M 431: Assignment 4

Nathan Stouffer

Page 55 — Problem 21

Problem. If A, B are subgroups of G such that $b^{-1}Ab \subset A$ for all $b \in B$, show that AB is a subgroup of G.

Proof. Our task is to show that $AB \leq G$. Note that $AB := \{ab \mid a \in A, b \in B\}$. We are also given the conditions that $A, B \leq G$ and $b^{-1}Ab = \{b^{-1}ab \mid a \in A\} \subset A$ for all $b \in B$. We get associativity for free and we can quickly deduce that $e \in AB$ since $e \in A, B$ and $e = ee \in AB$.

We now show that AB is closed under the group operation in G. Pick any $x, y \in AB$ then $x = a_1b_1$ and $y = a_2b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $xy = a_1b_1a_2b_2 = b_1b_1^{-1}a_1b_1a_2b_2$ where the existence of b_1^{-1} is guaranteed since G is a group. But then $b_1^{-1}a_1b_1 \in A$ so let $b_1^{-1}a_1b_1 = a_3 \in A$. Then we have $xy = b_1a_3a_2b_2$. Setting $a_4 = a_3a_2$ ($a_4 \in A$ by closedness), we get $xy = b_1a_4b_2$. Again since G is a group, we can say that $b_1a_4b_2 = b_1a_4b_1^{-1}b_1b_2 = (b_1^{-1})^{-1}a_4b_1^{-1}b_1b_2$. Then let $a = (b_1^{-1})^{-1}a_4b_1^{-1}$ (a member of A by our assumed property) and $b = b_1b_2$ (a member of B by closedness) and then we have $xy = ab$ where $a \in A, b \in B$. Thus $cd \in AB$.

As the final step towards proving $AB \leq G$, we must verify that every element has an inverse in AB. Fix any $x = ab \in AB$, we wonder if $x^{-1} = (ab)^{-1} \in AB$. Certainly $(ab)^{-1} \in G$ and $(ab)^{-1} = b^{-1}a^{-1} = b^{-1}a^{-1}bb^{-1}$ by properties of G. But then we know $b^{-1}a^{-1}b \in A$ (since a^{-1} is an element of A so the property applies). Letting $b^{-1}a^{-1}b = a' \in A$ we have $x^{-1} = a'b^{-1}$ which must be a member of AB since $a' \in A$ and $b^{-1} \in B$ (since B is subgroup). Therefore, AB is a subgroup of G.

$\square$

Page 65 — Problem 19

Problem. Find all the distinct conjugacy classes of S_3.

Proof.

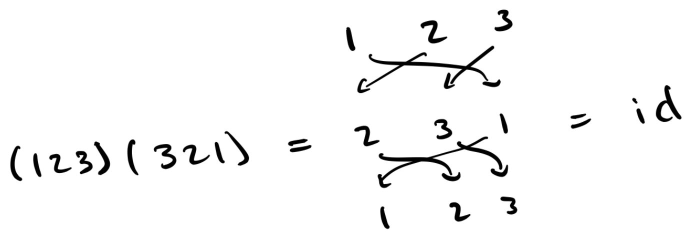

Figure 1: Necessary closure for 3 cycles

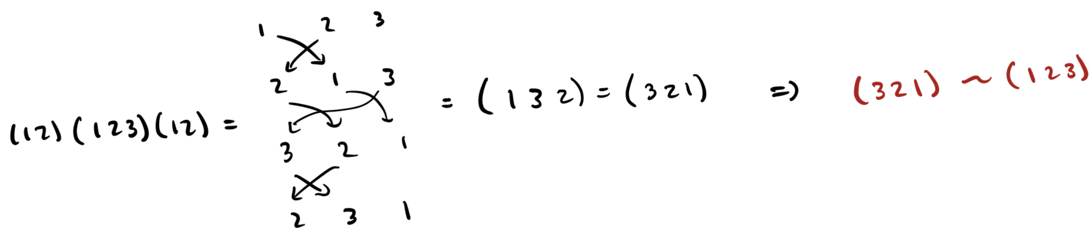

Figure 2: Necessary closure for 3 cycles

□

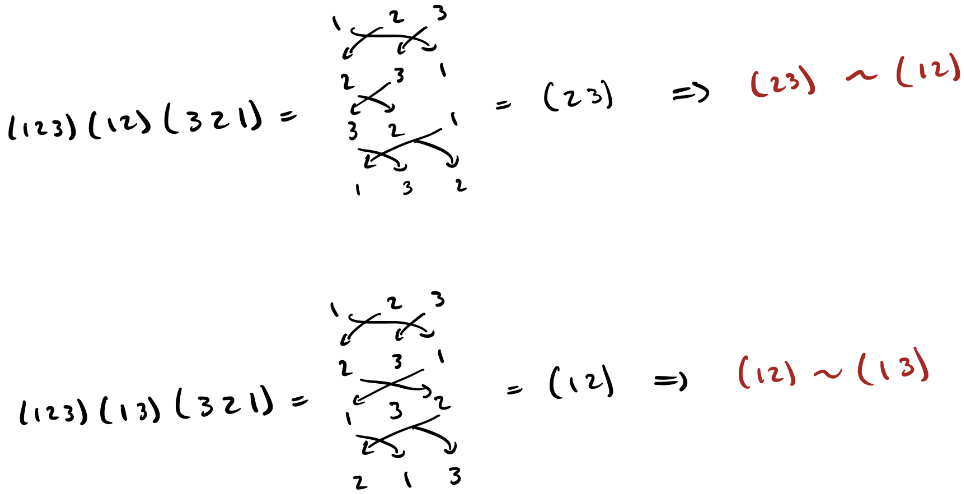

Figure 3: Necessary closure for 3 cycles

Page 65 — Problem 21

Problem. Let G be the dihedral group of order 8. Find the conjugacy classes in G.

Proof.

$\square$

Heisenberg group problem

Problem. Find the center of our new friend, the Heisenberg group,

$$\mathbb{H}_3(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

Proof.

□

Crayon the clock

Problem. Let G be equal to $\mathbb{Z}_{15} := \{0, 1, 2, ..., 14\}$. Draw it some way. Find a subgroup H of order $|H| = 5$ and then color differently all the different subsets of $\mathbb{Z}_{15}$ of the form aH. (How many are there?) If you have more crayons, do another drawing for an H with $|H| = 3$.

Proof.

$\square$