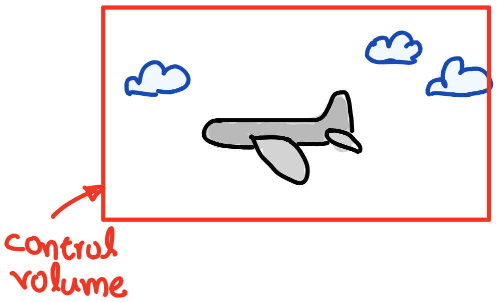
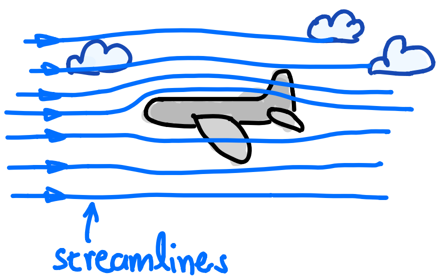
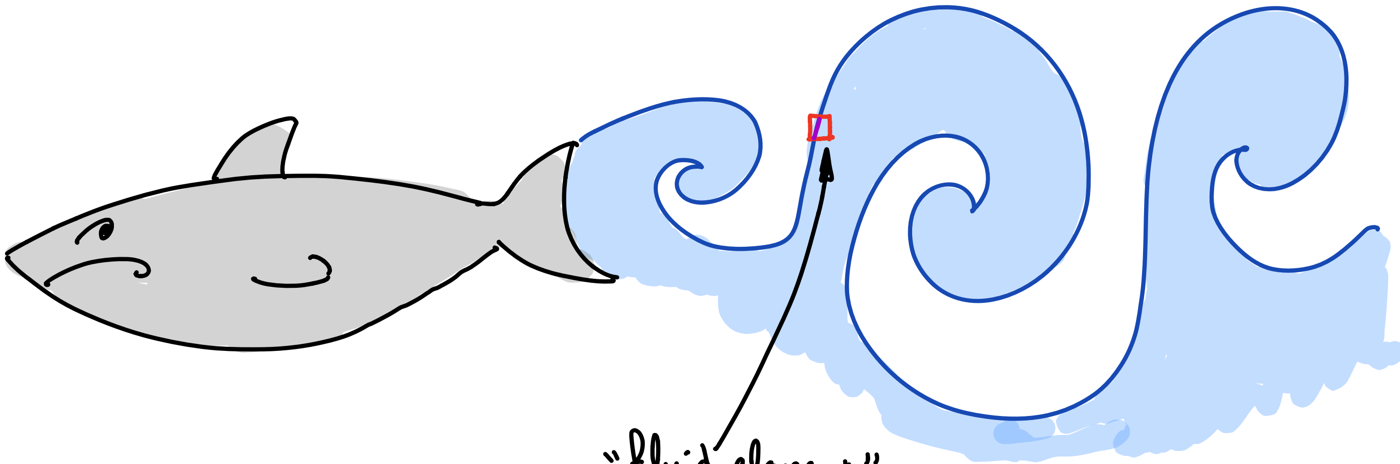
# Differential Analysis of Fluid Flow

## Introduction

| Module 5 | Module 6 |
|---|---|
|  |  |
| **Question:** What is the drag force? | **Question:** What is the flow around the plane? |
| **Method:** Control Volume Analysis | **Method:** Differential analysis |
| **Solution:** A few lines on paper | **Solution:** Typically numerical computer simulations. There are very few exact solutions |
| **Pro:** Simple and fast | **Pro:** "everything" and exactly what engineers want |
| **Con:** Only overall mass, momentum budget, no details | **Con:** Computationally expensive and only possible with some further assumptions, aka "modeling" |

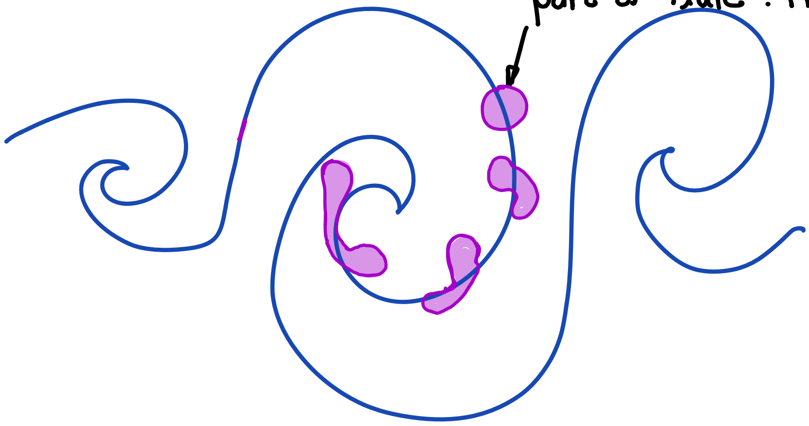
# Fluid element kinematic



"fluid element"
dimensions are sufficiently small so that "everything" can be approximated with straight lines

