AN INTRODUCTION TO CHARACTERISTIC CLASSES

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1. Introduction

In algebraic topology, we wish to use tools from abstract algebra to help us classify topological spaces up to homomorphism by way of algebraic invariants. One way to do this is through chracteristic classes. In this paper, we will give a basic roadmap to an axiomatic approach to Stiefel-Whitney classes. We will first study vector bundles and then cohomology groups which will then be combined to discover characteristic classes. After giving some applications and examples, mainly about Stiefel-Whitney numbers, we hope to motivate the study of other characteristic classes such as the Chern or Pontrjagin classes.

2. Vector Bundles

Vector bundles are important to the study of topological spaces as they give a precise idea about families of vector spaces parametrized by some other space. Through their applications, we can find deeper properties about the spaces themselves and also maps between spaces. (maybe something on finitely generated projective modules)

Definition 1. *A* ***real vector bundle*** ξ *over* B *consists of the following:*

1. *A topological space* $E = E(\xi)$ *called the total space;*
2. *A continuous map* $\pi : E \to B$ *called the projection map;*
3. *For each* $b \in B$*, the structure of a vector space over the real numbers in the set* $\pi^{-1}(b)$*.*

We can think of vector bundles visually as some bridging structure defined by the projection map from one space to another. To help digest this definition, let us give some simple examples starting off with the trivial bundle.

Example 1. *The trivial bundle with total space* $B \times \mathbb{R}^n$*, with projection map* $\pi(b, x) = b$*, and with the vector space structures in the fibers defined by*

$$t_1(b, x_1) + t_2(b, x_2) = (b, t_1x_1 + t_2x_2),$$

will be denoted by ϵ^n_B*.*

Example 2. *The tangent bundle* τ_M *of a smooth manifold* M *has the total space* DM *consisting of all pairs* (x, v) *with* $x \in M$ *and* v *tangent to* M *at* x*. The projection map is defined by* $\pi(x, v) = x$ *and the vector space structure in* $\pi^{-1}(x)$ *is defined by*

$$t_1(x, v_1) + t_2(x, v_2) = (x, t_1v_1 + t_2v_2).$$

vector fields are sections of vector bundles

Example 3. *The normal bundle of a smooth manifold* $M \subset \mathbb{R}^n$ *has the total space* $E \subset M \times \mathbb{R}^n$*, which is the set of all pairs* (x, v) *such that* v *is orthogonal to the tangent space* DM_X*. The projection map and the vector space structure is similarly defined as the tangent bundle.*

Below is a hand-drawing from Milnor-Stasheff [**MilnorStasheff**] to help visualize these examples.

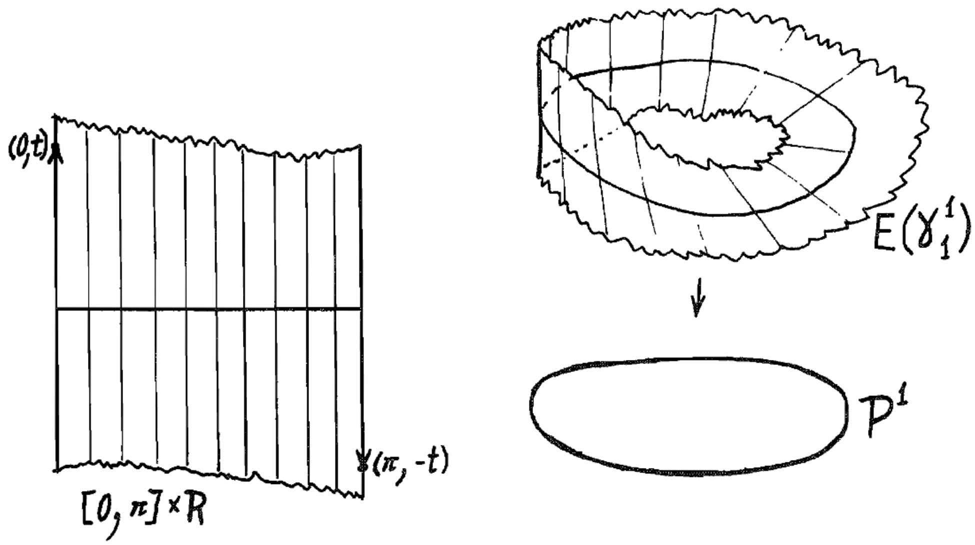

Figure 2.

