Streamfunction (Chapter 6.2.3 of textbook)

For two-dimensional flow we can define a scalar function $\psi(x,y)$

greek letter psi

with the property:

$$u(x,y) = \frac{\partial \psi(x,y)}{\partial y}$$

$$v(x,y) = -\frac{\partial \psi(x,y)}{\partial x}$$

for incompressible flow: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$

$\Rightarrow \psi$ satisfies continuity equation

We can do that for three-dimensional flow but then the streamfunction is a vector function, and no one want to deal with that...

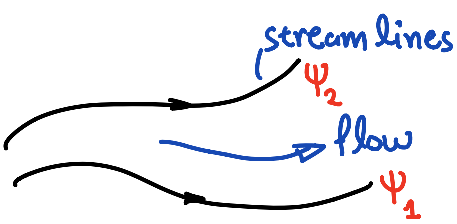
What's the big deal with ψ?

- ψ is a scalar that can generate a vector field

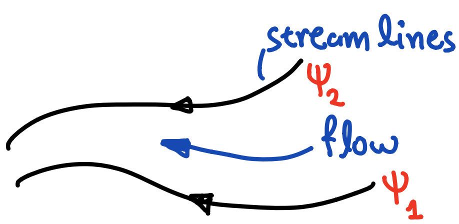
  it is as if we are packing (or condensing) information into one variable

Important Properties

- ψ is constant on streamlines
- $\Delta\psi$ is the volumetric flow rate



$\psi_2 > \psi_1$



$\psi_2 < \psi_1$

Velocity Potential

If the flow is
- incompressible
- inviscid
- irrotational (vorticity = 0 everywhere)

We can define a scalar function $\phi(t,x,y,z)$ such that $\vec{V} = \nabla\phi$

greek letter phi

in Cartesian: $u = \dfrac{\partial \phi}{\partial x}$

$v = \dfrac{\partial \phi}{\partial y}$

$w = \dfrac{\partial \phi}{\partial z}$

Continuity equation: $\nabla \cdot \nabla\phi = \nabla^2\phi = 0$ Laplace equation

by construction flow is irrotational because $\nabla \times \nabla\phi = 0$

- Also, for 2D irrotational flow: $\zeta_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$

$$= - \frac{\partial}{\partial x}\frac{\partial \psi}{\partial x} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial y} = 0$$

$\nabla^2\psi = 0$ Laplace equation

Important: both ϕ and ψ satisfy Laplace equation

but are not the same because the boundary conditions are different

we can show that ϕ and ψ are actually orthogonal

Potential Flow

Why are $\phi(x,y)$ and $\psi(x,y)$ cool?

- We can solve $\nabla^2\phi = 0$ instead of $\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} = -\frac{1}{\rho}\nabla p$ and $\nabla\cdot\vec{V} = 0$
- $\nabla^2\phi = 0$ is a linear equation

  thus... if ϕ_1 and ϕ_2 are solutions to $\nabla^2\phi_1 = 0$ and $\nabla^2\phi_2 = 0$

  then $\phi_1 + \phi_2$ is also a solution to $\nabla^2(\phi_1 + \phi_2) = 0$

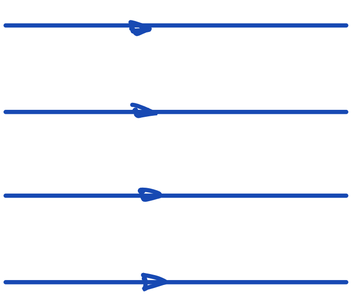
  So we can create flow C by adding flow A and flow B together

  - Can add ϕ, ψ, and velocity components
  - Do NOT add the pressures. This is wrong!

