# Fluid Statics   Chapter 2

1. Preliminaries: "static" usualy means "not moving"
   but in fluids means "no relative motion"

2. Fluid statics are closely related to pressure (chapter 2.1)

   a. pressure is a kind of $\boxed{\text{Stress}}$

   b. pressure is special because:

   - It does <u>not</u> depend on direction   Pascal's Law
   - pressure is the isotropic part of the stress tensor
   - pressure is a scalar : it's just a number, e.g., 10100 Pa

   REMEMBER: Force from pressure = pressure × area

3. Equation for pressure (Chapter 2.2)

$$\underbrace{\nabla p}_{\text{pressure gradient}} = -\rho \underbrace{\vec{a}_t}_{\text{total fluid acceleration}}$$

negative sign means that vector $\nabla p$ and $\vec{a}_t$ point in opossite directions

**IMPORTANT:**
- Vector equation
- $\nabla p$ → says take the gradient of pressure

  e.g. in Cartesion $(x,y,z)$:

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

  in cylindrical $\nabla p$ is different

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

$$\vec{a}_t = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

Cartesian

Components: x-component: $\frac{\partial p}{\partial x} = -\rho a_x$

y-component: $\frac{\partial p}{\partial y} = -\rho a_y$

z-component: $\frac{\partial p}{\partial z} = -\rho a_z$

Gravitational field is part of $a_z$, so... z-component is:

$$\frac{\partial p}{\partial z} = -\rho a_{z,ext} - \rho g$$

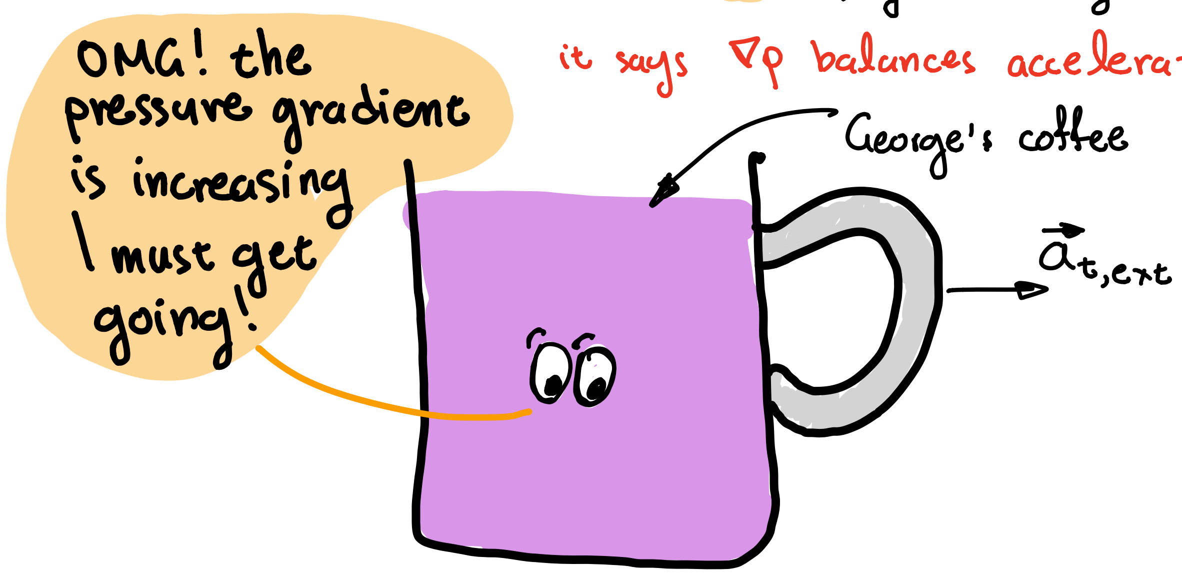
acceleration of gravity

4. Physical interpretation:

$\nabla p = -\rho \vec{a}_t$

- describes conservation of momentum in fluid
- special case for "static" fluid
- it does not imply causality

it says $\nabla p$ balances acceleration




5. Finally: what is the point of all these math?

Solving $\nabla p = -\rho \vec{a}_t$ can give $p(x,y,z)$ in the fluid

from $p$ engineers can compute forces

# Cases of use of the pressure equation

## Case 1: no motion

### Case 1a: no motion and incompressible fluid

means $\rho$ = constant

no motion $\Rightarrow$ $\vec{a}_{ext} = 0$

$$\frac{\partial p}{\partial x} = -\rho \cancel{a_{x,ext}}^{0} = 0 \rightarrow p \text{ does not vary with } x$$