There is much to learn about vector bundles, but for the sake of this paper, we will give only a fraction of the important definitions.

Definition 2. *Two vector bundles, ξ and η, are isomorphic $\xi \cong \eta$ if there exists a homeomorphism*

$$f : E(\xi) \to E(\eta)$$

between the total spaces which maps each vector space $F_b(\xi)$ isomorphically onto the corresponding vector space $F_b(\eta)$.

Theorem 1. *The bundle γ_n^1 over P^n is not trivial, for $n \geq 1$.*

To prove this theorem, we first require the notion of a cross-section.

Definition 3. *A* ***cross-section*** *of a vector bundle ξ with base space B is a continuous function*

$$s : B \to E(\xi)$$

which takes each $b \in B$ into the corresponding fiber $F_b(\xi)$. A common example is the cross-section of the tangent bundle of a smooth manifold M, which is usually called a vector field on M.

Definition 4. *A cross-section is* ***nowhere zero*** *if $s(b)$ is a non-zero vector of $F_b(\xi)$ for each b. The cross-sections $s_1, \ldots, s_n$ are* ***nowhere dependent*** *if for each $b \in B$, the vectors $s_1(b), \ldots, s_n(b)$ are linearly independent.*

Proof. Since a trivial $\mathbb{R}^1$-bundle has a cross-section that is nowhere zero, we wish to show that γ_n^1 has no such cross-section. Let

$$s : \mathbf{P}^n \to E(\gamma_n^1)$$

be any cross-section, and consider the composition

$$S^n \to \mathbf{P}^n \to^n E(\gamma_n^1)$$

carrying each point $x \in S^n$ to some pair

$$(\{\pm x\}, t(x)x) \in E(\gamma_n^1).$$

$t(x)$ is a continuous real-valued function of x with the property that $t(-x) = -t(x)$. Since S^n is connected, by the intermediate value theorem $t(x_0) = 0$ for some point x_0. Thus, $s(\{\pm x_0\}) = (\{\pm x_0\}, 0)$, and so γ_n^1 is not trivial. $\square$

Now, we further motivate the importance of vector bundles with some significant results.

Theorem 2. *An $\mathbb{R}^n$ bundle ξ is trivial if and only if ξ admits n cross-sections $s_1, \dots, s_n$ which are nowhere dependent.*

Lemma 1. *Let ξ and η be vector bundles over B and let $f : E(\xi) \to E(\eta)$ be a continuous function which maps each vector space $F_b(\xi)$ isomorphically onto the corresponding vector space $F_b(\eta)$. Then f is necessarily a homeomorphism. Hence ξ is isomorphic to η.*

3. Cohomology

With a great phonetic similarity to homology, cohomology groups also share similar axioms to homology groups. However, the benefit of computing cohomology groups is that the natural "cup" product of cohomology groups provides an additional algebraic structure that can be more useful. Let us first begin with some definitions.

Definition 5. *Given groups $(A, \circ)$ and $(B, \cdot)$, a* ***group homomorphism*** *is a function $h : A \to B$ such that for all $x, y \in A$, $h(x \circ y) = h(x) \cdot h(y)$.*

The building blocks for these homology and cohomology groups lies in connecting spaces through homomorphisms in two similar, yet distinct manners.

Definition 6. *A* ***chain complex*** *$(A_\bullet, d_\bullet)$ is a sequence of abelian groups or modules $\dots, A_0, A_1, \dots$ connected by group homomorphisms (called* ***boundary operators*** *or* ***differentials****) $d_n : A_n \to A_{n-1}$ such that the compositions of any two maps is the zero map ($d_n \circ d_{n+1} = 0$).*

$$\cdots \xleftarrow{d_{n-1}} A_{n-1} \xleftarrow{d_n} A_n \xleftarrow{d_{n+1}} \cdots$$

