

M 431: Assignment 4

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Page 55 — Problem 21

Problem. If A, B are subgroups of G such that $b^{-1}Ab \subset A$ for all $b \in B$, show that AB is a subgroup of G .

Proof. Our task is to show that $AB \leq G$. Note that $AB := \{ab \mid a \in A, b \in B\}$. We are also given the conditions that $A, B \leq G$ and $b^{-1}Ab = \{b^{-1}ab \mid a \in A\} \subset A$ for all $b \in B$. We get associativity for free and we can quickly deduce that $e \in AB$ since $e \in A, B$ and $e = ee \in AB$.

We now show that AB is closed under the group operation in G . Pick any $x, y \in AB$ then $x = a_1b_1$ and $y = a_2b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $xy = a_1b_1a_2b_2 = b_1b_1^{-1}a_1b_1a_2b_2$ where the existence of b_1^{-1} is guaranteed since G is a group. But then $b_1^{-1}a_1b_1 \in A$ so let $b_1^{-1}a_1b_1 = a_3 \in A$. Then we have $xy = b_1a_3a_2b_2$. Setting $a_4 = a_3a_2$ ($a_4 \in A$ by closedness), we get $xy = b_1a_4b_2$. Again since G is a group, we can say that $b_1a_4b_2 = b_1a_4b_1^{-1}b_1b_2 = (b_1^{-1})^{-1}a_4b_1^{-1}b_1b_2$. Then let $a = (b_1^{-1})^{-1}a_4b_1^{-1}$ (a member of A by our assumed property) and $b = b_1b_2$ (a member of B by closedness) and then we have $xy = ab$ where $a \in A, b \in B$. Thus $cd \in AB$.

As the final step towards proving $AB \leq G$, we must verify that every element has an inverse in AB . Fix any $x = ab \in AB$, we wonder if $x^{-1} = (ab)^{-1} \in AB$. Certainly $(ab)^{-1} \in G$ and $(ab)^{-1} = b^{-1}a^{-1} = b^{-1}a^{-1}bb^{-1}$ by properties of G . But then we know $b^{-1}a^{-1}b \in A$ (since a^{-1} is an element of A so the property applies). Letting $b^{-1}a^{-1}b = a' \in A$ we have $x^{-1} = a'b^{-1}$ which must be a member of AB since $a' \in A$ and $b^{-1} \in B$ (since B is subgroup). Therefore, AB is a subgroup of G .

□

Page 65 — Problem 19

Problem. Find all the distinct conjugacy classes of S_3 .

Proof. Our goal is to find all the distinct conjugacy classes of S_3 . Recall that $S_3 = \{id, (12), (13), (23), (123), (321)\}$ and note that $(123)^{-1} = (321)$ (see Figure 1 for proof). Given $f, g \in S_3$ we have $f \sim g$ (f and g are conjugate) if there exists some $h \in G$ such that $g = h^{-1}fh$. The textbook also notes that \sim is an equivalence relation.

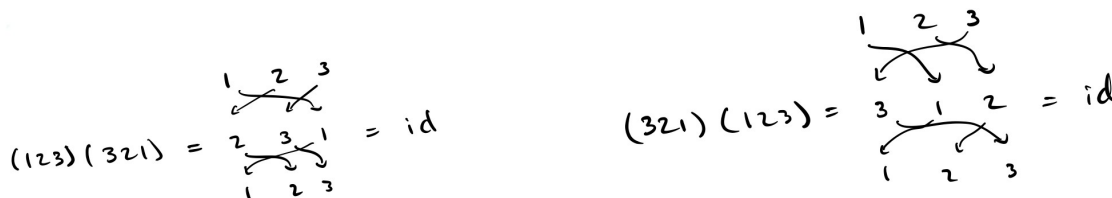


Figure 1: Showing $(123)^{-1} = (321)$

First, we show that $\{id\}$ is a conjugacy class. Since \sim is an equivalence relation, we necessarily have $id \sim id$. Now pick f a member of the conjugacy class of id . Then $id = g^{-1}fg$ for some $g \in S_3$. Left multiplying by g and right multiplying by g^{-1} , we obtain $f = id$, therefore the conjugacy class of id is the singleton $\{id\}$.

With the definition for \sim , certainly (123) and (321) are in the same conjugacy class (Figure 2 and the fact that \sim is symmetric). Additionally, the maps (12) , (13) , and (23) are in the same conjugacy class (Figure 3 and the symmetric/transitive properties of equivalence relations).

