Midterm 1 solutions

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1

Every Hamiltonian graph is 2-connected. True or False?

True, because there is a subgraph which is a cycle. So whenever you remove an edge, either that edge is in a Hamiltonian cycle, in which case the Hamiltonian cycle becomes a path which still connects all vertices, or the edge was not in the Hamiltonian cycle, in which case the same Hamiltonian cycle still connects all vertices.

2

Every Eulerian simple graph with an even number of vertices has an even number of edges. True or False?

False. One example is the graph formed by taking C_3 and C_4 and joining them at a vertex.

3

If a graph has a closed Eulerian walk, then it has a Hamiltonian cycle. True or False?

False. Consider any bipartite graph with an odd number of vertices, each of which has even degree.

4

There exists a (not necessarily simple) graph G with no loops and more than one vertex such that its degree sequence contains no repetitions. True or False?

True. For example, take the path graph P_2 and replace one of the edges with a double edge. Then we have a multigraph with degree sequence $(1, 2, 3)$.

5

If two simple graphs have the same degree sequence, then they are isomorphic. True or False?

False. For example, the disjoint union of two 3-cycles has degree sequence $(2, 2, 2, 2, 2, 2)$, which is the same as the degree sequence of C_6.

6

How many simple graphs are there with vertex set $[n]$ (where $[n] = \{1, 2, \ldots, n\}$) and m edges (not up to isomorphism)?

If there are n labeled nodes, there are $\binom{n}{2}$ places where an edge could go, therefore

$$\binom{\binom{n}{2}}{m}$$

ways to choose which m edges to put in.

7

How many functions are there from $[n]$ to $[n]$ (where $[n] = \{1, 2, \ldots, n\}$) for which there is exactly one i such that $f(i) = i$?

First, choose any $i \in [n]$ to be the one such that $f(i) = i$. Then for each $j \in [n] \setminus \{i\}$, choose some $f(j) \in [n] \setminus \{j\}$ ($n - 1$ options for each j). In total, there are $n \cdot (n-1)^{n-1}$ such functions.

8

How many four-digit odd numbers are there that do not contain any digit more than once? (The first digit cannot be zero, so for example, 0123 is not considered a four-digit number).

- The last digit can be 1, 3, 5, 7, or 9, so the first choice we make has 5 options
- Second, choose the first digit. It can be any integer between 1 and 9, except the one we chose for the last digit, so there are 8 options
- Next, choose the second digit. It can be any integer 0-9 except the two we've already chosen, so there are 8 options
- Lastly, choose the third digit, which can be any integer 0-9 except the 3 we've already chosen, so there are 7 options.

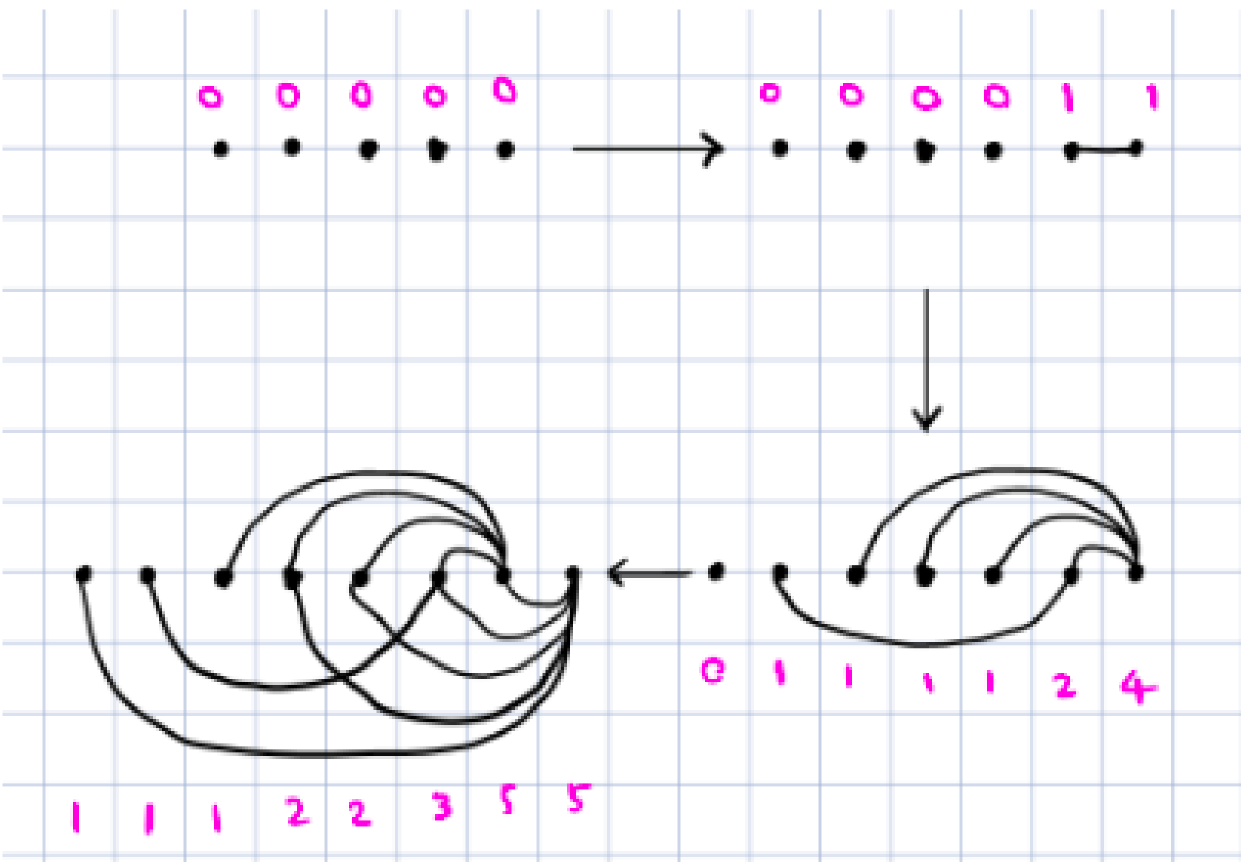
In total, there are $5 \times 8 \times 8 \times 7 = 2240$ options.

9

Which of the following are degree sequences of simple graphs? Provide a construction or a proof of impossibility for each.

1. $(1, 1, 1, 2, 2, 3, 5, 5)$
2. $(1, 1, 1, 2, 4, 5, 5, 5)$

For number (1):



For number (2): 11124555 $\rightarrow$ 0111344 $\rightarrow$ 000123 $\rightarrow$ (-1)0001 so it is not a degree sequence.