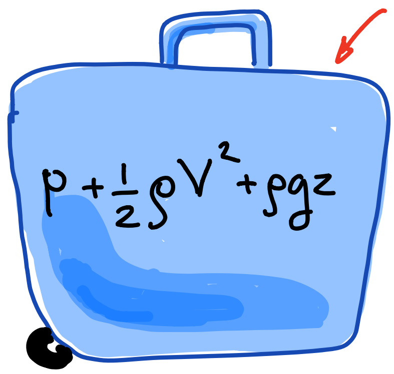
The Bernoulli Equation (Chapter 3)

Module 2: momentum conservation when no relative motion $\nabla p = -\rho \vec{a}_t$

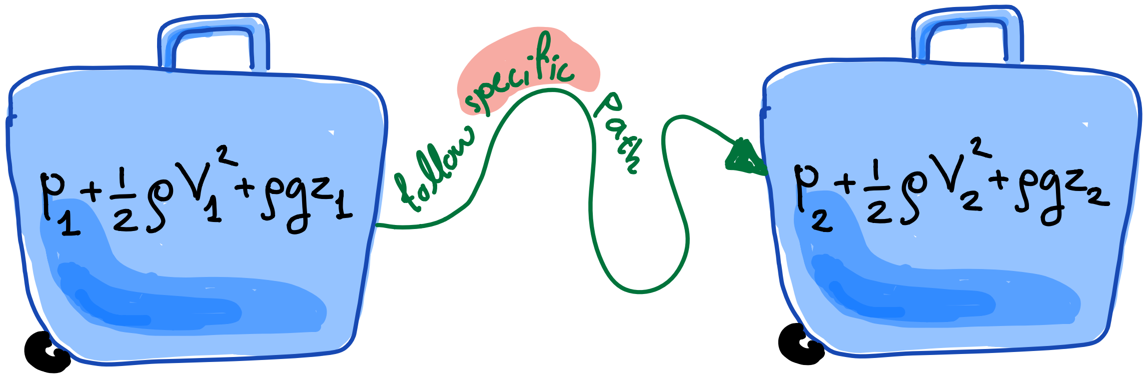
Module 3 (this module):
- Allow for fluid motion
- **BUT** restrict our application to
  - → specific locations
  - → and other assumptions

The Bernoulli Equation:



whatever is in the bag is conserved

Location 1 Location 2



Math: $p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2$

Important: Assumptions:
1. Viscous effects are negligible
2. Flow is steady
3. Flow is incompressible: $\rho = \text{const}$
4. Bernoulli eq. valid along a streamline

Assumptions:
1. Viscous effects are negligible ✓
2. Flow is steady **NEW**
3. Flow is incompressible : $\rho = \text{const}$ ✓
4. Bernoulli eq. valid along a streamline **NEW**

The Velocity Field

1. Vector field representation $\vec{u}(t, \underbrace{x, y, z})$

at each location in space

and every time instance

we have a vector

$$\vec{u} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k}$$

components

unit vectors

so... $\vec{u}(t,x,y,z) = u(t,x,y,z)\,\hat{i} + v(t,x,y,z)\,\hat{j} + w(t,x,y,z)\,\hat{k}$

Velocity vector magnitude $V = \sqrt{u^2 + v^2 + w^2}$ ← scalar non-negative quantity

2. Steady and unsteady flow

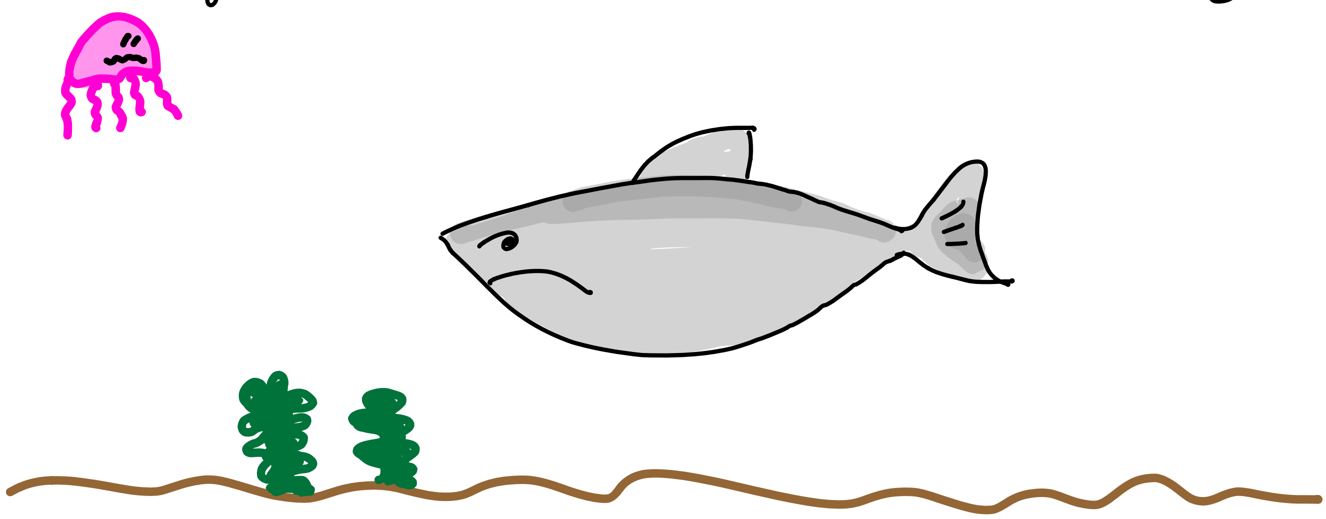
$$\frac{\partial \vec{u}}{\partial t} = \frac{\partial u}{\partial t}\hat{i} + \frac{\partial v}{\partial t}\hat{j} + \frac{\partial w}{\partial t}\hat{k} \begin{cases} = 0 & \text{steady flow} \\ \neq 0 & \text{unsteady flow} \end{cases}$$