  → Superposition of "basic" flows
- We can use Bernoulli to find pressure

Basic Flows (Table 6.1)

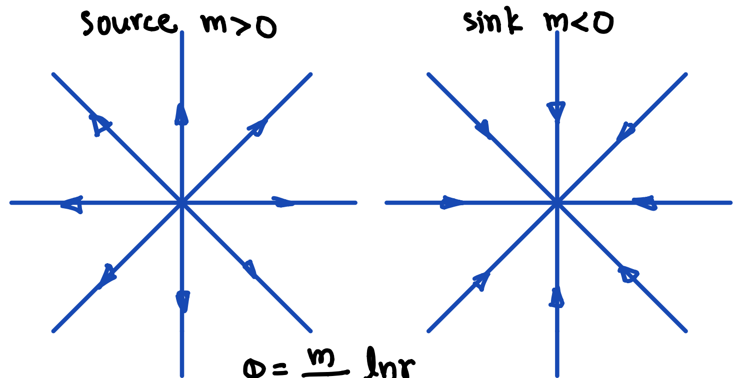
Uniform flow



$\phi = Ux$

$\psi = Uy$

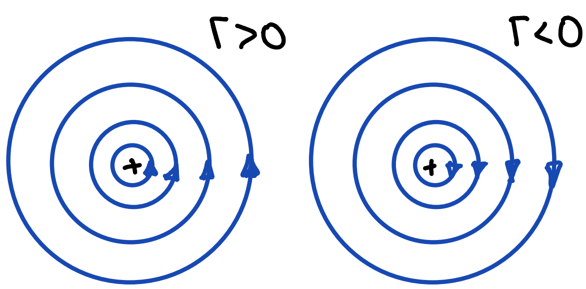
Source or Sink



$\phi = \frac{m}{2\pi}\ln r$

$\psi = \frac{m}{2\pi}\theta$

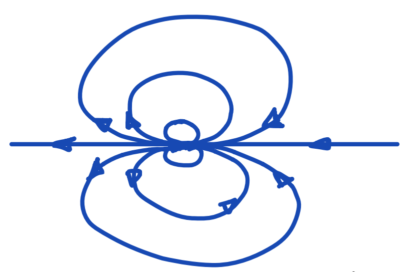
Free Vortex



$\phi = \frac{\Gamma}{2\pi}\theta$

$\psi = -\frac{\Gamma}{2\pi}\ln r$

Doublet



$\phi = \frac{K\cos\theta}{r}$ $\psi = \frac{K\sin\theta}{r}$

■ Table 6.1

Summary of Basic, Plane Potential Flows

Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see Fig. 6.16b)	$\phi = U(x\cos\alpha + y\sin\alpha)$	$\psi = U(y\cos\alpha - x\sin\alpha)$	$u = U\cos\alpha$ $v = U\sin\alpha$
Source or sink (see Fig. 6.17) $m > 0$ source $m < 0$ sink	$\phi = \dfrac{m}{2\pi}\ln r$	$\psi = \dfrac{m}{2\pi}\theta$	$v_r = \dfrac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see Fig. 6.18) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \dfrac{\Gamma}{2\pi}\theta$	$\psi = -\dfrac{\Gamma}{2\pi}\ln r$	$v_r = 0$ $v_\theta = \dfrac{\Gamma}{2\pi r}$
Doublet (see Fig. 6.23)	$\phi = \dfrac{K\cos\theta}{r}$	$\psi = -\dfrac{K\sin\theta}{r}$	$v_r = -\dfrac{K\cos\theta}{r^2}$ $v_\theta = -\dfrac{K\sin\theta}{r^2}$

this is Cartesian

These are cylindrical

[a]Velocity components are related to the velocity potential and stream function through the relationships:

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}.$$

IMPORTANT

How to get velocity from ϕ and ψ

ALSO, IMPORTANT:

these relations are for basic flows located at the coordinate system origin

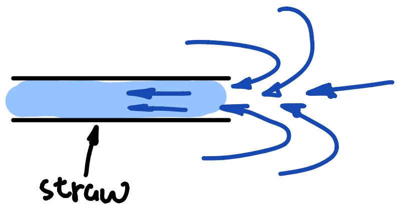
must shift the coordinates if basic flow is not at the origin

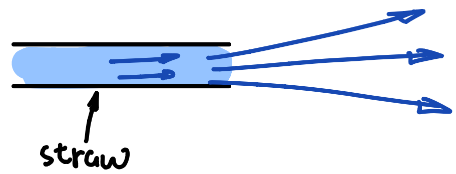
A few important questions...

