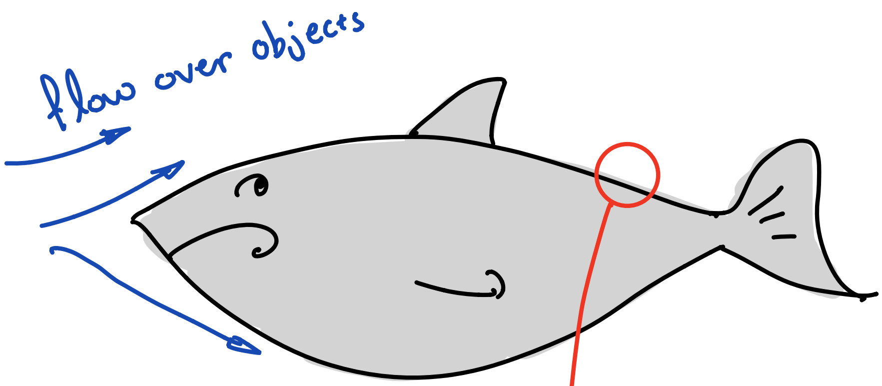
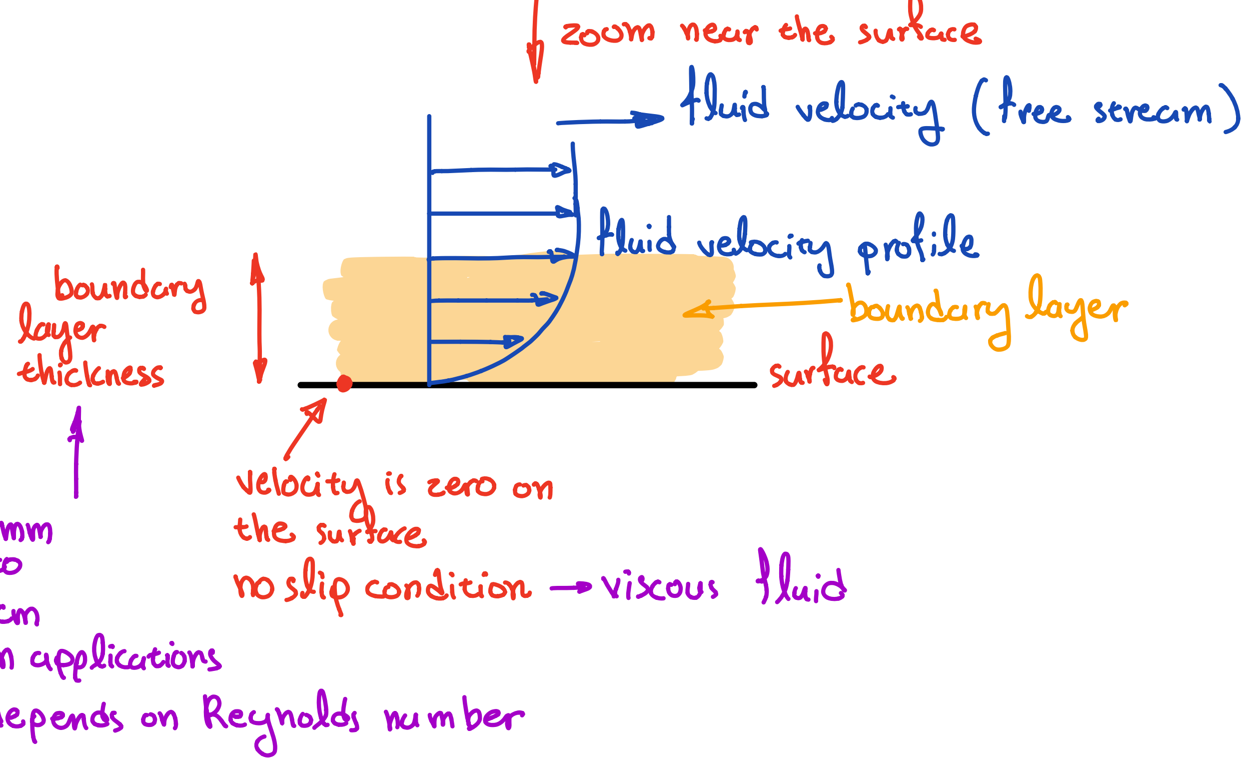
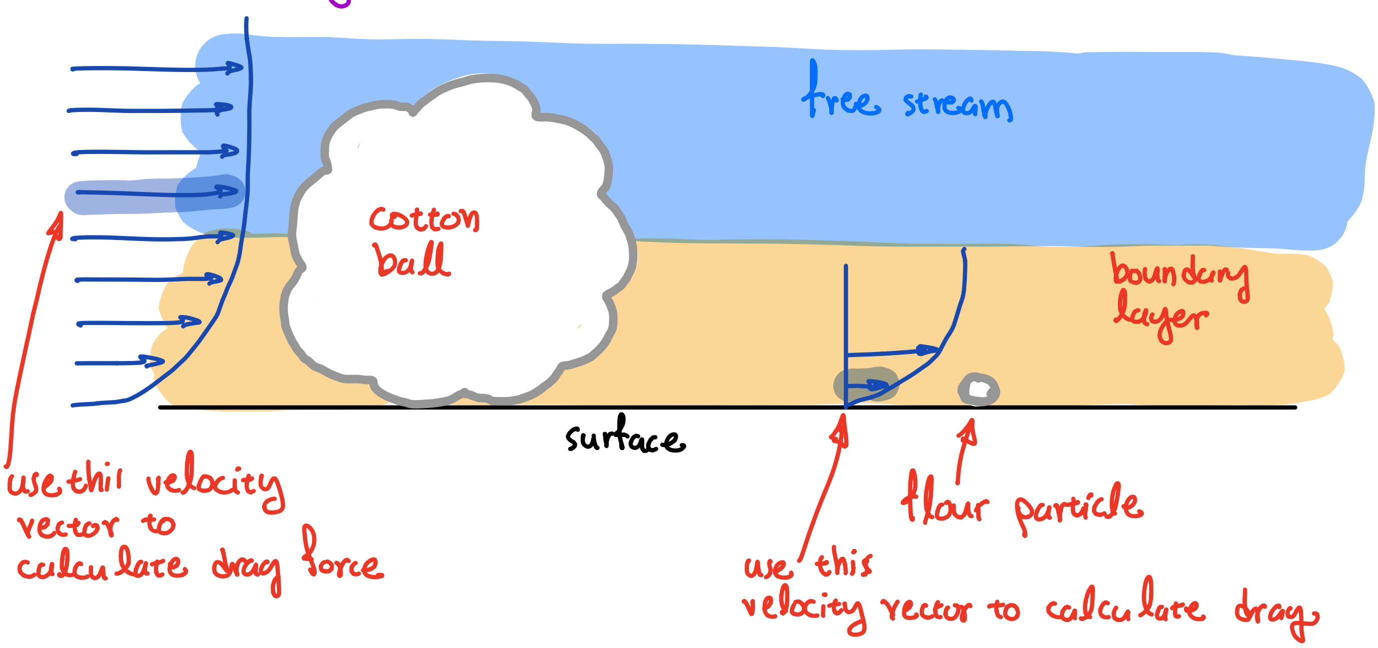
Boundary layer (simplified discussion)

