Review

partial derivative

functions of multiple varibles

$f(x,y) = x^2+y^2 \qquad f(x,y,z) = x^2+y^2+z^2$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2) = 2x; \quad \frac{\partial f}{\partial y} = 2y$

chain rule

composition of 2 functions

$f(x(t), y(t))$

$t \mapsto (x(t), y(t)) \mapsto f(x(t), y(t))$

ex: $f(x,y) = x^2+y^2 \qquad x(t) = e^t, y(t) = e^{-t}$

$f(x(t), y(t)) = e^{2t} + e^{-2t}$

$\frac{\partial}{\partial t} f(x(t), y(t)) = \frac{\partial f}{\partial x} \cdot x'(t) + \frac{\partial f}{\partial y} \cdot y'(t)$

$\frac{\partial f}{\partial t} = 2x(t) \cdot e^t + 2y(t) \cdot -e^{-t}$

$= 2e^t \cdot e^t - 2e^{-t} \cdot e^{-t} = 2e^{2t} - 2e^{-2t}$

more complex chain rule

$f(x(s,t), y(s,t))$

$f(x,y) = x^2+y^2, \quad x(s,t) = s^2+t^2 \qquad y(s,t) = s^2-t^2$

$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$

stationary points

gradient $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \qquad \nabla$: del: nabla

stationary point : c.p. : $\nabla f(x,y) = \vec{0}$

First order equation (ODE)

only first derivative is involved

seperable 1st order ODE: $\frac{dy}{dx} = f(x) \cdot g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$

$g(y) \equiv 0$ is missed

second order linear equations

general form: $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=d$

if $d=0$, equation is homogenious

ex: $y''+y'-2y=0$

characteristic equation: $y^2+y-2=0$

$y=-2,1$

$y=C_1e^{-2x}+C_2e^{x}$ is gen. sol.

integrating factor

ex: $y(x)$ $\frac{dy}{dx}+f(x)y=g(x)$

I.F. $=e^{\int f(x)dx}$ multiplied to both sides

$$e^{\int f(x)dx}\frac{dy}{dx}+e^{\int f(x)dx}f(x)y=e^{\int f(x)dx}g(x)$$

say $F(x)=\int f(x)dx$

$$e^{F}\frac{dy}{dx}+e^{F}f(x)y=e^{F}g(x)$$

$$\int\frac{d}{dx}\left(e^{F}y\right)=\int e^{F}g$$

because of product rule

$$e^{F}y=\int e^{F}g,\quad y=e^{-F}\int e^{F}g$$

ex: $y'+2y=e^{x}$

I.F. $=e^{2x}$

$$e^{2x}y=\int e^{2x}\cdot e^{x}dx=\int e^{3x}dx$$

$$\boxed{y=\frac{e^{x}}{3}+Ce^{-2x}}$$

Chapter 1. Introduction

1.1 PDE motivations and content

what is a PDE? why care?

$u(x,y)$ $\qquad \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y}$ $\qquad$ 1st order partial derivatives

$\frac{\partial^2 u}{\partial x^2} \quad \frac{\partial^2 u}{\partial y \partial x} \quad \frac{\partial^2 u}{\partial x \partial y} \quad \frac{\partial^2 u}{\partial y^2}$ $\qquad$ 2nd order

PDE: an equation involving u and its partial derivatives

general form:

$$F\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \dots\right) = 0$$

simpler notations:

$u_x : \frac{\partial u}{\partial x}$ $\qquad u_{xy} : (u_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial y \partial x}$

$$u_{xyx} = u_{xxy} = u_{yxx}$$

Transport equation: $a u_x + b u_y = 0$

Laplace equation: $u_{xx} + u_{yy} = 0$

Wave equation: $u_{tt} - u_{xx} = 0$ (1D and time) $\qquad u_{tt} - (u_{xx} + u_{yy}) = 0$ (2D and time)

heat equation: $u_t - u_{xx} = 0$ $\qquad u_t - (u_{xx} + u_{yy}) = 0$

solving simple PDEs

ex: $u(x,y)$ $\qquad u_x = 0$

$\frac{\partial u}{\partial x} = 0$ $\qquad$ For each y, u is a constant w.r.t. x

$\therefore \boxed{u = u(y)}$

ex: $u_x = xyu$

treat y as a constant and $u_x = y \cdot xu$ ← ODE

$\frac{du}{dx} = y\,xu \quad \therefore \quad u = \pm C(y) \cdot e^{yx^2/2}$

and $u \equiv 0$

ex: $u_{xy} = 0$

$= (u_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0$ $\qquad$ let $v = \frac{\partial u}{\partial x}$

$\frac{\partial v}{\partial y} = 0 \Rightarrow v_y = 0$

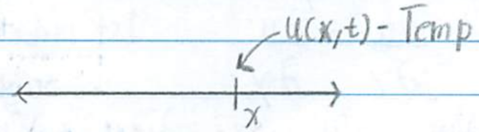
$v = v(x)$

$\frac{\partial u}{\partial x} = v(x) \quad = du = v(x)dx$

$$u = V(x) + C(y)$$

1.2 Initial and Boundary Value Problems

ex: Heat e.q. on the $\mathbb{R}$



time : t

$u_t - u_{xx} = 0$

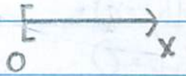
Find general sol

We can know an initial heating condition; $u(x,0) = g(x)$

IVP: $\begin{cases} u_t - u_{xx} = 0 & t \in \mathbb{R},\ x \in \mathbb{R} \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases}$

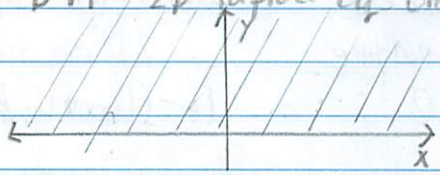
should output a specific solution

ex: BVP: laplace eq on the half line



$\begin{cases} u_{xx} = 0 \\ u(0) = u_0 \end{cases}$

ex: BVP: 2D laplace eq on the half-plane



$\begin{cases} u_{xx} + u_{yy} = 0 & x \in \mathbb{R},\ y > 0 \\ \text{B.C. } u(x,0) = g(x) & x \in \mathbb{R} \end{cases}$

IVP is time boundary BVP is space boundary

IVBP

ex: Heat eq on half line $u(x,t)$

$\begin{cases} u_t - u_{xx} = 0 \\ \text{I.C. } u(x,0) = g(x) & x > 0 \\ \text{B.C. } u(0,t) = h(t) & t \in \mathbb{R} \end{cases}$

Well-posedness: given a problem (IVP, BVP, or IBVP)

Question 1: does sol. exist?

Question 2: is the sol. unique? How many sols.?

Question 3: Does changing right side of equations matter? Do the solutions depend on R.H.S. continously? No chaos?

Homework #1

Problem 1: Determine if the following are linear homogenous, linear inhomogenous, or nonlinear (specify if semi-or quasi-linear)

(2) $u_t + uu_x = 0$

$\boxed{\text{nonlinear, quasilinear}}$

(4) $u_t + uu_x + x = 0$

$\boxed{\text{nonlinear, quasilinear}}$

(6) $u_t^2 - u_x^2 - 1 = 0$

$\boxed{\text{nonlinear}}$

(8) $xu_x + yu_y + zu_z = 0$

$\boxed{\text{linear homogenous.}}$

(10) $u_t + u_x^2 + u_y^2 = 0$

$\boxed{\text{nonlinear}}$

Problem 2: Determine the order of the following. Determine if they are linear homogenous, linear inhomogenous, or non-linear.

(12) $u_t - (1+u^2)u_{xx} = 0 \Rightarrow u_{xx} = \dfrac{u_t}{1+u^2}$ $\boxed{\text{2nd order nonlinear, semilinear}}$

(14) $u_t + uu_x + u_{xxx} = 0$ $\boxed{\text{3rd order nonlinear, semilinear}}$

(16) $u_{tt} + u_{xxxx} + u = 0$ $\boxed{\text{4th order linear homogenous}}$

(18) $u_{tt} + u_{xxxx} + \sin(x)\sin(u) = 0$ $\boxed{\text{4th order nonlinear, semilinear}}$

Problem 3: Find the general solution to the following

(20) $u_{xy} = 2u_x$

$$\frac{\partial}{\partial y}\left(u_x\right) = 2u_x \qquad \text{let } u_x = v(x,y)$$

$$\frac{\partial}{\partial y}(v) = 2v(x,y) \Rightarrow v_y = 2v(x,y) \quad \text{Hold } x \text{ constant}$$

$$\frac{dv}{dy} = 2v(y) \Rightarrow \int \frac{dv}{2v(y)} = \int dy \Rightarrow \frac{\ln|v|}{2} = y + c_1(x)$$

$$v = e^{2y+c_1(x)}$$

$$\frac{\partial u}{\partial x} = e^{2y+c_1(x)} \quad \text{Hold } y \text{ constant}$$

$$u = \int e^{2y+c_1(x)}dx = \boxed{\frac{c_1(x)}{c_1'(x)} e^{2y+c_1(x)} + c_2(y)}$$

$$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial x}\left(\frac{c_1}{c_1'(x)} e^{2y+c_1(x)} + c_2(y)\right)$$

$$u_x = e^{2y+c_1(x)}$$

$$u_{xy} = 2e^{2y+c_1(x)} = 2u_x$$

$$u_y = \frac{2}{c_1'(x)} e^{2y+c_1(x)} + c_2'(y)$$

$$u_{yx} = 2e^{2y+c_1(x)} = u_{xy}$$

(22) $u_{xy} = 2u_x + e^{x+y}$ let $u_x = v(x,y)$

$\frac{\partial}{\partial y}(v) = 2v + e^{x+y}$ Hold x constant

$\frac{dv}{dy} - 2v = e^{x+y}$ Integrating factor: $e^{\int -2dy}$

e^{-2y}

$$e^{-2y} v = \int e^{x+y} \cdot e^{-2y} dy$$

$$= \int e^{x+y-2y} dy = \int e^{x-y} dy$$

$$e^{-2y} v = -e^{x-y} + c_1(x)$$

$$v = -e^{x-y} \cdot e^{2y} + e^{2y} c_1(x)$$

$$v = -e^{x+y} + e^{2y} c_1(x)$$

$u_x = -e^{x+y} + e^{2y} c_1(x)$ Hold y constant

$\boxed{u = -e^{x+y} + e^{2y} C_1(x) + c_2(y)}$ * $\int c_1(x) dx = C_1(x)$ *

Check: $u_x = -e^{x+y} + e^{2y} c_1(x)$

$u_{xy} = -e^{x+y} + 2e^{2y} c_1(x)$ $e^{2y} c_1(x) = u_x + e^{x+y}$

$= -e^{x+y} + 2u_x + 2e^{x+y}$

$u_{xy} = 2u_x + e^{x+y}$

check again: $u_y = -e^{x+y} + 2e^{2y} C_1(x) + c_2'(y)$

$u_{yx} = -e^{x+y} + 2e^{2y} c_1(x) = u_{xy}$

Classification of 2nd order linear PDE

General form of 2nd order linear PDE:

$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g$

· constants can be dependent on x and/or y

Discriminant: $\Delta = b^2 - 4ac$

if: $\Delta > 0$ Hyperbolic eq

$\Delta = 0$ Parabolic eq

$\Delta < 0$ Elliptic eq

2.1 First order PDE's

$F(u, u_x, u_t) = f(x,t)$: General form

linear form: $au + bu_x + cu_t = f$

ex: solve $au_t + bu_x = 0$ $\quad u(x,t)$

Fact: $du = u_x dx + u_t dt$

$$= \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial t} dt$$

$u_t dt + u_x dx = du$ $\qquad a \alpha dt \quad b \alpha dx$

$u_t a + u_x b = 0$ $\qquad \varepsilon a = dt \quad \varepsilon b = dx$

$$\frac{dt}{a} = \varepsilon = \frac{dx}{b}$$

$$du = u_t \frac{dt}{a} \cdot a + u_x \frac{dx}{b} \cdot b$$

$du = u_t \varepsilon \cdot a + u_x \varepsilon \cdot b$

$du = \varepsilon(\underbrace{au_t + bu_x}_{\to 0})$

$\therefore$ if $\frac{dt}{a} = \frac{dx}{b}$ then $du = 0$

$$\frac{dt}{a} = \frac{dx}{b} \Rightarrow \frac{t}{a} = \frac{x}{b} + C$$

$$x = \frac{b}{a} t - bC$$

← eq of a line

$du = 0$ and $C = \frac{t}{a} - \frac{x}{b}$

$u = D\left(\frac{t}{a} - \frac{x}{b}\right)$

↑ any function

IVP

PDE and initial condition $\begin{cases} au_t + bu_x = 0 \iff u = D(\frac{t}{a} - \frac{x}{b}) \\ u(x,0) = f(x) \end{cases}$

$u(x,0) = D(0 - \frac{x}{b}) = f(x)$

say: $y = -x/b$ then $D(y) = f(-by)$

$u = D(\frac{t}{a} - \frac{x}{b}) = \quad = \boxed{f(-b \cdot \frac{t}{a} + x)}$

now $y = \frac{t}{a} - \frac{x}{b}$ and $f(-by) =$

ex: solve $\begin{cases} 2u_t + 3u_x = 0 \iff u = D(t/2 - x/3) \\ u(x,0) = e^x \end{cases}$

$u(x,0) = D(0 - x/3) = e^x \qquad y = -x/3, \quad x = -3y$

$D(x) = e^{-3y}$

$\therefore u = e^{-3(t/2 - x/3)} \Rightarrow \boxed{u = e^{(-3t/2 + x)}}$

Variable coefficents

$au_t + bu_x = 0$

where $a(x,t)$ and $b(x,t)$

ex: $u_t - tu_x = 0$

$\frac{dt}{1} = \frac{dx}{t} \Rightarrow \int t\,dt = \int dx$

$\frac{t^2}{2} = x + C \Rightarrow C = \frac{t^2}{2} - x$

$du = 0$ if $\boxed{u = D(t^2/2 - x)}$

Inhomogenious: $au_t + bu_x = f(x,t)$

$du = u_t a \cdot \frac{dt}{a} + u_x b \cdot \frac{dx}{b}$ if $\frac{dt}{a} = \varepsilon = \frac{dx}{b}$

$du = \varepsilon(\underbrace{au_t + bu_x}_{f(x,t)})$

$du = \varepsilon \cdot f(x,t) \Rightarrow \boxed{\varepsilon = \frac{du}{f(x,t)} = \frac{dt}{a} = \frac{dx}{b}}$ ☆

ex: $u_t + u_x = x$

$$\frac{dt}{1} = \frac{dx}{1} = \frac{du}{x} \qquad \frac{dt}{1} = \frac{du}{x \leftarrow t-c}$$

$$t = x + c \qquad \int (t-c)\,dt = \int du$$

$$c = t - x \qquad \frac{(t-c)^2}{2} = u + D(c)$$

$$u = \frac{(t-(t-x))^2}{2} - D(t-x) = \boxed{\frac{x^2}{2} - D(t-x)}$$

ex: $\begin{cases} u_t + u_x = x \\ u(x,0) = e^x \end{cases} \qquad u = \frac{x^2}{2} - D(t-x)$

$$u(x,0) = \frac{x^2}{2} - D(0-x) = e^x$$

condition: $y = -x$, $-y = x$

$\therefore (-y)^2/2 - D(y) = e^{-y}$

$\therefore D(y) = -e^{-y} + \frac{y^2}{2}$

always: $u(x,t) = \frac{x^2}{2} - D(y)$

$y = t - x$

$$u(x,t) = \frac{x^2}{2} - \frac{y^2}{2} + e^{-y} = \boxed{\frac{x^2}{2} - \frac{(t-x)^2}{2} + e^{-(t-x)}}$$

IBVP: $\begin{cases} u_t + e u_x = 0, & x > 0,\ t > 0 \\ u(x,0) = f(x) & \text{I.C. } (t=0) \\ u(0,t) = g(t) & \text{B.C. } (x=0) \end{cases}$

$$\frac{dt}{1} = \frac{dx}{e} \iff c = t - \frac{x}{e} \qquad du = 0 \Rightarrow u = D(t - x/e)$$

I.C. $u(x,0) = D(0 - \frac{x}{e}) = f(x)$

@ $t=0$, $y = -x/e$, $x = -ey$ $\therefore D(y) = f(-ey)$

@ $t = t$ $\quad y = t - \frac{x}{e}$ and $u(x,t) = f(-et+x)$ when $y < 0$

B.C. $u(0,t) = D(t) = g(t)$

$D(y) = g(y) \quad y > 0 \qquad D(x) \begin{cases} g(x) & y > 0 \\ f(-ey) & y < 0 \end{cases}$

$$u = D(t - \tfrac{x}{e}) = \begin{cases} g(t - \frac{x}{e}), & t - \frac{x}{e} > 0 \\ f(-et+x), & t - \frac{x}{e} < 0 \end{cases}$$

Multidemensional equations

ex: $u_t + u_x + u_y + u_z = 0$

$$\frac{dt}{1} = \frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$$

$t = x + c_1 \qquad\qquad c_1 = t - x$

$t = y + c_2 \qquad\qquad c_2 = t - y$

$t = z + c_3 \qquad\qquad c_3 = t - z$

$du = 0 \therefore u = D(c_1, c_2, c_3) = D(x-t, y-t, z-t)$

Homework #2

problem 1:

(2) $u_t - t u_x = 0$

a. Draw characteristics and find gen. sols.

characteristic: $\frac{dt}{1} = \frac{dx}{-t} = \varepsilon$

$du = u_t dt + u_x dx, \quad 0 = u_t - t u_x$

$du = u_t \cdot 1 \cdot \frac{dt}{1} + u_x \cdot -t \cdot \frac{dx}{-t}$

$-t^2/2 = x + C$

$\boxed{C = \frac{t^2}{2} + x}$

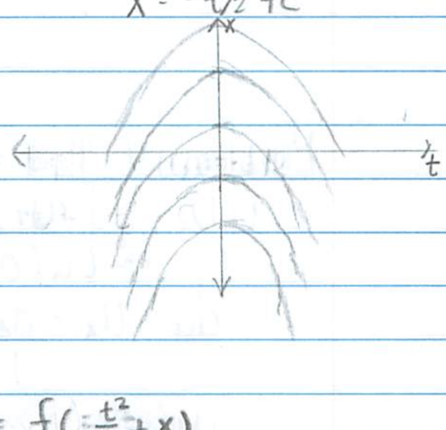
$x = -t^2/2 + C$

$du = (u_t + u_x \cdot t)\,\varepsilon$

$\hookrightarrow = 0$

$du = \varepsilon \cdot 0 = 0$

$\boxed{u = D(\frac{t^2}{2} + x)}$



b. I.V.P. $u(x,0) = f(x)$

$u(x,0) = D(0 + x) = f(x)$

$D(y) = f(y)$

$y = x$ @ $t = 0$

$u(x,t) = D(\frac{t^2}{2} + x) = D(y) = f(y) = f(\frac{t^2}{2} + x)$

$y = \frac{t^2}{2} + x$ @ $\mathbb{R}^2$ of t, x

$\boxed{u(x,t) = f(\frac{t^2}{2} + x)}$ this is unique

check: $u_t = f' \cdot (t) \qquad u_x = f' \cdot 1$

$u_t - t u_x = 0$

$t \cdot f' - t(f') = 0$ ✓

(10) $(x+1)u_t + u_x = 0$

a. Find characteristics and gen. sol.

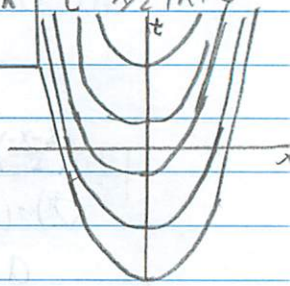
$du = u_t dt + u_x dx$ $\qquad$ $0 = (x+1)u_t + u_x$

$du = u_t(x+1)\frac{dt}{(x+1)} + u_x \frac{dx}{1}$ $\qquad$ characteristic $\frac{dt}{x+1} = \frac{dx}{1} = \varepsilon$

$du = ((x+1)u_t + u_x)\varepsilon$ $\qquad$ $t = \int x\,dx + \int 1\,dx$

$\quad\quad\quad\quad\hat{}\ 0$ $\qquad$ $t = x^2/2 + x + c$

$\boxed{u = D\left(t - \frac{x^2}{2} - x\right)}$ $\qquad$ $\boxed{C = t - \frac{x^2}{2} - x}$ $\qquad$ $t = x^2/2 + x + c$



b. I.V.P. $u(x,0) = f(x)$

$u(x,0) = D\left(0 - \frac{x^2}{2} - x\right) = f(x)$

@ $t=0$, $y = -x^2/2 - x \Rightarrow 0 = x^2 + 2x + 2y$

$x = -1 \pm \sqrt{1-2y}$

$D(y) = f(-1 \pm \sqrt{1-2y})$

$u(x,t) = D(y) = f(-1 \pm \sqrt{1-2t+x^2+2x})$

$\boxed{u(x,t) = f(-1 \pm \sqrt{x^2+2x+1-2t})}$

This is not unique, a solution with plus and one with minus exist.

Problem 4: Find the solutions

(21) $\begin{cases} u_x + 3u_y = u \\ u(0,y) = y \end{cases}$

$du = u_x \cdot \frac{dx}{1} + u_y \cdot 3 \cdot \frac{dy}{3}$ $\qquad$ $\frac{dx}{1} = \frac{dy}{3} = \varepsilon$

$du = (u_x + 3u_y)\,\varepsilon$ $\qquad$ $3\,dx = dy$

$du = u\,\varepsilon, \quad \varepsilon = \frac{du}{u}$ $\qquad$ $3x = y + c$

$C = 3x - y$

$\frac{du}{u} = dx$

$\ln|u| = x + D$

$\boxed{u = e^x \cdot D(3x-y)}$

$u(0,y) = e^0 D(3(0)-y) = y$

$D(-y) = y \quad , \quad D(z) = -z$

$z = -y \qquad y = -z$

$u(x,y) = e^x D(3x-y) = e^x D(z)$

$z = 3x-y$

$u(x,y) = e^x(-z) \qquad \boxed{u(x,y) = e^x(-3x+y)}$

check: $u_x = e^x(-3) + (-3x+y)e^x$

$u_y = e^x$

$u_x + 3u_y = u$

$\cancel{-3e^x} - 3xe^x + ye^x + \cancel{3e^x} = u$

$ye^x - 3xe^x = u = (-3x+y)e^x$ ✓

problem 1:

(1) $u_t - 3u_x - 2u_y = 0$

1. Find gen. sol.

$du = u_t\,dt - 3u_x \frac{dx}{-3} - 2u_y \frac{dy}{-2}$ characteristic: $dt = \frac{dx}{-3}$, $dt = \frac{dy}{-2}$

$du = \varepsilon(u_t - 3u_x - 2u_y)$ $\qquad t = -\frac{1}{3}x + c_1 \qquad t = -\frac{1}{2}y + c_2$

$du = 0, \quad u = D(c_1, c_2)$ $\qquad c_1 = t + \frac{x}{3} \qquad c_2 = t + \frac{y}{2}$

$\boxed{u = D(t + \frac{x}{3}, t + \frac{y}{2})}$

2. I.V.P. $u(x,y,0) = f(x,y)$

$u(x,y,0) = D(0 + \frac{x}{3}, 0 + \frac{y}{2}) = f(x,y)$

@$t=0$: $p = \frac{x}{3}$, $q = y/2$

$D(p,q) = f(3p, 2q)$

$u(x,y,t) = D(p,q) \qquad \boxed{u(x,y,t) = f(\underbrace{3t+x}_{r}, \underbrace{2t+y}_{s})}$

$p = t + \frac{x}{3}, \; q = t + y/2$

check: $u_t = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial t} = f_r(3) + f_s(2)$

$u_x = f_r \cdot r_x + f_s \cdot s_x = f_r(1) + f_s(0)$

$u_y = f_r \cdot r_y + f_s \cdot s_y = f_r(0) + f_s(1)$

$0 = \underbrace{3f_r + 2f_s}_{u_t} - 3(\underbrace{f_r}_{u_x}) - 2(\underbrace{f_s}_{u_y})$

$0 = 0$ ✓

2.3 Homogenous 1D wave equations

$u_{tt} - c^2 u_{xx} = 0$; $c > 0$. constant "speed of wave"

it is second order and hyperbolic

solving: $u_{tt} - c^2 u_{xx} = 0$

say: $t^2 - cx^2 = 0$

$(t - cx)(t + cx) = 0$

$\Downarrow \quad \Downarrow$

$u_t - cu_x \quad u_t + cu_x$

maybe: $u_{tt} - c^2 u_{xx}$ relates to $u_t - cu_x$, $u_t + cu_x$

we can solve these

How to factor with PDE's ?

Remember: $\partial_t = \frac{\partial}{\partial t}$, $\partial_t^2 = \frac{\partial^2}{\partial t^2}$

Then: $u_{tt} - c^2 u_{xx} = 0 \Rightarrow \partial_t^2 u - c^2 \partial_x^2 u = 0$

$0 = u(\partial_t^2 - c^2 \partial_x^2)$

$0 = (\partial_t - c\partial_x)(\partial_t + c\partial_x) u$

$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x) u$

Define: $v = (\partial_t + c\partial_x) u = u_t + cu_x$

$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x) v = v_t - cv_x = 0$

now $u_{tt} - c^2 u_{xx} = 0 \Rightarrow \begin{cases} v = u_t + cu_x & \text{eq. 1} \\ 0 = v_t - cv_x & \text{eq. 2} \end{cases}$

solving eq. 2: $0 = v_t - cv_x$ characteristic: $\frac{dt}{1} = \frac{dx}{-c}$

$dv = 0$ allong $ct = -x + C_2$

$t = -x/c + C_1$

$v = D(C_2) = D(ct + x)$

$ct = -x + C_2$

solving eq. 1: $u_t + cu_x = v = D(ct + x)$

characteristic: $\frac{dt}{1} = \frac{dx}{c}$

$C_3 = ct - x$

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{D(ct+x)}$$

note: $x = ct - C_3$

$$dt = \frac{du}{D(ct+x)}$$

$$dt = \frac{du}{D(2ct - C_3)}$$

$$dt = \frac{du}{D(2ct - C_3)}$$

$$\int D(2ct - C_3)\, dt = \int du$$

say: $y = 2ct - C_3$

$dy = 2c\,dt \therefore dt = \frac{dy}{2c}$ $\quad \int du = \int D(y) \frac{dy}{2c}$

say: $h(y) = \int D(y)dy$

then: $u = \frac{h(y)}{2c} = \frac{1}{2c} \cdot h(2ct - C_3) = u + D(C_3)$

$$u = \frac{1}{2c} h(2ct - C_3) - D(C_3)$$

$$u = \frac{1}{2c} h(2ct - ct + x) - D(ct - x)$$

$$\boxed{u = \frac{1}{2c} h(ct+x) - D(ct - x)}$$

say: $f(ct+x) = \frac{h(ct+x)}{2}$ then: $\boxed{u = f(ct+x) + g(ct-x)}$

$g = -D$

1D wave IVP:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) & \text{initial position} \\ u_t(x,0) = \psi(x) & \text{initial velocity} \end{cases}$$

gen sol was $u = f(x+ct) + g(ct-x)$

$$u(x,0) = f(x) + g(-x) = \phi(x)$$

$$u_t(x,0) = f'(x+ct)\cdot c + g'(ct-x)\cdot c$$

$$= f'(x)\cdot c + g'(-x)\cdot c = \psi(x)$$

$$\begin{cases} f(x) + g(-x) = \phi(x) & \text{eq. 1} \\ (f'(x) + g(-x))\cdot c = \psi(x) & \text{eq. 2} \end{cases}$$

solve eq. 2:

$$\int f'(x) + g'(-x)\,dx = \frac{1}{c}\int \psi(x)\,dx$$

$$f(x) - g(-x) = \frac{1}{c}\int_a^x \psi(y)\,dy$$

$$\begin{cases} f(x) + g(-x) = \phi(x) \\ f(x) - g(-x) = \frac{1}{c}\int_a^x \psi(y)\,dy \end{cases}$$ add together and subtract together

$$f(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int_a^x \psi(x)\,dx\right)$$

$$g(-x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int_a^x \psi(y)\,dy\right)$$

$$g(x) = \frac{1}{2}\left(\phi(-x) - \frac{1}{c}\int_a^{-x} \psi(y)\,dy\right)$$

say $y = -(ct-x)$

$$u = \frac{1}{2}\left(\phi(ct+x) + \phi(-(ct-x))\right) + \frac{1}{2c}\left(\int_a^{x+ct} \psi(y)\,dy - \int_a^{-(ct-x)} \psi(y)\,dy\right)$$

recall: $\int_a^b - \int_a^c = \int_c^b$

$$\boxed{u = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy}$$

d'Alembert's formula

Characteristic coordinates

Another way to solve the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Idea: change of varibles

$(x,y) \in \mathbb{R}^2 \longleftrightarrow (\theta, \psi) \in \mathbb{R}^2$

complicated here easy here

multi review:

ex: rect → polar $x = \rho\cos\theta$

$(x,y) \to (\rho, \theta)$ $y = \rho\sin\theta$

$\frac{\partial x}{\partial \rho} = \cos\theta \quad \frac{\partial x}{\partial \theta} = -\rho\sin\theta$

$\frac{\partial y}{\partial \rho} = \sin\theta \quad \frac{\partial y}{\partial \theta} = \rho\cos\theta$

Jacobian matrix: $\begin{pmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}$

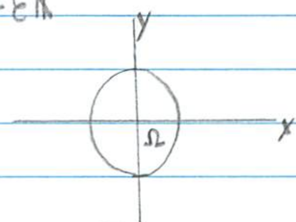
J: Jacobian det: $ad - bc = \rho$

integral review:

Double integral $\Omega \in \mathbb{R}^2$

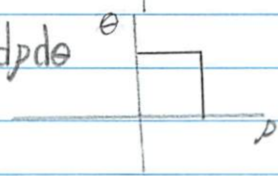
ex: Ω = unit disk

$\iint_\Omega f(x,y)\,dx\,dy$



$\parallel$

$\iint_S f(x(\rho,\theta), y(\rho,\theta)) \cdot J\,d\rho\,d\theta$



$0 \le \theta \le 2\pi$

$0 \le \rho \le 1$

Back to $u_{tt} - c^2 u_{xx} = 0$

$\rightsquigarrow u_t - c u_x, \quad u_t + c u_x$

characteristics:

$\frac{dt}{1} = \frac{dx}{-c} \qquad \frac{dt}{1} = \frac{dx}{c}$

$x + ct = -C \qquad x - ct = C$

Define: $\xi = x+ct$ are related to the curves of characteristics
$\eta = x-ct$

$\xi = \text{Constant}, \eta = \text{Constant}$

Rewrite equation in new coordinates

$U(x,t) = U(x(\xi,\eta), t(\xi,\eta)) \Rightarrow U(\xi,\eta)$

$x = \frac{1}{2}(\xi+\eta)$

$t = \frac{1}{2c}(\xi-\eta)$

recalculate U_{tt}, U_{xx} as ξ and η

$U_t = \frac{\partial}{\partial t} U(\xi(x,t), \eta(x,t))$

chain rule

$\frac{\partial U}{\partial t} = U_\xi \cdot \xi_t + U_\eta \cdot \eta_t$ $\quad\quad \xi_t = c$, $\eta_t = -c$

$= cU_\xi - cU_\eta$

$U_{tt} = \frac{\partial U_t}{\partial t} = c\left(\frac{\partial}{\partial t}U_\xi - \frac{\partial}{\partial t}U_\eta\right)$

now need: $\frac{\partial U_\xi}{\partial t} = (U_\xi)_\xi \cdot \xi_t + (U_\xi)_\eta \cdot \eta_t$

$= c(U_{\xi\xi} - U_{\xi\eta})$

$\frac{\partial U_\eta}{\partial t} = U_{\eta\xi} \cdot \xi_t + U_{\eta\eta} \cdot \eta_t$

$= c(U_{\eta\xi} - U_{\eta\eta})$

$U_{tt} = c^2(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta})$

$U_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$

$U_{tt} - c^2U_{xx} = (c^2(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta})) - c^2(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta})$

$= \boxed{-4c^2U_{\xi\eta}}$

if $U_{tt} - c^2U_{xx} = 0$

then $U_{\xi\eta} = 0$

then $U = f(\xi) + g(\eta)$

$U = f(x+ct) + g(x-ct)$ which we got earlier

Applications of inhomogenous wave equation:

$$u_{tt} - c^2 u_{xx} = f(x,t)$$

$$-4c^2 u_{\xi\eta} = f(x(\xi,\eta), t(\xi,\eta)) = f(\xi,\eta)$$

$$u_{\xi\eta} = \frac{-1}{4c^2} f(\xi,\eta)$$

solve IVP:
homo - I.C.

