Relativity Physics 127 Homework 3

Due Wednesday April 23rd 2025, 11:59pm on gradescope.
Note that there is no grace period this time due to Quiz no 2 on April 24th.

1.) Coleman, problem (2.1)

2.) Coleman, problem (2.2)

3.) In class we discussed conservation laws that were only functions of the asymptotic particle velocities $v^{(a)\mu}_{\text{in/ out}}$ and transformed as Lorentz tensors. (See section 2.2.1 of Coleman's book.) In addition we imposed the requirement of locality, which told us that the conserved quantities all take the form $F\{v^{(a)}\} = \sum_a f_a(v^{(a)})$, i.e. no cross terms involving different particles $a \neq b$ are allowed. Based on this we showed that there were no interesting conserved Lorentz scalars, and that there was an interested conserved Lorentz vector, the 4-momentum $P^\mu = \sum_a m_a v^{(a)\mu}$.

a. Argue that Lorentz invariance and locality allow for a possible conserved rank-2 symmetric traceless quantity of the form

$$S^{\mu\nu} = \sum_a \mu_a \left(v^{(a)\mu} v^{(a)\nu} - \frac{1}{4} g^{\mu\nu} \right) \ .$$

Explain why this would lead to nine conservation laws.

b. Together P^μ and $S^{\mu\nu}$ comprise 13 conserved quantities. Argue by counting that this prohibits any elastic scattering of 2 particles going into two particles. Do this by showing that both the initial and the final state of such a process is only characterized by 12 numbers. Thus the process is over-constrained by the conservation laws and can (generically) not happen.

4.) In class I argued that the anti-symmetric second rank Lorentz tensor

$$J^{\mu\nu} = -J^{\nu\mu} = \sum_a \left(x^\mu p^\nu - x^\nu p^\mu \right)^{(a)}$$

is a conserved quantity. Note that it is clearly local (i.e. it has no $a \neq b$ cross terms) and a tensor.

(4.1) Because $J^{\mu\nu}$ explicitly depends on x^μ, it is not invariant under translations $x^{(a)\mu} \to x^{(a)\mu} + \varepsilon^\mu$. Show that such a translation shifts $J^{\mu\nu}$ by an amount involving the constant vector ε^μ, as well as the conserved 4-momentum P^μ, thus leading to another conserved quantity without introducing a new conservation law.

(4.2) Show that the spatial components J^{ij} coincide with the standard angular momentum in the non-relativistic limit where all particles move with speeds much less than 1.

(4.3) Show that the conservation equation J^{0i} = const. implies that the system's center of energy X^i (generalizing the non-relativistic center of mass) moves at constant speed according to the following formula,

$$X^i = \frac{P^i}{E} t + \text{const.} \ , \qquad E = \sum_a p^{(a)0} \ , \qquad X^i = \frac{1}{E} \sum_a \left(x^i p^0\right)^{(a)} \ .$$

5. Work through example (not problem) 2.3 in Coleman and reproduce equation 1.93 and 1.94 from the special case alluded to at the end of the example.