CO2: Advanced Combinatorics

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1 Homework 1

Problem 1.1. Prove the following two properties of linear spaces $(V, \mathcal{E})$.

1. $\sum_{e \in \mathcal{E}} \binom{|e|}{2} = \binom{|V|}{2}$
2. $\sum_{e \ni v} (|e| - 1) = |V| - 1$

1. The left hand side

Problem 1.2. The Fano plane has cyclic representation: shift the set $\{1, 2, 4\}$ (by adding 1 to its elements) six times, using arithmetic (mod 7). Find similar representations for the finite plane of order 3 and order 4.

Order	Vertices per edge	Total vertices
2	3	7
3	4	13
4	5	21
n	$n+1$	n^2+n+1

```python
#!/usr/bin/env python3

def works(edge):
    """ Given an edge, represented as a set of vertex indices,
    returns true iff it is a valid cyclic representation of a finite plane.
    """
    n = len(edge) - 1
    num_vertices = n*n + n + 1
    for i in range(1, num_vertices):
            new_edge = {(v+i) % num_vertices for v in edge}
            if len(edge.intersection(new_edge)) != 1:
                    return False
    return True

def find_solution(n):
    """ Iterates through sets {a_i} of the form
    0 <= a_1 < a_2 < ... < a_{n+1} < num_vertices
    and returns the first one which is a valid cyclic representation of
    a finite plane. Adds one to the indices to match the way humans count.
    """
```

```
    edge = list(range(n+1))
    num_vertices = n*n + n + 1
    while not works(set(edge)):
        index = n
        edge[index] += 1
        while edge[index] > num_vertices - 1:
            index -= 1
            edge[index] += 1
        for j in range(index, n+1):
            edge[j] = edge[index] + (j - index)
    edge = {v + 1 for v in edge}
    return sorted(edge)

for bleh in range(2,6):
    print(find_solution(bleh))
```

The output of this program is

```
[1, 2, 4]
[1, 2, 4, 10]
[1, 2, 5, 15, 17]
[1, 2, 4, 9, 13, 19]
```

Problem 1.3. Work out the details of the proof of Theorem 1.4!

Problem 1.4. Prove or give counterexample for the following two statements: Regular linear spaces are uniform. Uniform linear spaces are regular.

Problem 1.5. Give a catalogue of linear spaces with six vertices. (Follow the convention of Figure 1.6.)

2 Homework 2

Problem 2.1. Prove that fiite planes of order at least 3 have proper 2-colorings.

Problem 2.2. Suppose that $\mathcal{H} = (V, \mathcal{E})$ has no singleton edges and $|e \cap f| \neq 1$ for all $e, f \in \mathcal{E}$. Prove that the greedy algorithm colors V with at most 2 colors (in every ordering of V).

Problem 2.3. Prove that Steiner triple systems have no proper 2-colorings.

Problem 2.4. Prove that the n-element subsets of a $(2n+k)$-element ground set can be partitioned into $k+2$ classes so that each class has pairwise intersecting sets.

Problem 2.5. Prove that $R(3, 4) = 9$.

3 Homework 3

Problem 3.1. Show that $R(3, 3, \ldots, 3) \leq 3r!$ (assuming r arguments).

Problem 3.2. Show that $R(3, 3, \ldots, 3) > 2^r$ (assuming r arguments). Extra: any ideas for improvement?

Problem 3.3. Show that $R_3(n, n)$ is also a good choice for $f(n)$ in Theorem 3.6. Hint: color a triple of points $\{a, b, c\}$ according to the parity of the number of points inside the triangle abc.

Problem 3.4. Prove that $R_3(p, q) \leq R_2(R_3(p - 1, q), R_3(p, q - 1)) + 1$.

Problem 3.5. Show that if the edges of a countable infinite complete graph are colored with red or blue in any fashion then there is a monochromatic infinite complete subgraph in it.

4 Extra Problems

Problem 4.1. Show that if a Steiner triple system $S(2, 3, n)$ exists then $n \equiv 1$ or $n \equiv 3 \pmod 6$.

Problem 4.2. In a group of 70 students, ofr every choice of distinct students A, B, students A knows a language which student B does not know. At least how many languages do they know together?

Problem 4.3. Use the Fano plane for making a schedule for 8 bridge players to play at 2 tables (4 at each) on 7 consecutive days so that any 3 players are together at the same table exactly once.

Problem 4.4. Modify the definition of Zykov graph Z_{n+1} so that instead of n coplies of Z_n, one copy of Z_i is used for is used for $i = 1, 2, \ldots, n$. Show that $\chi(Z_n) = n$ is still true.

Problem 4.5. Prove that the edges of the complete graph $K_{\mathbb{R}}$ can be 2-colored so that the largest complete monochromatic subgraph has size at most $\aleph_0$.

Problem 4.6. Show that if the edges of a countable infinite 3-uniform hypergraph are colored with red or blue in any fashion then there is a monochromatic infinite complete 3-uniform subhypergraph in it.

Problem 4.7. Prove that $R(n) \leq R_3(6, n)$.