Physics 245 Homework #5

Nathan Solomon

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Problem 0.1.

(a) To show that both sides of the equation are equal, we just need to show that they act the same way on each $|n\rangle$ in the number basis:

$$\begin{aligned}
\exp\left(\frac{iH_0t}{\hbar}\right) a \exp\left(-\frac{iH_0t}{\hbar}\right) |n\rangle &= \exp\left(i\omega t\left(N+\frac{1}{2}\right)\right) a \exp\left(-i\omega t\left(N+\frac{1}{2}\right)\right) |n\rangle \\
&= \exp(i\omega tN)\, a \exp(-i\omega tN) |n\rangle \\
&= \exp(i\omega tN)\, a e^{-i\omega tn} |n\rangle \\
&= e^{-i\omega tn} \exp(i\omega tN) \sqrt{n} |n-1\rangle \\
&= \sqrt{n} e^{-i\omega tn} e^{i\omega t(n-1)} |n-1\rangle \\
&= e^{-i\omega t} \sqrt{n} |n-1\rangle \\
&= e^{-i\omega t} a |n\rangle \\
\exp\left(\frac{iH_0t}{\hbar}\right) a \exp\left(-\frac{iH_0t}{\hbar}\right) &= e^{-i\omega t} a.
\end{aligned}$$

(b) Taking the adjoint of both sides of the equation from part (a):

$$\begin{aligned}
\exp\left(\frac{iH_0t}{\hbar}\right) a \exp\left(-\frac{iH_0t}{\hbar}\right) &= a \exp(-i\omega t) \\
\exp\left(-\frac{iH_0t}{\hbar}\right)^\dagger a^\dagger \exp\left(\frac{iH_0t}{\hbar}\right)^\dagger &= \exp(-i\omega t)^\dagger a^\dagger \\
\exp\left(\frac{iH_0t}{\hbar}\right) a^\dagger \exp\left(-\frac{iH_0t}{\hbar}\right) &= \exp(i\omega t)\, a^\dagger \\
&= a^\dagger \exp(i\omega t).
\end{aligned}$$

Problem 0.2.

See the jupyter notebook below.

Problem 0.3.

See the jupyter notebook below.

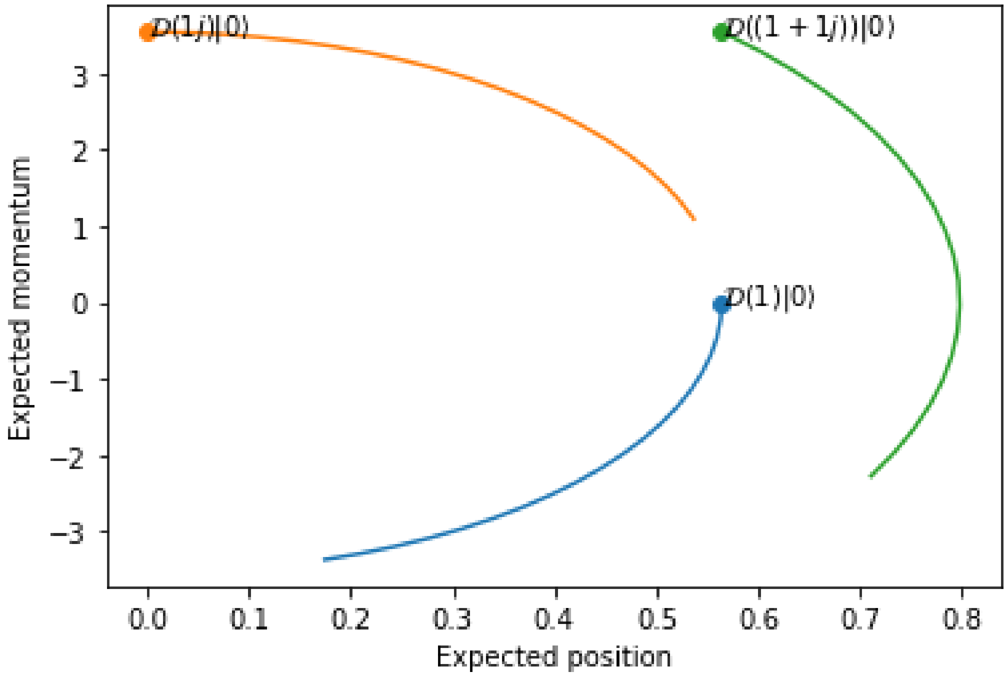
notebook

November 5, 2024

[1]:
```python
import qutip as qt
import matplotlib.pyplot as plt
import numpy as np
```