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

$$u_{\xi\eta} = \frac{-1}{4c^2} f(\xi,\eta)$$

@ $t=0 \Rightarrow u(x,0)$ $\quad \xi = x+0 \quad \eta = x+0 = \xi$ $\quad$ when $t=0$, $\xi=\eta$

$$u(\xi,\xi) = 0$$

$u_t = c(u_\xi - u_\eta) = 0$ @ $t=0$

$$u_\xi(\xi,\xi) = u_\eta(\xi,\xi)$$

Rearranged IVP:

$$\begin{cases} u_{\xi\eta} = -\frac{1}{4c^2} f(\xi,\eta) \\ u(\xi,\xi) = 0 \\ u_\xi(\xi,\xi) = u_\eta(\xi,\xi) \end{cases}$$

note: $u_\xi = \int u_{\xi\eta}\, d\eta$

$$= \int_{a(\xi)}^{\eta} u_{\xi\eta}(\xi,\tilde{\eta})\, d\tilde{\eta}$$

constraint on $a(\xi)$, $\quad$ because $\xi=\eta$

$$u_\xi(\xi,\xi) = u_\eta(\xi,\xi) = \int_{a(\xi)}^{\xi} u_{\xi\eta}(\xi,\tilde{\eta})\, d\tilde{\eta}$$

$u(\xi,\xi) = 0$ implies $u_\xi(\xi,\xi) = 0$ also

$\therefore u_\eta(\xi,\xi) = 0$ also also

$$u_\xi(\xi,\xi) = \int_{a(\xi)}^{\xi} u_{\xi\eta}(\xi,\tilde{\eta})\, d\tilde{\eta} = 0$$

$$u_\xi(\xi,\eta) = \underbrace{\int_{a(\xi)}^{\xi} u_{\xi\eta}(\xi,\tilde{\eta})\, d\tilde{\eta}}_{=0} + \int_{\xi}^{\eta} u_{\xi\eta}(\xi,\tilde{\eta})\, d\tilde{\eta}$$

$$u_\xi(\xi,\eta) = \int_{\xi}^{\eta} u_{\xi\eta}(\xi,\tilde{\eta})\, d\tilde{\eta}$$

Homework 3

problem 1: Find the gen. sol.

(5) $4u_{tt} - 9u_{xx} = 0$

$u_{tt} - \frac{9}{4}u_{xx} = 0$ $\qquad \xi = x+ct$ $\qquad c^2 = \frac{9}{4} \therefore c = \frac{3}{2}, \frac{-3}{2}$

$\qquad \eta = x-ct$

$u_{\xi\eta} = 0$

$u_\xi = f(\xi)$

$u(\xi,\eta) = F(\xi) + g(\eta)$

$u(\xi(x,t), \eta(x,t)) = u(x,t) = F(x+ct) + g(x-ct)$

$\boxed{u(x,t) = F(x \pm \tfrac{3}{2}t) + g(x \pm \tfrac{3}{2}t)}$

problem 2: solve the IVP $\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$ with

(8) $\varphi(x) = \begin{cases} 1 & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$, $\psi(x) = 0$

gen. sol: $u(x,t) = f(x+ct) + g(-x+ct)$

$u_t(x,t) = cf'(x+ct) + cg'(-x+ct)$

$u(x,0) = \varphi(x) = f(x) + g(-x) = \begin{cases} 1, & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$

$u_t(x,0) = \psi(x) = cf'(x) + cg'(-x) = 0$

$\int f'(x) + g'(-x)\,dx = \int 0\,dx = \int_a^x 0\,dt$

$f(x) - g(-x) = 0$

$\underline{|x|<1}$ $\qquad y = |x|$ $\qquad \underline{|x| \geq 1}$

$f(x) + g(-x) = 1$ $\qquad f(x) + g(-x) = 0$

$f(x) - g(-x) = 0$ $\qquad f(x) - g(-x) = 0$

$\qquad + \qquad -$

$2f(x) = 1$ $\qquad 2g(-x) = 1$ $\qquad f(x) = 0$ $\qquad g(-x) = 0$

$f(x) = \frac{1}{2}$ $\qquad g(-x) = \frac{1}{2}$ $\qquad y = x$ $\qquad z = -x,\ x = -z$

$|x| = |y|$ $\qquad |x| = |-z|$

$y = x+ct$ $\qquad -z = x-ct$

$$\boxed{u(x,t) = \frac{1}{2}\left(\begin{cases} 1, & |x+ct|<1 \\ 0, & |x+ct| \geq 1 \end{cases} + \begin{cases} 1, & |x-ct|<1 \\ 0, & |x-ct| \geq 1 \end{cases}\right)}$$

$$u(x,t) = \begin{cases} 1, & |x+ct|<1 \text{ and } |x-ct|<1 \\ 1/2, & (|x+ct|<1 \text{ and } |x-ct|\geq 1) \text{ or } (|x+ct|\geq 1 \text{ and } |x-ct|<1) \\ 0, & |x+ct|\geq 1 \text{ and } |x-ct| \geq 1 \end{cases}$$

(16) $\varphi(x) = 0$, $\psi(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$

gen. sol. $u(x,t) = f(x+ct) + g(-x+ct)$

$u_t(x,t) = cf'(x) + cg'(-x)$

$u(x,0) = f(x) + g(-x) = \varphi(x) = 0$

$u_t(x,0) = cf'(x) + cg'(-x) = \psi(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$

$$\int f'(x) + g'(-x)\,dx = \frac{1}{c}\int \psi(x)\,dx$$

$$f(x) - g(-x) = \frac{1}{c}\int_a^x \psi(q)\,dq$$

$f(x) + g(-x) = 0$

$f(x) - g(-x) = \frac{1}{c}\int_a^x \psi(q)\,dq$

\+ −

$2f(x) = \frac{1}{c}\int_a^x \psi(q)\,dq$

$f(x) = \frac{1}{2c}\int_a^x \psi(q)\,dq$

$f(z) = \frac{1}{2c}\int_a^z \psi(q)\,dq$

$2g(-x) = \frac{-1}{c}\int_a^x \psi(q)\,dq$

$g(-x) = \frac{-1}{2c}\int_a^x \psi(q)\,dq$

$y = -x$, $x = -y$

$g(y) = -\frac{1}{2c}\int_a^{-y} \psi(q)\,dq$

@ $t \neq 0$, $z = x+ct$

@ $t \neq 0$ $y = -x+ct$

$$u(x,t) = f(z) + g(y)$$

$$u(x,t) = \frac{1}{2c}\left(\int_a^{x+ct} \psi(q)\,dq - \int_a^{x-ct} \psi(q)\,dq\right)$$

$$u(x,t) = \frac{1}{2c}\left(\int_{x-ct}^{x+ct} \psi(q)\,dq\right) = \frac{1}{2c}\left(\int_{x-ct}^{1} 1-q^2\,dq + \int_1^{x+ct} 0\,dq\right)$$

$$= \frac{1}{2c}\left(q - \frac{q^3}{3}\Big|_{x-ct}^{1}\right) = \frac{1}{2c}\left(1 - \frac{1}{3} - \left((x-ct) + \frac{(x-ct)^3}{3}\right)\right)$$

$$\boxed{u(x,t) = \frac{1}{2c}\left(\frac{2}{3} - x + ct + \frac{(x-ct)^3}{3}\right)}$$

problem 3: Find $u(x,t)$ and describe the domain where the function is unique.

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x, x^2/2) = x^3 \\ u_t(x, x^2/2) = 2x \end{cases}$$

$* \; c = 1$

$$u(x,t) = f(x+t) + g(-x+t)$$

$$u_t(x,t) = f'(x+t) + g'(-x+t)$$

$$u(x, x^2/2) = f(x + x^2/2) + g(-x + x^2/2) = x^3$$

$$u_t(x, x^2/2) = f'(x+x^2/2) + g'(-x + x^2/2) = 2x$$

$$f'(x+x^2/2)\cdot(1+x) + g'(-x+x^2/2)\cdot(-1+x) = 3x^2$$

$$f'(x+x^2/2)\cdot(-1-x) + g'(-x+x^2/2)\cdot(-1-x) = 2x\cdot(-1-x)$$

$$g'(-x+x^2/2)(-2) = (-2x-2x^2)+3x^2$$

$$g'(-x+x^2/2) = x + x^2 - \tfrac{3}{2}x^2$$

$$g'(-x+x^2/2) = x - \tfrac{1}{2}x^2$$

$* \; n = x - \frac{x^2}{2}$

$$\int g'(-n)\, dn = \int n\, dn$$

$$-g(-n) = \frac{n^2}{2}, \quad g(-n) = \frac{-n^2}{2}$$

$$m = -n, \; n = -m$$

$$g(m) = -\frac{m^2}{2}$$

$$(1+x)A + (-1+x)B = 3x^2$$

$$(1-x)A + (1-x)B = 2x\cdot(1-x)$$

$$2A = 2x - 2x^2 + 3x^2$$

$$A = x - x^2 + \tfrac{3}{2}x^2$$

$$A = f'(x+x^2/2) = (x + x^2/2)$$

$$h = x + x^2/2$$

$$\int f'(h)\, dh = \int h\, dh$$

$$f(h) = \frac{h^2}{2}$$

$$\boxed{u(x,t) = \frac{(x+t)^2}{2} - \frac{(-x+t)^2}{2}}$$

problem 4: Prove that if u solves $\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty \\ u(x,0) = \varphi(x) \\ u_t(x,0) = 0 \end{cases}$

then $v = \int_0^t u(x,t')dt'$ solves $\begin{cases} v_{tt} - c^2 v_{xx} = 0 & -\infty < x < \infty \\ v(x,0) = 0 \\ v_t(x,0) = \varphi(x) \end{cases}$

Known: $u(x,t) = \frac{1}{2}\Big(\varphi(x+ct) + \varphi(x-ct)\Big) + 0$

check: $u(x,0) = \frac{1}{2}(\varphi(x) + \varphi(x)) = \varphi(x)$ ✓

$u_t(x,0) = \frac{1}{2}(c\varphi'(x) - c\varphi'(x)) = 0$ ✓

$u_{tt} = \frac{1}{2}(c^2\varphi''(x+ct) + c^2\varphi''(x-ct))$ $\quad u_x = \frac{1}{2}(\varphi'(x+ct) + \varphi'(x-ct))$

$u_{xx} = \frac{1}{2}(\varphi''(x+ct) + \varphi''(x-ct))$

$\frac{1}{2}c^2(\varphi''(x+ct) + \varphi''(x-ct)) = c^2 \cdot \frac{1}{2}(\varphi''(x+ct) + \varphi''(x-ct))$ ✓

$$v = \int_0^t \tfrac{1}{2}(\varphi(x+ct') + \varphi(x-ct'))dt' = f(x+ct) + g(-x+ct)$$

$v(x,0) = f(x) + g(-x) = 0$

$v_t(x,0) = cf'(x) + cg'(-x) = \varphi(x)$

$f(x) - g(-x) = \frac{1}{c}\int_a^x \varphi(x')dx'$

$f(x) = \frac{1}{2c}\int_a^x \varphi(x')dx'$ $\quad g(-x) = \frac{-1}{2c}\int_a^x \varphi(x')dx'$

$g(x) = \frac{-1}{2c}\int_a^{-x} \varphi(x')dx'$

$$v = \frac{1}{2c}\int_a^{x+ct}\varphi(x')dx' - \frac{1}{2c}\int_a^{x-ct}\varphi(x')dx$$ point a.

$$= \frac{1}{2c}\left(\int \varphi(x+ct)d(x+ct) - \int \varphi(x-ct)\,d(x-ct)\right)$$

$m = x+ct$ $\quad n = x-ct$

$dm = cdt$ $\quad dn = -ct$

$$v = \frac{1}{2c}\left(\int \varphi(m)dm - \int \varphi(n)dn\right) = \frac{1}{2c}\left(c\int \varphi(x+ct)dt + c\int \varphi(x-ct)dt\right)$$

$\therefore \boxed{v = \frac{1}{2}\int_a^t \varphi(x+ct') + \varphi(x-ct')\,dt'}$ $\quad v(x,0) = \int_a^0 \sim\sim = 0$

$\therefore a = 0$

(2) $V(x,t) = \int_0^t \varphi(x+ct') + \varphi(x-ct')dt' \cdot \frac{1}{2}$

$$V_t(x,t) = \frac{\partial}{\partial t}\int_0^t \frac{\varphi(x+ct') + \varphi(x-ct')dt'}{2}$$

$$= \frac{\varphi(x+ct) + \varphi(x-ct)}{2} = u(x,t) \checkmark$$

(3) starting from point a of part (1)

$$V = \frac{1}{2c}\int_a^{x+ct} \varphi(x')\,dx' - \frac{1}{2c}\int_a^{x-ct} \varphi(x')\,dx'$$

fact: $\int_a^b - \int_a^c = \int_b^c$

$$\boxed{V = \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(x')\,dx'}$$

(4) $V(x,0) = \frac{1}{2c}\int_x^x \varphi(x')\,dx' = 0$ ✓

$$V_t(x,t) = \frac{1}{2c}\int_a^{x+ct} \varphi(x')dx' - \frac{1}{2c}\int_a^{x-ct} \varphi(x')dx'$$

$$= \frac{1}{2c}\left(\int \varphi(m)\,dm - \int \varphi(n)\,dn\right)$$

$$= \frac{1}{2}\left(\int \varphi(x+ct)dt + \int \varphi(x-ct)dt\right)$$

$$V_t(x,0) = \frac{1}{2}\left(\varphi(x+c\cdot 0) + \varphi(x-c\cdot 0)\right) = \varphi(x) \checkmark$$

where we left off: $u_\xi(\xi,\eta) = \int_\xi^\eta u_{\xi\eta}(\xi,\eta')\,d\eta'$

where: $\xi = x+ct$

$\eta = x-ct$

$u = \int u_\xi(\xi,\eta)\,d\xi$ Fix η

$u = \int_{b(\eta)}^{\xi} u_\xi(\xi',\eta)\,d\xi'$ so what is constant $b(\eta)$?

$u(\xi,\xi)=0$, aka $\xi=\eta$

$u(\xi,\xi) = 0 = \int_{b(\xi)}^{\xi} u_\xi(\xi',\xi)\,d\xi'$

$\xi=\eta$

$0 = \int_{b(\eta)}^{\eta} u_\xi(\xi',\eta)\,d\xi'$

now $u(\xi,\eta) = \underbrace{\int_{b(\eta)}^{\eta} u_\xi(\xi',\eta)\,d\xi'}_{\text{but } =0} + \int_\eta^\xi u_\xi(\xi',\eta)\,d\xi'$ because $\int_a^c = \int_a^b + \int_b^c$

so now $u(\xi,\eta) = \int_\eta^\xi u_\xi(\xi',\eta)\,d\xi'$

plug in version from first line but $u(\xi',\eta)$

$u(\xi,\eta) = \int_\eta^\xi \int_{\xi'}^{\eta} u_{\xi\eta}(\xi',\eta')\,d\eta'\,d\xi'$

remember $u_{\xi\eta} = \frac{-1}{4c} f(\xi,\eta)$

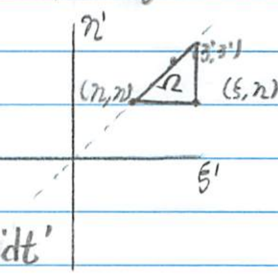
$u(\xi,\eta) = \int_\eta^\xi \int_{\xi'}^{\eta} -\frac{1}{4c} f(\xi',\eta')\,d\eta'\,d\xi'$

negative sign lets us flip integral

$$\boxed{u(\xi,\eta) = \int_\eta^\xi \int_\eta^{\xi'} \frac{1}{4c} f(\xi',\eta')\,d\eta'\,d\xi'}$$

say: $\begin{cases} \eta \le \xi' \le \xi \\ \eta \le \eta' \le \xi' \end{cases}$ from bounds

region of integration:



transfer $\iint_\Omega f(\xi',\eta')\,d\eta'\,d\xi' = \iint_\Delta f(\xi'(x',t'),\eta'(x',t'))\,|J|\,dx'\,dt'$

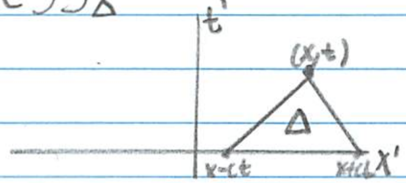
recall: $\xi' = x'+ct'$, $\eta' = x'-ct'$

$J = \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} = -2c$ $|J| = 2c$

conclusion: $\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

$$u = \frac{1}{4c^2} \iint_{\Omega} f(\xi', \eta')\, d\eta'\, d\xi'$$

$$= \frac{1}{4c^2} \iint_{\Delta} f(x', t') \cdot 2c\, dx'\, dt' = \frac{1}{2c} \iint_{\Delta} f(x', t')\, dx'\, dt'$$



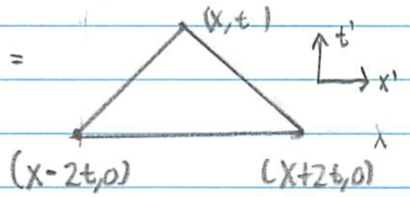
$$\Delta = \begin{cases} (x-ct) \le (x'+ct') \le (x+ct) \\ (x-ct) \le (x'-ct') \le (x'+ct') \end{cases}$$

example: solve: $\begin{cases} u_{tt} - 4u_{xx} = \sin(x)\cos(t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$ $c = 2$

applic formula:

$$u = \frac{1}{2c} \iint_{\Delta} \sin(x')\cos(t')\, dx'\, dt'$$

Δ =



left line: $x' = x - c(t-t')$ Right line: $x' = x + c(t-t')$

$$u = \frac{1}{2c} \int_0^t \int_{x-c(t-t')}^{x+c(t-t')} f(x', t')\, dx'\, dt'$$

in the example: $u = \frac{1}{4} \int_0^t \int_{x-2(t-t')}^{x+2(t-t')} \sin(x')\cos(t')\, dx'\, dt'$

2.5 IVP with inhomogenous initials and inhomogemes wave e.q.

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

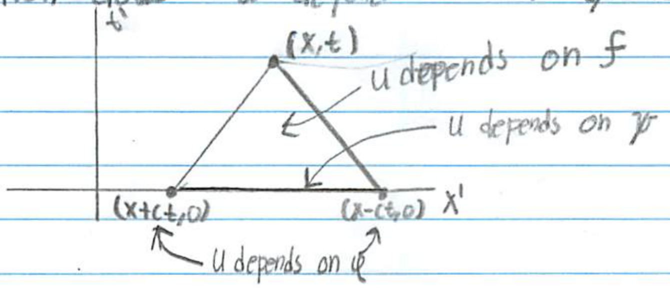
Just add our homo wave/inhomo initial with our inhomo wave/homo initial

$$u(x,t) = \frac{1}{2}\left(\varphi(x-ct) + \varphi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx' + \frac{1}{2c}\iint_{\Delta(x,t)} f(x',t')\,dx'dt'$$

$$\iint_{\Delta(x,t)} = \int_0^t \int_{x-c(t-t')}^{x+c(t-t')}$$

Domains of dependence and infuence:

how does u depend on f, φ, and ψ



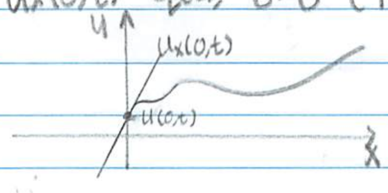
2.6 1D Wave equation: IBVP

Wave on the half line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & x>0,\ t>0 \\ u(x,0) = \varphi(x) & x>0 \\ u_t(x,0) = \psi(x) & x>0 \\ \text{Boundry conditions} & \end{cases}$$

(the second and third lines: Initial conditions)

Boundry conditions $\begin{cases} u(0,t) = p(t), & t>0 \quad (\text{Dirichlet}) \\ \text{or} \quad u_x(0,t) = q(t), & t>0 \quad (\text{Neumann}) \end{cases}$



For simplicity, let look at homogenous Dirichlet, $f(x,t)=0$

Recall: $u = f(x+ct) + g(x-ct)$

I.C. $u(x,0) = \varphi(x)$

$$u(x,0) = f(x) + g(x) = \varphi(x)$$

$u_t(x,0) = \psi(x)$

$$u_t(x,0) = c \cdot f'(x) - c \cdot g'(x) = \psi(x)$$

integrate

$$f(x) - g(x) = \frac{1}{c}\int_0^x \psi(x')\,dx' + C$$

because $\int_a^x = \int_0^x + C$

We can ignore $+C$ because

$$f(0) - g(0) = 0 + C$$

$$u = f(x+ct) + \frac{C}{2} - g(x-ct) + \frac{C}{2} = f(x+ct) + g(x-ct)$$

back to solving I.C.

$$\begin{cases} f(x) - g(x) = \frac{1}{c}\int_0^x \psi(x')\,dx' & x>0 \\ f(x) + g(x) = \varphi(x) & x>0 \end{cases}$$

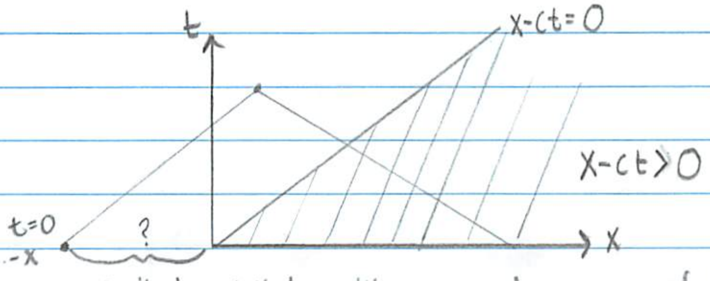
$f(x) = \frac{1}{2c}\int_0^x \psi(x')\,dx' + \frac{1}{2}\varphi(x)$ $\quad g(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(x')\,dx'$

$$u = f(x+ct) + g(x-ct)$$

issue if $x - ct < 0$ $\quad x < 0$

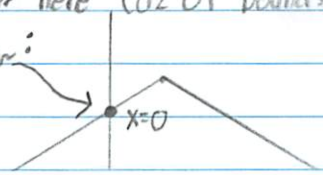
we can only have $x - ct > 0$

if $x-ct>0$: $u = \frac{1}{2}\left(\varphi(x+ct) - \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')\,dx'$



can't have initial conditions over here cuz of bounds

we can however know:



Dirichlet: $u(0,t) = P(t) = f(ct) + g(-ct), \quad t>0$

say $y = ct$ then

$$g(-y) = P(y/c) - f(y)$$

now lets use this relation for our boundry issue with $x = y$

$$g(-x) = P(x/c) - \left(\frac{1}{2c}\int_0^x \psi(x')\,dx + \frac{1}{2}\varphi(x)\right)$$

now do $g(-(ct-x))$

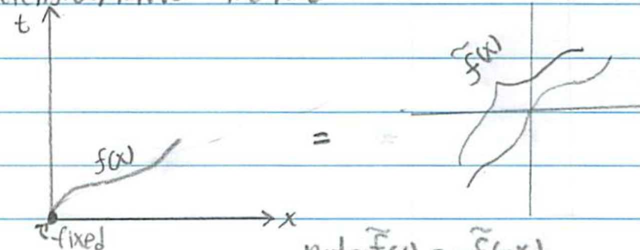
$$u = \frac{1}{2c}\int_0^{x+ct}\psi(x')\,dx' + \frac{1}{2}\varphi(x+ct) + P\left(\frac{ct-x}{c}\right) - \frac{1}{2c}\int_0^{ct-x}\psi(x')\,dx' - \frac{1}{2}\varphi(ct-x)$$

simplify

$$u = \frac{1}{2}\left(\varphi(x+ct) - \varphi(ct-x)\right) + P\left(\frac{ct-x}{c}\right) + \frac{1}{2c}\int_{ct-x}^{x+ct}\psi(x')\,dx' \quad \text{for } x-ct<0$$

Another method to solve B.C. extension/mirror method

$$\begin{cases} u_{tt} - c\,u_{xx} = f(x,t) & x>0,\ t>0 \\ u(x,0) = \varphi(x) & x>0, \\ u_t(x,0) = \psi(x) & x>0, \\ u(0,t) = 0 & t>0 \end{cases}$$



note: $\tilde{f}(x) = -\tilde{f}(-x)$ because odd function

expand f φ and ψ by an odd expansion

$$\tilde{f}(x,t) = \begin{cases} f(x,t), & x>0 \\ -f(-x,t), & x<0 \end{cases} \qquad \tilde{\varphi}(x) = \begin{cases} \varphi(x), & x>0 \\ -\varphi(-x), & x<0 \end{cases} \qquad \tilde{\psi}(x) = \begin{cases} \psi(x) & x>0 \\ -\psi(-x), & x<0 \end{cases}$$

also note odd functions have $\tilde{f}(0) = 0$

if all pieces are odd the $\tilde{u}$ must be odd function too

$$\tilde{u}(x,t) = -\tilde{u}(-x,t)$$

and $\tilde{u}$ automatically solves $\tilde{u}(0,t) = 0$

Try mirror with $\begin{cases} u_{tt} - c\,u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = 0 \end{cases}$

Since $\tilde{0} = 0$, we can ignore it.

