Fluid Dynamics Equation Sheet

- Chapter 1: Introduction
  - Gage Pressure
    - $P_{gage} = P_{abs} - P_{ambient}$
      - p_{abs} = absolute pressure
      - $p_{ambient}$ = local pressure
      - p_{gage} = what engineers measure
  - Specific weight
    - $\gamma = \rho g$
  - Specific Gravity
    - $SG = \frac{\rho}{\rho_{H_2O@4^oC}}$
  - Ideal Gas Law
    - $\rho = \frac{p}{RT}$
      - p = absolute pressure
      - R =a substance specific gas constant
      - T= absolute temperature
  - Shear Stress
    - $\tau = \frac{F}{A}$
  - Newtonian Fluid Shear Stress
    - $\tau = \mu\frac{\partial u}{\partial y}$
      - μ = dynamic viscosity
      - u = fluid velocity
      - y =the y direction
  - $\frac{\partial}{\partial t}(Momentum) = \nabla \cdot \tau + "other\ stuff"$
    - Divide by density on both sides
  - $\frac{\partial}{\partial t}(Velocity) = \nabla \cdot \frac{\tau}{\rho}$
    - Where $\tau$is stress
- Chapter 2: Fluid Statics
  - Equation of Pressure Gradient
    - $\vec{\nabla} p =- \rho \vec{a}_t$
      - $\vec{\nabla} p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$
      - $\vec{a}_t$ =acceleration

- Special Pressure Equation Cases 1a:
  - No motion and incompressible
    - $\frac{\partial p}{\partial x} = 0$
    - $\frac{\partial p}{\partial y} = 0$
    - $\frac{\partial p}{\partial z} = -\rho g$
      - $\vec{a}_{t,ext} = 0$
        - External forces are 0 and fluid is incompressible
      - $-\rho g$is from gravity
- Special Pressure Equation Cases 1b:
  - No motion and compressible
    - $\frac{\partial p}{\partial z} = -\rho(z)g$
      - Most gasses are this way
      - Density is a function of position
- Special Pressure Equation Cases 2a:
  - Constant acceleration and incompressible
    - $\frac{\partial p}{\partial x} = -\rho a_x$
    - $\frac{\partial p}{\partial y} = -\rho a_y$
    - $\frac{\partial p}{\partial z} = -\rho g - \rho a_z$
      - $\vec{a}_t = \langle a_x, a_y, a_z \rangle$

- Buoyancy Force
  - $F_b = \rho g V = \gamma V$
    - ρ =the density of the fluid
    - V = the volume that is submerged
    - $\gamma = \rho g$ = specific weight
- Line of constant pressure
  - Multivariable Calculus review:
    - $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = 0$
      - For constant pressure, derivative $dp = 0$
      - This come from the 2d equation:
        - $y - y_0 = \frac{dy}{dx}(x - x_0)$

            - $dy = \frac{dy}{dx}dx$

- Chapter 3: The Bernoulli Equation
    - Bernoulli's Equation
        - $p + \frac{1}{2}\rho V^2 + \rho gz = \text{Constant}$
            - p = pressure
            - ρ =fluids density
            - V = Velocity **magnitude**
            - $V = \sqrt{x^2 + y^2 + z^2}$
            - z = height
            - Assumptions
                - No Viscosity
                - Steady flow
                - Incompressible
                - Only along streamlines
    - Steady State flow definition
        - $\frac{\partial \vec{V}}{\partial t} = \vec{0}$
    - Conservation of mass flow
        - $\rho V_1 A_1 = \rho V_2 A_2$
            - V = Velocity
            - A = Pipe area
    - Volumetric flow rate
        - $Q = VA$
            - Q =volumetric flow rate
    - Mass flow rate
        - $\dot{M} = \rho VA = \rho Q$
            - $\dot{M}$ = mass flow rate
- Chapter 4:Fluid Kinematics
    - Equation for Streamlines
        - $\frac{dx}{u} = \frac{dy}{v} = \frac{dw}{z}$
            - For two dimensions:
                - $\frac{dy}{dx} = \frac{v}{u}$
                - Because stream line is always **tangent** to velocity field

