What is a signal

any function of time: $f(t)$

Force: $F(t)$ Displacement: $x(t)$ Voltage: $V(t)$

What are systems:

processing of signals

$$u(t) \longrightarrow \boxed{\text{System}} \longrightarrow x(t)$$

static systems:

only depend on current values,

Dynamic systems:

Depends on current and previous values, integration and differenciation

linear rules:

$u(t) \rightarrow y(t)$ $\qquad$ $u_1(t) \rightarrow y_1(t)$, $u_2(t) \rightarrow y_2(t)$

$a\,u(t) \rightarrow a\,y(t)$ $\qquad$ $u_1(t) + u_2(t) \rightarrow y_1(t) + y_2(t)$

Complex plain:

Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Rectangular form: $a + ib$ $\qquad$ Polar form: $Ae^{i\theta}$

$$a^2 + b^2 = A^2$$

$$\theta = \cos^{-1}(a/A) \text{ or } \sin^{-1}(b/A)$$

complex varible s, $f(s)$

Laplace transformation:

$$\mathcal{L}\{f(t)\} = F(t) = \int_0^{\infty} f(t)e^{-st}\,dt$$

common $\mathcal{L}$:

$f(t) = e^{-at}$ $\qquad$ $F(t) = \dfrac{1}{s+a}$ $\qquad$ existence condition: $Re[s+a] > 0$

Laplace transformation property

Superposition property:

$$\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} = \alpha F_1(s) + \beta F_2(s) = \alpha \mathcal{L}\{f_1(t)\} + \beta \mathcal{L}\{f_2(t)\}$$

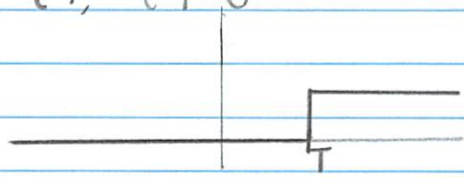
example: $\mathcal{L}\{3e^{2t} + 2\}$

$$= 3 \cdot \frac{1}{s-2} + 2 \cdot \frac{1}{s}$$

Translated signal: $1(t-T)$

$\uparrow$ constant

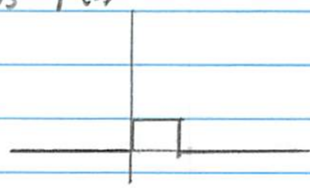
$$1(t-T) = \begin{cases} 0, & t-T<0 \\ 1, & t-T \geq 0 \end{cases} \Rightarrow 1(t-T) = \begin{cases} 0, & t<T \\ 1, & t \geq T \end{cases}$$



$\mathcal{L}\{f(t-T)1(t-T)\} = e^{-sT}F(s)$ assuming $\mathcal{L}\{f(t)\} = F(s)$

Pulse function:

$$A(t) = \begin{cases} 0, & t<0 \\ 1 & 0 \leq t < T \\ 0 & t > T \end{cases}$$



$$A(t) = 1(t) - 1(t-T)$$

$$\mathcal{L}\{A(t)\} = \mathcal{L}\{1(t)\} - \mathcal{L}\{1(t-T)\}$$

$$= \frac{1}{s} - \frac{e^{-sT}}{s}$$

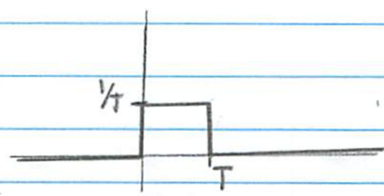
ex: $\mathcal{L}\{e^{-2t}\, 1(t-3)\}$

$$e^{-2t} = e^{-2[t-3+3]} = e^{-6} \cdot e^{-2(t-3)} = f(t-3), \quad f(t) = e^{-6} \cdot e^{-2t}$$

$$\mathcal{L} = e^{-6} e^{-3s} \cdot \frac{1}{s+2}$$

Impulse signal:

$$A(t,T) = \begin{cases} 0, & t<0 \\ 1/T, & 0 \leq t < T \\ 0, & t \geq T \end{cases}$$



$\boxed{\text{Impulse signal} \quad \delta(t) = \lim_{T \to 0} A(t,T)}$

area = 1

$$\mathcal{L}\{\delta(t)\} = \mathcal{L}\{\lim_{T\to 0} A(t,T)\} = \lim_{T\to 0} \mathcal{L}\{A(t,T)\} = \lim_{T\to 0} \frac{1}{T}\left(\frac{1}{s} - \frac{e^{-sT}}{s}\right)$$

$$= \frac{1}{s} \lim_{T\to 0} \frac{1}{T}\left(1-e^{-sT}\right)$$

$$e^{-sT} = 1 - sT - s\otimes T^2 - s\otimes T^3 \underbrace{+\dots}_{O(T)}$$

$$= \frac{1}{s} \lim_{T\to 0} \frac{1}{T}\left(1-(1-sT+\otimes T^2+\dots)\right)$$

$$= \frac{1}{s} \lim_{T\to 0} \frac{1}{T}\left(sT+\otimes T^2+\dots\right)$$

$$= \frac{1}{s} \lim_{T\to 0} \left(s + \frac{\otimes T^2 + \otimes T^3 + \dots}{T}\right) = \frac{1}{s} \lim_{T\to 0}\left(s + \otimes T + \otimes T^2 + \dots\right)$$

$$= \frac{1}{s} \cdot s = 1$$

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

Laplace transformation of sinusoidal signals:

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

Signal multiplied by e^{-at}:

given $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$

Differenciation and integration theorem:

given: $\mathcal{L}\{f(t)\} = F(s)$

then $\mathcal{L}\{f'(t)\} = s \cdot F(s) - f(0)$

then $\mathcal{L}\{\int_0^t f(\tau)\, d\tau\} =$

say $\int_0^t f(\tau)\, d\tau = g(t)$

$g(0) = \int_0^0 = 0$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s \cdot \mathcal{L}\{g(t)\} - g(0)$$

$$F(s) = s \cdot \mathcal{L}\{g(t)\} \quad \therefore \quad \boxed{\mathcal{L}\{g(t)\} = \frac{F(s)}{s}}$$

polynomial signals: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

also $\mathcal{L}\left\{\frac{t^n}{n!}\right\} = \frac{1}{s^{n+1}}$

Homework 2

1. $f(t) = e^{-3t} + 3$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{s+3} + \frac{3}{s}}$$

2. $f(t) = 5 + 4\delta(t)$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{5}{s} + 4}$$

3. $f(t) = 3\sin(3t+2) = 3\sin(3t)\cos(2) + 3\cos(3t)\sin(2)$

$$\boxed{\mathcal{L}\{f(t)\} = 3\cos(2) \cdot \frac{3}{3^2+s^2} + 3\sin(2) \cdot \frac{s}{3^2+s^2}}$$

4. $f(t) = 6\cos(t-2) = 6\cos(t)\cos(2) + 6\sin(t)\sin(2)$

$$\boxed{\mathcal{L}\{f(t)\} = 6\cos(2) \cdot \frac{s}{1+s^2} + 6\sin(2) \cdot \frac{1}{1+s^2}}$$

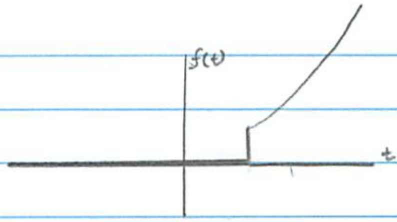
5. $f(t) = 1(t-4) = 1 \cdot 1(t-4)$

$g(t-\tau) = 1 \quad \therefore g(t) = 1 \quad \therefore G(s) = \frac{1}{s}$

$\mathcal{L}\{g(t-\tau)1(t-\tau)\} = e^{-s\tau}G(s)$

$$\boxed{\mathcal{L}\{f(t)\} = e^{-4s} \cdot \frac{1}{s}}$$

6. $f(t) = \mathbb{1}(t-2)e^t = e^t \cdot \mathbb{1}(t-2)$



$g(t-2) = e^t$

$x = t-2$

$t = x+2$

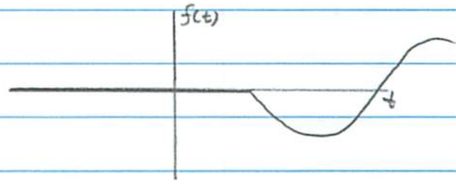
$g(x) = e^{x+2} = e^x \cdot e^2$

$G(s) = \mathcal{L}\{g(x)\} = e^2 \cdot \frac{1}{s-1}$

$\mathcal{L}\{g(t-2) \cdot \mathbb{1}(t-2)\} = e^{-2s} \cdot G(s)$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{e^{-2s} \cdot e^2}{s-1}}$$

7. $f(t) = \mathbb{1}(t-3)\sin(t) = \sin(t) \cdot \mathbb{1}(t-3)$



$g(t-3) = \sin(t)$

$u = t-3$

$t = u+3$

$g(u) = \sin(u+3)$

$= \sin(u)\cos(3) + \cos(u)\sin(3)$

$G(s) = \cos(3) \cdot \frac{1}{s^2+1} + \sin(3) \cdot \frac{s}{s^2+1}$

$\mathcal{L}\{g(t-3) \cdot \mathbb{1}(t-3)\} = e^{-3s} G(s)$

$$\boxed{\mathcal{L}\{f(t)\} = e^{-3s}\left(\frac{\cos(3)}{s^2+1} + \frac{s\sin(3)}{s^2+1}\right)}$$

Final/initial value theorms:

given $\mathcal{L}\{f(t)\} = F(s)$

condition: if and only if the limits exist

initial value is:

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s \cdot F(s)$$

Final value is:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s \cdot F(s)$$

ex: $f(t) = 1(t)$ $\quad F(s) = \frac{1}{s}$

$$\lim_{t\to 0^+} 1(t) = 1 = \lim_{s\to\infty} \frac{s}{s} = 1 \text{ also}$$

$$\lim_{t\to\infty} 1(t) = 1 = \lim_{s\to 0} \frac{s}{s} = 1 \text{ also}$$

ex: $f(t) = e^{-2t}\cos(t)$ $\quad F(s) = (s+2)/((s+2)^2+1)$

$$\lim_{t\to 0^+} f(t) = 1 \qquad \lim_{s\to\infty} s \cdot F(s) = 1 \checkmark$$

$$\lim_{t\to\infty} f(t) = 0 \qquad \lim_{s\to 0} s \cdot F(s) = \frac{0}{5} = 0 \checkmark$$

Property $f(t/a)$ $\quad$ given $\mathcal{L}\{f(t)\} = F(s)$

then $\mathcal{L}\{f(t/a)\} = a\,F(a \cdot s)$

Convolution theorem: given $\mathcal{L}\{f_1(t)\} = F_1(s)$

$\mathcal{L}\{f_2(t)\} = F_2(s)$

then: $\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s) \cdot F_2(s)$

also note: $f_1 * f_2 = f_2 * f_1$

Solution to a LTI system:

given: system and $u(t)$ input and initial conditions

you can find $y(t)$ output for $t \geq 0$

Example: $2\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = u(t)$

$$u(t) = e^{-5t} \quad y(0) = 5 \quad \dot{y}(0) = -1$$

$\mathcal{L}$ both sides

$$2\left(s^2 Y(s) - s\,y(0) - \dot{y}(0)\right) + 3\left(sY(s) - y(0)\right) + 4Y(s) = \frac{1}{s+5}$$

$$Y(s) = \frac{1 + (10s + 13)(s+5)}{(s+5)(2s^2 + 3s + 4)}$$

transfer function: zero's initial conditions

$$\frac{Y(s)}{U(s)} = \underbrace{\frac{b_m s^m + \dots + b_0}{a_n s^n + \dots + a_0}}_{T(s)}$$

$$Y(s) = U(s)\,T(s)$$

Homework 3

1. $f(t) = e^{-3t} + 3 + 2t + 4t^2 + 5t^3$

$$\mathcal{L}\{f(t)\} = \frac{1}{s+3} + \frac{3}{s} + \frac{2}{s^2} + \frac{8}{s^3} + \frac{30}{s^4}$$