We need to solve the full line extension

$$\begin{cases} \tilde{u}_{tt} - c\tilde{u}_{xx} = 0 \\ \tilde{u}(x,0) = \tilde{\varphi}(x) \\ \tilde{u}_t(x,0) = \tilde{\psi}(x) \end{cases}$$

Since these are full line we can use d'Alembert formula

$$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{\varphi}(x+ct) + \tilde{\varphi}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(x')\,dx'$$

rewrite as φ and ψ using piecewise functions

$\tilde{\varphi}(x+ct) = \tilde{\varphi}(x+ct)$ for $x>0, t>0$ becaus $x+ct$ will always be positive

$$\tilde{\varphi}(x-ct) = \begin{cases} \varphi(x-ct), & x-ct>0 \\ -\varphi(\underbrace{-(x-ct)}_{ct-x}) & x-ct<0 \end{cases}$$

$$\int_{x-ct}^{x+ct} \tilde{\psi}(x')\,dx' = \int_{x-ct}^{x+ct} \psi(x')\,dx' \quad x-ct>0$$

if $x-ct<0$ then $\int_{-(x-ct)}^{0} -\psi(-x')\,dx' + \int_0^{x+ct} \psi(x')\,dx'$ sax $x''=-x'$, $dx''=-dx'$

$$\int_{ct-x}^{0} \psi(x'')\,dx'' + \int_0^{x+ct} \psi(x')\,dx'$$

you can see this is all the same as we got before

Homework 4

problem 1: solve the IVP

(1) $\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$

$$u(x,t) = \frac{1}{2}\left(g(x+ct)+g(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct} h(x')dx' + \frac{1}{2c}\int_0^t\int_{x-c(t-t')}^{x+c(t-t')} f(x',t')\,dx'dt'$$

(2) with $f(x,t)=\sin(\alpha x)$, $g(x)=0$, $h(x)=0$

$f(x',t') = \sin(\alpha x')$

$$u(x,t) = \int_0^t\int_{x-c(t-t')}^{x+c(t-t')} \sin(\alpha x')\,dx'dt' \cdot \frac{1}{2c}$$

$$\int_0^t \left(\frac{-1}{\alpha}\cos(\alpha x')\Big/_{x-c(t-t')}^{x+c(t-t')} \right) dt' \cdot \frac{1}{2c}$$

$$-\frac{1}{\alpha}\int_0^t \cos(\alpha x+\alpha ct-\alpha ct') - \cos(\alpha x-\alpha ct+\alpha ct')\,dt' \cdot \frac{1}{2c}$$

$$-\frac{1}{\alpha 2c}\left(\sin(\alpha x+\alpha ct-\alpha ct')\cdot\frac{-1}{\alpha c} - \sin(\alpha x-\alpha ct+\alpha ct')\cdot\frac{1}{\alpha c} \right)\Big/_0^t$$

$$\frac{-1}{\alpha 2c}\left(\frac{-1}{\alpha c}\sin(\alpha x) - \frac{1}{\alpha c}\sin(\alpha x) - \sin(\alpha x+\alpha ct)\cdot\frac{-1}{\alpha c} + \sin(\alpha x-\alpha ct)\cdot\frac{1}{\alpha c}\right)$$

$$-\frac{1}{\alpha 2c}\left(\frac{1}{\alpha c}\left(-\sin(\alpha x) - \sin(\alpha x) + \sin(\alpha x+\alpha ct) + \sin(\alpha x-\alpha ct)\right)\right)$$

$$\boxed{-\frac{1}{2\alpha^2 c^2}\left(-2\sin(\alpha x)+\sin(\alpha x+\alpha ct)+\sin(\alpha x-\alpha ct)\right)}$$

(3) with $f(x,t)=\sin(\alpha x)\sin(\beta t)$ $\quad g(x)=0 \quad h(x)=0$

$$u(x,t) = \int_0^t\int_{x-c(t-t')}^{x+c(t-t')} \sin(\alpha x)\sin(\beta t)\,dx'dt' \cdot \frac{1}{2c}$$

$$= \int_0^t -\frac{1}{\alpha}\cdot\sin(\beta t')\cos(\alpha x')\Big/_{x-c(t-t')}^{x+c(t-t')} dt' \cdot \frac{1}{2c}$$

$$= -\frac{1}{\alpha}\int_0^t \sin(\beta t')\cos(\alpha x+\alpha ct-\alpha ct') - \sin(\beta t')\cos(\alpha x-\alpha ct+\alpha ct')\,dt' \cdot \frac{1}{2c}$$

*note: $\sin(a)\cos(b) = \frac{1}{2}\left(\sin(a+b)+\sin(a-b)\right)$

$$u(x,t) = \frac{-1}{2\alpha}\int_0^t \sin(\alpha x+\alpha ct+t'(\beta-\alpha c)) + \sin(\alpha x+\alpha ct+t'(-\beta-\alpha c))\,dt' \cdot \frac{1}{2c}$$

$$+\frac{-1}{2\alpha}\int_0^t -\sin(\alpha x-\alpha ct+t'(\beta+\alpha c)) - \sin(\alpha x-\alpha ct+t'(-\beta+\alpha c))\,dt' \cdot \frac{1}{2c}$$

$$u(x,t) = \frac{-1}{4\alpha c}\left(\frac{\cos(\alpha x + \alpha c t + t'(\beta - \alpha c))}{\beta - \alpha c} + \frac{\cos(\alpha x + \alpha c t + t'(-\beta - \alpha c))}{-\beta - \alpha c} - \frac{\cos(\alpha x - \alpha c t + t'(\beta + \alpha c))}{\beta + \alpha c} - \frac{\cos(\alpha x - \alpha c t + t'(-\beta + \alpha c))}{-\beta + \alpha c} \right)\Bigg|_0^t$$

$$\boxed{= \frac{-1}{4\alpha c}\left(\frac{\cos(\alpha x + \beta t) - \cos(\alpha x + \alpha c t)}{\beta - \alpha c} + \frac{\cos(\alpha x - \beta t) - \cos(\alpha x - \alpha c t)}{\beta - \alpha c} + \frac{\cos(\alpha x - \beta t) - \cos(\alpha x + \alpha c t)}{-\beta - \alpha c} + \frac{\cos(\alpha x + \beta t) - \cos(\alpha x - \alpha c t)}{-\beta - \alpha c} \right)}$$

(4) $f(x,t) = f(x) \quad g(x) = 0 \quad h(x) = 0$

$$u(x,t) = \int_0^t \int_{x-c(t-t')}^{x+c(t-t')} f(x)\,dx\,dt' \cdot \frac{1}{2c} \qquad \text{say } \frac{d}{dx}F(x) = f(x)$$

$$= \int_0^t F(x + c(t-t')) - F(x - c(t-t'))\,dt' \cdot 1/2c \qquad \text{say } \frac{d}{dx}Y(x) = F(x)$$

$$= \frac{-1}{c}Y(x+ct-ct') - \frac{1}{c}Y(x-ct+ct')\Big|_0^t \cdot \frac{1}{2c}$$

$$= \boxed{\frac{-1}{2c^2}\left(2Y(x) - Y(x+ct) - Y(x-ct)\right)}$$

(5) $f(x,t) = t \cdot f(x) \quad g(x) = 0 \quad h(x) = 0$

$$u(x,t) = \int_0^t \int_{x-c(t-t')}^{x+c(t-t')} t' f(x)\,dx\,dt' \cdot \frac{1}{2c} \qquad \text{say } \frac{d}{dx}Z(x) = Y(x)$$

$$= \int_0^t t' \cdot F(x+ct-ct') - t' \cdot F(x-ct+ct')\,dt' \cdot \frac{1}{2c}$$

$$= \frac{t'Y(x+ct-ct')}{-c} + \int \frac{Y(x+ct-ct')}{-c}dt' - \frac{t'Y(x-ct+ct')}{-c} + \frac{1}{c}\int Y(x-ct+ct')\,dt' \Big|_0^t \cdot \frac{1}{2c}$$

$$= \left(\frac{t'Y(x+ct-ct')}{-c} + \frac{Z(x+ct-ct')}{-c^2} - \frac{t'Y(x-ct+ct')}{-c} + \frac{Z(x-ct+ct')}{-c^2} \right)\Big|_0^t \cdot \frac{1}{2c}$$

$$= \boxed{\left(\frac{tY(x)}{-c} + \frac{tY(x)}{-c} + \frac{Z(x)}{-c^2} + \frac{Z(x)}{c^2} + \frac{Z(x+ct)}{+c^2} - \frac{Z(x-ct)}{c^2} \right)} \cdot \frac{1}{2c}$$

$$= \frac{-1}{2c^2}\left(2tY(x) + \frac{Z(x)+Z(x)}{c^2} - \frac{Z(x+ct)}{c} + \frac{Z(x-ct)}{c} \right)$$

$$= \boxed{\frac{-1}{2c^2}\left(2tY(x) + \frac{Z(x-ct) + Z(x+ct) + 2Z(x)}{c} \right)}$$

problem 4:

(a) Find solution $u(x,t)$ to $\begin{cases} u_{tt} - u_{xx} = (x^2-1)e^{-x^2/2} \\ u(x,0) = -e^{-x^2/2},\ u_t(x,0) = 0 \end{cases}$

$$u(x,t) = \frac{1}{2}\left(-e^{\frac{-(x+ct)^2}{2}} - e^{\frac{-(x-ct)^2}{2}}\right) + \frac{1}{2c}\int_0^t\int_{x-c(t-t')}^{x+c(t-t')} (x'^2-1)e^{-x'^2/2}\,dx'\,dt'$$

From problem 1 (4) but $f(x) = (x^2-1)e^{-x^2/2}$

$F(x) = -xe^{-x^2/2}$ and $Y(x) = e^{-x^2/2}$

$$u(x,t) = \frac{1}{2}\left(-e^{\frac{-(x+ct)^2}{2}} - e^{\frac{-(x-ct)^2}{2}}\right) - \frac{1}{2c^2}\left(2e^{-x^2/2} - e^{\frac{-(x+ct)^2}{2}} - e^{\frac{-(x-ct)^2}{2}}\right) \quad *c = 1$$

$$\boxed{u = -e^{-x^2/2}}$$

$$\lim_{t\to\infty} u(x,t) = -e^{-x^2/2}$$

problem 1: Find the solution $\begin{cases} u_{tt} - c^2u_{xx} = 0 & t>0,\ x>0 \\ u(x,0) = \varphi(x) & x>0 \\ u_t(x,0) = c\varphi'(x) & x>0 \\ u(0,t) = \chi(t) & t>0 \end{cases}$

$$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2}\int_{x-ct}^{x+ct}\varphi'(x')\,dx' \qquad x-ct>0$$

$$= \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2}\left(\varphi(x+ct) - \varphi(x-ct)\right)$$

$$= \varphi(x+ct)$$

$$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) - \varphi(ct-x)\right) + \frac{1}{2}\int_{ct-x}^{x+ct}\varphi'(x')\,dx' + \chi\left(\tfrac{ct-x}{c}\right) \qquad x-ct<0$$

$$= \frac{1}{2}\left(\varphi(x+ct) - \varphi(ct-x)\right) + \frac{1}{2}\left(\varphi(x+ct) - \varphi(ct-x)\right) + \chi\left(\tfrac{ct-x}{c}\right)$$

$$\boxed{u(x,t) = \begin{cases} \varphi(x+ct) & x>ct \\ \varphi(x+ct) - \varphi(x-ct) + \chi\left(\frac{ct-x}{c}\right) & 0<x<ct \end{cases}}$$

2. solve for $u(x,t)$ for
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & t>0,\ x>0 \\ u(x,0) = \varphi(x) & x>0 \\ u_t(x,0) = c\,\varphi'(x) & x>0 \\ u_x(0,t) = \chi(t) & t>0 \end{cases}$$

$$u_x(0,t) = \chi(t) = f'(ct) + g'(-ct)$$

$$y = ct$$

$$\chi(y/c) = f'(y) + g'(-y)$$

$$\chi(x/c) = f'(x) + g'(-x)$$

$$f(x) = \frac{1}{2}\int_0^x \varphi'(x')dx' + \frac{1}{2}\varphi(x) \qquad g(x) = \frac{1}{2}\varphi(-x) - \frac{1}{2}\int_0^{-x} \varphi'(x)dx'$$

$$= \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x) \qquad = \frac{1}{2}\varphi(-x) - \frac{1}{2}\varphi(-x) = 0$$

$$f'(x) = \frac{1}{2}\varphi'(x) + \frac{1}{2}\varphi'(x) = \varphi'(x)$$

$$\chi(x/c) = \varphi'(x) \qquad \varphi'(x-ct) = \chi\left(\frac{ct-x}{c}\right)$$

$$\frac{1}{-c}\varphi(x-ct) = \int_0^{\frac{ct-x}{c}} \chi(x')dx'$$

$$\boxed{u(x,t) = \begin{cases} \varphi(x+ct) & x>ct \\ -c\int_0^{\frac{ct-x}{c}} \chi(x')dx' + \varphi(x+ct) + \varphi(x-ct) & 0<x<ct \end{cases}}$$

Problem 8: solve

(10)
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x>0 \\ u(x,0) = 0 & x>0 \\ u_t(x,0) = \cos(x) & x>0 \\ u_x(0,t) = 0 & t>0 \end{cases}$$

$$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) - \varphi(ct-x)\right) - c\int_0^{\frac{ct-x}{c}} \chi(x)dx' + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(x')dx' \quad x+ct<0$$

$$u(x,t) = \frac{1}{2c}\int_{ct-x}^{x+ct} \cos(x')dx' = \frac{1}{2c}\sin(x')\Big|_{ct-x}^{x+ct} = \frac{\sin(x+ct) - \sin(ct-x)}{2c}$$

$$x-ct>0 \Rightarrow u(x,t) = \int_{x-ct}^{x+ct} \frac{1}{2c}\cdot\cos(x')dx' = \frac{\sin(x+ct) - \sin(x-ct)}{2c}$$

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2c}\cdot\left(\sin(x+ct) - \sin(x-ct)\right) & x>ct \\ \frac{1}{2c}\cdot\left(\sin(x+ct) - \sin(ct-x)\right) & 0<x<ct \end{cases}}$$

(14) $\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x > 0 \\ u(x,0) = \cos(x) & x > 0 \\ u_t(x,0) = 0 & x > 0 \\ u_x(0,t) = 0 & t > 0 \end{cases}$

for $x - ct < 0$: $u(x,t) = \frac{1}{2}\left(\cos(x+ct) - \cos(ct-x)\right)$

for $x > ct$: $u(x,t) = \frac{1}{2}\left(\cos(x+ct) + \cos(x-ct)\right)$

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}\left(\cos(x+ct) + \cos(x-ct)\right) & x > ct \\ \frac{1}{2}\left(\cos(x+ct) - \cos(ct-x)\right) & 0 < x < ct \end{cases}}$$

Method of extension/continuation

$$\text{IVP}\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

claim: if $f(x,t)$, $\varphi(x)$, and $\psi(x)$ are odd functions of x, the $u(x,t)$ is also odd in x

Let $v(x,t) = -u(-x,t)$

prove: $v(x,t) = u(x,t)$

$$v_{tt} - c^2 v_{xx} = -u_{tt} - c^2(-u_{xx})$$
$$= -(u_{tt} - c^2 u_{xx}) = -f(-x,t)$$

this is what we want to see

$-f(-x,t) = f(x,t)$ because it is assumed odd

1D wave equation on the finite interval

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & 0<x<l\,,\ t>0 \\ u(x,0) = \varphi(x) & 0<x<l \\ u_t(x,0) = \psi(x) & 0<x<l \\ \text{B.C. can be any combo of Dirichlet and Newman @0 and @}l \end{cases}$$

let use 2 newman for today: $q_0(t)$, $q_l(t)$

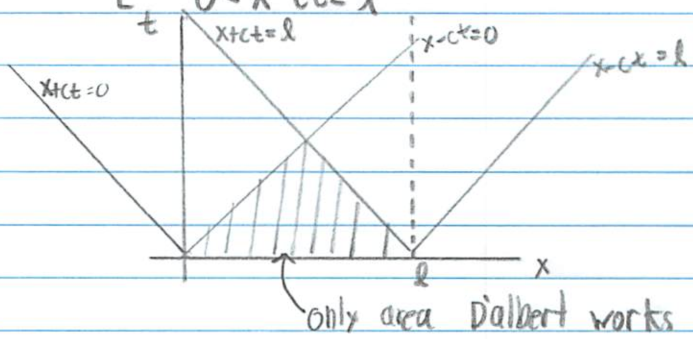
also today lets let $f(x,t)=0$

recall: $u = f(x+ct) + g(x+ct)$

$$f(x) = \frac{1}{2}\left(\frac{1}{c}\int_0^x \psi(x')dx' + \varphi(x)\right) \quad \text{for } 0<x<l$$

$$g(x) = \frac{1}{2}\left(\varphi(x) - \frac{1}{c}\int_0^x \psi(x')dx'\right) \quad u(x)$$

$$u(x,t)\ \begin{cases} 0<x+ct<l \text{ and} \\ 0<x-ct<l \end{cases}$$



$u_x(0,t) = q_0(t)$

$= f'(ct) + g'(-ct) = q_0(t)$

$\int \quad f'(x) + g'(-x) = \int q_0(x/c)$

$f(x) - g(-x) = \int_0^x q_0(x'/c)\,dx' + C \quad$ but $C = 0$

Recall: $u = f(\underbrace{x+ct}) + g(x-ct)$

always positive $x>0$ ← can be any number $<\ell$

max case is $x=\ell, t=0$. yields ℓ

we need to find $f(x)$ for any $x>0$

and $g(x)$ for $x<\ell$

we already know $f(x)$ and $g(x)$ when $0<x<\ell$

so what is $f(x)$ for $x \geq \ell$

and $g(x)$ for $x \leq 0$

we know $\int q_0(x'/c)\,dx'$ is always positive bc $q_0(t)$ $t>0$

$g(-x) = -\int_0^x q_0(x'/c)\,dx' + f(x)$

↑ $x>0$ ↑ $0<x<\ell$

so we know $g(-x)$ $0<x<\ell$

aka $g(x)$ $-\ell<x<0$

what about $g(x)$ $x<-\ell$?

lets use our other boundry $u_x(\ell,t) = q_\ell(t)$, $t>0$

$u_x(\ell,t) = f'(\ell+ct) + g'(\ell-ct) = q_\ell(t)$

$x = \ell+ct \qquad x>\ell$ bc $t>0$

$\int \quad f'(x) + g'(2\ell-x) = \int q_\ell\left(\frac{x-\ell}{c}\right)$

$f(x) - g(2\ell-x) = \int_\ell^x q_\ell\left(\frac{x'-\ell}{c}\right)dx' + C$ ~~but C=0~~

@ $x=\ell$, $f(\ell) - g(\ell) = C = \frac{1}{c}\int_0^\ell \psi(x')\,dx'$

$$\boxed{f(x) - g(2\ell-x) = \int_\ell^x q_\ell\left(\frac{x'-\ell}{c}\right)dx' + \frac{1}{c}\int_0^\ell \psi(x')\,dx'}$$

this makes sence for $x>\ell$

$f(x) = g(2\ell-x) + \int_\ell^x q_\ell\left(\frac{x'-\ell}{c}\right)dx' + \frac{1}{c}\int_0^\ell \psi(x')\,dx'$

↑ we found $-\ell < 2\ell-x < 0$ and $0 < 2\ell-x < \ell$

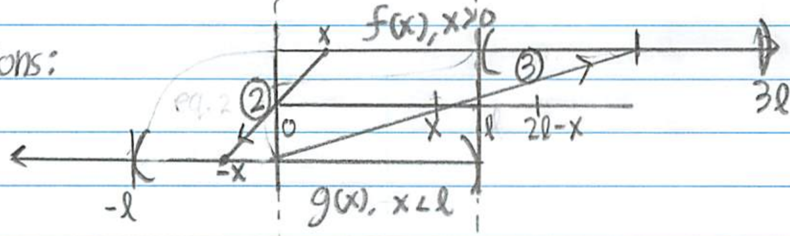
combind $-\ell < 2\ell - x < \ell \Rightarrow \ell < x < 3\ell$

So far we have: $f(x)$ and $g(x)$ $\quad 0<x<l$ ①

Left mirror $\quad f(x)-g(-x)=\int_0^x q_0\left(\frac{x'}{c}\right)dx' \quad x<0$ ②

Right mirror $\quad f(x)-g(2l-x)=\int_l^x q_l\left(\frac{x'-l}{c}\right)dx' + \frac{1}{c}\int_0^l r(x')\,dx' \quad x>l$ ③

reflections:



ex: solve
$$\begin{cases} u_{tt}-4u_{xx}=0 & 0<x<1 \quad t>0 \\ u(x,0)=1 & u_t(x,0)=1 \\ u_x(0,x)=1 & u_x(l,t)=1 \end{cases}$$

Find $u(x,1)$

$u=f(x+2t)+g(x-2t)$ @ $t=1$ $\quad u=f(x+2)+g(x-2)$

$0<x<1 \Rightarrow 2<x+2<3 \quad -2<x-2<-1$

so we needa find $f(x)$, $2<x<3$ and $g(x)$, $-2<x<-1$

① $f(x)=\frac{1}{2}+\frac{1}{4}\int_0^x 1\,dx'=\frac{1}{2}+\frac{1}{4}x \qquad g(x)=\frac{1}{2}-\frac{1}{4}x \quad 0<x<1$

② $f(x)-g(-x)=x \quad x>0$

③ $f(x)-g(2-x)=x-\frac{1}{2} \quad x>1$

Step 1: use left mirror to reflect (0,1) on f to (-1,0) on g

② $g(-x)=f(x)-x \qquad 0<x<1 \qquad 0<u<1$

$=\frac{1}{2}+\frac{1}{4}x-x=\frac{1}{2}-\frac{3}{4}x \qquad u\Rightarrow -x \quad 0\Rightarrow 0 \quad 1\Rightarrow -1$

$g(x)=\frac{1}{2}+\frac{3x}{4} \quad -1<x<0 \qquad 0<-x<-1 \Rightarrow -1<x<0$

Step 2: use right mirror to reflect $g(x)$, $-1<x<1$ to $f(x)$, $1<x<3$

③ $f(x)=g(2-x)+x-\frac{1}{2}$

$$\begin{cases} 1<x<2 \Leftrightarrow 0<2-x<1 \\ 2<x<3 \Leftrightarrow -1<2-x<0 \end{cases}$$

$$\left\{ \frac{1}{2}+\frac{3(2-x)}{4}+x-\frac{1}{2} \quad -1<2-x<0 \right.$$

$f(x)=\frac{3}{2}+\frac{1}{4}x \quad 2<x<3$ ✓

Step 3 Reflect $f(x)$, $1<x<2$ to $g(x)$, $-2<x<1$

② $g(-x) = f(x) - x$ $\qquad$ $f(x)$, $1<x<2$

$$= \frac{1}{2} - \frac{1}{4}(2+x) + x - \frac{1}{2} - x$$

$$= -\frac{1}{2} + \frac{1}{4}x$$

$g(x) = -\frac{1}{2} + \frac{1}{4}x$ $\qquad$ $-2<x<-1$

Final solution: $u = f(x+2) + g(x-2)$ $\quad 0<x<1$

$$u(x,1) = \frac{3}{2} + \frac{1}{4}(x+2) - \frac{1}{2} - \frac{1}{4}(x-2)$$

$$= 2$$

Homework 5

problem 8: Solve

(16)
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x>0 \\ u(x,0) = \sin(x) & x>0 \\ u_t(x,0) = 0 & x>0 \\ u_x(0,t) = 0 & t>0 \end{cases}$$

$$\tilde{u}(x,t) = \frac{1}{2}\left(\widetilde{\sin}(x+ct) + \widetilde{\sin}(x-ct)\right)$$

$$= \begin{cases} \frac{1}{2}(\sin(x+ct) + \sin(x-ct)) & x-ct>0 \\ \frac{1}{2}(-\sin(-(x+ct)) - \sin(-(x-ct))) & x-ct<0 \end{cases}$$

* note $\sin(-\theta) = -\sin\theta$

$$\boxed{u(x,t) = \frac{1}{2}(\sin(x+ct) + \sin(x-ct)) \quad x>0,\ t>0}$$

(20)
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x>0 \\ u(x,0) = 0 & x>0 \\ u_t(x,0) = 1 & x>0 \\ u(0,t) = 0 & t>0 \end{cases}$$

$$\tilde{u}(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{1}\, dx'$$

$2ct$

$$= \begin{cases} \frac{1}{2c}\int_{x-ct}^{x+ct} 1\, dx' \\ \frac{1}{2c}\int_{x-ct}^{x+ct} -1\, dx' \end{cases} \Rightarrow \begin{cases} \frac{1}{2c}(x+ct-x+ct) \\ \frac{1}{2c}(-x-ct+x-ct) \end{cases} \Rightarrow \boxed{\begin{cases} t & x>ct \\ -t & 0<x<ct \end{cases}}$$

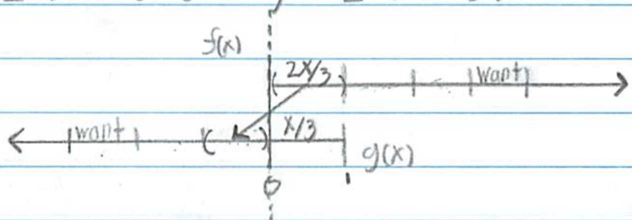
2. Find $u(x,1)$: $\begin{cases} u_{tt} - 9u_{xx} = 0 & 0<x<1, \quad t>0 \\ u(x,0) = x & 0<x<1 \\ u_t(x,0) = 1 & 0<x<1 \\ u_x(0,t) = 1 & t>0 \\ u_x(1,t) = 1 & t>0 \end{cases}$ $\quad {}^*c = 3$

$u(x,1) = f(x+3) + g(x-3)$ $\quad 0<x+3<1 \quad 0<x-3<1$

$3<x<4 \quad -3<x<-2 \quad \Leftarrow \quad 3<x<1+3 \quad -3<x<1-3$

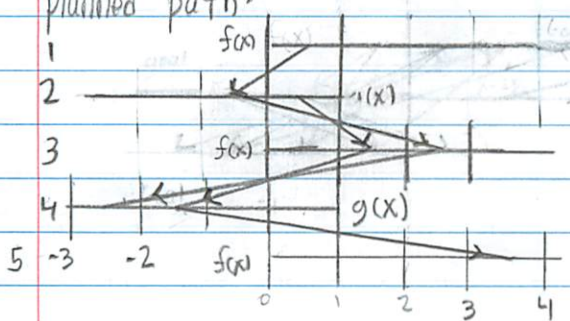
$$f(x) = \frac{1}{2}\left(\frac{1}{3}\int_0^x 1\,dx' + x\right) = \frac{1}{2}\left(\frac{x}{3} + x\right) = \frac{2x}{3} \quad 0<x<1$$

$$g(x) = \frac{1}{2}\left(x - \frac{1}{3}\int_0^x 1\,dx'\right) = \frac{1}{2}\left(x - \frac{x}{3}\right) = \frac{x}{3} \quad 0<x<1$$



$$g(-x) = -\int_0^x 1\,dx' + f(x) = -x + \frac{2x}{3} \quad 0<x<1$$
$$g(x) = x - \frac{2x}{3} = \frac{x}{3} \quad -1<x<0$$

planned path:



$0<x<1$

$-x \quad -1<x<0 \quad 0<x<1$

$2-x \quad 2<x<3 \quad 1<x<2$

$-x \quad -3<x<-2 \quad -2<x<-1$

$2-x \quad 3<x<4$

2-3: $f(x) = g(2-x) + \int_\ell^x 1\,dx' + \frac{1}{3}\int_0^\ell 1\,dx' = (x-2) - \frac{2}{3}(x-2) + x - \ell + \frac{1}{3}\ell = \frac{2}{3}(2x-\ell-1) \quad 2<x<3$

$= \frac{x-2}{3} + x - \ell + \frac{1}{3}\ell = \frac{2}{3}(2x-\ell-1) \quad 1<x<2$

3-4: $g(-x) = -x + \frac{2}{3}(2x-\ell-1)$

$\boxed{g(x) = x + \frac{2}{3}(-2x-\ell-1) \quad -3<x<-2}$, $g(x) = x + \frac{2}{3}(-2x-\ell-1)$, $-2<x-1$

4-5: $f(x) = g(2-x) + x - \frac{2}{3}\ell = (2-x) + \frac{2}{3}(-2(2-x)-\ell-1) + x - \frac{2}{3}\ell$

$= \boxed{\frac{4}{3}(x-\ell-1), \quad 3<x<4}$

$u(x,1) = \frac{4}{3}(x-2) + x + \frac{2}{3}(-2x-2) = \cancel{\frac{4}{3}(x-2)} + x + \cancel{\frac{4}{3}(-x-1)} = x-4$

$$\boxed{u(x,1) = x-4}$$

3. solve $\begin{cases} u_{tt} - u_{xx} = 1 & x>0,\ t>0 \\ u(x,0) = 1 & x>0 \\ u_t(x,0) = 1 & x>0 \\ u(0,t) = t+1 & t>0 \end{cases}$

(1) let $v = u - t - 1$

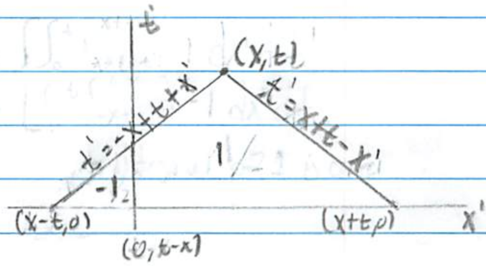
$V_t = U_t - 1$ $\quad V_x = U_x$ $\quad V(x,0) = \overset{1}{U(x,0)} - 0 - 1$

$V_{tt} = U_{tt}$ $\quad V_{xx} = U_{xx}$ $\quad V_t(x,0) = \underset{1}{U_t(x,0)} - 1$

$\begin{cases} V_{tt} - V_{xx} = 1 \\ V(x,0) = 0 \\ V_t(x,0) = 0 \\ V(0,t) = 0 \end{cases}$ $\quad V(0,t) = \underset{t+1}{U(0,t)} - t - 1$

$$\tilde{V}(x,t) = \frac{1}{2}\int_0^t \int_{x-t+t'}^{x+t-t'} \tilde{1}\, dx'\, dt'$$

$\tilde{1} = \begin{cases} 1 & x>t \\ -1 & 0<x<t \end{cases}$



$V(x,t) = \begin{cases} \frac{1}{2}t^2 & x>t \\ -x^2/2 - xt + t^2/2 & 0<x<t \end{cases}$

$U = V + t + 1$

$$\boxed{U(x,t) = \begin{cases} \frac{1}{2}t^2 + t + 1 & x>t \\ \frac{1}{2}x^2 - xt + \frac{1}{2}t^2 + t + 1 & 0<x<t \end{cases}}$$

(2) let $v = u - t^2/2$ $\quad V_t = U_t - t$ $\quad V_x = U_x$ $\quad V(x,0) = U(x,0)$

$V_{tt} = U_{tt} - 1$ $\quad V_{xx} = U_{xx}$ $\quad V_t(x,0) = U_t(x,0)$

$\begin{cases} V_{tt} - V_{xx} = 0 \\ V(x,0) = 1 \\ V_t(x,0) = 1 \\ V(0,t) = -t^2/2 + t + 1 \end{cases}$ $\quad V(0,t) = \underset{t+1}{U(0,t)} - t^2/2$

$$V(x,t) = \begin{cases} \frac{1}{2}(2) + \frac{1}{2}\int_{x-t}^{x+t} 1\, dx' \\ \frac{1}{2}(-2) + \frac{1}{2}\int_{-x+t}^{x+t} -1\, dx' + \left(\frac{-1}{2}(t-x)^2 + (t-x) + 1\right) \end{cases}$$

$$V(x,t) = \begin{cases} 1 + t \\ -\frac{1}{2}x^2 + xt + \frac{1}{2}t^2 + t - x + 1 - x \end{cases}$$

$$= \begin{cases} 1 + t \\ -\frac{1}{2}x^2 - xt + t + 1 \end{cases}$$

$u = v + t^2/2$

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}t^2 + t + 1 & x > t \\ \frac{1}{2}x^2 - xt + \frac{1}{2}t^2 + t + 1 & 0 < x < t \end{cases}}$$

chapter 3: 1D Heat equation

$u(x,t), \quad x \in \mathbb{R}, \quad t > 0$

$u_t - k u_{xx} = 0$ $\qquad$ k constant > 0

We find special solutions first:

self-similar solution: (geometrically very symetric

assume $u(x,t)$ is a solution

$u_{\alpha\beta\gamma}(x,t) = \gamma u(\alpha x, \beta t)$

self similar means: $u_{\alpha\beta\gamma}(x,t) = u(x,t)$ for all α, β, γ

so what values of $\alpha\ \beta\ \gamma$ solve the heat eq.

