Physics 231B Homework #3

Nathan Solomon

June 14, 2024

Problem 0.1. Prove that $A_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$.

Since S_4 is generated by transpositions (12), (23), and (34), and A_4 is the subgroup of S_4 equal to the product of an even number of transpositions, A_4 is generated by pairs of transpositions. In particular, the permutations (12)(34) and (13)(24) commute with each other, because

$$(12)(34) \circ (13)(24) = (14)(23) = (13)(24) \circ (12)(34).$$

If we let $g_1 = (12)(34)$ and $g_2 = (13)(24)$, then $g_1^2 = e = g_2^2$ and $g_1 g_2 = g_2 g_1$, so $\langle g_1, g_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now define an action $\alpha : \mathbb{Z}_3 \to \mathrm{Aut}(A_4)$ such that $\alpha(1)$ is conjugation by (123).

$$\begin{aligned}
\alpha(1) &= (x \mapsto (321)x(123)) \\
\alpha(1)(g_1) &= (13)(24) = g_2 \\
\alpha(1)(g_2) &= (14)(23) = g_1 g_2 \\
\alpha(1)(g_1 g_2) &= (12)(34) = g_1.
\end{aligned}$$

Now consider a homomorphism f from A_4 to $\langle g_1, g_2 \rangle \rtimes_\alpha \mathbb{Z}_3$. To define any homomorphism, we only need to say how it acts on a set of generators, which I will choose to be $\{g_1, g_2, (123)\}$.

$$\begin{aligned}
f(g_1) &= ((g_1, e), 0) \\
f(g_2) &= ((e, g_2), 0) \\
f((123)) &= ((e, e), 1).
\end{aligned}$$

By Lagrange's theorem, there are 12 elements in the image of f, which means f is injective and surjective (because $|A_4|$ is also 12). Since we have found a bijective homomorphism from A_4 to $\langle g_1, g_2 \rangle \rtimes_\alpha \mathbb{Z}_3 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$, we can conclude those groups are isomorphic.

Problem 0.2. Prove that A_4 has no subgroup of size 6.

Using the notation from my answer to problem 1, any element of $A_4 \cong \langle g_1, g_2 \rangle \rtimes_\alpha \mathbb{Z}_3$ can be written as $((a, b), c)$, where a is either e or g_1, b is either e or g_2, and c is either 0, 1, or 2.

Now consider a subgroup of order 6. If every element in that subgroup has $c = 0$, then the subgroup would have order 1, 2, or 4, because it would be a subgroup of $\langle g_1, g_2 \rangle$. If every element in that subgroup has $a = e = b$, then that subgroup would either be the trivial group or it would be $\mathbb{Z}_3$. Therefore, to contain at least 6 elements, our subgroup must contain one element where $c \neq 0$ and at least one element where either $a \neq e$ or $b \neq e$. However, those/that element(s) alone are enough to generate a group with 12 elements. Therefore there is no subgroup of A_4 with exactly 6 elements.

Problem 0.3. Artin Chapter 6 Problem 12.2, page 193. Describe the orbits of poles for the group of rotations of an octahedron.

Poles are defined on page 184 of Artin as points which are fixed by a non-identity element of the group. Artin also notes that every pole is collinear with the center of the octahedron and either (1) a vertex of the octahedron, (2) the center of a face of the octahedron, or (3) the center of an edge of the octahedron.

In the first case, the orbit is 6 points in the shape of an octahedron, because the orbit of any vertex is the set of all vertices of the octahedron.

In the second case, the orbit is 8 points in the shape of the cube, because the orbit of any point in the center of a face is the set of points in the center of any other face.

In the third case, the orbit is 12 points in the shape of a cuboctahedron, since the orbit of any point in the center of an edge is the set of points in the centers of all other edges.

Problem 0.4. Artin Chapter 6 Problem M.1, page 193. Let G be a two-dimensional crystallographic group such that no element $g \neq 1$ fixes any point in the plane. Prove that G is generated by two translations, or else by one translation and one glide.

Suppose B is a basis for G. B cannot contain either a reflection or a rotation, because every rotation fixes one point, and every reflection fixes an entire line. Since every isometry of the plane is either a translation, a glide, a reflection, or a rotation, we can conclude that the elements of B are all translations or glides.

In order for G to be two dimensional, B must have exatly two elements. If B contains two glides, we can compose the two glides to form a translation. Replacing one of those two glides in B with their composition would not change the group generated by B.

Therefore B can be either be chosen to have two translations, or one glide and one translation.

Problem 0.5. Show that $\mathbb{R}^d \rtimes O(d)$ is isomorphic to the subgroup of $GL(d+1,\mathbb{R})$ consisting of matrices of the form
$$\begin{pmatrix} M & v \\ 0\cdots 0 & 1 \end{pmatrix},$$
where $v \in \mathbb{R}^d$ is a column vector and $M \in O(d)$.

Let A be the set of matrices of that form, and let $f : \mathbb{R}^d \rtimes O(d) \to A$ be the function which takes (v, M) to
$$\begin{pmatrix} M & v \\ 0\cdots 0 & 1 \end{pmatrix}.$$
Then f is clearly bijective, so we just need to show that it's a homomorphism.

For any $v, v' \in \mathbb{R}^d$ and $M, M' \in O(d)$,
$$\begin{aligned} f((v,M)\cdot(v',M')) &= f((v+Mv', MM')) \\ &= \begin{pmatrix} MM' & Mv'+v \\ 0\cdots 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} M & v \\ 0\cdots 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} M' & v' \\ 0\cdots 0 & 1 \end{pmatrix} \\ &= f((v,M))\cdot f((v',M')), \end{aligned}$$
so f is an isomorphism.

Problem 0.6. Consider the wallpaper group $\mathbb{Z}^2 \rtimes C_4$ we discussed in class acting on $\mathbb{R}^2$. What are the points in $\mathbb{R}^2$ which have non-trivial stabilizers? What are their stabilizer groups?

Every point in $\mathbb{Z}^2$ has a non-trivial stabilizer, because for each of those points, the stabilizer is the set of rotations about that point. Additionally, a point $(x, y) \in \mathbb{R}^2$ has a non-trivial stabilizer iff a rotation by 90°, 180°, or 270° followed by a translation by $(a, b) \in \mathbb{Z}^2$ brings it back to (x, y) – that is, iff one of the following is true:

- $x = a - y$ and $y = b + x$
- $x = a - x$ and $y = b - y$
- $x = a + y$ and $y = b - x$

If either the first of those criteria is true, then we would also have $x = (a - b) - x$ and $y = (b + a) - y$, and if the third criterion is true, then we would also have $x = (a + b) - x$ and $y = (b - a) - y$. Putting those together, we see that (x, y) has a non-trivial stabilizer if and only if both $2x$ and $2y$ are integers.