Control Volume Analysis

- Conservation of mass and linear momentum in a region (control volume) of fluid
- Typical use is to find forces on objects or fluid volumes
- Integral form → only need information at volume surface
  ↘ "budget" of mass and momentum

1. Mass conservation (aka continuity)

$$\underbrace{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall}_{\substack{\text{unsteady term} \\ \text{total accumulation} \\ \text{or depletion of} \\ \text{fluid mass in CV}}} + \underbrace{\int_{CS} \rho \, \vec{V}\cdot\hat{n} \, dA}_{\substack{\text{fluid mass} \\ \text{flux through} \\ \text{the CV boundary}}} = 0 \quad \leftarrow \text{Scalar equation}$$

2. Momentum conservation

$$\underbrace{\frac{\partial}{\partial t}\int_{CV} \vec{V} \rho \, d\forall}_{\substack{\text{unsteady term} \\ \text{total accumulation} \\ \text{or depletion of} \\ \text{fluid momentum in CV}}} + \underbrace{\int_{CS} \vec{V} \rho \, \vec{V}\cdot\hat{n} \, dA}_{\substack{\text{fluid momentum} \\ \text{flux through} \\ \text{the CV boundary}}} = \underbrace{\sum \vec{F}}_{\substack{\text{total force} \\ \text{from stresses}}} \quad \leftarrow \textbf{VECTOR}\text{ equation}$$

x-component: $$\frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \, \vec{V}\cdot\hat{n} \, dA = \sum F_x$$

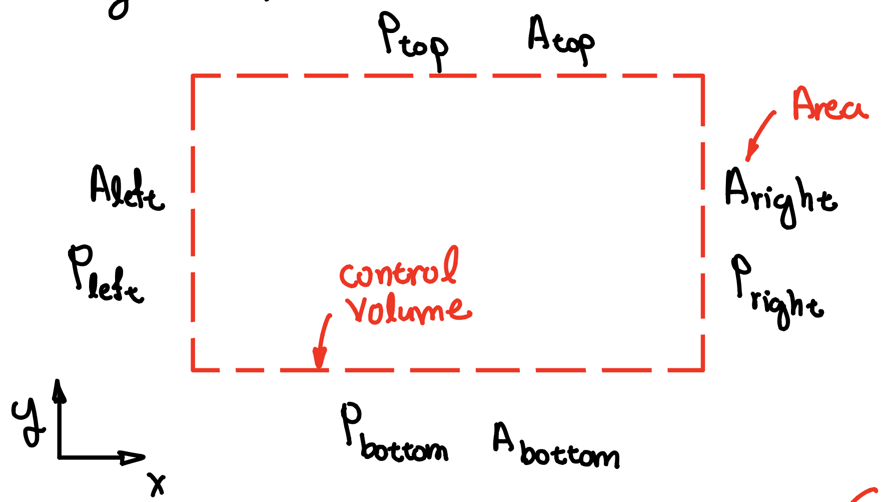
y-component: $$\frac{\partial}{\partial t}\int_{CV} v \rho \, d\forall + \int_{CS} v \rho \, \vec{V}\cdot\hat{n} \, dA = \sum F_y$$

remember: $\vec{V} = u\hat{i} + v\hat{j}$ in Cartesian (x, y)

What's the deal with $\sum \vec{F}$ on the RHS?

$\vec{F}$ contains the force from pressure (and potentialy other stresses)

Dealing with pressure:

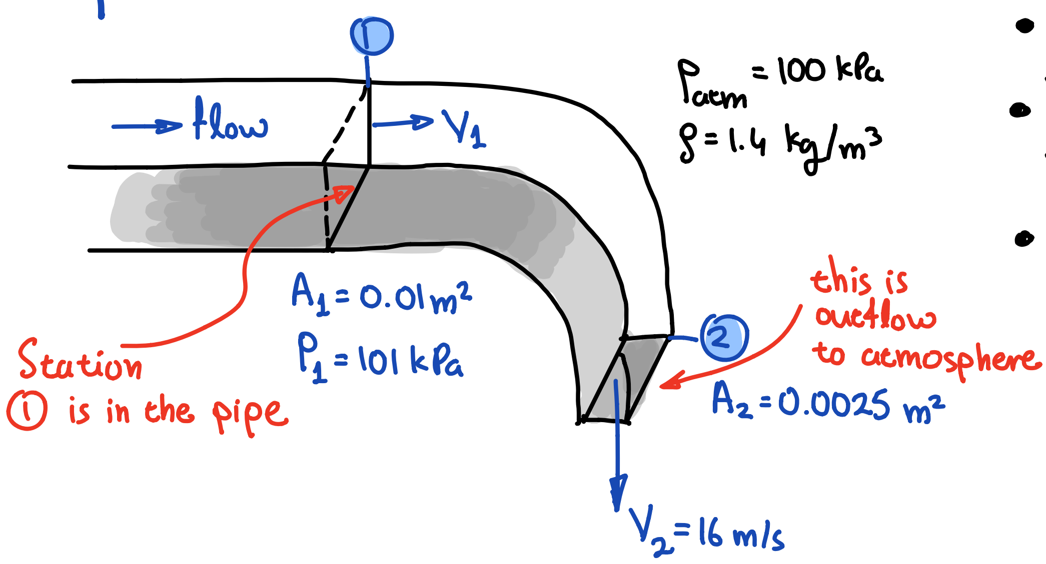


other forces on CV

$$\sum F_x = P_{left} A_{left} - P_{right} A_{right} + F_x$$

$$\sum F_y = P_{bottom} A_{bottom} - P_{top} A_{top} + F_y$$

Example 1:



Assume:

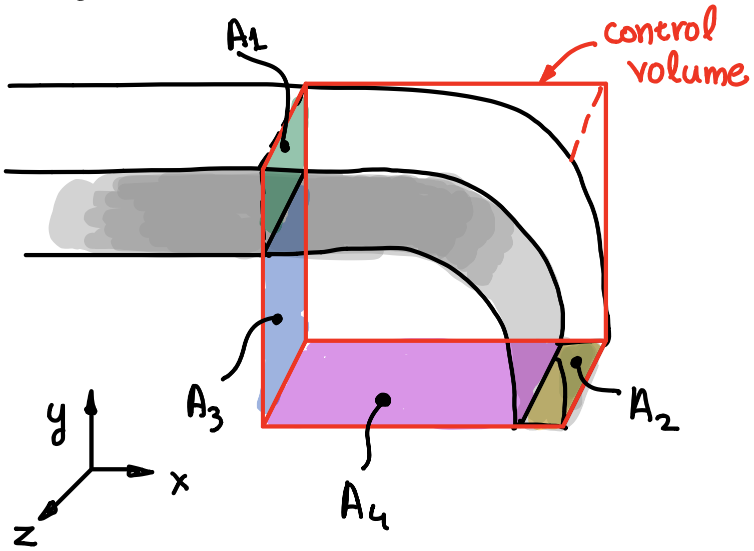
- Steady flow
- Incompressible fluid
- Uniform profiles at stations 1 and 2

- Find the force required to support the L-shaped pipe at location 1.
- Solution:

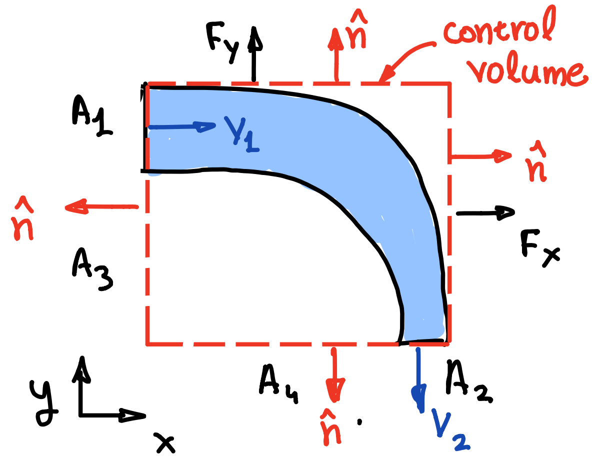
Steps:

1. Draw the control volume
2. Draw unit normal vectos and write velocity as vector
3. Apply mass conservation in CV
3. Apply momentum conservation in CV (all components as nessesary)