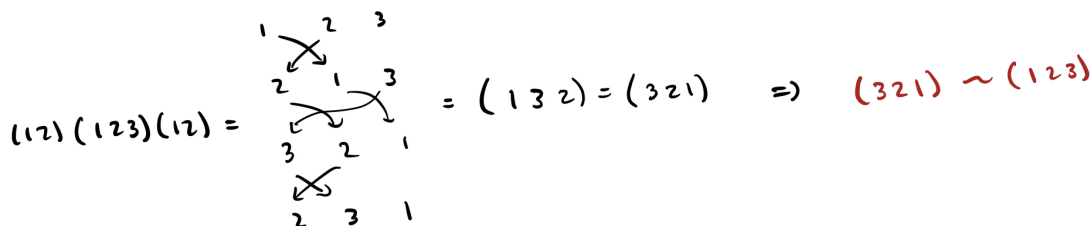


Figure 2: Showing $(123) \sim (321)$

$$(123)(12)(321) = \begin{array}{c} 1 \quad 2 \quad 3 \\ \swarrow \quad \searrow \quad \swarrow \\ 2 \quad 3 \quad 1 \\ \swarrow \quad \searrow \quad \swarrow \\ 3 \quad 2 \quad 1 \\ \swarrow \quad \searrow \quad \swarrow \\ 1 \quad 3 \quad 2 \end{array} = (23) \Rightarrow (23) \sim (12)$$

$$(123)(13)(321) = \begin{array}{c} 1 \quad 2 \quad 3 \\ \swarrow \quad \searrow \quad \swarrow \\ 2 \quad 3 \quad 1 \\ \swarrow \quad \searrow \quad \swarrow \\ 1 \quad 3 \quad 2 \\ \swarrow \quad \searrow \quad \swarrow \\ 2 \quad 1 \quad 3 \end{array} = (12) \Rightarrow (12) \sim (13)$$

Figure 3: Showing $(12) \sim (13) \sim (23)$

We now show that the conjugacy classes of S_3 are $A_1 = \{id\}$, $A_2 = \{(12), (13), (23)\}$, and $A_3 = \{(123), (321)\}$. We already know $\{id\}$ to be its own conjugacy class. So we only need show that A_2, A_3 are not part of the same conjugacy class. Since \sim is an equivalence relation, it suffices to show that a single pair of elements from A_2, A_3 are not in the same conjugacy class. Can we show (12) and (123) are not in the same conjugacy class? Suppose $(12) \sim (123)$, then there exists $h \in G$ such that $(12) = h^{-1}(123)h$. Then it must also be true that $(12)^3 = (h^{-1}(123)h)^3$. Simplifying, $(12)^3 = (12)$ and $(h^{-1}(123)h)^3 = h^{-1}(123)^3h = h^{-1}idh = h^{-1}h = id$. But $(12) \neq id$ so we have reached a contradiction and the elements (12) and (123) cannot be part of the same conjugacy class. Then their conjugacy classes must be distinct (since \sim is an equivalence relation). Therefore the conjugacy classes of S_3 are $A_1 = \{id\}$, $A_2 = \{(12), (13), (23)\}$, and $A_3 = \{(123), (321)\}$.

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Page 65 — Problem 21

Problem. Let G be the dihedral group of order 8. Find the conjugacy classes in G .

Proof. I claim the conjugacy classes of D_8 are $\{id\}$, $\{r, r^3\}$, $\{r^2\}$, $\{f, fr^2\}$, and $\{fr, fr^3\}$. Certainly $\{id\}$ is an equivalence class since $id \in [id]$ and any other element $a \in [id]$ satisfies $id = x^{-1}ax$ for some $x \in D_8$. But then left multiplying by x and right multiplying by x^{-1} gives $a = id$.

We now show that each of the above sets certainly contain elements in the same equivalence class. We know $fr^3 \sim fr$ because $ffrf = rf = fr^{-1} = fr^3$. Additionally, $f \sim fr^2$ since $r^3fr^2r = f$. And finally, $r^3 \sim r$ since $frf = ffr^{-1} = r^3$. I tried to come up with a slick proof for why no other elements are related to each other but I couldn't find one. I leave this problem saying the since D_8 is finite, the sets I have presented can verified to be equivalence classes by running through all $x \in D_8$.