$$\vec{0} \equiv 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$$

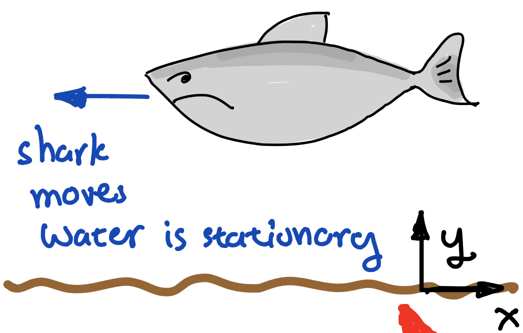
remember : $\frac{\partial}{\partial t}$ is partial derivative w.r.t. time

3. Frame of reference

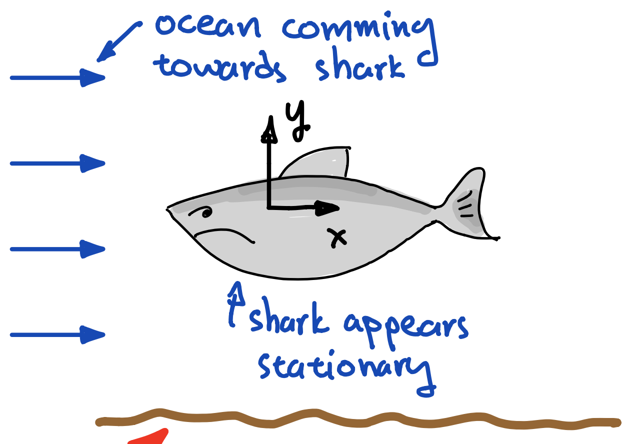
usually fluid mechanics are done in the frame of the moving object