1. Draw control volume



2. Draw unit normals



$\vec{V}_1 = u_1\hat{i}$

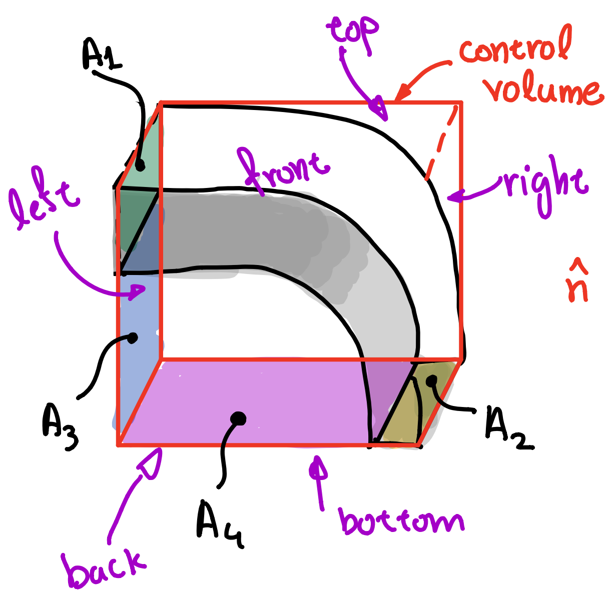
$\vec{V}_2 = -V_2\hat{j}$

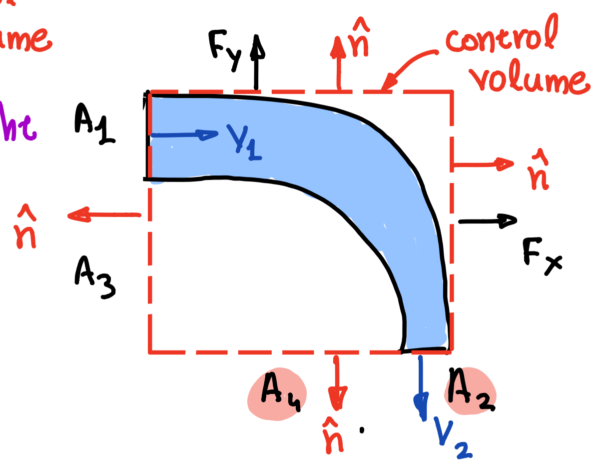
signs are important

3. Apply mass conservation

a. Use what we know: $A_1 V_1 = A_2 V_2 \Rightarrow V_1 = \frac{A_2}{A_1} V_2 = 4\ \text{m/s}$

b. Use control volume analysis





$\vec{V}_1 = u_1\hat{i}$

$\vec{V}_2 = -V_2\hat{j}$

$$\cancel{\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall} + \int_{cs} \rho \, \vec{V}\cdot\hat{n} \, dA = 0$$

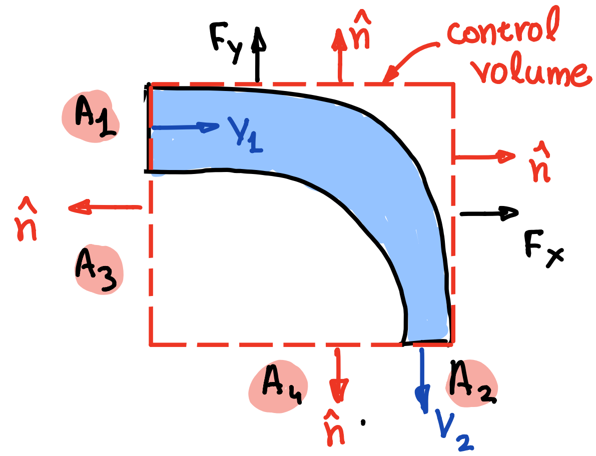
0 steady flow

$$\int_{front} + \int_{back} + \int_{top} + \int_{bottom} + \int_{left} + \int_{right}$$

$$\int_{front} \rho \vec{V}\cdot\hat{n}\, dA = \int_{front} \rho \vec{0}\cdot\hat{n}\, dA = 0$$

$$\int_{back} \rho \vec{V}\cdot\hat{n}\, dA = \int_{back} \rho \vec{0}\cdot\hat{n}\, dA = 0$$

