ME 3253 Final Project

Nathan Stenseng

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Part 1:

1) Derive the transfer function

In order to derive the transfer function, we will use physical laws of motion, specifically Newton's second law for one dimensional motion which is as follows:

$$\sum F = m\ddot{x}(t)$$

We have three forces acting on the cart: the input force, the force of the spring, and the force of the damper:

$$F_{Input} = f(t)$$
$$F_{Spring} = -k_{eq}x(t)$$
$$F_{Damper} = -b_{eq}\dot{x}(t)$$

Plugging the forces into our first equation yields the following differential equation:

$$m\ddot{x}(t) + b_{eq}\dot{x}(t) + k_{eq}x(t) = f(t)$$

In order to solve this, we will first take the Laplace transform of both sides:

$$ms^2X(s) - msx(0) - m\dot{x}(0) + b_{eq}sX(s) - b_{eq}x(0) + k_{eq}X(s) = F(s)$$

For finding the transfer function, we will assume the initial conditions $x^{(n)}(0) = 0$. This simplifies the above equation to:

$$X(s)\left(ms^2 + b_{eq}s + keq\right) = F(s)$$

And that means that our transfer function will be:

$$\frac{X(s)}{F(s)} = T(s) = \frac{1}{ms^2 + b_{eq}s + k_{eq}}$$

Next, we will be finding that k_{eq} and b_{eq}. We know that the two springs and the two dampers are in parallel so that means that:

$$k_{eq} = k_1 + k_2$$

$$b_{eq} = b_1 + b_2$$

Our given values are $m = 2kg, b_1 = 4\frac{N\cdot s}{m}, b_2 = 8\frac{N\cdot s}{m}, k_1 = 10\frac{N}{m}, k_2 = 20\frac{N}{m}$. Plugging in these values into our transfer equation yields:

$$\boxed{T(s) = \frac{1}{2s^2 + 12s + 30} = \frac{1}{2}\cdot\frac{1}{(s+3)^2 + 6}}$$

2) Derive the step function response

We can say the if the input force is a step function, then $F(s) = \frac{1}{s}$, and therefore:

$$X(s) = \frac{1}{2s} \cdot \frac{1}{s^2 + 6s + 15}$$

We have to do some rearranging before we can take the inverse Laplace transform of both sides, getting the output $x(0)$. We will use partial fraction decomposition to rearrange the fraction as the summation of fractions:

$$\frac{\frac{1}{2}}{(s)(s^2 + 6s + 15)} = \frac{A}{s} + \frac{Bs + C}{(s^2 + 6s + 15)}$$

$$\frac{1}{2} = As^2 + 6As + 15A + Bs^2 + Cs$$

The following can be broken up into a system of three equations:

$$0 = A + B$$

$$0 = 6A + C$$

$$\frac{1}{2} = 15A$$

Solving this system of equations gives us values of $A = \frac{1}{30}$, $B = -\frac{1}{30}$, $C = -\frac{6}{30}$. This means we can rewrite $X(s)$ as:

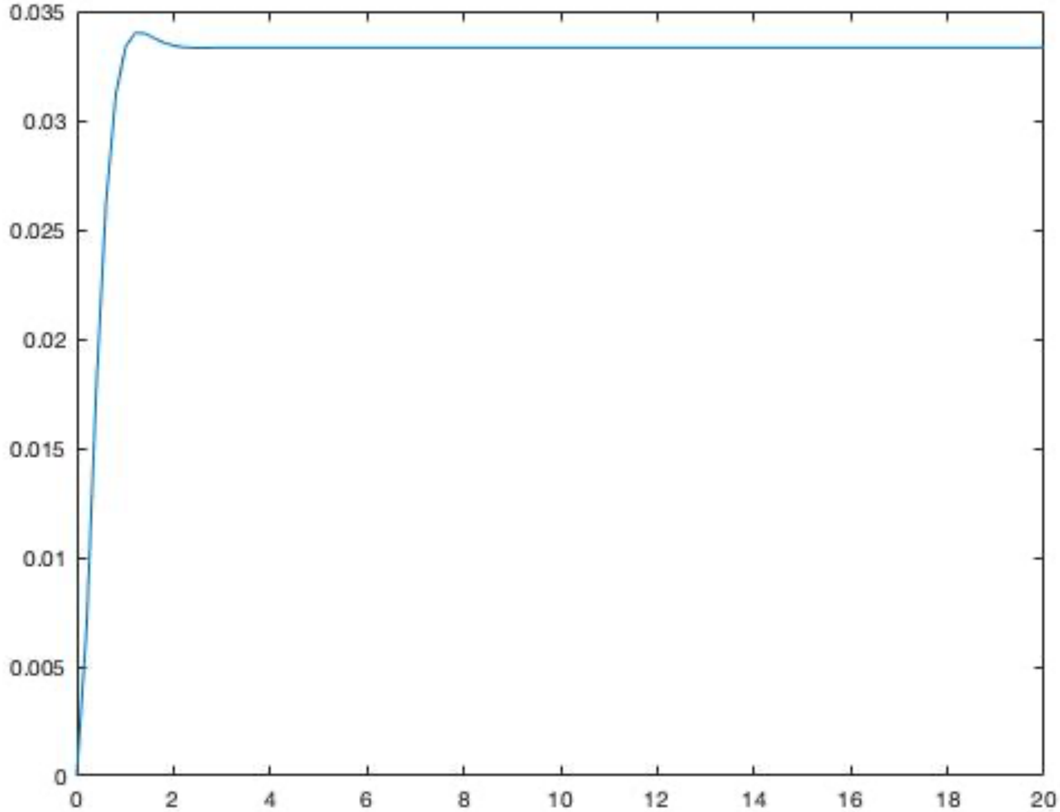
$$X(s) = \frac{1}{30s} - \frac{1}{30} \cdot \frac{s + 6}{(s + 3)^2 + 6}$$

One last rearranging step needs to be made to take the Laplace transform:

$$X(s) = \frac{1}{30s} - \frac{1}{30}\left(\frac{s + 3}{(s + 3)^2 + 6} + \frac{\sqrt{6}}{\sqrt{6}} \cdot \frac{3}{(s + 3)^2 + 6}\right)$$

Now we can easily see that the inverse Laplace transform of $X(s)$ is:

$$\boxed{x(t) = \frac{1}{30} - \frac{1}{30}e^{-3t}cos\left(\sqrt{6}t\right) - \frac{1}{10\sqrt{6}}e^{-3t}sin\left(\sqrt{6}t\right)}$$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3) Find the response if the input is $f(t) = exp(-2t)$

If this is our input, we first need to find it's Laplace transform:

$$F(s) = \frac{1}{(s+2)}$$

Now we can plug this into our transfer function equation and rearrange to get:

$$X(s) = \frac{1}{2} \cdot \frac{1}{(s+2)(s^2+6s+15)}$$

Again, we will need some rearranging to be able to find the inverse Laplace transform, and therefore find our response $x(t)$. We again will preform partial fraction decomposition to split the fraction into an addition of two fractions which have simpler inverse Laplace transforms:

$$\frac{\frac{1}{2}}{(s+2)(s^2+6s+15)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+6s+15}$$

$$\frac{1}{2} = As^2 + 6As + 15A + Bs^2 + Cs + 2Bs + 2C$$

The following can be broken up into a system of equations:

$$0 = A + B$$

$$0 = 6A + 2B + C$$

$$\frac{1}{2} = 15A + 2C$$

And solving this system of equations yields $A = \frac{1}{14}, B = \frac{-1}{14}, C = \frac{-4}{14}$. This means we can rewrite $X(s)$as:

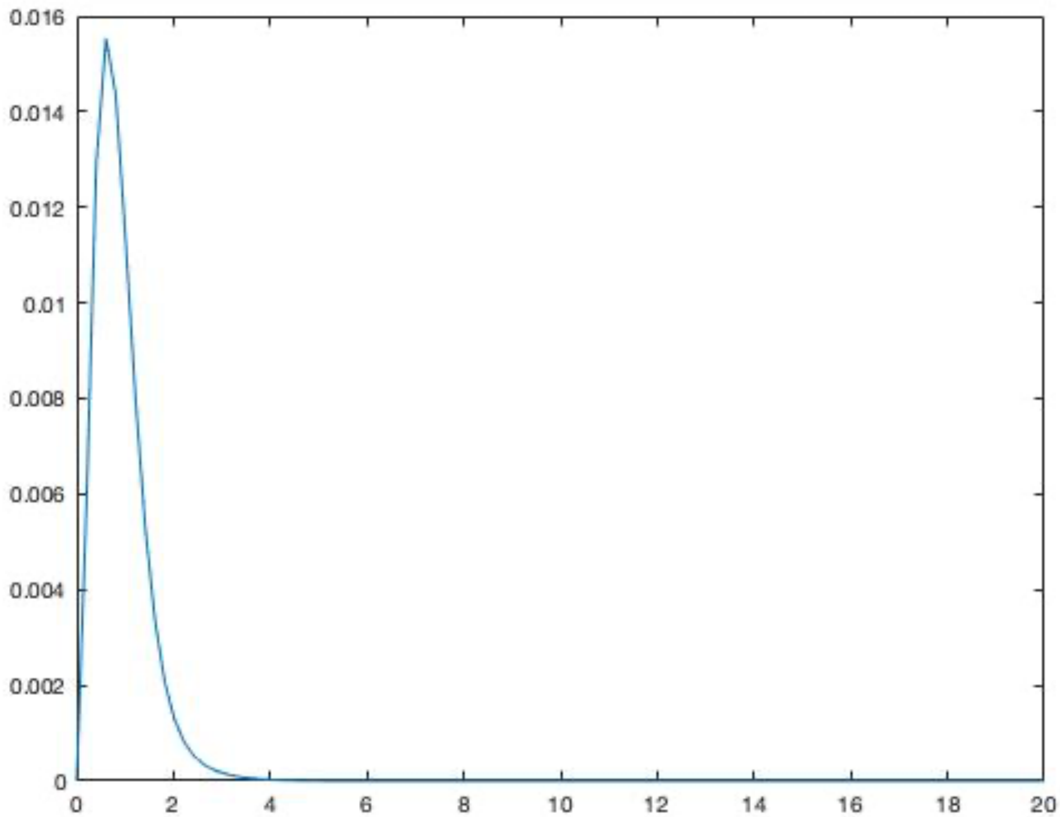
$$X(s) = \frac{1}{14} \cdot \frac{1}{s+2} - \frac{1}{14}\left(\frac{s+4}{(s+3)^2+6}\right)$$