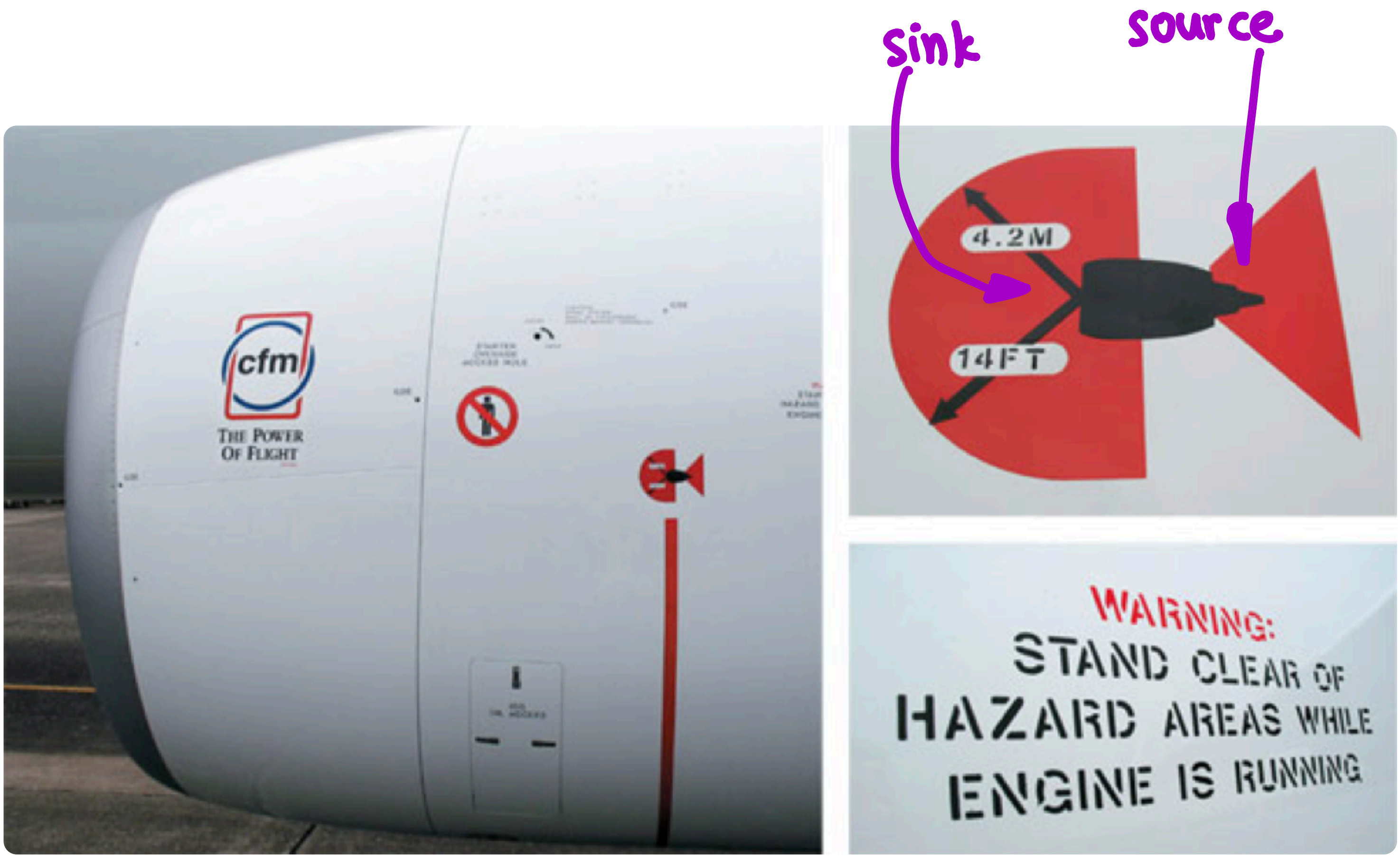
- Are these flows realistic?

  some more than others...

For example: sink is ok | source... not so much...