$(u_{\alpha\beta\gamma})_t = \beta\gamma u_t(\alpha x, \beta t)$

$(u_{\alpha\beta\gamma})_{xx} = \alpha^2\gamma u_{xx}(\alpha x, \beta t)$

$\beta\gamma u_t(\alpha x, \beta t) - k\alpha^2\gamma u_{xx}(\alpha x, \beta t) = 0$ $\qquad$ we can say: $u_t = k u_{xx}$

$\therefore$

$\gamma k u_{xx}(\alpha x, \beta t)(\beta - \alpha^2) = 0$

$\hookrightarrow$ assume $\neq 0$ $\quad \therefore \beta = \alpha^2$

Finiteness of the heat mass: $I(t) = \int_{-\infty}^{\infty} u(x,t)\,dx < \infty$

so $I(t)$ must be constant

$\frac{d}{dt} I(t) = 0$

$u_{\alpha\beta\gamma}(x,t) = \gamma u(\alpha x, \alpha^2 t)$

$I(t) = \int \gamma u(\alpha x, \alpha^2 t)\,dx$ $\qquad$ $x' = \alpha x$ $\qquad$ $\frac{dx'}{\alpha} = dx$

$= \int \gamma u(x', \alpha^2 t) \frac{dx'}{\alpha}$

$\Rightarrow \gamma/\alpha \int u(x', t)\,dx' = \gamma/\alpha\, I(t) = I(t)$

$(1 - \gamma/\alpha) I(t) = 0$

$\therefore \gamma = \alpha$

$$\boxed{u(x,t) = \alpha u(\alpha x, \alpha^2 t)}$$

α can be anything so lets say $\alpha = t^{-1/2}$

$u(x,t) = t^{-1/2}\, u(t^{-1/2}x, 1)$ say: $\varphi(x) = u(x,1)$

$= t^{-1/2}\, \varphi(x t^{-1/2})$

$u_t = -\frac{1}{2} t^{-3/2} \varphi(x t^{-1/2}) + t^{-1/2} \varphi'(x t^{-1/2})\, x(-\frac{1}{2} t^{-3/2})$

$-k u_{xx} = -k t^{-1/2} \varphi''(x t^{-1/2}) \cdot t^{-1/2} \cdot t^{-1/2}$

$u_t - k u_{xx} = -\frac{1}{2} t^{-3/2} \varphi(x t^{-1/2}) - \frac{1}{2} x t^{-2} \varphi'(x t^{-1/2}) - k t^{-3/2} \varphi''(x t^{-1/2}) = 0$

say $x t^{-1/2} = y$

$\Rightarrow \varphi(y) + y\varphi'(y) + 2k\varphi''(y) = 0$

$(y\varphi(y))' + 2k\varphi''(y) = 0$

$(y\varphi(y) + 2k\varphi'(y))' = 0$

$y\varphi(y) + 2k\varphi'(y) = C$

assume $C=0$ because $\varphi(y)$'s shold decay fast as $y \to \infty$

$y\varphi + 2k\frac{d\varphi}{dy} = 0$

$\varphi = Ce^{\frac{-y}{4k}}$

$u(x,t) = C t^{-1/2} e^{-x^2/4kt}$

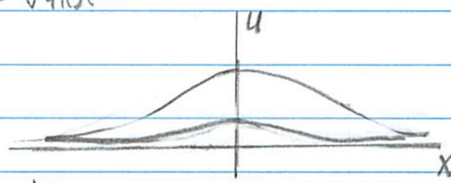
pick C suchthat $I(t) = 1$

$\int_{-\infty}^{\infty} u(x,t)\,dx = 1$

$C\int_{-\infty}^{\infty} t^{-1/2} e^{-x^2/4kt}\,dx = 1$

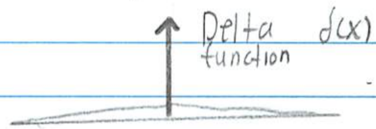
$C = \frac{1}{\sqrt{4k\pi}}$

$$\boxed{u(x,t) = \frac{e^{-x^2/4kt}}{\sqrt{4k\pi t}}}$$



$\lim_{t\to\infty} = 0$

$\lim_{t\to 0^+}$



IVP: $\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = g(x) \end{cases}$

claim: $u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} g(y)\,dy$

say $G_0(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{\frac{-x^2}{4kt}}$

$(G_0(x-y,t))_t - k(G_0(x-y,t))_{xx} = 0$

$u(x,0) = \lim_{t \to 0} \int_{-\infty}^{\infty} G_0(x-y,t) g(y)\,dy$

we know $G_0(x,0) = \delta(x)$

$\therefore \int_{-\infty}^{\infty} \delta(x-y) g(y)\,dy = g(x)$

What about heat on the half line:

$\begin{cases} u_t - k u_{xx} = 0 & x>0,\ t>0 \\ u(x,0) = g(x) \\ u(0,t) = p(t) \text{ or } u_x(0,t) = q(t) \end{cases}$ lets assume 0

Direchlet case: $u(0,t) = 0$

make an odd extension:

$\tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0 \end{cases}$

now $\begin{cases} u_t - k u_{xx} = 0 & -\infty < x < \infty \\ u(x,0) = \tilde{g}(x) \end{cases}$

$u(x,t) = \int_{-\infty}^{\infty} G_0(x-y,t)\, \tilde{g}(y)\,dy$

$u(x,0) = \tilde{g}(x)$
$= g(x) \quad x \geq 0$

lets prove $u(0,t) = 0,\ t > 0$

$u(0,t) = \int_{-\infty}^{\infty} \frac{1}{4\pi k t} e^{-\frac{(-y)^2}{4kt}} \tilde{g}(y)\,dy$ ← this whole thing is still odd

integral of an odd function is 0

$\therefore u(0,t)$ does $= 0$

lets simplify: $\int_{-\infty}^{\infty} G_0(x-y,t)\tilde{g}(y)\,dy$

$$u(x,t)=\int_{-\infty}^{0} G_0(x-y,t)\cdot -g(-y)\,dy + \int_{0}^{\infty} G_0(x-y,t)\cdot g(y)\,dy$$

$$y=-y$$

$$dy=-dy \quad y(-\infty)=\infty \quad y(0)=0$$

$$u(x,t)=\int_{\infty}^{0} G_0(x+y,t)\,g(y)\,dy + \int_{0}^{\infty} G_0(x-y,t)\,g(y)\,dy$$

$$=\int_{0}^{\infty}\underbrace{\left(G_0(x-y,t)-G_0(x+y,t)\right)}_{G_D(x,y,t)\text{ : heat kernal function}}g(y)\,dy$$

D for Direchlet boundry condition

lets now do $u_x(0,t)=0$ (Newman)

This time we will use an even extension

$$\tilde{g}(x)=\begin{cases} g(x) & x\geq 0 \\ g(-x) & x<0 \end{cases}$$

$$u(x,t)=\int_{-\infty}^{\infty} G_0(x-y,t)\,\tilde{g}(y)\,dy$$

odd   even when $x=0$

$\therefore$ whole integral is odd and $\int_{-\infty}^{\infty}=0$ so $u_x=0$

simplify: $u(x,t)=\int_{0}^{\infty}\left(G_0(x-y,t)+G_0(x+y,t)\right)g(y)\,dy$

$(\qquad\qquad) \Rightarrow G_N(x,y,t)$

Inhomogenous boundry conditions:

$$\begin{cases} u_t - k u_{xx} = 0 & x>0,\ t>0 \\ u(x,0) = g(x) \\ u(0,t) = p(t) \text{ or } u_x(0,t) = q(t) \end{cases}$$

skip!

Inhomogenous heat equation

$$\begin{cases} u_t - k u_{xx} = f(x,t) & -\infty < x < \infty,\ t>0 \\ u(x,0) = g(x) \end{cases}$$

☆ lets look at $g(x)=0$ ☆

claim: $u(x,t) = \int_0^t \int_{-\infty}^{\infty} G_0(x-y, t-\tau) f(y,\tau)\, dy\, d\tau$

proof $u(x,0) = \int_0^0 = 0$

What about: $\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0) = 0 \\ u(0,t) = 0 \end{cases}$

claim: $u(x,t) = \int_0^t \int_0^{\infty} G_D(x,y,t-\tau) f(y,\tau)\, dy\, d\tau$

Homework 6

Problem 1: $\begin{cases} u_t - ku_{xx} = 0 & t>0,\ -\infty<x<\infty \\ u(x,0) = g(x) \end{cases}$

(2) $g(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & |x|\geq 1 \end{cases} \iff \begin{cases} 0 & x<-1 \\ x+1 & -1<x<0 \\ -x+1 & 0<x<1 \\ 0 & x>1 \end{cases}$

$$u(x,y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{\frac{-(x-y)^2}{4kt}} \cdot g(y)\,dy \qquad w = \frac{x-y}{\sqrt{4kt}} \quad y = -w\sqrt{4kt}+x$$

$$= \int_{-\infty}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{\infty} \qquad dw = \frac{-1}{\sqrt{4kt}}\,dy$$

$$= \int_{\infty}^{\frac{x+1}{\sqrt{4kt}}} \frac{-1}{\sqrt{\pi}} e^{-w^2}\cdot 0\,dw + \int_{\frac{x+1}{\sqrt{4kt}}}^{0} \frac{-1}{\sqrt{\pi}} e^{-w^2}(y+1)\,dw + \int_{0}^{\frac{x+1}{\sqrt{4kt}}} \frac{-1}{\sqrt{\pi}} e^{-w^2}(-y+1)\,dw + 0$$

$$= \int_{\frac{x+1}{\sqrt{4kt}}}^{0} \frac{-1}{\sqrt{\pi}} e^{-w^2} - \frac{y}{\sqrt{\pi}} \cdot e^{-w^2}\,dw + \int_{0}^{\frac{x+1}{\sqrt{4kt}}} \frac{-1}{\sqrt{\pi}} e^{-w^2} + \frac{y}{\sqrt{\pi}} e^{-w^2}\,dw$$

$$= \int_{\frac{x+1}{\sqrt{4kt}}}^{\frac{x+1}{\sqrt{4kt}}} \frac{-1}{\pi} e^{-w^2}\,dw + \int_{0}^{\frac{x+1}{\sqrt{4kt}}} \frac{y}{\sqrt{\pi}} \cdot e^{-w^2}\,dw + \int_{0}^{\frac{x+1}{\sqrt{4kt}}} \frac{y}{\sqrt{\pi}} e^{-w^2}\,dw$$

$$= \int_{0}^{\frac{x+1}{\sqrt{4kt}}} \frac{2}{\sqrt{\pi}} (-w\sqrt{4kt}+x) e^{-w^2}\,dw$$

$$= x\,\mathrm{erf}\left(\frac{x+1}{\sqrt{4kt}}\right) + \frac{2}{\sqrt{\pi}} \cdot -\sqrt{4kt} \int_{0}^{\frac{x+1}{\sqrt{4kt}}} we^{-w^2}\,dw$$

$u=-w^2 \quad du = -2w\,dw$

$$-\sqrt{4kt} \cdot \frac{2}{\sqrt{\pi}} \int_{0}^{-\left(\frac{x+1}{\sqrt{4kt}}\right)^2} \frac{1}{-2} e^{u}\,du$$

$$\frac{\sqrt{4kt}}{\sqrt{\pi}} \cdot \left(\exp\left(\frac{-(x+1)^2}{4kt}\right) - 1\right)$$

$$\boxed{u(x,y) = -2\sqrt{\frac{kt}{\pi}} + 2\sqrt{\frac{kt}{\pi}} \exp\left(\frac{-(x+1)^2}{4kt}\right) + x\,\mathrm{erf}\left(\frac{x+1}{\sqrt{4kt}}\right)}$$

(6) $g(x) = \begin{cases} x^2 e^{-ax} & x > 0 \\ x^2 e^{ax} & x < 0 \end{cases}$

$u(x,y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k t \pi}} e^{\frac{-(x-y)^2}{4kt}} g(y)dy \qquad w = \frac{x-y}{\sqrt{4kt}} \qquad dw = \frac{-1}{\sqrt{4kt}} dy$

$u(x,y) = \int_{\infty}^{-\infty} \frac{-1}{\sqrt{\pi}} e^{-w^2} g(y) dw = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-w^2} g(y) dw \qquad w(0) = \frac{x}{\sqrt{4kt}}$

$= \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} \frac{x^2}{\sqrt{\pi}} e^{-w^2} e^{ay} dw + \int_{\frac{x}{\sqrt{4kt}}}^{\infty} \frac{y^2}{\sqrt{\pi}} e^{-w^2} e^{-ay} dw \qquad y = x - w\sqrt{4kt}$

$e^{y} = e^{x} e^{-w\sqrt{4kt}} \quad e^{ay} = e^{ax - aw}$

$= \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} \frac{(x - w\sqrt{4kt})^2}{\sqrt{\pi}} e^{ax} e^{-w^2 - aw\sqrt{4kt}} dw + \int_{\frac{x}{\sqrt{4kt}}}^{\infty} \frac{(x - w\sqrt{4kt})^2}{\sqrt{\pi}} e^{-ax} e^{-w^2 + aw\sqrt{4kt}} dw$

note: $(x - w\sqrt{4kt})(x - w\sqrt{4kt}) = x^2 - 2w\sqrt{4kt} + 4ktw^2$

$= w^2 - \frac{w\sqrt{4kt}}{2kt} + x^2/4kt$

$= \frac{x^2}{4kt\sqrt{\pi}} \left(e^{ax} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} \left(w^2 - \frac{w}{\sqrt{kt}} \right) e^{-w^2 - aw\sqrt{4kt}} dw + e^{-ax} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} \left(w^2 - \frac{w}{\sqrt{kt}} \right) e^{-w^2 + aw\sqrt{4kt}} dw \right)$

problem 3: $\begin{cases} u_t - ku_{xx} = 0 & t>0, \quad 0<x<\infty \\ u(x,0) = g(x) \\ u(0,t) = h(t) \end{cases}$

(2) $g(x) = \begin{cases} 1 & x<1 \\ 0 & x \geq 1 \end{cases} \qquad h(t) = 0$

$$u(x,t) = \int_0^\infty \frac{1}{\sqrt{4kt\pi}} e^{-\frac{(x-y)^2}{4kt}} g(y)\,dy - \int_0^\infty \frac{1}{\sqrt{4kt\pi}} e^{-\frac{(x+y)^2}{4kt}} g(y)\,dy$$

$$= \int_1^\infty \frac{1}{\sqrt{4kt\pi}} e^{-\frac{(x-y)^2}{4kt}}\,dy - \int_1^\infty \frac{1}{\sqrt{4kt\pi}} e^{-\frac{(x+y)^2}{4kt}}\,dy$$

$w = \frac{x-y}{\sqrt{4kt}} \qquad u = \frac{x+y}{\sqrt{4kt}}$

$dw = \frac{-1}{\sqrt{4kt}}dy \qquad du = \frac{du}{\sqrt{4kt}}$

$$= \int_{\frac{x-1}{\sqrt{4kt}}}^{-\infty} \frac{-1}{\sqrt{\pi}} e^{-w^2}\,dw - \int_{\frac{x+1}{\sqrt{4kt}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u^2}\,du$$

$$= \int_{-\infty}^{\frac{x-1}{\sqrt{4kt}}} \frac{1}{2}\cdot\frac{2}{\sqrt{\pi}} e^{-w^2}\,dw - \int_{\frac{x+1}{\sqrt{4kt}}}^{\infty} \frac{1}{2}\cdot\frac{2}{\sqrt{\pi}} e^{-u^2}\,du$$

$$= \boxed{\frac{1}{2}\left(\operatorname{erf}\left(\frac{x-1}{\sqrt{4kt}}\right) - \operatorname{erf}(-\infty) - \operatorname{erf}(\infty) + \operatorname{erf}\left(\frac{x+1}{\sqrt{4kt}}\right)\right)}$$

problem 4: $\begin{cases} u_t - ku_{xx} = 0 & t>0, \quad 0<x<\infty \\ u(x,0) = g(x) \\ u_x(0,t) = h(t) \end{cases}$

(5) $g(x) = xe^{-x}, \quad h(t) = 0$

$$u(x,t) = \int_0^\infty \frac{1}{\sqrt{4kt\pi}} e^{-\frac{(x-y)^2}{4kt}} \cdot ye^{-y}\,dy + \int_0^\infty \frac{1}{\sqrt{4kt\pi}} e^{-\frac{(x+y)^2}{4kt}} ye^{-y}\,dy$$

note: $\frac{(x-y)(x-y)}{4kt} = \frac{x^2}{4kt} - \frac{2xy}{4kt} + \frac{y^2}{4kt}$

$-\frac{4kty}{4kt}$

$$= \frac{x^2 - y(2x+4kt) + y^2}{4kt}$$

complete the square:

$$\frac{-1}{4kt}\left(y^2+(-2x+4kt)y+x^2\right) \qquad \frac{-1}{4kt}\left(y^2+(2x+4kt)y+x^2\right)$$

$$\left(\frac{b}{2a}\right)^2 = \frac{(-2x+4kt)^2}{4} \qquad \left(\frac{b}{2a}\right)^2 = \frac{(2x+4kt)^2}{4}$$

$$\frac{-1}{4kt}\left(\left(y+\underbrace{\left(\frac{-2x+4kt}{2}\right)}_{B}\right)^2+x^2-\frac{(-2x+4kt)^2}{4}\right)$$

$$\frac{-1}{4kt}\left(\left(y+\overbrace{\frac{2x+4kt}{2}}^{\eta}\right)^2+x^2-\left(\frac{2x+4kt}{2}\right)^2\right)$$

$$u(x,t) = \frac{1}{\sqrt{4kt}}\cdot\exp\left(\frac{-x^2}{4kt}+\frac{(-2x+4kt)^2}{16kt}\right)\int_0^\infty y e^{\frac{-(y+B)^2}{4kt}}\,dy$$
$$+\frac{1}{\sqrt{4kt}}\cdot\exp\left(\frac{-x^2}{4kt}+\frac{(2x+4kt)^2}{16kt}\right)\int_0^\infty y e^{\frac{-(y+\eta)^2}{4kt}}\,dy$$

solving integral:

$$\int_0^\infty (y+B)e^{\frac{-(y+B)^2}{4kt}} - B\, e^{\frac{-(y+B)^2}{4kt}}\,dy \qquad \text{same process for } \eta$$

$$u=\frac{(y+B)^2}{-4kt} \quad -2kt\,du=(y+B)\,dy \qquad w=\frac{y-B}{\sqrt{4kt}} \quad dy=\sqrt{4kt}\,dw$$

$$-2kt\int_{\frac{B^2}{-4kt}}^{-\infty} e^u\,du \qquad -B\sqrt{4kt}\int_{\frac{-B}{\sqrt{4kt}}}^{\infty} e^{-w^2}\,dw$$

$$2kt\left(\exp\left(\frac{B^2}{-4kt}\right)-\overset{0}{\exp(-\infty)}\right)-\frac{B\sqrt{4kt\pi}}{2}\left(\overset{1}{\mathrm{erf}(\infty)}-\mathrm{erf}\left(\frac{-B}{\sqrt{4kt}}\right)\right)$$

$$\boxed{\begin{aligned}u(x,t)=\frac{1}{\sqrt{4kt}}\exp\left(\frac{-x^2}{4kt}\right)&\left(\exp\left(\frac{(-2x+4kt)^2}{16kt}\right)\left(2kt\exp\left(\frac{(-x+2kt)^2}{4kt}\right)-\right.\right.\\ &\left.(-x+2kt)\sqrt{kt\pi}\left(1-\mathrm{erf}\left(\frac{x-2kt}{4kt}\right)\right)\right)+\left(2kt\exp\left(\frac{(x+2kt)^2}{4kt}\right)-\right.\\ &\left.\left.(x+2kt)\sqrt{kt\pi}\left(1-\mathrm{erf}\left(\frac{-x-2kt}{4kt}\right)\right)\right)\exp\left(\frac{(x+2kt)^2}{4kt}\right)\right)\end{aligned}}$$

2. $\begin{cases} u_t - ku_{xx} = f(x,t) & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = g(x) & -\infty < x < \infty \end{cases}$

where $f(x,t) = 1$ and $g(x) = x$.

superposition: $\begin{cases} u_{1t} - ku_{1xx} = 0 \\ u_1(x,0) = g(x) \end{cases} + \begin{cases} u_{2t} - ku_{2xx} = f(x,t) \\ u_2(x,0) = 0 \end{cases}$

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

$$u_1(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} \cdot y\,dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} (y-x) e^{-\frac{(x-y)^2}{4kt}} + x e^{-\frac{(x-y)^2}{4kt}}\,dy$$

$u = -\frac{(x-y)^2}{4kt}$ $\qquad$ $w = \frac{x-y}{\sqrt{4kt}}$ $\quad$ $dw = \frac{-1}{\sqrt{4kt}}\,dy$

$du = \frac{x-y}{2kt}\,dy$ $\qquad$ $-x \int_{\infty}^{-\infty} e^{-w^2}\,dw \cdot \sqrt{4kt}$

$-2kt \int_{-\infty}^{-\infty} e^{u} = 0$ $\qquad$ $\sqrt{4kt}\left(x\int_{-\infty}^{0} e^{-w^2}\,dw + x\int_{0}^{\infty} e^{-w^2}\,dw\right)$

$$\frac{2x\sqrt{kt\pi}}{2}\left(\mathrm{erf}(\infty) - \mathrm{erf}(-\infty)\right)$$

$$u_1(x,t) = \frac{x}{2}\left(\underset{1}{\mathrm{erf}(\infty)} - \underset{-1}{\mathrm{erf}(-\infty)}\right) = x$$

$$u_2(x,t) = \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4k(t-\tau)}} \cdot 1\,dy\,d\tau$$

$w = (x-y)/\sqrt{4k(t-\tau)}$

$$\frac{1}{2}\int_{-\infty}^{\frac{x}{\sqrt{4k(t-\tau)}}} \frac{2}{\sqrt{\pi}} e^{-w^2}\,dw = \frac{1}{2}\left(\mathrm{erf}\left(\frac{x}{\sqrt{4k(t-\tau)}}\right) + 1\right)$$

$$u_2(x,t) = \frac{t}{2} + \frac{1}{2}\int_0^t \mathrm{erf}\left(\frac{x}{\sqrt{4k(t-\tau)}}\right) d\tau$$

$$\boxed{u(x,t) = \frac{xt}{2} + \frac{x}{2}\int_0^t \mathrm{erf}\left(\frac{x}{\sqrt{4k(t-\tau)}}\right) d\tau}$$

Chapter 4: seperation of variables and Fourier series

4.1: separation of varibles: solving finite interval problems

$$u(x,t) \Rightarrow X(x)\cdot T(t)$$

lets do it with the wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & u(x,0)=g(x) & u(0,t)=0 \\ & u_t(x,0)=h(x) & u(l,t)=0 \end{cases}$$

First look at PDE

$u_t = X(x)\,T'(t)$ $\qquad$ $u_x = X'(x)\,T(t)$

$u_{tt} = X(x)\,T''(t)$ $\qquad$ $u_{xx} = X''(x)\,T(t)$

$$X(x)\,T''(t) - c^2 X''(x)\,T(t) = 0$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} \longrightarrow \text{both have to be equal constants: } -\lambda$$

$$\frac{X''(x)}{X(x)} = -\lambda \quad \text{and} \quad \frac{T''(t)}{c^2 T(t)} = -\lambda$$

$$X'' + \lambda X(x) = 0 \qquad T'' + \lambda c^2 T(t) = 0$$

Boundry conditions: $u(0,t)=0 \Rightarrow X(0)\,T(t) = 0 \;\therefore X(0)=0$

similarly $X(l)=0$

New system from what we gathered
$$\begin{cases} X''(x) + \lambda X(x) = 0 & 0<x<l \\ X(0)=0 & \leftarrow \text{Bounded ODE} \\ X(l)=0 \end{cases}$$
called "eigenvalue problem"

rewrite ODE as: $-X''(x) = \lambda X(x)$ ← eigen function

like a "matrix" $\qquad$ eigen value

but lets solve $X'' + \lambda X = 0$

$$z^2 + \lambda = 0 \qquad z = \sqrt{-\lambda}$$

case 1 $\quad$ if $\lambda > 0$, $\; z = \pm\sqrt{\lambda}\,i \;\therefore X = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

case 2 $\quad$ if $\lambda = 0$, $\; z=0$ and $z=0 \;\therefore X = C_1 e^0 + C_2 x e^0 = C_1 + C_2 X$

case 3 $\quad$ if $\lambda < 0$, $\; z = \pm\sqrt{-\lambda} \;\therefore X = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$

case 1: $\lambda > 0$, $X = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

$X(0) = 0 \Rightarrow C_1 = 0$

$X(\ell) = 0 \Rightarrow C_2\sin(\sqrt{\lambda}\ell) = 0$

$\therefore \lambda\ell = n\pi$ for $n = 1, 2, 3, 4$

$\sqrt{\lambda} = n\pi/\ell$

$X = \sin(\frac{n\pi}{\ell}x)$ or written $X_n = \sin(\sqrt{\lambda_n}x)$

case 2: $\lambda = 0$, $X = C_1 + C_2X$

$X(0) = 0$, $X(\ell) = 0$ gives trivial C_1 and C_2

so nothing meaningful.

case 3: $\lambda < 0$, $X = C_1e^{\sqrt{-\ell}x} + C_2e^{-\sqrt{-\ell}x}$

$X(0) = 0 \Rightarrow C_2 = -C_1$

$X(\ell) = 0 \Rightarrow X = C_1(e^{\sqrt{-\ell}X} + e^{-\sqrt{-\ell}X}) = 0$

$C_1 = 0$, trivial

Therefore case 1 is our only real solution

solution to eigenvalue problem:

$$\begin{cases} \sqrt{\lambda_n} = n\pi/\ell \\ X_n = C_n\sin(\sqrt{\lambda_n}X) \end{cases}$$

Now back to T:

$T_n''(t) + c^2\lambda_n T_n(t) = 0$

$z^2 + c^2\lambda_n = 0$ $\quad z^2 = -c^2(n\pi/\ell)^2$

$z = \frac{cn\pi}{\ell}\cdot i, -\frac{cn\pi}{\ell}\cdot i$

$T_n = A_n\cos(\frac{n\pi c}{\ell}t) + B_n\sin(\frac{n\pi c}{\ell}t)$

Now all together: $u_n(x,t) = X_n(x)\cdot T_n(t)$

$u_n(x,t) = (A_n\cos(\frac{n\pi c}{\ell}t) + B_n\sin(\frac{n\pi c}{\ell}t))\sin(\frac{n\pi}{\ell}x)$

$u_n(x,0) = A_n\sin(\frac{n\pi}{\ell}x) \neq g(x)$

1 n cannot solve the I.V.P.'s

What if we use superposition to sum U_n for different n and have it solve IVP's because the bondry and PDE equal 0

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$u(x,0) = \sum A_n \sin(\frac{n\pi}{\ell}x)$ and $u_t(x,0) = \sum B_n \frac{n\pi c}{\ell} \sin(\frac{n\pi}{\ell}x)$

$= g(x)$ $= h(x)$

Goal: Pick A_n and B_n equations that make the equations hold.

4.2 Eigenvalue problem

From wave equation:

$u = X(x)\,T(t)$

eigenvalue problem $\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \quad X(\ell) = 0 \end{cases}$ BVP

$T''(t) + c^2 \lambda T(t) = 0$

$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \qquad X_n(x) = \sin\frac{n\pi x}{\ell}$

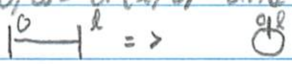
What if we change our boundry condition:

1. "Dirclet Dirclet"
2. "Newman Newman"
3. "D N"
4. "N D"

always homogenous

1. $X(0) = X(\ell) = 0$
2. $X'(0) = X'(\ell) = 0$
3. $X(0) = X'(\ell) = 0$
4. $X'(0) = X(\ell) = 0$

5. Periodic B.C. : $u(0,t) = u(\ell,t)$ and $u_x(0,t) = u_x(\ell,t)$

so imagine 

5. $X(0) = X(\ell)$, $X'(0) = X'(\ell)$

Condition 1-5 solutions: We already found 1.

2. $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \quad X'(\ell) = 0 \end{cases}$

subcase 1: $\lambda > 0$, $X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

2: $\lambda = 0$, $X = c_1 + c_2 x$

3. $\lambda < 0$, $X = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

check B.C. don't use trivial sols.

1. $\lambda = \left(\frac{n\pi}{\ell}\right)^2$ $X = c_1 \cos(\frac{n\pi x}{\ell})$
2. $\lambda = 0$ $X = c_1$
3. only trivial

notice subcase 1 and 2 are the same
if $n=0$, in case 1 $\lambda = 0$ and $X = C_1$
which is the same as case 2
now n can be a positive integer or $n=0$
sol to case 2.
$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$, $n = 0, 1, 2, 3 \dots$
$X_n = \cos\left(\frac{n\pi x}{\ell}\right)$ ignore constant C_1 because it is accounted for in $T(t)$ eq.