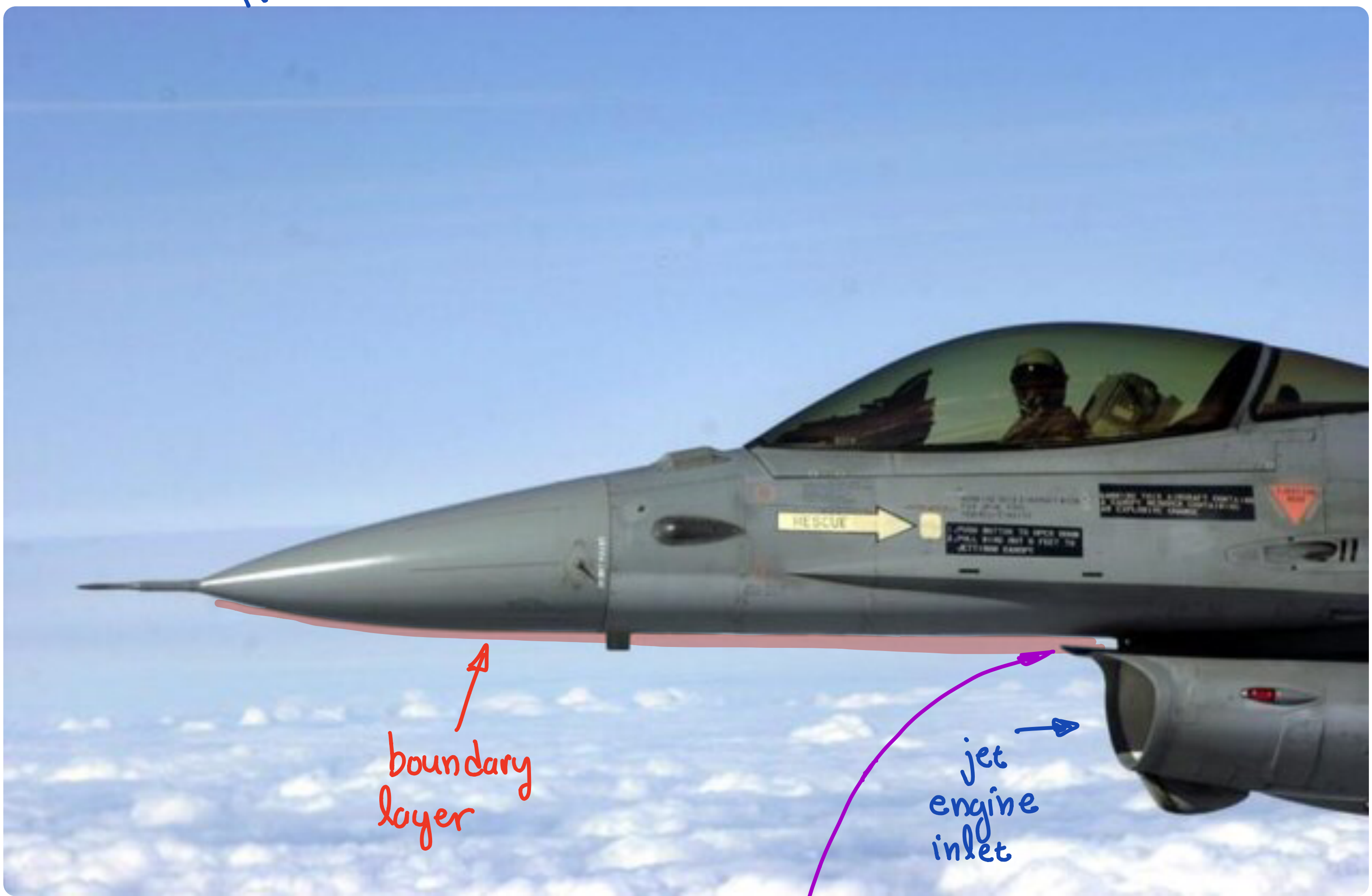
see ch. 9.1 and 9.2 of textbook







Another application



boundary layer is "pealed off" and not injested into jet engine

Drag force - boundary layer - flow separation

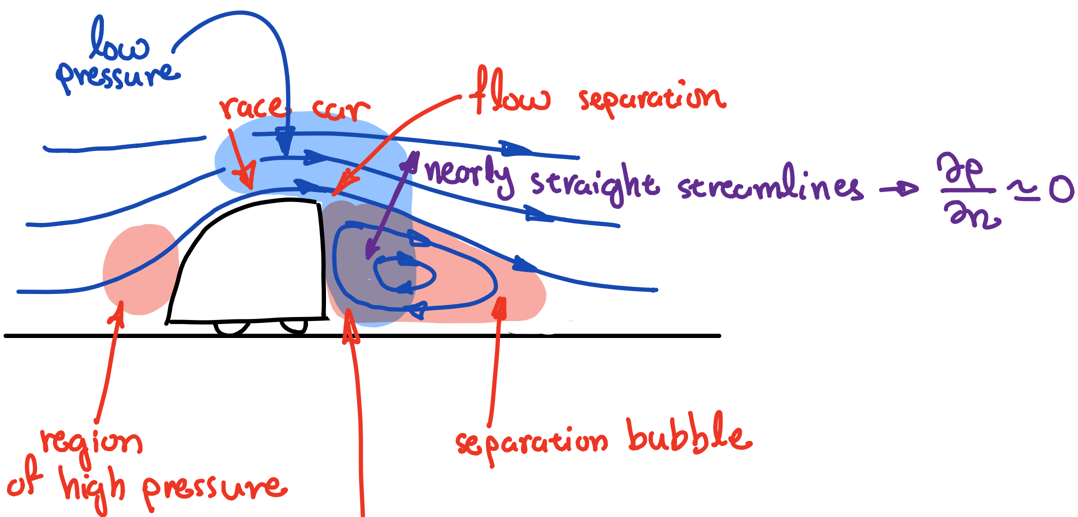
$$\overrightarrow{\text{Force on object}} = \oint_{\substack{\text{surface}\\ \text{of object}}} T_{ij} \cdot \hat{n}\, dA$$ ← vector

integral of stresses over area

$$= \oint_S \text{pressure} \cdot \hat{n}\, dA + \oint_S \underbrace{\tau_{ij}}_{\text{shear stresses}} \cdot \hat{n}\, dA$$

