Math 246A HW 3

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Notes 1, Exercises 23, 26, 27; Notes 2, Exercise 3 (i)-(iv); Stein-Shakarchi Chapter 1, Exercises 9, 25. Due Friday, October 20th.

Notes 1, Exercise 23 (Wirtinger derivatives). Let U be an open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be a Fréchet differentiable function. Define the Wirtinger derivatives $\frac{\partial f}{\partial z} : U \to \mathbb{C}$, $\frac{\partial f}{\partial \overline{z}} : U \to \mathbb{C}$ by the formulae

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)$$

- (i) Show that f is holomorphic on U if and only if the Wirtinger derivative $\frac{\partial f}{\partial \overline{z}}$ vanishes identically on U.
- (ii) If f is given by a polynomial

$$f(z) = \sum_{n,m \geq 0 : n+m \leq d} c_{n,m} z^n \overline{z}^m$$

  in both z and $\overline{z}$ for some complex coefficients $c_{n,m}$ and some natural number d, show that

$$\frac{\partial f}{\partial z}(z) = \sum_{n,m \geq 0 : n+m \leq d} c_{n,m} (n z^{n-1}) \overline{z}^m$$

  and

$$\frac{\partial f}{\partial \overline{z}}(z) = \sum_{n,m \geq 0 : n+m \leq d} c_{n,m} z^n (m \overline{z}^{m-1})$$

  (*Hint:* first establish a Leibniz rule for Wirtinger derivatives.) Conclude in particular that f is holomorphic if and only if $c_{n,m}$ vanishes whenever $m \geq 1$ (i.e. f does not contain any terms that involve $\overline{z}$).
- (iii) If z_0 is a point in U, show that one has the Taylor expansion

$$f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0)(z - z_0) + \frac{\partial f}{\partial \overline{z}}(z_0)\overline{(z - z_0)} + o(|z - z_0|)$$

  as $z \to z_0$, where $o(|z - z_0|)$ denotes a quantity of the form $|z - z_0| c(z)$, where $c(z)$ goes to zero as z goes to z_0 (compare with equation (1) in Notes 1). Conversely, show that this property determines the numbers $\frac{\partial f}{\partial z}(z_0)$ and $\frac{\partial f}{\partial \overline{z}}(z_0)$ uniquely (and thus can be used as an alternate definition of the Wirtinger derivatives).

- (i) If $\frac{\partial f}{\partial \overline{z}}$ vanishes identically on U, then at every point in U,

$$\frac{1}{2}\frac{\partial f}{\partial x} = \frac{1}{2i}\frac{\partial f}{\partial y},$$

  so f satisfies the Cauchy-Riemann equations. Since f is also Fréchet differentiable, that means f is holomorphic.

If f is holomorphic, then the partial derivatives are defined and satisfy the Cauchy-Riemann equations, which implies

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) = 0$$

at every point in U, so the Wirtinger derivative $\frac{\partial f}{\partial \overline{z}}$ vanishes identically on U if an only f is holomorphic.

- (ii) Since x and y are real numbers satisfying $z = x + iy$, we can expand the definition of f to

$$f(x+iy) = \sum_{n,m\geq 0:n+m\leq d} c_{n,m}(x+iy)^n(x-iy)^m.$$

For each term in that sum, we can use the power rule and product rule to take the partial derivatives:

$$\frac{\partial f}{\partial x} = \sum_{n,m\geq 0:n+m\leq d} c_{n,m}\left(n(x+iy)^{n-1}(x-iy)^m + m(x+iy)^n(x-iy)^{m-1}\right)$$
$$\frac{\partial f}{\partial y} = \sum_{n,m\geq 0:n+m\leq d} c_{n,m}\left(in(x+iy)^{n-1}(x-iy)^m - im(x+iy)^n(x-iy)^{m-1}\right).$$

Then applying the definitions of the Wirtinger derivatives, that becomes

$$\frac{\partial f}{\partial z} = \sum_{n,m\geq 0:n+m\leq d} c_{n,m}(nz^{n-1})\overline{z}^m$$
$$\text{and } \frac{\partial f}{\partial \overline{z}} = \sum_{n,m\geq 0:n+m\leq d} c_{n,m}z^n(m\overline{z}^{m-1}).$$

If $c_{n,m}$ vanishes whenever $m \geq 1$, then f is a polynomial in z, so f is holomorphic.