$$\frac{\partial p}{\partial y} = -\rho \cancel{a_{y,ext}}^{0} = 0 \rightarrow p \text{ does not vary with } y$$

$$\frac{\partial p}{\partial z} = -\rho \cancel{a_{z,ext}}^{0} - \rho g = -\rho g \rightarrow \frac{\partial p}{\partial z} = -\rho g$$

we are looking for $p(z)$ ... solve

unknown

$$\frac{\partial p}{\partial z} = -\rho g$$

known parameters

independent variable: coordinate z

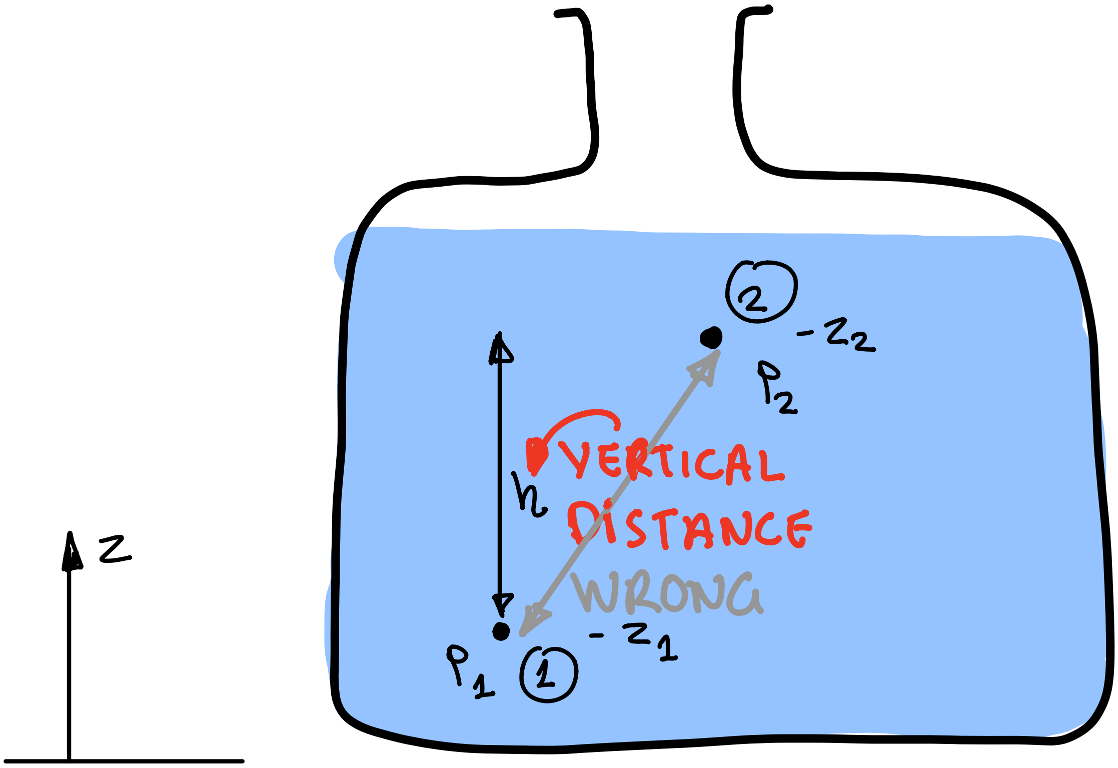
$$\int_{p_1}^{p_2} dp = -\int_{z_1}^{z_2} \rho g \, dz$$