2. $f(t) = 2te^{3t} + 4t^2 e^{-5t}$ * $\mathcal{L}\{e^{-at} g(t)\} = G(s+a)$

$$\mathcal{L}\{f(t)\} = \frac{2}{(s-3)^2} + \frac{8}{(s+5)^3}$$

3. $f(t) = \sin(\omega t) e^{-at}$

$$\mathcal{L}\{f(t)\} = \frac{\omega}{(s+a)^2 + \omega^2}$$

4. $f(t) = \cos(\omega t) e^{-at}$

$$\mathcal{L}\{f(t)\} = \frac{s+a}{(s+a)^2 + \omega^2}$$

5. $f(t) = 3\sin(4t+2)e^{-6t} = 3e^{-6t}\sin(4t)\cos(2) + 3e^{-6t}\cos(4t)\sin(2)$

$\mathcal{L}\{f(t)\} = 3\cos(2)\cdot\dfrac{4}{(s+6)^2+4^2} + 3\sin(2)\cdot\dfrac{(s+6)}{(s+6)^2+4^2}$

6. $f(t) = \cos(2t-2)e^{3t} = e^{3t}\cos(2t)\cos(2) + e^{3t}\sin(2t)\sin(2)$

$\mathcal{L}\{f(t)\} = \cos(2)\cdot\dfrac{(s-3)}{(s-3)^2+2^2} + \sin(2)\cdot\dfrac{2}{(s-3)^2+2^2}$

7. $f(t) = t^3e^{-0.1t}$

$\mathcal{L}\{f(t)\} = \dfrac{6}{(s+0.1)^4}$

8. $f(t) = 3 + e^{-0.1t}\cos(4t)$

$\mathcal{L}\{f(t)\} = \dfrac{3}{s} + \dfrac{(s+0.1)}{(s+0.1)^2+4^2}$

9. find $\lim\limits_{t\to 0^+}$ and $\lim\limits_{t\to\infty}$ for $f(t)$ in question 7.

$\lim\limits_{t\to 0^+} f(t) = \lim\limits_{s\to\infty} s\cdot F(s) = \dfrac{\infty}{\infty^4} = 0$

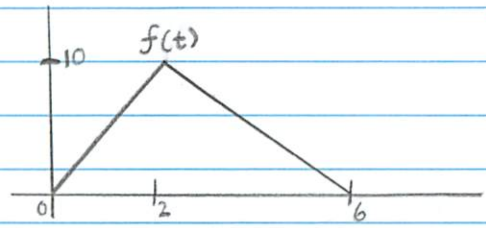
$\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} s\cdot F(s) = \dfrac{0}{(0.1)^4} = 0$

10. find $\lim\limits_{t\to 0^+}$ and $\lim\limits_{t\to\infty}$ for $f(t)$ in quesion 8.

$\lim\limits_{t\to 0^+} f(t)\ \lim\limits_{s\to\infty} s\cdot F(s) = 3 + \dfrac{s^2}{s^2} = 4$

$\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} s\cdot F(s) = 3 + \dfrac{0}{4} = 3$

11.



$m = \frac{-10}{4} = \frac{-5}{2}$

$$f(t) = \begin{cases} 0 & t<0 \\ 5t & 0 \le t \le 2 \\ 0 & t>2 \end{cases} + \begin{cases} 0 & t<2 \\ -\frac{5}{2}t+15 & 2<t\le 6 \\ 0 & t>6 \end{cases}$$

$$= \begin{cases} 0 & t<0 \\ 5t & 0\le t<2 \\ 5t+(-5t-\frac{5}{2}(t)+15) & 2\le t\le 6 \\ 0 & t>6 \end{cases}$$

$$= \left(5t\cdot 1(t) + (-5t-\tfrac{5}{2}t+15)\cdot 1(t-2) - (-\tfrac{5}{2}t+15)\, 1(t-6)\right)$$

$$f(t) = 5t\cdot 1(t) - 5(\tfrac{3}{2}t-3)\cdot 1(t-2) + 5(\tfrac{1}{2}t-3)\cdot 1(t-6)$$

$$\mathcal{L}\{g(t-T)\,1(t-T)\} = e^{-sT}G(s)$$

$$\mathcal{L}\{5(t-0)\cdot 1(t-0)\} = \frac{5}{s^2}$$

$$-5(\tfrac{3}{2}t-3) = -\tfrac{15}{2}(t-2)$$

$$\mathcal{L}\{-\tfrac{15}{2}(t-2)\cdot 1(t-2)\} = -e^{-2s}\frac{15}{2s^2}$$

$$5(\tfrac{1}{2}t-3) = \tfrac{5}{2}(t-6)$$

$$\mathcal{L}\{\tfrac{5}{2}(t-6)\cdot 1(t-6)\} = e^{-6s}\cdot\frac{5}{2s^2}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2}\left(5 - \frac{15}{2e^{2s}} + \frac{5}{2e^{6s}}\right)$$

When given an LTI system, assume zero's for I.C.

Transfer function $T(s) = \frac{Y(s)}{U(s)}$

relationship between input and output of LTI system

Order: highest order in denominator

Relative order: highest order between denominator and numerator

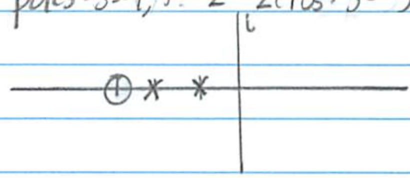
Proper LTI system: Relative order ≥ 0

ex: roots of denominator: poles: X

roots of numator: zeros: O

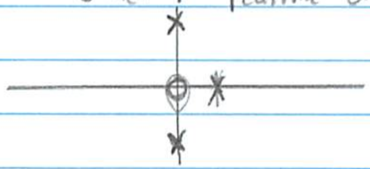
ex: $T(s) = \frac{s+3}{(s+1)(s+2)}$

order: 2 relative order: 1 poles: $s=-1, s=-2$ zeros: $s=-3$



ex: $T(s) = \frac{s^2}{(s^2+4)(s-1)^2}$

order: 4 realtive order: 2 poles: $\underbrace{s=1,\ s=2j,\ s=-2j,\ s=1}_{\text{plural poles}}$ zeros: $s=0, s=0$



impulse responce:

output when input is impulse: $\delta(t)$

$Y(s) = T(s)U(s)$ $U(s) = \mathcal{L}\{u(t)\} = \mathcal{L}\{\delta(t)\} = 1$

$Y(s) = T(s)$

$\therefore y(t) = h(t)$ where $\mathcal{L}\{h(t)\} = T(s)$

Step responce

input is $1(t)$

$Y(s) = T(s)\,U(s)$ $U(s) = \frac{1}{s}$

$Y(s) = T(s) \cdot \frac{1}{s}$

$y(t) = \mathcal{L}^{-1}\{T(s) \cdot \frac{1}{s}\}$

Ways to describe an LTI system:

relationship between input and output:

1. ODE — time domain
2. T(s) — s (frequency) domain

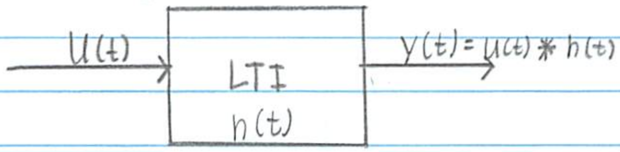
} physics

3. h(t) — impulse responce
4. step responce

} experimental

5. bode plots

Why convolutions again?



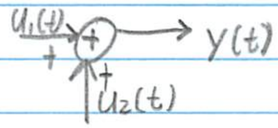
Block diagram of dynamical system

directed lime: signal flow: ⟶

Box: LTI system: ☐

Static summation system:

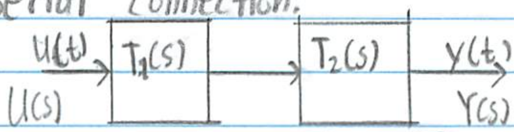
if $y = u_1 + u_2$ then



Connections:

connection:
1. Serial
2. parallel
3. feedback

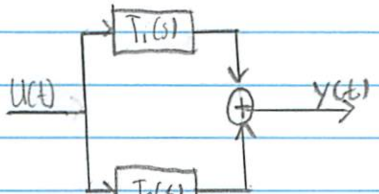
serial connection:



$Y(s) \cdot T_2(s) \cdot (T_1(s) \cdot U(s))$

$T(s) = T_2(s) \cdot T_1(s)$

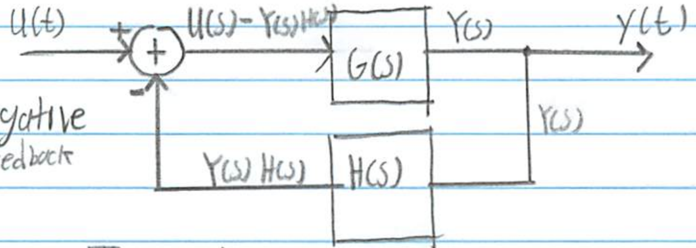
parrellel connection:



$Y(s) = U(s)(T_1(s) + T_2(s))$

$T(s) = T_1(s) + T_2(s)$

closed loop feedback:



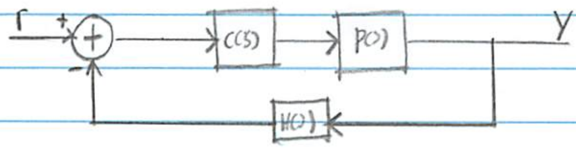
$$Y(s) = G(s)\left(U(s) - H(s)Y(s)\right)$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)} = T(s)$$

positive feedback

$$T(s) = \frac{G(s)}{1 - G(s)H(s)}$$

Automatic control



		cruise control
$P(s)$: plant - system to be controlled		car speed
$y(t)$: physical varible to be controlled		speed
$r(t)$: command		set speed
$H(s)$: sensor data	ideal sensor → 1	1
$C(s)$: controller		we design this

State space model → modern control

plant is not LTI => nonlinear control

Homework 4

1. $2\ddot{y}(t)+5\dot{y}(t)+3y(t)=4\dot{u}(t)+6u(t)$

where $u(t)=2\sin(3t)$, $y(0)=1$, $\dot{y}(0)=-1$

$\mathcal{L}\{L.H.S.\}=2(s^2Y(s)-sy(0)-y'(0))+5(sY(s)-y(0))+3Y(s)$

$=2s^2Y(s)-2s+2+5sY(s)-5+3Y(s)$

$=Y(s)(2s^2+5s+3)-2s-3$

$\mathcal{L}\{R.H.S.\}=4(sU(s)-u(0))+6U(s)$

$u(0)=0 \qquad U(s)=\frac{2\cdot 3}{s^2+9}$

$\mathcal{L}\{R.H.S.\}=\frac{24s}{s^2+9}+\frac{36}{s^2+9}=\frac{50s}{s^2+9}$

$Y(s)=\left(\frac{24s+36}{s^2+9}+2s+3\right)\cdot\frac{1}{2s^2+5s+3}$

$*\ 2s^2+5s+3=(s+1)(2s+3)$

$*\ s^2+9=(s+3i)(s-3i)$

$=\frac{24s+36}{(s+3i)(s-3i)(s+1)(2s+3)}+\frac{2s+3}{(s+1)(2s+3)}$

$=\frac{12\cancel{(2s+3)}}{(s^2+9)(s+1)\cancel{(2s+3)}}+\frac{\cancel{2s+3}}{(s+1)\cancel{(2s+3)}}=\frac{12+(s^2+9)}{(s^2+9)(s+1)}$

$\boxed{Y(s)=\frac{s^2+21}{(s^2+9)(s+1)}}$

2. $2y^{(5)}(t)+5y^{(3)}(t)+2\ddot{y}(t)+3y(t)=2u^{(3)}(t)+4\dot{u}(t)+6u(t)$

$\mathcal{L}\{L.H.S.\}=2(s^5Y(s)-s^4y(0)-s^3\dot{y}(0)-s^2\ddot{y}(0)-sy^{(3)}(0)-y^{(4)}(0))$

