M 431: Assignment 3

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Problem. Let G be a finite, nonempty set with an operation $*$ such that:

1. G is closed under $*$
2. $*$ is associative
3. Given $a, b, c \in G$ with $a * b = a * c$, then $b = c$
4. Given $a, b, c \in G$ with $b * a = c * a$, then $b = c$

Prove that G must be a group under $*$.

Proof. To prove that $(G, *)$ is a group, we must show that G contains an identity element and an inverse for each element. Let's begin with the identity element. We must find an element $e \in G$ such that $x*e = e*x = x$ for all $x \in G$. Let $|G| = n < +\infty$ and fix an element $g \in G$ and consider $g, g^2, g^3, ..., g^{n+1}$. Since G is closed under $*$, every g^k is an element of G, yet G has only n elements so we must have $g^i = g^j$ for some $1 \leq i < j \leq n + 1$.

Now let $l = j - i > 0$ (which means $j = l + i = i + l$) and we can say that $g^j = g^{l+i} = g^l * g^i$ and $g^j = g^{i+l} = g^i * g^l$. But then $g^j = g^i$ so we have $g^i = g^i * g^l = g^l * g^i$. Letting $g^i = \bar{g}$ and $g^l = \bar{e}$ (both elements of G by closure under $*$) gives us $\bar{g} = \bar{g} * \bar{e} = \bar{e} * \bar{g}$ for the specific element $\bar{g} \in G$.

We now show that $\bar{e}$ is an identity element for every element of G. Fix any $x \in G$, then $\bar{g} * x = \bar{g} * \bar{e} * x$ since $\bar{g} = \bar{g} * \bar{e}$. But then property 3 says that $x = \bar{e} * x$. Further, $x * \bar{g} = x * \bar{e} * \bar{g}$ since $\bar{e} * \bar{g} = \bar{g}$ and then property 4 allows us to say that $x * \bar{e} = x$. So we have just shown that $x * \bar{e} = \bar{e} * x = x$ for an arbitrary $x \in G$. In other words, $\bar{e}$ is an identity element for G.

Now we must show an inverse element exists for every element of G: that there exists some element $g' \in G$ such that $g * g' = g' * g = \bar{e}$. To do this, we take g and $g^l = \bar{e}$ as before. Pick $g' = g^{l-1} \in G$ then $g * g' = g * g^{l-1} = g^{1+l-1} = g^{l+1-1} = g^l = \bar{e}$ and $g' * g = g^{l-1} * g = g^{l-1+1} = g^l = \bar{e}$ as desired. Since we chose g arbirarily, we have just shown every element has an inverse in G.

So we have showed that an identity exists in G and each element has an inverse so G satisfies the conditions of a group.

□

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Problem. Let S_3 be the symmetric group of degree 3. Find all the subgroups of S_3.

Proof. First note that we always have the trivial subgroups $\{id\}$ and S_3. We must now find the remaining subgroups of S_3. Note that $|S_3| = 3! = 6$ and for any subgroup $H \leq S_3$, we must have $id \in H$. This leaves us only 5 options to include in a subgroup H. If we choose one of the elements $(12), (23), (13)$ to accompany id then we have a subgroup for each of those elements is it's own inverse. So in addition to the trivial subgroups we also have $\{id, (12)\}, \{id, (23)\}, \{id, (13)\}$.

Note that the remaining elements we can choose to accompany id are $(123) = (312) = (231)$ and $(132) = (213) = (321)$. Now suppose we choose one of $(123), (132)$ to accompany id. Then Figure 1 shows that the other must also be in the subgroup to satisfy closure. Also each of them applied 3 times to the themselves so we have inverses as well. Therefore $\{id, (123), (132)\}$ constitutes a subgroup as well.

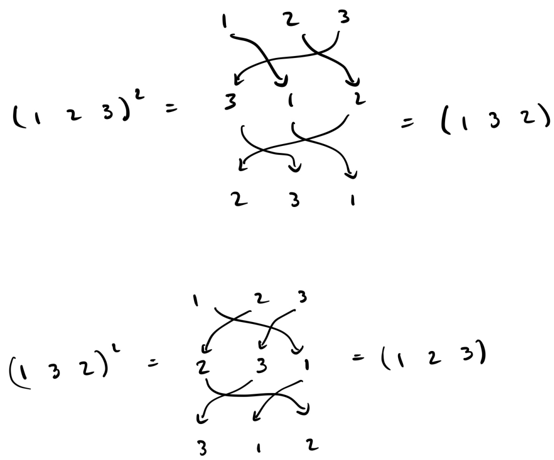

Figure 1: Necessary closure for 3 cycles

We now show that we have listed all subgroups of S_3. We can deduce this from the requirement that a subgroup requires closure. Of the remaining combinations of elements we could choose to accompany id in a subgroup, all of them require including the whole group. This is shown in Figures 2 and 3. As an example pick elements (12), (23) to accompany id. To satisfy closure, we must include (123) (Figure 2) but then we must include (132) (Figure 1) and then we must include (13) (Figure 3) which is the entire set.