$$p_2 - p_1 = -\rho g (z_2 - z_1)$$

$$p_2 = p_1 + \rho g \underbrace{(z_1 - z_2)}_{\text{vertical distance}}$$

$$p_2 = p_1 + \rho g h$$ Hydrostatic distribution

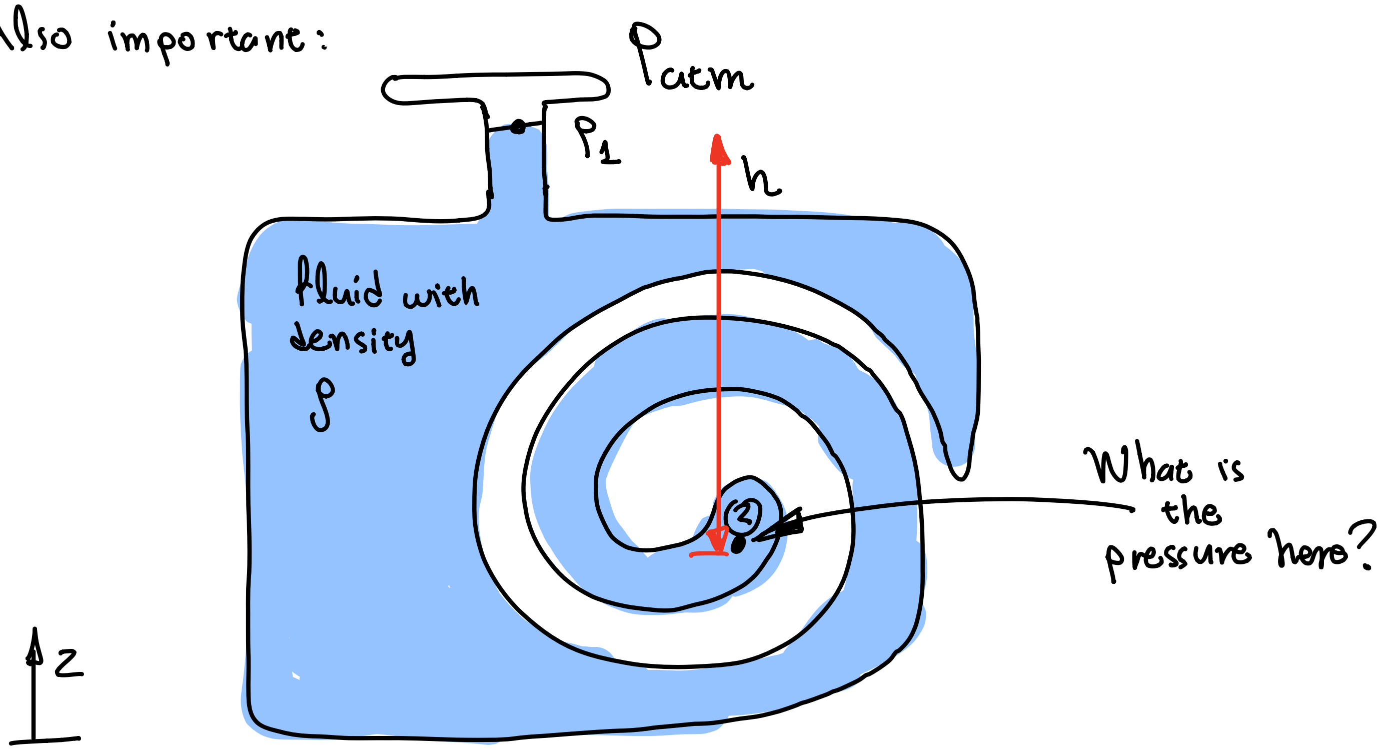
$$p_2 = p_1 + \rho g (z_1 - z_2)$$




$$p_2 = p_1 + \rho g \underbrace{(z_1 - z_2)}_{\text{negative}} \qquad z_1 < z_2$$

$\Rightarrow p_2 < p_1$

Also important:



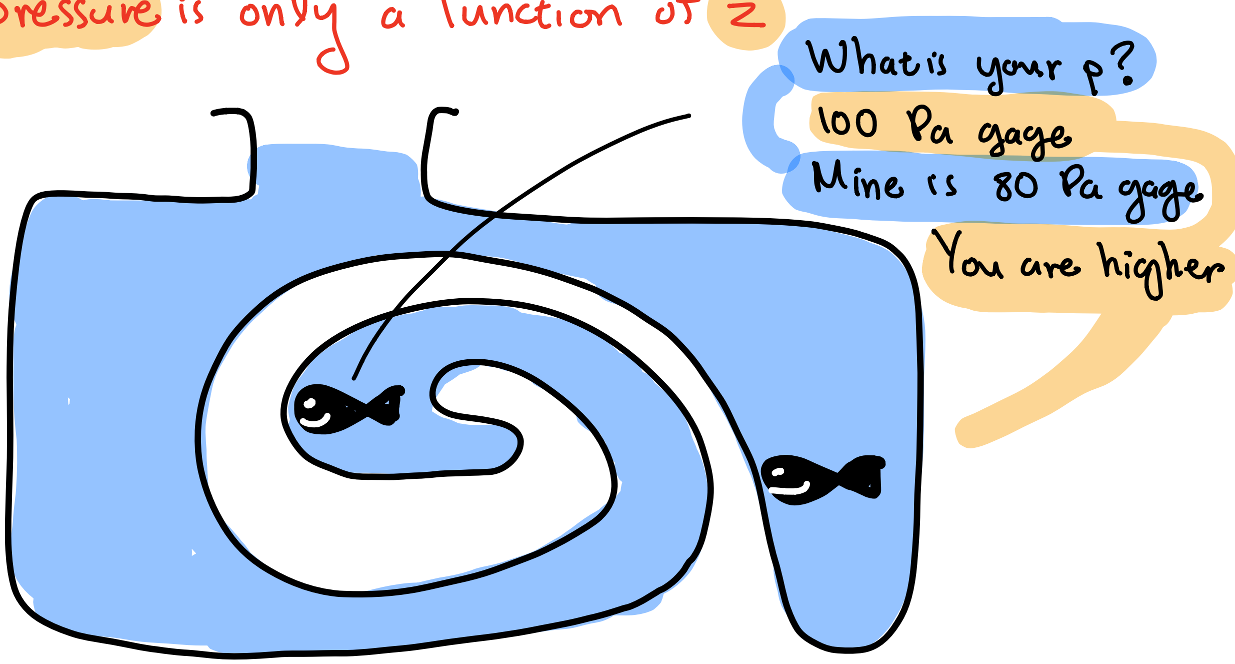

$$P_2 = P_1 + \rho g (z_1 - z_2)$$

$$P_2 = P_{atm} + \rho g h$$

$$P_2 - P_{atm} = \rho g h$$

$$P_{2,gage} = \rho g h$$

Again pressure is only a function of z



Case 1b: no motion and compressible fluid

$\rho$ = function of z

Pressure equation: $\frac{dp}{dz} = -\rho g$

$$\int_{p_1}^{p_2} dp = -\int_{z_1}^{z_2} \underbrace{\rho(z)}_{\text{need this function}} g\, dz$$

## Usefull problem-solving strategy

1. Check that assumptions of Case 1a are true
2. Relation: $p_2 - p_1 = \rho g (z_1 - z_2)$

5 variables

usually $\rho$ is known

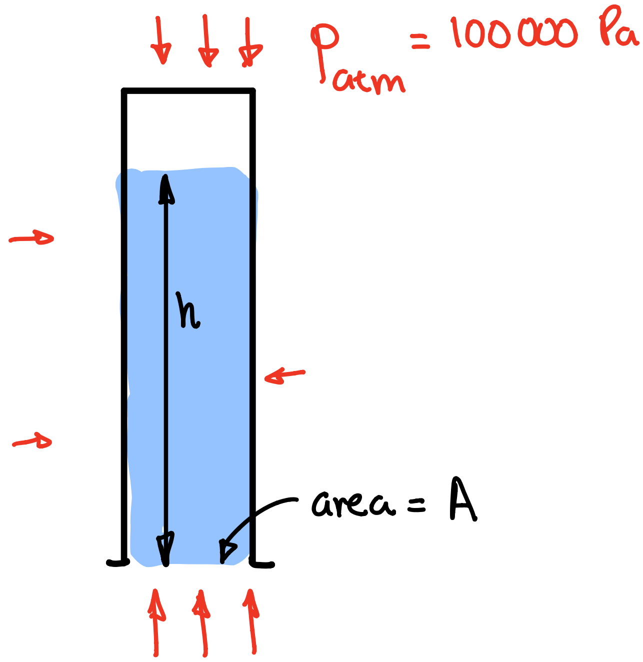
so,.. 4 variables 1 equation

if any three of $p_1, p_2, z_1, z_2$ are given can find the fourth

Please be carefull with signs

# Atmospheric Pressure

What is the maximum fluid column height h?