$+5(s^3Y(s)-s^2y(0)-s\dot{y}(0)-\ddot{y}(0))$

$+2(s^2Y(s)-sy(0)-\dot{y}(0))$

$+3(Y(s))$

$Y(s)(2s^5+5s^3+2s^2+3)$ assuming I.C. = 0

$\mathcal{L}\{R.H.S.\}=2s^3U(s)+4sU(s)+6U(s)$

$Y(s)(2s^5+5s^3+2s^2+3)=U(s)(2s^3+4s+6)$

$\frac{Y(s)}{U(s)}=T(s)=\frac{2s^3+4s+6}{2s^5+5s^3+2s^2+3}$

order is 5, relative order is 2. The positive relative order means the system is proper

3. $8y^{(3)}(t) + 4\ddot{y}(t) + 6\dot{y}(t) + 3y(t) = 2u^{(3)}(t) + 3\ddot{u}(t) + 4\dot{u}(t) + 6u(t)$

$\mathcal{L}\{L.H.S.\} = 8s^3Y(s) + 4s^2Y(s) + 6sY(s) + 3Y(s)$

$\mathcal{L}\{R.H.S.\} = 2s^3U(s) + 3s^2U(s) + 4sU(s) + 6U(s)$

$$\frac{Y(s)}{U(s)} = T(s) = \frac{2s^3 + 3s^2 + 4s + 6}{8s^3 + 4s^2 + 6s + 3}$$

Order: 3 Relative order: 0 The system is proper, not strictly

4. $g(t) = 2te^{-3t}$, $h(t) = 3e^{-4t}\sin(5t)$

$$\mathcal{L}\{g(t) * h(t)\} = \boxed{\left(\frac{2}{(s+3)^2}\right)\left(\frac{3 \cdot 5}{(s+4)^2 + 25}\right)}$$

5. $Y(s) = T(s)\,U(s)$

when: $u(t) = \delta(t)$, $U(s) = 1$, $y(t) = e^{-4t}\cos(5t)$

$$\therefore Y(s) = \frac{(s+4)^2}{(s+4)^2 + 25} = T(s)$$

when: $u(t) = 2t\,e^{-3t}$, $U(s) = \frac{2}{(s+3)^2}$

$$Y(s) = T(s)\,U(s)$$

$$= \boxed{\frac{2(s+4)^2}{(s+4)^2(s+3)^2 + 25(s+3)^2}}$$

Inverse laplace transform:

In this class we will have $\mathcal{L}^{-1}$ of fractions of polynomials

1st order fraction general:

$$\mathcal{L}^{-1}\left\{\frac{b}{s+a}\right\} = be^{-at}$$

ex: $\mathcal{L}^{-1}\left\{\frac{2s+5}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+2)+1}{s+2}\right\}$

$\mathcal{L}^{-1}\left\{2+\frac{1}{s+2}\right\} = 2\delta(t) + e^{-2t}$

Second order fraction: $Y(s) = \dfrac{cs+d}{s^2+as+b}$

Find denominator roots: $s^2+as+b=0$

cases: $\begin{cases} \text{2 diff. real roots} \\ \text{2 same real roots} \\ \text{complex conjugate pair} \end{cases}$

case 1: roots P_1 and P_2 $\quad P_1 \neq P_2$

partial fraction: $Y(s) = \dfrac{cs+d}{(s-P_1)(s-P_2)} = \dfrac{e_1}{s-P_1} + \dfrac{e_2}{s-P_2}$

$(e_1+e_2)s + (-e_1P_2 - e_2P_2) = cs+d$

$\begin{cases} e_1+e_2 = c \\ -e_1P_1 - e_2P_2 = d \end{cases}$

$e_2 = \dfrac{P_2c+d}{P_2-P_1}$, $e_1 = c - e_2$

$\mathcal{L}^{-1}\{F(s)\} = e_1e^{P_1t} + e_2e^{P_2t}$

case 2: $P_1 = P_2$

$\dfrac{cs+d}{(s-P)^2} = \dfrac{X_1}{s-p} + \dfrac{X_2}{(s-p)^2}$

$X_1 = C \qquad X_2 = d+cP$

$\mathcal{L}^{-1}\{F(s)\} = X_1e^{Pt} + X_2te^{Pt}$

case 3: no real roots

$$p_{1,2} = \frac{-a \pm j\sqrt{4b-a^2}}{2} \qquad p_1 = \frac{a}{2} \quad p_2 = \sqrt{b - a^2/4}$$

$$\frac{cs+d}{s^2+as+b} = \frac{X_1(s+p_1)}{(s+p_1)^2+p_2^2} + \frac{X_2}{(s+p_1)^2+p_2^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = X_1 e^{-p_1 t}\cos(p_2 t) + \frac{X_2}{p_2} e^{-p_1 t}\sin(p_2 t)$$

Higher order fractions:

$$Y(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

set denominator $=0$ and solve for s

we should get n roots

$$n \begin{cases} 1. \text{ single real roots} \\ 2. \text{ plural real roots} \\ 3. \text{ combo} \end{cases}$$

$$s^n + a_{n-1}s^{n-1} + \cdots + a_0 = \prod_{i=1}^{n_1}(s-p_i) \cdot \prod_{j=1}^{n_2}(s-q_j)^{p_j} \cdot \prod_{k=1}^{n_3}(s^2+d_1 s+d_2)$$

ex: $\mathcal{L}^{-1}\{F(s)\}$ where $F(s) = \frac{3}{(s+1)(s-2)^2}$

$$\frac{3}{(s+1)(s-2)^2} = \frac{X_1(s-2)^2 + Y_{11}(s+1)(s-2) + Y_{12}(s+1)}{(s+1)(s-2)^2}$$

Homework 5

1. $2\ddot{y}(t) - 3\dot{y}(t) + 4y(t) = 5\dot{u}(t) - 6u(t)$

$\mathcal{L}\{L.H.S.\} = 2s^2 Y(s) - 3s Y(s) + 4Y(s)$

$\mathcal{L}\{R.H.S.\} = 5s U(s) - 6U(s)$

$$Y(s)(2s^2 - 3s + 4) = U(s)(5s - 6)$$

$$\frac{Y(s)}{U(s)} = T(s) = \frac{5s-6}{2s^2-3s+4}$$

$$Y(s) = T(s)U(s)$$

if $u(t) = \delta(t)$ then $U(s) = 1$

and $Y(s) = \dfrac{5s-6}{2s^2-3s+4}$

if $u(t) = 1(t)$ then $U(s) = \frac{1}{s}$

and $Y(s) = \dfrac{5s-6}{2s^3-3s^2+4s}$

2. $2y^{(5)}(t) + 5y^{(3)}(t) - 2\ddot{y}(t) - 3y(t) = 2u^{(3)}(t) + 4\dot{u}(t) - 6u(t)$

$\mathcal{L}$ $\mathcal{L}$

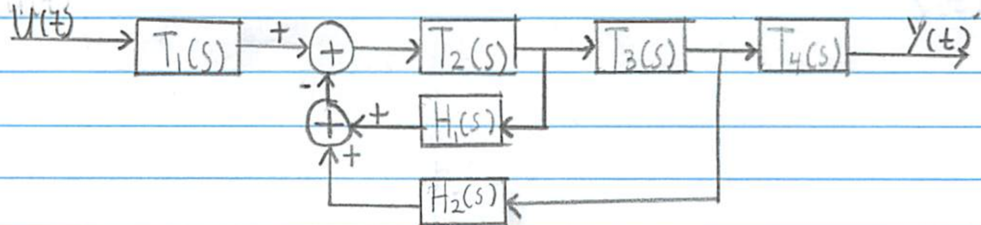
$$Y(s)(2s^5 + 5s^3 - 2s^2 - 3) = U(s)(2s^3 + 4s - 6)$$

$$T(s) = \frac{2s^3+4s-6}{2s^5+5s^3-2s^2-3}$$

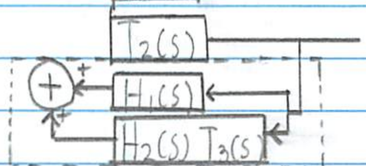
if $u(t) = \delta(t)$, $Y(s) = \dfrac{2s^3+4s-6}{2s^5+5s^3-2s^2-3}$

if $u(t) = 1(t)$, $Y(s) = \dfrac{2s^3+4s-6}{2s^6+5s^4-2s^3-3}$

3.

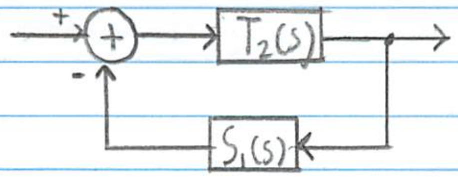


Redraw



$(H_1(s) + H_2(s)T_3(s)) = S_1(s)$

redraw



$$S_2(s) = \frac{T_2(s)}{1+T_2(s)\,S_1(s)}$$

$$u(t) \to \boxed{T_1(s)} \to \boxed{S_2(s)} \to \boxed{T_3(s)} \to \boxed{T_4(s)} \to y(t)$$

$$T(s) = T_1(s)\,S_2(s)\,T_3(s)\,T_4(s)$$

$$\boxed{T(s) = \frac{T_1(s)\,T_2(s)\,T_3(s)\,T_4(s)}{1+T_2(s)\left(H_1(s)+H_2(s)T_3(s)\right)}}$$

4. $T_1(s) = \frac{2}{s+3}$ $T_2(s) = \frac{1}{s-2}$ $T_3(s) = \frac{1}{s}$ $T_4(s) = 1$

$H_1(s) = \frac{s+1}{s+3}$ $H_2(s) = \frac{6}{s+1}$

$$T(s) = \frac{2\cdot 1\cdot 1\cdot 1}{(s+3)(s-2)(s)} \cdot \frac{1}{1+\left(\frac{1}{s-2}\right)\left(\frac{s+1}{s+3}+\frac{6}{s+1}\cdot\frac{1}{s}\right)}$$

$$= \frac{2}{(s+3)(s-2)(s)\left(1+\frac{1}{s-2}\left(\frac{s+1}{s+3}+\frac{6}{(s+1)s}\right)\right)}$$

$$\frac{s^3+2s^2+7s+18}{(s+3)(s-2)(s+1)(s)}+1 = \frac{s^4+3s^3-3s^2+s+18}{(s+3)(s-2)(s+1)(s)}$$

$$\boxed{T(s) = \frac{2(s+1)}{s^4+3s^3-3s^2+s+18}}$$

5. $\dot{y}(t)+4y(t) = 4u(t) \Rightarrow Y(s)(s+4) = U(s)\cdot 4$

$$\boxed{T(s) = \frac{4}{s+4}} \qquad Y(s) = T(s)\,U(s)$$

$u(t) = \delta(t) \Rightarrow Y(s) = \frac{4}{s+4}$ $\boxed{y(t) = 4e^{-4t}}$

$u(t) = 1(t)$, $U(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{4}{s^2+4s}$

complete square: $s^2+4s+\underbrace{4}_{\left(\frac{4}{2}\right)^2}-4 = (s+2)^2-4$

$$Y(s) = 2\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2-4}\right\} \Rightarrow \boxed{y(t) = 2e^{-2t}\sinh(2t)}$$

Homework 6

1. $F(s) = \dfrac{3}{s-2}$ $\boxed{\mathcal{L}^{-1}\{F(s)\} = 3e^{2t}}$

2. $F(s) = \dfrac{4s+2}{2s+3} = 2\dfrac{(2s+1)}{2s+3} = 2\cdot\dfrac{(2s+\overset{3}{1+2}-2)}{2s+3}$

$= 2\cdot\dfrac{(2s+3)}{2s+3} - \dfrac{2\cdot(2)}{2(s+\frac{3}{2})} = 2 - \dfrac{2}{s+3/2}$

$\boxed{\mathcal{L}^{-1}\{F(s)\} = 2\delta(t) - 2e^{-3t/2}}$

3. $F(s) = \dfrac{3}{s^2-3s+2} = \dfrac{3}{(s-\frac{3}{2})^2+2-(\frac{3}{2})^2} = \dfrac{3}{(s-\frac{3}{2})^2+\frac{8}{4}-\frac{9}{4}} = \dfrac{3}{(s-\frac{3}{2})^2-\frac{1}{4}}$