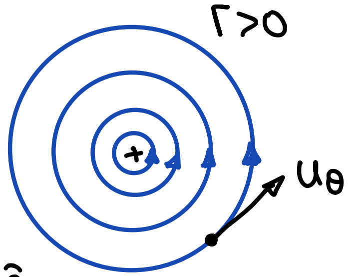






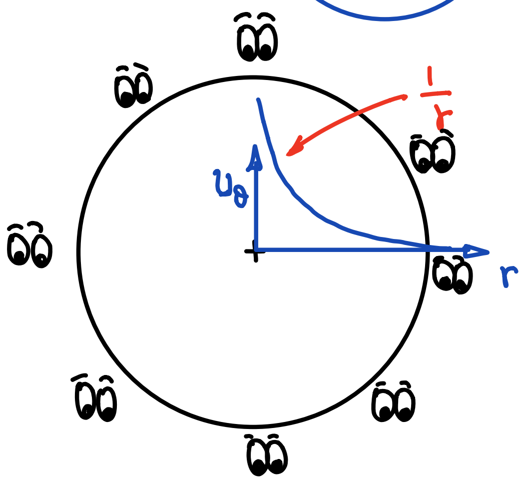
- How can you have a vortex without vorticity?

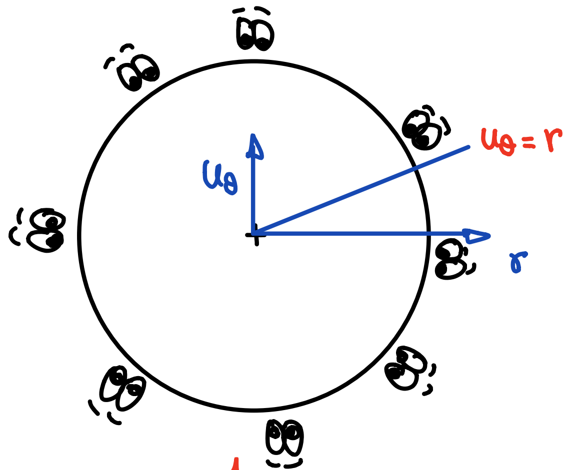
Free Vortex



Velocity field $\vec{V} = u_\theta \hat{\theta}$

$$u_\theta = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{r}$$





fluid element travels along a circular path but does not rotate around itself

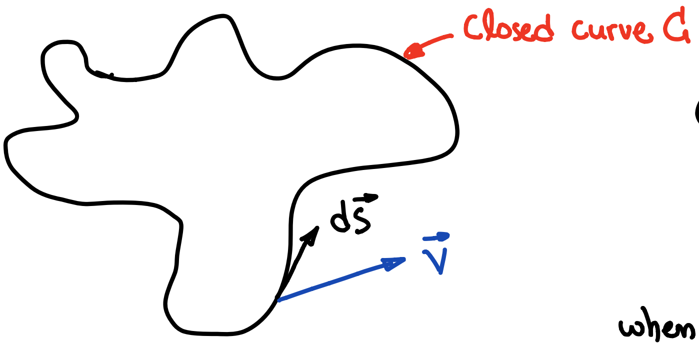
zero vorticity

fluid element travels along a circular path and rotates around itself

Vorticity has to do with the fluid element rotation and not the path followed by the element!

- Most "basic" potential flows are weird at the origin

yes... the origin is "special"



Circulation in fluids:

$$\Gamma = \oint_C \vec{V} \cdot d\vec{s}$$

when potential flow:

$$\Gamma = \oint \nabla\varphi \cdot d\vec{s} = 0$$

However... $\Gamma = \oint \vec{V} \cdot d\vec{s}$

$$= \oint_{circle} \frac{\Gamma}{2\pi r} \, r \, d\theta = \frac{\Gamma}{2\pi} \int_0^{2\pi} d\theta = \Gamma \neq 0$$