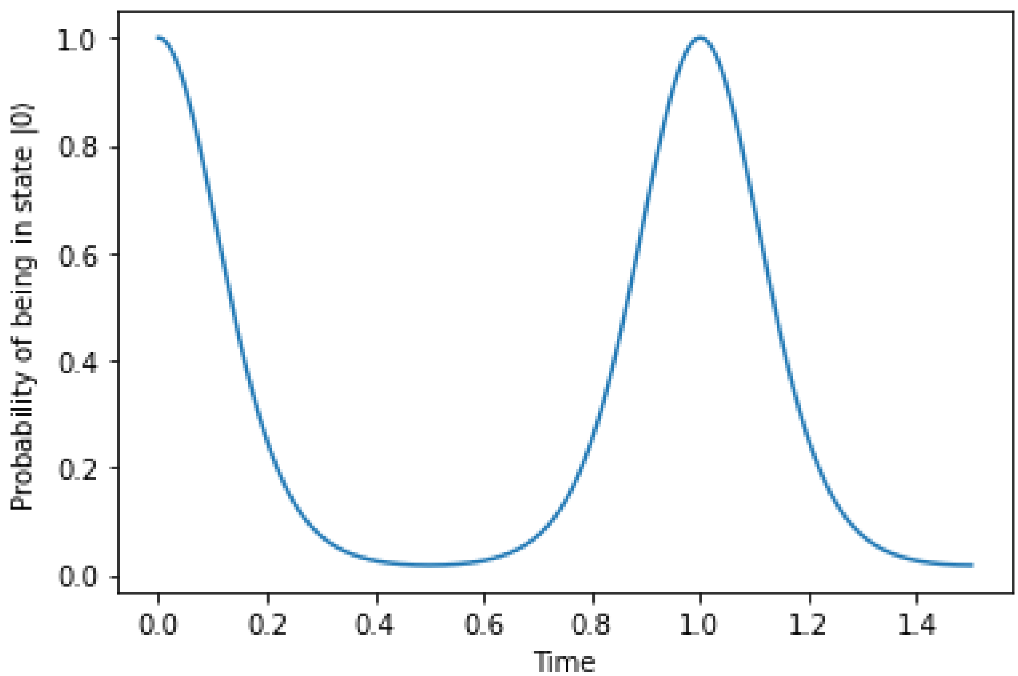
[2]:
```python
# Problem 2
N = 10
m = 1
omega = 2 * np.pi
hbar = 1

a = qt.destroy(N)
x = np.sqrt(hbar / 2 / m / omega) * (a + a.dag())
p = (-1j) * np.sqrt(hbar * m * omega / 2) * (a - a.dag())

for alpha in [1, 1j, 1+1j]:
    Psi = qt.coherent(N, alpha)
    expected_x = qt.expect(x, Psi)
    expected_p = qt.expect(p, Psi)
    plt.scatter(expected_x, expected_p)
    plt.annotate(f"$\\mathcal {{ D }} ({alpha})| 0 \\rangle$", [expected_x,␣
↪expected_p])
# Don't call plt.show() yet, because we want to see how these states change in␣
↪time

def U(t):
    return (-1j * omega * t * (qt.num(N) + qt.qeye(N) / 2)).expm()

for alpha in [1, 1j, 1+1j]:
    expected_x = []
    expected_p = []
    for t in np.linspace(0, 0.2, 20):
        Psi = U(t) * qt.coherent(N, alpha)
        expected_x.append(qt.expect(x, Psi))
        expected_p.append(qt.expect(p, Psi))
    plt.plot(expected_x, expected_p)
plt.xlabel("Expected position")
plt.ylabel("Expected momentum")
```