- Acceleration
  - $\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$
    - $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$W
    - $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$
- Material Derivative
  - $\frac{D()}{Dt} = \frac{\partial()}{\partial t} + (\vec{V} \cdot \vec{\nabla})()$
    - Acceleration $\vec{a} = \frac{D\vec{V}}{Dt}$
- Extensive/Intensive relationship
  - $B = m \cdot b$
    - $B$is an extensive property
    - b is an intensive property
    - m =mass
- Amount of B in V
  - $B = \int_{\forall} \rho b d\forall$
    - $\forall$ =Volume
    - b =intensive property
    - B =extensive property
- Flux of extensive property
  - $\dot{B} = \int_{S} \rho b\vec{V} \cdot \vec{n} dS$
    - S =a surface
    - b =intensive property
    - B =extensive property
    - $\vec{V}$ =velocity vector
    - $\vec{n}$ =unit vector normal to the surface
- Reynolds Transport Theorem
  - $\frac{\partial B_{cv}}{\partial t} + \dot{B}_{cs} = 0$
    - $\frac{\partial B_{cv}}{\partial t}$ = the unsteady term. It should be zero in steady flow
    - $\dot{B}$ = the fluid flux flowing through the surface

- Chapter 5: Finite Control Volume Analysis
  - Mass Conservation
    - $\frac{\partial}{\partial t}\int_{c\forall} \rho d\forall + \int_{cs} \rho\vec{V} \cdot \hat{n}ds = 0$
      - The first term is the unsteady term
      - $c\forall$ =is the control Volume
      - cs = control surface
      - You can split the control surface into multiple flat plans and add it together
  - Momentum Conservation
    - $\frac{\partial}{\partial t}\int_{c\forall} \vec{V}\rho d\forall + \int_{cs} \vec{V}\rho\vec{V} \cdot \hat{n}ds = \sum F_{ext}$
      - $F_{ext}$is all external forces
      - Like pressure and stress and such
    - External Forces x direction (typically)
      - Since $F = pA$
      - $\sum F_{ext} = p_{left}A_{left} - p_{right}A_{right} + F_x$
    - External Forces y direction (typically)
      - $\sum F_{ext} = p_{bottom}A_{bottom} - p_{top}A_{top} + F_y$
- Chapter 6: Differential Analysis of Fluid Flow
  - Volumetric Dilation
    - $div(\vec{V})$
      - $div(\vec{V}) = \vec{\nabla} \cdot \vec{V}$watch your coordinate system tho
  - Rotation Vector $\vec{\omega}$
    - $\frac{1}{2}\, curl(\vec{V})$
      - $curl(\vec{V}) = \vec{\nabla} \times \vec{V}$watch your coordinate system tho
  - Vorticity $\vec{\zeta}$
    - $curl(\vec{V})$ or $2\vec{\omega}$

- Gradient Tensor
  - $\vec{\nabla}\vec{V} = S_{ij} + \Omega_{ij}$
    - $S = \frac{1}{2}(\vec{\nabla}\vec{V} + (\vec{\nabla}\vec{V})^T)$
      - the rate of stress tensor
    - $\Omega = \frac{1}{2}(\vec{\nabla}\vec{V} - (\vec{\nabla}\vec{V})^T)$
      - Rotation tensor
- $trace(\vec{\nabla}\vec{V}) = div(\vec{V})$
  - $trace$is the sum of the diagonal components of a square matrix
- Mass conservation
  - $\vec{\nabla} \cdot \vec{V} = 0$
- “Newton's Second Law”
  - $ma = \sum F$
    - $ma = \rho \frac{D\vec{V}}{Dt}$
    - $F_{int} = \vec{\nabla} \cdot T_{ij}$
    - $F_{ext} = \rho\vec{g}$
- Stress tensor
  - T_{ij} =isotropic tensor + deviatoric stress tensor
    - isotropic tensor = $-\ pI_{33}$
    - Where $I_{ij}$is an $i \times j$identity matrix
    - deviatoric stress tensor = τ_{ij} the typical stress tensor from mechanics of materials
- Deviatoric Stress Tensor
  - $\tau_{ij} = \frac{-2}{3}\mu(\vec{\nabla} \cdot \vec{V})I_{ij} + 2\mu S_{ij}$
    - $S_{ij} = 0$ in incompressible fluids
    - μ =dynamic viscosity
- Naviar Stokes
  - $\vec{\nabla} \cdot \vec{V} = 0$