We can rearrange the final term to makes the equations easier to take the inverse Laplace transform of:

$$X(s) = \frac{1}{14} \cdot \frac{1}{s+2} - \frac{1}{14}\left(\frac{s+3}{(s+3)^2+6} + \frac{\sqrt{6}}{\sqrt{6}} \cdot \frac{1}{(s+3)^2+6}\right)$$

Taking the inverse Laplace transform of this gives us out output function:

$$\boxed{x(t) = \frac{1}{14}e^{-2t} - \frac{1}{14}e^{-3t}cos(\sqrt{6}t) - \frac{1}{14\sqrt{6}}e^{-3t}sin(\sqrt{6}t)}$$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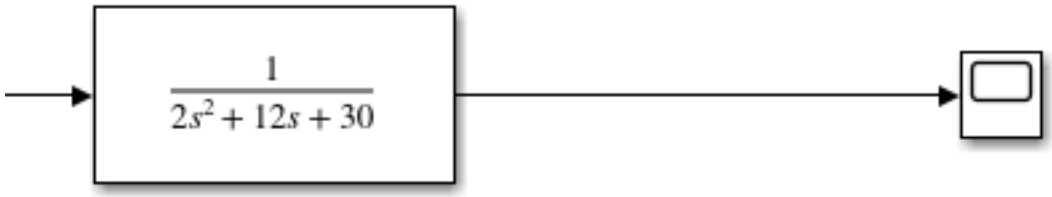
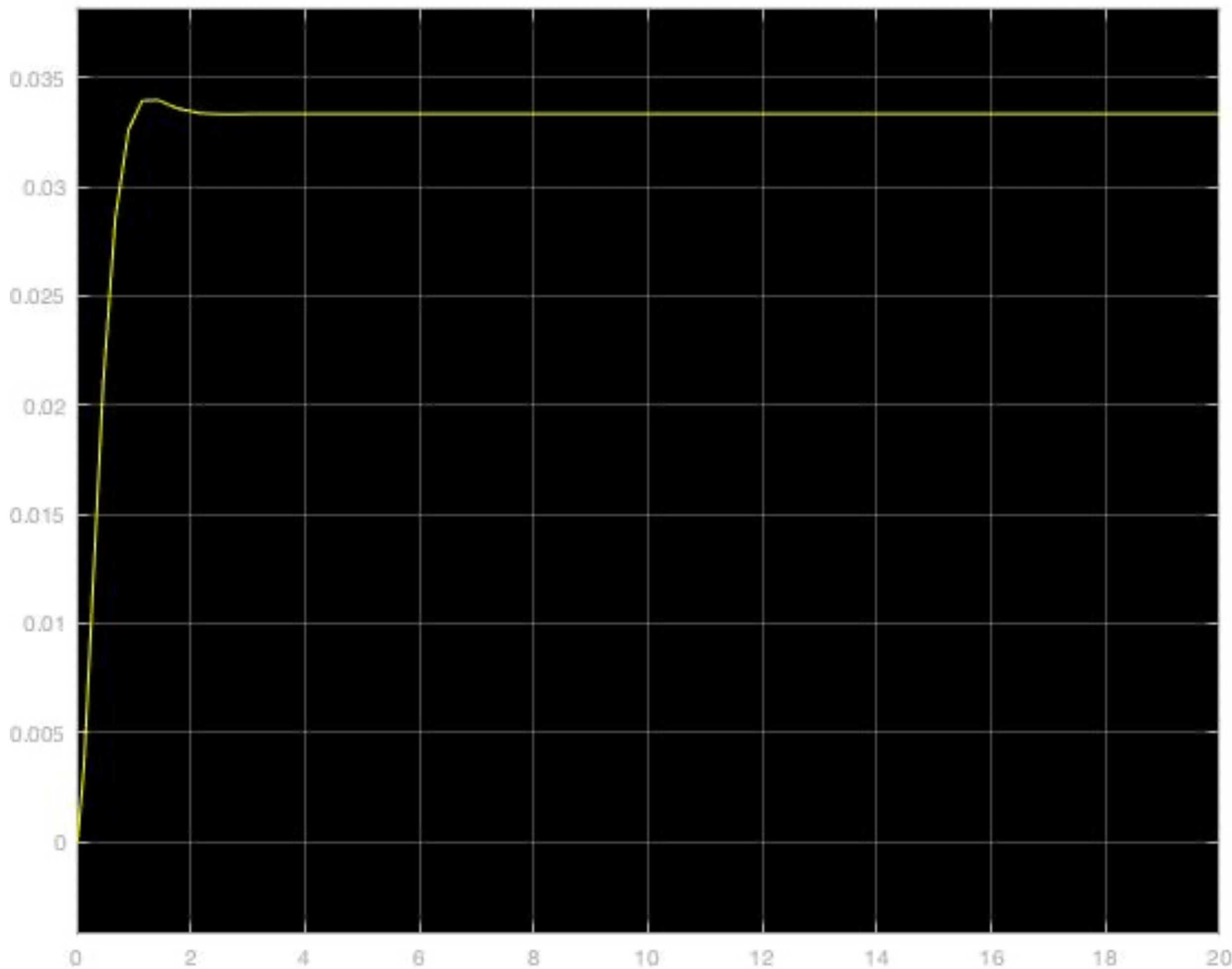
Part 2:

4) Build the model using the transfer function block.

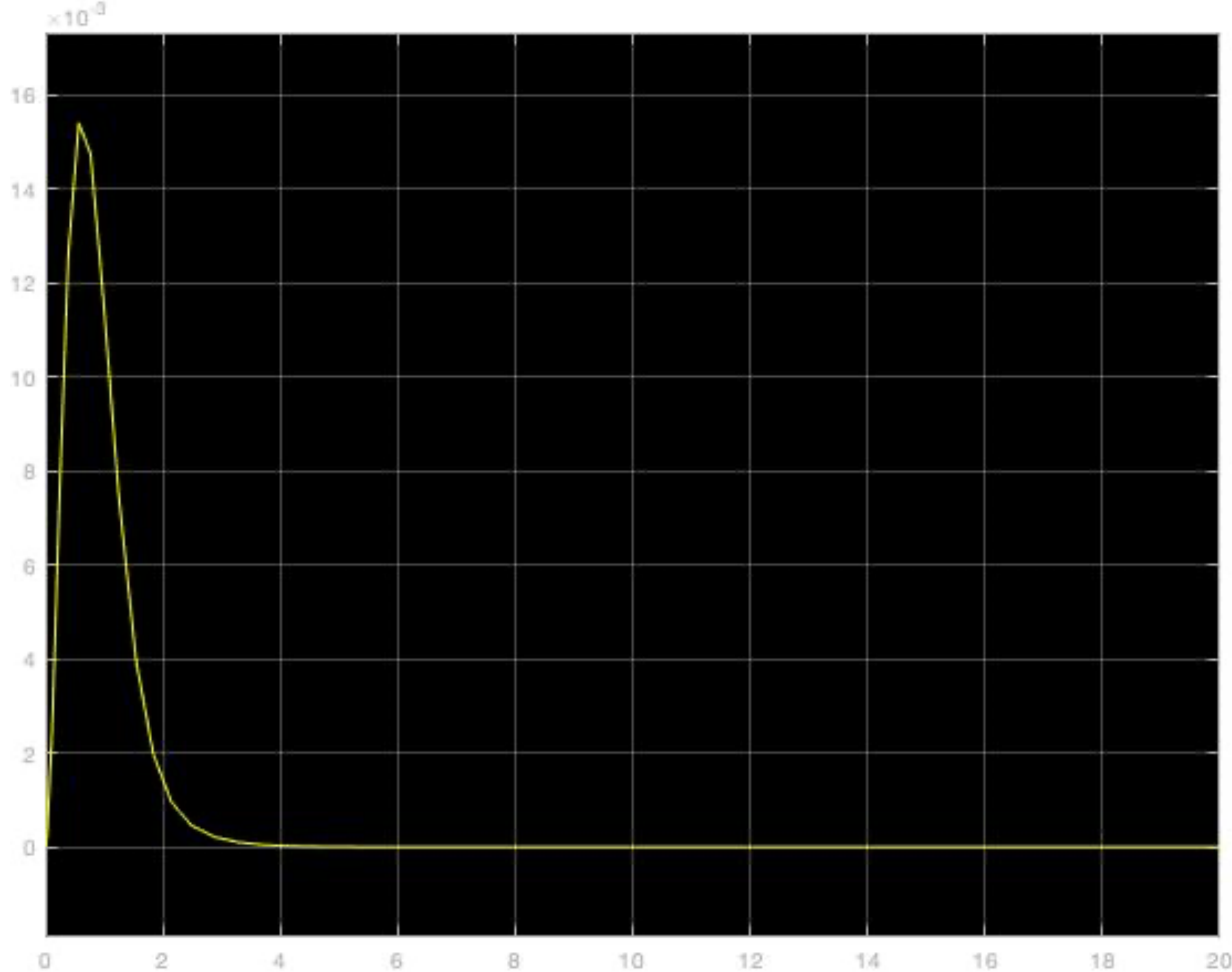


5) Using the Simulink model above, generate the step response and plot it. Compare it to part



This plot is the same as what we saw in part 2).