Frame of ocean floor



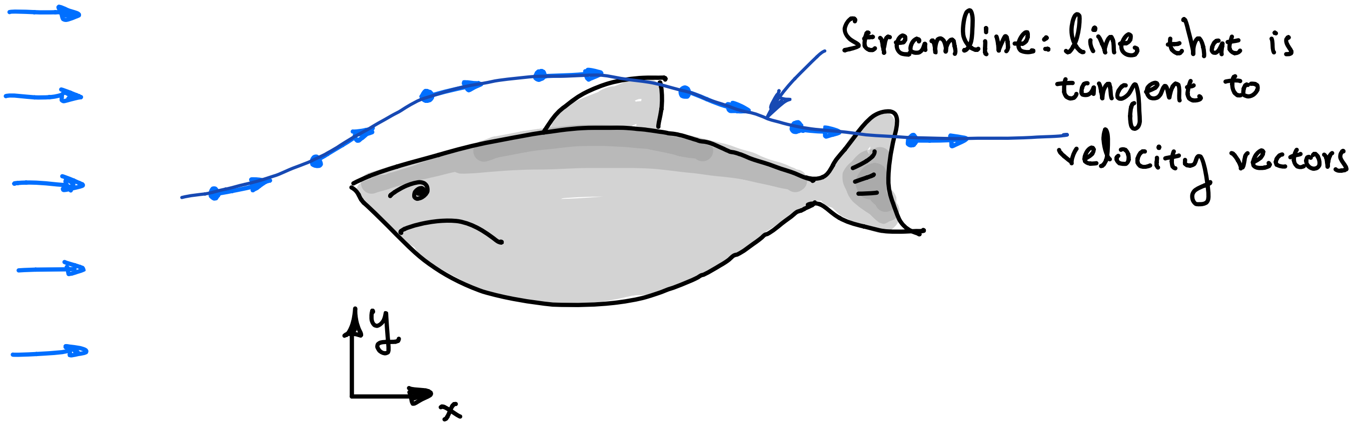
Frame of shark



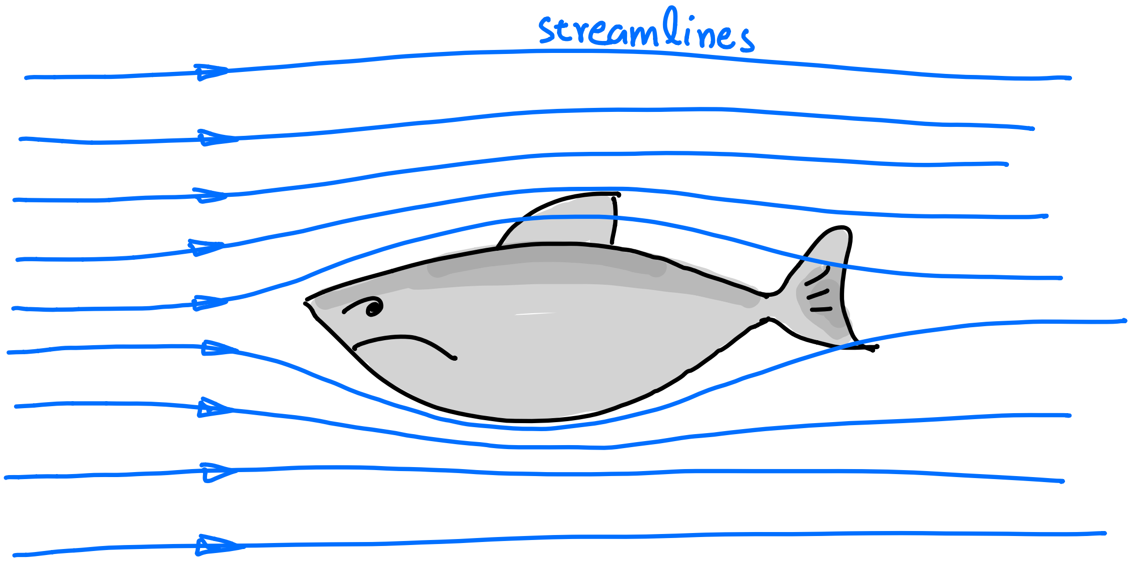
SAME FLUID DYNAMICS!

4. Streamlines

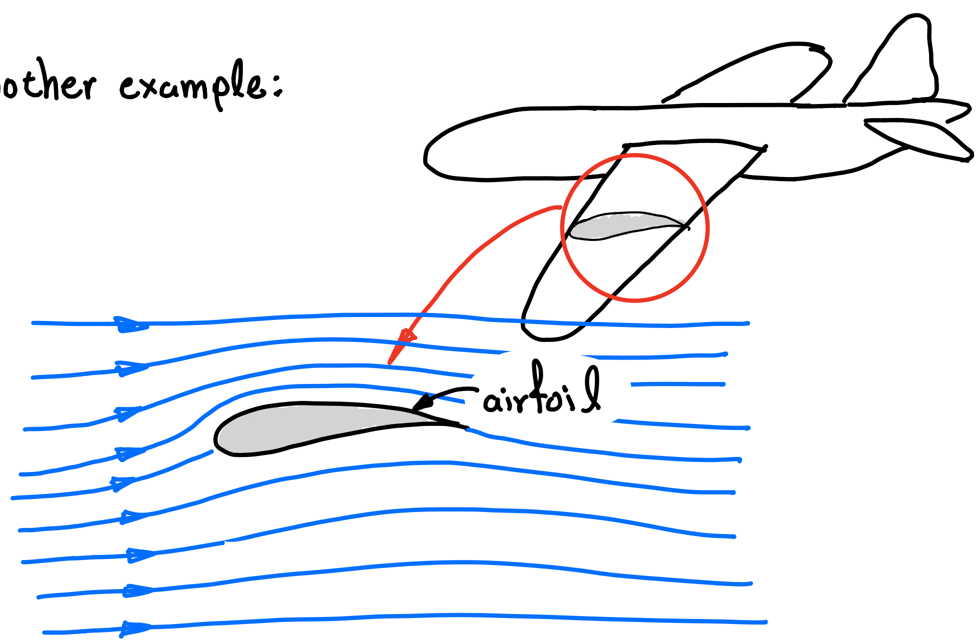
Shark frame of reference



- Streamlines are used to visualize the flow field
- For steady flow streamlines correspond to particle paths