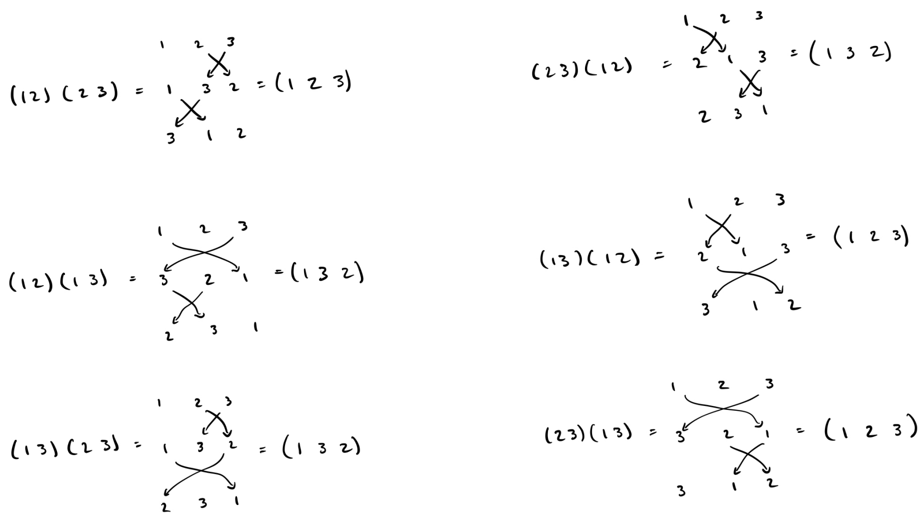

Figure 2: Necessary closures

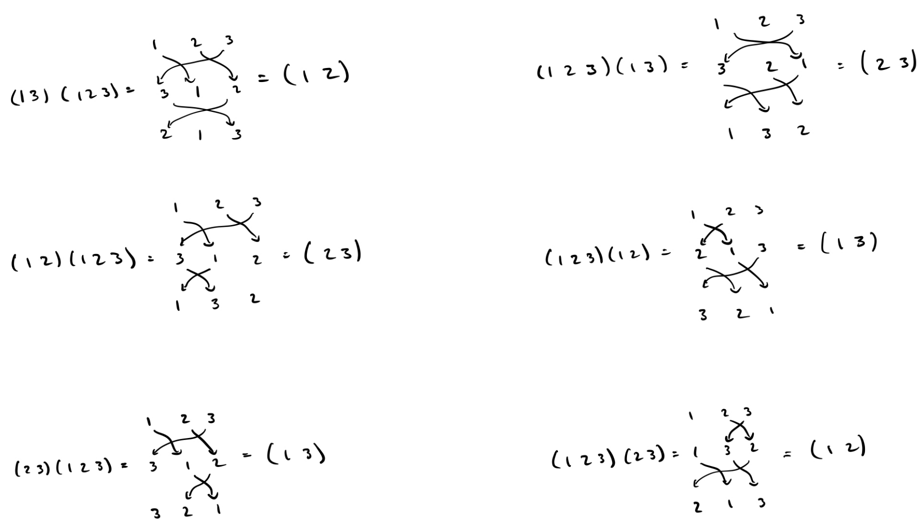

Figure 3: Necessary closures

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Problem. Prove that a cyclic group is abelian.

Proof. Let's first introduce the definition of a cyclic group. A group G is cyclic if there exists some $a \in G$ such every $x \in G$ is a power of a ($x = a^j$ for some $j \in \mathbb{Z}$). Additionally, a group is abelian if $ab = ba$ for all $a, b \in G$.

Suppose we have some cyclic group G. Since G is cyclic, there is some $g \in G$ such that every $x \in G$ is of the form $x = g^j$ for some $j \in \mathbb{Z}$. Now take any $a, b \in G$ and note that $a = g^i$ and $b = g^k$ for some $i, k \in Z$. Then $ab = g^i g^k = g^{i+k} = g^{k+i} = g^k g^i = ba$ as desired. Therefore, every cyclic group is abelian.