3. $\lambda_n = \left(\frac{2n+1}{2} \cdot \frac{\pi}{\ell}\right)^2$, $n = 0, 1, 2, 3 \dots$
$X_n = \sin(\sqrt{\lambda_n} \cdot x)$

4. $\lambda_n = \left(\frac{2n+1}{2} \cdot \frac{\pi}{\ell}\right)^2$, $n = 0, 1, 2 \dots$
$X_n = \cos(\sqrt{\lambda_n} \cdot x)$

5. $\lambda_n = \left(\frac{2n\pi}{\ell}\right)^2$ $n = 0, 1, 2 \dots$
if $n=0$, $\lambda_0 = 0$ and $X_0(x) = 1$
if $n \neq 0$, $X_n(x) = A_n \cos(\sqrt{\lambda_n}\, x) + B_n \sin(\sqrt{\lambda_n}\, x)$

Fourier series:
lets do the periodic case 5 first
but replace ℓ with 2ℓ
λ_n becomes $\left(\frac{n\pi}{\ell}\right)^2$
if $n=0$, $X_0(x) = \frac{1}{2}$ ← constant don't matter
$n \neq 0$, $X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right)$ or $\sin\left(\frac{n\pi x}{\ell}\right)$

$$\left\{\frac{1}{2}, \cos\left(\frac{n\pi x}{\ell}\right), \sin\left(\frac{n\pi x}{\ell}\right)\right\}$$

Fourier series is a linear combination of all these

$$\frac{1}{2} \cdot a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

Would this be any arbitrary function: $f(x)$

Homework 7

1. solve $\begin{cases} u_t - k u_{xx} = f(x,t) & x>0, t>0 \\ u(x,0) = g(x) & x>0 \\ u(0,t) = 0 & t>0 \end{cases}$

We will use superposition saying $u = u_1 + u_2$ where:

$$\begin{cases} u_{1t} - k u_{1xx} = 0 \\ u_1(x,0) = g(x) \\ u_1(0,t) = 0 \end{cases} \qquad \begin{cases} u_{2t} - k u_{2xx} = f(x,t) \\ u_2(x,0) = 0 \\ u_2(0,t) = 0 \end{cases}$$

$$u_1(x,t) = \int_0^\infty \frac{1}{2\sqrt{k\pi t}} e^{\frac{-(x-y)^2}{4kt}} g(y)dy - \int_0^\infty \frac{1}{2\sqrt{k\pi t}} e^{\frac{-(x+y)^2}{4kt}} g(y)dy$$

$$u_2(x,t) = \int_0^t \int_0^\infty \frac{1}{2\sqrt{k\pi}\sqrt{t-\tau}} \left(\exp\left(\frac{-(x-y)^2}{4k(t-\tau)}\right) - \exp\left(\frac{-(x+y)^2}{4k(t-\tau)}\right)\right) f(y,\tau)dyd\tau$$

and solve $f(x,t) = 1$ and $g(x) = x$

$$u_1 = \frac{1}{2\sqrt{k\pi t}} \int_{\frac{x}{2\sqrt{kt}}}^{-\infty} -2\sqrt{kt}\, y e^{-w^2} dw + \frac{1}{2\sqrt{k\pi t}} \int_{\frac{x}{2\sqrt{kt}}}^{\infty} 2\sqrt{kt}\, y e^{-u^2} du$$

$w = \frac{x-y}{2\sqrt{kt}}$, $y = -2w\sqrt{kt} + x$ $\qquad y = 2u\sqrt{kt} + x$

$dw = \frac{-1}{2\sqrt{kt}}dy$, $dy = -2\sqrt{kt}dw$ $\qquad dy = 2\sqrt{kt}du$

$$= \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{\frac{x}{2\sqrt{kt}}} ye^{-w^2}dw + \int_{\frac{x}{2\sqrt{kt}}}^{\infty} ye^{-u^2}du\right)$$

$$\cancel{-2}w\cancel{\sqrt{kt}} \cancel{+x} = \cancel{2}u\cancel{\sqrt{kt}} \cancel{+x}$$

$u = -w \qquad -dw = du$

$$= \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{\frac{x}{2\sqrt{\pi t}}} ye^{-w^2}dw + \int_{\frac{-x}{2\sqrt{kt}}}^{-\infty} -ye^{-(-w)^2}dw\right)$$

$$+ \int_{-\infty}^{\frac{-x}{2\sqrt{kt}}} ye^{-w^2}dw$$

$$\int ye^{-w^2}dw = \int -2\sqrt{kt}\cdot we^{-w^2}dw + \int xe^{-w^2}dw$$

$v = -w^2 \quad dv = -2wdw, \quad \frac{-1}{2}dv = wdw$

$$= \frac{-2\sqrt{kt}}{-2}\int e^v dv + \frac{x\sqrt{\pi}}{2}\int \frac{2}{\sqrt{\pi}} e^{-w^2}dw$$

$$= \sqrt{kt}\, e^v + \ldots$$

$$= \sqrt{kt}\, e^{-w^2} + \ldots$$

$*\exp(-\infty)=0$

$$u_1=\frac{1}{\sqrt{\pi}}\left(\sqrt{kt}\exp\left(-w^2\right)\Big|_{-\infty}^{\frac{x}{2\sqrt{kt}}}+\frac{x\sqrt{\pi}}{2}\left(\operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)-\operatorname{erf}(-\infty)\right)\right.$$

$$\left.+\sqrt{kt}\exp\left(-w^2\right)\Big|_{-\infty}^{\frac{-x}{2\sqrt{kt}}}+\frac{x\sqrt{\pi}}{2}\left(\operatorname{erf}\left(\frac{-x}{2\sqrt{kt}}\right)-\operatorname{erf}(-\infty)\right)\right)$$

$^{-1}$ $^{-1}$

$$u_1=\frac{1}{\sqrt{\pi}}\left(\sqrt{kt}\exp\left(\frac{x}{2\sqrt{kt}}\right)+\sqrt{kt}\exp\left(\frac{-x}{2\sqrt{kt}}\right)+\frac{x\sqrt{\pi}}{2}\left(\operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)+\operatorname{erf}\left(\frac{-x}{2\sqrt{kt}}\right)\right)\right)$$

u_2: first solve:

$$\frac{1}{2\sqrt{k\pi}\sqrt{t-\tau}}\left(\int_0^\infty \exp\left(\frac{-(x-y)^2}{4k(t-\tau)}\right)dy-\int_0^\infty \exp\left(\frac{-(x+y)^2}{4k(t-\tau)}\right)dy\right)$$

$y=-2w\sqrt{k(t-\tau)}+x$ $\qquad$ $y=2u\sqrt{k(t-\tau)}+x$

$dy=-2\sqrt{k(t-\tau)}\,dw$ $\qquad$ $dy=2\sqrt{k(t-\tau)}\,du$

$$=\left(\int_{\frac{x}{2\sqrt{k(t-\tau)}}}^{-\infty}-2\sqrt{k(t-\tau)}\,e^{-w^2}dw+\int_{\frac{x}{2\sqrt{k(t-\tau)}}}^{\infty}2\sqrt{k(t-\tau)}\,e^{-u^2}du\right)$$

$$=\frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{\frac{x}{2\sqrt{k(t-\tau)}}}e^{-w^2}dw+\int_{\frac{x}{2\sqrt{k(t-\tau)}}}^{\infty}e^{-u^2}du\right)$$

$$=\frac{1}{2}\left(\operatorname{erf}\left(\frac{x}{2\sqrt{k(t-\tau)}}\right)-(-1)+\underset{1}{\operatorname{erf}(\infty)}-\operatorname{erf}\left(\frac{x}{2\sqrt{k(t-\tau)}}\right)\right)$$

$=1$

$$u_2=\int_0^t 1\,d\tau=t$$

$\therefore$

$$\boxed{u=t+\sqrt{\frac{kt}{\pi}}\left(\exp\left(\frac{x}{2\sqrt{kt}}\right)+\exp\left(\frac{-x}{2\sqrt{kt}}\right)\right)+\frac{x}{2}\left(\operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)+\operatorname{erf}\left(\frac{-x}{2\sqrt{kt}}\right)\right)}$$

erf is odd

$h(x)=-h(-x)$

$-h(x)=h(-x)$

$\therefore \operatorname{erf}(-x)=-\operatorname{erf}(x)$

$$+\frac{x}{2}(0)$$

problem 6: $u_t + cu_x = ku_{xx}$

convection term

(2) lets let $u(x,t) = U(x-vt, t)$ call $U(x'(x,t), t'(x,t))$

$x' = x - vt \quad t' = t$

then: $\frac{\partial U}{\partial t} = \frac{\partial U}{\partial x'} \cdot \frac{\partial x'}{\partial t} + \frac{\partial U}{\partial t'} \cdot \frac{\partial t'}{\partial t}$

$= U_{x'} \cdot x'_t + U_{t'} \cdot t'_t = -vU_{x'} + U_{t'} = U_t$

$\frac{\partial U}{\partial x} = U_{x'} \cdot \underbrace{x'_x}_{1} + U_{t'} \cdot \underbrace{t'_x}_{0} = U_{x'} = U_x$

rewrite: $-vU_{x'} + U_{t'} + cU_{x'} = kU_{x'x'}$

if $c = v$ then:

$U_{t'} - kU_{x'x'} = 0$

$\therefore U(x', t') = \frac{1 \cdot e^{\frac{-x'^2}{4kt}}}{\sqrt{4k\pi t}}$

$$\therefore \boxed{U(x+ct, t) = u(x,t) = \frac{1}{2\sqrt{k\pi t}} \exp\left(\frac{-(x+ct)^2}{4kt}\right)}$$

3. solve the eigenvalue problems

(3) (DN) $\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l \\ X(0) = X'(l) = 0 \end{cases}$

ODE: $X'' + \lambda X = 0$

characteristic: $z^2 + \lambda = 0 \Rightarrow z^2 = -\lambda$

1. $\lambda > 0$, $z = \pm i\sqrt{\lambda}$ $\therefore X = C_1 \cos(x\sqrt{\lambda}) + C_2 \sin(x\sqrt{\lambda})$
2. $\lambda = 0$, $z = 0$ and $z = 0$ $\therefore X = C_1 + C_2 x$
3. $\lambda < 0$, $z = \pm\sqrt{-\lambda}$ $\therefore X = C_1 e^{x\sqrt{-\lambda}} + C_2 e^{-x\sqrt{-\lambda}}$

1. $X(0) = C_1 = 0$

$X'(x) = 0 + C_2\sqrt{\lambda}\cos(x\sqrt{\lambda})$

$X'(l) = C_2\sqrt{\lambda}\cos(l\sqrt{\lambda}) = 0$

assume C_2 non trivial $\therefore \sqrt{\lambda}\cos(l\sqrt{\lambda}) = 0$

$l\sqrt{\lambda} = \underbrace{\cos^{-1}(0)}_{n\pi/2}$

$\lambda = \left(\frac{n\pi}{2l}\right)^2 \quad n = 1, 2, 3 \ldots$

2. $X(0) = C_1 = 0$
$X'(x) = C_2, \ C_2 = 0$ $\Rightarrow$ trivial

3. $X(0) = C_1 + C_2 = 0$
$X'(x) = C_1\sqrt{-\lambda}\, e^{x\sqrt{-\lambda}} - C_2\sqrt{-\lambda}\, e^{-x\sqrt{-\lambda}}$
$X'(l) = C_1\sqrt{-\lambda}\, e^{l\sqrt{-\lambda}} - C_2\sqrt{-\lambda}\, e^{-l\sqrt{-\lambda}} = 0$

$$\begin{cases} C_1 e^{l\sqrt{-\lambda}} - C_2 e^{-l\sqrt{-\lambda}} = 0 \\ C_1 + C_2 = 0 \end{cases} \quad \therefore -C_2 = C_1$$

$C_1(e^{l\sqrt{-\lambda}} + e^{-l\sqrt{-\lambda}}) = 0$

nontrivial C_1 $\therefore e^{l\sqrt{-\lambda}} + e^{-l\sqrt{-\lambda}} = 0$

$\ln(e^{l\sqrt{-\lambda}}) = \ln(-e^{-l\sqrt{-\lambda}})$

$l\sqrt{-\lambda}\,\underbrace{\ln(e)}_{1} = \ln(-e^{-l\sqrt{-\lambda}} \cdot -1)$

$e \cdot -1 = e \cdot (\underbrace{\cos(\pi + 2n\pi)}_{-1} + i\underbrace{\sin(\pi + 2n\pi)}_{0}) = \underbrace{e \cdot e^{i(\pi + 2n\pi)}}_{-1}$ $\quad n = 0, 1, 2, 3 \dots$

$\ln(-e^{-l\sqrt{-\lambda}}) = -l\sqrt{-\lambda}\ln(e) + \ln(e^{i(\pi+2n\pi)})$

$l\sqrt{-\lambda} = -\lambda\sqrt{-\lambda} + i(\pi + 2n\pi)$

$2l\sqrt{-\lambda} = (2\pi n + \pi)i$

$\sqrt{-\lambda} = \dfrac{\pi(2n+1)}{2l} i$, $\quad -\lambda = \left(\dfrac{\pi(2n+1)}{2l}\right)^2 \cdot (i)^2$

$\therefore \boxed{\lambda = \left(\dfrac{(2n+1)}{2} \cdot \dfrac{\pi}{l}\right)^2} = \left(\dfrac{n\pi}{2l}\right)^2$ ✓

because $2n+1$ is alway an n ✓
so this λ also satisfies 1.

$\boxed{X_n = C_n \sin(x\sqrt{\lambda})}$

(4) (ND) $\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < \lambda \\ X'(0) = X(\ell) = 0 \end{cases}$

charictaristic to ODE: $z^2 + \lambda = 0$ $\quad z^2 = -\lambda$

if: 1. $\lambda > 0$, $z = \pm i\sqrt{\lambda}$ $\therefore X = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$

2. $\lambda = 0$, $z = 0, z = 0$ $\therefore X = C + Dx$

3. $\lambda < 0$, $z = \pm\sqrt{-\lambda}$ $\therefore X = Ee^{x\sqrt{-\lambda}} + Fe^{-x\sqrt{-\lambda}}$

1. $X'(x) = -A\sqrt{\lambda}\sin(x\sqrt{\lambda}) + B\sqrt{\lambda}\cos(x\sqrt{\lambda})$

$X'(0) = 0 = B\sqrt{\lambda}$ $\therefore B = 0$

lets assume non-trivial

$X(x) = A\cos(x\sqrt{\lambda})$, $\quad X(\ell) = 0 = A\cos(\ell\sqrt{\lambda})$

$0 = \cos(\ell\sqrt{\lambda})$

$\ell\sqrt{\lambda} = \frac{n\pi}{2}$ $\therefore \lambda = \left(\frac{n\pi}{2\ell}\right)^2$

2. trivial

3. $X'(0) = 0 = E\sqrt{-\lambda} - F\sqrt{-\lambda} \Rightarrow E - F = 0$ $\therefore E = F$

$X(\ell) = 0 = Ee^{\ell\sqrt{-\lambda}} + Ee^{-\ell\sqrt{-\lambda}}$

assume non triv $\rightarrow$ $E(e^{\ell\sqrt{-\lambda}} + e^{-\ell\sqrt{-\lambda}}) = 0$

$e^{\ell\sqrt{-\lambda}} = -e^{-\ell\sqrt{-\lambda}}$

$\ell\sqrt{-\lambda} = -\ell\sqrt{-\lambda} + i(2\pi n + \pi)$

$2\ell\sqrt{-\lambda} = i\pi(2n+1)$

$-\lambda = i^2\left(\frac{(2n+1)}{2} \cdot \frac{\pi}{\ell}\right)^2$

$\lambda = \left(\frac{(2n+1)}{2} \cdot \frac{\pi}{\ell}\right)^2$ if $\lambda < 0$

but hey: $2n+1$ is still another n

so $\left(\frac{(2n+1)}{2} \cdot \frac{\pi}{\ell}\right)^2$ satisfies what $\left(\frac{n\pi}{2\ell}\right)^2$ satisfied

so: $\boxed{\lambda = \left(\frac{2n+1}{2} \cdot \frac{\pi}{\ell}\right)^2, \quad X(x) = C_n\cos(2\sqrt{\lambda})}$

(5) (periodic) $\begin{cases} X''(x) + \lambda X(x) = 0, & -l < x < l \\ X(-l) = X(l), & X'(-l) = X'(l) \end{cases}$

characteristic: $z^2 + \lambda = 0 \Rightarrow z^2 = -\lambda$

1. $\lambda > 0$, $z = \pm i\sqrt{\lambda}$ $\therefore X = C_1 \cos(x\sqrt{\lambda}) + C_2 \sin(x\sqrt{\lambda})$
2. $\lambda = 0$, $z = \pm 0$ $\therefore X = C_3 + C_4 x$
3. $\lambda < 0$, $z = \pm\sqrt{-\lambda}$ $\therefore X = C_5 e^{x\sqrt{-\lambda}} + C_6 e^{-x\sqrt{-\lambda}}$

1. $X(l) = C_1 \cos(l\sqrt{\lambda}) + C_2 \sin(l\sqrt{\lambda}) = C_1 \cos(-l\sqrt{\lambda}) + C_2 \sin(-l\sqrt{\lambda}) = X(-l)$
$= C_1 \cos(l\sqrt{\lambda}) - C_2 \sin(l\sqrt{\lambda})$

nontrivial C_2 $\quad 2C_2 \sin(l\sqrt{\lambda}) = 0$

$\sin(l\sqrt{\lambda}) = 0$

$l\sqrt{\lambda} = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{l}\right)^2$

2. $X(-l) = C_3 - C_4 l = C_3 + C_4 l = X(l) \Rightarrow 2C_4 l = 0$
$X'(-l) = C_4 = C_4 = X'(l)$
only trivials exist

3. $X(-l) = C_5 e^{-l\sqrt{-\lambda}} + C_6 e^{l\sqrt{-\lambda}}$
$= X(l) = C_5 e^{l\sqrt{-\lambda}} + C_6 e^{-l\sqrt{-\lambda}}$

$X(-l) - X(l) = 0$

$e^{-l\sqrt{-\lambda}}(C_5 - C_6) + e^{l\sqrt{-\lambda}}(C_6 - C_5) = 0$

$(e^{-l\sqrt{-\lambda}} - e^{l\sqrt{-\lambda}})(C_5 - C_6) = 0$

$X'(-l) = C_5\sqrt{-\lambda}\, e^{-l\sqrt{-\lambda}} - C_6\sqrt{-\lambda}\, e^{l\sqrt{-\lambda}}$
$= X'(l) = C_5\sqrt{-\lambda}\, e^{l\sqrt{-\lambda}} - C_6\sqrt{-\lambda}\, e^{-l\sqrt{-\lambda}}$

$X'(-l) - X'(l) = 0$

$\sqrt{-\lambda}\, e^{-l\sqrt{-\lambda}}(C_5 + C_6) + \sqrt{-\lambda}\, e^{l\sqrt{-\lambda}}(-C_6 - C_5) = 0$

$(e^{-l\sqrt{-\lambda}} - e^{l\sqrt{-\lambda}})(C_5 + C_6) = 0$

if non trivial C_5, C_6

$e^{-l\sqrt{-\lambda}} - e^{l\sqrt{-\lambda}} = 0$

$e^{-l\sqrt{-\lambda}} = e^{l\sqrt{-\lambda}} \Rightarrow -l\sqrt{-\lambda} = l\sqrt{-\lambda} \Rightarrow 2l\sqrt{-\lambda} = 0 \quad \lambda = 0$

$\therefore$

$$\boxed{X_n(x) = C_n\left(\cos(x\sqrt{\lambda_n}) + D_n \sin(x\sqrt{\lambda_n})\right), \quad \lambda = \left(\frac{n\pi}{l}\right)^2}$$

Back to Periodic condition

$$\begin{cases} X'' + \lambda X = 0 \\ X(x_0) = X(x_1) \\ X'(x_0) = X'(x_1) \end{cases}$$

swap 0 and $2l$ bounds for more general terms

has solutions made up of

$$\left\{ \frac{1}{2}, \cos\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right) \right\} \quad n = 1, 2, 3\ldots$$

think of this as an orthogonal system

Inner product: $\langle f(x), g(x) \rangle = \int_0^{2l} f(x) \cdot g(x)\,dx$

Orthogonality: $f \perp g$: $\langle f, g \rangle = 0$

Orthogonal system: any two functions are orthogonal

check with our case

1. $\langle \frac{1}{2}, \cos(\frac{n\pi}{l}x) \rangle$
2. $\langle \frac{1}{2}, \sin(\frac{m\pi}{l}x) \rangle$
3. $\langle \cos(\frac{n\pi x}{l}), \sin(\frac{m\pi x}{l}) \rangle$
4. $\langle \cos(\frac{n\pi x}{l}), \cos(\frac{m\pi x}{l}) \rangle$ check $n = m$ and $n \neq m$
5. $\langle \sin(\frac{n\pi x}{l}), \sin(\frac{m\pi x}{l}) \rangle$ " $n = m$ and $n \neq m$

1. $\int_0^{2l} \frac{1}{2} \cos(\frac{n\pi x}{l})\,dx = \frac{1}{2} \cdot \frac{l}{n\pi} \sin(\frac{n\pi x}{l}) \Big|_0^{2l} = 0$ ✓ $n \neq 0$ or else undefined

3. $\int_0^{2\pi} \cos\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx$ use trig multiply rule $\cos(\alpha)\sin(\beta) = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$= 0$

it all checks out $\therefore$ our system is orthogonal

We want $f(x)$ for

$$f(x) = \frac{1}{2}a_0 + \sum \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

can this be done for any $f(x)$ on $(0, 2l)$

What should a_0, a_n, b_n, n be?

consider: $\langle f(x), \frac{1}{2} \rangle$

$\langle f(x), \cos(\frac{n\pi x}{l}) \rangle$

$\langle f(x), \sin(\frac{n\pi}{l}x) \rangle$

since $f(x)$ is 3 terms, you can integrate by each term individually and sum.

$\langle f(x), \frac{1}{2} \rangle = \langle \frac{1}{2}a_0, \frac{1}{2} \rangle + \sum_{n=1}^{\infty} \left(\langle a_n \cos(\frac{n\pi x}{\ell}), \frac{1}{2} \rangle + \langle b_n \sin(\frac{n\pi x}{\ell}), \frac{1}{2} \rangle \right)$

$a_0 \langle \frac{1}{2}, \frac{1}{2} \rangle = \langle \frac{1}{2}a_0, \frac{1}{2} \rangle = a_0 \int_0^{2\ell} \frac{1}{2} \cdot \frac{1}{2} \, dx = a_0 \ell/2$

$\langle a_n \cos(\frac{n\pi x}{\ell}), \frac{1}{2} \rangle = a_n \langle \cos(\frac{n\pi x}{\ell}), \frac{1}{2} \rangle = a_n \langle \frac{1}{2}, \cos(\frac{n\pi x}{\ell}) \rangle$

we know this $= 0$

similar reason $\langle b_n \sin(\frac{n\pi x}{\ell}), \frac{1}{2} \rangle = 0$

$\therefore \langle f(x), \frac{1}{2} \rangle = a_0 \ell/2 = \int_0^{2\ell} f(x) \cdot \frac{1}{2} \, dx$

$\therefore a_0 = \frac{1}{\ell} \int_0^{2\ell} f(x) \, dx$

Now $\langle f(x), \cos(\frac{n\pi x}{\ell}) \rangle$

$= \underbrace{\langle \frac{1}{2}a_0, \cos(\frac{n\pi x}{\ell}) \rangle}_{0} + \sum \left(\langle a_m \cos(\frac{m\pi x}{\ell}), \cos(\frac{n\pi x}{\ell}) \rangle + \underbrace{\langle b_m \sin(\frac{m\pi x}{\ell}), \cos(\frac{n\pi x}{\ell}) \rangle}_{0} \right)$

$\langle a_m \cos(\frac{m\pi x}{\ell}), \cos(\frac{n\pi x}{\ell}) \rangle = \begin{cases} \ell, & n = m \\ 0, & n \neq m \end{cases}$

$= a_n \ell$

$\therefore a_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \cos(\frac{n\pi x}{\ell}) \, dx \qquad n = 0, 1, 2, 3 \ldots$

$\langle f(x), \sin(\frac{n\pi x}{\ell}) \rangle$

$= b_n \ell$

$\therefore b_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \sin(\frac{n\pi x}{\ell}) \, dx \qquad n = 1, 2, 3 \ldots$

Assume $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell})) = 0$

call whole left hand side $\tilde{f}(x)$

clearly a_0 a_n and $b_n = 0$

Completness: $f(x) = \tilde{f}(x)$ if this is true, all our pervious calcs work

$a_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \cos(\frac{n\pi x}{\ell}) \, dx \qquad n = 0, 1, 2 \ldots$

$b_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \sin(\frac{n\pi x}{\ell}) \, dx$

$f(x) = \tilde{f}(x) \Rightarrow f(x) - \tilde{f}(x) = 0 \quad \therefore \langle f(x) - \tilde{f}(x), f(x) - \tilde{f}(x) \rangle = 0$

$\langle f(x), f(x) - \tilde{f}(x) \rangle - \underbrace{\langle \tilde{f}(x), f(x) - \tilde{f}(x) \rangle}_{\text{claim } = 0}$ $= 0$

because

$= \langle f(x) - \tilde{f}(x), \tilde{f}(x) \rangle = \langle f(x) - \tilde{f}(x), \frac{1}{2}a_0 + \sum(\cos + \sin) \rangle$

we know $\langle f(x), \frac{1}{2} \rangle = \langle \tilde{f}(x), \frac{1}{2} \rangle \quad \therefore \langle f(x) - \tilde{f}(x), \frac{1}{2} \rangle = 0$

$\therefore a_0 \underbrace{\langle f(x) - \tilde{f}(x), \frac{1}{2} \rangle}_{0} + \sum_{n=1}^{\infty} \left(a_n \underbrace{\langle f(x) - \tilde{f}(x), \cos \rangle}_{0} + b_n \underbrace{\langle f(x) - \tilde{f}(x), \sin \rangle}_{0} \right)$

so claim is true

$\therefore\ \tilde{f}(x) \perp f(x) - \tilde{f}(x)$

and $\langle f(x) - \tilde{f}(x), f(x) - \tilde{f}(x) \rangle = \langle f(x), f(x) - \tilde{f}(x) \rangle$

$= \langle f(x), f(x) \rangle - \langle f(x), \tilde{f}(x) \rangle$

claim: $\langle f(x), \tilde{f}(x) \rangle = \langle \tilde{f}(x), \tilde{f}(x) \rangle$

$\Rightarrow \langle f(x), \tilde{f}(x) \rangle - \langle \tilde{f}(x), \tilde{f}(x) \rangle = 0$

$\Rightarrow \langle f(x) - \tilde{f}(x), \tilde{f}(x) \rangle = 0$

hey, we already proved this
so claim holds

$\langle f(x) - \tilde{f}(x), f(x) - \tilde{f}(x) \rangle = \langle f(x), f(x) \rangle - \langle \tilde{f}(x), \tilde{f}(x) \rangle$

Bessel's inequalty: $\langle f(x), f(x) \rangle \geq \langle \tilde{f}(x), \tilde{f}(x) \rangle$

because L.H.S. is ≥ 0

but L.H.S. is actually 0 $\therefore \langle f(x), f(x) \rangle = \langle \tilde{f}(x), \tilde{f}(x) \rangle$

Parseval's equalty

if $f(x) = \tilde{f}(x)$ then Parseval's equality is true.
but we won't prove it here, just trust lol

$\langle \tilde{f}(x), \tilde{f}(x) \rangle = \langle f(x), \tilde{f}(x) \rangle$

$= \langle f(x), \frac{1}{2} a_0 \rangle + \sum_{n=1}^{\infty} \left(\langle f(x), a_n \cos\left(\frac{n\pi x}{\ell}\right) \rangle + \langle f(x), b_n \sin\left(\frac{n\pi x}{\ell}\right) \rangle \right)$

$= a_0 \langle f(x), \frac{1}{2} \rangle + \sum \left(a_n \langle f(x), \cos\left(\frac{n\pi x}{\ell}\right) \rangle + b_n \langle f(x), \sin\left(\frac{n\pi x}{\ell}\right) \rangle \right)$

$= a_0 \cdot \frac{a_0 \ell}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot a_n \ell + b_n \cdot b_n \ell \right)$

$= \ell a_0^2 / 2 + \ell \sum_{n=0}^{\infty} \left(a_n^2 + b_n^2 \right)$

Parseval's becomes

$\langle f(x), f(x) \rangle = \int_0^{2\ell} f(x)^2 \, dx = \ell a_0^2 / 2 + \ell \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right)$

Other Fourier Series for other eigenvalue problems

1. "DD"
2. "NN"
3. "DN"
4. "ND"
5. "Periodic BC" ← Just did this

1. $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \quad X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right) \quad n = 1, 2, 3$

our system $\left\{ \sin\left(\frac{n\pi x}{\ell}\right), n = 1, 2, 3 \right\}$

orthogonality $\sin\left(\frac{n\pi x}{\ell}\right) \perp \sin\left(\frac{m\pi x}{\ell}\right)$

$$\int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right) dx \qquad {}^{*}\sin(\alpha+\beta) = \tfrac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$= \frac{1}{2}\int_0^\ell \cos\left(\frac{(n-m)\pi x}{\ell}\right) - \cos\left(\frac{(n+m)\pi x}{\ell}\right) dx$$

$$= \frac{1}{2}\begin{cases} \ell, & n = m \\ \frac{\ell}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{\ell}\right)\Big|_0^\ell = 0, & n \neq m \end{cases} - \underbrace{\frac{\ell}{2(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{\ell}\right)\Big|_0^\ell}_{0}$$

$$\therefore \left\langle \sin\left(\frac{n\pi x}{\ell}\right), \sin\left(\frac{m\pi x}{\ell}\right) \right\rangle = \begin{cases} \ell/2, & n = m \\ 0, & n \neq m \end{cases}$$

Example Fourier series:

$$\left\{\frac{1}{2}, \cos\left(\frac{n\pi x}{\ell}\right), \sin\left(\frac{n\pi x}{\ell}\right)\right\} \quad -\ell < x < \ell$$

$$f(x) = \begin{cases} 1 & 0 < x < \ell \\ -1 & -\ell < x < 0 \end{cases}$$ *odd function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$$ odd · even = odd

integral $\int_{-\ell}^{\ell}$ odd $= 0 \quad \therefore a_n = 0$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$ odd · odd = even

$$= \frac{2}{\ell}\int_{0}^{\ell} 1 \cdot \sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{2}{\ell} \cdot \frac{\ell}{n\pi}\left(-\cos\left(\frac{n\pi x}{\ell}\right)\right)\Big|_0^{\ell}$$

$$= \frac{-2}{n\pi}\left(\cos(n\pi) - 1\right) = \frac{-2}{n\pi}\left((-1)^n - 1\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-2}{n\pi}\left((-1)^n - 1\right)\sin\left(\frac{n\pi x}{\ell}\right)$$ valid but ugly

$$n = \begin{cases} 2k \\ 2k-1 \end{cases} \qquad k = 1, 2, 3\ldots$$

if $n = 2k$ aka n is even $b_n = 0$

$\therefore$ only reasonable sol is n is odd aka $n = 2k-1$

$$f(x) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi}\sin\left(\frac{(2k-1)\pi x}{\ell}\right)$$

note at discontinuities located at $(x=a)$

$$\bar{f}(x) = \frac{1}{2}\left[\lim_{x\to a^+} f(x) + \lim_{x\to a^-} f(x)\right]$$

Homework 8

Problem 2: Decompose the following into a Fourier series on the interval $[-\ell, \ell]$ and sketch the graph

1. x

$x = \tilde{f}(x)$

$\tilde{f}(x) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \right) \qquad n = 1, 2, 3 \ldots$