$$\int_{top} \rho \vec{V}\cdot\hat{n}\, dA = \int_{top} \rho \vec{0}\cdot\hat{n}\, dA = 0$$



$$\int_{bottom} \rho \vec{V}\cdot\hat{n}\, dA = \int_{A_4} \rho \vec{0}\cdot\hat{n}\, dA + \int_{A_2} \rho (-V_2\hat{j})\cdot(-\hat{j})\, dA$$

$$\vec{V}_1 = u_1\hat{i}$$

$$\vec{V}_2 = -V_2\hat{j}$$

$$\int_{left} \rho \vec{V}\cdot\hat{n}\, dA = \int_{A_1} \rho (u_1\hat{i})\cdot(-\hat{i})\, dA + \int_{A_3} \rho \vec{0}\cdot\hat{n}\, dA$$

$$\int_{right} \rho \vec{V}\cdot\hat{n}\, dA = \int_{right} \rho \vec{0}\cdot\hat{n}\, dA = 0$$

$$\int_{A_2} \rho (-V_2\hat{j})\cdot(-\hat{j})\, dA + \int_{A_1} \rho (V_1\hat{i})\cdot(-\hat{i})\, dA = 0$$

$$\int_{A_2} \rho V_2\, dA - \int_{A_1} \rho V_1\, dA = 0$$

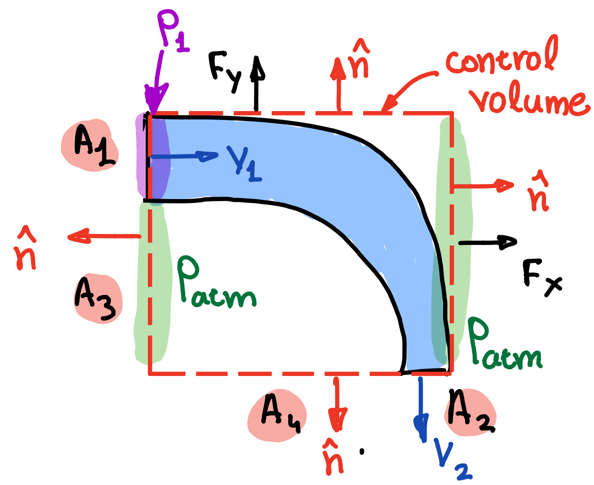
constant quantities
do not depent on (x,y,z)

$$\rho V_2 \int_{A_2} dA - \rho V_1 \int_{A_1} dA = 0$$

$$\rho V_2 A_2 - \rho V_1 A_1 = 0$$

$$V_1 = \frac{A_2}{A_1} V_2 = 4 \text{ m/s}$$

4a. Apply x-momentum conservation → find F_x



x-components of $\vec V$

x-component: $\frac{\partial}{\partial t}\cancel{\int_{CV} u\rho\, d\forall} + \int_{CS} u\rho\, \vec V\cdot\hat n\, dA = \sum F_x$

0 steady flow

$\vec V_1 = u_1\hat i$

$\vec V_2 = -v_2\hat j$

$\int_{CS} u\rho\, \vec V\cdot\hat n\, dA =$ pressure forces + F_x

$\int_{top+bottom+left+right+front+back}$

we will only write $\int_{cs}$ at places where $\vec V$ is non-zero to save time and effort

$$\int_{A_1} u\rho\, \vec V\cdot\hat n\, dA + \int_{A_2} u\rho\, \vec V\cdot\hat n\, dA = P_{left} A_{left} - P_{right} A_{right} + F_x$$

positive because pushes to the right

all left area

all right area

negative: pushes to the left

$$\int_{A_1} u_1\rho(u_1\hat i)\cdot(-\hat i)\, dA + \int_{A_2} 0\, \rho(-v_2\hat j)\cdot(-\hat j)\, dA = P_1 A_1 + P_{atm} A_3 - P_{atm}(A_1+A_3) + F_x$$

