this reduces to the index-free matrix equation $\Lambda^T g \Lambda = g$.

c.) Show that the Minkowski inner product $v \cdot w$ defined in 1a.) above is Lorentz invariant, i.e. if $v' = \Lambda v$ and $w' = \Lambda w$, where Λ is a Lorentz transformation satisfying the condition reviewed in 1b.) above, then show that $v' \cdot w' = v \cdot w$.

2.) A boost along the x-axis is given and a rotation around the x-axis is given by

$$\Lambda^{\mu}{}_{\nu}(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R^{\mu}{}_{\nu}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

a) Verify that both matrices are Lorentz transformations, i.e. $\Lambda^T \cdot g \cdot \Lambda = g$ and $R^T \cdot g \cdot R = g$

b) Find the inverse transformations of Λ and R and the parameter transformation of α and θ which gives the inverse.

c) Calculate $\Lambda(\alpha_1) \cdot \Lambda(\alpha_2)$ and $R(\theta_1) \cdot R(\theta_2)$ and show that they can be again written as $\Lambda(\alpha')$ and $R(\theta')$ and find α' and θ'

3.) Consider the action of a boost on a spacetime point x^μ. For simplicity we consider boosts in the x-direction, and thus also take $x^\mu = (t, x, 0, 0)$ to lie in the $t - x$ plane. If x^μ is timelike, show that you can boost to a frame where $x = 0$. Similarly, if x^μ is spacelike, show that you can boost to a frame where $t = 0$.

4.) In class we briefly discussed the fact that the Lorentz group has four disconnected components and is parameterized by six continuous parameters [See also section 1.2.4 in Coleman]. Respeat the analysis for the three dimensional Lorentz group, i.e 3×3 matrices which leave the three dimensional Minkoswski metric tensor in variant

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

i.e. satisfy

$$\Lambda^T \cdot g \cdot \Lambda = g$$

You should find that there are still four disconnected components, but it's not exactly the same due to the fact that the parity transformation

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

is actually a rotation (which one ?) and hence connected to the identity.

5.) Coleman problem 1.1 (You can either attempt a proof or just do an example path).