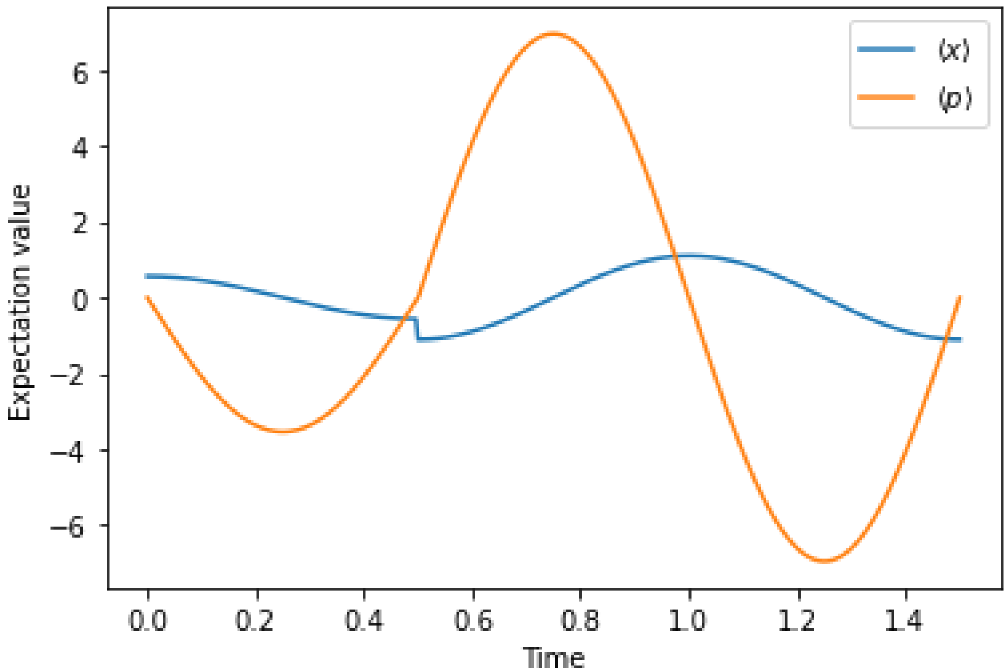
```
plt.show()
```



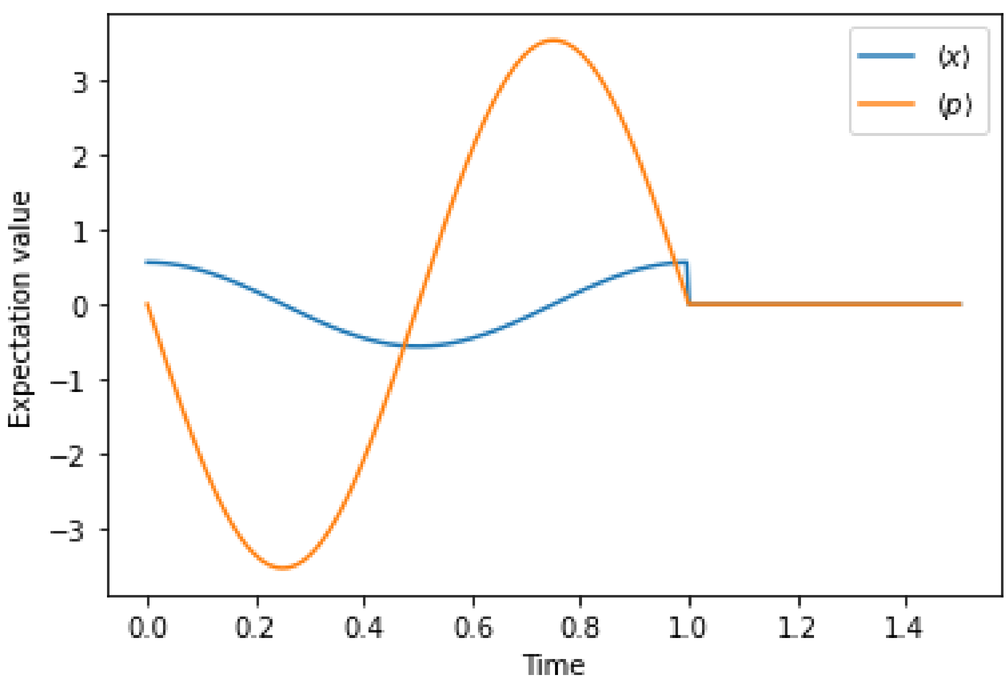
[3]:
```
# Problem 3
N = 10
m = 1
omega = 2 * np.pi
hbar = 1

a = qt.destroy(N)
x = np.sqrt(hbar / 2 / m / omega) * (a + a.dag())
p = -1j * np.sqrt(hbar * m * omega / 2) * (a - a.dag())
H = hbar * omega * (qt.num(N) + qt.qeye(N) / 2)

Psi = qt.coherent(N, 1)

def U(t, t_w):
    if t < t_w:
        return (-1j*H*t/hbar).expm()
    return (-1j*H*(t-t_w)/hbar).expm() * qt.displace(N,-1) * (-1j*H*t_w/hbar).
 ↪expm()

times = np.linspace(0, 1.5, 400)
probabilities = [qt.expect(qt.basis(N, 0).proj(), U(t_w, t_w) * Psi) for t_w in␣
 ↪times]
```

```
plt.plot(times, probabilities)
plt.xlabel("Time")
plt.ylabel("Probability of being in state $|0\\rangle$")
plt.show()

def plot(t_w):
    states = [U(t, t_w) * Psi for t in times]
    positions = [qt.expect(x, state) for state in states]
    momenta = [qt.expect(p, state) for state in states]
    plt.xlabel("Time")
    plt.ylabel("Expectation value")
    plt.plot(times, positions, label = "$\\langle x \\rangle$")
    plt.plot(times, momenta, label = "$\\langle p \\rangle$")
    plt.legend()
    plt.show()

plot(0.5)
plot(1)
```