6) Using the Simulink model above, generate the response to $f(t) = exp(-2t)$ and plot it. Compare it to part 3).



This plot is the same as what we saw in part 3).

Part 3:

7) For the following scenarios, derive the system response and use the plot function in Matlab to plot the response.

Recall from before that the Laplace transform for our equation is:

$$2s^2X(s) - 2sx(0) - 2\dot{x}(0) + 12sX(s) - 12x(0) + 30X(s) = F(s)$$

a) $x(0) = 1, \dot{x}(0) = -2, f(t) = 0$

Plugging in these values to our main equation yields:

$$2s^2X(s) - 2s + 4 + 12sX(s) - 12 + 30X(s) = 0$$

Rearranging this yields:

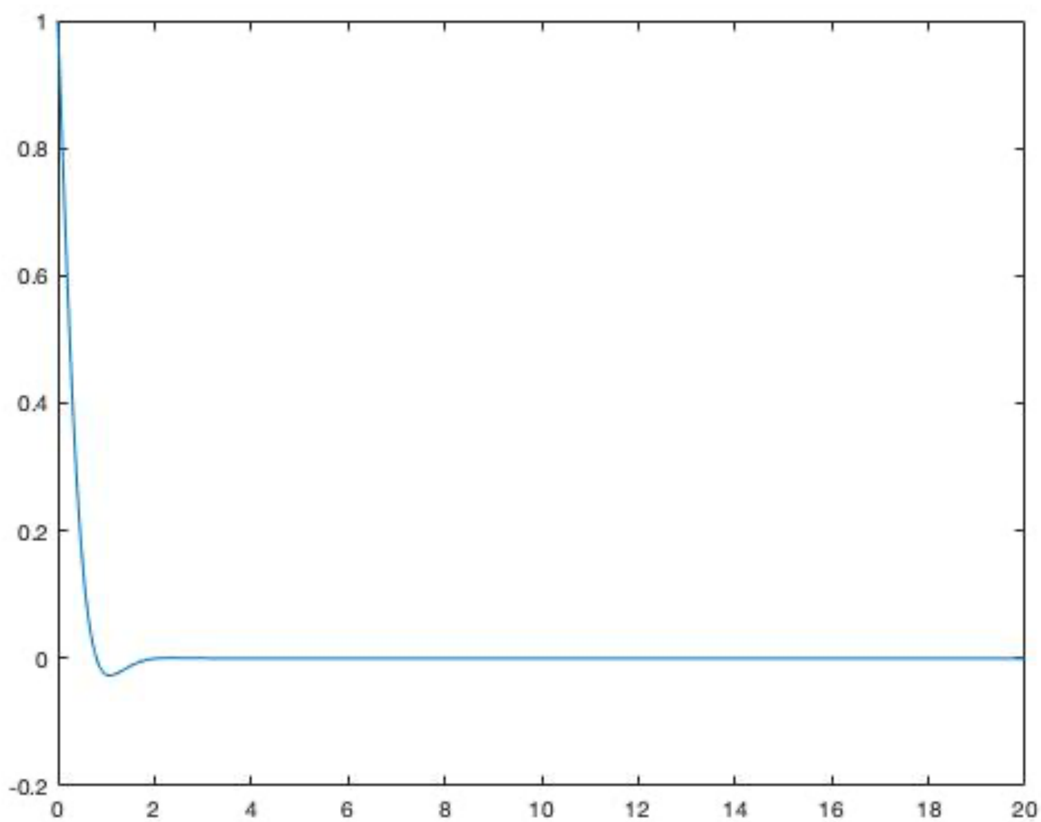
$$X(s) = \frac{2s + 8}{2s^2 + 12s + 30}$$

We can further transform this into an equivalent statement which has a simpler inverse Laplace transform:

$$X(s) = \frac{s + 3}{(s + 3)^2 + 6} + \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{6}}{(s + 3)^2 + 6}$$

Taking the inverse Laplace transform of both sides gives us:

$$\boxed{x(t) = e^{-3t}cos(\sqrt{6}t) + \frac{1}{\sqrt{6}}e^{-3t}sin(\sqrt{6}t)}$$



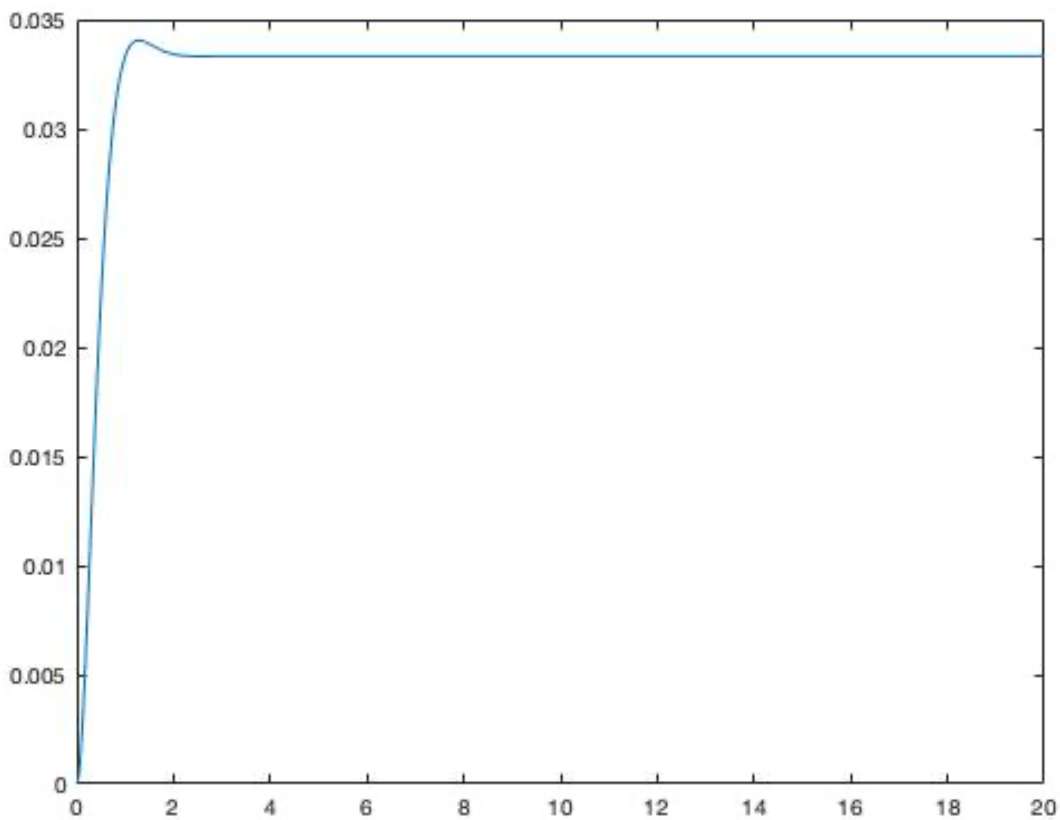
b) $x(0) = 0, \dot{x}(0) = 0, f(t) = 1$

Since this part has zeros for our initial conditions, we can use our previously derived transfer function as:

$$X(s) = T(s)F(s)$$

We know that the Laplace transform of 1 is $\frac{1}{s}$. This is just the step response which was derived in part 2).

$$\boxed{x(t) = \frac{1}{30} - \frac{1}{30}e^{-3t}cos(\sqrt{6}t) - \frac{1}{10\sqrt{6}}e^{-3t}sin(\sqrt{6}t)}$$



c) $x(0) = 1, \dot{x}(0) = -2, f(t) = 1$

Plugging into our Laplace transform for our system gives:

$$2s^2X(s) - 2s + 4 + 12sX(s) + 12 + 30X(s) = \frac{1}{s}$$

Rearranging this yields:

$$X(s) = \frac{1}{2s} \cdot \frac{1}{s^2 + 6s + 15} + \frac{2s - 16}{2s^2 + 12s + 30}$$

We can simplify this into an easier form to find the Laplace transform of:

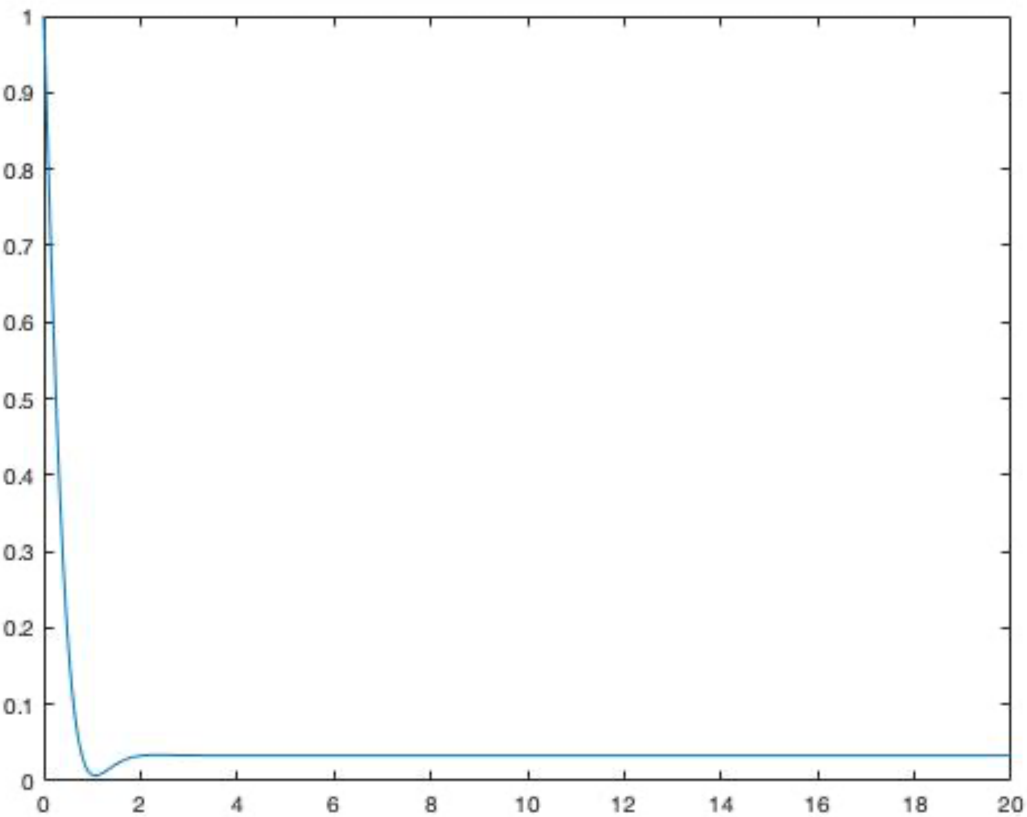
$$X(s) = \frac{1}{30s} - \frac{1}{30}\left(\frac{s+3}{(s+3)^2 + 6} + \frac{\sqrt{6}}{\sqrt{6}} \cdot \frac{3}{(s+3)^2 + 6}\right) + \frac{s+3}{(s+3)^2 + 6} - \frac{11}{\sqrt{6}} \cdot \frac{\sqrt{6}}{(s+3)^2 + 6}$$

Taking the inverse Laplace transform of both sides:

$$x(t) = \frac{1}{30} - \frac{1}{30}e^{-3t}cos(\sqrt{6}t) - \frac{1}{10\sqrt{6}}e^{-3t}sin(\sqrt{6}t) + e^{-3t}cos(\sqrt{6}t) - \frac{11}{\sqrt{6}}e^{-3t}sin(\sqrt{6}t)$$

This can be simplified to:

$$\boxed{x(t) = \frac{1}{30} + \left(\frac{-1}{30} + 1\right)e^{-3t}cos(\sqrt{6}t) + \left(\frac{-1}{10\sqrt{6}} + \frac{1}{\sqrt{6}}\right)e^{-3t}sin(\sqrt{6}t)}$$



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d) $x(0) = 0, \dot{x}(0) = 0, f(t) = sin(5t)$

Since we have zeros as initial conditions, we will use our transfer function to find $X(s)$. But first we will say that $F(s) = \frac{25}{s^2+25}$. Now we can say that:

$$X(s) = \frac{1}{2} \cdot \frac{1}{s^2 + 6s + 15} \cdot \frac{5}{s^2 + 25}$$

Combining these terms gives:

$$X(s) = \frac{5}{2} \cdot \frac{1}{(s^2 + 6s + 15)(s^2 + 25)}$$

Splitting this up via partial fraction decomposition gives:

$$\frac{5}{2} \cdot \frac{1}{(s^2 + 6s + 15)(s^2 + 25)} = \frac{As + B}{(s^2 + 6s + 15)} + \frac{Cs + D}{s^2 + 25}$$

$$\frac{5}{2} = (As + B)(s^2 + 25) + (Cs + D)(s^2 + 6s + 15)$$

$$\frac{5}{2} = As^3 + Bs^2 + 25As + 25B + Cs^3 + 6Cs^2 + 15Cs + Ds^2 + 6Ds + 15D$$

Splitting this up gives the following system of equations:

$$0 = A + C$$

$$0 = B + 6C + D$$

$$0 = 25A + 15C + 6D$$

$$\frac{5}{2} = 25B + 15D$$

Solving this gives $A = \frac{3}{200}, B = \frac{23}{200}, C = \frac{-3}{200}$ and $D = \frac{-1}{40}$. Plugging these in means we can rearrange our equation to read:

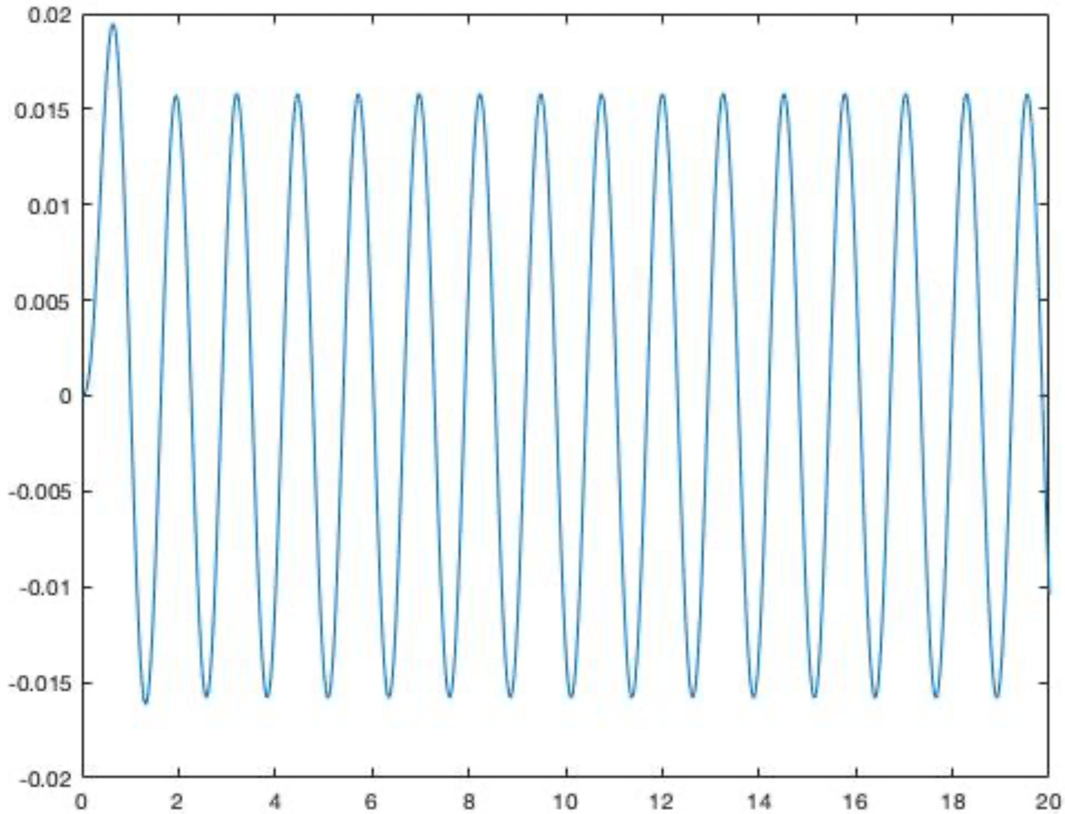
$$X(s) = \frac{1}{200} \cdot \frac{3s + 23}{(s + 3)^2 + 6} - \frac{1}{200} \cdot \frac{3s + 5}{s^2 + 25}$$

We can further rearrange into a form that has a simpler inverse Laplace transform:

$$X(s) = \frac{3}{200}\left(\frac{s + 3}{(s + 3)^2 + 6} + \frac{1}{3} \cdot \frac{14}{(s + 3)^2 + 6}\right) - \frac{3}{200} \cdot \left(\frac{s}{s^2 + 25} + \frac{1}{3} \cdot \frac{5}{s^2 + 25}\right)$$

Taking the inverse Laplace transform gives our response equation:

$$x(t) = \frac{3}{200}\left(\mathrm{e}^{-3\mathrm{t}}\cos\left(\sqrt{6}\mathrm{t}\right) + \frac{14}{3\sqrt{6}}\mathrm{e}^{-3\mathrm{t}}\sin\left(\sqrt{6}\mathrm{t}\right)\right) - \frac{3}{200}\left(cos(5t) + \frac{1}{3}sin(5t)\right)$$



e) $x(0) = 1, \dot{x}(0) = -2, f(t) = \sin(5t)$

The Laplace transform of our general equation with these initial conditions gives:

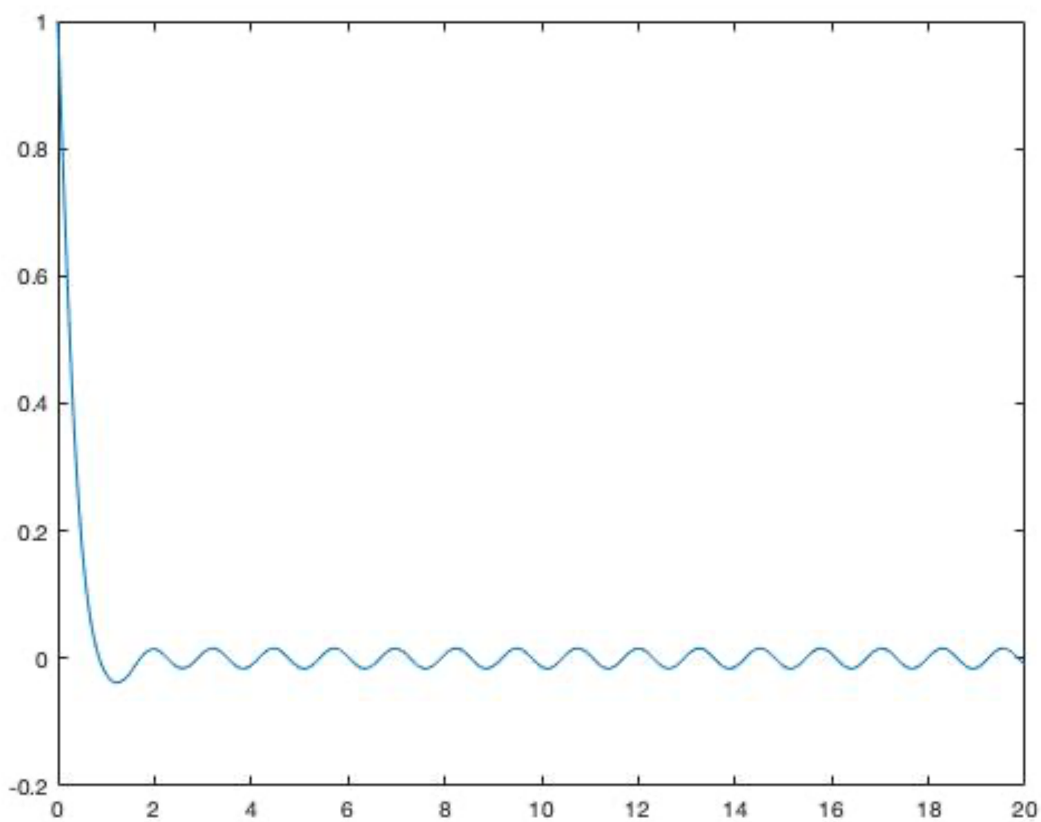
$$2s^2X(s) - 2s + 4 + 12sX(s) + 12 + 30X(s) = \frac{25}{s^2 + 25}$$

Rearranging terms and solving for $X(s)$ yields:

$$X(s) = \frac{2s - 16}{2s^2 + 12s + 30} + \frac{1}{2s^2 + 12s + 30} \cdot \frac{25}{s^2 + 25}$$

Seeing as this is a combination of part a) and part d), we will say the inverse Laplace transform of both sides is:

$$\boxed{x(t) = \left(1 + \frac{3}{200}\right) e^{-3t} cos(\sqrt{6}t) + \left(\frac{-11}{\sqrt{6}} + \frac{7}{100}\right) e^{-3t} sin(\sqrt{6}t) - \frac{3}{200}\left(cos(5t) + \frac{1}{3} sin(5t)\right)}$$



8) Does the response in (c) equal the sum of (a) and (b)? Does the response in (e) equal the sum of (a) and (d)? Why?

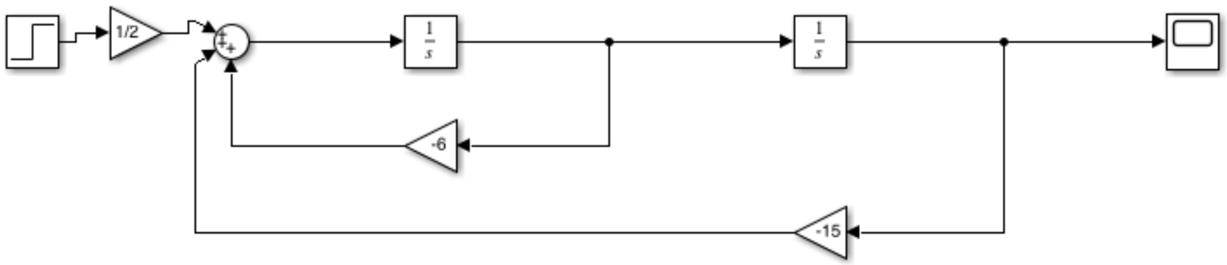
We do see that the sum of the responses are the same as the individual changes to the initial conditions. This is due to the linearity of Laplace and inverse Laplace transforms but mainly because of the linearity of our system.

Part 4:

9) Use only integrators to build the system. Attach the Simulink model.

Rearranging our initial equation gives us an integral model of our system as follows:

$$\ddot{x}(t) = \frac{1}{2}f(t) - 6\dot{x}(t) - 15x(t)$$



10) Derive the equivalent transfer function from f and x for the model in 9). Is it the same as the transfer function in 1)?

To simplify the above transfer function, we will treat it as multiple positive feedback loops. For a general positive feedback loop with $H(s)$ looping back on $G(s)$, the equivalent transfer function would be:

$$T_{eq}(s) = \frac{G(s)}{1 - G(s)H(s)}$$

Our first equivalent transfer function will be with $G(s) = \frac{1}{s}$ and $H(s) = -6$, giving us:

$$T_{eq1}(s) = \frac{1}{s + 6}$$

Our second equivalent transfer function will be $G(s) = T_{eq1}(s) \cdot \frac{1}{s}$ and $H(s) = -15$, giving us:

$$T_{eq2}(s) = \frac{1}{s^2 + 6s + 15}$$

Lastly, including our $\frac{1}{2}$ gain gives a total equivalent transfer function of:

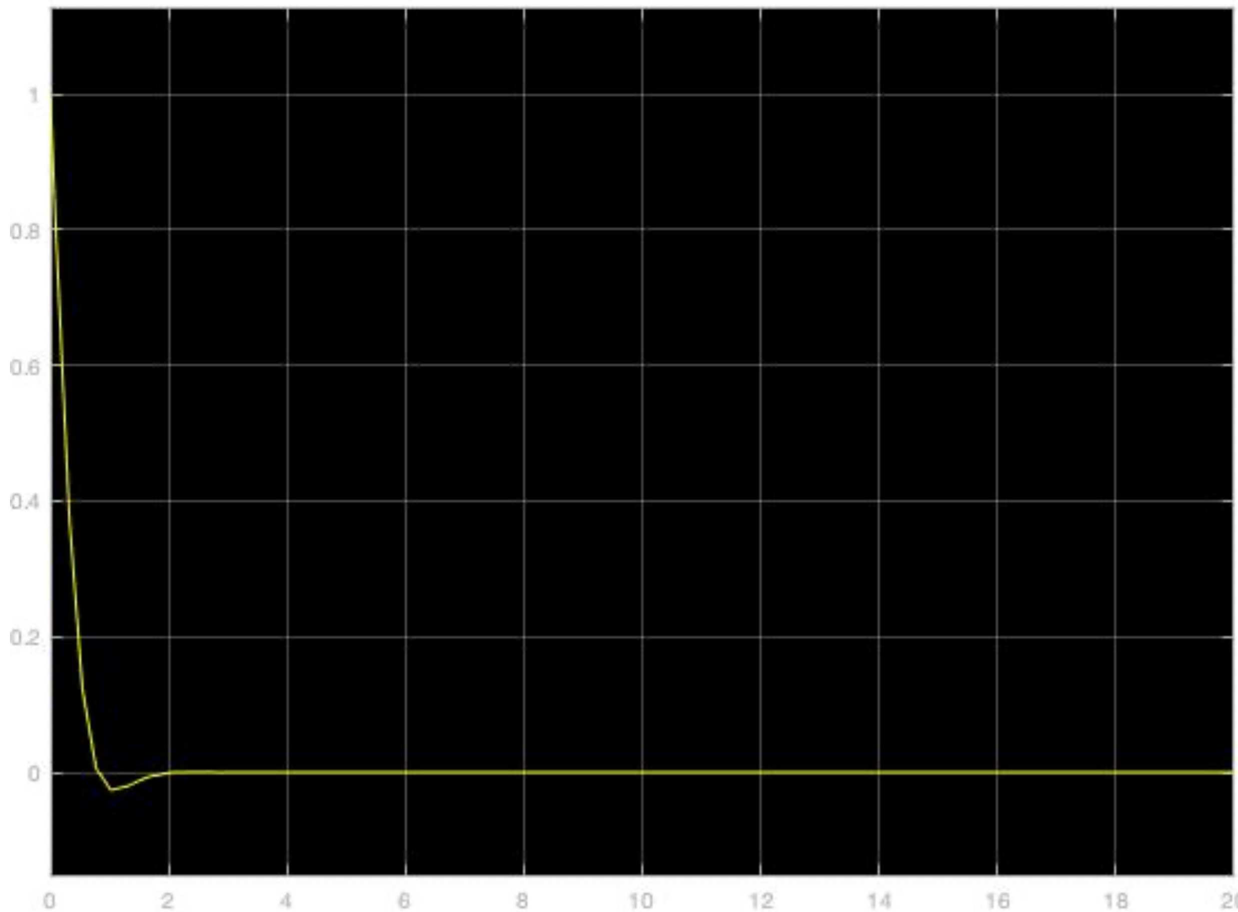
$$T_{eq}(s) = \frac{1}{2} \cdot \frac{1}{s^2 + 6s + 15}$$

Which is the same transfer function we found in 1).