Specifically for Drag $= \left[\oint_S \text{pressure} \cdot \hat{n}\, dA + \oint_S \tau_{ij} \cdot \hat{n}\, dA \right] \cdot \dfrac{\vec{V}}{|V|}$

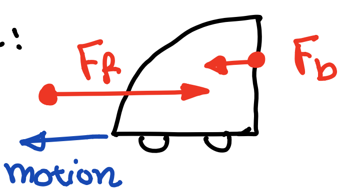
$= \text{form drag} + \text{skin friction}$



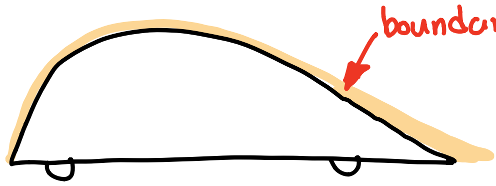
pressure behind the car is low because of Bernoulli normal to streamline

$$\frac{\partial p}{\partial n} = -\frac{\rho V^2}{R}$$

Because of flow separation:



= large force to the right



remember: $F = \tau A = \mu \frac{\partial u}{\partial y} \text{Area}$

this is a very small number 10^{-5} for air

What we need to know:

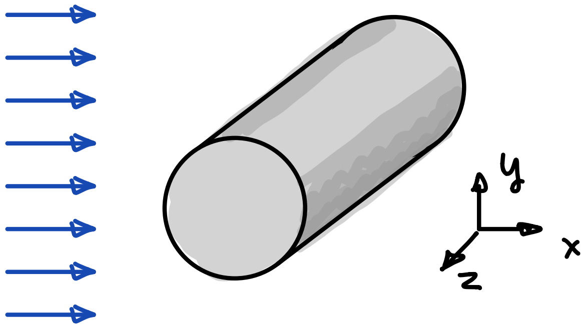
- Form drag (because of pressure) is large

