Physics 127 Homework #3

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April 23, 2025

Problem 0.1. Coleman problem 2.1 Prove the contraction law (Eq. 2.21). That is, show that the contraction of a tensor with n indices (n_u up and n_d down) is a tensor with $n-2$ indices ($n_u - 1$ up and $n_d - 1$ down).

If you let $n = n_u - 1$ and $m = n_d - 1$, then

$$\sum_{\lambda=0}^{3} T^{\mu_1 \cdots \mu_n \lambda}_{\nu_1 \cdots \nu_m \lambda} = T^{\mu_1 \cdots \mu_n \lambda}_{\nu_1 \cdots \nu_m \lambda} \delta^{\lambda}_{\lambda} = S^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}.$$

Problem 0.2. Coleman problem 2.2

(a) If $S^{\mu\nu}$ is a symmetric (rank 2) tensor, $A^{\mu\nu}$ is an antisymmetric (rank 2) tensor, and Λ is a Lorentz transform which maps $S^{\mu\nu} \to S^{\sigma\tau}$ and $A^{\mu\nu} \to A^{\sigma\tau}$, then

$$\begin{aligned} S^{\sigma\tau} &= \Lambda^{\sigma}_{\mu} \Lambda^{\tau}_{\nu} S^{\mu\nu} = \Lambda^{\tau}_{\nu} \Lambda^{\sigma}_{\mu} S^{\nu\mu} = S^{\tau\sigma} \\ A^{\sigma\tau} &= \Lambda^{\sigma}_{\mu} \Lambda^{\tau}_{\nu} A^{\mu\nu} = \Lambda^{\tau}_{\nu} \Lambda^{\sigma}_{\mu} (-A^{\nu\mu}) = -A^{\tau\sigma}. \end{aligned}$$

(b) If T^{ν}_{μ} is a traceless tensor, meaning $T^{\nu}_{\mu} \delta^{\mu}_{\nu} = 0$, and Λ is a Lorentz transform which maps T^{ν}_{μ} to $T^{\tau}_{\sigma} = (\Lambda^{-1})^{\mu}_{\sigma} T^{\nu}_{\mu} \Lambda^{\tau}_{\nu}$, then

$$\mathrm{Tr}(T^{\tau}_{\sigma}) = T^{\tau}_{\sigma} \delta^{\sigma}_{\tau} = (\Lambda^{-1})^{\mu}_{\sigma} T^{\nu}_{\mu} \Lambda^{\tau}_{\nu} \delta^{\sigma}_{\tau} = (\Lambda^{-1})^{\mu}_{\sigma} \Lambda^{\sigma}_{\nu} T^{\nu}_{\mu} = \delta^{\mu}_{\nu} T^{\nu}_{\mu} = \mathrm{Tr}(T^{\nu}_{\mu}) = 0.$$

(c) If $s_{\mu\nu} = s_{\nu\mu}$, then

$$\begin{aligned} s^{\mu\nu} &= g^{\mu\rho} s_{\rho}^{\nu} \\ &= g^{\mu\rho} s_{\rho\sigma} g^{\sigma\nu} \\ &= g^{\nu\sigma} s_{\sigma\rho} g^{\rho\mu} \\ &= s^{\nu\mu}. \end{aligned}$$

(d) You can visualize this as a $d \times d$ matrix, so every symmetric matrix is uniquely determined by the entries on or above the main diagonal, which there are $d(d+1)/2$ of. Similarly, each antisymmetric $d \times d$ matrix must have all zeros on the main diagonal, so it is uniquey determined by the $d(d-1)/2$ entries above the main diagonal.

(e) When you contract indices, it doesn't matter whether you sum over μ first or ν first.

Problem 0.3.

(a) This expression is symmetric, because $v^\mu v^\nu = v^\nu v^\mu$ and $g^{\mu\nu} = f^{\nu\mu}$, and traceless, because $\text{Tr}(v^\mu v^\nu) = 1$ and $\text{Tr}(g^{\mu\nu}) = 4$.