## Break-down of fluid motion:

part of fluid : finite deformation

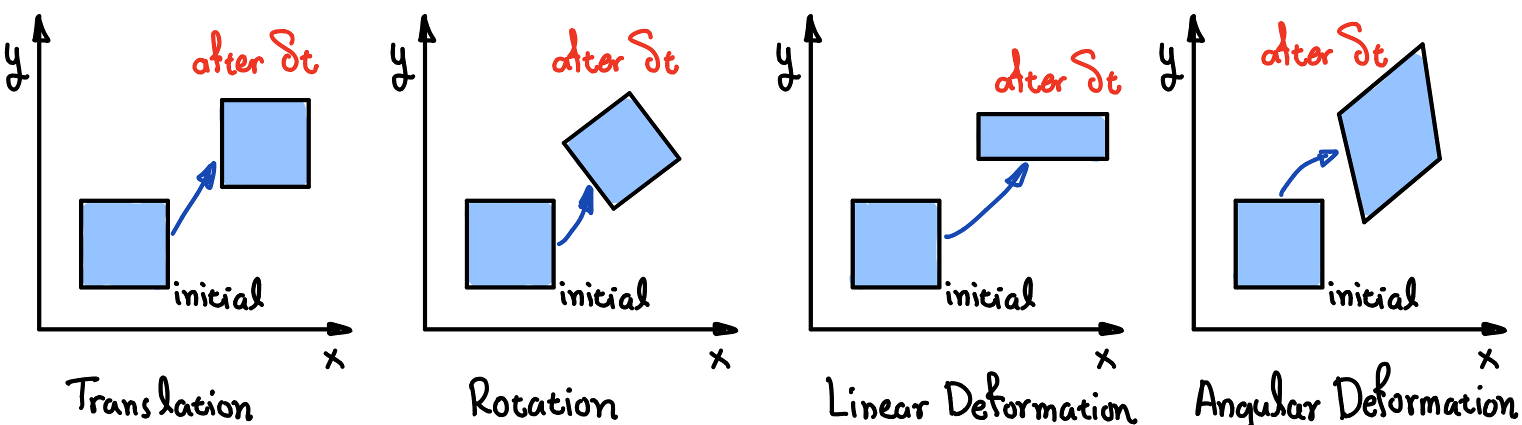


the deformation of a finite volume of fluid is very complex

⋮

we use analysis of infinitesimal elements to derive differential equations to describe the flow

## Infinitesimal element motion

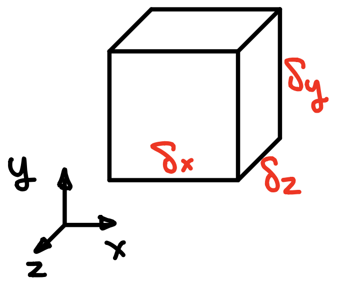



Sides remain parallel

angles between sides change

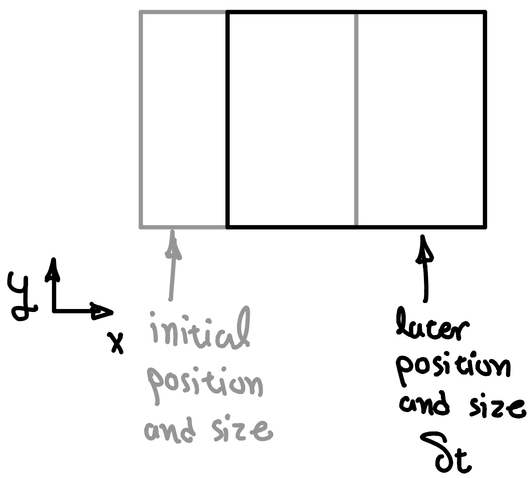
# Linear Motion and Deformation (ch. 6.1.2 of textbook)