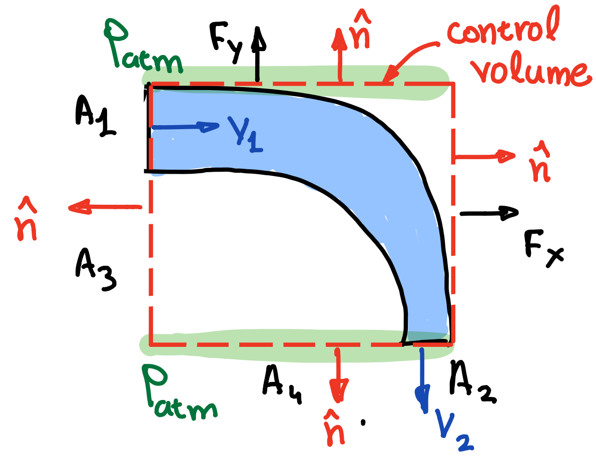
$$\int_{A_1} -V_1^2\rho\, dA = (P_1 - P_{atm})A_1 + F_x$$

constant

$$-V_1^2\rho A_1 = (P_1 - P_{atm})A_1 + F_x$$

$$F_x = -10.2\text{ N}$$

4b. Apply y-momentum conservation → find F_y



y-component: $\cancel{\frac{\partial}{\partial t}\int_{CV} v \rho \, d\forall} + \int_{CS} v \rho \, \vec{V}\cdot\hat{n} \, dA = \sum F_y$

0 steady flow

$$\int_{CS} v \rho \, \vec{V}\cdot\hat{n} \, dA = \text{pressure forces} + F_y$$

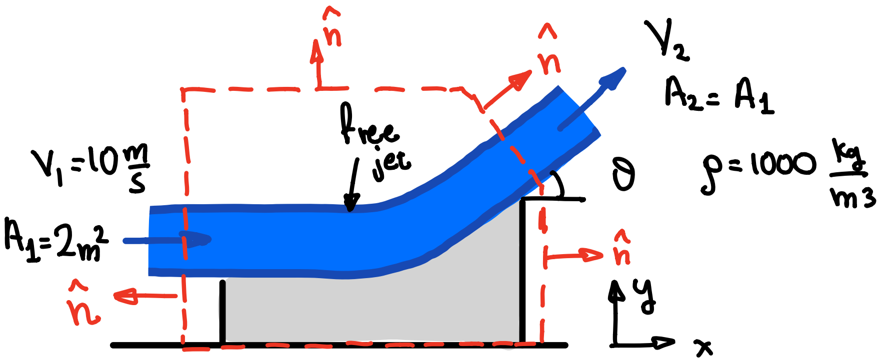
$$\int_{A_1} 0 \, \rho \, (u_1 \hat{i})\cdot(-\hat{i}) \, dA + \int_{A_2} -V_2 \, \rho \, (-V_2 \hat{j})\cdot(-\hat{j}) \, dA =$$

$$\underbrace{P_{atm}(A_2 + A_4)}_{\text{bottom}} - \underbrace{P_{atm}(A_2 + A_4)}_{\text{top}} + F_y$$

$$\int_{A_2} -V_2^2 \rho \, dA = F_y$$

$$-V_2 \rho A_2 = F_y \rightarrow F_y = -0.896 \text{ N}$$

Example 2 (example 3.10 of textbook)



A vane deflects a free jet at an angle θ.
Calculate the force required to hold the vane in place

Find F_x and F_y on vane

Assume:
- steady flow
- incompressible flow
- uniform velocity $\vec{V}_1$ and $\vec{V}_2$ profiles

Solution: Step 1: draw CV

Step 2: draw $\hat{n}$ and write $\vec{V}$

$$\vec{V}_1 = 10\,\hat{i} = V_1\,\hat{i}$$

$$\vec{V}_2 = u_2\,\hat{i} + v_2\,\hat{j} = V_2 \cos\theta\,\hat{i} + V_2 \sin\theta\,\hat{j}$$

