M 431: Assignment 2

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Problem. If G is a group in which $a^2 = e$ for all $a \in G$, show that G is abelian.

Proof. First note that $a^2 = e$ for all $a \in G$ implies that for every $a \in G$ $a^{-1} = a$ (for a^{-1} is the element that satisifes $aa^{-1} = e$). Now pick any $x, y \in G$. Since G is a group, we have $(xy)(xy)^{-1} = e$ and $(xy)^{-1} = y^{-1}x^{-1}$. Our assumption that $a^2 = e$ for all $a \in G$ allows us to say that $y^{-1}x^{-1} = yx$. Additionally, that same assumption tells us that $(xy)^{-1} = xy$. This gives the chain of equalities

$$xy = (xy)^{-1} = y^{-1}x^{-1} = yx$$

Since x, y were arbitrary members of G, we have just shown that G is abelian.

$\square$

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Problem. If G is a finite group of even order, show that there must be an element $a \neq e$ such that $a = a^{-1}$.

Proof. Suppose not, that is, suppose we have a finite group G of even order such that $a = e$ is the only element to satisfy $a = a^{-1}$ for any $a \in G$. Let's define a map $f : G \longrightarrow G$ which takes $x \in G$ to $x^{-1} = y \in G$. The map f is well defined because each element in a group has an inverse. Note that f must be a bijection (f is 1-1 because inverses are unique in a group and f is onto because each element in a group is an inverse). Further, $ff(x) = f(f(x)) = x$ because $f(x)x = yx = e$.

Since $ff(x) = x$ for all $x \in G$, every element in G is part of a 2-cycle or a 1-cycle under f. We assumed that e is the only element to be its own inverse, which is equivalent to saying $e = f(e)$ is the only 1-cycle. Therefore, the remaining $|G| - 1$ elements must be part of 2-cycles. But $|G| - 1$ is odd so it cannot be broken into disjoint pairs. So we have reached a contradiction and it must be the case that some $a \in G$ ($a \neq e$) satisfies $a = a^{-1}$.

$\square$

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Problem. If G is the dihedral group of order $2n$ as defined in Example 10, prove that

(a) If n is odd and $a \in G$ is such that $a * b = b * a$ for all $b \in G$, then $a = e$.

(b) If n is even, show that there is an $a \in G$, $a \neq e$, such that $a * b = b * a$ for all $b \in G$.

(c) If n is even, find all the elements $a \in G$ such that $a * b = b * a$ for all $b \in G$.

Proof.

$\square$

Custom Problem

Problem. Find the order of the group $Sym(\mathbb{Q})$ of the (rigid) symmetries of the cube $Q := [-1, 1]^3 \subset \mathbb{R}^3$. The task is not to just get the number but to organize all the possible symmetries and build a narrative that gets one to the answer in a rigorous way resting on few, easy to grasp geometric facts.

Proof.

$\square$