Example 4. *$A_1 = (\mathbb{Z}, +)$ is the group of integers under addition*
$A_0 = (\mathbb{Z}/2\mathbb{Z}, +)$ is the cyclic group of integers mod 2 under addition
$A_n = 0$ is the zero group for all $n \notin \{0, 1\}$. $d_1(x) = x \pmod 2$
$d_n(x) = 0$ for all $n \neq 1$

$$\cdots 0 \leftarrow 0 \leftarrow (\mathbb{Z}/2\mathbb{Z}, +) \xleftarrow{\text{mod } 2} (\mathbb{Z}, +) \leftarrow 0 \leftarrow 0 \cdots$$

Since $d_2 \circ d_1(x) = d_1(0) = 0$, and $d_n \circ d_{n+1} = 0$ by inspection for all other n, this is clearly a chain complex.

The dual of a chain complex is appropriately named the cochain complex.

Definition 7. *A* ***cochain complex*** *$(A^\bullet, d^\bullet)$ is a sequence of abelian groups or modules $\dots, A^0, A^1, \dots$ connected by homomorphisms $d^n : A^n \to A^{n+1}$ such that the compositions of any two maps is the zero map ($d^{n+1} \circ d^n = 0$).*

$$\cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \cdots$$

Example 5. *For any $p \in \mathbb{Z}^{\geq 2}$*
$A^0 = (\mathbb{Z}, +)$
$A^1 = (\mathbb{Z}, +)$
$A^2 = (\mathbb{Z}/p\mathbb{Z}, +)$
$A^n = 0$ *for all* $n \notin \{0, 1, 2\}$
$d^0(x) = p \times x$

$d^1(x) = x \pmod p$
$d^n(x) = 0$ *for all* $n \notin \{0, 1\}$

$$\cdots 0 \to 0 \to (\mathbb{Z}, +) \xrightarrow{\times p} (\mathbb{Z}, +) \xrightarrow{\text{mod } p} (\mathbb{Z}/p\mathbb{Z}, +) \to 0 \to 0 \to \cdots$$

Remark 1. *Here are some clarifying remarks about (co)chain complexes:*

- *They are infinitely long in both directions. If you call a (co)chain complex finitely long, then you are suggesting that everything that comes before and after in the chain is the zero group.*
- *As a consequence, they have no beginning or end unless you are not counting the infinitely long chains of zero groups that come before or after.*
- *(Co)chain groups don't need to have the zero group at all! They can be infinitely long and just have non-zero groups that extend in both directions.*

4. Cohomology Groups

Now, we show how these groups can be computed in order to study topological spaces.

Definition 8. *The* ***nth homology group*** *of a chain complex denoted* H_n*, is*

$$H_n = \frac{\ker d_n}{\text{im } d_{n+1}}$$

Definition 9. *The* ***nth cohomology group*** *of a cochain complex denoted* H^n*, is*

$$H^n = \frac{\ker d^n}{\text{im } d^{n-1}}$$

Example 6. *Using the cochain complex example from above, we see that the kernel of* $d^1(x) = x \pmod p$ *is just the set* $p\mathbb{Z} \subset \mathbb{Z}$*. The image of* $d^1(x) = px$ *is* $p\mathbb{Z}$*. That means the quotient group* H^1 *is*

$$H^1 = \frac{\ker d^1}{\text{im } d^0} = \frac{p\mathbb{Z}}{p\mathbb{Z}} = p\mathbb{Z}$$

Definition 10. *The* ***standard n-simplex*** *is the convex set* $\Delta^n \subset \mathbb{R}^{n+1}$ *consisting of all* $(n+1)$*-tuples* $(t_0, \ldots, t_n)$ *of real numbers with*

$$t_i \geq 0 \text{ and } t_0 + t_1 + \cdots + t_n = 1.$$

Simplices show up quite often in the study of geometric structures, and their visualizations are very friendly.

Example 7. *The* 0*-simplex consists of just the one point* $x = 0$ *on the real number line* $\mathbb{R}^1$*.*

Example 8. *The* 1*-simplex consists of the line segment connecting the points* $(1, 0)$ *and* $(0, 1)$ *including the endpoints.*

Now, let us build some fundamental concepts for characteristic classes.