It is allowed to be a conserved quantity because it contains no terms that depend on multiple particles, so being conserved would not violate locality, and because it is Lorentz invariant, because

$$\begin{aligned}\Lambda^\sigma_\mu S^{\mu\nu}\Lambda^\tau_\nu &= \Lambda^\sigma_\mu \sum_a \mu_a \left(v^{(a)\mu}v^{(a)\nu} - \frac{1}{4}g^{\mu\nu}\right)\Lambda^\tau_\nu \\ &= \sum_a \mu_a \left(\Lambda^\sigma_\mu v^{(a)\mu}v^{(a)\nu}\Lambda^\tau_\nu - \frac{1}{4}\Lambda^\sigma_\mu g^{\mu\nu}\Lambda^\tau_\nu\right) \\ &= \sum_a \mu_a \left(v^{(a)\sigma}v^{(a)\tau} - \frac{1}{4}g^{\sigma\tau}\right) \\ &= S^{\sigma\tau}.\end{aligned}$$

Since $S^{\mu\nu}$ can be any symmetric and traceless 4×4 matrix, the dimension of the space of possible values of $S^{\mu\nu}$ is 9. Therefore, if that matrix/tensor is conserved, we get 9 conserved scalars.

(b) Each particle's initial and final trajectories can be defined by the linear equation $x^\mu(s) = x^\mu(0) + sv^\mu$ parameterized by s. That depends on two constant vectors, $x^\mu(0)$ and v^μ, which each contain 4 scalars, so describing the collision between 2 particles requires 16 scalars. However, we don't actually care about the value of $x^\mu(0)$ for both particles, we only care about the difference between $x^\mu(0)$ for the first particle and for the second. Therefore, we really only need 12 scalars to describe the collision, which means if we had 13 conserved scalars, the collision would be so over-constrained that it would be infinitesimally likely to occur.

Problem 0.4.

(a) After applying the translation $x^\mu \to x^\mu + \varepsilon^\mu$ for each particle a, we get

$$\begin{aligned}J^{\mu\nu} &\to (x^\mu + \varepsilon^\mu)\,p^\nu - (x^\nu + \varepsilon^\nu)\,p^\mu \\ &= J^{\mu\nu} + \varepsilon^\mu p^\nu + \varepsilon^\nu p^\mu.\end{aligned}$$

(b) In the non-relativistic limit, $x^\mu = (t, x^i)$ and $p^\mu = m(1, v^i)$, so the spatial components of $J^{\mu\nu}$ are

$$J^{ij} = x^i m v^j - x^j m v^i = \varepsilon^{ijk} x_i m v_j = (x \times mv)^k = (x \times p)^k = L^k.$$

(c) If J^{0i} is equal to some constant vector C^i, then so is

$$J^{0i} = x^0 p^i - x^i p^0 = tp^i - x^i E = C^i,$$

where t is time and E is energy. Therefore $x^i = tp^i/E - C^i/E$ (C^i/E is also a constant). Since the energy of particle a is $p^{(a)0}$, the total energy is $E = \sum_a p^{(a)0}$, and putting those equations together gives $x^i = \sum_a (x^i p^0)^{(a)}/E$.

Problem 0.5. Equation 1.93 says

$$\Delta t = \frac{1-v}{\sqrt{1-v^2}} = \sqrt{\frac{1-v}{1+v}}$$

when the star is moving towards us, and equation 1.94 says

$$\Delta t = \sqrt{\frac{1+v}{1-v}}$$

when the star is moving away from us.