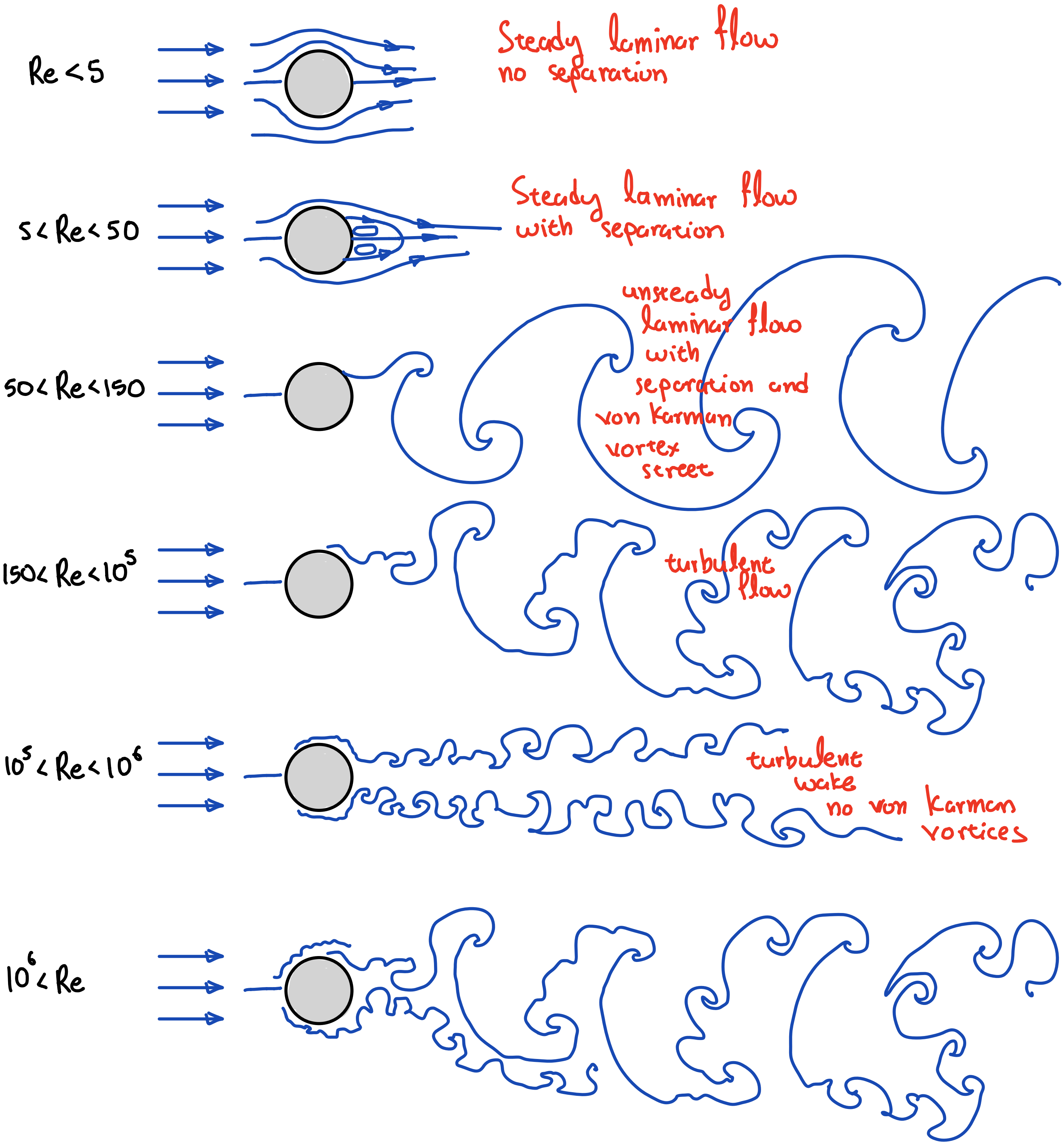
  it is very nasty avoid always

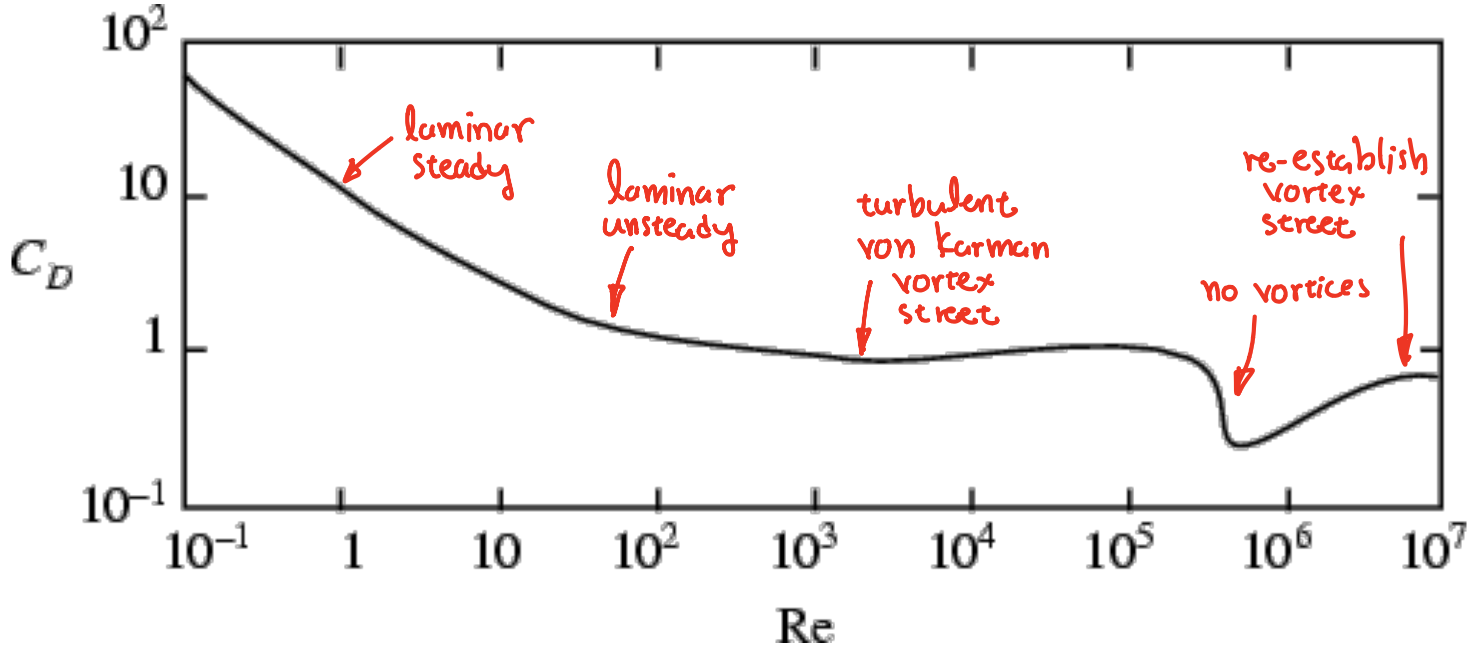
- Skin friction is small – cannot be avoided

  must very cleverly manipulate the flow to reduce skin friction
  → aerodynamics

Flow around cylinder





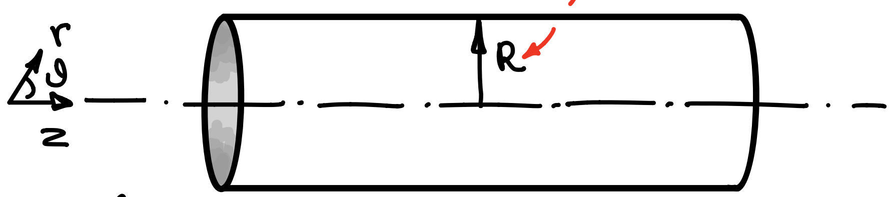


laminar steady
laminar unsteady
turbulent von karman vortex street
no vortices
re-establish vortex street
C_D
Re

Viscous flow in circular pipe

Chapter 6.9.3 but read all 6.8 and 6.9

upper case R: radius, it's a constant



cylindrical-polar coordinates r, θ, z

lower case r: independent variable

$$\vec{V} = u_r \hat{r} + u_\theta \hat{\theta} + u_z \hat{k}$$

We want to find the "flow" in the pipe

"flow" means determine $\vec{V}(t, r, \theta, z)$ and $p(t, r, \theta, z)$

Assumptions

English	Math
① Steady flow	$\frac{\partial \bullet}{\partial t} = 0$
② Straight flow along pipe	$u_r = 0$
③ No swirl	$u_\theta = 0$
④ Axisymmetric flow	$\frac{\partial \bullet}{\partial \theta} = 0$
⑤ Laminar flow (no turbulence)	check that $Re < 1000$

- flow is "smooth"
- Reynolds number is low < 1000

"Solution" approach

because we are asked for $\vec{V}(t,r,\theta,z)$ and $p(t,r,\theta,z)$ we need to use the differential equations of motion

Eq 6.33 and 6.128:

Assumptions

① $\frac{\partial \bullet}{\partial t} = 0$

② $u_r = 0$

③ $u_\theta = 0$

④ $\frac{\partial \bullet}{\partial \theta} = 0$

Continuity, a.k.a. "Mass Conservation"

$$\overset{①}{\cancel{\frac{\partial \rho}{\partial t}}} + \frac{1}{r}\overset{②}{\cancel{\frac{\partial}{\partial r} r u_r}} + \frac{1}{r}\overset{③}{\cancel{\frac{\partial u_\theta}{\partial \theta}}} + \frac{\partial u_z}{\partial z} = 0$$

r-momentum:

$$\rho\left(\overset{①}{\cancel{\frac{\partial u_r}{\partial t}}} + \overset{②}{\cancel{u_r\frac{\partial u_r}{\partial r}}} + \overset{②}{\cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}} - \overset{③}{\cancel{\frac{u_\theta^2}{r}}} + \overset{②}{\cancel{u_z\frac{\partial u_r}{\partial z}}}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\overset{②}{\cancel{\frac{\partial u_r}{\partial r}}}\right) - \overset{②}{\cancel{\frac{u_r}{r}}} + \frac{1}{r^2}\overset{②}{\cancel{\frac{\partial^2 u_r}{\partial \theta^2}}} - \frac{2}{r^2}\overset{③}{\cancel{\frac{\partial u_\theta}{\partial \theta}}} + \overset{②}{\cancel{\frac{\partial^2 u_r}{\partial z^2}}}\right]$$

θ-momentum:

$$\rho\left(\overset{①}{\cancel{\frac{\partial u_\theta}{\partial t}}} + \overset{②}{\cancel{u_r\frac{\partial u_\theta}{\partial r}}} + \overset{③}{\cancel{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}}} + \overset{②}{\cancel{\frac{u_r u_\theta}{r}}} + u_z\frac{\partial u_\theta}{\partial z}\right) = \overset{④}{\cancel{-\frac{1}{r}\frac{\partial p}{\partial \theta}}} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\overset{③}{\cancel{\frac{\partial u_\theta}{\partial r}}}\right) - \overset{③}{\cancel{\frac{u_\theta}{r}}} + \frac{1}{r^2}\overset{③}{\cancel{\frac{\partial^2 u_\theta}{\partial \theta^2}}} + \frac{2}{r^2}\overset{②}{\cancel{\frac{\partial u_r}{\partial \theta}}} + \overset{③}{\cancel{\frac{\partial^2 u_\theta}{\partial z^2}}}\right]$$

z-momentum:

$$\rho\left(\overset{①}{\cancel{\frac{\partial u_z}{\partial t}}} + \overset{②}{\cancel{u_r\frac{\partial u_z}{\partial r}}} + \overset{③}{\cancel{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\overset{④}{\cancel{\frac{\partial^2 u_z}{\partial \theta^2}}} + \frac{\partial^2 u_z}{\partial z^2}\right]$$

