Equivalence Relations on Algebraic Cycles

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History and Motivation

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- Riemann-Roch Theorem, Abel-Jacobi Theorem major contributions using divisors (algebraic cycles of codimension 1)
- In higher dimensions, behavior of varieties becomes more complicated and these theorems don't nicely extend

Algebraic Cycles

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Definition

A **cycle** Z **of codimension** r on X is an element of the free abelian group generated by the closed irreducible subvarieties of codimension r of X. It is a finite formal sum $\sum n_i[V_i]$ where n_i are integers and V_i are subvarieties.

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We denote by $C^r(X)$ the group of all cycles of codimension r on X.

Algebraic Cycles

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- $C^r(X)$ is naturally quite large
- Use equivalence relations to develop more intuition on the geometry of X.

A Nice Example

Let $r = 1$. A cycle of codimension one is a divisor.

Definition

Two divisors D_1 and D_2 on X are linearly equivalent if there exists a rational function on X such that $D_1 - D_2 = (f)_0 - (f)_\infty$, where $(f)_0$ denotes the divisor of zeros and $(f)_\infty$ denotes the divisor of poles.

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Example

Let $C^1_{\text{lin}}(X)$ denote the group of divisors linearly equivalent to 0. Then, the quotient group $C^1(X)/C^1_{\text{lin}}(X)$ is Pic X, the group of linear equivalence classes of divisors on X.

A Nice Example

For $X = \mathbb{P}^n$, $C^1(X)/C^1_{\mathrm{lin}}(X) \cong \mathbb{Z}$.

Rational Equivalence

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Definition

Two cycles Z_1 and Z_2 of codimension r on X are **rationally equivalent** if there is a cycle Z on $X \times \mathbb{P}^1$, which intersects each fiber $X \times \{t\}$ in something of codimension r, and such that Z_1 and Z_2 are obtained respectively by intersecting Z with the fibers $X \times \{0\}$ and $X \times \{1\}$.

Rational Equivalence is an Equivalence Relation

- Reflectivity: $Z_1 \sim_{rat} Z_1$. Take $Z = Z_1 \times \mathbb{P}^1$

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- Symmetry: $Z_1 \sim_{rat} Z_2 \implies Z_2 \sim_{rat} Z_1$. Apply automorphism of $\mathbb{P}^1$ that interchanges 0 and 1.
- Transivity: $Z_1 \sim_{rat} Z_2, Z_2 \sim_{rat} Z_3 \implies Z_1 \sim_{rat} Z_3$. Let $Z \subseteq X \times \mathbb{P}^1$ give $Z_1 \sim_{rat} Z_2$ and $Z' \subseteq X \times \mathbb{P}^1$ give $Z_2 \sim_{rat} Z_3$. Then, $Z + Z' - Z_2 \times \mathbb{P}^1$ gives $Z_1 \sim_{rat} Z_3$.

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Definition

Let C be an irreducible curve, and $a, b \in C$ be any two points. Two cycles Z_1 and Z_2 of codimension r on X are **algebraically equivalent** if there is a cycle Z on $X \times C$, which intersects each fiber $X \times \{t\}$ in something of codimension r, and such that Z_1 and Z_2 are obtained respectively by intersecting Z with the fibers $X \times \{a\}$ and $X \times \{b\}$.

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Remark

$C^r_{rat}(X) \subset C^r_{alg}(X) \subset C^r(X)$

Other Adequate Equivalence Relations

- Numerical equivalence, torsion equivalence, homological equivalence

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- $C^r \supseteq C^r_{num} \supseteq C^r_{hom} \supseteq C^r_{\tau} \supseteq C^r_{alg}$

Further Study and Applications

- intermediate jacobians, k-theoretic and cohomology methods, relative cycles

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- intermediate jacobians, k-theoretic and cohomology methods, relative cycles
- behavior of algebraic cycles useful in intersection theory, algebraic k-theory, and hodge conjecture

References

- Spencer Bloch (1980) "Lectures on Algebraic Cycles", Mathematics Department Duke University.
- Robin Hartshorne (1974) "Equivalence Relations on Algebraic Cycles and Subvarieties of Small Codimension", AMS.