Physics 127 Homework #1

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Problem 0.1.

(a) The "square matrix g" is actually g^μ_ν, and the transpose of v^μ is v_μ. So including the indices, $v \cdot w$ can be written as

$$v \cdot w = v^T g w = v^\mu g_{\mu\nu} w^\nu.$$

This is symmetric because

$$w \cdot v = w^\mu g_{\mu\nu} v^\nu = w^\nu g_{\nu\mu} v^\mu = v^\mu g_{\nu\mu} w^\nu = v^\mu g_{\mu\nu} w^\nu = v \cdot w.$$

Note that I used the fact that $g_{\nu\mu} = g_{\mu\nu}$, which is valid because both sides are equal to

$$g_{\mu\nu} = e_0 \otimes e_0 - e_1 \otimes e_1 - e_2 \otimes e_2 - e_3 \otimes e_3 = g_{\nu\mu}.$$

If $v = w = x$, then

$$x \cdot x = x_0 x^0 - x_1 x^1 - x_2 x^2 - x_3 x^3.$$

(b) Let

$$\mathcal{B} = \left\{ e^0, e^1, e^2, e^3 \right\}$$

be a basis for $\mathbb{R}^4$, and for each basis vector e^μ, denote the dual basis vector as $(e^*)^\mu = e_\mu$. Then Λ^μ_ν is a linear combination of terms of the form $e^\mu_\nu := e_\nu \otimes e^\mu$. The transpose of that is $(e^T)_\nu \otimes (e^T)^\mu = e^\nu \otimes e_\mu = e^\nu_\mu$.

Since we have $(e^\mu_\nu)^T = e^\nu_\mu$ for any basis element e^μ_ν of $(\mathbb{R}^4)^* \otimes \mathbb{R}^4$, the equation

$$(\Lambda^T)^\mu_\nu = \Lambda^\nu_\mu$$

is true for any $\Lambda : \mathbb{R}^4 \to \mathbb{R}^4$.

The condition for Λ to be a Lorentz transformation can be written as

$$\Lambda^\mu_\rho g_{\mu\nu} \Lambda^\nu_\sigma = g_{\rho\sigma},$$

since matrix multiplication commutes in Einstein notation. If we want to write $g_{\mu\nu}$ as a square matrix, then that equation becomes

$$\Lambda^\rho_\mu g^\mu_\nu \Lambda^\nu_\sigma = g^\rho_\sigma,$$

which is equivalent to $\Lambda^T g \lambda = g$, where Λ and g wihtout indices denote type (1, 1) tensors (aka square matrices).

(c)

$$\begin{aligned}
v' \cdot w' &= (\Lambda v) \cdot (\Lambda w) \\
&= (\Lambda v)^T g (\Lambda w) \\
&= v^T (\Lambda^T g \Lambda) w \\
&= v^T g w \\
&= v \cdot w.
\end{aligned}$$

Problem 0.2.

(a)

$$\begin{aligned}
\Lambda^T g \Lambda &= \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ \sinh\alpha & -\cosh\alpha & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cosh^2\alpha - \sinh^2\alpha & 0 & 0 & 0 \\ 0 & \sinh^2\alpha - \cosh^2\alpha & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
&= g. \\
R^T g R &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\cos^2\theta - \sin^2\theta & 0 \\ 0 & 0 & 0 & -\sin^2\theta - \cos^2\theta \end{bmatrix} \\
&= g.
\end{aligned}$$

(b) $R(\theta)$ is a rotation of the yz-plane about the x-axis by an angle θ, so $R(\phi)$ is an inverse of $R(\theta)$ iff $\phi + \theta$ is an integer multiple of 2π. Also, given a rotation matrix $R(\theta)$, you can take the transpose to get $R(\theta)^{-1}$, since $R(\theta)$ is an orthogonal matrix.

To take the inverse of a block-diagonal matrix, you can just take the inverse of each block, so the inverse of $\Lambda(\alpha)$ is

$$\begin{aligned}
\Lambda(\alpha)^{-1} &= \begin{pmatrix} \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & 1_{2\times 2}^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{bmatrix} & 0_{2\times 2} \\ 0_{2\times 2} & 1_{2\times 2} \end{pmatrix} \\
&= \begin{pmatrix} \cosh(\alpha) & \sinh(-\alpha) & 0 & 0 \\ \sinh(-\alpha) & \cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \Lambda(-\alpha).
\end{aligned}$$

In conclusion, $R(\theta)^{-1} = R(-\theta)$ and $\Lambda(\alpha)^{-1} = \Lambda(-\alpha)$.