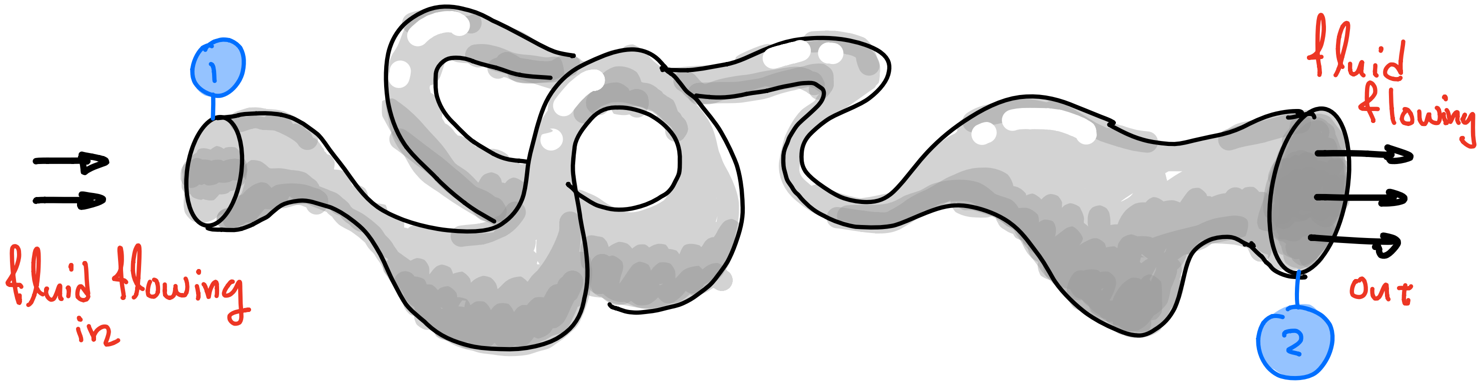
Another example:



5. Conservation of mass

••• Simplified ••• discussion •••

general case discussed in Module 4 (Chapter 4)



velocity vector
- normal to Area 1
- uniform over Area 1
- magnitude V_1

same for (2)

Conservation of mass implies :

fluid mass flowing in = fluid mass flowing out

$$\rho V_1 A_1 = \rho V_2 A_2$$

incompressible fluid $\rho = \text{const}$

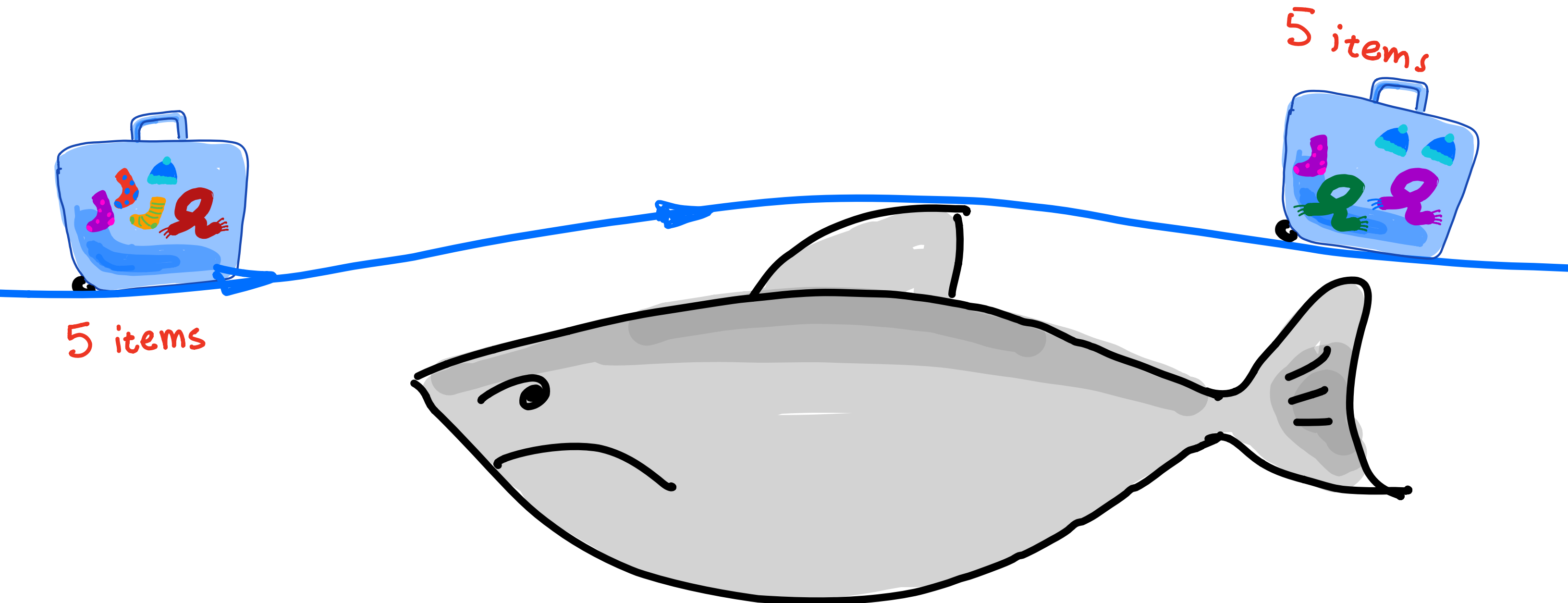
$$\Rightarrow V_1 A_1 = V_2 A_2$$

Volumetric flow rate : $Q = VA$ Volume per unit time $\frac{m^3}{s}$

Mass flow rate : $\dot{M} = \rho V A$ Mass per unit time kg/s

The Bernoulli equation along streamlines

<u>Assumptions</u>:
1. Viscous effects are negligible
2. Flow is steady
3. Flow is incompressible : $\rho = \text{const}$
4. Bernoulli eq. valid along a streamline



