A Solution to Problem 4-3

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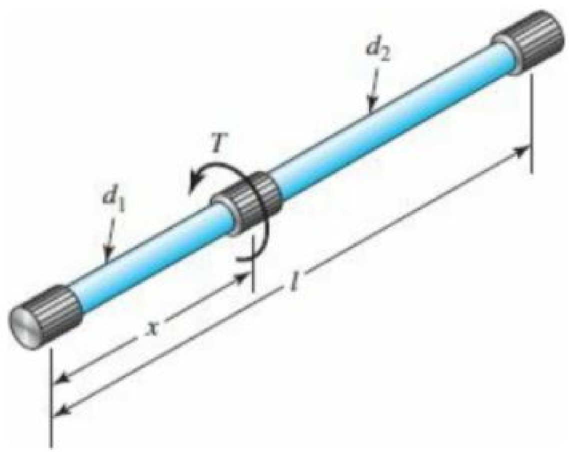
The Problem Statement

A torsion-bar spring consist of a prismatic bar, usually of round cross section, that is twisted at one end and held fast at the other to form a stiff spring. An engineer needs a stiffer one than usual and so considers building in both ends and applying the torque somewhere in the central portion of the span, as shown in the figure. This effectively creates two springs in parallel. If the bar is uniform in diameter, that is, if $d = d_1 = d_2$,

(a) determine how the spring rate and the end rection depend on the location x at which the torque is applied,

(b) determine the spring rate, the end reaction, and the maximum shearing stress, if $d = 0.5$ in, $x = 5$ in, $l = 10$ in, $T = 1500$ lbf·in, and $G = 11.5$ Mpsi



Part (a)

Our first step is to find an equation for our spring rate. We will say that spring rate k of a bar with angular deflection θ under torque T is as follow:

$$k = \frac{T}{\theta} \tag{1}$$

We also know that the angular deflection for a circular cross section rod under applied torque T with length l, material shear modulus G and polor moment of inertia J is:

$$\theta = \frac{Tl}{GJ} \tag{2}$$

Substituting eq.(2) into eq.(1) yields the final equation we will be using for calulating our spring rate:

$$k = \frac{GJ}{l} \tag{3}$$

In our case however, we will be varying the length of the rod sections. Rod section 1 will have length x and rod section 2 will have length $l - x$. This means that the coorisponding spring rates will be:

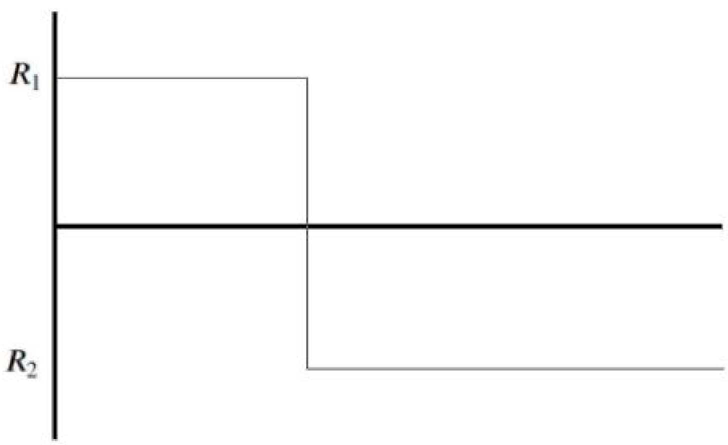
$$k_1 = \frac{GJ}{x} \tag{4}$$

$$k_2 = \frac{GJ}{l - x} \tag{5}$$

Linearity applies to spring constants of connected bars so we can say $k = k_1 + k_2$. Our equation for k will then be:

$$k = GJ\left(\frac{1}{x} + \frac{1}{l - x}\right) \tag{6}$$

Next we will solve for the reactions at end 1 and end 2, R_1 and R_2 respectively. A torque momement diagram will help us understand the forces at play in at each point of the rod:



Also using our static equalibrium equation we can say that:

$$R_1 + R_2 = T \tag{7}$$

.

This single equation isn't enough alone to define our two variables, but we can use eq.(2) to help us out. We know that from $[0, x)$ the rod is expriencing torque R_1 and $(x, l]$ the torque experienced is R_2. We also know that the section from $[0, x)$ must experience the same angular deflection as the section from $(x, l]$. Therefore we can plug each section into eq.(2) and set them equal to eachother:

$$\frac{R_1 x}{GJ} = \frac{R_2(l - x)}{GJ} \tag{8}$$

Combinding eq.(7) and eq.(8) yields:

$$R_1 = \frac{T(l - x)}{l} \tag{9}$$

$$R_2 = \frac{Tx}{l} \tag{10}$$

Part (b)

Now we will plug in the given values into eq.(6), eq.(9), and eq.(10) to get our spring rate, and reaction forces.

[1]:
```python
from thermostate import Q_, units
from numpy import pi
d = 0.5*units.inches
x = 5*units.inches
l = 10*units.inches
T = Q_(1500, "lbf*in")
G = Q_(11.5, "Mpsi")
J = pi*d**4/32

k = G*J*(1/x+1/(l-x))
R_1 = T*(l-x)/l
R_2 = T*x/l
print(k.to("kip*in/rad").round(2))
print(R_1.to("kip*in"))
print(R_2.to("kip*in"))
```

```
28.23 inch * kip / radian
0.75 inch * kip
0.75 inch * kip
```

$k = 28.23$ kip·in/rad
$R_1 = 0.75$ kip·in
$R_2 = 0.75$ kip·in

Looking at the torque diagram, we know that the maximum torques occur at 0 and l. But as we saw, R_1 and R_2 are the same, so we can only evaluate stresses at 0. We also know the the maximum stress under torque is:

$$\tau = \frac{Tr}{J} \tag{11}$$

Plugging in values yields:

[2]:
```
tau = R_1*(d/2)/J
print(tau.to("kpsi").round(2))
```

```
30.56 kilopound_force_per_square_inch
```

$\tau_{max} = 30.56$ kpsi

And since their are no normal stresses, we know that the τ experiences is the maximum τ.