$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} x \cdot \cos\left(\frac{n\pi x}{\ell}\right) dx \quad \Rightarrow \text{odd} \cdot \text{even} = \text{odd}$

$\int_{-\ell}^{\ell} \text{odd} = 0 \quad \therefore a_n = 0$

$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} x \cdot \sin\left(\frac{n\pi x}{\ell}\right) dx \quad \Rightarrow \text{odd} \cdot \text{odd} = \text{even}$

$= \frac{2}{\ell} \int_{0}^{\ell} x \sin\left(\frac{n\pi x}{\ell}\right) dx \qquad u = x \quad dv = \sin\left(\frac{n\pi x}{\ell}\right) dx$

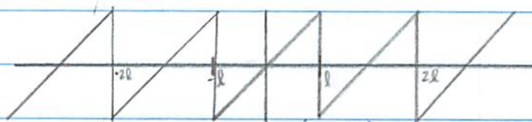
$du = dx \quad v = \frac{-\ell}{n\pi} \cos\left(\frac{n\pi x}{\ell}\right)$

$= \frac{2}{\ell} \left(\frac{-\ell x}{n\pi} \cos\left(\frac{n\pi x}{\ell}\right) \Big|_0^{\ell} - \int_0^{\ell} \frac{-\ell}{n\pi} \cos\left(\frac{n\pi x}{\ell}\right) dx \right)$

$= \frac{2}{\ell} \left(\frac{-\ell^2}{n\pi} \underbrace{\cos(n\pi)}_{-1 \text{ or } 1} - (0) + \frac{\ell^2}{n^2\pi^2} \left(\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin(0)}_{0} \right) \right)$

$= \frac{-2\ell}{n\pi} \left((-1)^n \right)$

$$\boxed{\tilde{f}(x) = \sum_{n=1}^{\infty} \frac{-2\ell}{n\pi} (-1)^n \sin\left(\frac{n\pi x}{\ell}\right)}$$



2. $|x| = \begin{cases} -x, & x < 0 \\ x & x > 0 \end{cases} \qquad \tilde{f}(x) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \right)$

$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} |x| \cos\left(\frac{n\pi x}{\ell}\right) dx \quad \Rightarrow \text{even} \cdot \text{even} = \text{even}$

$= \frac{2}{\ell} \int_0^{\ell} x \cos\left(\frac{n\pi x}{\ell}\right) dx \qquad u = x \quad dv = \cos\left(\frac{n\pi x}{\ell}\right) dx$

$du = dx \quad v = \frac{\ell}{n\pi} \sin\left(\frac{n\pi x}{\ell}\right)$

$= \frac{2}{\ell} \left(\frac{\ell x}{n\pi} \sin\left(\frac{n\pi x}{\ell}\right) \Big|_0^{\ell} - \int_0^{\ell} \frac{\ell}{n\pi} \sin\left(\frac{n\pi x}{\ell}\right) dx \right)$

$= \frac{2}{\ell} \left(0 - 0 + \frac{\ell^2}{n^2\pi^2} \cos\left(\frac{n\pi x}{\ell}\right) \Big|_0^{\ell} \right)$

$$= \frac{2\ell}{n^2\pi^2}\left(\underset{-1 \text{ or } 1}{\cos(n\pi)} - 1\right) = \frac{2\ell}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} |x| \sin\left(\frac{n\pi x}{\ell}\right)dx = 0 \quad \text{because even·odd = odd}$$

$$\tilde{f}(x) = \sum_{n=1}^{\infty} \frac{2\ell}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{\ell}\right) + \frac{1}{2}a_0$$

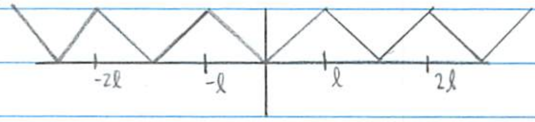
if n is even aka $n = 2k$, $a_n = 0$

so $n = 2k-1$ is only solutions

$$\tilde{f}(x) = \sum_{k=1}^{\infty}\left(\frac{-4\ell}{(2k-1)^2\pi^2}\cos\left(\frac{(2k-1)\pi x}{\ell}\right)\right) + \frac{1}{2}a_0$$

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} |x|\,dx = \frac{1}{\cancel{\ell}}\cdot \cancel{2}\left(\frac{1}{\cancel{2}}\ell\cdot\cancel{\ell}\right) = \ell$$

$$\boxed{\tilde{f}(x) = \frac{\ell}{2} + \sum_{k=1}^{\infty}\frac{-4\ell}{(2k-1)^2\pi^2}\cos\left(\frac{(2k-1)\pi x}{\ell}\right)}$$



3. x^2

$$x^2 = \tilde{f}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{\ell}\right) + b_n\sin\left(\frac{n\pi x}{\ell}\right)\right)$$

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} x^2\,dx \quad \text{even}$$

$$= \frac{2}{\ell}\int_0^{\ell} x^2\,dx = \frac{2}{\ell}\cdot\frac{1}{3}x^3\Big|_0^{\ell} = \frac{2\ell^2}{3}$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} x^2\cos\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{2}{\ell}\int_0^{\ell} x^2\cos\left(\frac{n\pi x}{\ell}\right)dx$$

even · even = even

		$\cos\left(\frac{n\pi x}{\ell}\right)$
$+$	x^2	$\frac{\ell}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right)$
$-$	$2x$	$\frac{-\ell^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{\ell}\right)$
$+$	2	$\frac{-\ell^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{\ell}\right)$

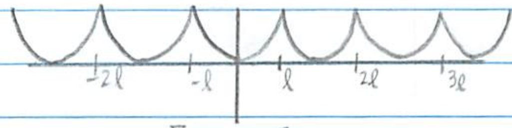
$$= \frac{2}{\ell}\left(\frac{\ell x^2}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right) + \frac{2\ell^2 x}{n^2\pi^2}\cos\left(\frac{n\pi x}{\ell}\right) - \frac{2\ell^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{\ell}\right)\right)\Bigg|_0^{\ell}$$

$$= \frac{2}{\ell}\left(0 + \frac{2\ell^3}{n^2\pi^2}\cos(n\pi) - 0 - (0 + 0 - 0)\right)$$

$$= \frac{4\ell^2}{n^2\pi^2}(-1)^n$$

$$b_n = \frac{1}{l}\int_{-l}^{l} x^2 \sin\left(\frac{n\pi x}{l}\right)dx = 0$$ because even·odd = odd

$$\boxed{F(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}(-1)^n \cos\left(\frac{n\pi x}{l}\right)}$$



problem 3: Decompose into a Fourier series on $[-\pi,\pi]$

1. $|\sin(x)|$ ← is even

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\,dx$$ even

$$= \frac{2}{\pi}\int_{0}^{\pi} |\sin(x)|\,dx$$ from $[0,\pi]$ $\sin(x)$ is all positive so $|\sin(x)| = \sin(x)$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{2}{\pi}\left(-\cos(x)\right)\Big|_0^{\pi} = \frac{2}{\pi}\left(-(-1)+1\right) = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)| \cos\left(\frac{n\pi x}{l}\right)dx$$ even · even = even

$$= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos\left(\frac{n\pi x}{l}\right)dx$$ $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\beta+\alpha) - \sin(\beta-\alpha)\right)$

$$= \frac{1}{\pi}\int_0^{\pi} \sin\left(\frac{(n\pi+l)x}{l}\right)dx - \frac{1}{\pi}\int_0^{\pi}\sin\left(\frac{(n\pi-l)x}{l}\right)dx$$

$$= \frac{1}{\pi}\left(\frac{-l}{n\pi+l}\cos\left(\frac{(n\pi+l)x}{l}\right) + \frac{l}{n\pi-l}\cos\left(\frac{(n\pi-l)x}{l}\right)\right)\Bigg|_0^{\pi}$$ *forgot l is π

$$= \frac{1}{\pi}\left(\frac{-1}{n+1}\cos((n+1)x) + \frac{1}{n-1}\cos((n-1)x)\right)\Bigg|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{-1}{n+1}\Big(\underset{-1 \text{ or } 1}{\cos(n\pi+\pi)} - \underset{1}{\cos(0)}\Big) + \frac{1}{n-1}\Big(\underset{-1 \text{ or } 1}{\cos(n\pi-\pi)} - \underset{1}{\cos(0)}\Big)\right)$$

$$= \frac{1}{\pi}\left(\frac{-1\cdot(-1)^{n-1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right) = \frac{1}{\pi}\left(\frac{(-1)^n}{n+1} + \frac{(-1)^{n-1}}{n-1}\right)$$

b_n = even · odd = odd ∴ $b_n = 0$

$$\boxed{F(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{1}{\pi}\left(\frac{(-1)^n}{n+1} + \frac{(-1)^{n-1}}{n-1}\right)\cos(nx)}$$

$$\boxed{= \frac{2}{\pi} + \sum_{k=1}^{\infty}\frac{2}{\pi}(-1)^k\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right)\cos\left(k(2x+\pi)\right)}$$

2. $|\cos(x)|$

$\cos(x) = 0$

$x = \cos^{-1}(0)$

$x = \frac{\pi}{2}, -\frac{\pi}{2}, \quad [-\pi, \pi]$

$$|\cos(x)| = \begin{cases} -\cos(x) & -\pi < x < -\pi/2 \\ \cos(x) & -\pi/2 < x < \pi/2 \\ -\cos(x) & \pi/2 < x < \pi \end{cases}$$

$^*\cos(x)$ is even

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos(x)|\,dx$$

$$= \frac{2}{\pi}\left(\int_0^{\pi/2} \cos(x)\,dx + \int_{\pi/2}^{\pi} -\cos(x)\,dx\right)$$

$$= \frac{2}{\pi}\left(1 - 0 - (0 - 1)\right) = \frac{2}{\pi}(1+1) = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos(x)|\cos(nx)\,dx$$

even

$^*\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$

$$= \frac{1}{\pi}\left(\int_0^{\pi/2} \cos((n+1)x) + \cos((n-1)x)\,dx - \int_{\pi/2}^{\pi} \cos((n+1)x) + \cos((n-1)x)\,dx\right)$$

$$= \frac{1}{\pi}\left(\left(\frac{\sin((n+1)x)}{n+1} + \frac{\sin((n-1)x)}{n-1}\right)\Bigg|_0^{\frac{\pi}{2}} - \left(\frac{\sin((n+1)x)}{n+1} + \frac{\sin((n-1)x)}{n-1}\right)\Bigg|_{\pi/2}^{\pi}\right)$$

$\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right) = 0, 1, -1 \qquad \sin(0) = 0 \qquad \sin(n\pi + \pi) = 0$

$\sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) = -1, 0, 1 \qquad \sin(n\pi - \pi) = 0$

$$= \frac{1}{\pi}\left(\frac{\sin((n+1)\frac{\pi}{2})}{n+1} + \frac{\sin((n-1)\pi/2)}{n-1} + \frac{\sin((n+1)\frac{\pi}{2})}{n+1} + \frac{\sin((n-1)\pi/2)}{n-1}\right)$$

$$= \frac{2}{\pi}\left(\frac{\sin(\frac{\pi}{2}(n+1))}{n+1} + \frac{\sin(\frac{\pi}{2}(n-1))}{n-1}\right)$$

if $n = 2k+1$, $\sin\left(\frac{2k\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{2k\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}\right) = 0$ $\quad \therefore n = 2k$

$$= \frac{2}{\pi}\left(\frac{(-1)^k}{2k+1} + \frac{(-1)^{k-1}}{2k-1}\right) = \frac{2}{\pi}(-1)^k\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right)$$

$$\boxed{\tilde{f}(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right)\cos(2kx)}$$

Parseval's ideniity: $\langle f(x), f(x) \rangle = \langle \tilde{f}(x), \tilde{f}(x) \rangle$

$$\int_{-l}^{l} f(x)^2 dx = \frac{l}{2} a_0^2 + l \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

from Hw question 1:

$$f(x) = x \quad a_0 = 0 \quad a_n = 0 \quad b_n = \frac{-2l(-1)^n}{n\pi}$$

$$\int_{-l}^{l} x^2 dx = l \sum_{n=1}^{\infty} \left(\frac{-2l(-1)^n}{n^2\pi^2} \right) = \frac{4l^3}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{x^3}{3} \Big|_{-l}^{l} = \frac{1}{3} \left(l^3 - (-l)^3 \right) = \frac{2l^3}{3}$$

$$\frac{2l^3}{3} = \frac{4l^3}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \therefore \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Hw question 2:

$$f(x) = |x| \quad a_0 = l \quad a_n = \frac{-4l}{(2k-1)^2\pi^2}$$

$$\int_{-l}^{l} |x|^2 dx = \int_{-l}^{l} x^2 dx = \frac{l^3}{2} + l \sum_{k=1}^{\infty} \frac{16l^2}{(2k-1)^4 \pi^4}$$

$$\frac{2l^3}{3} = \frac{l^3}{2} + \frac{16l^3}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}}$$

q3: $f(x) = x^2 \quad a_0 = \frac{2l^2}{3} \quad a_n = \frac{4l^2}{n^2\pi^2}(-1)^n$

$$\int_{-l}^{l} x^4 dx = 2\int_{0}^{l} x^4 dx = \frac{4l^5}{18} + l \sum_{n=1}^{\infty} \frac{16l^4}{n^4\pi^4}$$

$$\frac{2}{5} l^5 - \frac{4l^5}{18} = \frac{16l^5}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

q5.

$f(x) = |\cos(x)|$ $\qquad a_0 = \frac{4}{\pi}$ $\qquad a_n = \frac{2}{\pi}\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right)(-1)^k$

$$= \frac{2}{\pi}\left(\frac{2k-1}{4k^2-1^2} - \frac{2k+1}{4k^2-1^2}\right)(-1)^k$$

$$= \frac{2}{\pi}\left(\frac{-2}{4k^2-1}\right)(-1)^k$$

$$\int_{-\pi}^{\pi} \cos^2(x)\,dx = 2\int_0^{\pi} \cos^2(x)\,dx \qquad * \cos^2(x) = \tfrac{1}{2}(1+\cos(2x))$$

$$= \int_0^{\pi} 1\,dx + \int_0^{\pi} \cos(2x)\,dx = \pi + \frac{\sin(2x)}{2}\Big|_0^{\pi} = \pi$$

$$\pi = \frac{\cancel{\pi}}{\cancel{2}} \cdot \frac{\cancel{16}^{8}}{\pi^2} + \cancel{\pi}\sum_{k=1}^{\infty} \frac{16}{\pi^2}\left(\frac{1}{(4k^2-1)^2}\right)$$

$$\pi = \frac{8}{\pi} + \frac{16}{\pi}\sum \frac{1}{(4k^2-1)^2}$$

$$\frac{\left(\pi - \frac{8}{\pi}\right)\pi}{16} = \boxed{\frac{\pi^2-8}{16} = \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^2}}$$

3. solve
$$\begin{cases} u_{tt} - u_{xx} = 0 & -\pi < x < \pi \\ u(x,0) = |\sin(x)| & -\pi < x < \pi \\ u_t(x,0) = |\cos(x)| & -\pi < x < \pi \\ u(-\pi,t) = u(\pi,t) \\ u_x(-\pi,t) = u_x(\pi,t) \end{cases}$$

$u = X(x)T(t)$

$$X(x)T''(t) - c^2X''(x)T(t) = 0$$

$$X(x)T''(t) = c^2X''(x)T(t)$$

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$X''(x) + \lambda X(x) = 0 \qquad T''(t) + c^2\lambda T(t) = 0$$

$$u(-\pi,t) = X(-\pi)\cancel{T(t)} = X(\pi)\cancel{T(t)} = u(\pi,t)$$

$$X(-\pi) = X(\pi)$$

$$X'(-\pi) = X'(\pi)$$

Eigenvalue problem: $\begin{cases} X''(x) + \lambda X(x) = 0 & -\pi < 0 < \pi \\ X(-\pi) = X(\pi) \\ X'(-\pi) = X(\pi) \end{cases}$ $\quad l = \pi$

$X_n(x) = A_n \cos(nx) + B_n \sin(nx) \qquad \lambda_n = n^2$

$T''(t) - c^2 \lambda_n T(t) = 0$

$T_n(t) = C_n \cos(n\sqrt{c}\, t) + D_n \sin(n\sqrt{c}\, t)$

$u(x,t) = (A_n \cos(nx) + B_n \sin(nx))(C_n \cos(n\sqrt{c}\, t) + D_n \sin(n\sqrt{c}\, t))$

$u(x,t) = A_n \cos(nx)\cos(n\sqrt{c}\, t) + B_n \cos(nx)\sin(n\sqrt{c}\, t)$
$\quad + C_n \sin(nx)\cos(n\sqrt{c}\, t) + D_n \sin(nx)\sin(n\sqrt{c}\, t)$ ☆new constants☆

$u(x,0) = A_n \cos(nx) + C_n \sin(nx) = |\sin(x)|$

$A_n = \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^k \left(\frac{-2}{4k^2-1}\right), \qquad C_n = 0$

$u_t(x,t) = -n\sqrt{c} A_n \cos(nx)\sin(n\sqrt{c}\, t) + n\sqrt{c} B_n \cos(nx)\cos(n\sqrt{c}\, t) + n\sqrt{c}\, t D_n \sin(nx)\cos(n\sqrt{c}\, t)$

$u_t(x,0) = n\sqrt{c} B_n \cos(nx) + n\sqrt{c} D_n \sin(nx) = |\cos(x)|$

$B_n = \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^k \left(\frac{-4\sqrt{c}}{4k^2-1}\right)$

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \frac{2(-1)^k}{\pi} \left(\frac{-2\cos(nx)\cos(n\sqrt{c}\, t) - 4\sqrt{c}\cos(nx)\sin(n\sqrt{c}\, t)}{4k^2-1}\right)}$$

Fourier series for other orthoginal systems.

1. "DD" $\left\{\sin\left(\frac{n\pi x}{l}\right),\ n=1,2,3\ldots\right\}$ $(0<x<l)$

2. "NN" $\left\{\cos\left(\frac{n\pi x}{l}\right),\ n=0,1,2,3\ldots\right\}$ $(0<x<l)$

3. "DN" $\left\{\sin\left(\frac{(2n+1)\pi x}{2}\right),\ n=0,1,2\ldots\right\}$ $(0<x<l)$

4. "ND" $\left\{\cos\left(\frac{(2n+1)\pi x}{2}\right),\ n=0,1,2\ldots\right\}$ $(0<x<l)$

Fourier series theory

i. orthoginal system check

ii. Find coefficents

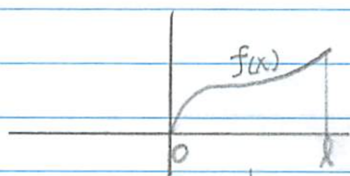
iii. conpleteness and Parseval's identy

note: interval $x_0<x<x_1$ must be $x_1-x_0=2l$

1. i. $\left\langle \sin\left(\frac{n\pi x}{l}\right), \sin\left(\frac{m\pi x}{l}\right)\right\rangle_{0<x<l} = \begin{cases} l/2, & n=m \\ 0, & n\neq m \end{cases}$

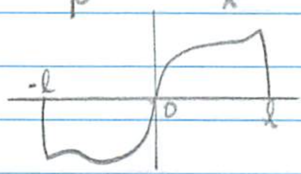
ii. $\tilde{f}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

we only have



but we can odd extension:

$$f^{odd}(x) = \begin{cases} f(x), & 0<x<l \\ -f(-x), & -l<x<0 \end{cases}$$



We can find series of f^{odd}

$$f^{odd}(x) = \tilde{f}^{odd}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f^{odd}(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f^{odd}(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

this reduces, for case 1, to

$a_n = 0$ because odd · odd

$$b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

on $0<x<\ell$, $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/\ell)$

check parsevel on full range, $-\ell<x<\ell$,
should be even and work out to 2 $0<x<\ell$

2. "NN"

i. use even extension

$$0<x<\ell \quad \left\langle \cos\left(\frac{n\pi x}{\ell}\right), \cos\left(\frac{m\pi x}{\ell}\right)\right\rangle = \begin{cases} \ell/2 & n=m \\ 0 & n\neq m \end{cases}$$

ii. $f(x) = \frac{1}{2}a_0 + \sum a_n \cos\left(\frac{n\pi x}{\ell}\right)$

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$$

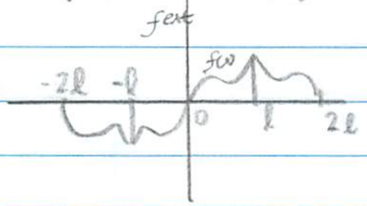
3. "DN"

i. $$\left\langle \sin\left(\frac{2n+1}{2\ell}\pi x\right), \sin\left(\frac{2m+1}{2\ell}\pi x\right)\right\rangle_{\text{exend } -2\ell<x<2\ell} = \begin{cases} 2\ell & n=m \\ 0 & n\neq m \end{cases}$$

using extensions

$$\left\langle \sin\left(\frac{2n+1}{2\ell}\pi x\right), \sin\left(\frac{2m+1}{2\ell}\pi x\right)\right\rangle_{0<x<\ell} = \begin{cases} \ell/2, & n=m \\ 0, & n\neq m \end{cases}$$

ii. odd extension at D, 0 even at N, ℓ



$$f^{ext}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{2\ell}\right) + b_n \sin\left(\frac{n\pi x}{2\ell}\right)\right)$$

$a_n = 0$

$$b_n = \begin{cases} 0, & 2k \\ \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{2k+1}{2\ell}\pi x\right)dx, & 2k+1 \end{cases}$$

$$f(x) = \sum_{k=0}^{\infty} b_k \sin\left(\frac{2k+1}{2\ell}\pi x\right)$$

4. "ND"

$$\left\langle \qquad \right\rangle = \begin{cases} \ell/2, & n=m \\ 0, & n\neq m \end{cases}$$

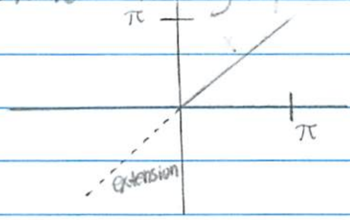
$$\tilde{f}(x) = \sum_{n=0}^{\infty} a_{2n+1} \cos\left(\frac{(2n+1)\pi x}{2\ell}\right)$$

$$a_{2n+1} = \frac{2}{\ell}\int_0^{\ell} f(x) \cos\left(\frac{(2n+1)\pi x}{2\ell}\right)$$

HW 9

1. Problem 4: Decompose into sin-Fourier series on interval $[0,\pi]$ and sketch the graph.

2. $f(x) = x$



odd extension:

$$[-\pi,\pi]: f_{odd}(x) = \begin{cases} x, & 0<x<\pi \\ -(-x), & -\pi<x<0 \end{cases} = x$$

$$f_{odd}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(x)\cos(nx)\,dx \Rightarrow \text{odd}\cdot\text{even} = \text{odd}$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(x)\sin(nx)\,dx = \text{odd}\cdot\text{odd} = \text{even}$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

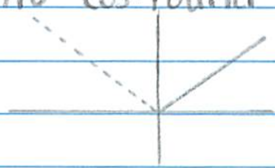
$$= \frac{2}{n^2\pi}\left(\sin(nx) - nx\cos(nx)\right)\Big|_{x=0}^{x=\pi} \qquad 0 - n\pi - 0 + 0$$

$$= -\frac{2}{n}(-1)^n$$

$$\boxed{\tilde{f}(x) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n\sin(nx)}$$

Problem 5: Decompose into cos-Fourier series on $[0,\pi]$

2. $f(x) = x$



even extension:

$$f_{even}(x) = \begin{cases} x, & 0<x<\pi \\ (-x), & -\pi<x<0 \end{cases}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{even}(x)\cos(nx)\,dx \Rightarrow \text{even}\cdot\text{even} = \text{even}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(n\pi)\,dx$$

$$= \frac{2}{n^2\pi}\left(nx\sin(nx) + \cos(nx)\right)\Big|_{x=0}^{x=\pi} \qquad 0 + (-1)^n - 0 - 1$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right) \qquad * \text{ if } (-1)^n = 1 \text{ then } a_n = 0$$

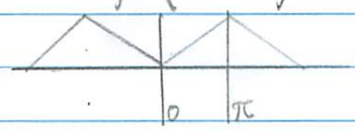
$n = 2k$

only valid solution is $n = 2k-1$

$$a_n = \frac{2}{(2k-1)^2\pi}(-2) = \frac{-4}{(2k-1)^2\pi}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{even}(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left(\pi^2 - 0\right) = \pi$$

$$\boxed{\tilde{f}(x) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{-4}{(2k-1)^2\pi}\cos((2k-1)x)}$$



problem 6: Decompose to a Fourier series with the set $\left\{\sin\left(\left(n+\frac{1}{2}\right)x\right),\ n = 0,1,2\ldots\right\}$ on $[0,\pi]$

compare to the set

$\left\{\sin\left(\frac{n\pi x}{l}\right),\ n = 0,1,2\ldots\right\}$ on $[-\pi,\pi]$ $\quad l = \pi$

$\left\{\sin\left(\frac{(2n+1)x}{2}\right),\ n = 0,1,2\ldots\right\}$ on $[-2\pi, 2\pi]$ $\quad l = 2\pi$

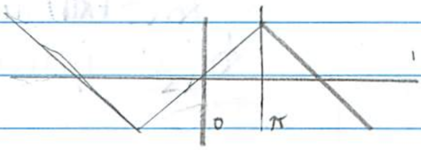
because $= \frac{2}{2}\left(n+\frac{1}{2}\right)$

$$\text{now, } b_n = \frac{2}{\pi}\int_0^{\pi} x\sin\left(\frac{(2n+1)x}{2}\right)dx$$

$$= \frac{2}{\pi}\left(\frac{4\sin\left(nx+\frac{x}{2}\right)}{(2n+1)^2} - \frac{2x\cos\left(nx+\frac{x}{2}\right)}{2n+1}\right)\Bigg|_{x=0}^{x=\pi}$$

$$= \frac{2}{\pi}\left(\frac{4\,(-1)^n}{(2n+1)^2}\right)$$

$$\boxed{\tilde{f}(x) = \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{4\cdot(-1)^n}{(2n+1)^2}\sin\left(\left(n+\frac{1}{2}\right)x\right)}$$



2. Write Parseval's identity for the above.

Problem 4:

$$\int_0^{\pi} x^2 dx = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{x^3}{3}\Big/_0^{\pi} = \frac{\pi^3}{3} =$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

problem 5:

$$\int_0^{\pi} x^2 dx = \frac{\pi}{4}(\pi^2) + \sum_{n=1}^{\infty} \frac{16}{(2k-1)^4\pi^2} \cdot \frac{\pi}{2}$$

$$\left(\frac{\pi^3}{3} - \frac{\pi^3}{4}\right)\left(\frac{\pi}{8}\right) = \sum_{n=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}$$

Problem 6:

$$\int_0^{l} x^2 dx = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{4}{\pi^2} \cdot \frac{16}{(2n+1)^4}\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

3. consider the system: $\left\{\cos\left(\frac{(2n+1)\pi x}{2l}\right), n = 0, 1, 2, \ldots\right\}$ on $0 < x < l$

$$\left\langle \cos\left(\frac{(2n+1)\pi x}{2l}\right), \cos\left(\frac{(2m+1)\pi x}{2l}\right)\right\rangle$$

We know

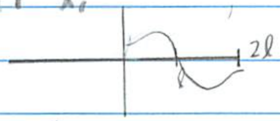
$$\left\langle \cos\left(\frac{n\pi x}{l}\right), \cos\left(\frac{m\pi x}{l}\right)\right\rangle \text{ on } -l < x < l = \begin{cases} l, & n = m \\ 0, & n \neq m \end{cases}$$

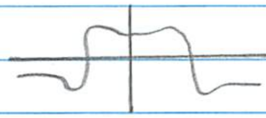
transform to $\left\langle \cos\left(\frac{n\pi x}{2l}\right), \cos\left(\frac{m\pi x}{2l}\right)\right\}$ on $-2l < x < 2l = \begin{cases} 2l \\ 0 \end{cases}$

on a quatar of that interval:

our system is $\boxed{\begin{cases} l/2, & m = n \\ 0, & m \neq n \end{cases}}$

ii. given $f(x)$ on $0<x<l$. $f_{odd}(x)$ is an odd extension at l.

$f_{odd}(X) =$ 

$f_{ext}(X) =$ 

$$\bar{f}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{2l}\right)$$

$$a_n = \frac{1}{2l}\int_{-2l}^{2l} f_{ext}(x)\cos\left(\frac{\pi n x}{2l}\right)dx$$

about 0

even · even = even

$$= \frac{2}{2l}\int_0^{2l} f_{odd}(x)\cos\left(\frac{\pi n x}{2l}\right)dx$$

about l

odd · odd = even

$$= \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{\pi x}{2l}(2n+1)\right)dx$$

iii. $\cos\left(\frac{(2m+1)\pi x}{2l}\right) \cdot \cos\left(\frac{(2m+1)\pi x}{2l}\right) = 1$ since $m=m$

$$\therefore \int_0^{l} f(x)^2\,dx = \frac{l}{2}\sum_{n=0}^{\infty} a_n^2$$

iv. $f(x) = x$

$$a_n = \frac{2}{l}\int_0^{l} x\cos\left(\frac{(2n+1)\pi x}{2l}\right)dx$$

$$= 4\left(\frac{x\sin\left(\frac{(2n+1)\pi x}{2l}\right)}{\pi(2n+1)} + \frac{2l\cos\left(\frac{(2n+1)\pi x}{2l}\right)}{\pi^2(2n+1)^2}\right)\Bigg|_0^{l}$$

$*\ \frac{n\pi x}{l} + \frac{\pi x}{2l}$

$$= 4\left(\frac{(-1)^n l}{\pi(2n+1)} + \frac{2l}{\pi^2(2n+1)^2}\right)$$

Chapter 6: Seperation of variables

6.1: For heat equation

$$\begin{cases} u_t - k u_{xx} = 0 & 0<x<\ell, \quad t>0 \\ u(x,0) = g(x) = 0 & 0<x<\ell \\ u(0,t) = 0 = u(\ell,t) & t>0 \end{cases}$$

say: $u(x,t) = X(x)\,T(t)$

then: $u_t = X(x)\,T'(t)$, $\quad u_{xx} = X''(x)\,T(t)$

$$X(x)\,T'(t) - k\,X''(x)\,T(t) = 0$$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{k\,T(t)} = -\lambda$$

since one is (x) and other is (t) they must equal a constant

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X(\ell) = 0 \end{cases} \qquad T_n' + \lambda_n k T_n = 0$$

$$\frac{dT_n}{dt} = -\lambda_n k T_n$$

$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$, $n = 1, 2, \ldots$ $\qquad \int \frac{1}{T_n}\,dT_n = \int -\lambda_n k\,dt$

$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right)$ $\qquad T_n = C_n e^{-\lambda_n k t}$

$$u_n(x,t) = C_n e^{-\lambda_n k t} \sin\left(\frac{n\pi x}{\ell}\right), \qquad \lambda_n = (n\pi/\ell)^2$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n k t} \sin\left(\frac{n\pi x}{\ell}\right)$$

remember, you can only sum u_n because equation is homogenous, and B.C.