Let m_{max} be the highest m for which there exists an n such that $n + m \leq d$ and $c_{n,m} \neq 0$. When $m = m_{max}$,

$$\frac{\partial^m f}{(\partial \overline{z})^m} = \sum_{n=0}^{d-m} c_{n,m}(m!)z^n$$

which is a nonzero polynomial in z. The statement we want to prove is vacuously true if U is empty, so we assume U is nonempty. Since U is open, it must contain infinitely many points, but because it's a finite degree polynomial, by the fundamental theorem of algebra, it has finitely many roots. This implies

$$\frac{\partial f}{\partial \overline{z}} \neq 0$$

so f is holomorphic if and only if $c_{n,m} = 0$ whenever $m \geq 1$.

- (iii) Notation: let x, y, x_0, y_0 be real numbers such that

$$z = x + iy \qquad \text{and} \qquad z_0 = x_0 + iy_0.$$

COMPARE WITH DEFINITION OF FRECHET DIFFERENTIABLE

Notes 1, Exercise 26 (Maximum principle for holomorphic functions). If $f : U \to \mathbb{C}$ is a continuously twice differentiable holomorphic function on an open set U, and K is a compact subset of U, show that

$$\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|.$$

(*Hint:* use Theorem 25 and the fact that $|w| = \sup_{\theta \in \mathbb{R}} \operatorname{Re}(we^{i\theta})$ for any compex number w.) What happens if we replace the suprema on both sides by infima? This result is also known as the maximum modulus principle.

Consider the function $g : \mathbb{C} \to \mathbb{R}$ defined by $g(x+iy) = \ln|f(x+iy)|$. One can verify that

$$\begin{aligned}
\Delta g(x+iy) &= \frac{\partial^2}{\partial x^2} g(x+iy) + \frac{\partial^2}{\partial y^2} g(x+iy) \\
&= \frac{\partial^2}{\partial x^2} \ln|f(x+iy)| + \frac{\partial^2}{\partial y^2} \ln|f(x+iy)| \\
&= \frac{\partial}{\partial x} \frac{f'(x+iy)}{f(x+iy)} + \frac{\partial}{\partial y} \frac{if'(x+iy)}{f(x+iy)} \\
&= \frac{f''(x+iy)f(x+iy) - (f'(x+iy))^2}{f(x+iy)^2} + \frac{-f''(x+iy)f(x+iy) - (if'(x+iy))^2}{f(x+iy)^2} \\
&= 0.
\end{aligned}$$

Therefore g is a harmonic function. So by theorem 25 from Notes 1,

$$\sup_{z \in K} g(z) = \sup_{z \in \partial K} g(z).$$

Since $\exp : \mathbb{R} \to \mathbb{R}$ is strictly increasing, it is valid to apply it to both sides of that equation, so we get

$$\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|.$$

Note that if we replace the suprema with infima, the statement is no longer true – for example, if f is the identity map on $\mathbb{C}$ and K is the unit disc, then

$$\inf_{z \in K} |f(z)| = 0$$

but since ∂K is the circle of radius 1 around the origin,

$$\inf_{z \in \partial K} |f(z)| = 1.$$

Notes 1, Exercise 27. Recall the Wirtinger derivatives defined in Exercise 23(i).

- (i) If $f : U \to \mathbb{C}$ is twice continuously differentiable on an open subset U of $\mathbb{C}$, show that
$$\Delta f = 4\frac{\partial}{\partial z}\frac{\partial f}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial f}{\partial z}.$$
Use this to give an alternate proof that (C^2) holomorphic functions are harmonic.

- (ii) If f is given by a polynomial
$$f(z) = \sum_{n,m\geq 0: n_m \leq d} c_{n,m} z^n \overline{z}^m$$
in both z and $\overline{z}$ for some complex coefficients $c_{n,m}$ and some natural number d, show that f is harmonic on $\mathbb{C}$ if and only if $c_{n,m}$ vanishes whenever n and m are both positive (i.e. f only contains terms $c_{n,0}z^n$ or $c_{0,m}\overline{z}^m$ that only involve one of z or $\overline{z}$).

- (iii) If $u : U \to \mathbb{R}$ is a real polynomial
$$u(x+iy) = \sum_{n,m\geq 0: n_m \leq d} a_{n,m} x^n y^m$$
in x and y for some real coefficients $a_{n,m}$ and some natural number d, show that u is harmonic if and only if it is the real part of a polynomial $f(z) = \sum_{n=0}^{d} c_n z^n$ of one complex variable z.

Notes 2, Exercise 3. Let $\gamma_1, \gamma_2, \gamma_3, \widetilde{\gamma}_1, \widetilde{\gamma}_2$ be continuous curves. Suppose that the terminal point of γ_1 equals the initial point of γ_2, and the terminal point of γ_2 equals the initial point of γ_3.