We are left with:

$$\frac{\partial u_z}{\partial z} = 0 \quad ⑥ \qquad \text{continuity}$$

$$\frac{\partial p}{\partial r} = 0 \qquad \text{r-momentum}$$

z-momentum

$$\rho u_z \underset{⑥}{\cancel{\frac{\partial u_z}{\partial z}}} = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \underset{⑥}{\cancel{\frac{\partial^2 u_z}{\partial z^2}}}\right]$$

Then: $\frac{\partial p}{\partial r} = 0 \Rightarrow$ pressure does not vary with r

$\frac{\partial p}{\partial \theta} = 0$ because of ④ $\Rightarrow$ pressure does not vary with θ

what we need

$$\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{\partial p}{\partial z}$$

- what we need is under two derivatives, so we need to integrate twice w.r.t. r

Integrate once

$$\int \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)dr = \int \frac{r}{\mu}\frac{\partial p}{\partial z}\,dr$$

$$r\frac{\partial u_z}{\partial r} = \frac{r^2}{2\mu}\frac{\partial p}{\partial z} + c_1$$

$$\frac{\partial u_z}{\partial r} = \frac{r}{2\mu}\frac{\partial p}{\partial z} + \frac{c_1}{r}$$

Integrate twice

$$\int \frac{\partial u_z}{\partial r}\,dr = \int\left(\frac{r}{2\mu}\frac{\partial p}{\partial z} + \frac{c_1}{r}\right)dr$$

$$u_z(r) = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + c_1 \ln r + c_2$$

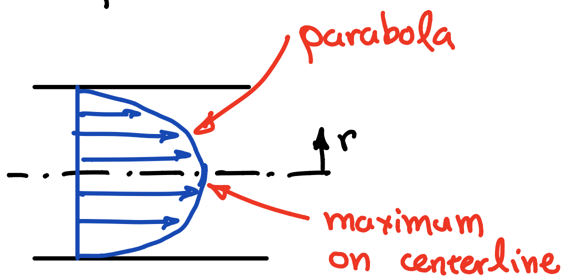
We need boundary conditions to determine c_1 and c_2

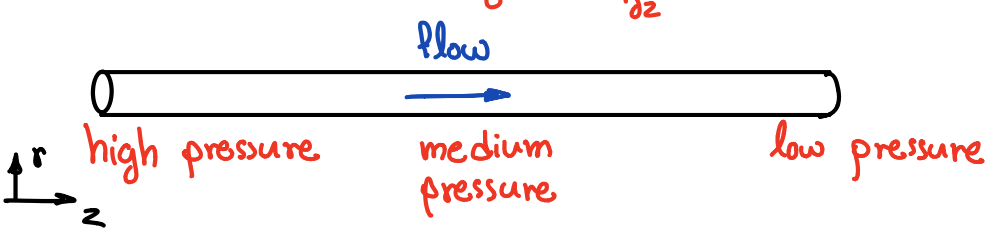
1. $u_z(r)$ must be finite in $0<r<R$ so $c_1 \ln r < \infty \Rightarrow c_1 = 0$
2. No slip condition for $r=R \Rightarrow u_r(r=R)=0$

$$0 = \frac{R^2}{4\mu}\frac{\partial p}{\partial z} + c_2 \Rightarrow c_2 = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z}$$

$$\underbrace{u_z(r)}_{\text{positive}} = \underbrace{\frac{1}{4\mu}}_{\text{positive}}\frac{\partial p}{\partial z}\underbrace{(r^2-R^2)}_{\substack{\text{negative}\\ r<R}}$$

thus this is negative $\frac{\partial p}{\partial z} < 0$





How does pressure vary along the pipe? 👀

Volumetric flow rate: $$Q = \int_{\text{area}} u_z(r)\, dA$$

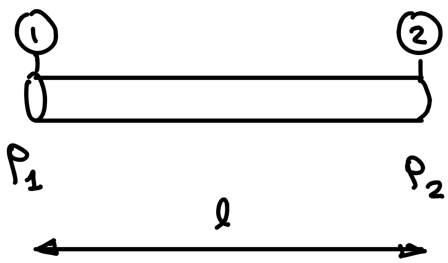
$$= \int_0^R \int_0^{2\pi} \frac{1}{4\mu} \frac{\partial p}{\partial z} (r^2 - R^2)\, d\theta\, r\, dr$$

$$= -\frac{\pi R^4}{8\mu} \frac{\partial p}{\partial z}$$

$$\frac{\partial p}{\partial z} = -\underbrace{\frac{8\mu Q}{\pi R^4}}$$

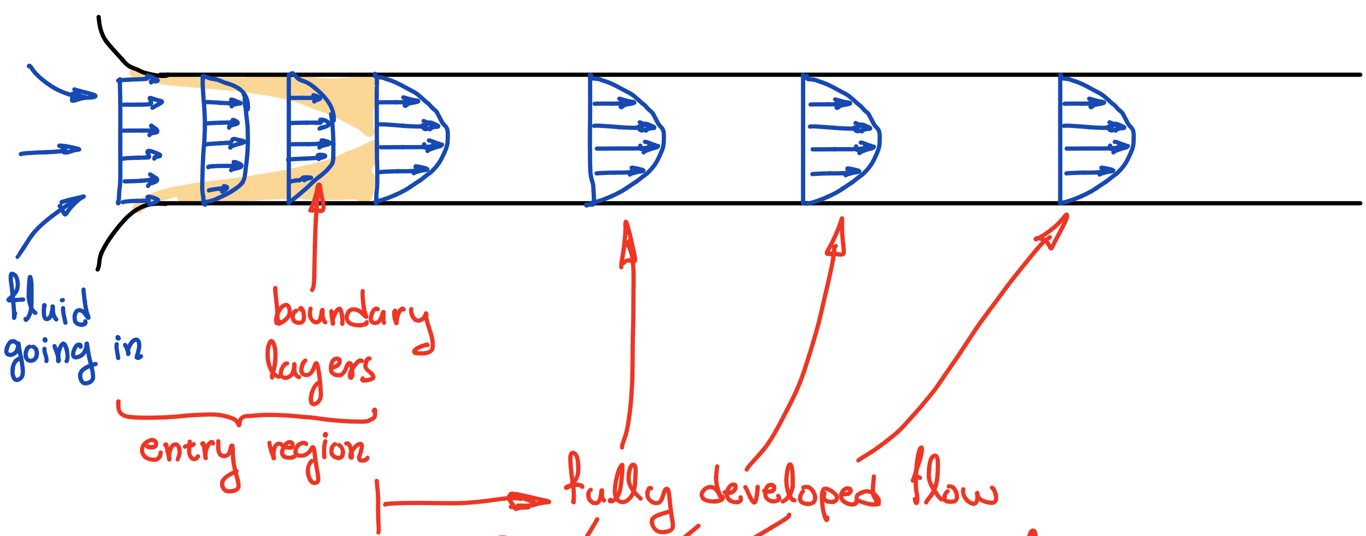
for given pipe geometry and flow rate this is constant

so... $\frac{\partial p}{\partial z}$ = constant $\Rightarrow$ pressure varies linearly with distance!