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{\ell}\right), \quad 0<x<\ell \qquad {}^*e^0 = 1$$

$= g(x)$

$$C_n = \frac{2}{\ell}\int_0^{\ell} g(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

example: $g(x) = 1$, $0 < x < 1$

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = g(x) \\ u(0,t) = 0 = u(1,t) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

$$C_n = 2\int_0^1 \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{-2((-1)^n - 1)}{n\pi}$$

$$\boxed{u(x,t) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} e^{-(2m\pi+\pi)^2 t} \sin((2m+1)\pi x)}$$

Stabalizing to the stationary solution:

$$\begin{cases} u_t - k u_{xx} = f(x) & \leftarrow \text{independent of } t \\ u(x,0) = g(x) \\ u(0,t) = \varphi \\ u(l,t) = \gamma \end{cases} \Big\} \text{ constants}$$

Stationary solution $v(x)$ (independent of t)

$$\begin{cases} -kv''(x) = f(x) & 0 < x < l \\ v(0) = \varphi \\ v(l) = \gamma \end{cases}$$

say: $w(x,t) = u(x,t) - v(x)$

$$= \begin{cases} u_t - k u_{xx} + k v''(x) = 0 \\ u(x,0) - v(x) = g(x) - v(x) \\ u(0,t) - v(0) = 0 \\ u(l,t) - v(l) = 0 \end{cases}$$

$$u(x,t) = w(x,t) + v(x)$$

solving $v(x)$:

$$v''(x) = \frac{-1}{k} f(x)$$

$$v'(x) = \frac{-1}{k}\int_0^x f(x'')\,dx'' + C$$

$$v(x) = \frac{-1}{k}\int_0^x\int_0^{x'} f(x'')\,dx''\,dx' + Cx + D$$

$$v(0) = D = \varphi$$

$$v(l) = \frac{-1}{k}\int_0^l\int_0^{x'} f(x'')\,dx''\,dx' + C\cdot l \cdot \varphi = \gamma$$

$$C = \frac{1}{l}\left(\gamma - \varphi + \frac{1}{k}\int_0^l\int_0^{x'} f(x'')\,dx''\,dx'\right.$$

$$W(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 kt}$$

$$C_n = \frac{2}{l}\int_0^l (g(x) - v(x)) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$u(x,t) = w(x,t) + v(x)$$

Stabalizing is what happens as $t \to \infty$

$$\lim_{t\to\infty} u(x,t) = 0 + v(x)$$

because $e^{-\infty}$ in $w(x,t) = 0$ at ∞

similarly $e^{-\infty} = 0$ as $n \to \infty$

therefore dominante term is @ $n=1$

1D wave equation with Newman conditions

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \ell \quad t > 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \\ u_x(0,t) = 0 \\ u_x(\ell,t) = 0 \end{cases}$$

$$u(x,t) = X(x)\,T(t)$$

$$X(x)T''(t) - c^2 X''(x)T(t) = 0$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = 0 = X'(\ell) \end{cases}$$

$$T''(t) + \lambda c^2 T(t) = 0$$

$$z^2 + \lambda_n c^2 = 0$$

$$z = \pm \frac{n\pi c}{\ell} i$$

$n = 0, \quad \lambda_0 = 0, \quad X_0(x) = \frac{1}{2}$

$n \geq 0, \quad \lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \quad X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right)$

$$T_n(t) = A_n \cos\left(\frac{n\pi c t}{\ell}\right) + B_n \sin\left(\frac{n\pi c t}{\ell}\right)$$

$$T_0(t) = A_0 + B_0 t$$

$$u_n(x,t) = \begin{cases} (A_0 + B_0 t)\cdot \frac{1}{2}, & n = 0 \\ \left(A_n \cos\left(\frac{n\pi c t}{\ell}\right) + B_n \sin\left(\frac{n\pi c t}{\ell}\right)\right)\cos\left(\frac{n\pi x}{\ell}\right), & n > 0 \end{cases}$$

$$u = \sum_{n=0}^{\infty} u_n(x,t)$$

$$u(x,t) = \frac{1}{2}\left(A_0 + B_0 t\right) + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi c t}{\ell}\right) + B_n \sin\left(\frac{n\pi c t}{\ell}\right)\right)\cos\left(\frac{n\pi x}{\ell}\right)$$

$$u_t(x,t) = \frac{B_0}{2} + \sum_{n=1}^{\infty}\left(-A_n \frac{n\pi c}{\ell}\sin\left(\frac{n\pi c t}{\ell}\right) + B_n \frac{n\pi c}{\ell}\cos\left(\frac{n\pi c t}{\ell}\right)\right)\cos\left(\frac{n\pi x}{\ell}\right)$$

$$u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) = g(x)$$

$$u_t(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{\ell}\cos\left(\frac{n\pi x}{\ell}\right) = h(x)$$

$$A_n = \frac{2}{\ell}\int_0^\ell g(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$$

$$B_n \frac{n\pi c}{\ell} = \frac{2}{\ell}\int_0^\ell h(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \qquad B_0 = \frac{2}{\ell}\int_0^\ell h(x)\,dx$$

Laplace equation in half-strip:

2D Laplace equation $u(x,y)$ $(x,y) \in \mathbb{R}^2$

$u_{xx} + u_{yy} = 0$ inhomogenous works too

Δ is laplace operator

$\Delta u = u_{xx} + u_{yy} = 0$

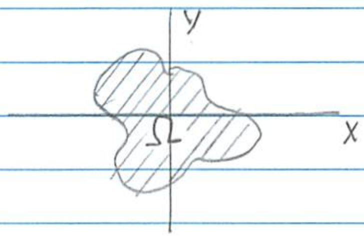
examples of solution:

any linear function: $u(x,y) = ax + by$

or: $u(x,y) = a(x^2 - y^2)$ is a solution too

$u(x,y) = axy$

Boundry conditions:

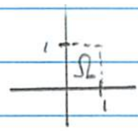


Domain Ω

boundry of domain is: $\partial\Omega$

B.V.P: $\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = g \end{cases}$

ex: $\Omega = [0,1]^2$



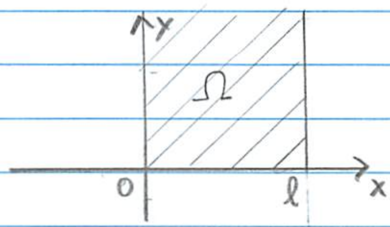
$$\begin{cases} \Delta u = 0 \\ u(0,y) = g(y), & 0 \le y \le 1 \\ u(1,y) = h(y), & 0 \le y \le 1 \\ u(x,0) = \varphi(x), & 0 \le x \le 1 \\ u(x,1) = \psi(x), & 0 \le x \le 1 \end{cases}$$

Dirichlet boundry

if Ω is the half-strip:

$[0,\ell] \times [0,\infty)$

$0 < x < \ell, \quad y > 0$



B.C. $\begin{cases} \Delta u = 0 \\ u(x,0) = g(x), & 0 < x < \ell \\ u(0,y) = 0, & y > 0 \\ u_x(\ell,y) = 0, & y > 0 \end{cases}$

when $\Delta u = 0$ we say it is a harmonic function

can be mixes of D and N

lets also assume: $u(x,t)$ is bounded on Ω

$|u(x,t)| \le M, \quad (x,y) \in \Omega$

Bound

seperation of varibles: $u(x,y) = X(x)\,Y(y)$

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

$$X''(x) + \lambda X(x) = 0 \qquad\qquad Y''(y) - \lambda Y(y) = 0$$

$u(0,y) = X(0)\,Y(y) \Rightarrow X(0) = 0$

don't want trivial

$u_x(l,y) = X'(l)\,Y(y) \Rightarrow X'(l) = 0$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X'(l) = 0 \end{cases} \qquad \lambda_n = \left(\frac{(2n+1)\pi}{2l}\right)^2$$

$$X_n(x)\ \sin\left(\frac{(2n+1)\pi x}{2l}\right) \quad n = 0,1,2\ldots$$

always positive

$$Y_n''(y) - \lambda_n Y(y) = 0$$

$$z^2 - \lambda_n = 0, \quad z = \pm\sqrt{\lambda_n}$$

$$Y_n(y) = C_n e^{\sqrt{\lambda_n}\,y} + D_n e^{-\sqrt{\lambda_n}\,y}$$

now ensure $u \not\to \infty$ as $y \to \infty$

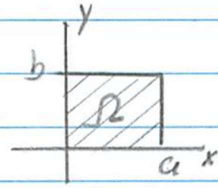
We need to say $C_n = 0$ to get rid of exponential

$$u(x,y) = \sum_{n=1}^{\infty} D_n e^{-\sqrt{\lambda_n}\,y} \sin\left(\frac{(2n+1)\pi}{2l}x\right)$$

$$u(x,0) = \sum_{n=0}^{\infty} D_n \sin\left(\frac{(2n+1)\pi x}{2l}\right) = g(x)$$

$$D_n = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{(2n+1)\pi x}{2l}\right)dx$$

Laplace equation in rectangle:



$$\begin{cases} u_{xx}+u_{xy}=0 \\ u(x,0)=g(x) & 0<x<a \\ u(x,b)=h(x) & 0<x<a \\ u_x(0,y)=0 & 0<y<b \\ u(a,y)=0 & 0<y<b \end{cases} \qquad \Omega=(0,a)\times(0,b)$$

Use any combo of D and N

We get:

$$\begin{cases} X''+\lambda X=0 & 0<x<a \\ X'(0)=0 \\ X(a)=0 \end{cases} \qquad Y''-\lambda Y=0$$

$$\lambda_n=\left(\frac{(2n+1)\pi}{2\ell}\right)^2 \qquad X_n(x)=\cos\left(\frac{(2n+1)\pi x}{2\ell}\right)$$

$$Y_n(y)=C_n e^{\sqrt{\lambda_n}\,y}+D_n e^{-\sqrt{\lambda_n}\,y}$$

Don't assume M boundness

$$u(x,y)=\sum_{n=0}^{\infty}\left(C_n e^{\sqrt{\lambda_n}\,y}+D_n e^{-\sqrt{\lambda_n}\,y}\right)\cos\left(\frac{(2n+1)\pi x}{2\ell}\right)$$

$$u(x,0)=\sum_{n=0}^{\infty}(C_n+D_n)\cos\left(\frac{(2n+1)\pi x}{2\ell}\right)=g(x)$$

$$u(x,b)=\sum_{n=0}^{\infty}\left(C_n e^{b\sqrt{\lambda_n}}+D_n e^{-\sqrt{\lambda_n}\cdot b}\right)\cos\left(\frac{(2n+1)\pi x}{2\ell}\right)=h(x)$$

$$C_n+D_n=\frac{2}{\ell}\int_0^{\ell} g(x)\cos\left(\frac{(2n+1)\pi x}{2\ell}\right)dx$$

$$C_n e^{b\sqrt{\lambda_n}}+D_n e^{-b\sqrt{\lambda_n}}=\frac{2}{\ell}\int_0^{\ell} h(x)\cos\left(\frac{(2n+1)\pi x}{2\ell}\right)dx$$

Hw 10

1. solve then find $\lim_{t\to\infty} U(x,t)$

$$\begin{cases} U_t - 2U_{xx} = 1, & 0<x<1, \ t>0 \\ U(x,0) = x, & 0<x<1 \\ U(0,t) = 0, & t>0 \\ U(1,t) = 1, & t>0 \end{cases}$$

Stationary solution

$$\begin{cases} -2v''(x) = 1 \\ v(0) = 0 \\ v(1) = 1 \end{cases}$$

$$v''(x) = -1/2$$

$$v'(x) = \frac{-x}{2} + C$$

$$v(x) = \frac{-x^2}{4} + Cx + D$$

$$v(0) = 0 = D \qquad v(1) = \frac{-1}{4} + C = 1 \ \therefore \ C = \frac{5}{4}$$

$$v(x) = \frac{-x^2}{4} + \frac{5x}{4}$$

$$W(x,t) = U(x,t) - v(x)$$

$$W_t(x,t) = U_t \qquad W_{xx}(x,t) = U_{xx} + \frac{1}{2}$$

$$W_t - 2W_{xx} = U_t - (2U_{xx} + 1) = 1$$

$$W_t - 2W_{xx} = 0$$

$$W(x,0) = U(x,0) - v(x) = x + \frac{x^2}{4} - \frac{5x}{4}$$

$$w(0,t) = U(0,t) - v(0) = 0$$

$$w(1,t) = U(1,t) - v(1) = 1 - 1 = 0$$

now we solve:

$$\begin{cases} W_t - 2W_{xx} = 0 \\ W(x,0) = x^2/4 - x/4 \\ W(0,t) = 0 \\ W(1,t) = 0 \end{cases}$$

say $w(x,t) = X(x)\,T(t)$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X(1) = 0 \end{cases}$$

then: $\lambda_n = (n\pi)^2 \quad n = 0,1,2,3$

$X_n = \sin(n\pi x)$

$T'(t) + \lambda 2 T(t) = 0$

$T_n(t) = C_n e^{-(n\pi)^2 \cdot 2t}$

$$w(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \exp(-2n^2\pi^2 t)$$

$$w(x,0) = \frac{x^2}{4} - \frac{x}{4} = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

$$C_n = \frac{2}{4}\left(\int_0^1 x^2 \sin(n\pi x)\,dx - \int_0^1 x \sin(n\pi x)\,dx\right)$$

+	x^2	$\sin(n\pi x)$		+	x	$\sin(n\pi x)$
−	$2x$	$\frac{-1}{n\pi}\cos(n\pi x)$		−	1	$\frac{-1}{n\pi}\cos(n\pi x)$
+	2	$\frac{-1}{n^2\pi^2}\sin(n\pi x)$		+	0	$\frac{-1}{n^2\pi^2}\sin(n\pi x)$
−	0	$\frac{1}{n^3\pi^3}\cos(n\pi x)$				

$$C_n = \frac{1}{2}\left(\frac{-x^2}{n\pi}\cos(n\pi x) + \frac{2x}{n^2\pi^2}\sin(n\pi x) + \frac{2}{n^3\pi^3}\cos(n\pi x) + \frac{x}{n\pi}\cos(n\pi x) - \frac{1}{n^2\pi^2}\sin(n\pi x)\right)\Bigg|_{x=0}^{x=1}$$

$\cos(n\pi) = (-1)^n$

$\sin(n\pi) = 0$

$$= \frac{1}{\cancel{2}}\left(\cancel{\frac{-(-1)^n}{n\pi}} + \frac{\cancel{2}(-1)^n}{n^3\pi^3} + \cancel{\frac{(-1)^n}{n\pi}} - \frac{\cancel{2}}{n^3\pi^3}\right) = \frac{(-1)^n - 1}{n^3\pi^3}$$

if $n = 2x$

if $= 0$

$$= \frac{-2}{\pi^3(2k-1)^3}$$

$$W(x,t) = \sum_{k=1}^{\infty}\left(\frac{-2}{\pi^3(2k-1)^3}\right)\sin((2k-1)\pi x)\exp(-2\pi^2(2k-1)^2 t)$$

$u(x,t) = w(x,t) + v(x)$

$$\boxed{u(x,t) = \frac{-x^2}{4} + \frac{5x}{4} - \sum_{k=1}^{\infty}\left(\frac{2}{\pi^3(2k-1)^3}\right)\sin(\pi(2k-1)x)\exp(-2\pi^2(2k-1)^2 t)}$$

$$\boxed{\lim_{t\to\infty} u(x,t) = \frac{-x^2}{4} + \frac{5x}{4}}$$

2. solve:
$$\begin{cases} u_{tt} - 4u_{xx} = 0, & 0<x<1 \\ u(x,0) = -2020\sin\left(\frac{\pi x}{2}\right) + \sin\left(\frac{3435\pi x}{2}\right), & 0<x<1 \\ u_t(x,0) = \sin\left(\frac{2021\pi x}{2}\right), & 0<x<1 \\ u(0,t) = 0, \\ u_x(1,t) = 0, \end{cases}$$

characteristic:
$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X'(1) = 0 \end{cases} \qquad T''(t) + \lambda_n c^2 T(t) = 0$$

$$\lambda_n = \left(\frac{2n+1}{2} \cdot \pi\right)^2 \qquad X_n(x) = \sin\left(\frac{2n+1}{2} \cdot \pi x\right)$$

$$T_n(t) = C_n \cos\left(\frac{2n+1}{2} \cdot 2\pi t\right) + D_n \sin\left(\frac{2n+1}{2} \cdot 2\pi t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\left(\pi(2n+1)t\right) + D_n \sin\left(\pi(2n+1)t\right)\right) \sin\left(\frac{2n+1}{2} \cdot \pi x\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{2n+1}{2} \cdot \pi x\right) = -2020\sin\left(\frac{\pi x}{2}\right) + \sin\left(\frac{3435\pi x}{2}\right)$$

$$C_n = 2\left(-2020\int_0^1 \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{2n+1}{2} \cdot \pi x\right)dx + \int_0^1 \sin\left(\frac{3435\pi x}{2}\right)\sin\left(\frac{2n+1}{2} \cdot \pi x\right)dx\right)$$

$$C_n = -2020\int_0^1 \cos\left(\frac{2n\pi x}{2}\right) - \cos\left(\frac{2n+2}{2} \cdot \pi x\right)dx + \int_0^1 \cos\left(\frac{2n-3434}{2} \cdot \pi x\right) - \cos\left(\frac{2n+3436}{2} \cdot \pi x\right)dx$$

$$= -2020\left(\frac{\sin(n\pi x)}{n\pi} - \frac{\sin(\pi(n+1)x)}{(n+1)\pi}\right)\Bigg|_{x=0}^{x=1} + \left(\frac{\sin(\pi(n-1717)x)}{(n-1717)\pi} - \frac{\sin(\pi(n+1718)x)}{(n+1718)\pi}\right)\Bigg|_0^1$$

$$= -2020((0-0)-(0-0)) + (0-0-(0-0)) = 0$$

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(\frac{D_n \cos(\pi(n+1)t)}{\pi(n+1)}\right)\sin\left(\frac{2n+1}{2} \cdot \pi x\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{D_n}{\pi(n+1)} \sin\left(\frac{2n+1}{2} \cdot \pi x\right)$$

$$\frac{D_n}{\pi(n+1)} = 2\int_0^1 \sin\left(\frac{2021\pi x}{2}\right)\sin\left(\frac{2n+1}{2} \cdot \pi x\right)dx$$
$$= 0$$

$\boxed{u(x,t) = 0}$?

3. solve: $\begin{cases} u_{xx}+u_{yy}=0 & 0<x<\pi,\ y>0 \\ u(x,0)=e^x & 0<x<\pi \\ u_x(0,y)=0 & y>0 \\ u_x(\pi,y)=0 & y>0 \\ |u(x,y)|\le M \end{cases}$

$X''(x)+\lambda_n X(x)=0$ $\qquad$ $Y''(y)-\lambda Y(y)=0$

$\lambda_n = n^2 \qquad X_n(x)=\cos(nx)$ $\qquad$ $Y_n(y)=D_n e^{-ny}$

$$u(x,y)=\sum_{n=0}^{\infty} D_n e^{-ny}\cos(nx)$$

$$u(x,0)=e^x=\sum_{n=0}^{\infty} D_n\cos(nx)$$

$$D_n=\frac{2}{\pi}\int_0^{\pi} e^x\cos(nx)\,dx$$

$+\ -\ e^x \qquad \cos(nx)$

$-\ -\ e^x \qquad \frac{1}{n}\sin(nx)$

$+\ -\ e^x - \int \frac{-1}{n^2}\cos(nx)$

$$\int e^x\cos(nx)\,dx = \frac{e^x}{n}\sin(nx)+\frac{e^x}{n^2}\cos(nx)-\frac{1}{n^2}\int e^x\cos(nx)\,dx$$

$$n^2\int e^x\cos(nx)\,dx+\int e^x\cos(nx)\,dx = ne^x\sin(nx)+e^x\cos(nx)$$

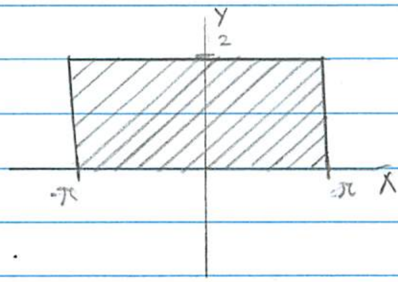
$$\int e^x\cos(nx)\,dx = \frac{ne^x\sin(nx)+e^x\cos(nx)}{n^2+1}\Bigg|_{x=0}^{x=\pi}$$

$$D_n=\frac{(-1)^n e^{\pi}-1}{n^2+1}\cdot\frac{2}{\pi}$$

$$\boxed{u(x,y)=\frac{2}{\pi}\sum_{n=0}^{\infty}\left(\frac{(-1)^n e^{\pi}-1}{n^2+1}\right)e^{-ny}\cos(nx)}$$

$$\boxed{\lim_{y\to\infty} u(x,y)=0}$$

4. Solve: $\begin{cases} u_{xx}+u_{yy}=0, & -\pi<x<\pi,\ 0<y<2 \\ u(x,0)=x, & -\pi<x<\pi \\ u(x,2)=-x, & -\pi<x<\pi \\ u(-\pi,y)=u(\pi,y), & 0<y<2 \\ u_x(-\pi,y)=u_x(\pi,y), & 0<y<2 \end{cases}$



$u(x,y)=X(x)\cdot Y(y)$

$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(-\pi)=X(\pi) \\ X'(-\pi)=X'(\pi) \end{cases}$

$Y''(y)-\lambda_n Y(y)=0$

$Y_n(y)=C_n e^{\sqrt{\lambda} y}+D_n e^{-\sqrt{\lambda} y}$

$\lambda_n=(2n)^2$

$X_n(x)=A_n\cos(2nx)+B_n\sin(2nx)$

$$U(x,y)=\sum_{n=1}^{\infty}\left(C_n e^{2ny}+D_n e^{-2ny}\right)\left(\cos(2nx)+\sin(2nx)\right)$$

$$u(x,0)=x=\sum_{n=1}^{\infty}(C_n+D_n)\cos(2nx)+(C_n+D_n)\sin(2nx)$$

$$C_n+D_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos(2nx)\,dx=0$$

$$C_n+D_n=\frac{2}{\pi}\int_0^{\pi}x\sin(2nx)\,dx=\frac{\sin(2nx)-2nx\cos(2nx)}{2n^2\pi}\Bigg|_{x=0}^{x=\pi}$$

$$=\frac{-2n\pi}{2n^2\pi}=\frac{-1}{n}$$

$$u(x,2)=C_n e^{4n}+D_n e^{-4n}=\frac{-2}{\pi}\int_0^{\pi}x\sin(2nx)\,dx=\frac{1}{n}$$

$\begin{cases} C_n+D_n=-1/n \\ C_n e^{4n}+D_n e^{-4n}=1/n \end{cases}$

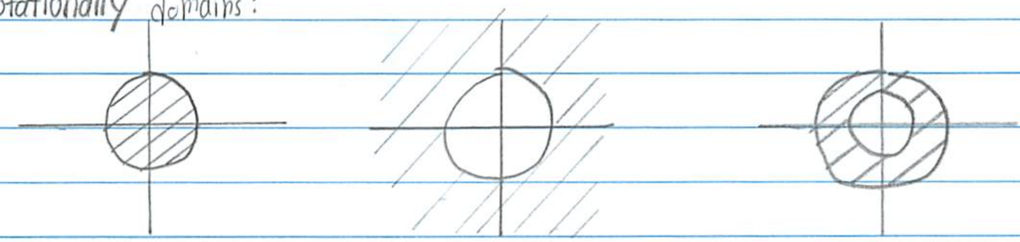
6.3: Laplace operator in polar coordinates

previouly we could only say $\swarrow x \quad \swarrow y$

$$U = X(x)\,Y(y) \quad \text{because } \Omega = [0,a] \times [0,b]$$

now lets look at radially symetric domains
rotationally symetric domain

rotationally domains:



$x^2+y^2<a^2$ — $0<r<a,\ 0\le\theta<2\pi$

$x^2+y^2>a^2$ — $r>a,\ 0\le\theta\le2\pi$

$a^2<x^2+y^2<b^2$ — $a<r<b,\ 0\le\theta\le2\pi$

$$\Delta U = U_{xx} + U_{yy} = \ ?$$

Proof that: $u_{xx}+u_{yy} = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}$

$$U_x = \frac{\partial U}{\partial x} = \frac{\partial}{\partial x} U(r(x,y), \theta(x,y)) = \partial_x U \qquad r = \sqrt{x^2+y^2}, \quad \theta = \arctan(y/x)$$

$$= U_r \cdot \frac{dr}{dx} + U_\theta \cdot \frac{\partial\theta}{\partial x}$$

$$U_x = U_r \cdot \frac{x}{\sqrt{x^2+y^2}} + U_\theta \cdot \frac{-y}{x^2+y^2}$$

$$U_{xx} = \partial_x U_r \left(\frac{x}{\sqrt{x^2+y^2}}\right) + U_r \cdot \partial_x\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \partial_x U_\theta \left(\frac{y}{x^2+y^2}\right) - U_\theta \cdot \partial_x\left(\frac{y}{x^2+y^2}\right)$$

$$= U_{rr} \cdot \frac{x^2}{x^2+y^2} + U_{r\theta} \cdot \left(\frac{-y}{x^2+y^2} \cdot \frac{x}{\sqrt{x^2+y^2}}\right) + U_r \cdot \left(\frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}\right)$$

$$- U_{\theta r} \cdot \left(\frac{xy}{\sqrt{x^2+y^2}\,(x^2+y^2)}\right) - U_{\theta\theta} \cdot \left(\frac{-y^2}{(x^2+y^2)^2}\right) + U_\theta \cdot \left(\frac{2xy}{(x^2+y^2)^2}\right)$$

$$U_{yy} = U_{rr} \cdot \frac{y^2}{x^2+y^2} + U_{r\theta} \cdot \left(\frac{xy}{\sqrt{x^2+y^2}\,(x^2+y^2)}\right) + U_r \cdot \left(\frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}\right)$$

$$+ U_{\theta r} \cdot \left(\frac{xy}{\sqrt{x^2+y^2}\,(x^2+y^2)}\right) + U_{\theta\theta} \cdot \frac{x^2}{(x^2+y^2)^2} + U_\theta \cdot \left(\frac{-2xy}{(x^2+y^2)^2}\right)$$

$$\boxed{U_{xx} + U_{yy} = U_{rr} + \frac{1}{r} \cdot U_r + \frac{1}{r^2} \cdot U_{\theta\theta}}$$

BVP: $\begin{cases} \Delta u = 0, & x^2+y^2 < a^2 \\ u(a\cos\theta, a\sin\theta) = g(\theta), & 0 \le \theta < 2\pi \end{cases}$

in polar: $\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, & 0 \le \theta \le 2\pi, \quad 0 < r < a \\ u(a,\theta) = g(\theta), & 0 \le \theta \le 2\pi \end{cases}$

now our bound is seperable so our u is also

$u(r,\theta) = R(r)\,\Theta(\theta)$

$\Delta u = R''(r)\Theta(\theta) + \frac{1}{r} R'(r)\Theta(\theta) + \frac{1}{r^2} R(r)\Theta''(\theta) = 0$

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{-r^2 R''(r) - r R'(r)}{R(r)} = -\lambda$$

$\Theta''(\theta) + \lambda\Theta(\theta) = 0$ $\qquad$ $r^2 R''(r) + r R'(r) - \lambda R(r) = 0$

for our circle $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$ $\qquad$ Euler equation:

$R_n(r) = r^n$ or r^{-n}

because $0 = 2\pi$ $\qquad$ $\boxed{R_n(r) = A_n r^n + B_n r^{-n}}$ for $\lambda_n = n^2$

$\lambda_n = \left(\frac{2n\pi}{2l}\right)^2$, $l = \pi$ $\qquad$ $\boxed{R_0 = C\ln(r) + D}$

$\lambda_n = n^2$, $\lambda_0 = 0$

$\Theta_0(\theta) = \frac{1}{2}$

$\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$

$$u(r,\theta) = \frac{1}{2}\left(C_0 + D_0\ln(r)\right) + \sum_{n=1}^{\infty}\left(C_n r^n + D_n r^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

in order to have $r=0$ be valid: $D_0 = 0$ and $D_n = 0$

because $D_0\ln(0)$ doesn't make sense

$$u(r,\theta) = \frac{C_0}{2} + \sum_{n=1}^{\infty}\left(a_n r^n\cos(n\theta) + b_n r^n\sin(n\theta)\right)$$

$$u(a,\theta) = g(\theta) = \frac{C_0}{2} + \sum\left(a_n a^n\cos(n\theta) + b_n a^n\sin(n\theta)\right)$$

$$C_0 = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\,d\theta$$

$$a_n = \frac{1}{\pi a^n}\int_0^{2\pi} g(\theta)\cos(n\theta)\,d\theta$$

$$b_n = \frac{1}{\pi a^n}\int_0^{2\pi} g(\theta)\sin(n\theta)\,d\theta$$

Different boundry: annulus

Carry out same method for disk

For: $\begin{cases} \Delta u = 0 & x^2+y^2>a^2 \\ u = g & x^2+y^2=a^2 \end{cases}$

$$u(r,\theta) = \frac{1}{2}\left(C_0 + D_0 \ln(r)\right) + \sum \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

can't say $D=0$ cuz $r \neq 0$

instead add $|u| \leq M$ boundedness ar $r \to \infty$

$r \to \infty$ still implies $D_0 \ln(\infty)$ and $C_n r^n = 0$ to ensure $u \not\to \infty$

$$u(r,\theta) = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} \left(a_n r^{-n} \cos(n\theta) + b_n r^{-n} \sin(n\theta)\right)$$

same as before but r^{-n} istead

For: $\begin{cases} \Delta u = 0, & a < x^2+y^2 < b \\ u = g, & x^2+y^2 = a \\ u = h, & x^2+y^2 = b \end{cases}$

still get:

$$u(r,\theta) = \frac{1}{2}\left(C_0 + D_0 \ln(r)\right) + \sum \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

foil

$$= \frac{1}{2}\left(C_0 + D_0 \ln(r)\right) + \sum \left(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) + c_n r^{-n} \cos(n\theta) + d_n r^{-n} \sin(n\theta)\right)$$

$$u(a,\theta) = g(\theta) = \frac{1}{2}\left(C_0 + D_0 \ln(a)\right) + \sum \left(a_n a^n + b_n a^{-n}\right)\cos(n\theta) + \left(c_n a^n + d_n a^{-n}\right) \sin(n\theta)$$

$$u(b,\theta) = h(\theta) = \frac{1}{2}\left(C_0 + D_0 \ln(b)\right) + \sum \left(a_n b^n + b_n b^{-n}\right)\cos(n\theta) + \left(c_n b^n + d_n b^{-n}\right) \sin(n\theta)$$