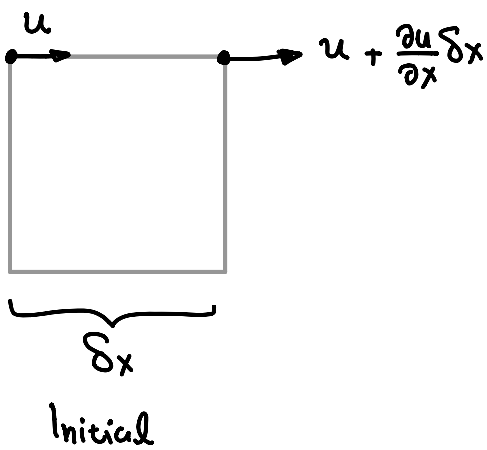
Fluid element:

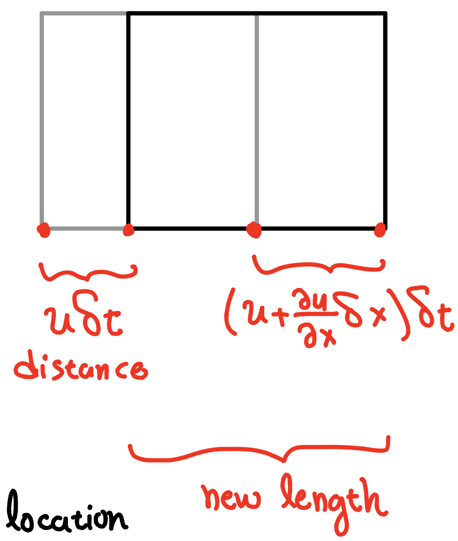



initial volume $= \delta x \delta y \delta z$

- Fluid element is in a flow field with $\frac{\partial u}{\partial x} =$ constant (not $u$ constant)




new length = right side location − left side location

$$= \delta x + \left(u+\frac{\partial u}{\partial x}\delta x\right)\delta t - u\delta t$$

$$= \delta x + \frac{\partial u}{\partial x}\delta x\,\delta t$$

New volume (after time $\delta t$) $= \left(\delta x + \frac{\partial u}{\partial x}\delta x\,\delta t\right)\delta y\,\delta z$

$$= \delta x\,\delta y\,\delta z + \frac{\partial u}{\partial x}\delta x\,\delta y\,\delta z$$

rate of change of volume per unit volume $= \dfrac{\frac{d}{dt}\,\delta\,\text{Volume}}{\text{initial volume}}$

take $\frac{d}{dt}$

divide by initial volume

$$= \frac{\frac{d}{dt}(\text{new} - \text{initial volume})}{\text{initial}} = \frac{\frac{d}{dt}\left(\delta x\,\delta y\,\delta z + \frac{\partial u}{\partial x}\delta x\,\delta y\,\delta z - \delta x\,\delta y\,\delta z\right)}{\delta x\,\delta y\,\delta z}$$

$$= \frac{d}{dt}\frac{\partial u}{\partial x}\delta t$$

take limit for small $\delta t$ :

$$\text{rate of change of volume per unit volume} = \lim_{\delta t \to 0} \frac{d}{dt}\frac{\partial u}{\partial x}\delta t = \frac{\partial u}{\partial x}\lim_{\delta t \to 0}\frac{\delta t}{dt}$$

$$= \frac{\partial u}{\partial x}$$

Similarly for $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$

Remember: $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\text{rate of change of volume per unit volume} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$= \nabla \cdot \vec{V}$$

Volumetric dilatation rate $= \nabla \cdot \vec{V}$ important!!!

or just "dilatation"

Example: Given the two-dimensional velocity field:

$$\vec{V}(x,y) = (x+3y)\,\hat{i} + (3x-y)\,\hat{j}$$

Find the dilatation

Solution: dilatation $= \nabla \cdot \vec{V} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$

$$\vec{V}(x,y) = \underbrace{(x+3y)}_{u}\,\hat{i} + \underbrace{(3x-y)}_{v}\,\hat{j}$$

$$\text{dilatation} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

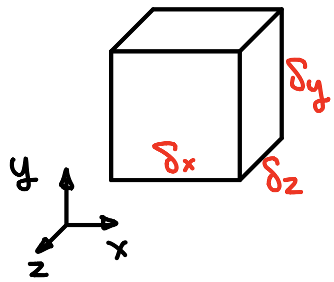
$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(3x-y)$$