□

Heisenberg group problem

Problem. Recall the general linear group $\mathbb{GL}_3(\mathbb{R})$ of 3×3 invertible matrices with real entries (taken with the matrix product). Verify that the following subset, called the Heisenberg group, is a subgroup of $\mathbb{GL}_3(\mathbb{R})$:

$$\mathbb{H}_3(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

Proof. First we note that $\mathbb{H}_3(\mathbb{R})$ is a (nonempty) subset of $\mathbb{GL}_3(\mathbb{R})$ since $I_3 \in \mathbb{H}_3(\mathbb{R})$ and every matrix in $\mathbb{H}_3(\mathbb{R})$ has rank 3. Then since $\mathbb{H}_3(\mathbb{R}) \subset \mathbb{GL}_3(\mathbb{R})$ we automatically inherit the associativity of matrix multiplication. So only two conditions remain for $\mathbb{H}_3(\mathbb{R})$ to be a subgroup: closure and the existence of an inverse in $\mathbb{H}_3(\mathbb{R})$. The previous two conditions imply $I_3 \in \mathbb{H}_3(\mathbb{R})$ but this was also verified by inspection.

Let's first prove closure. Pick $A, A' \in \mathbb{H}_3(\mathbb{R})$ and compute

$$AA' = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' + x & z' + xy' + z \\ 0 & 1 & y' + y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \bar{x} & \bar{z} \\ 0 & 1 & \bar{y} \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{H}_3(\mathbb{R})$$

where $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}$ by the closure of $\mathbb{R}$ under addition and multiplication. So $\mathbb{H}_3(\mathbb{R})$ is closed under matrix multiplication. We must now show that for every $A \in \mathbb{H}_3(\mathbb{R})$ we also have $B \in \mathbb{H}_3(\mathbb{R})$ such that $AB = BA = I_3$. For A, choose

$$B = \begin{bmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}$$

Certainly B is in the set $\mathbb{H}_3(\mathbb{R})$ and we have

$$AB = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = BA$$

So we have checked that $\mathbb{H}_3(\mathbb{R})$ is closed under matrix multiplication and every element has an inverse in the subset so $\mathbb{H}_3(\mathbb{R})$ is a subgroup of $\mathbb{GL}_3(\mathbb{R})$.

$\square$

Cube subgroups problem

Problem. Recall the group $Sym(Q)$ of the rigid symmetries of the cube $Q := [-1,1]^3$ in $\mathbb{R}^3$. Describe in words/pictures the following:

a subgroup of order 4
a subgroup of order 12
a subgroup of order 3
a subgroup of order 6
a subgroup of order 8

Proof. For the purposes of this problem, we label the initial faces of the cube as you would a die (with side one facing us and side 6 opposite and so on).

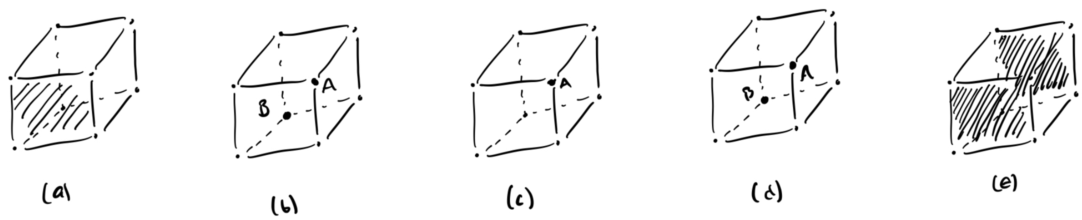

Figure 4: Subgroups of a cube

Order 4: We can make subgroup of order 4 by keeping all rigid symmetries of Q that keep us looking at the same face. In Figure 4a we shaded in this face. The subgroup is any symmetric rotation about the x axis, of which there are 4.

Order 12: I found and order 12 subgroup to be the most difficult to find. Here is what I have landed on: any rigid symmetry that preserves A's position in Figure 4b and a rotation by π about the z axis. The next part says that preserving A is a group of order 3 and then we get four distinct subgroups from the points that could end up in A's location (these are A and each of the points across the diagonal of a *square* touching A). Then the subgroup is of order 12.

Order 3: A subgroup of order 3 can be found by requiring that the vertex labed A in Figure 4c stays in the same location. This allows us three actions, synonomous with spinning the cube about A and the point diagonally through the cube from A.

Order 6: An order 6 subgroup is pretty similar to the order 3 subgroup we just discussed. Instead of requiring that A stays in the same location, we say that A and B in Figure 4d must remain on the diagonal they begin on. This gives the three actions when A is stationary and another three actions when we flip A and B (which can be done by a rotation by π about z axis and then a rotation by $\pi/2$ about x axis).

Order 8: For a subgroup of order 8, we require that we look at one of the two shaded faces shown in Figure 4e. This gives us four actions while looking at face 1 and another four while looking at face 6.

$\square$