$(\frac{b}{2})^2 = (\frac{-3}{2})^2$

$= \dfrac{6(\frac{1}{2})}{(s-\frac{3}{2})^2-(\frac{1}{2})^2}$ $\boxed{\mathcal{L}^{-1}\{F(s)\} = 6e^{3t/2}\sinh(t/2)}$

4. $F(s) = \dfrac{3s+5}{s^2+6s+9} = \dfrac{3s+5}{(s+3)(s+3)} = \dfrac{A}{s+3} + \dfrac{B}{(s+3)^2}$

$3s+5 = As+3A+B$ $\begin{cases} 3 = A \\ 5 = 3A+B \end{cases}$

$A=3,\ B=-4$

$F(s) = \dfrac{3}{s+3} - \dfrac{4}{(s+3)^2}$ $\boxed{\mathcal{L}^{-1} = 3e^{-3t} - 4te^{-3t}}$

5. $F(s) = \dfrac{2s+7}{s^2+4s+20} = \dfrac{2s+7}{(s+2)^2+20-4} = \dfrac{2s+7}{(s+2)^2+16} = \dfrac{2(s+2+\frac{7}{2}-2)}{(s+2)^2+16}$

$(\frac{4}{2})^2$

$= \dfrac{2(s+2)}{(s+2)^2+16} + \dfrac{7-4}{(s+2)^2+16}$

$\boxed{\mathcal{L}^{-1}\{F(s)\} = 2e^{-2t}\cos(4\cdot t) + \frac{3}{4}e^{-2t}\sin(4\cdot t)}$

6. $F(s) = \dfrac{3s^2+2s+7}{(s-2)(s^2+4s+20)} = \dfrac{A}{s-2} + \dfrac{Bs+C}{s^2+4s+20}$

$3s^2+2s+7 = As^2+4As+20A+Bs^2+Cs-2Bs-2C$

$\begin{cases} 3 = A+B \\ 2 = 4A-2B+C \\ 7 = 20A-2C \end{cases} \quad \therefore \; A = 23/32,\ B = 73/32,\ C = 59/16$

$F(s) = \dfrac{23}{32}\cdot\dfrac{1}{s-2} + \dfrac{1}{32}\cdot\dfrac{73s+118}{(s+2)^2+16}$

$= \dfrac{23}{32}\cdot\dfrac{1}{s-2} + \dfrac{1}{32}\left(\dfrac{73s+146}{(s+2)^2+16} - \dfrac{28}{4}\cdot\dfrac{4}{(s+2)^2+16}\right)$

$\boxed{\mathcal{L}^{-1}\{F(s)\} = \dfrac{23e^{2t}}{32} + \dfrac{73}{32}e^{-2t}\cos(4t) - \dfrac{7}{32}e^{-2t}\sin(4t)}$

7. $F(s) = \dfrac{2s+7}{(s-2)(s^2+4s+4)} = \dfrac{2s+7}{(s-2)(s+2)^2} = \dfrac{A}{(s-2)} + \dfrac{B}{(s+2)} + \dfrac{C}{(s+2)^2}$

$2s+7 = As^2+4As+4A+B(s-2)(s+2)+Cs-2c$

$\begin{cases} 0 = A+B \\ 2 = 4A+C \\ 7 = 4A-4B-2c \end{cases} \quad \therefore \; A = 11/16,\ B = -11/16,\ C = -3/4$

$\boxed{\mathcal{L}^{-1}\{F(s)\} = \dfrac{11}{16}e^{2t} - \dfrac{11}{16}e^{-2t} - \dfrac{3}{4}te^{-2t}}$

8. $T(s) = \dfrac{4s+5}{(s+1)(s+2)} \qquad u(t) = 2e^{-3t} \;\therefore\; U(s) = \dfrac{2}{s+3}$

$Y(s) = T(s)U(s) = \dfrac{8s+10}{(s+1)(s+2)(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{s+3}$

$8s+10 = As^2+5As+6A+Bs^2+4Bs+3B+Cs^2+3Cs+2C$

$\begin{cases} 0 = A+B+C \\ 8 = 5A+4B+3C \\ 10 = 6A+3B+2C \end{cases} \quad \therefore \; A=1,\ B=6,\ C=-7$

$\boxed{\mathcal{L}^{-1}\{Y(s)\} = y(t) = e^{-t}+6e^{-2t}-7e^{-3t}}$

9. $\dot{y}(t) + 3y(t) = 2u(t), \quad y(0) = 1, \quad u(t) = 2\cos(3t)$

$$sY(s) + 3Y(s) = 2U(s) \qquad U(s) = \frac{2s}{s^2+9}$$

$$\frac{Y(s)}{U(s)} = T(s) = \frac{2}{s+3}$$

$$Y(s) = \frac{4s}{(s+3)(s-3)(s+3)} = \frac{A}{s-3} + \frac{B}{s+3} + \frac{C}{(s+3)^2}$$

$$4s = As^2 + 6As + 9A + Bs^2 - 9B + Cs - 3C$$

$$\begin{cases} 0 = A + B \\ 4 = 6A + C \\ 0 = 9A - 9B - 3C \end{cases} \qquad \therefore \; A = 1/3, \; B = -1/3, \; C = 2$$

$$\boxed{y(t) = \frac{e^{3t}}{3} - \frac{e^{-3t}}{3} + 2te^{-3t}}$$

Linear system theory in practical problems
step 1: Define inputs and outputs
step 2: Modeling (Find ODE or transfer function)
    Modeling:
        1. Force principles (physics)
        2. Data driven (provide input and measure output)
            system identification modeling
                check if it is an LTI
                    frequency domain (sin signals) } measure output
                    impulse response/step response }
        3. combination of approaches
step 3: Mathematics
    Given specific input
        you should be able to find response
    Properties of the system
        stability
        transient performance
        natural frequency

Mechanical systems:

mass-spring-damper system: 1D, Horizontal

spring constant: k

damper coefficent: c

mass: m

Output: dispacement $x(t)$ - define positive and 0

Input: force $f(t)$ - define positive

Components:

mass → Newtons law $a = F/m$

spring → $|f| = k \cdot |\Delta l|$

damper → $|f| = c \cdot |v|$

$m\ddot{x}(t) = \sum f(t)$

$m\ddot{x}(t) = \underset{\text{input force}}{f(t)} + f_k(t) + f_c(t)$ Define varible directions

$m\ddot{x}(t) = f(t) - kx(t) - c\dot{x}(t)$

$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$

example: hanging mass-spring-damper

$m\ddot{x}(t) = f(t) + f_c(t) + f_k(t) + mg$

$m\ddot{x}(t) = f(t) - c\dot{x}(t) - k\Delta l(t) + mg$

@ equilibrium $mg = k\Delta l_0$

$\Delta l_0 = \frac{mg}{k}$

$x - \overset{0}{x_0} = \Delta l - \Delta l_0$

$\Delta l = x + \Delta l_0$

$m\ddot{x}(t) = f(t) - c\dot{x}(t) - kx(t) - k\left(\frac{mg}{k}\right) + mg$

$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$

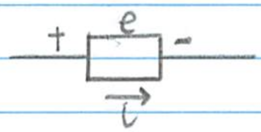
Spring connections:

series: $k = \frac{k_1 k_2}{k_1 + k_2}$

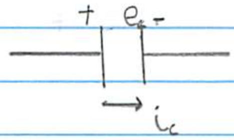
parallel: $k = k_1 + k_2$

Dynamic circuts:

recall: Voltage $= e$
Current $= i$



Capacitor: (C)



$q(t) = C \cdot e_c(t)$ ← Voltage$_{cap}$

↑ Charge ↑ capacitance

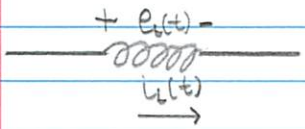
$\dot{q}(t) = i_c(t)$

$\therefore \; C\dot{e}_c(t) = i_c(t) \quad \Rightarrow e_c(t) = \frac{1}{C}\int_0^t i_c(\tau)\, d\tau$

Laplace: $CsE_c(s) - Ce_c(0) = I_c(s)$

aka: element description

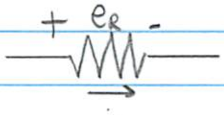
Inductor: (L)



$e_L(t) = L \cdot \dot{i}_L(t)$

$E_L(s) = LsI_L(s) - Li_L(0)$

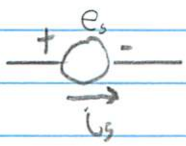
Resistor: (R)



$e_R(t) = R\, i_R(t)$

$E_R(s) = RI_R(s)$

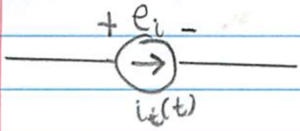
Voltage source: (S)



$e_s(t) = e_{in}(t)$ ← given/known

$E_s(t) = E_{in}(s)$

current source



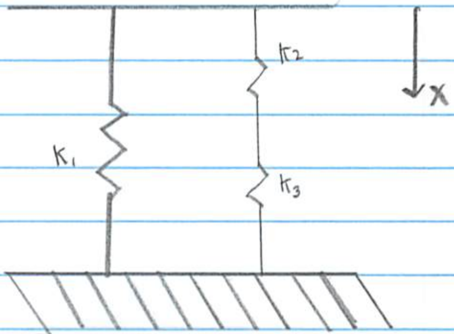
$i_i(t) = i_{in}(t)$ ← given/known

$I_i(s) = I_{in}(s)$

Homework 7

B-3-7:

Obtain the equivilent spring constant k_{eq} for the system shown.



2,3:

total extension: $\Delta l_{2,3} = \Delta l_2 + \Delta l_3$

$$F_{2,3} = -K_{2,3}(\Delta l_{2,3}) \qquad F_2 = F_3$$

$$\star\ F_{2,3} = F_2 = F_3\ \star \qquad -K_2 \Delta l_2 = -K_3 \Delta l_3$$

$$\Delta l_{2,3} = \frac{F_{2,3}}{-K_{2,3}} \qquad \Delta l_2 = \frac{F_2}{-K_2} \qquad \Delta l_3 = \frac{F_3}{-K_3}$$

$$\frac{1}{K_{2,3}} = \frac{1}{K_2} + \frac{1}{K_3} \Rightarrow K_{2,3} = \frac{K_2 K_3}{K_2 + K_3}$$

1, (2,3):

$$K_{1,(2,3)} = K_1 + K_{2,3}$$

$$\boxed{K_{eq} = K_1 + \frac{K_2 K_3}{K_2 + K_3}}$$

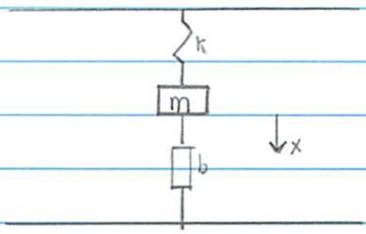
B-3-15

Derive a mathematical model for the system. Then find $x(t)$.