- (i) (Concatenation and reversal well defined up to equivalence) If $\gamma_1 \equiv \widetilde{\gamma}_1$ and $\gamma_2 \equiv \widetilde{\gamma}_2$, show that $\gamma_1 + \gamma_2 \equiv \widetilde{\gamma}_1 + \widetilde{\gamma}_2$ and $-\gamma_1 \equiv \widetilde{\gamma}_1$.

- (ii)

- (iii)

- (iv)

Stein-Shakarchi Chapter 1, Exercise 9. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \qquad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Begin with the equations

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

Differentiating u and v using the chain rule, we get

In the given region, we have $u = \log r$ and $v = \theta$, so

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial r}.$$

Also, the partial derivatives of u and v with respect to θ and r are all continuous, which implies log is Fréchet differentiable. Since log satisfies the Cauchy-Riemann equations, it is holomorphic (on the region where $r > 0$ and $-\pi < \theta < \pi$).

Stein-Shakarchi Chapter 1, Exercise 25. The next three calculations provide some insight into Cauchy's theorem, which we treat in the next chapter.

- (a) Evaluate the integrals

  $$\int_\gamma z^n dz$$

  for all integers n. Here γ is the circle centered at the origin with the positive (counterclockwise) orientation.

- (b) Same question as before, but with γ any circle not containing the origin.

- (c) Show that if $|a| < r < |b|$, then

  $$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

  where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

- (a) Suppose γ is a circle with radius $r > 0$, so it can be parameterized by

  $$z(t) = re^{it}$$

  for t in $[0, 2\pi]$. Then according the formula Stein-Shakarchi uses (on page 21) to define the integral along a curve in $\mathbb{C}$,

  $$\begin{aligned} \int_\gamma z^n dz &= \int_0^{2\pi} (z(t))^n z'(t) dt \\ &= \int_0^{2\pi} r^n e^{int}(ire^{it}) dt \\ &= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \end{aligned}$$

  which evaluates to zero when $n \neq -1$ and to $2\pi i$ when $n = -1$.

- (b) If γ does not contain or enclose the origin, then it has the parameterization

  $$z(t) = R + re^{it}$$

  for some $R > r > 0$. Using the same method as in part (a), we get

  $$\begin{aligned} \int_\gamma z^n dz &= \int_0^{2\pi} (z(t))^n z'(t) dt \\ &= \int_0^{2\pi} (R + re^{it})^n (ire^{it}) dt \end{aligned}$$

If $n \geq 0$ then we can use a binomial expansion on the integrand, and see that every term of that expansion is multiplied at least once by e^{it}, so the integral evaluates to zero. If n is negative, we instead use the negative binomial series (which is valid when $|re^{it}| < R$, according to `https://mathworld.wolfram.com/NegativeBinomialSeries.html`):

$$(R + re^{it})^n = \sum_{k=0}^{\infty} (-1)^k \binom{-n+k-1}{k} R^{n-k} (re^{it})^k.$$

Every term in the integrand is a constant multiple of $e^{it(k+1)}$, which when integrated from $t = 0$ to $t = 2\pi$, becomes zero unless $k = -1$. However, the summation does not include a $k = -1$ term, so that integral is zero no matter what n is.

- (c) Using the same method as in parts (a) and (b), suppose γ has the parameterization

$$z(t) = re^{it}$$

and then write

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_0^{2\pi} f(z(t))z'(t)dt \\
&= \int_0^{2\pi} \frac{ire^{it}}{(re^{it} - a)(re^{it} - b)} dt \\
&= i \int_0^{2\pi} \left[\frac{a}{(a-b)(re^{it} - a)} + \frac{b}{(a-b)(b - re^{it})} \right] dt \\
&= \left[\frac{ia}{a-b} \int_0^{2\pi} \frac{e^{-it}}{r} \cdot \frac{1}{1 - \frac{a}{r}e^{-it}} dt \right] + \left[\frac{i}{a-b} \int_0^{2\pi} \frac{1}{1 - \frac{r}{b}e^{it}} dt \right].
\end{aligned}$$

We know that the equation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

is true whenever $|x| < 1$, so that equation becomes

$$\begin{aligned}
\int_\gamma f(z)dz &= \left[\frac{ia}{a-b} \int_0^{2\pi} \frac{1}{r} \cdot \left(e^{-it} + \frac{a}{r}e^{-2it} + \frac{a^2}{r^2}e^{-3it} + \cdots \right) dt \right] + \left[\frac{i}{a-b} \int_0^{2\pi} \frac{1}{1 - \frac{r}{b}e^{it}} dt \right] \\
&= \frac{i}{a-b} \int_0^{2\pi} \left(1 + \frac{r}{b}e^{it} + \frac{r^2}{b^2}e^{2it} + \cdots \right) dt \\
&= \frac{2\pi i}{a-b}.
\end{aligned}$$