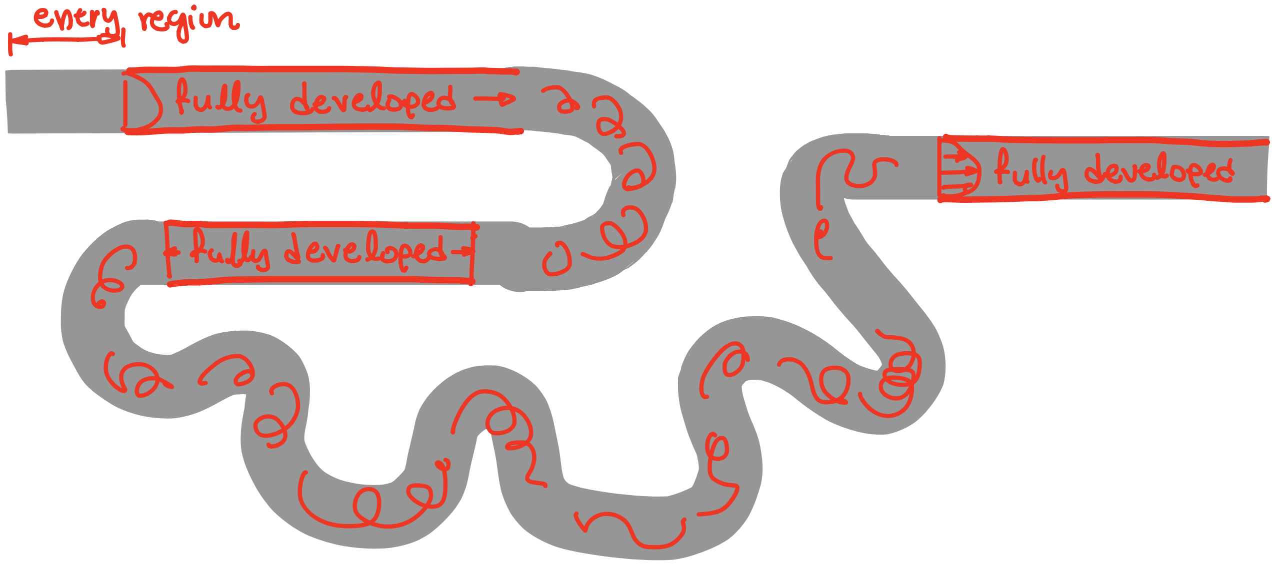


$$\frac{\partial p}{\partial z} = \frac{p_2 - p_1}{\ell} < 0$$ remember: $p_1 > p_2$

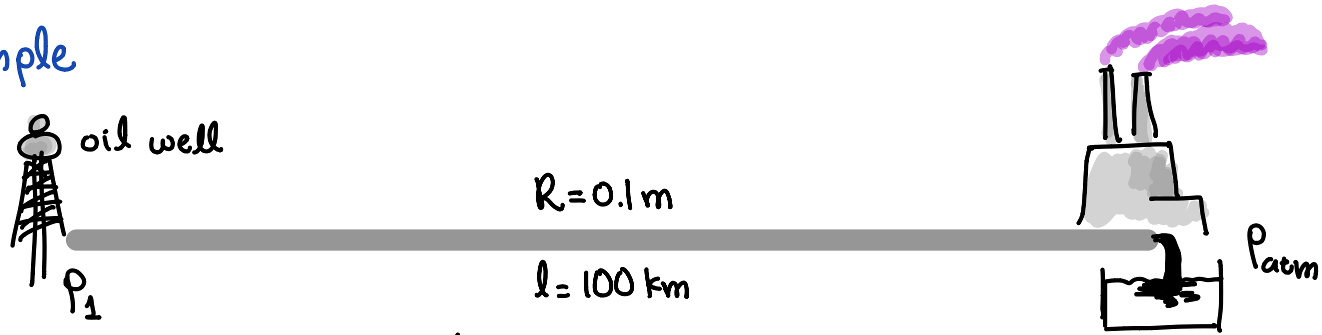
Where is $u_z(r) = \frac{1}{4\mu} \frac{\Delta p}{\ell} (R^2 - r^2)$ valid?