Definition 11. *A* ***singular n-simplex*** *in a topological space* X *is a continuous map* σ *from the standard n-simplex* Δ^n *to* X*, written* $\sigma : \Delta^n \to X$*. The* ***i-th face*** *of a singular n-simplex* σ *is the singular (n-1)-simplex*

$$\sigma \circ \phi_i : \Delta^{n-1} \to X$$

where the linear imbedding $\phi_i : \Delta^{n-1} \to \Delta^n$ *is defined by*

$$\phi_i(t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n).$$

We denote the set of all n-simplices in X *as* $\text{Sin}_n(X)$*. Note that* $\text{Sin}_n(X) = 0$ *for* $n < 0$*.*

Definition 12. *The **singular chain complex** of a topological space X, denoted $(C_\bullet(X), d_\bullet)$ is created where $C_n(X)$ is the free abelian group created by the basis* $\mathrm{Sin}_n(X)$ *and* $d_n : C_n(X) \to C_{n-1}(X)$ *is* $d_n(\sigma) = \sum_i (-1)^i \sigma \mid [v_0, \cdots, \hat{v}_i, \cdots, v_n]$.

Definition 13. *The **nth singular homology group** is the nth homology group of the nth singular chain complex.*

Definition 14. *The **cochain group** $C^n(X;\Lambda)$ is defined to be the dual module* $\mathrm{Hom}_\Lambda(C_n(X;\Lambda), \Lambda)$ *consisting of all Λ-linear maps from $C_n(X;\Lambda)$ to Λ.*

Definition 15. *The **coboundary** of a cochain $c \in C^n(X;\Lambda)$ is defined to be the cochain $\delta c \in C^{n+1}(X;\Lambda)$ whose value on each $(n+1)$-chain a is determined by the identity*

$$\langle \delta c, a\rangle + (-1)^n \langle c, da\rangle = 0.$$

Definition 16. *The **singular cohomology groups** of X is defined as*

$$H^n(X;\Lambda) = Z^n(X;\Lambda)/B^n(X;\Lambda) = (\ker d)/dC^{n-1}(X;\Lambda).$$

We can think of $B^n(X;\Lambda)$ as the coboundary of the cochain group $C^{n-1}(X;\Lambda)$ which is a subset of $C^n(X;\Lambda)$.

Remark 2. *Cohomology groups are determined algebraically by homology groups. Here are the steps to do so:*

1. *Take a chain complex of singular, simplicial, or cellular chains.*
2. *Take the homology groups of the chain complex,* $\ker d/\mathrm{im}\ d$.
3. *Replace the chain groups C_n by the dual groups* $\mathrm{Hom}(C_n, G)$ *and the boundary maps d by their dual maps δ.*
4. *Then, the cohomology groups are* $\ker \delta/\mathrm{im}\ \delta$.

5. Stiefel-Whitney Classes

Finally, we have a sufficient understanding to grasp what characteristic classes are. Primarily, we are interested in various characteristic classes because they are used to measure vector bundles of a topological space using cohomology. To do this, we map the principal bundles of a topological space X to its cohomology classes. This mapping can be done in many different ways, but in this paper we will look at the Steifel-Whitney Classes.

Definition 17. *$H^i(B;G)$ denotes the i-th singular cohomology group of topological space B with coefficients in G.*

For the Stiefel-Whitney classes, we give an axiomatic approach.

Axiom 1. *To each vector bundle ξ there corresponds a sequence of cohomology classes*

$$w_i(\xi) \in H^i(B(\xi);\mathbb{Z}/2), i = 0, 1, 2, \ldots$$

*called the **Stiefel-Whitney classes** of ξ. The class $w_0(\xi)$ is equal to the unit element*

$$1 \in H^0(B(\xi);\mathbb{Z}/2)$$

and $w_i(\xi)$ equals zero for i greater than n if ξ is an n-plane bundle.

Axiom 2. *(Naturality) If $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then*

$$w_i(\xi) = f^* w_i(\eta).$$

Axiom 3. *(The Whitney Product Theorem) If ξ and η are vector bundles over the same base space, then*

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta).$$

Axiom 4. *For the line bundle γ_1^1 over the circle P^1, the Stiefel-Whitney class $w_1(\gamma_1^1)$ is non-zero.*

Consequently from Axiom 2, we have the following.