        - $\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\, \vec{V} = \frac{-1}{\rho}\vec{\nabla}p + \nu \vec{\nabla}^2 \vec{V} + \vec{g}$
            - $\nu = \frac{\mu}{\rho}$kinematic viscosity
        - 2D Navier Stokes
            - Euler Equations
                - Naviar stokes without viscous terms
                - $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
                - $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-1}{\rho}\frac{\partial p}{\partial x}$
                - $\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{-1}{\rho}\frac{\partial p}{\partial y}$
    - Potential functions
        - Linearity applies
            - Assumptions incompressible
            - Inviscious
            - Irrotational
        - Stream functions $\psi(x, y)$
            - $u(x, y) = \frac{\partial \psi(x,y)}{\partial y}$
            - $v(x, y) = \frac{-\partial \psi(x,y)}{\partial x}$
            - In 2D, $\psi$is a scalar that can generate a vector field
                - $\psi$is constant along streamlines
                - $\Delta\psi$ is volumetric flow rate
        - Velocity potential ϕ
            - $\vec{V} = \vec{\nabla}\ \phi$
                - $u = \frac{\partial \phi}{\partial x}$
                - $v = \frac{\partial \phi}{\partial y}$
                - $w = \frac{\partial \phi}{\partial z}$
        - Basic flows
            - Uniform flow magnitude $U$and angle α
                - $\phi = Uxcos(\alpha) + Uysin(\alpha)$
                - $\psi = Uycos(\alpha) + Uxsin(\alpha)$
            - Source/Sink
                - $\phi = \frac{m}{2\pi}ln(r)$
                - $\psi = \frac{m}{2\pi}\theta$
            - Free vortex
                - $\phi = \frac{\Gamma}{2\pi}\theta$

            - $\psi = \frac{\Gamma}{2\pi} ln(r)$
        - Doublet
            - $\phi = \frac{K cos(\theta)}{r}$
            - $\psi = \frac{-K sin(\theta)}{r}$
    - Converting from cartesian to cylindrical
        - $r = \sqrt{(x - h)^2 + (y - k)^2}$
        - $\theta = tan^{-1}(\frac{y-k}{x-h})$
    - Recall:
        - $u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$
        - $v = \frac{\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x}$
        - $u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$
        - $u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{-\partial\psi}{\partial r}$
        - Where u_r and u_θ are the velocities in those given directions
- Circulation
    - $\Gamma = \oint_{cs} \vec{V} \cdot d\vec{s}$

- Dimensional Analysis
    - Buckingham-pi theorem
        - A problem can be reduced to $k - r$ nondimensional groups
            - k: Number of variables
            - r: Number of dimensions (units) used
    - Reynolds number
        - $Re = \frac{\rho dV}{\mu} = \frac{dV}{\nu}$
    - Drag coefficient
        - $C_d = \frac{F_d}{\rho V^2 d^2}$ or $\frac{F_d}{\frac{1}{2}\rho V^2 A_c}$
    - Mach number
        - $\frac{V}{\sqrt{\gamma RT}}$
        - γ: Ratio of specific heats = 1.17 for air
- Boundary layer
    - Skin friction
        - $F = \tau A = \mu\frac{\partial u}{\partial y}A$
            - From Newtonian shear stress

- Viscous Flow in a Smooth Laminar Circular Pipe
  - Assumptions
    - Steady flow: $\frac{d}{dt} = 0$
    - Straight lines along pipe: $u_r = 0$
    - No swirls/spirals down the pipe: $u_\theta = 0$
    - Axisymmetric flow: $\frac{\partial}{\partial \theta} = 0$
    - Laminar flow: $Re < 1000$
  - After Navier-Stokes with simplifications
    - Continuity: $\frac{\partial u_z}{\partial z} = 0$
    - $r -$ component: $\frac{\partial p}{\partial r} = 0$
    - $z -$ component: $\frac{\mu}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}) = \frac{\partial p}{\partial z}$
  - After boundary conditions
    - $u_z(r) = \frac{1}{4\mu}\frac{\partial p}{\partial z}(r^2 - R^2)$
      - Where R is the radius of the pipe
    - $u_z(r) = \frac{1}{4\mu}\frac{\Delta p}{l}(R^2 - r^2)$
      - Pressure drop: $\Delta p = p_1 - p_2$
  - Volumetric flow rate
    - $Q = V_{mean}A = \int u_z dA = \frac{\pi R^4}{8\mu}\frac{\Delta p}{l}$
      - Where Δp is pressure drop $(p_1 - p_2)$
- Viscous Flow in a Rough Circular Pipe
  - Friction factor
    - $f = \frac{\Delta p}{\frac{1}{2}\rho V_{mean}^2} \cdot \frac{D}{l}$
  - Moody diagram

- Shows f as a function of Re and $\frac{\epsilon}{D}$
    - ϵ is the roughness of the pipe
    - If Laminer, $f = \frac{64}{Re}$