$\langle x \rangle$
$\langle p \rangle$
Expectation value
Time



$\langle x \rangle$
$\langle p \rangle$
Expectation value
Time

Phys 245 Quantum Computation
Homework 5

1. *[30] Filling in the details*. In class, we derived the displacement operator by first transforming the driven harmonic oscillator Hamiltonian, $H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\beta\sin(\omega t + \phi)(\hat{a} + \hat{a}^\dagger)$ into the interaction picture with respect to the bare harmonic oscillator Hamiltonian, $H_o = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$. Prove the following identities which we used in class:
   a. [15] $e^{\frac{\imath H_o t}{\hbar}}\hat{a}\, e^{-\frac{\imath H_o t}{\hbar}} = \hat{a}\, e^{-\imath\omega t}$
   b. [15] $e^{\frac{\imath H_o t}{\hbar}}\hat{a}^\dagger e^{-\frac{\imath H_o t}{\hbar}} = \hat{a}^\dagger\, e^{\imath\omega t}$

2. *[40] Displacement operator via QuTip.* For this problem, use QuTip to make a coherent state via the displacement operator (it's a built-in operator) acting on the vacuum state as $|\alpha\rangle = \widehat{D}(\alpha)|0\rangle$.
   a. [15] Plot the expectation value of position and momentum in the (x,p) plan for the following states:
      i. $\alpha = 1$
      ii. $\alpha = i$
      iii. $\alpha = 1 + i$
   b. [25] Now, define a Python function that that takes time, t, as its input and returns the time evolution operator a time t for the Hamiltonian $H_o = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$. Use that function to calculate $|\alpha(t)\rangle = \widehat{U}(t)|\alpha\rangle$ for various times during the oscillation period of the oscillator. Plot the expectation value of position and momentum in the (x,p) plane for the following states (you can choose m = 1 and $\omega = 2\pi$ for simplicity):
      i. $\alpha = 1$
      ii. $\alpha = i$
      iii. $\alpha = 1 + i$

3. *[40] "Ramsey" spectroscopy with a QHO?!* Suppose at t = 0 a vacuum state is displaced by a displacement oscillator with $\alpha = 1$. Then after a variable time t_w a second displacement operator with $\alpha = -1$ is applied. For m = 1 kg, $\omega = 2\pi$ rad/s do the following (HINT: For this problem I recommend using QuTiP and the time evolution operator you constructed in Prob. 2.)
   a. [20] Plot the probability of being in the vacuum state as a function of t_w.
   b. [10] Fix t_w = 0.5 and make a plot of the oscillator's evolution as a function of time. Make sure to include the effects of the displacement operator, however, you may assume the displacements happen instantaneously.
   c. [10] Fix t_w = 1 and make a plot of the oscillator's evolution as a function of time. Make sure to include the effects of the displacement operator, however, you may assume the displacements happen instantaneously.

4. No more problems. Just study for the midterm on Friday! It will cover Qubits and Harmonic Oscillators.