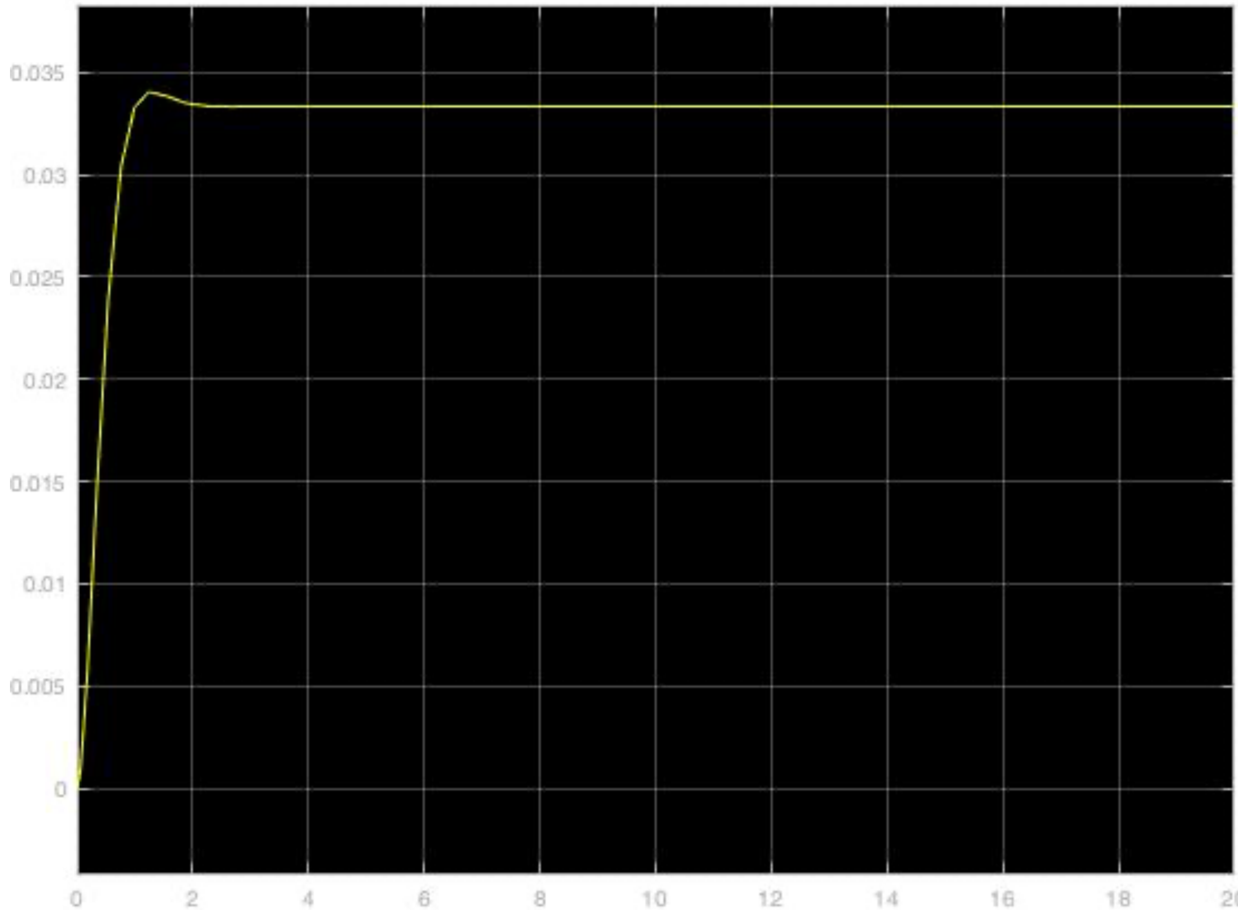
11) Run simulations for the five scenarios from 7) and plot the responses.

We do see the same graphs for all 5 scenarios.

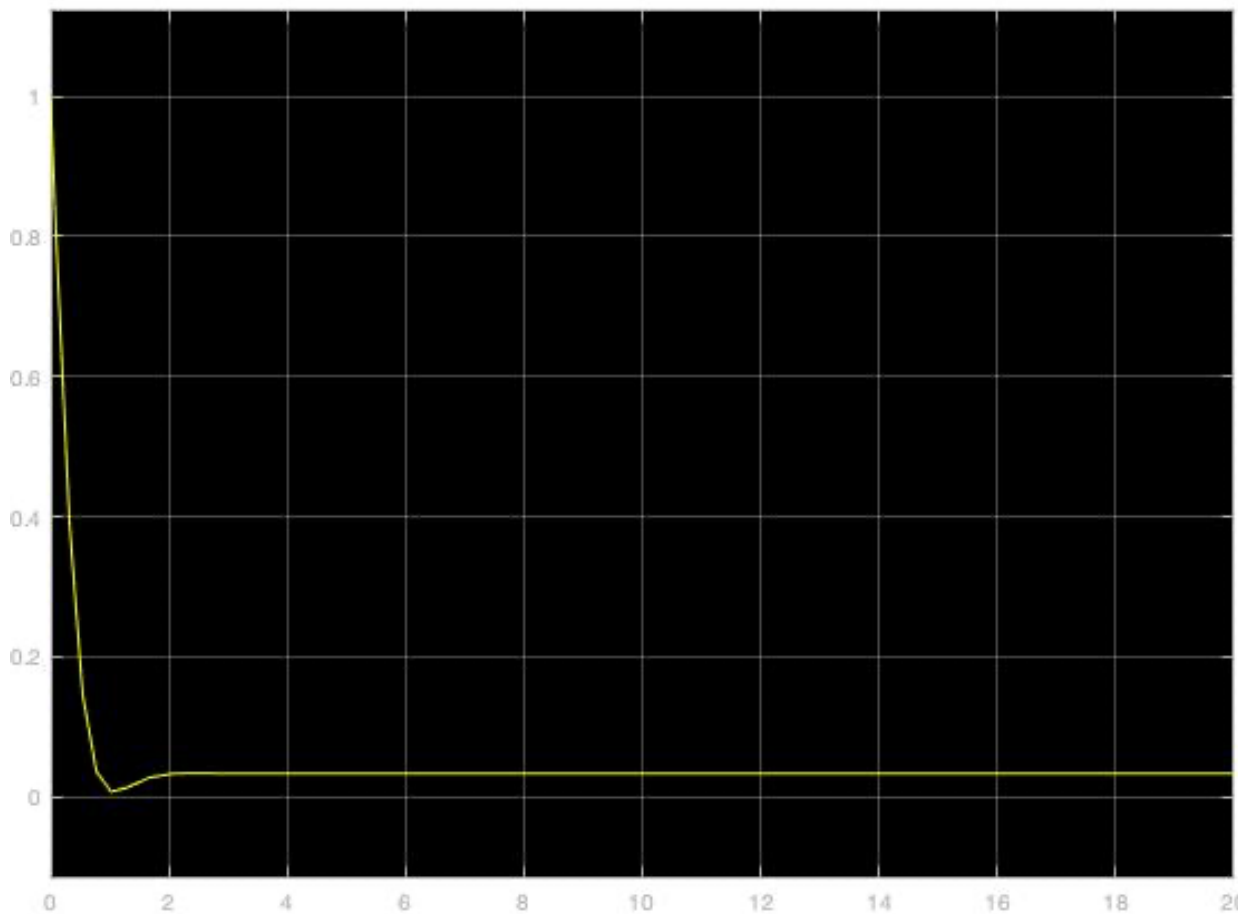
a)



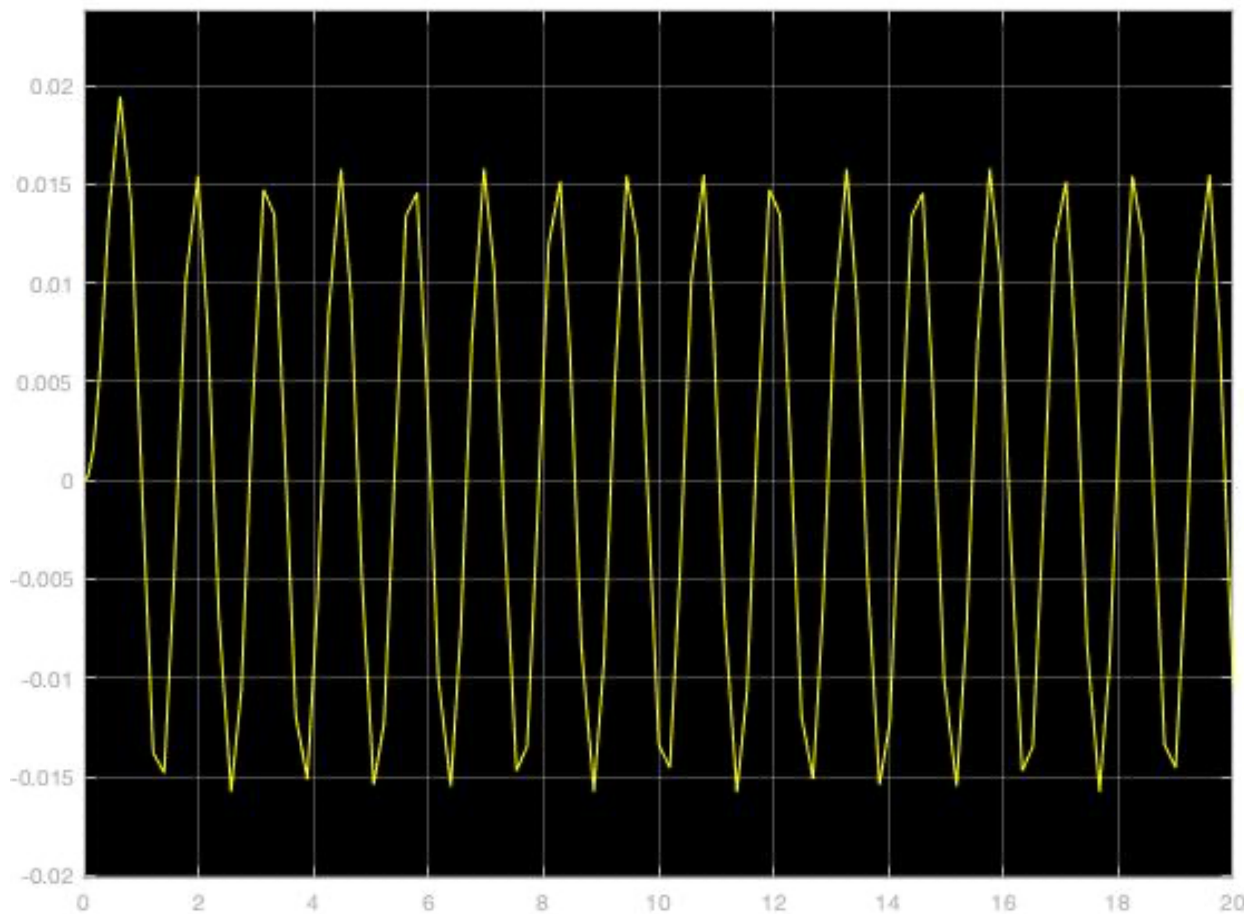
b)