hydrostatic term

$$p + \frac{1}{2}\rho V^2 + \rho g z = \text{constant along streamline}$$

Scalar equation

pressure
density
Velocity MAGNITUDE
acceleration of gravity
vertical coordinate

Strategy for solving problems
→ How to use the Bernoulli equation

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2$$

- 6 variables → know 5 can find the 6th
- usually z is known

  → 2 velocities and 2 pressures

Typically both velocity and pressure are unknown

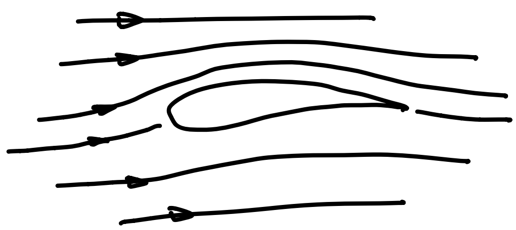
in this case: use mass conservation to find v : $Q_1 = Q_2$
$A_1 v_1 = A_2 v_2$

use Bernoulli to find pressure

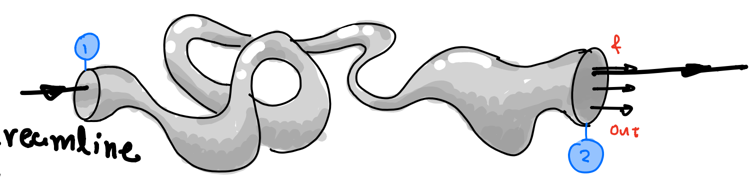
How do I know what is a streamline?

Case A: "external" flow

streamlines are given to you



Case B: internal flow



you can connect inflow and outflow yourselves with a streamline

Absolute or gage pressure?

$$P_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = P_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2$$

$$P_{1,absolute} + \frac{1}{2}\rho V_1^2 + \rho g z_1 = P_{2,absolute} + \frac{1}{2}\rho V_2^2 + \rho g z_2$$

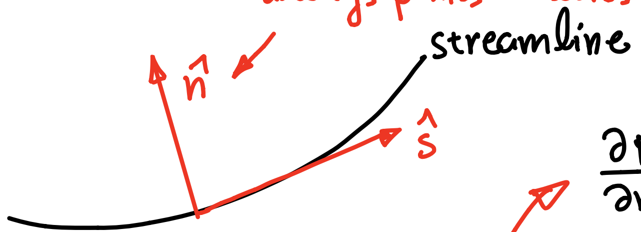
$$P_{1,gage} + \cancel{P_{atm}} + \frac{1}{2}\rho V_1^2 + \rho g z_1 = P_{2,gage} + \cancel{P_{atm}} + \frac{1}{2}\rho V_2^2 + \rho g z_2$$

$$P_{1,gage} + \frac{1}{2}\rho V_1^2 + \rho g z_1 = P_{2,gage} + \frac{1}{2}\rho V_2^2 + \rho g z_2$$

Use can use either absolute or gage but both sides of the equation must be the same kind of pressure

Bernoulli equation normal to streamline

always points towards the "inside"



$$\frac{\partial p}{\partial n} = -\frac{\rho V^2}{\mathcal{R}}$$

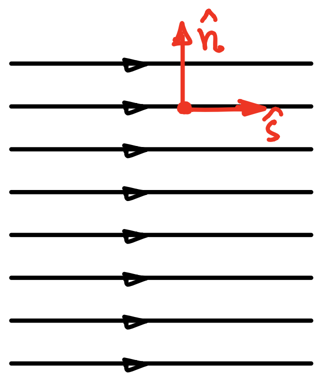
differential form

radious of curvature

$\mathcal{R} = R$ for circular streamlines

Important limiting case:

straight streamlines : $\mathcal{R} \to \infty$



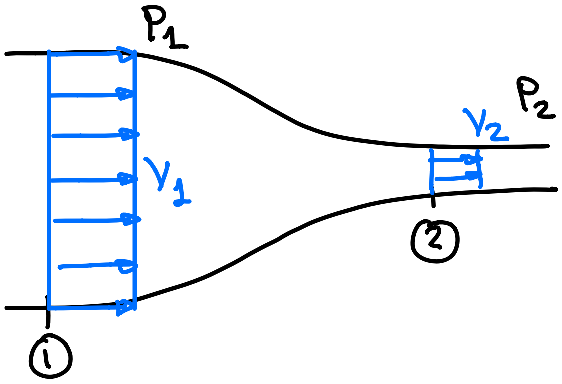
$$\Rightarrow \frac{\partial p}{\partial n} = -\underbrace{\frac{\rho V^2}{\mathcal{R} \to \infty}}_{\to 0}$$