$$= 1 - 1 = 0$$

in this case dilatation does not depent on x,y

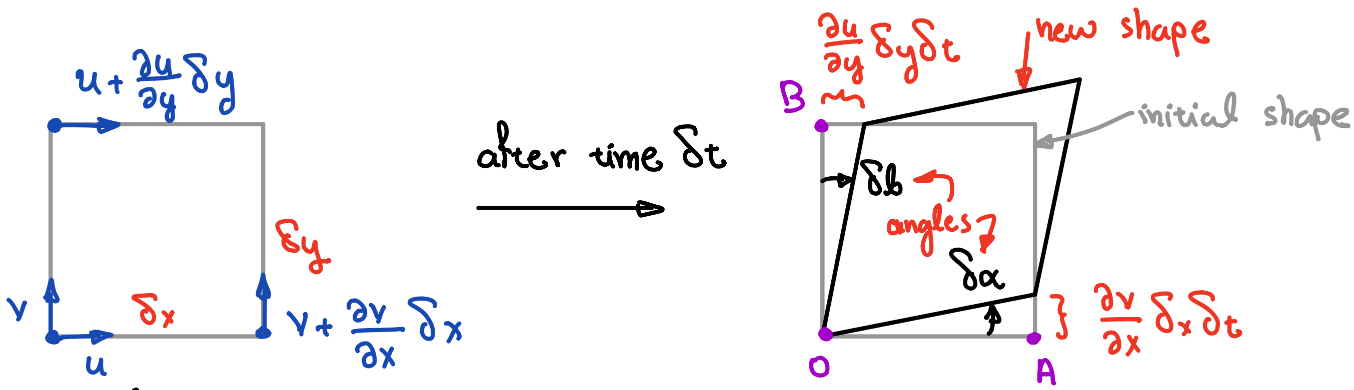
# Angular Motion and Deformation (ch. 6.1.3 of textbook)

Fluid element:




- Fluid element is in a flow field with
  - $\frac{\partial u}{\partial y}$ = constant
  - $\frac{\partial v}{\partial x}$ = constant

cross derivatives




initial time

Angular velocity of side OA : $\omega_{OA} = \lim_{\delta t \to 0} \underbrace{\frac{\delta\alpha}{\delta t}}_{\text{angular rate of change}}$

greek letter "omega"

For small angles $\tan \delta\alpha \simeq \delta\alpha$

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\tan \delta\alpha}{\delta t} = \lim_{\delta t \to 0} \frac{\frac{\partial v}{\partial x}\delta x \delta t}{\delta x\, \delta t} = \frac{\partial v}{\partial x}$$

... similarly ... $\omega_{OB} = \frac{\partial u}{\partial y}$

We define positive rotation counter clockwise
- $\omega_{OA}$ is negative
- $\omega_{OB}$ is positive

Total rotation is the average of OA and OB:

$$\omega_z = \frac{1}{2}(\omega_{OA} + \omega_{OB}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

rotation around z axis

Similarly: $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Rotation "vector" $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

$$\vec{\omega} = \frac{1}{2} \nabla \times \vec{V}$$

Also...

Vorticity "vector" $\vec{\zeta} = 2\vec{\omega} = \nabla \times \vec{V}$

greek letter zeta

$\vec{\zeta}$ quatifies the local $(t, x, y, z)$ solid body rotation of the fluid

Two types of flows:
- $\vec{\zeta} \neq 0$ anywhere $\Rightarrow$ rotational
- $\vec{\zeta} = 0$ everywhere $\Rightarrow$ irrotational

Importance terms

The relation between $\vec{\zeta}$ and $\vec{V}$ is complex
we cannot figure out their relation by just looking at $\vec{V}$

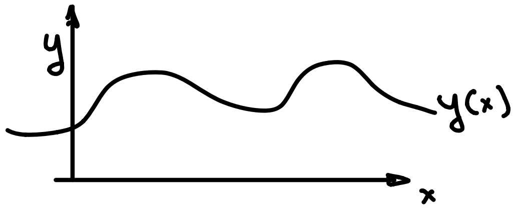
The truth about rotation and vorticity vectors...


rotation vector does <u>not</u> reflect in the mirror!

it is not a proper vector

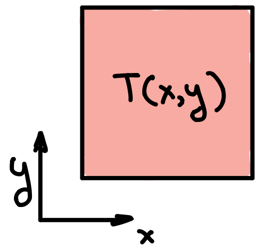
# Gradient of Stuff...