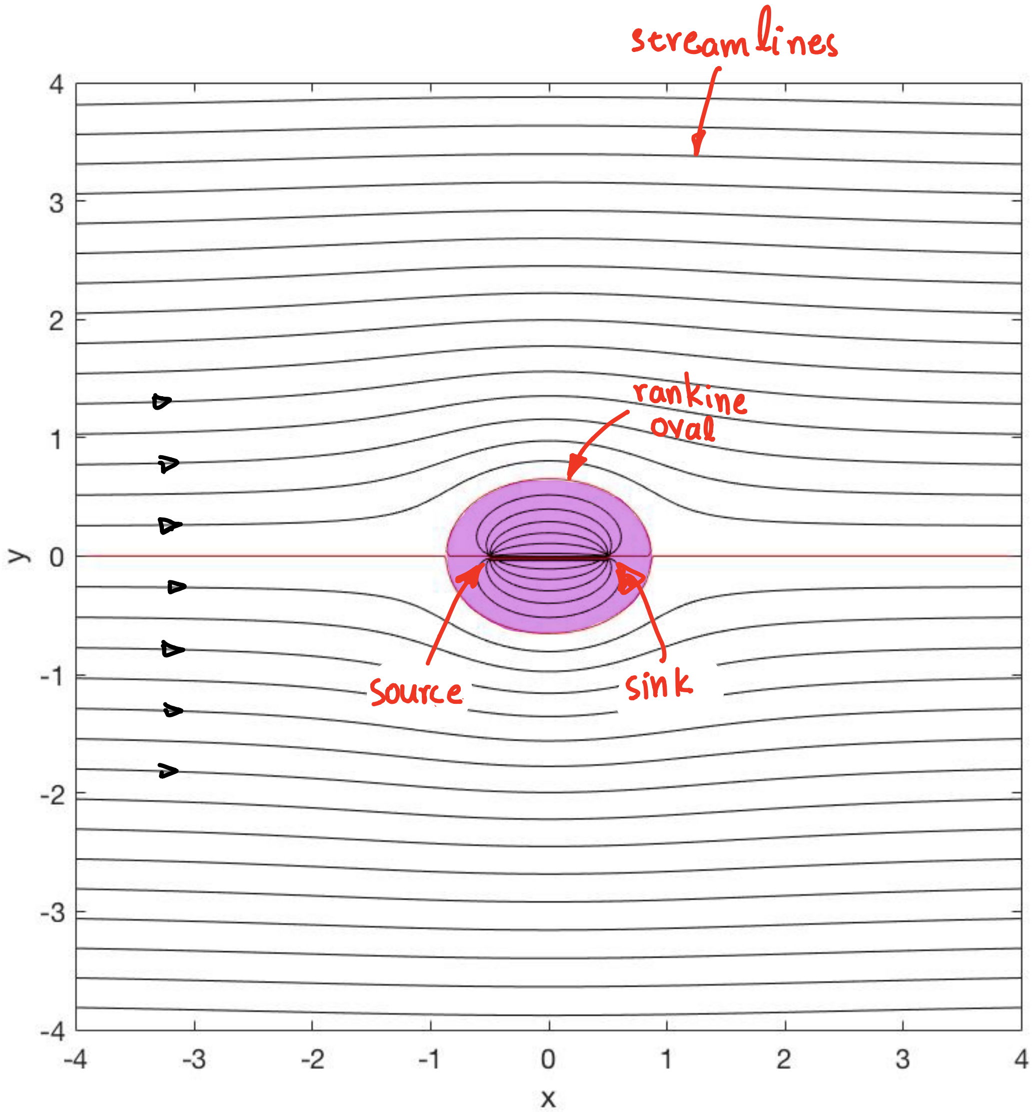
because flow is singular at the origin!