$m = 2\,kg \quad b = 4\,\frac{Ns}{m} \quad k = 20\,N/m$

$x(0) = 0.1\,m,\ \dot{x}(0) = 0$

$x = 0$ is equalibrium



$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t) \Rightarrow m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0$$

$$m(s^2X(s) - sx(0) - \dot{x}(0)) + b(sX(s) - x(0)) + kX(s) = 0$$

$$X(s)(ms^2 + bs + k) = msx(0) + m\dot{x}(0) + bx(0) = 2 \cdot 0.1\,s + 4 \cdot 0.1 = 0.2s + 0.4$$

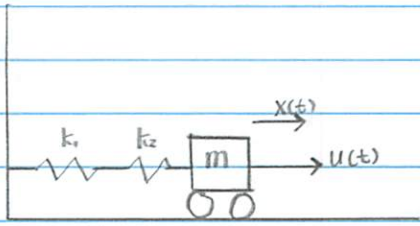
$$X(s) = \frac{0.2(s+0.2)}{2s^2+4s+20} = \frac{0.2}{2} \cdot \frac{s+0.2}{s^2+2s+10} = \frac{1}{10}\left(\frac{s+1}{(s+1)^2-9} - \frac{0.8}{(s+1)^2-9}\right)$$

$$\left(\frac{2}{2 \cdot 1}\right)^2$$

$$\boxed{x(t) = \frac{e^{-t}\cos(3t)}{10} + \frac{4e^{-t}\sin(3t)}{150}}$$

B-4-3

Looking at the following mechanical system, assume $u(t)$ is an input force, $x(t)$ is output. Find the transfer function.



$$m\ddot{x}(t) = -k_{eq}x(t) + u(t)$$

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

$$m\ddot{x}(t) + k_{eq}x(t) = u(t)$$

$$ms^2X(s) + msx(0) + m\dot{x}(0) + k_{eq}X(s) = U(s)$$

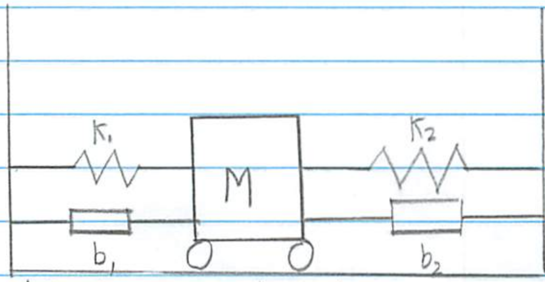
$$X(s)(ms^2 + k_{eq}) = U(s) - msx(0) - m\dot{x}(0)$$

$x(0) \leftarrow 0$, $\dot{x}(0) \leftarrow 0$ transfer has 0's I.C.

$$\frac{X(s)}{U(s)} = \boxed{T(s) = \frac{1}{ms^2 + \frac{k_1 k_2}{k_1 + k_2}}}$$

B-4-17

Displacement output $x(t)$ is measured from equalibrium. At $t=0$, $x(0) = x_0$, $\dot{x}(0) = v_0$. Obtain the output function if $m = 10\,kg$, $b_1 = 50\,N\cdot s/m$, $b_2 = 70\,N\cdot s/m$, $k_1 = 400\,N/m$, and $k_2 = 600\,N/m$



$k_{eq} = k_1 + k_2 = 1000$ $\quad b_{eq} = b_1 + b_2 = 120$

$$m\ddot{x}(t) = -k_{eq}x(t) - b_{eq}\dot{x}(t)$$

$$ms^2X(s) - msx(0) - m\dot{x}(0) + bsX(s) + bx(0) + kX(s) = 0$$

$$X(s)(ms^2 + bs + k) = msx_0 + mv_0 + bx_0$$

$$X(s) = \frac{10x_0s + 120x_0 + 10v_0}{10s^2 + 120s + 1000} = \frac{x_0(s+12) + 10v_0}{s^2 + 12s + 100} = \frac{x_0(s+6) + 10v_0 - 6x_0}{(s+6)^2 + 64}$$

$$\left(\frac{b}{2a}\right)^2 = (6)^2$$

$$= \frac{x_0(s+6)}{(s+6)^2 + 64} + \frac{(10v_0 - 6x_0)}{8} \cdot \frac{8}{(s+6)^2 + 8^2}$$

$$\boxed{x(t) = x_0e^{-6t}\cos(8t) + \frac{(10v_0 - 6x_0)}{8}e^{-6t}\sin(8t)}$$

Complex impedance:

Applies to Resistor, Capacitor, inductor

ohm's law

$$\frac{e_r(t)}{i_r(t)} = R$$

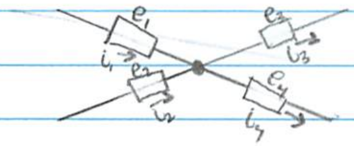
$$\frac{E_r(s)}{I_r(s)} = R \qquad \frac{E_c(s)}{I_c(s)} = \frac{1}{Cs} \qquad \frac{E_L(s)}{I_L(s)} = Ls$$

this assumes zero's I.C.

Kirchhoff's current law (node law)

a node (wire) cannot hold charge:

aka all current in must go out

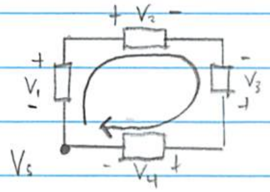


$i_1 + i_2 - i_3 - i_4 = 0$

One node in a loop is dependent

Kirchhoff's voltage law (loop law)

V and e are interchangeable



$$\cancel{V_s} + V_1 - V_2 + V_3 - V_4 = \cancel{V_s}$$

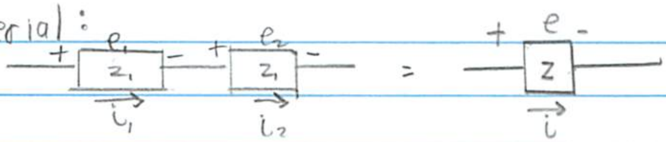
$$V_1 - V_2 + V_3 - V_4 = 0$$

Complex impedence approach to solution: (z)

1. zero's I.C.
2. s domain

resistor	cap.	inducter	general
$z = R$	$z = \frac{1}{Cs}$	$z = Ls$	$\frac{E(s)}{I(s)} = Z(s)$

serial:



$E(s) = E_1(s) + E_2(s)$

$Z = Z_1 + Z_2$

$I(s) = I_1(s) = I_2(s)$

parrellel: $Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$

$E(s) = E_1(s) = E_2(s) \qquad I(s) = I_1(s) + I_2(s)$

Filters:

Analog are most common:

these are LTI systems

- low pass
- High pass
- band pass
- notch

Low pass: Low frequency passes, high does not

allows $\sin(\omega t)$ when ω is small

High pass: only lets high frequency

Band pass: lets a certain range go through

Notch: doesn't let certain range go through

Frequency responce (steady state)

$$u(t) \longrightarrow \boxed{T(s)} \longrightarrow y(t)$$

$u(t) = \sin(\omega t)$

$\omega = 2\pi f$

send a frequency in

responce: $y(t)$ as $t \to \infty$

If $T(s)$ is stable:

if: $u(t) = \sin(\omega t)$ then: $y(t) = A \sin(\omega t + \varphi)$

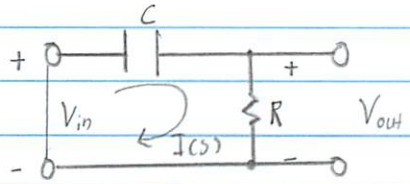
$T(s)$: $s = j\omega$

$$T(j\omega) = \underbrace{A(\omega)}_{A} \, e^{\overbrace{j\theta(\omega)}^{\varphi}}$$

if $A(\omega)$ is big, ω passes through

use this to determine filter

RC Filter example: find $T(s)$, assume 0's I.C.



$$V_{out} = I(s)R = \frac{V_{in}(s)}{R + \frac{1}{Cs}} \cdot R$$

$$T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R + \frac{1}{Cs}} = \frac{RCs}{1 + RCs}$$

$$|T(j\omega)| = \left|\frac{jRC\omega}{1 + jRC\omega}\right|$$

if $\omega \to \infty$, $T(j\omega) \to 1$

if $\omega \to 0$, $T(j\omega) \to 0$

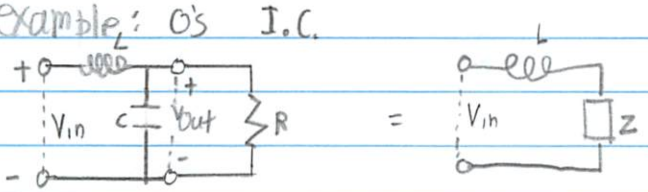
High f goes through so it is a high pass filter

1 - high pass = low pass

Bandwidth: $RC\omega = 1$

$$\omega = \frac{1}{RC} = \text{bandwidth}$$

example: 0's I.C.



$$Z = \frac{\frac{1}{Cs} \cdot R}{\frac{1}{Cs} + R} = \frac{R}{1 + RCs}$$

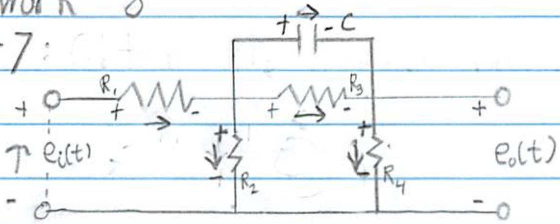
$$V_{out}(s) = I(s) \cdot Z = \frac{V_{in}(s)}{Ls + Z} \cdot Z$$

$$T(s) = \frac{Z}{Ls + Z} = \frac{R}{R + Ls(1 + RCs)} = \frac{R}{LRCs^2 + Ls + R}$$

Low pass

Homework 8

B-6-7:



Determine $T(s) : \dfrac{E_o(s)}{E_0(s)}$

$E_o(s) = E_{R4}(s)$

components: $E_{R1}(s) = R_1 I_{R1}(s)$ $\quad E_{R4}(s) = R_4 I_{R4}(s)$

$E_{R2}(s) = R_2 I_{R2}(s)$ $\quad Cs\,E_c(s) = I_c(s)$

$E_{R3}(s) = R_3 I_{R3}(s)$

Loop:

$E_{in}(s) - E_{R1}(s) - E_{R2}(s) = 0$ 1.

$E_{R2}(s) - E_{R3}(s) - E_{R4}(s) = 0$ 2.

$-E_c(s) + E_{R3}(s) = 0$ 3.

node:

$\cancel{I_{in}(s) - I_{R1}(s) = 0}$

$I_{R1}(s) - I_c(s) - I_{R3}(s) - I_{R2}(s) = 0$ 4.

$I_c(s) + I_{R3}(s) - I_{R4}(s) = 0$ 5.

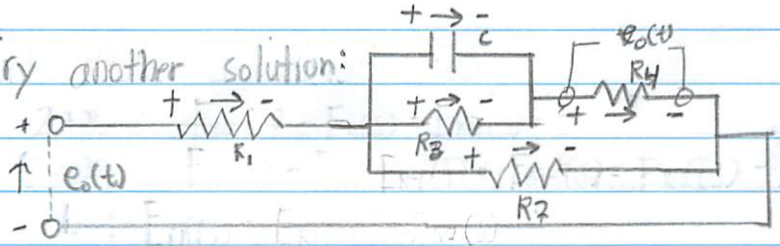
$\cancel{I_{in}(s) + I_{R2}(s) + I_{R3}(s) = 0}$

using 1 and 4: $I_{R1}(s) + I_{R2}(s) + I_{R3}(s) = 0$ 6.

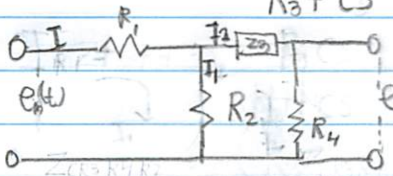
solve:

$$\begin{array}{c}
\begin{matrix} E_{in} & E_{R1} & I_{R1} & E_{R2} & I_{R2} & E_{R3} & I_{R3} & E_{R4} & I_{R4} & E_c & I_c & \end{matrix} \\
\left[\begin{array}{ccccccccccc|c}
0 & 1 & -R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -R_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -R_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Cs & -1 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}$$

Try another solution:



$$Z_{CR_3} = \frac{R_3CS}{R_3+CS} = Z_3$$



$$I = I_1 + I_2$$

$$R_2I_1 = (Z_3+R_4)\,I_2$$

Rearrange:

$$I = \frac{(Z_3+R_4)}{R_2}I_2 + I_2 = \left(\frac{Z_3+R_4}{R_2} + \frac{R_2}{R_2}\right)I_2 \quad \therefore\; I_2 = \left(\frac{R_2}{R_2+Z_3+R_4}\right)I$$

$$I = I_1 + \left(\frac{R_2}{Z_3+R_4}\right)I_1 = \left(\frac{Z_3+R_4}{Z_3+R_4} + \frac{R_2}{Z_3+R_4}\right)I_1 \quad \therefore\; I_1 = \left(\frac{Z_3+R_4}{R_2+Z_3+R_4}\right)I$$

total voltage:

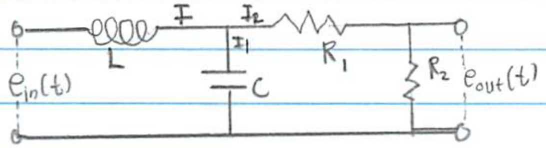
$$E_{in} = R_1I + R_2I_1 = \left(R_1 + R_2\left(\frac{Z_3+R_4}{R_2+Z_3+R_4}\right)\right)I = \frac{R_1R_2 + R_1Z_3 + R_1R_4 \cancel{+R_2Z_3+R_2R_4}}{R_2+Z_3+R_4}$$

$$E_{out} = R_4I_2 = \left(\frac{R_4R_2}{R_2+Z_3+R_4}\right)I$$

$$\boxed{\frac{E_{out}}{E_{in}} = \frac{R_2R_4}{R_1(R_2+Z_3+R_4)+R_2(Z_3+R_4)}}$$

$$= \frac{R_2R_4}{R_1\left(R_2+\left(\frac{R_3CS}{R_3+CS}\right)+R_4\right) + R_2\left(\left(\frac{R_3CS}{R_3+CS}\right)+R_4\right)}$$

B-6-8: Find $E_{out}(s)/E_{in}(s)$



$$I = I_1 + I_2$$

$$\frac{1}{CS} I_1 = R_1 I_2 + R_2 I_2 = (R_1 + R_2) I_2$$

Rearrange:

$$I = CS(R_1+R_2)I_2 + I_2 = I_2(CS(R_1+R_2)+1)$$

$$I_2 = \left(\frac{1}{CS(R_1+R_2)}\right) I$$

$$I = I_1 + \frac{1}{CS(R_1+R_2)} I_1 = \left(\frac{CS(R_1+R_2)+1}{CS(R_1+R_2)}\right) I$$

loop law

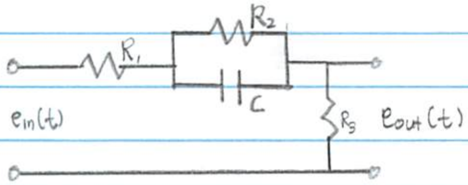
$$E_{in} = LsI + \frac{1}{Cs} I_1 = \left(\frac{Ls(C^2S^2(R_1+R_2))}{C^2S^2(R_1+R_2)} + \frac{Cs(R_1+R_2)+1}{C^2S^2(R_1+R_2)}\right) I$$

$$= \left(\frac{Lc^2(R_1+R_2)s^3 + C(R_1+R_2)s + 1}{c^2(R_1+R_2)s^2}\right) I$$

$$E_{out} = R_2 I_2 = \left(\frac{R_2 CS}{C^2(R_1+R_2)s^2}\right) I$$

$$\boxed{\frac{E_{out}}{E_{in}} = \frac{R_2 CS}{Lc^2(R_1+R_2)s^3 + C(R_1+R_2)s + 1}}$$

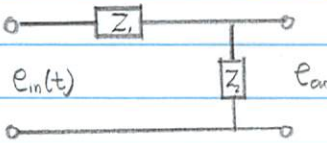
B-6-11: Find E_{out}/E_{in}



$$Z_1 = R_1 + \frac{R_2}{CS\left(R_2 + \frac{1}{CS}\right)}$$

$$= R_1 + \frac{R_2}{CR_2S+1} = \frac{CR_1R_2S + R_2 + R_1}{CR_2S+1}$$

$$Z_2 = R_3$$



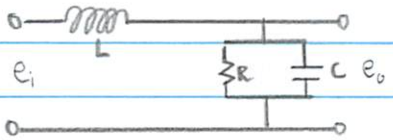
$$E_{in} = Z_1 I + Z_2 I$$

$$E_{out} = Z_2 I$$

$$\frac{E_{out}}{E_{in}} = \frac{Z_2}{Z_1+Z_2} = \frac{R_3}{\left(\frac{CR_1R_2S + R_2 + R_1}{CR_2S+1} + R_3\left(\frac{CR_2S+1}{CR_2S+1}\right)\right)}$$

$$\boxed{= \frac{CR_2R_3S + R_3}{CR_2(R_1+R_3)S + R_1 + R_2 + R_3}}$$

B-6-12: obtain E_{out}/E_{in}



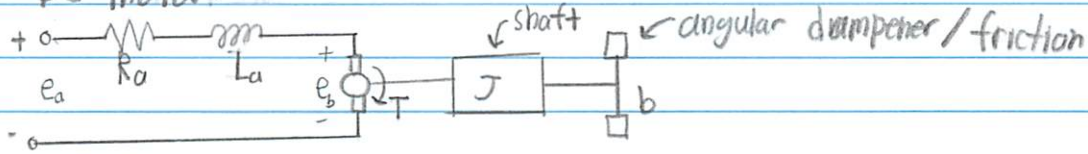
$$Z_1 = LS$$

$$Z_2 = \frac{R}{CS\left(R+\frac{1}{CS}\right)} = \frac{R}{RCS+1}$$

$$\frac{E_{out}}{E_{in}} = \frac{Z_2 I}{(Z_1+Z_2)I} = \frac{\frac{R}{RCS+1}}{\frac{LS(RCS+1)}{RCS+1} + \frac{R}{RCS+1}} = \boxed{\frac{R}{LRCS^2 + LS + R}}$$

Electro-mechanical systems:

DC motor:



Input: $e_a(t)$ voltage supplied

Output: $\theta(t)$ angular position of shaft

$T_M(s) = \dfrac{\Theta(s)}{E_a(s)}$ $T_M(s)$ is motor transfer function

$e_b(t)$: back emf caused by torque $T(t)$ not related to transfer function T_M

electrical equations:

$$E_a(s) - E_R(s) - E_L(s) - E_b(s) = 0$$

$$E_a(s) - E_b(s) = (R + Ls) I_a(s)$$

Mechanical equations

$$\Sigma T = J\ddot{\theta}(t) = T(t) - b\dot{\theta}(t)$$

$$T(t) = b\dot{\theta}(t) + J\ddot{\theta}(t) \Rightarrow T(s) = bs\Theta(s) + s^2 J\Theta(s)$$

combined system:

$$e_b(t) \propto \dot{\theta}(t) \Rightarrow e_b(t) = k_b\dot{\theta}(t) \Rightarrow E_b(s) = k_b s\Theta(s)$$

k_b is backemp coefficent

$$T(t) \propto i_a(t) \Rightarrow T(t) = k\, i_a(t) \Rightarrow T(s) = k I_a(s)$$

k is motor torque constant

solve:

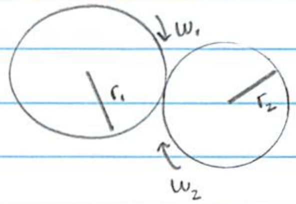
$$E_a(s) - k_b s\Theta(s) = (R + Ls) I_a(s)$$

$$Js^2\Theta(s) + bs\Theta(s) = k I_a(s)$$

$$E_a(s) - k_b s\Theta(s) = \frac{R + Ls}{k}\left(Js^2\Theta(s) + bs\Theta(s)\right)$$

$$T_M(s) = \frac{\Theta(s)}{E_a(s)} = \frac{k}{k_b s + (R + Ls)(Js^2 + bs)} = \boxed{\frac{k}{s(LJs^2 + (RJ + bL)s + bR + k_b)}}$$

Gear train:

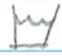


$\omega_1 r_1 = \omega_2 r_2$: no slip

$\frac{T_1}{r_1} = \frac{T_2}{r_2}$: Forces are equal

Thermal system: Dynamical system

Room : T_r

Hot water : $T_r + T_b(t)$ 

diff in temp

thermometer : $T_r + T_t(t)$

Input : $T_b(t)$ Output : $T_t(t)$

initially : thermometer = room temp $\therefore T_t(0) = 0$

$q(t)$: heat tranfer from enviorment (tub) to thermometer

$q(t) > 0$ means $T_t(t) \uparrow$

$q(t) \cdot dt = C\, dT_t(t)$

heat capacitance

$q(t) = C \cdot \dot{T}_t(t)$

also $q(t) = \frac{1}{R}(T_b(t) - T_t(t))$

thermal resistance

combind equations

$C \dot{T}_t(t) = (T_b(t) - T_t(t))/R$

Transfer function: $\frac{T_t(t)}{T_b(t)} = T(s) = \frac{1}{1+RCs}$

State space model:

- An ODE
- Describe LTI and non LTI
    - Standard form for LTI

$$\begin{cases} \dot{x}(t) = A x(t) + B u(t) \\ y(t) = C x(t) + D u(t) \end{cases}$$

A, B, C, D are matrix coefficents

input $u(t)$, output $y(t)$

$x(t)$: State vector/signals "States"

$$\dot{X}(t)_{n\times 1} = A_{n\times n} X(t)_{n\times 1} + B_{n\times 1} U(t)_{1\times 1}$$

$$y(t)_{1\times 1} = C_{1\times n} X(t)_{n\times 1} + D_{1\times 1} U(t)_{1\times 1}$$

How to obtain state space model from ODE

consider only $u(t)$, not $u^{(n)}(t)$

general:

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \ldots + a_0 y(t) = b_0 u(t)$$

step 1: define $x(t)$ with $n\times 1$ dimensions

$$X(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{bmatrix}$$

step 2: $\dot{X}(t)$

$$\dot{x}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \\ \vdots \\ y^{(n)}(t) \end{bmatrix} = \begin{bmatrix} X_2(t) \\ X_3(t) \\ \vdots \\ -a_0 X(t) - a_1 X_2(t) \ldots - a_{n-1} X_n(t) + b_0 u(t) \end{bmatrix}$$

in order to get $\dot{X}(t)$ from $X(t)$ and $u(t)$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \cdots & & \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & \ldots & & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix}$$

how to get $y(t)$ from $x(t)$ and $u(t)$
$= x_1(t)$

$C = [1\ 0\ 0\ 0 \ldots 0]$ $D = [0]$ for single inputs/outputs

Transfer function from state space model:
Take $\mathcal{L}$ of both equation and both sides

$$\mathcal{L}\{\dot{X}(t)\} = \mathcal{L}\{AX(t) + BU(t)\}$$
$$sX(s) - X(0) = AX(s) + BU(s)$$
$$sX(s) - A_{n\times n}X(s) = BU(s) + X(0)$$
$$(sI_{n\times n} - A_{n\times n})X(s) = BU(s) + X(0)$$
$$I_{n\times n}X(s) = (sI_{n\times n} - A_{n\times n})^{-1}(BU(s) + X(0))$$
$$X(s) = (sI - A)^{-1}(BU(s) + X(0))$$

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{Cx(t) + Du(t)\}$$
$$Y(s) = CX(s) + DU(s)$$

$$Y(s) = C(sI-A)^{-1}(BU(s) + X(0)) + DU(s)$$
$$Y(s) = (C_{1\times n}(sI-A)^{-1}_{n\times n}B_{n\times 1} + D_{1\times 1})U(s)_{1\times 1} + C(sI-A)^{-1}X(0)_{n\times 1}$$
$$Y(s) = (C(sI-A)^{-1}B + D)U(s) + C(sI-A)X(0)$$

Transfer function $X(0) = 0$

$$T(s) = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B$$

example: $\ddot{y}(t) + 2\dot{y}(t) - 3y(t) = 4u(t)$

$$X(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

$$\ddot{y}(t) = -2\dot{y}(t) + 3y(t) + 4u(t)$$
$$\dot{X}_2(t) = 3X_1(t) - 2X_2(t) + 4u(t)$$

cuz there is no X_3 so $X_3 = \dot{X}_2 = \ddot{y}$

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$C = [1\ \ 0] \quad D = [0]$$

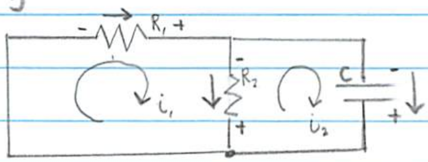
$$T(s) = C(sI-A)^{-1}B = [1\ 0]\begin{bmatrix} s & -1 \\ -3 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$= [1\ \ 0]\left(\frac{1}{s(s+2)-(-3)(-1)}\right)\begin{bmatrix} s+2 & 1 \\ 3 & s \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$= \frac{4}{s^2+2s-3}$$

Nonlinear state space model:

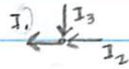
$$\dot{X}(t) = \begin{bmatrix} f_1(x(t), u(t)) \\ \vdots \\ f_n(x(t), u(t)) \end{bmatrix} \qquad y(t) = h(x(t), u(t))$$