Full Fourier series for g and h give 6 equations for 6 varibles

Poisson Formula:

$$\begin{cases} \Delta u = 0 & x^2+y^2<a^2 \\ u = g \end{cases}$$

$$u(r,\theta) = \int_0^{2\pi} G(r,\theta,\theta')\, g(\theta')\, d\theta'$$

where $G(r,\theta,\theta') = \frac{1}{2\pi} \cdot \frac{a^2 - r^2}{a^2 - 2ra\cos(\theta-\theta') + r^2}$

Just a rearranged version after plugging in Fourier coefficents

fun complex fact

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$ Euler's identity

$(e^{i\theta})^n = (\cos(\theta) + i\sin(\theta))^n$

$e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta)$ deMoivre's identity

Homework 11

Exercise 1:

prove that @ $r=0$ for a disk

$$u(0)=\frac{1}{2\pi}\int_0^{2\pi} g(\theta')\,d\theta'$$

and prove $u(0)=$ mean value of $u(a,\theta)$

For a disk:

$$u(r,\theta)=\frac{1}{2}C_0+\sum_{n=1}^{\infty}\left(a_n r^n\cos(n\theta)+b_n r^n\sin(n\theta)\right)$$

$$C_0=\frac{1}{\pi}\int_0^{2\pi} g(\theta')\,d\theta'$$

$$u(0,\theta)=\frac{1}{2}C_0+\sum 0$$

$$\boxed{u(0,\theta)=\frac{1}{2\pi}\int_0^{2\pi} g(\theta')\,d\theta'}$$

$$u(a,\theta)=g(\theta)$$

$$=\boxed{\frac{1}{2\pi}\int_0^{2\pi} g(\theta')\,d\theta'}$$

average value general form for $f(x)$

$\frac{1}{b-a}\int_a^b f(x)\,dx$ and we have $\frac{1}{2\pi-0}\int_0^{2\pi} g(\theta')\,d\theta'$

$a=0$ $b=2\pi$ $f(x)=g(\theta')$

at circumfrence

Exercise 2:

Prove that if:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & a < r \\ u = g(\theta) & r = a \\ \max|u| < \infty & \end{cases}$$

then:

$$G(r,\theta,\theta') = \frac{r^2 - a^2}{2\pi(a^2 - 2ra\cos(\theta-\theta') + r^2)}$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\theta')d\theta' + \sum_{n=1}^{\infty} \frac{2r^{-n}}{2\pi}\left(\int_0^{2\pi} g(\theta')\left(\frac{\cos(n\theta')\cos(n\theta) + \sin(n\theta')\sin(n\theta)}{a^{-n}}\right)d\theta'\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} G \cdot g(\theta')d\theta'$$

$$G = 1 + \sum_{n=1}^{\infty} 2r^{-n}a^n\left(\cos(n\theta')\cos(n\theta) + \sin(n\theta')\sin(n\theta)\right)$$

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$$= 1 + \sum_{n=1}^{\infty} \frac{2a^n}{r^n}\left(\cos(n(\theta - \theta'))\right)$$

de Moivre's theorem

$$= 1 + 2\sum_{n=1}^{\infty} \frac{2a^n}{r^n} \mathrm{Re}\left[e^{n(\theta-\theta')}\right]$$

$$= 1 + 2\sum_{n=1}^{\infty} \mathrm{Re}\left[\left(\frac{ae^{(\theta-\theta')}}{r}\right)^n\right]$$

$$\frac{ae^{(\theta-\theta')}}{r} = z$$

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$$

$$= 1 + 2\,\mathrm{Re}\left[\frac{ar^{-1}e^{(\theta-\theta')}}{1 - ar^{-1}e^{(\theta-\theta')}}\right] =$$

$$\frac{1 - ar^{-1}\cos(\theta-\theta') + 2ar^{-1}\cos(\theta-\theta')}{1 - ar^{-1}\cos(\theta-\theta')} = \frac{1 + ar^{-1}\cos(\theta-\theta')}{1 - ar^{-1}\cos(\theta-\theta')} \cdot \frac{1 + ar^{-1}\cos(\theta-\theta')}{(1 + ar^{-1}\cos(\theta-\theta'))}$$

$$= \frac{1 + 2ar^{-1}\cos(\theta-\theta') + a^2r^{-2}\cos^2(\theta-\theta')}{1 - a^2r^{-2}\cos^2(\theta-\theta')}$$

solve: $\begin{cases} \Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 1, & r<2 \\ u_r = 2\sin^2(\theta), & r=2 \\ u = 1, & r=0 \end{cases}$

let $v = u - \frac{r^2}{4}$

then: $v_r = u_r - r/2 \qquad v_{rr} = u_{rr} - 1/2$

$v_\theta = u_\theta \qquad v_{\theta\theta} = u_{\theta\theta}$

$v_r(2) = 2\sin^2(\theta) - 2^2/4 = 2\sin^2\theta - 1 = -\cos(2\theta)$

$v(0) = 1 - 0 = 1$

now: $\Delta v = (u_{rr} - 1/2) + (\frac{u_r}{r} - 1/2) + r^{-2}u_{\theta\theta}$

$\Delta v = \underbrace{\Delta u}_{1} - 1 \qquad \therefore \Delta v = 0$

solve: $\begin{cases} \Delta v = v_{rr} + r^{-1}v_r + r^{-2}v_{\theta\theta} = 0, & r<2 \\ v_r(2,\theta) = -\cos(2\theta), & r=2 \\ v(0,\theta) = 1, & r=0 \end{cases}$

$$v(r,\theta) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} r^n\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right)$$

$$c_0 = \frac{1}{\pi}\int_0^{2\pi} -\cos(2\theta')\,d\theta' = 0$$

$$a_n = \frac{1}{\pi\cdot 2^n}\int_0^{2\pi} -\cos(2\theta')\cos(n\theta')\,d\theta' = \frac{-1}{2\pi\cdot 2^n}\int_0^{2\pi}\cos(\theta'(2+n)) + \cos(\theta'(2-n))\,d\theta'$$

$$= \frac{-1}{2\pi\cdot 2^n}\left(\frac{\sin(\theta'(2+n))}{2+n} + \frac{\sin(\theta'(2-n))}{2-n}\right)\Bigg|_{\theta'=0}^{\theta'=2\pi} \qquad {}^*\sin(0)=0$$

$$= \frac{-1}{2\pi\cdot 2^n}\left(\frac{\sin(4\pi)\cos(2\pi n) + \cos(4\pi)\sin(2\pi n)}{2+n} + \frac{\sin(4\pi)\cos(2\pi n) - \cos(4\pi)\sin(2\pi n)}{2-n}\right)$$

$$= \frac{-1}{2\pi\cdot 2^n}\left(\frac{0+0}{2+n} + \frac{0-0}{2-n}\right) = 0$$

$$b_n = \frac{1}{\pi \cdot 2^n} \int_0^{2\pi} -\cos(2\theta')\sin(n\theta')\,d\theta'$$

$$= \frac{-1}{2\pi \cdot 2^n} \int_0^{2\pi} \sin(\theta'(2+n)) - \sin(\theta'(2-n))\,d\theta'$$

$$= \frac{1}{2\pi \cdot 2^n}\left(\frac{\cos(\theta'(2+n))}{2+n} - \frac{\cos(\theta'(2-n))}{2-n}\right)\Bigg|_{\theta'=0}^{\theta'=2\pi}$$

$$= \frac{1}{2\pi \cdot 2^n}\left(\frac{\cos(2\theta')\cos(n\theta') - \sin(2\theta')\sin(n\theta')}{2+n} - \frac{\cos(2\theta')\cos(n\theta') + \sin(2\theta')\sin(n\theta')}{2-n}\right)\Bigg|_{\theta'=0}^{\theta'=2\pi}$$

$\cos(4\pi) = 1 \quad \sin(4\pi) = 0$

$\cos(0) = 1 \quad \sin(0) = 0$

$\cos(2n\pi) = 1 \quad \sin(2n\pi) = 0$

$$= \frac{1}{2\pi \cdot 2^n}\left(\frac{1}{2+n} - \frac{1}{2-n} - \left(\frac{1}{2+n} - \frac{1}{2-n}\right)\right) = 0$$

$v(r,\theta) = 0$

$\therefore \quad 0 = u - \frac{r^2}{4} \quad \therefore \quad \boxed{u(r,\theta) = \frac{r^2}{4}}$

3. solve: $\begin{cases} \Delta u = 0, & 2 < r < 3 \\ u_r = \cos^2(\theta), & r = 2 \\ u = \sin^2(\theta), & r = 3 \end{cases}$

$$u(r,\theta) = \frac{1}{2}\left(C_0 + D_0\ln(r)\right) + \sum_{n=1}^{\infty}\left(a_n r^n + c_n r^{-n}\right)\cos(n\theta) + \left(b_n r^n + d_n r^{-n}\right)\sin(n\theta)$$

$$u_r(r,\theta) = \frac{1}{2r}D_0 + \sum_{n=1}^{\infty}\cos(n\theta)\left(n a_n r^{n-1} + n c_n r^{-n-1}\right) + \sin(n\theta)\left(n b_n r^{n-1} - n d_n r^{-n-1}\right)$$

$$\cos^2(\theta) = \frac{1}{2}\cdot\frac{D_0}{2} + \sum_{n=1}^{\infty}\frac{n}{3}\left(3^n a_n - \frac{c_n}{3^n}\right)\cos(n\theta) + \frac{n}{3}\left(3^n b_n - \frac{d_n}{3^n}\right)\sin(n\theta)$$

$$\sin^2(\theta) = \frac{1}{2}\left(C_0 + D_0\ln(2)\right) + \sum_{n=1}^{\infty}\left(2^n a_n + \frac{c_n}{2^n}\right)\cos(n\theta) + \left(2^n b_n + \frac{d_n}{2^n}\right)\sin(n\theta)$$

$$\frac{D_0}{2} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(\theta')\,d\theta'$$

$$D_0 = \frac{4}{\pi}\int_{0}^{\pi} \cos^2(\theta')\,d\theta' = 2$$

$$C_0 + D_0 \ln(2) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(\theta')\,d\theta'$$

$$C_0 + 2\ln(2) = 1 \quad \therefore \quad C_0 = 1 - 2\ln(2)$$

$$2^n a_n + \frac{C_n}{2^n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(\theta')\sin(n\theta')\,d\theta' \qquad \text{odd}\cdot\text{odd}\cdot\text{odd} = \text{odd}$$

$$2^n a_n + \frac{C_n}{2^n} = 0 \quad \therefore \quad a_n = \frac{-C_n}{2^{2n}} \qquad \frac{1}{2^n}\cdot\frac{1}{2^n} = 2^{-n}\cdot 2^{-n} = 2^{-2n}$$

$$\frac{n}{3}\left(3^n d_n - \frac{C_n}{3^n}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(\theta')\sin(n\theta')\,d\theta' \qquad \text{even}\cdot\text{even}\cdot\text{odd} = \text{odd}$$

$$3^n a_n - \frac{C_n}{3^n} = 0 \quad \therefore \quad a_n = \frac{C_n}{3^{2n}}$$

$$\frac{C_n}{3^{2n}} = \frac{-C_n}{2^{2n}} \Rightarrow C_n\left(\frac{1}{3^{2n}}\right) + C_n\left(\frac{1}{2^{2n}}\right) = 0$$

$$C_n\left(\frac{1}{3^{2n}} + \frac{1}{2^{2n}}\right) = 0 \quad \therefore C_n = 0$$

$$\therefore a_n = 0$$

$$2^n b_n + \frac{d_n}{2^n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(\theta')\cos(n\theta')\,d\theta' = \frac{2}{\pi}\int_{0}^{\pi} \sin^2(\theta')\cos(n\theta')\,d\theta'$$

$$= -\frac{4}{\pi}\left(\frac{\sin(n\pi)}{n(n^2-4)}\right) = 0$$

$$b_n = \frac{-d_n}{2^{2n}}$$

$$\frac{n}{3}\left(3^n b_n - \frac{d_n}{3^n}\right) = \frac{2}{\pi}\int_{0}^{\pi} \sin^2(\theta')\cos(n\theta')\,d\theta' = 0 \quad \therefore b_n = \frac{d_n}{3^{2n}} \quad \therefore \quad b_n = 0,\ d_n = 0$$

$$u(r,\theta) = \frac{1}{2} - \ln(2) + \ln(2) \quad \therefore \quad \boxed{u(r,\theta) = \frac{1}{2}}$$

Complex number stuff:

Conjugation $\overline{a+bi}$ means $a-bi$

so $(a+bi)\cdot\overline{a+bi} = a^2+b^2$ also $= |a+bi|^2$

Fourier series in the complex form:

recall:

$$\left\{\frac{1}{2}, \cos\left(\frac{n\pi x}{\ell}\right), \sin\left(\frac{n\pi x}{\ell}\right)\right\} \text{ on } -\ell < x < \ell$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{\ell}\right) + b_n\sin\left(\frac{n\pi x}{\ell}\right)\right)$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \qquad n=0,1,2,3\ldots$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx \qquad n=1,2,3\ldots$$

Euler identity:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$$

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta}-e^{i\theta}}{2i}$$

now:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\left(\frac{e^{\frac{in\pi x}{\ell}} + e^{-\frac{in\pi x}{\ell}}}{2}\right) + b_n\left(\frac{e^{\frac{in\pi x}{\ell}} - e^{-\frac{in\pi x}{\ell}}}{2i}\right)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)\exp\left(\frac{in\pi x}{\ell}\right) + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right)\exp\left(-\frac{in\pi x}{\ell}\right)$$

let: $C_n = a_n/2 + b_n/2i = \frac{1}{2}(a_n - ib_n)$

$C_{-n} = a_n/2 - b_n/2i = \frac{1}{2}(a_n + ib_n)$

$C_0 = a_0/2$

then: $f(x) = C_0 + \sum_{n=1}^{\infty} C_n\exp\left(\frac{in\pi x}{2}\right) + \sum_{n=1}^{\infty} C_{-n}\exp\left(\frac{-in\pi x}{\ell}\right)$

$\rightarrow = \sum_{n=-\infty}^{-1} C_n\exp(in\pi x/\ell)$

$= {}_0 + (1\to\infty) + (-\infty\to -1)$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{\ell}}}$$

$$C_n = \frac{1}{2}\left(\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx - \frac{i}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx\right)$$

$$\boxed{= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-\frac{in\pi x}{\ell}}dx}$$

Paseval's idenity:

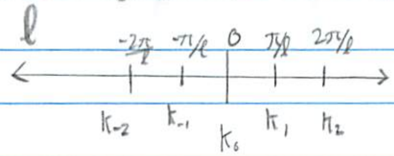
$$\int_{-\ell}^{\ell} |f(x)|^2 dx = 2\ell \sum_{n=-\infty}^{\infty} |C_n|^2$$

5.1: Fourier transform

take our interval $\ell \to \infty$

let:

$$k_n = \frac{n\pi}{\ell} \quad \text{and} \quad \Delta k_n = k_n - k_{n-1} = \frac{\pi}{\ell}$$



$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{ik_n x}$$

say $C_n = \frac{1}{2\pi} \cdot C(k_n) \cdot \Delta k_n \quad \therefore \quad C(k_n) = \frac{C_n}{\Delta k_n} \cdot 2\pi$

$$C(k_n) = \frac{\frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-ik_n x}dx}{\pi/\ell} \cdot 2\pi = \int_{-\ell}^{\ell} f(x)e^{-ik_n x}dx$$

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} C(k_n)e^{ik_n x}\Delta k_n$$ which is a reimann sum

as $\ell \to \infty$, $\Delta k_n \to 0$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(k)e^{ikx}dk} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dx \quad \text{inverse forier transform}$$

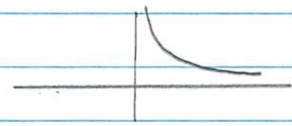
$$\boxed{C(k) = \int_{-\ell}^{\ell} f(x)e^{-ikx}dx = \hat{f}(k)} \quad \text{forier transform}$$

Paseval's idenity (Plancherel's formula)

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |C(k)|^2 dk$$

examples of Fourier transform

$$f(x) = \begin{cases} e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$-\infty < x < \infty$ and $-\infty < k < \infty$

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

$$= \int_0^{\infty} e^{-\alpha x} e^{-ikx}\, dx = \int_0^{\infty} e^{-(\alpha + ik)x}\, dx = \frac{-1}{\alpha + ik}\left(e^{-(\alpha+ik)\infty} - 1\right)$$

$$e^{-\alpha\infty} \cdot e^{ik\infty} = e^{-\alpha \cdot \infty}\left(\cos(k\cdot\infty) - i\sin(k\cdot\infty)\right) \Rightarrow \frac{\sim}{\infty}$$

$$\boxed{\hat{f}(k) = \frac{1}{\alpha + ik}}$$

inverse tells us

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha + ik} e^{ikx}\, dk = \begin{cases} e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Properties between $\underset{-\infty < x < \infty}{f(x)} \rightleftarrows \underset{-\infty < k < \infty}{\hat{f}(k)}$

linearity applies too

assume real constants a and b

just plug in to check
$$f(x-a) \longmapsto e^{-ika}\hat{f}(k)$$
modulation
$$e^{ibx}f(x) \longmapsto \hat{f}(k-b)$$

integration by parts to check
$$f'(x) \longmapsto ik\hat{f}(k)$$
assume $f(\infty) = f(-\infty) = 0$
$$xf(x) \longmapsto i\hat{f}'(k)$$

scale $\lambda \neq 0$ $\quad f(\lambda x) \longmapsto \frac{1}{|\lambda|}\hat{f}(k/\lambda)$

example: $f(x) = e^{-\frac{\alpha}{2}x^2}$ assume $\alpha > 0$ (constant)

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{\left(-\frac{\alpha}{2}x^2 - ikx\right)}\, dx$$

say $f'(x) = e^{-\frac{\alpha}{2}x^2} \cdot -\alpha x = -\alpha x f(x)$

$$ik\hat{f}(k) = -\alpha i \hat{f}'(k)$$

$$\Rightarrow \hat{f}'(k) = -\frac{1}{\alpha} k\hat{f}(k)$$

ODE $\boxed{\hat{f}(k) = Ce^{-k^2/2\alpha}}$ $\quad C = \hat{f}(0) = \int_{-\infty}^{\infty} f(x)\, dx$ $\quad \boxed{\hat{f}(k) = \frac{\sqrt{2\pi}}{\sqrt{\alpha}} e^{-k^2/2\alpha}}$

Application of Fourier transform

recall: IVP $\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = g(x) \end{cases}$

$$u(x,t) = \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{4\pi k t}} \exp\left(\frac{-(x-y)^2}{4kt}\right)}_{\lim_{t \to 0^+} = \delta(x-y)} g(y)\, dy$$

$$u(x,0) = \int_{-\infty}^{\infty} \delta(x-y) g(y)\, dy$$

this is the definition of convolution of $\delta * g$

convolution facts

$f * g = g * f$

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, g(y)\, dy$$

also a convolution

Fourier property

$f(x) * g(x) \longmapsto \hat{f}(k) \cdot \hat{g}(k)$

$f(x) \cdot g(x) \longmapsto \frac{1}{2\pi} \hat{f}(k) * \hat{g}(k)$

Heat equation

$\begin{cases} u_t - k u_{xx} = 0 & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = g(x) & -\infty < x < \infty \end{cases}$

$$u(x,t) \longmapsto \hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx$$

$-k \widehat{u_x} = -k\, ik\, \hat{u}$

$-k \widehat{u_{xx}} = k\, k^2 \hat{u}(k,t)$

$\widehat{u_t} - k \widehat{u_{xx}} = 0$

$\hat{u}_t + k k^2 \hat{u} = 0$

this is an ODE wrt t

$$\frac{d\hat{u}}{dt} + k k^2 \hat{u} = 0$$

$\hat{u} = C e^{-k k^2 t}$

$C(k)$

$u(x,0) = g(x)$

$\hat{u}(k,0) = \hat{g}(k)$

New system

$$\begin{cases} \hat{u}(k,t) = C(k)\, e^{-k k^2 t} \\ \hat{u}(k,0) = \hat{g}(k) \end{cases} \longrightarrow \therefore \; C(k) = \hat{g}(k)$$

$$\therefore \; \hat{u}(k,t) = \hat{g}(k)\, e^{-k k^2 t}$$

inverse fourier

$$\boxed{u(x,t) = g(x) * \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{4\pi k t}}}$$

HW 12

Problem 2: Let $\alpha > 0$. Find $\hat{f}(k)$

1. $f(x) = e^{-\alpha|x|}$

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-\alpha|x|} e^{-ikx}\,dx = \cancel{\int_{-\infty}^{\infty} e^{-(\alpha+ik)|x|}\,dx} \qquad e^{-\alpha|x|} = \begin{cases} e^{\alpha x} & x<0 \\ e^{-\alpha x} & x \geq 0 \end{cases}$$

$$= \int_{-\infty}^{0} e^{(\alpha-ik)x}\,dx + \int_{0}^{\infty} e^{-(\alpha+ik)x}\,dx$$

$$= \frac{1}{\alpha-ik} e^{(\alpha-ik)x}\Big|_{x=-\infty}^{x=0} - \frac{1}{\alpha+ik} e^{-(\alpha+ik)x}\Big|_{x=0}^{\infty}$$

$$= \frac{1}{\alpha-ik}(1-0) - \frac{1}{\alpha+ik}(0-1) = \frac{1}{\alpha-ik} + \frac{1}{\alpha+ik}$$

$$= \frac{\alpha+ik}{(\alpha-ik)(\alpha+ik)} + \frac{\alpha-ik}{(\alpha-ik)(\alpha+ik)} = \boxed{\frac{2\alpha}{\alpha^2+k^2}}$$

3. $f(x) = \begin{cases} xe^{-\alpha x}, & x>0 \\ 0, & x \leq 0 \end{cases}$

$$\hat{f}(k) = \int_{0}^{\infty} xe^{-\alpha x} e^{-ikx}\,dx = \int_{0}^{\infty} xe^{-(\alpha+ik)x}\,dx$$

$$u = x \qquad dv = e^{-(\alpha+ik)x}\,dx$$

$$du = dx \qquad v = \frac{-1}{\alpha+ik} e^{-(\alpha+ik)x}$$

$$\hat{f}(k) = \frac{-x}{\alpha+ik} e^{-(\alpha+ik)x}\Big|_{x=0}^{x=\infty} - \int_{0}^{\infty} \frac{-1}{\alpha+ik} e^{-(\alpha+ik)x}\,dx$$

$$= 0 - 0 - \frac{1}{(\alpha+ik)^2} e^{-(\alpha+ik)x}\Big|_{x=0}^{x=\infty}$$

$$= -\frac{1}{(\alpha+ik)^2}(0-1) = \boxed{\frac{1}{(\alpha+ik)^2}}$$

5. $f(x) = e^{-\alpha|x|}\sin(\beta x)$

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-\alpha|x|}\sin(\beta x)e^{-ikx}\,dx$$

$$\sin(\beta x) = \frac{e^{i\beta x} - e^{-i\beta x}}{2i}$$

$$= \frac{1}{2i}\left(\int_{-\infty}^{\infty} e^{-\alpha|x|}\cdot e^{i\beta x}\cdot e^{-ikx}\,dx - \int_{-\infty}^{\infty} e^{-\alpha|x|}\cdot e^{-i\beta x}\cdot e^{-ikx}\,dx\right)$$

$$= \frac{1}{2i}\left(\int_{-\infty}^{0} e^{(\alpha+i\beta-ik)x}\,dx + \int_{0}^{\infty} e^{(-\alpha+i\beta-ik)x}\,dx + \int_{-\infty}^{0} e^{(\alpha-i\beta-ik)x}\,dx + \int_{0}^{\infty} e^{(-\alpha-i\beta-ik)x}\,dx\right)$$

$$= \frac{1}{2i}\left(\frac{1-\frac{\infty}{\infty^2}\swarrow 0}{\alpha+i(\beta-k)} + \frac{\frac{\infty}{\infty^2}\swarrow 0 - 1}{-\alpha+i(\beta-k)} + \frac{1-\frac{\infty}{\infty^2}\swarrow 0}{\alpha-i(\beta+k)} + \frac{0-1}{-\alpha-i(\beta+k)}\right)$$

$$= \boxed{\frac{1}{2\alpha i-2\beta+2k} + \frac{1}{2\alpha i+2\beta-2k} + \frac{1}{2\alpha i+2\beta+2k} + \frac{1}{2\alpha i-2\beta-2k}}$$

Problem 3: Let $\alpha > 0$, find $\hat{f}(k)$

1. $f(x) = (x^2+\alpha^2)^{-1} = \dfrac{1}{\alpha^2+x^2}$

$$\hat{g}(k) = \frac{1}{\alpha^2+k^2}$$

$$\therefore\ g(x) = \frac{2\pi}{2\alpha}e^{-\alpha|x|}$$

since $f = \hat{g}$ then $\hat{f} = \hat{\hat{g}} = 2\pi g(-k)$

$$\hat{f}(k) = 2\pi\cdot\frac{\pi}{\alpha}e^{-\alpha|-k|} \qquad \boxed{\hat{f}(k) = \frac{2\pi^2}{\alpha}e^{\alpha k}}$$

2. $f(x) = \dfrac{x}{x^2+\alpha^2}$

$$h(x) = \frac{1}{x^2+\alpha^2}$$

$$\hat{h}(k) = \frac{2\pi^2}{\alpha} e^{\alpha k}$$ from last question

$f(x) = x\,h(x)$

$\widehat{xh(x)} = i\hat{h}'(x) = 2i\pi^2 e^{\alpha k}$

$\boxed{\hat{f}(x) = 2i\pi^2 e^{\alpha k}}$

3. $f(x) = \dfrac{1}{x^2+\alpha^2}\cos(\beta x)$

$\hat{f}(k) = \dfrac{2\pi^2}{\alpha} e^{\alpha k} * \widehat{\cos(\beta x)}$

$$\widehat{\cos(\beta x)} = \frac{1}{2i}\left(\int_{-\infty}^{\infty} e^{(\beta-k)ix}\,dx + \int_{-\infty}^{\infty} e^{-(\beta+k)ix}\,dx\right)$$

$$= \frac{1}{2i}\left(\frac{1}{i(\beta-k)} e^{(\beta-k)ix} - \frac{1}{i(\beta+k)} e^{(\beta+k)ix}\right)\Bigg|_{x=-\infty}^{x=\infty}$$

$$= \frac{1}{2i}\Big((1-\infty) - (-1+\infty)\Big) = \frac{1}{i} = -i$$

$$\boxed{\hat{f}(k) = -i * \frac{2\pi^2}{\alpha} e^{\alpha k}}$$

Problem 4: Based on $f(x) = e^{-\frac{\alpha x^2}{2}}$ find $\hat{f}(k)$

$$\hat{f}(x) = \frac{\sqrt{2\pi}}{\alpha} e^{-\frac{\alpha k^2}{2}}$$

1. $f(x) = xe^{-\alpha x^2/2}$ $\qquad -\alpha k$

$$\hat{f}(k) = -ki\sqrt{2\pi}\, e^{-\alpha k^2/2}$$

2. $f(x) = x^2 e^{-\alpha x^2/2}$

$$\hat{f}(k) = -\alpha k^2 i\sqrt{2\pi}\, e^{-\alpha k^2/2}$$

5. $f(x) = xe^{-\alpha x^2/2}\sin(\beta x)$

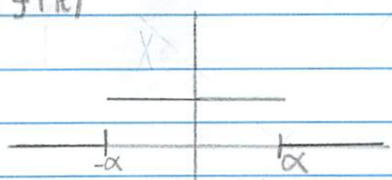
$$= \frac{xe^{-\alpha x^2/2}e^{ix}}{2i} - \frac{xe^{-\alpha x^2/2}e^{-ix}}{2i}$$

$$\hat{f}(k) = \frac{i\hat{f}'(k-1)}{2i} - \frac{i\hat{f}'(k+1)}{2i}$$

$$= \boxed{\frac{ki\sqrt{2\pi}}{2}\left(\exp\left(\frac{-\alpha(k+1)^2}{2}\right) - \exp\left(\frac{-\alpha(k-1)^2}{2}\right)\right)}$$

problem 5 knowing $\alpha > 0$, find $\hat{f}(k)$

1. $f(x) = \begin{cases} 1 & |x| \leq \alpha \\ 0 & |x| > \alpha \end{cases}$



$$\hat{f}(k) = \int_{-\alpha}^{\alpha} e^{-ikx}\,dx = \frac{1}{-ik} e^{-ikx}\Big|_{x=-\infty}^{x=\alpha} = \frac{1}{-ik}\left(e^{-ik\alpha} - e^{ik\alpha}\right)$$

$$= \boxed{\frac{2}{k}\sin(\alpha k)}$$

6. calculate $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx$ using part 1.

$$\frac{\hat{f}(k)}{2} = \frac{\sin(k)}{k} \qquad \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{2}\,dk$$

$$\pi f(x) = \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{2} e^{ikx}\,dk$$

$$\therefore \boxed{\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \pi e^{-ikx}\, U(x \pm 1)}$$