Proposition 1. *If ξ is isomorphic to η, then $w_i(\xi) = w_i(\eta)$.*

Proposition 2. *If ϵ is a trivial vector bundle, then $w_i(\epsilon) = 0$ for $i > 0$. This is because there exists a bundle map from ϵ to a vector bundle over a point.*

As a consequence of the Whitney Product theorem and the naturality axiom,

Proposition 3. *If ϵ is trivial, then $w_i(\epsilon \oplus \eta) = w_i(\eta)$.*

Proposition 4. *If ξ is an $\mathbb{R}^n$-bundle with a Euclidean metric which possesses a nowhere zero cross-section, then $w_n(\xi) = 0$. If ξ possesses k cross-sections which are nowhere linearly dependent, then*

$$w_{n-k+1}(\xi) = w_{n-k+2}(\xi) = \cdots = w_n(\xi) = 0.$$

Example 9. *Using the standard imbedding of $S^n \subset \mathbb{R}^{n+1}$, the normal bundle v is trivial. Let τ be the tangent bundle of the unit sphere S^n. Then by the Whitney Product theorem, $w(\tau)w(v) = 1$ and since v is trivial, $w(v) = 1$. Thus, $w(\tau) = 1$. Therefore, the class $w(\tau) = w(S^n)$ is equal to 1, and τ is indistinguishable from the trivial bundle over S^n by Steifel-Whitney classes.*

Theorem 3. *The Whitney sum $\tau \oplus \epsilon^1$ is isomorphic to the $(n+1)$-fold Whitney sum $\gamma_n^1 \oplus \gamma_n^1 \oplus \cdots \oplus \gamma_n^1$. Hence the total Stiefel-Whitney class of P^n is given by*

$$w(P^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1}a + \binom{n+1}{2}a^2 + \cdots + \binom{n+1}{n}a^n.$$

6. Applications and Examples

Definition 18. *A smooth map $f : M \to N$ between smooth manifolds is called an **immersion** if the Jacobian*

$$Df_X : DM_X \to DN_{f(x)}$$

maps the tangent space DM_X injectively for each $x \in M$.

If a manifold M of dimension n can be immersed in the Euclidean space $\mathbb{R}^{n+k}$, then the Whitney duality theorem implies that the dual Stiefel-Whitney classes are zero for $i > k$.

Let P^n denote the real projective space in n dimensions.

Theorem 4. *If P^{2^r} can be immersed in $\mathbb{R}^{2^r+k}$, then k must be at least $2^r - 1$.*

6.1. **Stiefel-Whitney Numbers.** Stiefel-Whitney numbers allows for a comparison between the Stiefel-Whitney classes of two different manifolds. In studying manifolds, we consider the collection of all possible Stiefel-Whitney numbers. We say that two different manifolds M and M' have the same Stiefel-Whitney numbers if

$$w_1^{r_1} \cdots w_n^{r_n}[M] = w_1^{r_1} \cdots w_1^{r_1}[M']$$

for every monomial $w_1^{r_1} \cdots w_n^{r_n}$ of total dimension n.

Theorem 5. *If B is a smooth compact $(n+1)$-dimensional manifold with boundary equal to M, then the Stiefel-Whitney numbers of M are all zero.*

Theorem 6. *If all the Stiefel-Whitney numbers of a manifold M are zero, then M can be realized as the boundary of some smooth compact manifold.*

Corollary 1. *Two smooth closed n-manifolds belong to the same cobordism class if and only if all of their corresponding Stiefel-Whitney numbers are equal.*

7. Closing Remarks

To close, vector bundles themselves are quite fascinating, however it's when we begin to construct groups with vector bundles that we see these magical algebraic properties of manifolds arise. Cohomology groups, while quite novel at first, do become a natural consideration. This is because we extract new relationships between spaces by way of the algebraic structure of cohomology classes of vector bundles. After the Stiefel-Whitney classes, there are many paths for further research after this paper. For example, one can explore vector bundles and manifolds in $\mathbb{C}$, which gives the added benefit of being algebraically closed. Then, one could study Chern classes and Chern numbers of a compact complex manifold.