Important conclusion: I can have a flow that has circulation using superposition of "basic" flows

Example : Rankine Oval

Flow around a "Rankine Oval" can be constructed by the superposition of uniform flow with $V = 2\ m/s$ in the horizontal direction, a source of strength $m = 2\pi \frac{1}{s}$ at $x = -0.5\ m$ and $y = 0\ m$, and a sink of strength $m = -2\pi \frac{1}{s}$ at $x = +0.5\ m$ and $y = 0$

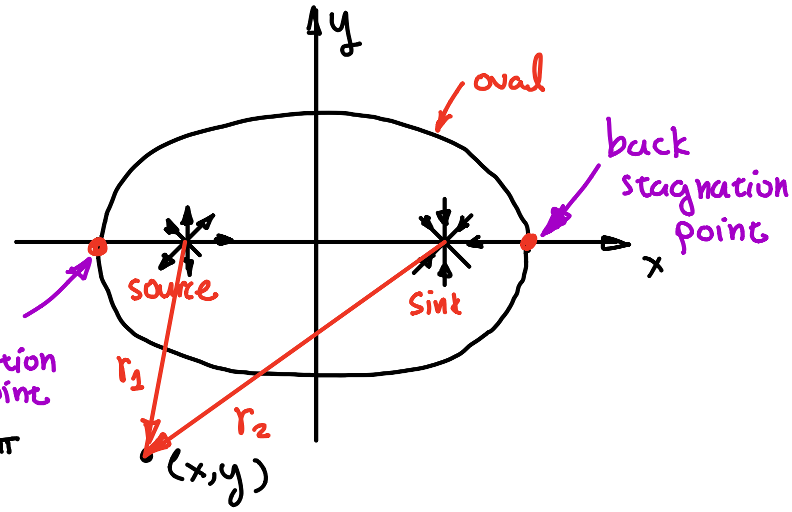
a. What is the velocity potential of the flow around the Rankin oval?
b. What is the length of the Rankin oval?



Solution

a. $\varphi_{oval} = \varphi_{uniform} + \varphi_{source} + \varphi_{sink}$

use Table 6.1 to find these



$m = 2\pi$ $\quad$ $m = -2\pi$

$$\varphi_{oval} = 2x + \frac{2\pi}{2\pi} \ln r_1 - \frac{2\pi}{2\pi} \ln r_2$$

$$= 2x + \ln r_1 - \ln r_2$$

these distances are from the source and sink locations $\Rightarrow$ THEY ARE DIFFERENT $r_1 \neq r_2$

to convert to Cartesian: $r_1 = \left[(x - x_{source})^2 + y^2\right]^{1/2}$

$$r_2 = \left[(x - x_{sink})^2 + y^2\right]^{1/2}$$

in Cartesian:

$$\varphi_{oval}(x,y) = 2x + \ln\left[(x - x_{source})^2 + y^2\right]^{1/2} - \ln\left[(x - x_{sink})^2 + y^2\right]^{1/2}$$

$$\varphi_{oval}(x,y) = 2x + \frac{1}{2}\ln\left[(x - x_{source})^2 + y^2\right] - \frac{1}{2}\ln\left[(x - x_{sink})^2 + y^2\right]$$

b. Length is the distance... between... the front and back stagnation points

stagnation point is where $u=0$ and $v=0$

$v=0$ on x-axis, so we will use $u=0$

We will find location where $u=0$ on x-axis:

$$u = \frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\left(2x + \frac{1}{2}\ln\left[(x - x_{source})^2 + y^2\right] - \frac{1}{2}\ln\left[(x - x_{sink})^2 + y^2\right]\right)$$

$$u(x,y) = 2 + \frac{x - x_{source}}{(x - x_{source})^2 + y^2} - \frac{x - x_{sink}}{(x - x_{sink})^2 + y^2}$$

on x-axis : $y=0$

$$u(x,0) = 2 + \frac{1}{x - x_{source}} - \frac{1}{x - x_{sink}}$$

$u=0$ where: $2 + \frac{1}{x - x_{source}} - \frac{1}{x - x_{sink}} = 0$

solve to find x :

$$2(\underbrace{x - x_{source}}_{=-\frac{1}{2}})(\underbrace{x - x_{sink}}_{=+\frac{1}{2}}) + \underbrace{x - x_{sink}}_{=+\frac{1}{2}} - \underbrace{x + x_{source}}_{=-\frac{1}{2}} = 0$$

$$2(x - \tfrac{1}{2})(x + \tfrac{1}{2}) - 1 = 0$$

$$x^2 - \tfrac{1}{4} = \tfrac{1}{2}$$

$$x = \pm \frac{\sqrt{3}}{2}$$

$x = +\frac{\sqrt{3}}{2}$ back stagnation point

$x = -\frac{\sqrt{3}}{2}$ front stagnation point

Rankine Oval length $= x_{back} - x_{front} = \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$ m