V_2 without $\rightarrow$ is the magnitude

Step 3: Apply mass conservation

magnitudes

$$A_1 V_1 = A_2 V_2 \Rightarrow V_2 = V_1 = 10 \text{ m/s}$$

Step 4: x-momentum conservation

$$\underbrace{\frac{\partial}{\partial t}\int_{CV} \rho u \, d\forall}_{0:\text{ steady flow}} + \int_{CS} \rho u \vec{V}\cdot\hat{n}\, dA = \Sigma F_x$$

$$\int_{A_1} V_1 \rho (V_1 \hat{i})\cdot(-\hat{i})\, dA + \int_{A_2} V_1 \cos\theta \, \rho \overbrace{(V_1 \cos\theta\,\hat{i} + V_1 \sin\theta\,\hat{j})\cdot(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})}^{V_1} dA$$

$$= \underbrace{\text{pressure forces}}_{=0} + F_x$$

$$\int_{A_1} -V_1^2 \rho \, dA + \int_{A_2} V_1 \cos\theta \, \rho \, V_1 \, dA = 0 + F_x$$

$$F_x = V_1^2 \cos\theta \, \rho A_2 - V_1^2 \rho A_1 \qquad (A_1 = A_2)$$

$$F_x = -V_1^2 \rho A_1 (1 - \cos\theta)$$ negative: to the left

no pressure contribution because pressure is atmospheric everywhere y-momentum conservation

$$\underbrace{\frac{\partial}{\partial t}\int_{CV} v\,\rho\, \cancel{dV}}_{0:\text{ steady flow}} + \int_{CS} v\,\rho\, \vec{V}\cdot\hat{n}\, dA = \Sigma F_y$$

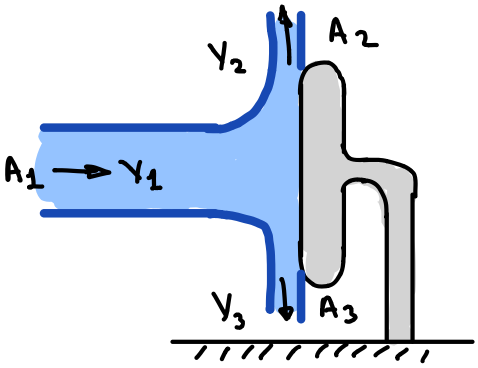
$$\int_{A_1} 0\, \rho\,(V_1\hat{i})\cdot(-\hat{i})\,dA + \int_{A_2} V_1\sin\theta\, \rho\,\overbrace{(V_1\cos\theta\,\hat{i} + V_1\sin\theta\,\hat{j})}^{V_1}\cdot(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\,dA = F_y$$

same as x-momentum

different than x-momentum

$$F_y = V_1^2\,\rho\,\sin\theta\, A_1$$

Example 3: the choise of CV does not matter but some CV choices are easier than others...



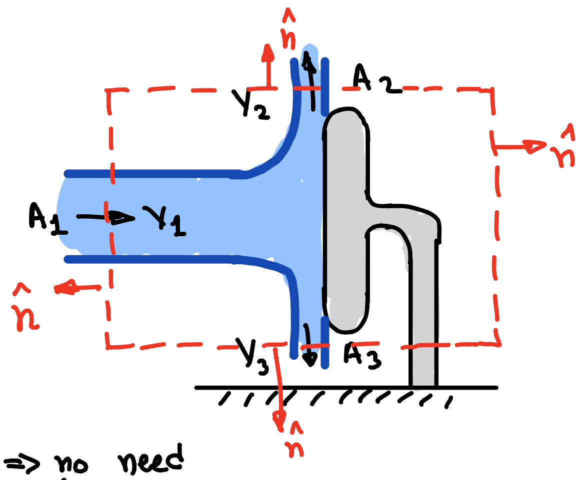
Incompressible and steady flow
Uniform velocity profiles

$A_2 = A_3$

$V_2 = V_3$

Find force on the stand

Control Volume 1:



$\vec{V}_1 = V_1 \hat{i}$

$\vec{V}_2 = V_2 \hat{j}$

$\vec{V}_3 = -V_3 \hat{j}$

- we know V_1, V_2, V_3 => no need for mass conservation
- the pressure is everywhere on CV surface atmospheric

x-momentum:

$$\cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV} + \int_{CS} u\rho\, \vec{V}\cdot\hat{n}\, dA = \Sigma F_x$$