Balance of forces:

Fluid Weight = $P_{atm}$ Force

$$mass * g = P_{atm} A$$

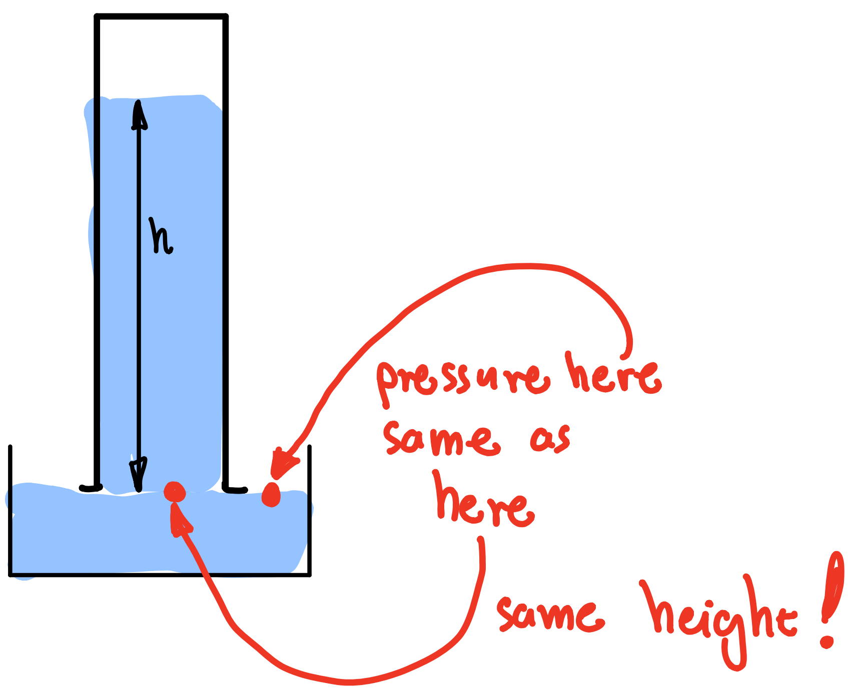
$$\rho \, Volume * g = P_{atm} A$$

$$\rho \cancel{A} h g = P_{atm} \cancel{A} \Rightarrow h = \frac{P_{atm}}{\rho g} = \frac{100\,000}{1000 * 10} = 10\text{ m}$$

water $\rho = 1000$ kg/m3

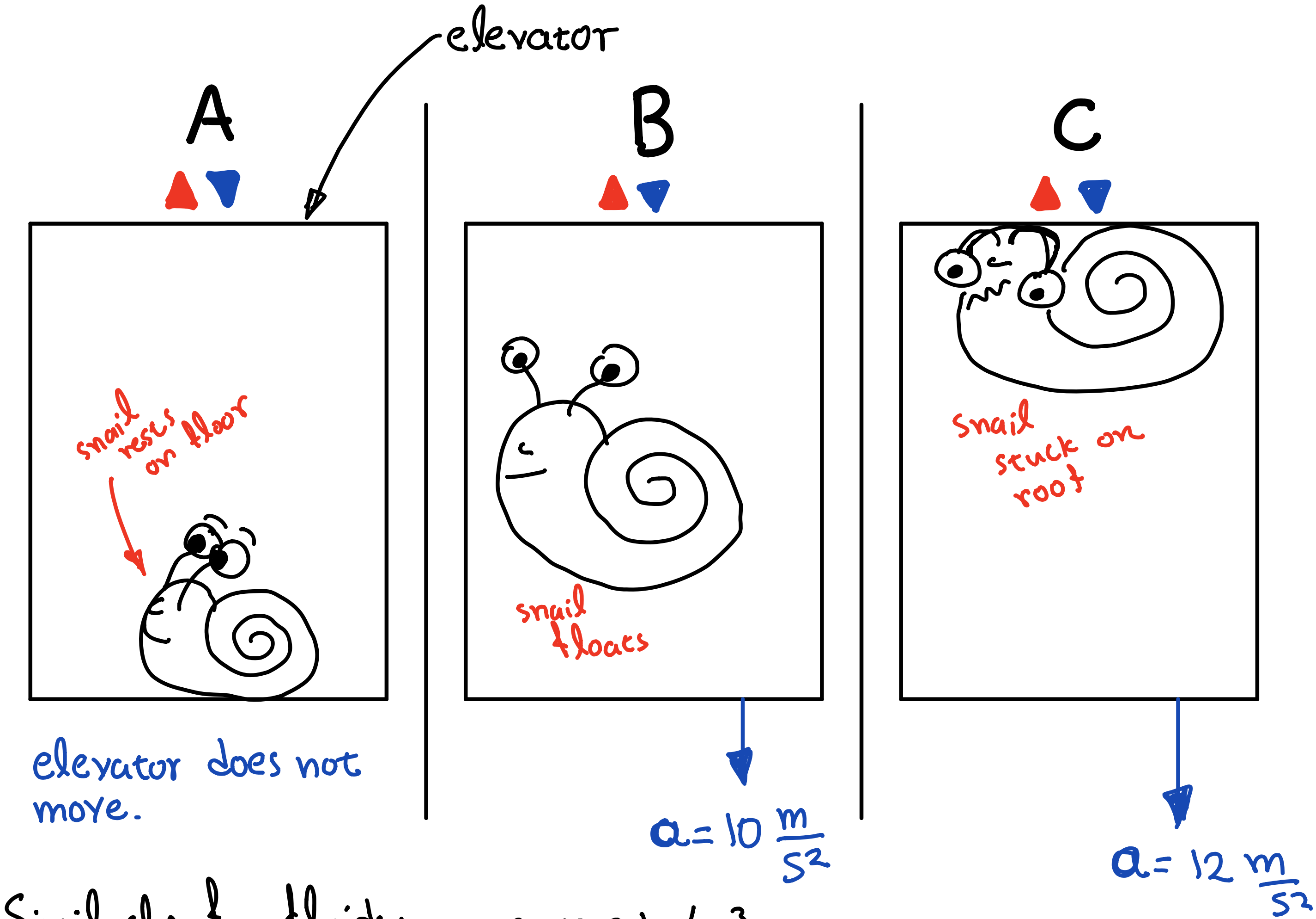
if fluid is mercury $\rho_{Hg} = 13\,534 \Rightarrow h = 100\,000 / (13534 * 9.81)$