$\frac{\partial p}{\partial n} = 0 \Rightarrow$ p not function of $\hat{n}$

IMPORTANT

$\Rightarrow$ pressure does not vary normal to straight streamlines

Example 1: Basic use of Bernoulli equation



Given:
- uniform $V_1 = 10$ m/s
- $P_1 = 2\times10^5$ Pa
- $\rho = 1000$ kg/m³
- Area $A_1 = 1\,m^2$
- Area $A_2 = 0.5\,m^2$

Find: V_2
P_2

Assume: uniform V_2 (1)
steady flow (2)
incompressible fluid (3)
inviscid fluid (4)
(1) and (2) at same height (5)
→ neglect hydrostatic part

Solution:

- We are asked for pressure so sounds like Bernoulli
- Check assumptions

1. Viscous effects are negligible (4) ✓
2. Flow is steady (2) ✓
3. Flow is incompressible: $\rho = \text{const}$ (3) ✓
4. Bernoulli eq. valid along a streamline (1) ✓

- Apply Bernoulli from (1) to (2):

$$P_1 + \frac{1}{2}\rho V_1^2 + \cancel{\rho g z_1} = P_2 + \frac{1}{2}\rho V_2^2 + \cancel{\rho g z_2}$$ because of (5)

two unknowns!

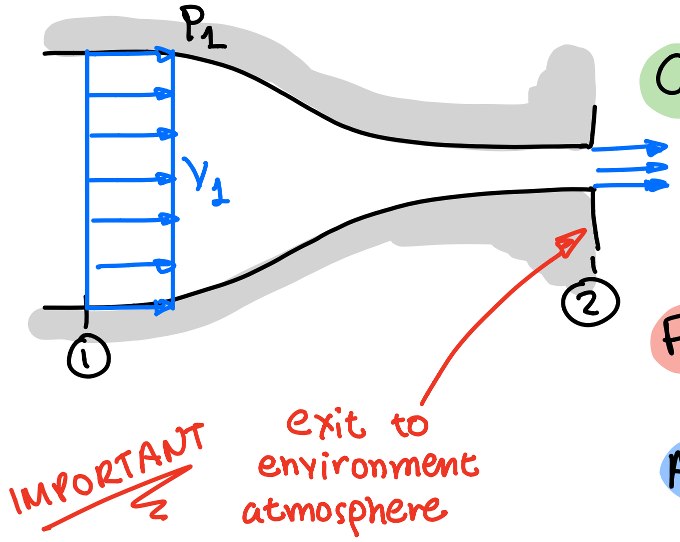
- use mass conservation to find V_2:

$$Q_1 = Q_2 \Rightarrow A_1V_1 = A_2V_2 \Rightarrow V_2 = \frac{A_1}{A_2}V_1 = \frac{1}{0.5}10 = 20 \text{ m/s}$$

- Back to Bernoulli: $P_2 + \frac{1}{2}\rho V_2^2 = P_1 + \frac{1}{2}\rho V_1^2$

$$P_2 = P_1 + \frac{1}{2}\rho(V_1^2 - V_2^2)$$
$$= 2\times10^5 + \frac{1}{2}1000(10^2 - 20^2)$$
$$= 0.5\times10^5 \text{ Pa}$$

Example 2 : Intermidiate use of Bernoulli equation



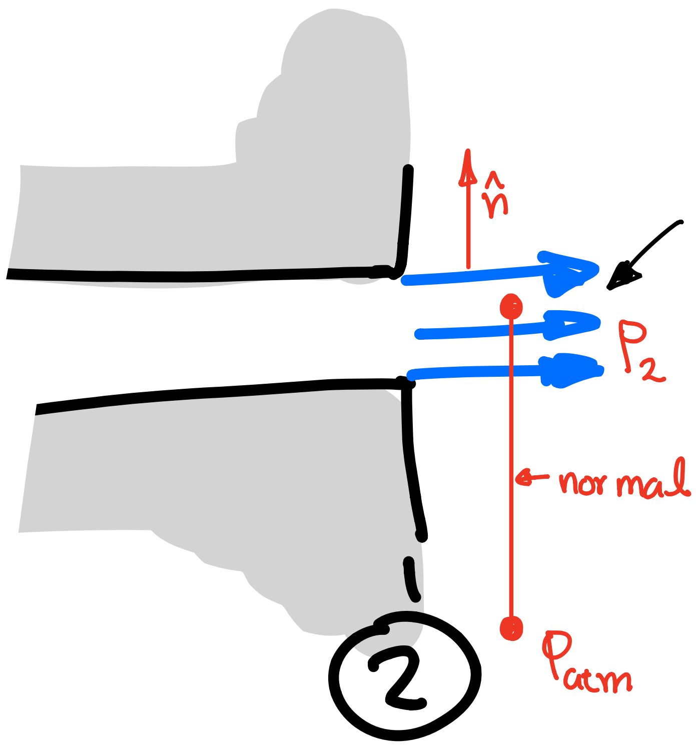
Given:
- uniform $V_1 = 10$ m/s
- $\rho = 1000$ kg/m³
- Area $A_1 = 1\,m^2$
- Area $A_2 = 0.5\,m^2$