Homework # 9

B-6-5



Knowing $e_0 = \frac{q(0)}{C}$ on the capacitor, calculate i_1 and i_2



$I_1 = I_2 + I_3$, $I_3 = I_1 - I_2$

$$E_{R1} + E_{R2} = 0 \qquad E_c - E_{R2} = 0$$

$$R_1 I_1 + R_2 I_3 = 0 \qquad \frac{I_2 + c\, e_c(0)}{CS} - R_2 I_3 = 0$$

$$R_1 I_1 + R_2 I_1 - R_2 I_2 = 0 \qquad \frac{I_2}{CS} + \frac{q(0)}{CS} - R_2 I_1 + R_2 I_2 = 0$$

$$(R_1 + R_2) I_1 - R_2 I_2 = 0 \qquad -R_2 I_1 + \left(R_2 + \frac{1}{CS}\right) I_2 = -\frac{q(0)}{CS}$$

$$I_1 = \left(\frac{CSR_2 + 1}{CSR_2}\right) I_2 + \frac{q(0)}{CSR_2}$$

$$(R_1 + R_2)\left(\frac{CSR_2 + 1}{CSR_2}\right) I_2 + \frac{q(0)(R_1 + R_2)}{CSR_2} - \frac{CSR_2^2 I_2}{CSR_2} = 0$$

$$\left((R_1 + R_2)(CSR_2 + 1) - CSR_2^2\right) I_2 = -q(0)(R_1 + R_2)$$

$$I_2 = \frac{-q(0)(R_1 + R_2)}{-CR_2^2 s + (R_1 + R_2) C R_2 s + (R_1 + R_2)}$$

$$I_2 = \frac{-q(0)(R_1 + R_2)}{(CR_1R_2 + \cancel{CR_2^2} - \cancel{CR_2^2}) s + (R_1 + R_2)} = -\frac{q(0)(R_1 + R_2)}{CR_1R_2} \cdot \frac{1}{s + \left(\frac{R_1 + R_2}{CR_1R_2}\right)}$$

$$\mathcal{L}^{-1}\{I_2\} = \boxed{i_2 = -\frac{q(0)(R_1 + R_2)}{CR_1R_2} \exp\left(\frac{-(R_1 + R_2)t}{CR_1R_2}\right)}$$

$$I_2 = \left(\frac{R_1+R_2}{R_2}\right) I_1$$

$$-R_2 I_1 + \left(R_2 + \frac{1}{CS}\right)\left(\frac{R_1+R_2}{R_2}\right) I_1 = \frac{-q(0)}{CS}$$

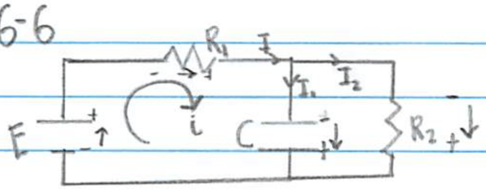
$$\left(\cancel{-R_2} + (R_1 + \cancel{R_2}) + \frac{R_1+R_2}{CSR_2}\right) I_1 = \frac{-q(0)}{CS}$$

$$\frac{CSR_1R_2 + R_1 + R_2}{\cancel{C}SR_2} I_1 = \frac{-q(0)}{\cancel{C}S}$$

$$I_1 = \frac{-q(0) R_2}{CSR_1R_2 + (R_1+R_2)} = \frac{-q(0)}{CR_1} \cdot \frac{1}{S + \frac{R_1+R_2}{CR_1R_2}}$$

$$\mathcal{L}^{-1}\left\{I_1\right\} = \boxed{\frac{-q(0)}{CR_1} \exp\left(\frac{-(R_1+R_2)t}{CR_1R_2}\right) = i_1}$$

B-6-6



Find i for $V_c(0)$ as cap voltage

$I = I_1 + I_2 \qquad I_2 = I - I_1$

$$E + E_{R1} + E_c = 0 \qquad E_c = E_{R2}$$

$$E + R_1 I + \frac{I_1 + cV_c(0)}{CS} = 0 \qquad \frac{I_1 + cV_c(0)}{CS} = I_2 R_2$$

$$\frac{I_1 + CV_c(0)}{CS} = IR_2 - I_1 R_2$$

$$E + R_1 I + R_2 I - R_2 I_1 = 0 \qquad \frac{I_1}{CS} + \frac{V_c(0)}{S} + R_2 I_1 - R_2 I = 0$$

$$-R_2 I_1 + (R_1+R_2) I = -E \qquad \left(R_2 + \frac{1}{CS}\right) I_1 - R_2 I = -\frac{V_c(0)}{S}$$

$$I_1 = \frac{E}{R_2} + \frac{(R_1+R_2) I}{R_2} \qquad \left(\frac{R_2CS+1}{R_2CS}\right) I_1 - I = -\frac{V_c(0)}{R_2 S}$$

$$\left(\frac{R_2CS+1}{R_2CS}\right)\left(\frac{E}{R_2}\right) + \left(\frac{R_2CS+1}{R_2CS}\right)\left(\frac{R_1+R_2}{R_2}\right) I - I = \frac{-V_c(0)}{R_2 S}$$

$$\frac{(R_2Cs+1)E}{R_2^2Cs} + \frac{(R_2Cs+1)(R_1+R_2)I}{R_2^2Cs} - \frac{R_2^2Cs}{R_2^2Cs}I = \frac{-R_2C\,V_c(0)}{R_2^2Cs}$$

$$I\left((R_2Cs+1)(R_1+R_2) - R_2^2Cs\right) = -(R_2Cs+1) - R_2C\,V_c(0)$$

$$I = \frac{-R_2Cs - 1 - R_2C\,V_c(0)}{-R_2^2Cs + R_1R_2Cs + (R_1+R_2)}$$

or

$$\left(R_2 + \frac{1}{Cs}\right)\left(\frac{E}{R_2} + \frac{(R_1+R_2)I}{R_2}\right) - R_2I = \frac{-V_c(0)}{R_2s}$$

$$E + (R_1+R_2)I + \frac{E}{R_2Cs} + \frac{(R_1-R_2)I}{R_2Cs} - R_2I = \frac{-V_c(0)}{R_2s}$$

$$R_2CsE + R_2Cs(R_1+R_2)I + E + (R_1+R_2)I - R_2^2CsI = -cV_c(0)$$

$$I\left(R_2Cs(R_1+R_2) + (R_1+R_2) - R_2^2Cs\right) = -cV_c(0) - E - ER_2Cs$$

$$I = \frac{-cV_c(0) - E - ER_2Cs}{(R_2C(R_1+R_2) - R_2^2C)s + (R_1+R_2)}$$

$$\boxed{\cancel{I = \frac{-cV_c(0) - E - ER_2Cs}{R_1R_2Cs + (R_1+R_2)}}}$$

or

$$E + E_{R_1} + E_c = 0 \qquad E + E_{R_1} + E_{R_2} = 0$$

$$E + R_1I + \frac{I + cV_c(0)}{cs} = 0 \qquad E + R_1I + R_2I = 0$$

$$R_1I = -E - R_2I$$

$$E - E - R_2I + \frac{I + cV_c(0)}{cs} = 0$$

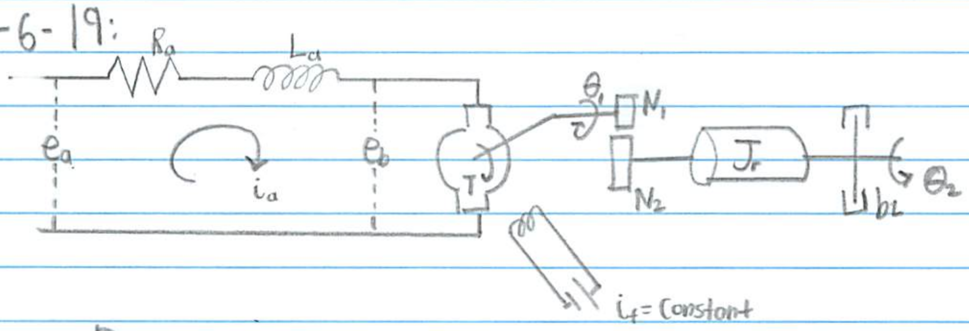
$$I - R_2CsI + cV_c(0) = 0$$

$$I(1 - R_2Cs) = -cV_c(0)$$

$$I = \frac{-cV_c(0)}{-R_2Cs + 1} = \frac{-CV_c(0)}{-R_2C} \cdot \frac{1}{s - \frac{1}{R_2C}}$$

$$\boxed{i = \frac{V_c(0)}{R_2}\exp\left(\frac{t}{R_2C}\right)}$$

B-6-19:



$R_a = 0.2\,\Omega \qquad L_a \approx 0$

$e_b = K_b\dot{\theta} \qquad K_b = 5.5\cdot10^{-2}\ V\cdot s/rad$

$T(t) = K\,i_a(t) \qquad K = 6\cdot10^{-5}\ lb_f\cdot ft/A$

$J_r = 1\cdot10^{-5}\,lbf\cdot ft\cdot s^2$

$J_L = 4.4\cdot10^{-3}\ lbf\cdot ft\cdot s^2$

$b_L = 4\cdot10^{-2}\ lbf\cdot ft/rad/s$

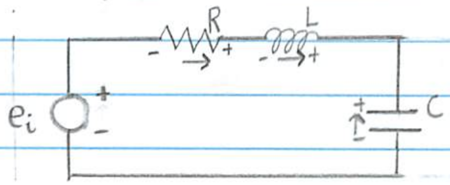
$n = N_1/N_2 = 0.1$

Obtain $\dfrac{\Theta(s)}{E_a(s)}$ $\qquad \dfrac{N_1}{N_2} = \dfrac{T_1}{T_2}$

$$T_M(s) = \frac{-6\cdot10^{-5}}{s(0.2\cdot(4.4\cdot10^{-3}\cdot0.1\cdot10^{-5})s + 4\cdot10^{-2}\cdot0.2 + 5.5\cdot10^{-2})}$$

$$= \boxed{\frac{-6\cdot10^{-5}}{8.8\cdot10^{-10}s^2 + 0.063s}}$$

4.



Find $e_c(t)$.

$E_i + E_R + E_L - E_c = 0 \qquad I = csE_c - ce_c(0)$

$E_i + RI + LsI - Li_L(0) - E_c = 0$

$E_i + RcsE_c - Rce_c(0) + Lcs^2E_c - Lcse_c(0) - Li_L(0) - E_c = 0$

$E_c(Lcs^2 + Rcs - 1) = (Lcs + Rc)e_c(0) + Li_L(0) - E_i$

$$\boxed{E_c = \frac{-(Lcs + Rc)e_c(0)}{(Lcs+Rc)s - 1} + \frac{Li_L(0) - E_i}{Lcs^2 + Rcs - 1}}$$

Stability:

Property of an LTI system itself

not dependent on input/I.C's

Stability means bounded input / bounded output

BIBO

for any bounded input, the response is bounded too

not stable $\begin{cases} \text{unstable : unbounded } y(t) \\ \text{marginally stable : sometime bounded, sometimes unbounded} \end{cases}$

How to check stability:

Depends on the poles of $T(s)$

$\hookrightarrow$ roots of denominator of $T(s)$

RHP: Right Half Plane

LHP: Left Half Plane

if pole in RHP, it is unstable

if pole in imaginary axis, it's marginally stable

else: stable

if in left plane, rougly form $T(s) = \frac{1}{s+a}$

which $\mathcal{L}^{-1} = e^{-at}$ which is bounded

Routh stability criterion

- it checks stability of LTI systems
- if ya don't wanna check for poles
- can give algebreic solution
  - works for undecided coefficents
- only checks stability, doesn't give poles

$$T(s) = \frac{N(s)}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n+1}}$$

- make $a_0 > 0$
- check if all coefficents are > 0
  - if not, $T(s)$ is not stable
- Construct table (first 2 rows)

s^n	a_0	a_2	$\cdots$	a_{n+1}
s^{n-1}	a_1	a_3	$\cdots$	
$\vdots$				
s^0				

- constructing more rows
  -

i	d_{i1}	$\dots$	$d_{i(j+1)}$
$i+1$	$d_{(i+1)1}$	$\dots$	$d_{(i+1)(j+1)}$
$i+2$		$d_{(i+2)i}$	

$$d_{(i+2)i} = \frac{d_{(i+1)1} \cdot d_{i(j+1)} - d_{i1} \cdot d_{(i+1)(j+1)}}{d_{(i+1)1}}$$