In the star's reference frame, the wavevector of the emitted light is $(\omega_0, k) = \omega_0(1, n)$, and the Earth's velocity is $v_E^\mu = \gamma(1, -\beta)$. In the Earth's referece frame, the Earth's velocity is $v_E^\mu = (1, 0, 0, 0)$, and the wavevector of the emitted light is $(\omega, \text{some spatial component that we don't care about})$. Since the dot product of two 4-vectors is invariant, $v_E^\mu \cdot k$ doesn't depend on the reference frame. In the Earth's reference frame, we have $v_E \cdot k = \omega$, and in the star's reference frame, we have $v_E \cdot k = \omega_0\gamma(1, n) \cdot (1, -\beta)$, so

$$\omega = \omega_0\gamma(1 + n \cdot \beta) = \frac{1 + n \cdot \beta}{\sqrt{1-\beta^2}}\omega_0 = \frac{1+v}{\sqrt{1-v^2}}\omega_0 = \sqrt{\frac{1+v}{1-v}}\omega_0.$$

That means the temporal frequency is multiplied by

$$\sqrt{\frac{1+v}{1-v}},$$

so the period is divided by that same number. If $v = n \cdot \beta$ is reversed, that factor becomes

$$\sqrt{\frac{1-v}{1+v}}.$$

Relativity Physics 127 Homework 3

Due Wednesday April 23rd 2025, 11:59pm on gradescope.
Note that there is no grace period this time due to Quiz no 2 on April 24th.

1.) Coleman, problem (2.1)

2.) Coleman, problem (2.2)

3.) In class we discussed conservation laws that were only functions of the asymptotic particle velocities $v^{(a)\mu}_{\text{in/ out}}$ and transformed as Lorentz tensors. (See section 2.2.1 of Coleman's book.) In addition we imposed the requirement of locality, which told us that the conserved quantities all take the form $F\{v^{(a)}\} = \sum_a f_a(v^{(a)})$, i.e. no cross terms involving different particles $a \neq b$ are allowed. Based on this we showed that there were no interesting conserved Lorentz scalars, and that there was an interested conserved Lorentz vector, the 4-momentum $P^\mu = \sum_a m_a v^{(a)\mu}$.

   a. Argue that Lorentz invariance and locality allow for a possible conserved rank-2 symmetric traceless quantity of the form

   $$S^{\mu\nu} = \sum_a \mu_a \left(v^{(a)\mu} v^{(a)\nu} - \frac{1}{4} g^{\mu\nu} \right) \ .$$

   Explain why this would lead to nine conservation laws.

   b. Together P^μ and $S^{\mu\nu}$ comprise 13 conserved quantities. Argue by counting that this prohibits any elastic scattering of 2 particles going into two particles. Do this by showing that both the initial and the final state of such a process is only characterized by 12 numbers. Thus the process is over-constrained by the conservation laws and can (generically) not happen.

4.) In class I argued that the anti-symmetric second rank Lorentz tensor

$$J^{\mu\nu} = -J^{\nu\mu} = \sum_a (x^\mu p^\nu - x^\nu p^\mu)^{(a)}$$

is a conserved quantity. Note that it is clearly local (i.e. it has no $a \neq b$ cross terms) and a tensor.

(4.1) Because $J^{\mu\nu}$ explicitly depends on x^μ, it is not invariant under translations $x^{(a)\mu} \to x^{(a)\mu} + \varepsilon^\mu$. Show that such a translation shifts $J^{\mu\nu}$ by an amount involving the constant vector ε^μ, as well as the conserved 4-momentum P^μ, thus leading to another conserved quantity without introducing a new conservation law.

(4.2) Show that the spatial components J^{ij} coincide with the standard angular momentum in the non-relativistic limit where all particles move with speeds much less than 1.

(4.3) Show that the conservation equation $J^{0i} = \text{const.}$ implies that the system's center of energy X^i (generalizing the non-relativistic center of mass) moves at constant speed according to the following formula,

$$X^i = \frac{P^i}{E} t + \text{const.} \ , \qquad E = \sum_a p^{(a)0} \ , \qquad X^i = \frac{1}{E} \sum_a \left(x^i p^0 \right)^{(a)} \ .$$

5. Work through example (not problem) 2.3 in Coleman and reproduce equation 1.93 and 1.94 from the special case alluded to at the end of the example.