Find: V_2
P_2

Assume:
uniform V_2 (1)
steady flow (2)
incompressible fluid (3)
inviscid fluid (4)
(1) and (2) at same height (5)
→ neglect hydrostatic part

Solution:
- check assumptions
- $Q_1 = Q_2 \Rightarrow V_2 = 20$ m/s
- Bernoulli (1)-(2) : $P_1 = P_2 + \frac{1}{2}\rho\left(V_2^2 - V_1^2\right)$

?



straight streamlines

$\Rightarrow$ pressure does not vary normal to streamlines

$\Rightarrow P_2 = P_{atm}$

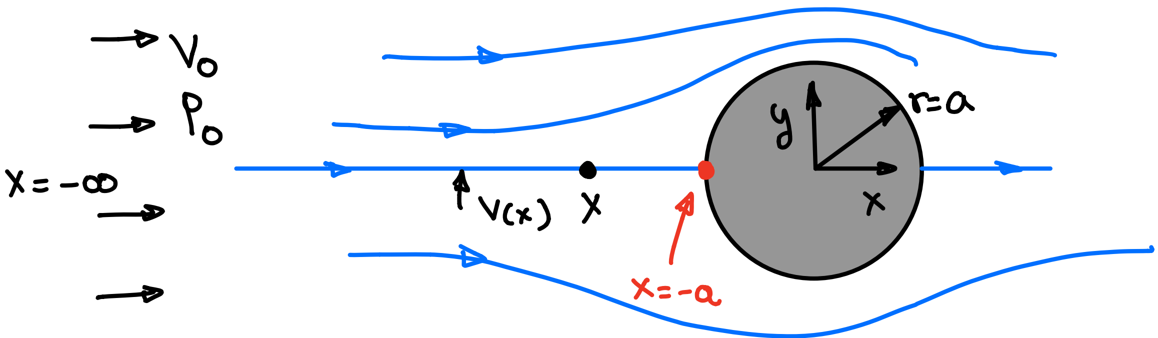
$$P_1 = P_2 + \frac{1}{2}\rho\left(V_2^2 - V_1^2\right)$$

$$P_1 - P_{atm} = \frac{1}{2}\rho\left(V_2^2 - V_1^2\right)$$

$$P_{1,gage} = \frac{1}{2}\rho\left(V_2^2 - V_1^2\right) = \frac{1000}{2}\left(20^2 - 10^2\right) = 150{,}000 \text{ Pa}$$

Example 3 (example 3.1 of textbook)

Flow around sphere of radius $r=a$



Assume: steady, incompressible ρ, inviscid flow

Given: $V(x) = V_0\left(1+\frac{a^3}{x^3}\right)$ velocity on streamline along x-axis

Find: $p(x)$ along streamline

Bernoulli between $x=-\infty$ and any point x

$$P_0 + \frac{1}{2}\rho V_0^2 = p(x) + \frac{1}{2}\rho V(x)^2$$

$$p(x) = P_0 + \frac{1}{2}\rho\left[V_0^2 - V(x)^2\right]$$

$$p(x) = P_0 + \frac{1}{2}\rho\left[V_0^2 - V_0^2\left(1+\frac{a^3}{x^3}\right)^2\right]$$

$$p(x) = P_0 + \frac{1}{2}\rho\left[V_0^2 - V_0^2 - 2V_0^2\frac{a^3}{x^3} - V_0\left(\frac{a}{x}\right)^6\right]$$

$$p(x) = P_0 - \rho V_0^2\left[\left(\frac{a}{x}\right)^3 + \frac{1}{2}\left(\frac{a}{x}\right)^6\right]$$

What is the pressure at $x=-a$?

$$p(-a) = P_0 - \rho V_0^2\left[-1+\frac{1}{2}\right] = P_0 + \frac{1}{2}\rho V_0^2$$

