Math 115B: Linear Algebra
Homework 2
Due: Thursday, January 23 at 8pm PT

- All answers should be accompanied with a full proof as justification unless otherwise stated.
- Homeworks should be submitted through Gradescope, which can be found on the course Canvas (Bruin Learn) page.
- As always, you are welcome and encouraged to collaborate on this assignment with other students in this course! However, answers must be submitted in your own words.
- Unless otherwise stated k denotes an arbitrary field and all vector spaces are over k.
- You are welcome to use results of previous problems on later problems, even if you do not solve the previous parts.

1. ($\frac{-}{5+5}$) For each of the following vector spaces V and each (ordered) basis $\mathcal{B}$, find an explicit formula for each vector in the dual basis $\mathcal{B}^*$.

   (a) $V = k^3, \mathcal{B} = \{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$.

   (b) $V = k[x]_{\leq 2}, \mathcal{B} = \{1, x, x^2\}$.

2. ($\frac{-}{5+10+5}$) Define some $f \in (\mathbb{R}^2)^*$ $f \begin{pmatrix} x \\ y \end{pmatrix} = 2x + y$ and a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ via the formula $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ x \end{pmatrix}$

   (a) Compute $T^*(f)$. (The book uses the term T^t for what we call T^*.)

   (b) Compute $[T^*]_{\mathcal{E}^*}$, where $\mathcal{E}$ is the standard ordered basis for $\mathbb{R}^2$ and $\mathcal{E}^* = \{\vec{e}_1^*, \vec{e}_2^*\}$ is the dual basis, explicitly by finding scalars a, b, c, d such that $T^*(\vec{e}_1^*) = a\vec{e}_1^* + c\vec{e}_2^*$ and $T^*(\vec{e}_2^*) = b\vec{e}_1^* + d\vec{e}_2^*$

   (c) Compute $[T]_{\mathcal{E}}$ and $([T]_{\mathcal{E}})^t$ and compare your result with your answer to the last question (you don't need to write anything about this comparison).

3. ($\frac{-}{5+5+5+5+5}$) Let V denote a finite dimensional k-vector space. For any subset $S \subseteq V$, define the *annihilator* S^0 of S as

$$S^0 := \{f \in V^* : f(x) = 0 \text{ for all } x \in S\}.$$

   (a) Prove that S^0 is a subspace of V^*. (Your proof will likely not use the fact that V is finite dimensional.)

   (b) If W is a subspace of V and $x \notin W$, prove that there exists some $f \in W^0$ such that $f(x) \neq 0$.

(c) In class, we constructed an isomorphism $\psi : V \to V^{**}$. Prove that $(S^0)^0 = \text{span}(\psi(S))$, where $\psi(S) := \{\psi(s) : s \in S\}$.

(d) For subspaces W_1 and W_2 of V, prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

(e) For subspaces W_1 and W_2, prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

4. $(\frac{-}{10})$ Prove that if W is a subspace of V, then $\dim(W) + \dim(W^0) = \dim(V)$. (For one point less: you may assume that $\dim(V) < \infty$. *Hint:* Extend an ordered basis $\{\vec{w}_1, ..., \vec{w}_k\}$ of W to an ordered basis $\mathcal{B} = \{\vec{w}_1, ..., \vec{w}_k, ..., \vec{w}_n\}$ of V. Let $\mathcal{B}^* = \{\vec{w}_1^*, ..., \vec{w}_k^*, ..., \vec{w}_n^*\}$. Prove that $\{\vec{w}_{k+1}^*, ..., \vec{w}_n^*\}$ is a basis for W^0.)

5. $(\frac{-}{15})$ Suppose that W is a finite dimensional vector space and $T : V \to W$ is a linear transformation. Prove that $\ker(T^*) = R(T)^0$.

   Here, the *kernel* of a linear transformation $U : X \to Y$ is $\{\vec{x} \in X : U(\vec{x}) = \vec{0}\}$ which is denoted as $N(U)$ in the textbook and is also referred to as the *null space* of U. Similarly, the *range* of U, written $R(U)$, is defined as $\{U(\vec{x}) : \vec{x} \in X\}$.

6. $(\frac{-}{5})$ Let R denote the 3×3 real matrix $\begin{pmatrix} -3 & -3 & -4 \\ 2 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$. Find all eigenvalues of R. For each eigenvalue, compute the corresponding eigen*space*.

7. $(\frac{-}{5})$ For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by the formula $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix}$, find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_{\mathcal{B}}$ is diagonal (and prove your answer is correct).[1]

8. $(\frac{-}{2+2+2+2+2})$ Given some vector space V and a linear *endomorphism* $T : V \to V$ (i.e. a linear transformation with the same domain and codomain, often also called a linear *operator*), we define a T*-invariant subspace* of V to be a subspace $W \subseteq V$ such that $T(W) \subseteq W$. For each of the following linear endomorphisms $T : V \to V$ determine whether the given subspace W is a T-invariant subspace of V.

   (a) $V = \mathbb{R}[x]$, $T(f(x)) = f'(x)$, $W = \mathbb{R}[x]_{\leq 2}$

   (b) $V = \mathbb{R}[x]$, $T(f(x)) = xf(x)$, $W = \mathbb{R}[x]_{\leq 2}$

   (c) $V = k^3$, $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}$, $W = \{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = x_2 = x_3\}$.

   (d) V is the set of all continuous functions $[0, 1] \to \mathbb{R}$, $T(f(t)) = (\int_0^1 f(x)dx)t$, $W = \{f \in V : f(t) = at + b \text{ for some } a, b \in \mathbb{R}\}$.

   (e) $V = k^{2\times 2}$, $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, W is the subspace of *symmetric* 2×2 matrices, i.e. those 2×2 matrices satisfying $A^t = A$.

[1] Note you don't have to 'show your work' as to how you got the answer, but make sure you are able to derive the answer on your own!