(c)

$$
\begin{aligned}
\Lambda(\alpha_1)\Lambda(\alpha_2) &= \begin{bmatrix} \cosh\alpha_1 & \sinh\alpha_1 & 0 & 0 \\ \sinh\alpha_1 & \cosh\alpha_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\alpha_2 & \sinh\alpha_2 & 0 & 0 \\ \sinh\alpha_2 & \cosh\alpha_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cosh\alpha_1\cosh\alpha_2 + \sinh\alpha_1\sinh\alpha_2 & \cosh\alpha_1\sinh\alpha_2 + \sinh\alpha_1\cosh\alpha_2 & 0 & 0 \\ \sinh\alpha_1\cosh\alpha_2 + \cosh\alpha_1\sinh\alpha_2 & \sinh\alpha_1\sinh\alpha_2 + \cosh\alpha_1\cosh\alpha_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cosh(\alpha_1+\alpha_2) & \sinh(\alpha_1+\alpha_2) & 0 & 0 \\ \sinh(\alpha_1+\alpha_2) & \cosh(\alpha_1+\alpha_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \Lambda(\alpha_1+\alpha_2). \\
R(\theta_1)R(\theta_2) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & 0 & -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \\ 0 & 0 & -\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ 0 & 0 & -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \\
&= R(\theta_1+\theta_2).
\end{aligned}
$$

Problem 0.3.

For simplicity, I'll ignore the x^2 and x^3 coordinates, so the point is $x^\mu = (t, x)$, and a Lorentz boost has the form

$$\Lambda^\nu_\mu = \begin{bmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{bmatrix}.$$

If x^μ is timelike, then $|t| > |x|$, so if you let $v = -x/t$, then v is a valid speed (that is, $|v| < c$), and the boosted event is

$$y^\nu = \Lambda^\nu_\mu x^\mu = \begin{bmatrix} \gamma & -\gamma x/t \\ -\gamma x/t & \gamma \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \gamma(t - x^2/t) \\ 0 \end{bmatrix},$$

which has a spatial coordinate of zero.

Similarly, if x^μ is spacelike, then $|t| < |x|$, so $v = -t/x$ is a valid speed, and the boosted event is

$$y^\nu = \Lambda^\nu_\mu x^\mu = \begin{bmatrix} \gamma & -\gamma t/x \\ -\gamma t/x & \gamma \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma(x - t^2/x) \end{bmatrix},$$

whose time coordinate is equal to zero.

Problem 0.4.

The Lorentz group in 3+1 dimensions, $O(1,3)$, is a 6-dimensional Lie group with 4 connected components.

In 2+1 dimensions, the Lorentz group $O(1,2)$ still has 4 connected components, because there is one component which contains the identity matrix, another for which time is reversed (proper & nonorthochronous), another for which the parity of space is reversed (improper & orthochronous), and another for which the parities of time and space are both reversed (improper & nonorthochronous). Those four components are all disconnected because there is no continuous way to interpolate between them. Some representative elements for the respective components are

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Note that to change the parity of space, we negate 1 of the spatial coordinates, because negating both spatial coordinates would be a proper transformation.

The dimension of each component is 3, because the identity component is generated by the three following types of boosts:

$$\begin{bmatrix}\gamma & \gamma v & 0\\ \gamma v & \gamma & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}\gamma & 0 & \gamma u\\ 0 & 1 & 0\\ \gamma u & 0 & \gamma\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\ 0 & a & 1-a^2\\ 0 & a^2-1 & a\end{bmatrix},$$

where $v, u, a \in \mathbb{R}$ and $|a| \leq 1$.

Problem 0.5. Alice and Bob both travel from spacetime point a to spacetime point b. Alice goes by the straight line path (in Minkowski space); Bob wanders around – his world line is curved. Which trip takes longer, according to each traveler's own watch?