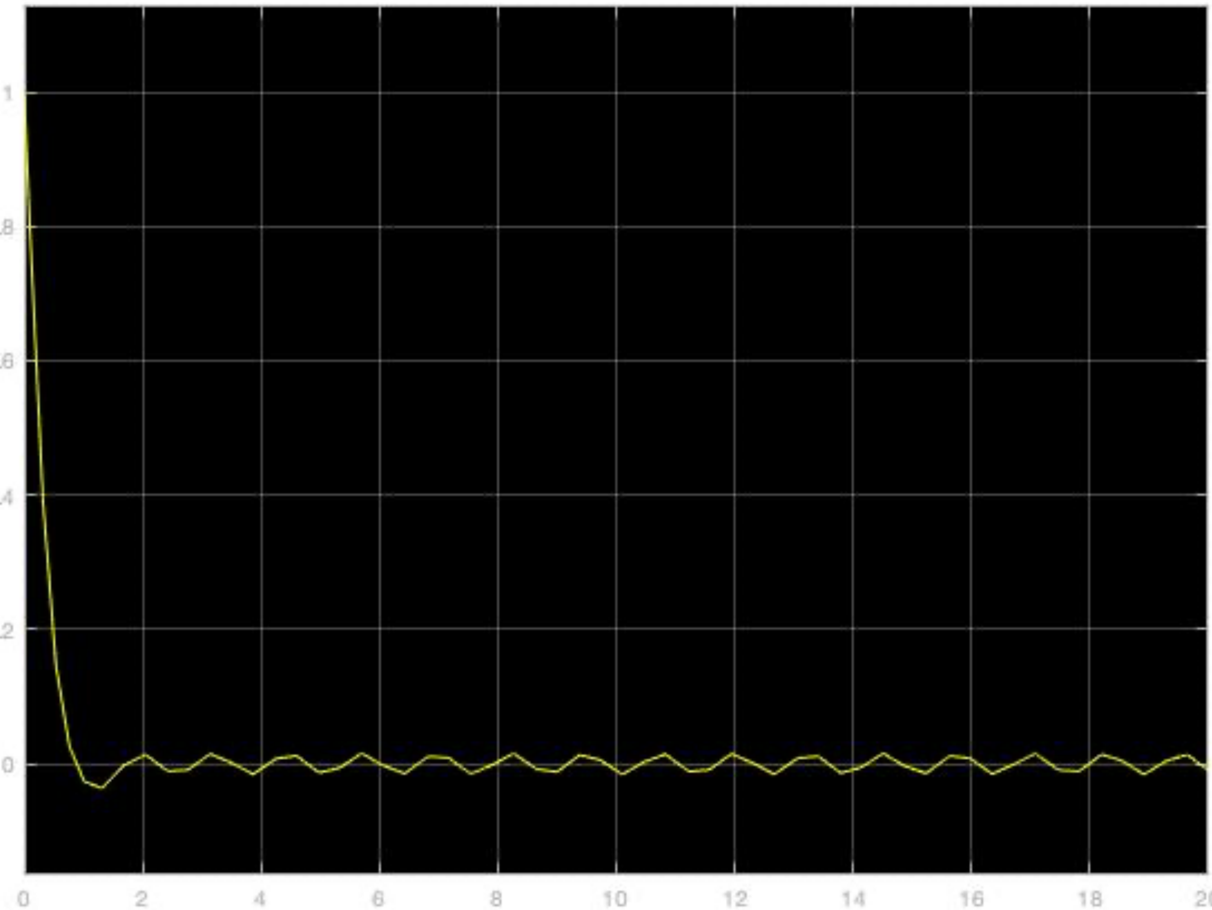
c)



d)



e)



12) Use state space model to build the system again. Attach the Simulink model.

We will say our state space model is as follows:

$$\dot{n}(t) = An(t) + Bf(t)$$

$$x(t) = Cn(t) + Du(t)$$

Where we have matrix:

$$n(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

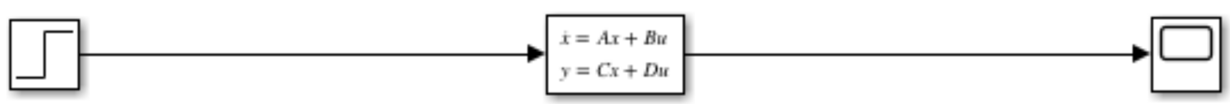
This means that the equation:

$$\ddot{x}(t) = \frac{1}{2}f(t) - 6\dot{x}(t) - 15x(t)$$

Will be a state space model with the following matrix coefficients:

$$\dot{n}(t) = \begin{bmatrix} 0 & 1 \\ -15 & -6 \end{bmatrix} n(t) + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} f(t)$$

$$x(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} n(t) + [0]u(t)$$



13) Use Matlab function “ss2ft” to find equivalent transfer function from f to x.

Using Matlab, we got the following transfer function:

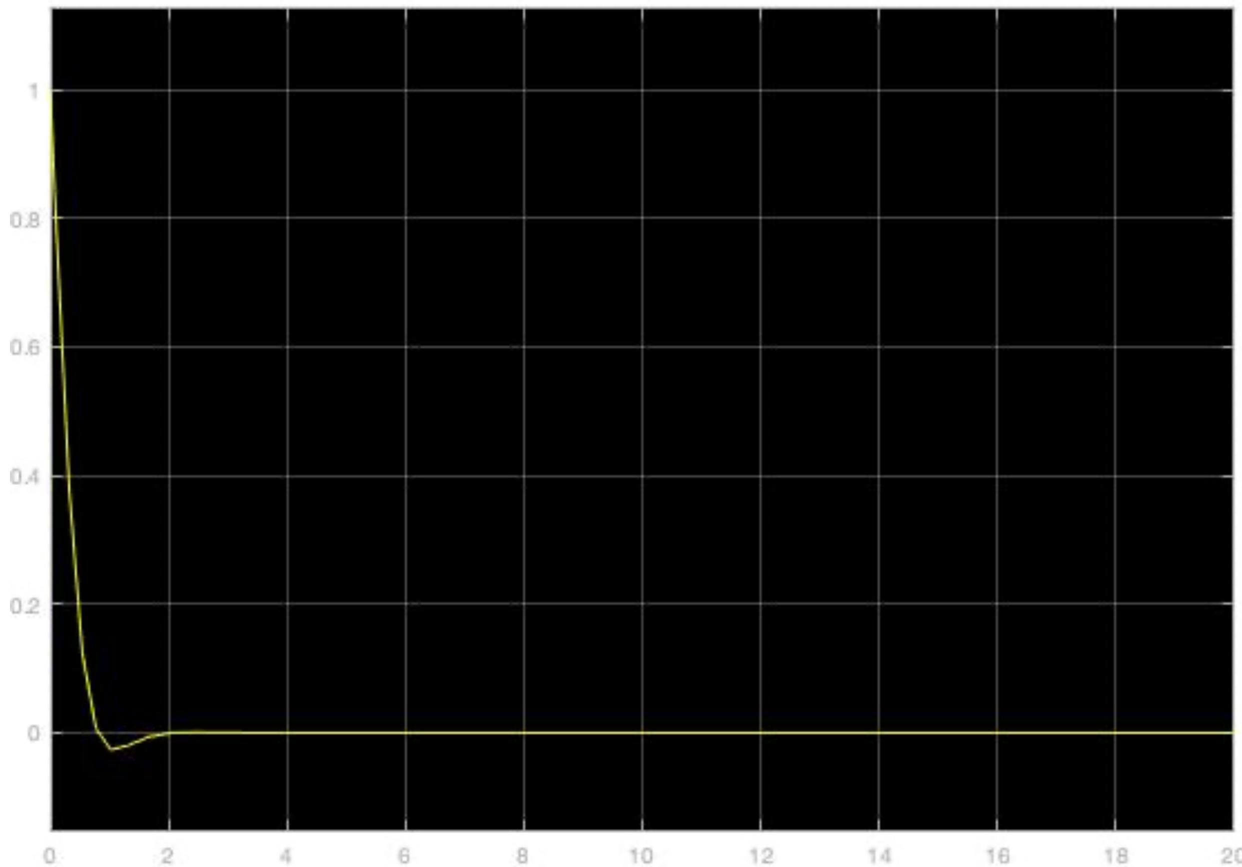
$$T(s) = \frac{0.500}{s^2 + 6s + 15}$$

This is the same transfer function we initially calculated because each system only has one transfer function.