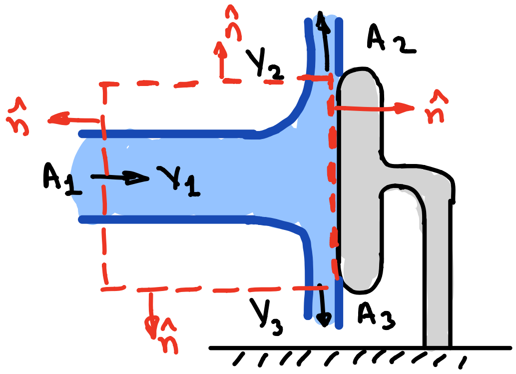
0 steady flow

$$\int_{A_1} V_1\, \rho\, (V_1\hat{i})\cdot(-\hat{i})\, dA + \int_{A_2} 0\, \rho\, \vec{V}\cdot\hat{n}\, dA + \int_{A_3} 0\, \rho\, \vec{V}\cdot\hat{n}\, dA = P_{atm} A_{left} - P_{atm} A_{right} + F_x$$

$$\int_{A_1} V_1\, \rho\, (-V_1)\, dA = F_x$$

$$F_x = -\rho V_1^2 A_1$$

Control Volume 2:



x - momentum:

$$\frac{\partial}{\partial t}\cancel{\int_{CV} u\rho\, dV} + \int_{CS} u\rho\, \vec{V}\cdot\hat{n}\, dA = \sum F_x$$

0 steady flow

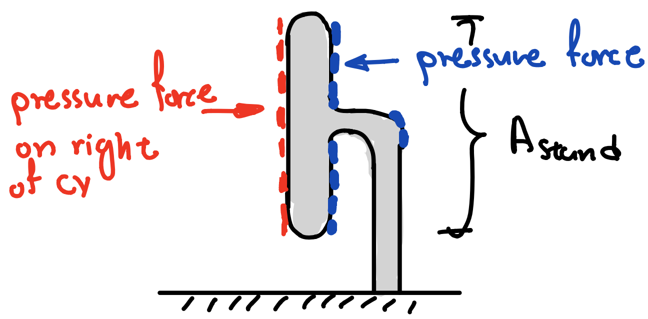
$$\int_{A_1} V_1\, \rho\, (V_1\hat{i})\cdot(-\hat{i})\,dA + \int_{A_2} 0\,\rho\, \vec{V}\cdot\hat{n}\, dA + \int_{A_3} 0\,\rho\,\vec{V}\cdot\hat{n}\,dA =$$

$$P_{atm} A_{left} - \text{pressure force on right}$$

$$-\rho V_1^2 A_1 = P_{atm} A_{left} - \text{pressure force on right}$$

$$\text{pressure force on right} = +\rho V_1^2 A_1 - P_{atm} A_{left}$$

Force on stand:



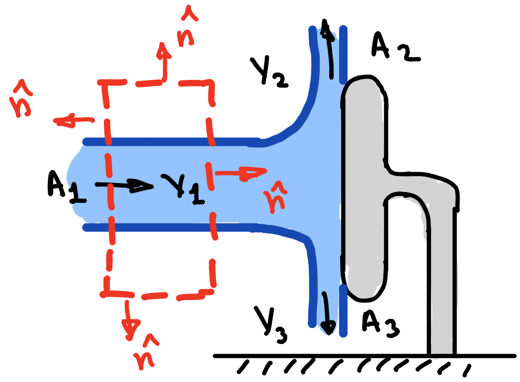
$$-(\text{pressure force on right of cv}) - P_{atm} A_{stand} = F_x$$

$$-\rho V_1^2 A_1 + P_{atm} A_{left} - P_{atm} A_{stand} = F_x$$

$$A_{left} = A_{stand}$$

$$F_x = -\rho V_1^2 A_1$$

Control Volume 3:



$$\int_{A_{1,left}} V_1 \, \rho \, \vec{V}_1 \cdot \hat{n} \, dA + \int_{A_{1,right}} V_1 \, \rho \, \vec{V}_1 \cdot \hat{n} \, dA \overset{?}{=} F_x$$

$$\int_{A_{1,left}} V_1 \, \rho \, (-V_1) \, dA + \int_{A_{1,right}} V_1 \, \rho \, (+V_1) \, dA \overset{?}{=} F_x$$

$$-V_1^2 \rho A_1 + V_1^2 \rho A_1 = F_x \Rightarrow F_x = 0$$ what???

Not a useful control volume!

Momentum does not change in CV!!!

We cannot find force on stand if CV does not include or touch the stand