The answer to this question doesn't change if we Lorentz boost the the frame where $a = (0,0,0,0)$ and $b = (T,0,0,0)$. This is Alice's frame, so she measures her own trip to take time T. If Bob's trip is parameterized by $x^\mu(t)$, where t goes from 0 to T, then he measures the duration of his own trip to be

$$\int_{t=0}^{t=T} \sqrt{\left(\frac{\partial x^0}{\partial t}\right)^2 - \left(\frac{\partial x^1}{\partial t}\right)^2} \mathrm{d}t.$$

But since the integrand is always positive, it's ≤ 1 everywhere, and < 1 somewhere, that integral must work out to be less than T.

Therefore, the duration Alice measures for her own trip (T) is greater than the duration Bob measures for his own trip.

Relativity Physics 127 Homework 1

Due Wednesday April 9th 2025, 11:59pm on gradescope.

1.) The Einstein summation convention is defined as follows: encountering a repeated spacetime index μ (one raised and one lowered) in an expression, we sum over that index from 0 to 3, e.g. $A^\mu B_\mu = \sum_{\mu=0}^{3} A^\mu B_\mu$.

a.) Consider four-vectors v^μ and w^μ in Mikowski space, and the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Think of v^μ, w^μ as column vectors v, w, and of $g_{\mu\nu}$ as a square matrix g.

The Mikowski inner (or dot) product $v \cdot w$ between v and w is defined by the index-free expression

$$v \cdot w = v^T g w \ .$$

Here T denotes the transpose. Express this in terms of the components $v^\mu, w^\mu, g_{\mu\nu}$ using Einstein summation convention. Then show that the inner product is symmetric, $v \cdot w = w \cdot v$. Recall that you are free to rename pairs of dummy indices that are summed over.

Set $v = w = x$, where x^μ is a spacetime point and show that $x \cdot x$ reduces to the distance-squared in Minkowski space we discussed in class.

b.) Consider a Lorentz transformation $\Lambda^\mu{}_\nu$, which you can think of as a 4×4 matrix Λ with entries $\Lambda^\mu{}_\nu$. Here the left index μ always denotes the row and the right index ν the column.

Argue that the transpose matrices Λ^T should have components

$$(\Lambda^T)_\mu{}^\nu = \Lambda^\nu{}_\mu \ .$$

The condition that Λ is a Lorentz transformation, expressed in components, takes the form

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma} \ .$$

Show that this reduces to the index-free matrix equation $\Lambda^T g \Lambda = g$.

c.) Show that the Minkowski inner product $v \cdot w$ defined in 1a.) above is Lorentz invariant, i.e. if $v' = \Lambda v$ and $w' = \Lambda w$, where Λ is a Lorentz transformation satisfying the condition reviewed in 1b.) above, then show that $v' \cdot w' = v \cdot w$.

2.) A boost along the x-axis is given and a rotation around the x-axis is given by

$$\Lambda^{\mu}{}_{\nu}(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R^{\mu}{}_{\nu}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

a) Verify that both matrices are Lorentz transformations, i.e. $\Lambda^T \cdot g \cdot \Lambda = g$ and $R^T \cdot g \cdot R = g$

b) Find the inverse transformations of Λ and R and the parameter transformation of α and θ which gives the inverse.

c) Calculate $\Lambda(\alpha_1)\cdot\Lambda(\alpha_2)$ and $R(\theta_1)\cdot R(\theta_2)$ and show that they can be again written as $\Lambda(\alpha')$ and $R(\theta')$ and find α' and θ'

3.) Consider the action of a boost on a spacetime point x^μ. For simplicity we consider boosts in the x-direction, and thus also take $x^\mu = (t, x, 0, 0)$ to lie in the $t - x$ plane. If x^μ is timelike, show that you can boost to a frame where $x = 0$. Similarly, if x^μ is spacelike, show that you can boost to a frame where $t = 0$.

4.) In class we briefly discussed the fact that the Lorentz group has four disconnected components and is parameterized by six continuous parameters [See also section 1.2.4 in Coleman]. Respeat the analysis for the three dimensional Lorentz group, i.e 3×3 matrices which leave the three dimensional Minkoswski metric tensor in variant

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

i.e. satisfy

$$\Lambda^T \cdot g \cdot \Lambda = g$$

You should find that there are still four disconnected components, but it's not exactly the same due to the fact that the parity transformation

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

is actually a rotation (which one ?) and hence connected to the identity.

5.) Coleman problem 1.1 (You can either attempt a proof or just do an example path).