Solution:

a. $\varphi_{oval} = \varphi_{uniform} + \varphi_{source} + \varphi_{sink}$

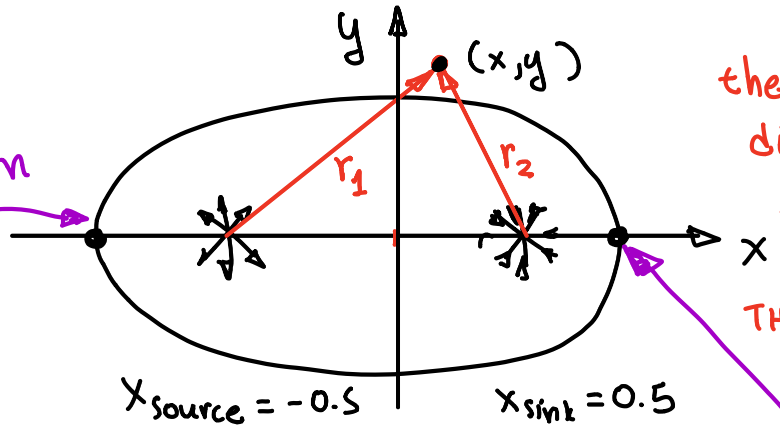
use Table 6.1 to find these

$$\varphi_{oval} = 2x + \frac{2\pi}{2\pi}\ln r_1 - \frac{2\pi}{2\pi}\ln r_2 = 2x + \ln r_1 - \ln r_2$$

these are distances from source and sink locations

THEY ARE DIFFERENT $r_1 \neq r_2$

front stagnation point



back stagnation point

$$r_1 = [(x - x_{source})^2 + y]^{1/2}$$

$$r_2 = [x - x_{sink})^2 + y]^{1/2}$$

In Cartesian:

$$\varphi_{oval}(x,y) = 2x + \frac{1}{2}\ln[(x - x_{source})^2 + y^2] - \frac{1}{2}\ln[(x - x_{sink})^2 + y^2]$$

b. Length = distance between stagnation points

stagnation point is where $u = 0$ on the x-axis

we cannot use $v=0$, see below...

$$u(x,y) = \frac{\partial \varphi}{\partial x} = 2 + \frac{x - x_{source}}{(x-x_{source})^2 + y^2} - \frac{x - x_{sink}}{(x-x_{sink})^2 + y^2}$$

find $u=0$ on x-axis $(y=0)$

$$2 + \frac{1}{x-x_{source}} - \frac{1}{x-x_{sink}} = 0$$

$$2(x-x_{source})(x-x_{sink}) + x - x_{sink} - x + x_{source} = 0$$

$$2\left(x-\tfrac{1}{2}\right)\left(x+\tfrac{1}{2}\right) - 1 = 0$$

$$x^2 - \frac{1}{4} = \frac{1}{2}$$

$$x = \pm \frac{\sqrt{3}}{2} \quad \longleftarrow \quad x = +\frac{\sqrt{3}}{2} \text{ back stagnation point}$$

$$x = -\frac{\sqrt{3}}{2} \text{ front stagnation point}$$

$$\text{Rankine Oval length} = x_{back} - x_{front} = \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3} \text{ m}$$

We cannot use the v-velocity component to solve the problem because $v(x, y=0) = 0$

Try with v-velocity

$$v(x,y) = \frac{\partial \varphi}{\partial y} = \frac{y}{(x-x_{source})^2 + y^2} - \frac{y}{(x-x_{sink})^2 + y^2}$$

$$v(x, y=0) = \frac{0}{(x-x_{source})^2} - \frac{0}{(x-x_{sink})^2} = 0$$

This is because all sources and sinks are on x-axis and flow is symmetric around the x-axis