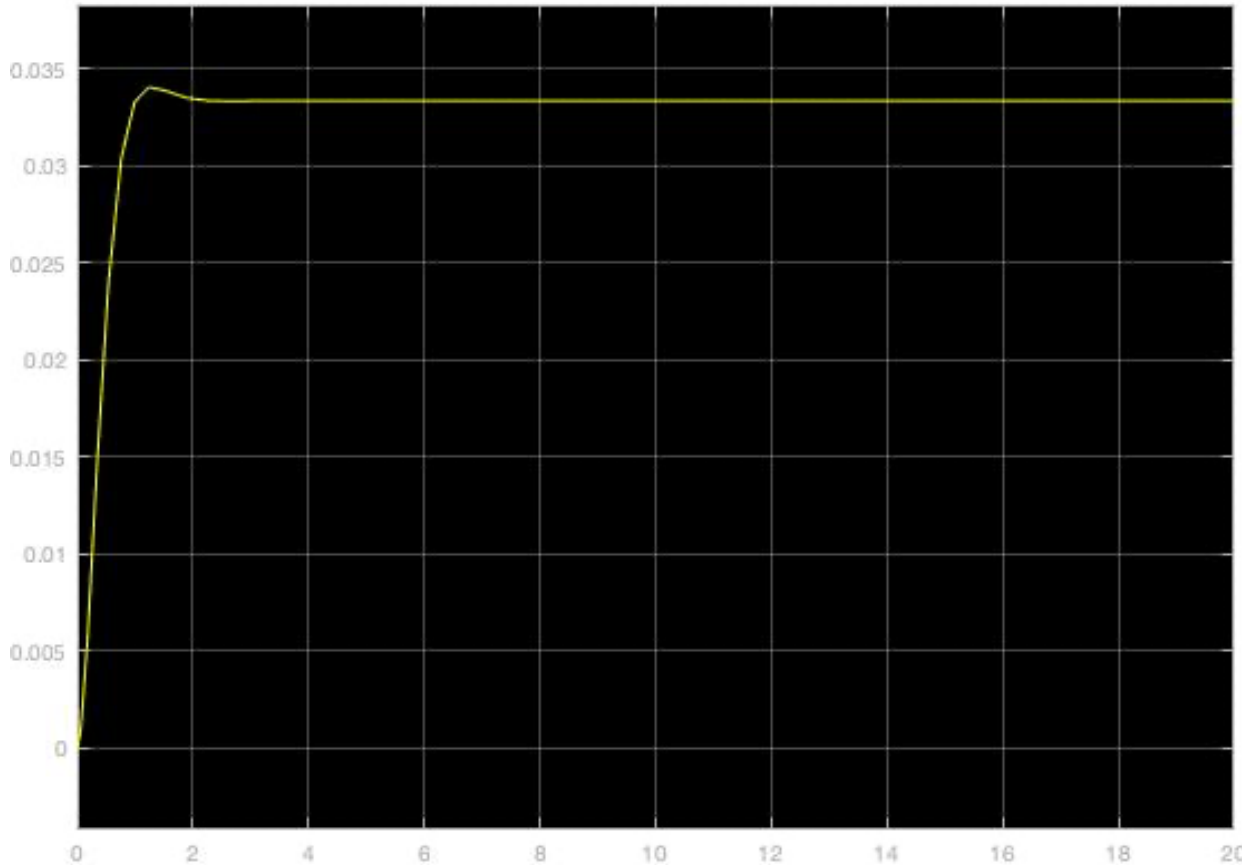
14) Run simulations for the five scenarios in 7) and plot the responses.

We do see the same graphs for all 5 scenarios.

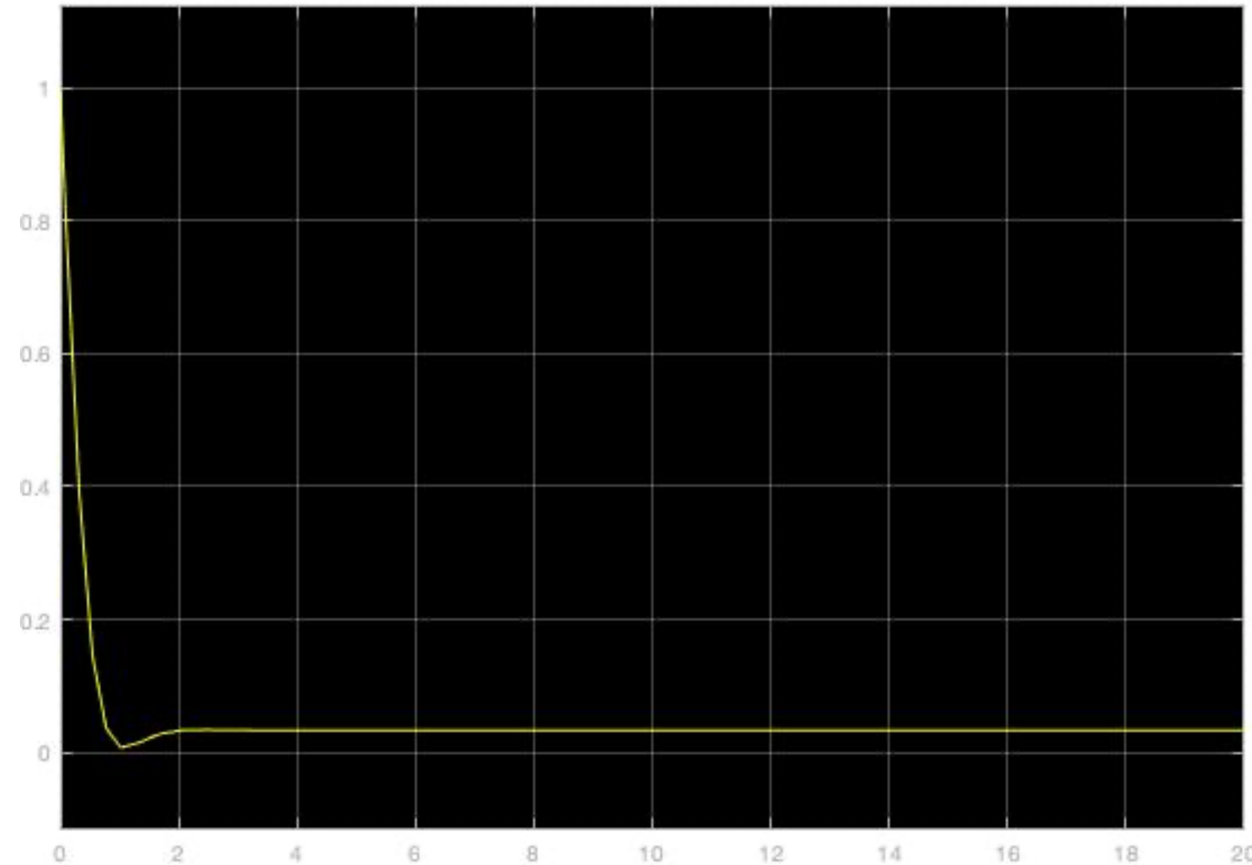
a)



b)



c)



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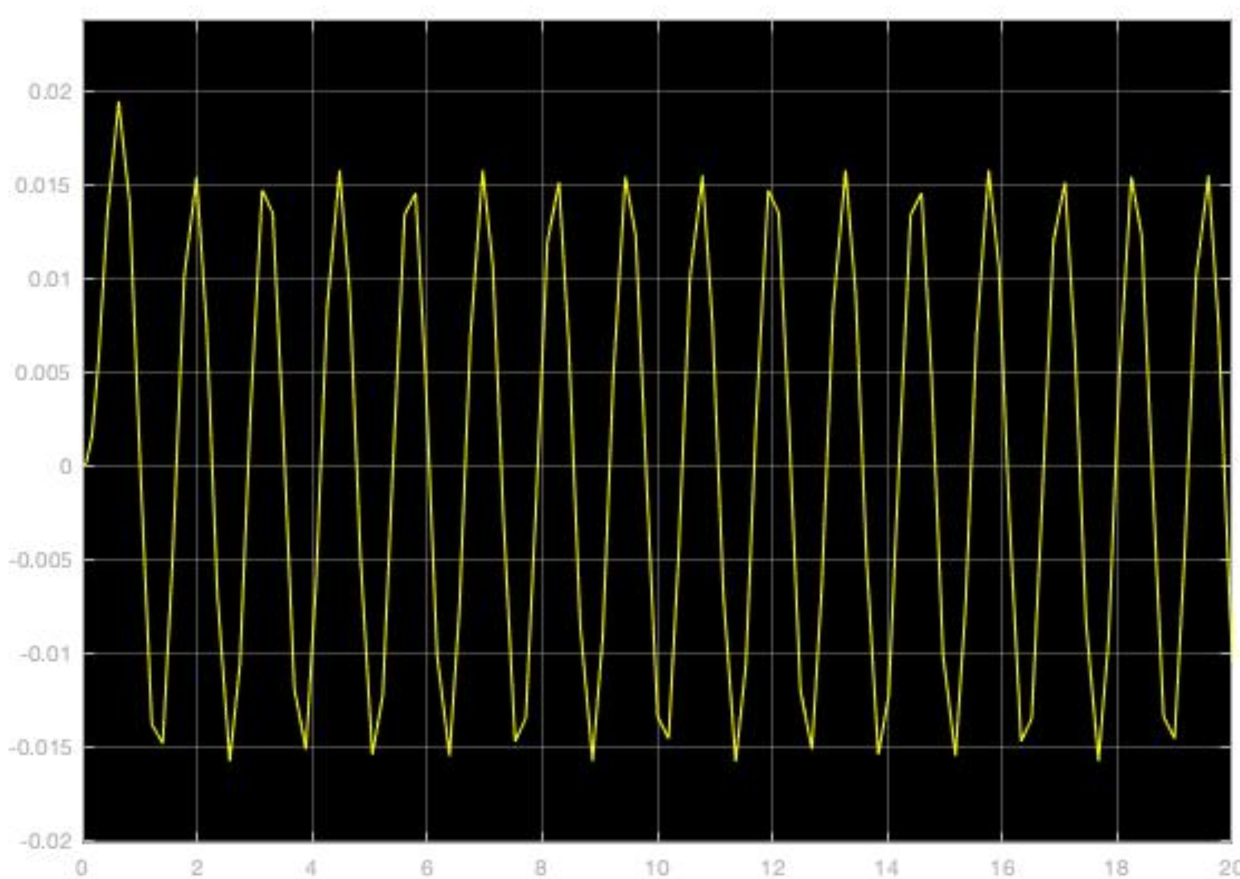

e)