$$h = 0.753\text{ m}$$

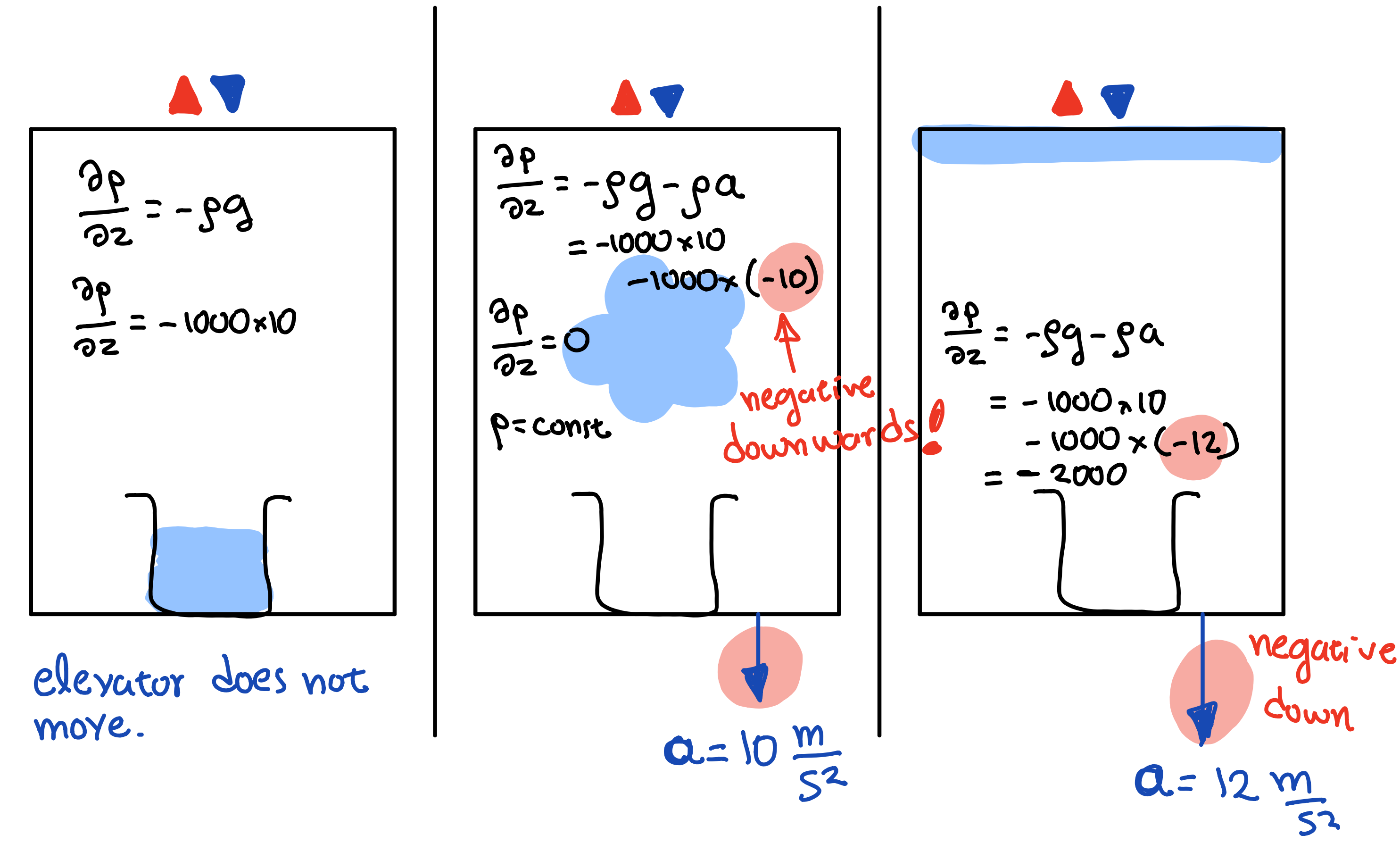


Case 2a: constant acceleration (motion) and incompressible ($\rho$ = const)

no Case 2b in this class




Similarly for fluids: $\rho = 1000\ kg/m^3$

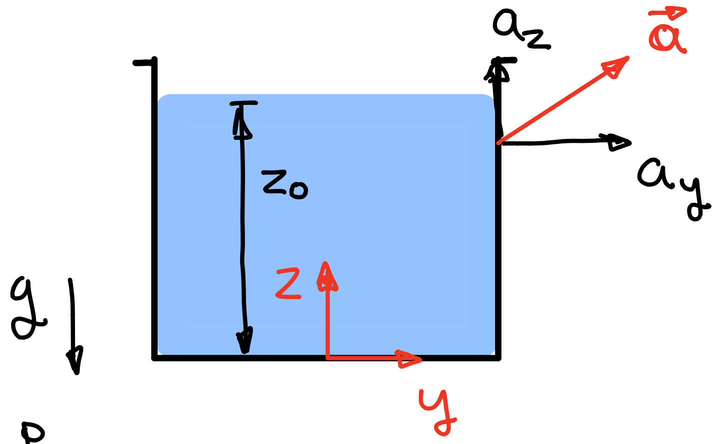


# Example 2.11

Acceleration $\vec{a} = a_y \hat{j} + a_z \hat{k}$

density $= \rho$

height of fluit at rest $= z_0$




**Find:** pressure in the fluid $p$

<u>Solution</u>: equation for pressure $\nabla p = -\rho \vec{a}_t$

x-comp: $\frac{\partial p}{\partial x} = \cancel{-\rho a_x} \to 0$

y-comp: $\frac{\partial p}{\partial y} = -\rho a_y$

z-comp: $\frac{\partial p}{\partial z} = -\rho a_z - \rho g$

2 equations $\Rightarrow p(y,z)$

We look for lines of constant pressure : $dp = 0$

$$dp = \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz = 0$$

$$dp = -\rho a_y \, dy - \rho(a_z + g) \, dz = 0$$

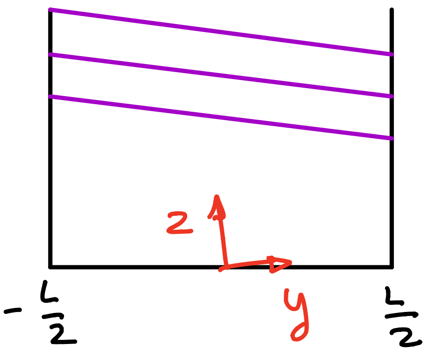
$$\frac{dz}{dy} = -\frac{a_y}{a_z + g}$$

← slope of lines of const pressure

slope is negative for $a_y > 0$ and $a_z > -g$

Need $z =$ function of $y$

$$\int dz = \int -\frac{a_y}{a_z + g} dy \Rightarrow z(y) = -\frac{a_y}{a_z + g} y + \text{constant}$$




What is the equation for the surface?

$$z_s(y) = -\frac{a_y}{a_z+g}\,y + c$$

How to find the constant?