The point at $x=-a$ is special because there $V=0$

it is called the stagnation point

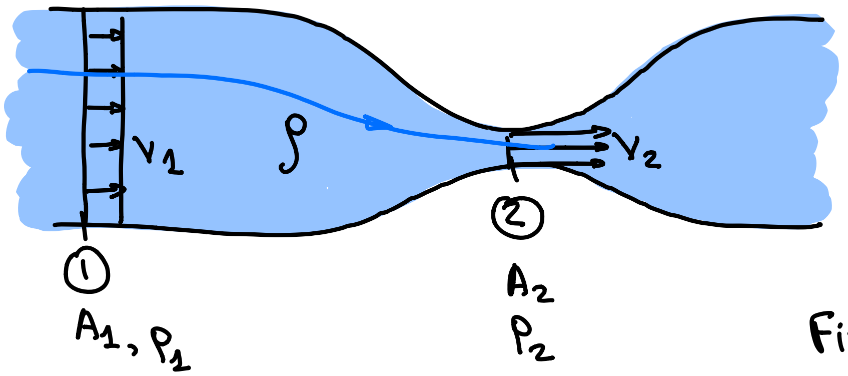
Actually, to find the pressure at the stagnation point we do not need $V(x)$ because we know $V=0$:

$$P_0 + \frac{1}{2}\rho V_0^2 = P_{stagnation} + \frac{1}{2}\rho \cancelto{0}{V_s^2}$$

stagnation pressure

$$P_{stagnation} = P_0 + \underbrace{\frac{1}{2}\rho V_0^2}_{\text{dynamic pressure}}$$

Example 4: Flowrate measurement: Venturi meter



Assume:
- steady
- incompressible
- inviscid

Find: flow rate Q

Given: p_1, p_2, ρ, A_1, A_2

Bernoulli (1) → (2)

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

Mass conservation: $Q = A_1 V_1 = A_2 V_2$ → $V_1^2 = \dfrac{Q^2}{A_1^2}$, $V_2^2 = \dfrac{Q^2}{A_2}$

$$p_1 + \frac{1}{2}\rho\frac{Q^2}{A_1^2} = p_2 + \frac{1}{2}\rho\frac{Q^2}{A_2^2}$$

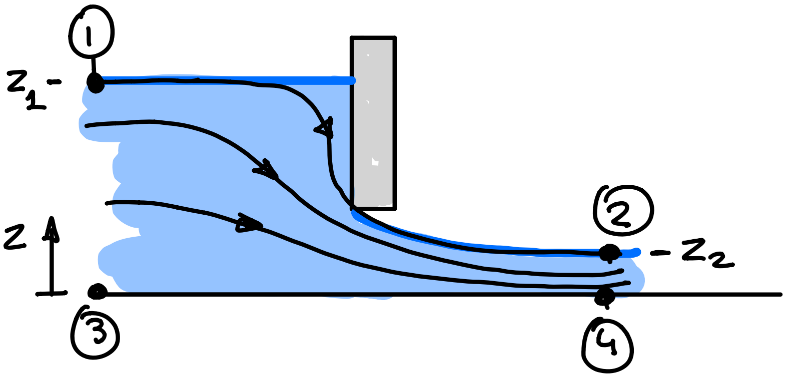
$$\frac{1}{2}\rho Q^2\left(\frac{1}{A_1^2} - \frac{1}{A_2}\right) = p_2 - p_1$$

$$Q^2 = \frac{2(p_2 - p_1)}{\rho\left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right)}$$

Flow rate Q is a function of Δp

Q is proportional to $\sqrt{\Delta p}$

Example 5: Sluice gate



Given: ρ, z_1, z_2

Find : Flowrate Q

All Bernoulli assumptions

Mass conservation: $Q = A_1 V_1 = A_2 V_2$

$= z_1 b V_1 = z_2 b V_2$ → $V_1 = \frac{Q}{b z_1}$ → $V_2 = \frac{Q}{b z_2}$

Bernoulli streamline: 1 → 2

$$P_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = P_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2$$

$$P_1 = P_2 = P_{atm}$$

$$\frac{1}{2}\rho \frac{Q^2}{b^2 z_1^2} + \rho g z_1 = \frac{1}{2}\rho \frac{Q^2}{b^2 z_2^2} + \rho g z_2$$

$$Q = b z_1 z_2 \sqrt{\frac{2g}{z_1 + z_2}}$$

Another way: Bernoulli streamline 3 → 4

$$P_3 + \frac{1}{2}\rho V_1^2 + \rho g z_3 = P_4 + \frac{1}{2}\rho V_2^2 + \rho g z_4$$

$$z_3 = z_4$$

$P_3 = ?$ $\qquad$ $P_4 = ?$

$P_3 = P_1 + \rho g z_1$

Hydrostatic pressure distribution: $\Delta p = -\rho g \, \Delta z$ → $P_4 = P_2 + \rho g z_2$

$$P_1 + \rho g z_1 + \frac{1}{2}\rho V_1^2 = P_2 + \rho g z_2 + \frac{1}{2}\rho V_2^2$$ ← same equation as before!