M 431: Assignment 10

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Page 137 — Problem 40

Problem. Prove that a finite domain is a division ring. As a consequence, show that $\mathbb{Z}_p$ is a field if p is prime.

Proof.

$\square$

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Problem. Let R be any ring with unit, and S the ring of 2×2 matrices over R.

(a) Check the associative law of multiplication in S.

(b) Show that $T = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \, a, b, c \in R \right\}$ is a subring of S.

(c) Show that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ as an inverse in T if and only if a and c have inverses in R. In that case, write down $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1}$ explicitly.

Proof.

$\square$

Page 135 — Problem 23

Problem. Define the map $*$ in the quaternions by taking

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \mapsto \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$$

Then show that:

(a) $x^{**} = (x^*)^* = x$
(b) $(x+y)^* = x^* + y^*$
(c) $xx^* = x^*x$ is real an nonnegative
(d) $(xy)^* = y^*x^*$

Proof.

□

Page 135 — Problem 24

Problem. Use $*$, define $|x| = \sqrt{xx^*}$. Show that $|xy| = |x||y|$ for any two quaternions x and y, by using parts (c) and (d) of problem 23.

Proof.

$\square$

Page 135 — Problem 25

Problem. Using the result of problem 24 to prove Lagrange's Identity.

Proof.

□

Subrings of $\mathbb{Q}$

Problem. The rationals are our best friends. Let's then try to understand all subrings (with unity) of $\mathbb{Q}$. Denote by $\mathbb{P}$ the set of all the primes in $\mathbb{N}$. Given a subset $P \subset \mathbb{P}$, set

$$\mathbb{Q}_P := \{m/n \mid \text{ prime factors of } n \text{ are in } P\}$$

with m/n being a reduce fraction: $(m, n) = 1$.

(i) Show that $\mathbb{Q}_P$ is a subring with unity of $\mathbb{Q}$. Reserve the letter R for subrings with unity, $R \subset \mathbb{Q}$. Define the denominator primes associated to such rings by

$$P_R := \{p \in \mathbb{P} \mid 1/p \in R\}$$

(ii) Show that if $P = P_R$ then $R = \mathbb{Q}_P$.

Proof.

$\square$