Volume of fluid is the same.

Volume of fluid when tank is at rest: $V_0 = z_0 L W$ (1)

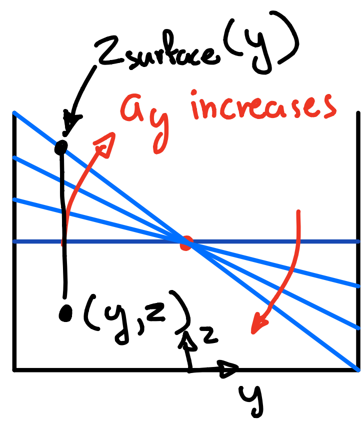
Volume of fluid when $\vec{a}$ = $\qquad V_1 = \int z_s(y)\,dy\,dx$

$$V_1 = \int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{0}^{W} -\frac{a_y}{a_z+g}\,y\,dy\,dx + \int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{0}^{W} c\,dy\,dx$$

$$= -\frac{a_y}{a_z+g}\,\underbrace{y^2\Big|_{-\frac{1}{2}}^{\frac{L}{2}}}_{=0}\,W + cLW = cLW \quad (2)$$

(1) = (2) $\rightarrow$ $c = z_0$

$$\boxed{Z_s = -\frac{a_y}{a_z+g}\,y + z_0}$$

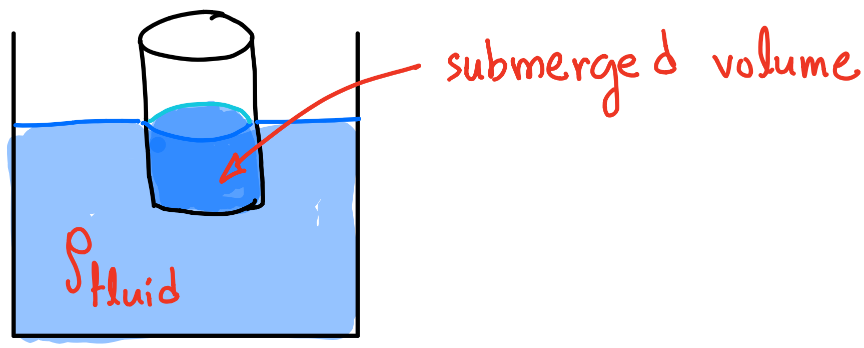



$$p(y,z) = p_{atm} + \rho g\left(Z_{surface}(y) - z\right)$$

$$p(y,z) = p_{atm} + \rho g\left(-\frac{a_y}{a_z+g}\,y + z_0 - z\right)$$ for $(y,z)$ in the fluid

# Buoyancy

$$\text{Buoyancy force} = \rho_{fluid}\, g\, V_{body\ submerged}$$



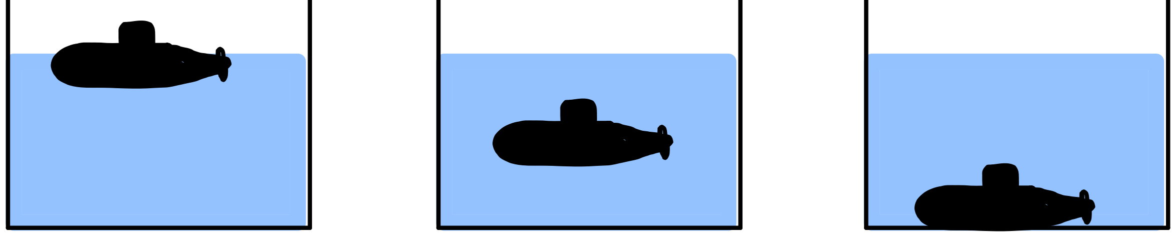

If something floats Weight = Buoyancy

$$\rho_{body}\, g\, V_{body} = \rho_{fluid}\, g\, V_{submerged}$$

these are different in general

not necessarily the same

$\rho_{sub} < \rho_{water}$ $\qquad$ $\rho_{sub} = \rho_{water}$ $\qquad$ $\rho_{sub} > \rho_{water}$



$V_{sub} > V_{submerged}$ $\qquad$ $V_{sub} = V_{submerged}$ $\qquad$ $V_{sub} = V_{submerged}$