Same velocity profile $u_z(r)$


entry region
fully developed
fully developed
fully developed

Example



$Q = 0.05\ m^3/s$

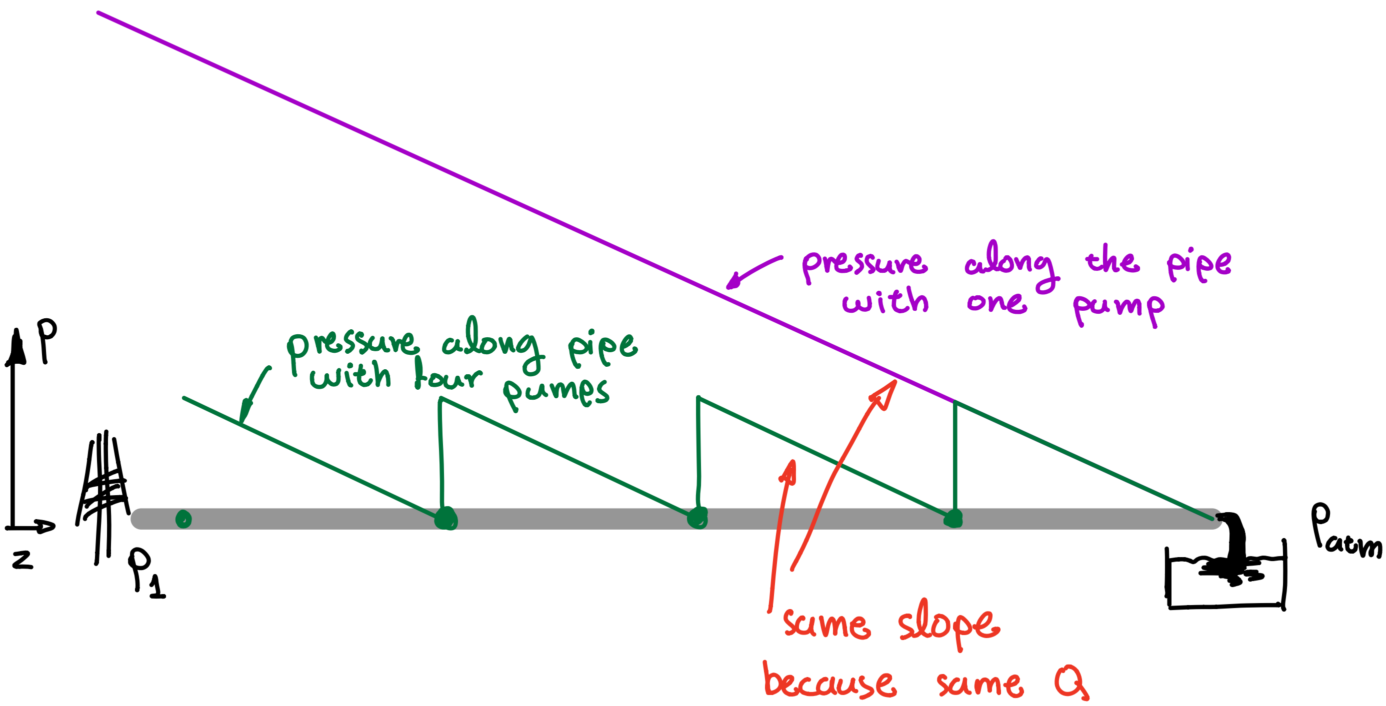
$\mu = 2\ Pas$

What is the pressure p_1 that we must pump the oil to have flow rate $Q = 0.05\ m^3/s$ in a pipe of radius $R = 0.1\ m$ and length $l = 100\ km$. Oil viscosity is $\mu = 2\ Pas$

Solution: $Q = \frac{\pi R^4}{8\mu}\frac{\Delta p}{l}$

$$\Delta p = p_1 - p_{atm} = \frac{8\mu l Q}{\pi R^4} = \frac{8 \times 2 \times 100000 \times 0.05}{\pi \times 0.1^4} = 254\ MPa$$

$p_1 = 254\ MPa$ gage   this is too high because pipe is long



Reynolds number

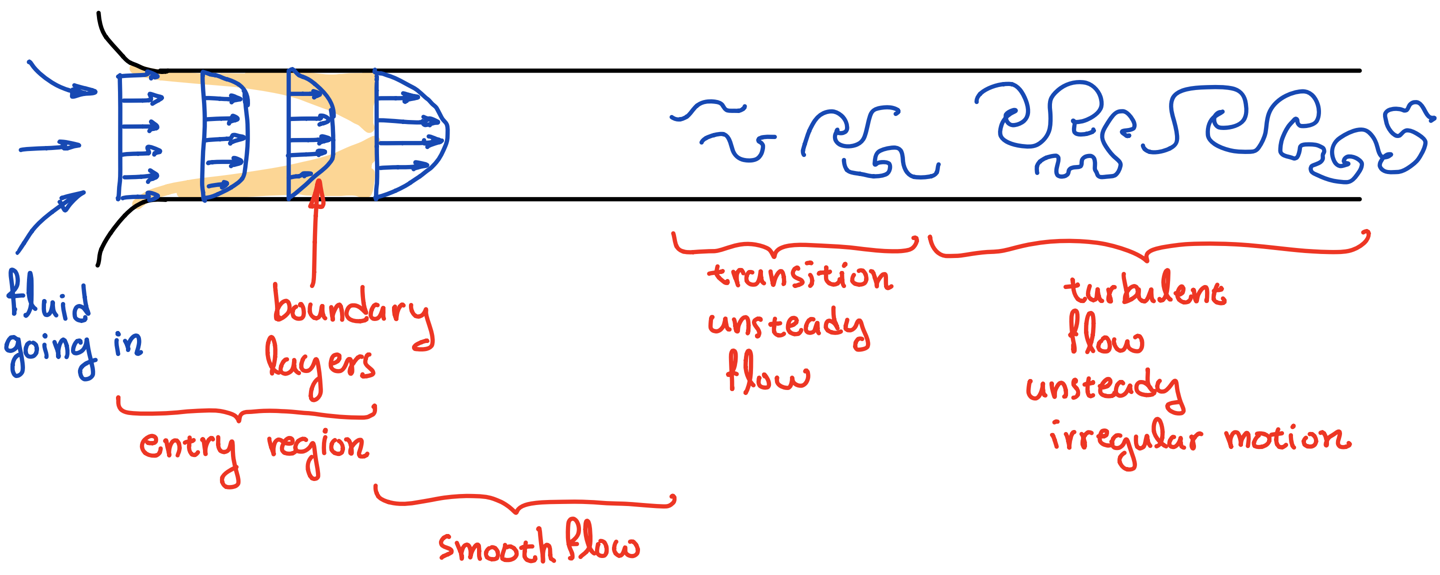
$$Re = \frac{\text{Velocity} * \text{Length}}{\text{kinematic viscosity}}$$

$$Re = \frac{?\ D}{\mu/\rho}$$

$$V_{mean} = \frac{Q}{\text{Area}} = \frac{Q}{\pi R^2} \rightarrow Re = \frac{V_{mean}\, D}{\mu/\rho}$$

Turbulence in the flow in the pipe

if $Re > 10{,}000$ then the flow is <u>not</u> laminar

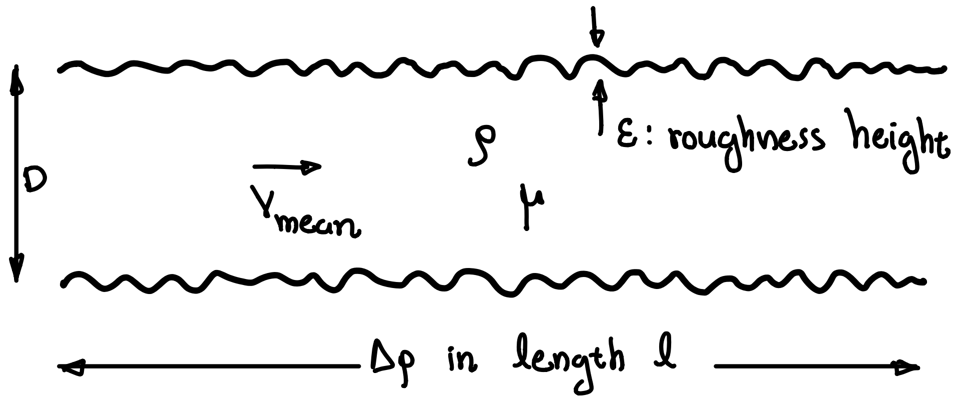


$$Q = \frac{\pi R^4}{8\mu} \frac{\Delta p}{\ell}$$

ONLY valid here!