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Heisenberg group problem

Problem. Find the center of our new friend, the Heisenberg group,

$$\mathbb{H}_3(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

Proof. We wish to find the center of $\mathbb{H}_3(\mathbb{R})$: $Z(\mathbb{H}_3(\mathbb{R})) = \{A \in \mathbb{H}_3(\mathbb{R}) \mid AB = BA \text{ for all } B \in \mathbb{H}_3(\mathbb{R})\}$. Note that $I_3 \in \mathbb{H}_3(\mathbb{R})$ so $\mathbb{H}_3(\mathbb{R})$ is nonempty. We first give a necessary condition for $A \in Z(\mathbb{H}_3(\mathbb{R}))$. Fix $A \in Z(\mathbb{H}_3(\mathbb{R}))$ and let $B \in \mathbb{H}_3(\mathbb{R})$, then we have $AB = BA$:

$$AB = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' + x & z' + xy' + z \\ 0 & 1 & y' + y \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x + x' & z + x'y + z' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{bmatrix}$$

Since $AB = BA$, their elements must match. From this, we gather that $xy' = x'y$ for all $x', y' \in \mathbb{R}$. This can only be true if $x = y = 0$ for $A \in Z(\mathbb{H}_3(\mathbb{R}))$. We now show that $x = y = 0$ is a sufficient condition for membership in $Z(\mathbb{H}_3(\mathbb{R}))$. Pick such a matrix A and then choose $B \in \mathbb{H}_3(\mathbb{R})$. Then we have

$$AB = \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' & z' + z \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = BA$$

Since B was arbitrary in $\mathbb{H}_3(\mathbb{R})$ we must have A in the center of $\mathbb{H}_3(\mathbb{R})$. Therefore $x = y = 0$ is a necessary and sufficient condition for members of $Z(\mathbb{H}_3(\mathbb{R}))$. Thus we have

$$Z(\mathbb{H}_3(\mathbb{R})) = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid z \in \mathbb{R} \right\}$$

□

Crayon the clock

Problem. Let G be equal to $\mathbb{Z}_{15} := \{0, 1, 2, \dots, 14\}$. Draw it some way. Find a subgroup H of order $|H| = 5$ and then color differently all the different subsets of \mathbb{Z}_{15} of the form aH . (How many are there?) If you have more crayons, do another drawing for an H with $|H| = 3$.

Proof. Here is a figure with my subgroups.

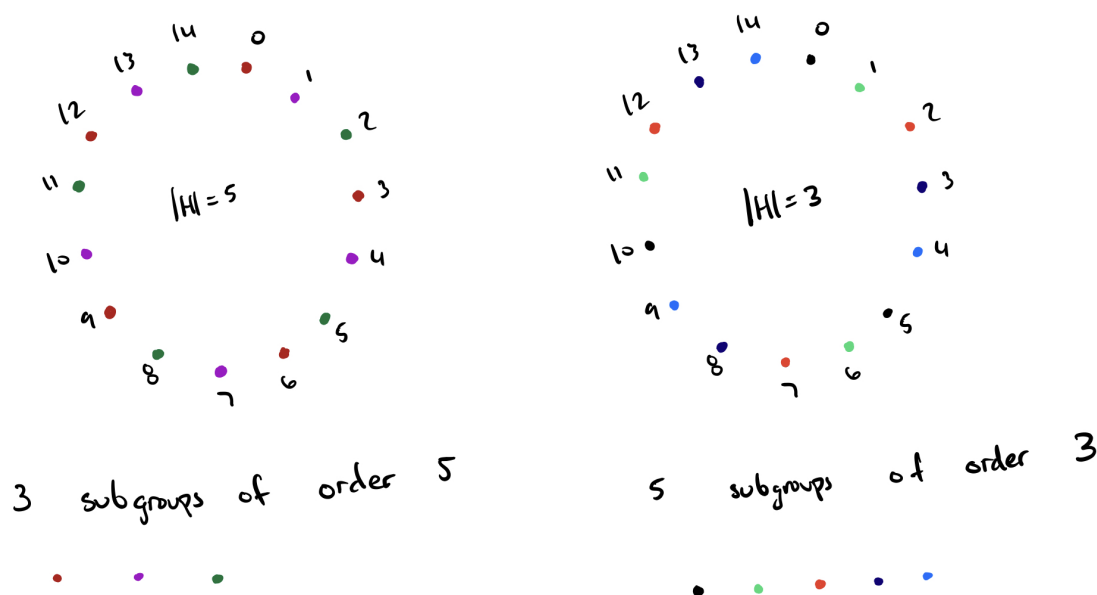


Figure 4: Subgroups of \mathbb{Z}_{15}

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