- emply elements $= 0$
- if all first coloum elements are > 0
  - stable
  - if all ≥ 0 => marginally stable

example

1. $T(s) = \dfrac{3s+6}{s^4+3s^3-2s^2+7s+6}$

not stable because -2

2. $T(s) = \dfrac{2s^2+3}{s^4+2s^3+3s^2+4s+5}$

s^4	1	3	5
s^3	2	4	0
s^2	$\frac{2\cdot3-1\cdot4}{2}=1$	$\frac{2\cdot5-1\cdot0}{2}=5$	$\frac{2\cdot0-1\cdot0}{2}=0$
s^1	-6	0	
s^0	5		

unstable because of -6

2 unstable poles because 2 sign switch

First order stable systems

proper: $T(s) = \dfrac{a_1 s + a_2}{b_1 s + b_2} = \dfrac{\frac{a_1}{b_1}(b_1 s + b_2) + (a_2 - \frac{a_1 b_2}{b_1})}{b_1 s + b_2}$

$$= \frac{a_1}{b_1} + \frac{a_2 - \frac{a_1 b_2}{b_1}}{b_2} \cdot \frac{\frac{b_2}{b_1}}{s + b_2/b_1}$$

stabdard form: $T(s) = \dfrac{\tau}{s+\tau}$ or $\dfrac{1}{1+Ts}$ where $T = \dfrac{1}{\tau}$

time constant

Second order stable systems:

$T(s) = \dfrac{a_1 s + b_0}{s^2 + b_1 s + b_0} = \dfrac{a_1 s + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$\xi = \dfrac{b_1}{2\sqrt{b_0}}$ $\quad \omega_n = \sqrt{b_0}$

Damping ratio:

ξ: damping ratio

ω_n: undamped natural frequency

Damped natural frequency

$\omega_d = \sqrt{1 - \xi^2} \cdot \omega_n$

if $\xi > 1$: overdamped

$\xi = 1$: critically damped

$0 < \xi < 1$: underdamped

$\xi = 0$: no damping

$\xi < 0$: unstable

Performance of an LTI

- stability
- steady state error / transient performance

$t \to \infty$ ↳ t is small

check step response

Homework 10

1. consider the transfer function: $\frac{Y(s)}{U(s)} = \frac{6}{s^2+3s+4}$

$\dot{x}(t) = Ax(t) + Bu(t)$

$y(t) = Cx(t) + Du(t)$

$s^2 Y(s) + 3s\,Y(s) + 4\,Y(s) = 6\,U(s)$

$y''(t) + 3y'(t) + 4y(t) = 6u(t)$

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$ $\quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \dot{x}_2(t) \end{bmatrix}$

$\ddot{y}(t) = -3\dot{y}(t) - 4y(t) + 6u(t)$

$= -3x_2(t) - 4x_1(t) + 6u(t)$

a. $$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]\,u(t) \end{aligned}}$$

From matlab

b. $$\boxed{\begin{aligned} A &= \begin{bmatrix} -3 & -4 \\ 1 & 0 \end{bmatrix} \quad & B &= \begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ C &= \begin{bmatrix} 0 & 6 \end{bmatrix} \quad & D &= [0] \end{aligned}}$$

c. The two models are not the same

d. $T(s) = C(sI - A)^{-1}B$

part a. $sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} s & -1 \\ -3 & s-4 \end{bmatrix}$

$(sI - A)^{-1} = \frac{1}{s(s-4) - (-1)(-3)} \begin{bmatrix} s-4 & 1 \\ 3 & s \end{bmatrix}$

$$\begin{array}{ccc} & 1\times 2 & 2\times 2 & \\ C(sI-A)^{-1}B = \frac{1}{s^2+3s+4} & \begin{bmatrix} 1 & 0 \end{bmatrix} & \begin{bmatrix} s-4 & 1 \\ 3 & 5 \end{bmatrix} & \begin{bmatrix} 0 \\ 6 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} s-4+0 & 1+0 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$\frac{1}{s^2+3s+4}\begin{bmatrix} 6 \end{bmatrix}$$

$$\boxed{T(s) = \frac{6}{s^2+3s+4}}$$

part b

$$(sI-A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & 4 \\ -1 & s \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s^2+3s+4}\begin{bmatrix} s & -4 \\ 1 & s+3 \end{bmatrix}$$

$$T(s) = C(sI-A)^{-1}B$$

$$= \frac{1}{s^2+3s+4}\begin{bmatrix} 0 & 6 \end{bmatrix}\begin{bmatrix} s & -4 \\ 1 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 6s+18 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 \end{bmatrix}$$

$$\boxed{T(s) = \frac{6}{s^2+3s+4}}$$

2. $\frac{Y(s)}{U(s)} = \frac{6}{s^3+2s^2+4}$

$$s^3Y(s) + 2s^2Y(s) + 4Y(s) = 6U(s)$$

$$y^{(3)}(t) + 2\ddot{y}(t) + 4y(t) = 6u(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} \qquad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \\ y^{(3)}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \dot{x}_3(t) \end{bmatrix}$$

$$y^{(3)}(t) = -2\ddot{y}(t) - 4y(t) + 6u(t)$$

$$\dot{x}_3(t) = -2x_3(t) - 4x_1(t) + 6u(t)$$

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]\, u(t) \end{aligned}}$$

matlab

$$A = \begin{bmatrix} -2 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 6 \end{bmatrix} \qquad D = [0]$$

These state space models are not the same part a:

checked using ss2tf ✓

part b:

checked using ss2tf ✓

3. What are the damping ratio and undamped natural frequency of $\frac{Y(s)}{U(s)} = \frac{4}{s^2+3s+4}$

$\zeta = \frac{3}{2\sqrt{4}} = \frac{3}{4}$ $\qquad \omega_n = \sqrt{4} = 2$

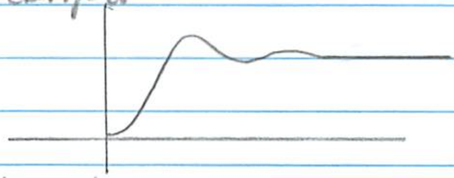
the system is underdamped

4. consider $\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$. Using matlab simulink

plot step response

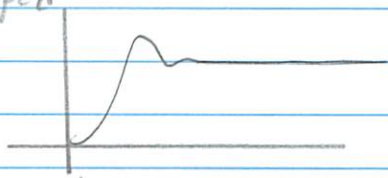
a. $\zeta = 0.5$, $\omega_n = 1$: underdamped

$\frac{1}{s^2+s+1}$



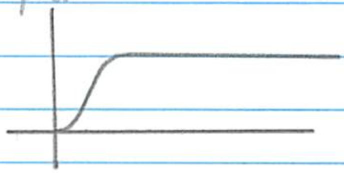
b. $\zeta = 0.5$, $\omega_n = 2$: underdamped

$\frac{4}{s^2+2s+4}$



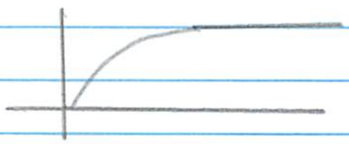
c. $\zeta = 1$, $\omega_n = 1$: critically damped

$\frac{1}{s^2+2s+1}$



d. $\zeta = 2$, $\omega_n = 1$: over damped

$\frac{1}{s^2+4s+1}$



5. check stability of $\frac{Y(s)}{U(s)} = \frac{s^3+3s^2+4s+2}{s^4+5s^3+3s^2+4s+6}$

s^4	1	3	6
s^3	5	4	0
s^2	$11/5$	6	0
s^1	$-106/11$	0	0
s^0	6	0	0

unstable
2 RHP poles

mat lab:

$-4.5+0i$ $\quad$ $0.28+1.1i$

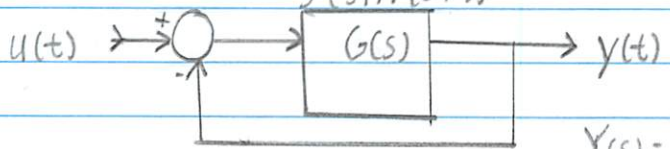
$-1.1+0i$ $\quad$ $0.28-1.1i$

6. B-10-14

Determine the range of k for a stable unity feedback control system with

$$G(s) = \frac{K}{s(s+1)(s+5)}$$

s^2+6s+5



$Y(s) = G(s)\,(U(s) - Y(s))$

$Y(s) = G(s)\,U(s) - Y(s)\,G(s)$

$Y(s)\,(1+G(s)) = G(s)\,U(s)$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{\frac{k}{s^3+6s^2+5s}}{1+\frac{k}{s^3+6s^2+5s}} = \frac{\frac{k}{s^3+6s^2+5s}}{\frac{s^3+6s^2+5s+k}{s^3+6s^2+5s}} = \frac{K}{s^3+6s^2+5s+K}$$

s^3	1	5
s^2	6	K
s^1	$\frac{30-k}{6}$	0
s^0	K	

$k>0$ and $\frac{30-k}{6} > 0$

$$\boxed{0 < k < 30}$$

Performance of LTI systems

- stability
  - For an LTI's Transfer function
    - solve for poles
    - Routh stability

if it is stable then look at it's performonce

- Time domain
  - impulse/step responce
    - steady state error

$$\lim_{t\to\infty} (u(t)-y(t))$$ use Final value theorem

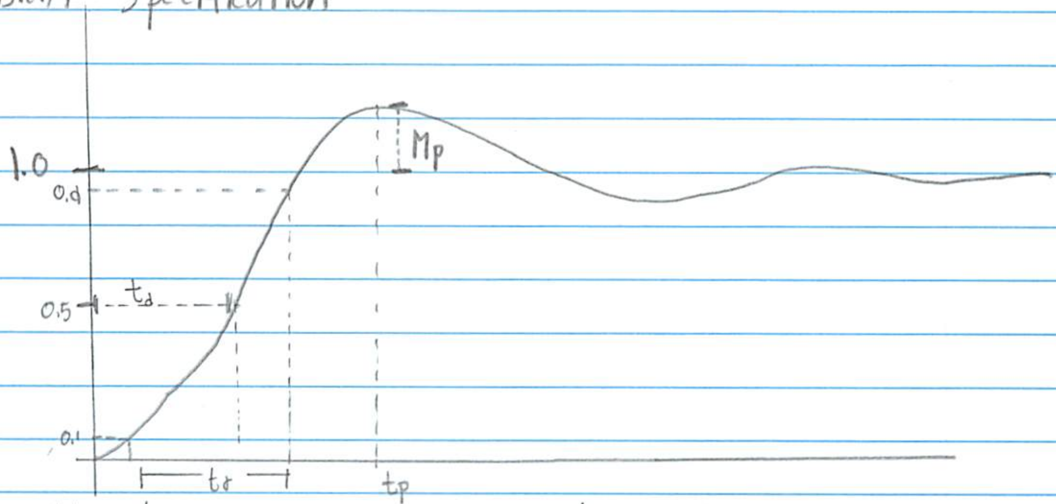
$$\lim_{s\to 0} (U(s)-Y(s))\cdot s$$

for step response: $\lim_{s\to 0} (1-T(s))$

$U(s) = \frac{1}{s}$

- Transient: what happens in finite time assuming 0 steady state error

- Transient specification



t_d: time dely, time to go to 0.5 t_r: rise time, time 0.1–0.9

t_p: time to peak if a peak exist M_p: Overshoot (as a percent)

t_s: settling time (tollerance 2%) inside region and never leaves

For: 1st order system: $M_p = 0\%$, $t_s \approx 4\cdot T$

2nd order system: $M_p = \exp\left(\frac{-\pi \xi}{\sqrt{1-\xi^2}}\right)$, $t_s = \frac{4}{\xi\omega_n}$