The real situation about pipe flow (chapter 8.3)

- The flow in the pipe is turbulent (not laminar) $Re > 10,000$
  $Q =$ blah blah blah Δp ... does not apply...
- The pipe is not smooth



	Parameters	Dimensions
flow	V_{mean}	$\frac{L}{T}$
	Δp	$\frac{M}{T^2 L}$
fluid	ρ	$\frac{M}{L^3}$
	μ	$\frac{M}{LT}$
geometry	D	L
	l	L
	ε	L
	$k = 7$	$r = 3$

Pi-Theorem : $k - r = 7 - 3 \rightarrow 4$ groups

Dimensionless groups: $\frac{\Delta p}{\frac{1}{2}\rho V_{mean}^2}$ $\quad Re = \frac{\rho V_{mean} D}{\mu}$ $\quad \frac{l}{D}$ $\quad \frac{\varepsilon}{D}$

In general $\dfrac{\Delta p}{\frac{1}{2}\rho V_{mean}^2} = \text{function}\left(Re, \dfrac{l}{D}, \dfrac{\varepsilon}{D}\right)$

We often work in terms of pressure drop per unit pipe length

so we rearrange to

$$\underbrace{\frac{\Delta p}{\frac{1}{2}\rho V_{mean}^2}\frac{D}{l}}_{\text{we call this the}} = \text{function}\left(Re, \frac{\varepsilon}{D}\right)$$

friction factor $f = \dfrac{2\,\Delta p\, D}{\rho V_{mean}^2\, l}$

before we had $C_D = \text{function}\,(Re)$

now we have $f = \text{function}\left(Re, \dfrac{\varepsilon}{D}\right)$

The Moody Diagram plots this function

Important: for laminar flow $f = \dfrac{64}{Re}$

Example: Moody Diagram

Water with density $\rho = 1000\ kg/m^3$ and viscosity $\mu = 8.9 \times 10^{-4}$ Pas flows in a cast iron pipe of diameter $D = 0.1\ m$. If the flow rate is $Q = 0.02\ m^3/s$, What is the pressure drop per meter?

Solution:

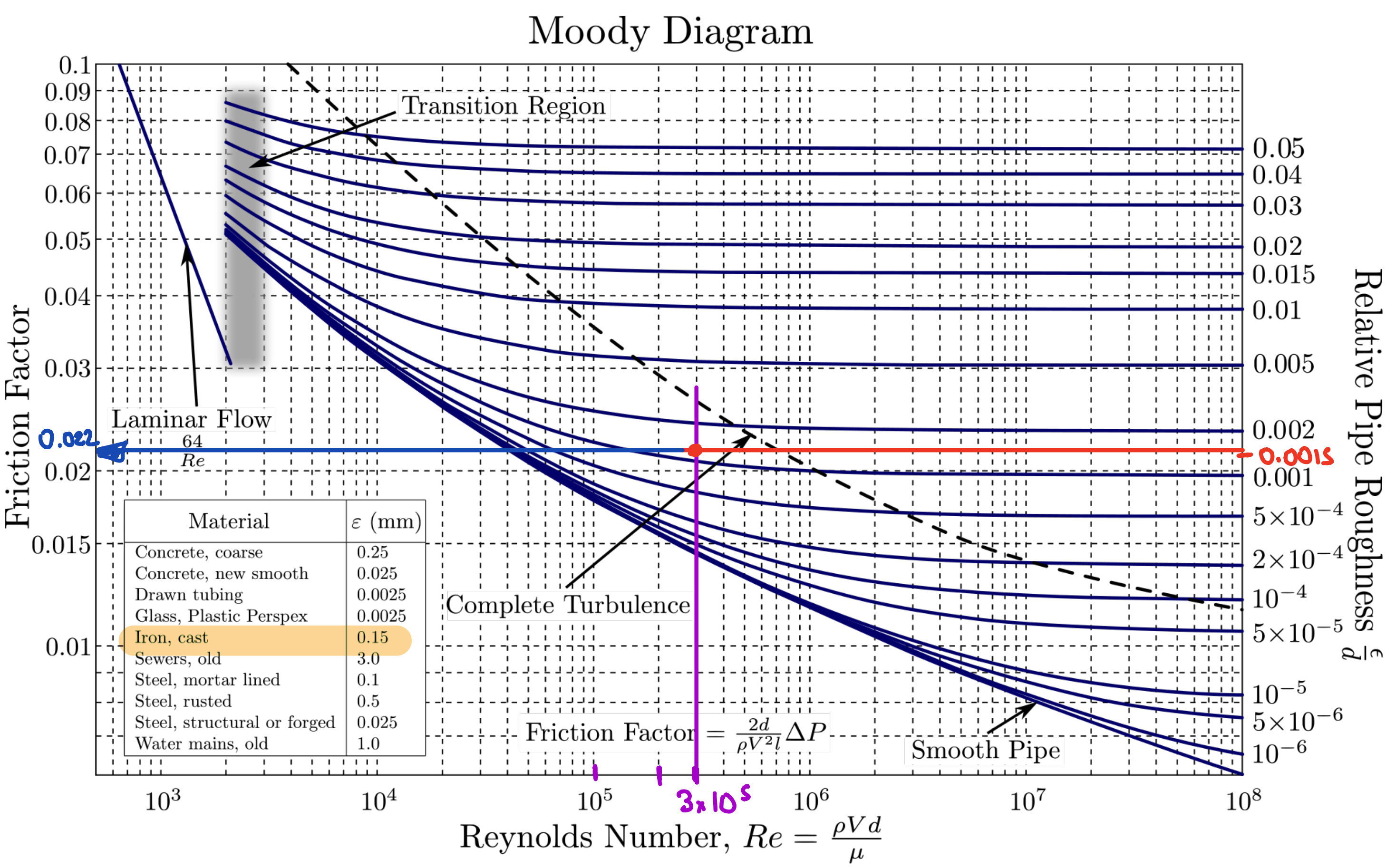
$$f = \frac{2 \Delta p D}{\rho V_{mean}^2 l} \Rightarrow \frac{\Delta p}{l} = \frac{f}{D} \frac{1}{2} \rho V_{mean}^2$$

unknowns

$$V_{mean} = \frac{Q}{Area} = \frac{0.02}{\pi R^2} = \frac{0.02}{\pi \times 0.05^2} = 2.55\ m/s$$

friction factor depends on Re and $\frac{\varepsilon}{D}$

$$Re = \frac{\rho V_{mean} D}{\mu} = \frac{1000 \times 2.55 \times 0.1}{8.9 \times 10^{-4}} = 2.86 \times 10^5$$



Cast iron pipe $\varepsilon = 0.15\ mm$

$$\frac{\varepsilon}{D} = \frac{0.15\ mm}{0.1\ m} = \frac{0.15}{100} = 0.0015$$

$f = 0.022$ from Moody diagram

$$\frac{\Delta p}{l} = \frac{1}{2} \frac{f}{D} \rho V_{mean}^2 = \frac{1}{2} \frac{0.022}{0.1} 1000 \times 2.55^2 = 715 \frac{Pa}{m}$$