- one-dimensional scalar




gradient : $\frac{dy}{dx}$

- multi-dimensional scalar




gradient: $\underbrace{\nabla T}_{\text{this is a vector}} = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j}$

- multi-dimensional vector field $\vec{V}(x,y,z) = u(x,y,z)\hat{i} + v(x,y,z)\hat{j} + w(x,y,z)\hat{k}$

$$\nabla\vec{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

Velocity gradient tensor

3x3 matrix : second-order tensor

## Velocity gradient, rotation, and strain

The velocity gradient $\nabla\vec{V}$ contains all the "information" about the local spatial derivatives of $\vec{V}$

Decomposition:

$$\nabla\vec{V} = \underbrace{\frac{1}{2}\left(\nabla\vec{V} + \nabla\vec{V}^{T}\right)}_{\substack{\text{symmetric} \\ 3\times 3 \text{ matrix}}} + \underbrace{\frac{1}{2}\left(\nabla\vec{V} - \nabla\vec{V}\right)}_{\substack{\text{anti-symmetric} \\ 3\times 3 \text{ matrix}}}$$

rate of strain tensor | rotation tensor

$$\nabla\vec{V} = S + \underline{\Omega}$$

six non-zero elements

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \text{e.g. } S_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\underline{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$$

← vorticity components

3 non-zero elements

Trace of $\nabla\vec{V}$ (and $S$) $= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ : dilatation

# Conservation of Mass – Continuity equation

Integral form $\qquad \dfrac{\partial}{\partial t}\int_{CV} \rho \, d\mathcal{V} + \int_{CS} \rho \, \vec{V}\cdot\hat{n} \, dA = 0$

$$\int_{CV} \frac{\partial}{\partial t}\rho \, d\mathcal{V} + \int_{CV} \nabla\cdot(\rho\vec{V}) \, d\mathcal{V} = 0$$

$$\int_{CV} \left[ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) \right] d\mathcal{V} = 0$$

Continuity equation: $\dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$

- scalar equation
- holds at any $(t, x, y, z)$

Steady flow : $\dfrac{\partial \bullet}{\partial t} = 0 \quad \Rightarrow \quad \nabla\cdot(\rho\vec{V}) = 0$

Incompressible flow $\rho = \text{const} \Rightarrow \underbrace{\cancel{\dfrac{\partial \rho}{\partial t}}}_{0} + \underbrace{\cancel{\nabla\rho}}_{0}\cdot\vec{V} + \rho \, \nabla\cdot\vec{V} = 0$

$\nabla\cdot\vec{V} = 0$ for both steady and unsteady

or... $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \quad \leftarrow$ continuity equation for incompressible

IMPORTANT → Dilatation is zero in an incompressible fluid!

# Momentum Conservation (ch. 6.3, 6.4.1, 6.8)

Fluid acceleration = body forces + surface forces

$$\frac{\partial \rho \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\rho\vec{V} = \rho\vec{g} + \nabla\cdot T_{ij}$$

unsteady term — convection term — gravity — divergence of the stress tensor

rate of change of momentum

transport of momentum by $\vec{V}$

$T_{ij}$ is a second order tensor

Where is pressure???

Definition of pressure: average normal stress

$$-p = \frac{1}{3}(T_{11} + T_{22} + T_{33}) = \frac{1}{3}\,\mathrm{trace}(T_{ij})$$

remember: positive pressure means compressive stress

identity second-order tensor $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus, $T_{ij} = -p\,I + \tau$

$$= \underbrace{\begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}}_{\text{isotropic tensor}} + \underbrace{\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}}_{\text{deviatoric stress tensor}}$$

ONLY for Newtonian fluid:

$$\tau_{ij} = \underbrace{-\frac{2}{3}\mu\,\nabla\cdot\vec{V}\,I}_{\text{diagonal part}} + 2\mu S_{ij}$$

rate of strain tensor

For incompressible fluid $\nabla\cdot\vec{V} = 0$

$$\rightarrow \tau_{ij} = 2\mu S_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

For example: $\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

- Stress tensor is symmetric
- 3 unique off-diagonal elements + 3 diagonal = 6 unique stresses

if $\mu$ = constant...

Momentum equation: $\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} = -\frac{1}{\rho}\nabla p + \underbrace{\frac{\mu}{\rho}}_{=\nu}\nabla^2\vec{V}$ ← VECTOR equation

$= \nu$ : kinematic viscosity

Continuity : $\nabla\cdot\vec{V} = 0$

Momentum equation: $\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{V}$

+ initial and boundary conditions

} Navier-Stokes equations

Continuity : $\nabla\cdot\vec{V} = 0$

Momentum equation: $\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} = -\frac{1}{\rho}\nabla p$

+ initial and boundary conditions

} Euler equations

no viscous terms

Momentum equation - Component form - Cartesian:

x- component

pressure, kinematic viscosity $\nu = \frac{\mu}{\rho}$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

density

y- component

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

z- component

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + g$$

gravity (if needed)