M 431: Assignment 6

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Homomorphisms between clocks

Problem. Find all homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{15}$.

Proof.

$\square$

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Problem. If G is a finite abelian group of order n and $\varphi : G \longrightarrow G$ is defined by $\varphi(a) = a^m$ for all $a \in G$, find the necessary and sufficient condition that φ be an isomorphism of G onto itself.

Proof.

$\square$

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Problem. If G is a group and $a \in G$, define $\sigma_a(g) = aga^{-1}$. We saw in Example 9 of this section that σ_a is an isomorphism of G onto itself, so $\sigma_a \in A(G)$, the group of all $1-1$ mappings of G (as a set) onto itself. Define $\psi : G \longrightarrow A(G)$ by $\psi(a) = \sigma_a$ for all $a \in G$. Prove that

(a) ψ is a homomorphism of G into $A(G)$.

(b) $\ker \psi = Z(G)$ the center of G.

Proof.

(a) We wish to show that ψ is a homomorphism. That is, we must show that $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in G$. On the LHS, we have $\psi(ab) = \sigma_{ab}$ and on the RHS we have $\psi(a)\psi(b) = \sigma_a \circ \sigma_b$. For the LHS to equal the RHS we must have $\sigma_{ab}(g) = (\sigma_a \circ \sigma_b)(g)$ for all $g \in G$. Fix any $g \in G$, then $\sigma_{ab}(g) = (ab)g(ab)^{-1} = abgb^{-1}a^{-1} = \sigma_a(bgb^{-1}) = \sigma_a(\sigma_b(g)) = (\sigma_a \circ \sigma_b)(g)$. Since g was arbitrary in G, we have shown the equality between maps $\sigma_{ab} = \sigma_a \circ \sigma_b$ which means ψ is a homomorphism.

(b) Now we want to show that $\ker \psi = Z(G)$. Let's write their definitions. We know $\ker \psi := \{a \in G \mid \psi(a) = id \in A(G)\} \subset G$ and that $Z(G) := \{a \in G \mid ag = ga \text{ for all } g \in G\}$. We will show equality by showing inclusion in both directions. Going to the left, pick any $a \in \ker \psi$, then $\psi(a) = id \in A(G)$. Also, by definition, $\psi(a) = \sigma_a$ this gives the equality of maps $\sigma_a = id$. Then we have $\sigma_a(g) = id(g) = g$ for all $g \in G$. By definition, $\sigma_a(g) = aga^{-1}$ so we also have $aga^{-1} = g$ for all $g \in G$. Left multiplying by a gives $ag = ga$ for all g, which is exactly the condition for membership in $Z(G)$. Therefore, $a \in Z(G)$ and since a was arbitray in $\ker \psi$ we have the inclusion $\ker \psi \subset Z(G)$.

We now show the other inclusion $Z(G) \subset \ker \psi$. Fix any $b \in Z(G)$ then $bg = gb$ for every $b \in G$. Left multplying by b^{-1} gives the equality $bgb^{-1} = g$ for all $b \in G$. The LHS is exactly $\sigma_b(g)$ so we have just shown that $\sigma_b(g) = g$ for every b. But then σ_b performs the same actions as the identity map $id \in A(G)$ so we have $\sigma_b = id$. Further, $\sigma_b = \psi(b)$ so we also have $\psi(b) = id \in A(G)$, which is the condition for membership in $\ker \psi$. By arbitrariness of b, we have shown $Z(G) \subset \ker \psi$.

$\square$

Heisenberg to plane

Problem. Find an epimorphism the Heisenberg group $\mathbb{H}_3(\mathbb{R})$ onto $\mathbb{R}^2$.

Proof. Consider the function $\varphi : \mathbb{H}_3(\mathbb{R}) \longrightarrow \mathbb{R}^2$ which maps a matrix $A = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{H}_3(\mathbb{R})$ to the point $(x, y) \in \mathbb{R}^2$. For φ to be an epimorphism, we must verify that φ is both surjective and a homomorphism. For φ to be surjective, for every point in $\mathbb{R}^2$ we must have a matrix in $\mathbb{H}_3(\mathbb{R})$ such that φ maps the matrix to the point. For the point $(a, b) \in \mathbb{R}^2$, choose the matrix $A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$. Then $\varphi(A) = (a, b)$ as desired. Now let's verify that φ is a homomorphism. We must check that $\varphi(AA') = \varphi(A)\varphi(A')$ for all $A, A' \in \mathbb{H}_3(\mathbb{R})$.

$$AA' = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' + x & z' + y'x + z \\ 0 & 1 & y' + y \\ 0 & 0 & 1 \end{bmatrix}$$

On the LHS, $\varphi(AA') = (x' + x, y' + x)$. Then on the RHS, we have $\varphi(A)\varphi(A') = (x, y) + (x', y') = (x + x', y + y') = (x' + x, y' + y)$. So the LHS equals the RHS and φ is a homomorphism.

$\square$

Klein group

Problem. Show that the group $Sym(R)$ where R is a rectangle that is not a square is isomorphic to the product $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. We know the elements of $\mathbb{Z}_2$ so we know the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the group opertion it is endowed with. We must explore the group $Sym(R)$. Certainly $Sym(R) \leq D_8$ (where S is a square) for every symmetry of a rectangle is also a symmetry of a square. Figure 1 depicts the axis of symmetries of a rectangle R. Together with the identity transformation, there are four. Since $Sym(R) \leq D_8$ let's give the four transformations the same names that we call them in D_8. Namely, $Sym(R) := \{e, r^2, f, fr^2\}$.

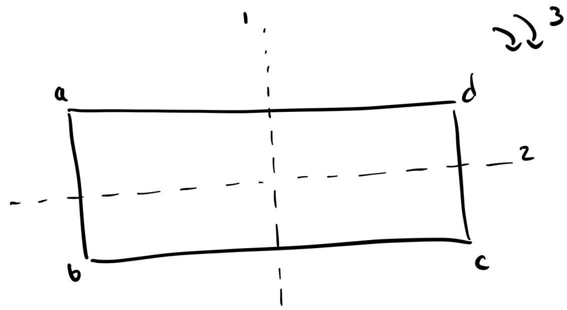

Figure 1: Symmetries of a rectangle R

Figure 2 defines a map $\varphi : Sym(R) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$. Just from the figure, we know that φ is a bijection.

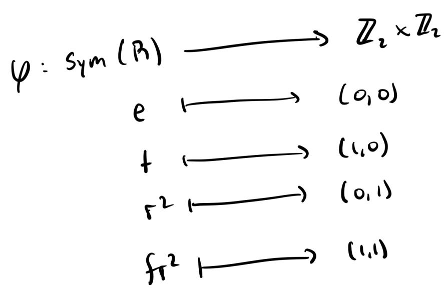

Figure 2: Isomorphism between $Sym(R)$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$

To verify that φ is an isomorphism, we must verify that φ is a homomorphism. Before doing this, note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $Sym(R)$ are abelian. We can check $Sym(R)$ by verifying that $fr^2 = r^2 f$. Since each group is abelian, we only need to check one "ordering" of each pair of elements. Now we verify that φ respects the group structure.

$$
\begin{aligned}
\varphi(e) + \varphi(x) = (0,0) + \varphi(x) = \varphi(x) = \varphi(ex) &\text{ for any } x \in Sym(R) \\
\varphi(f) + \varphi(r^2) = (1,0) + (0,1) = (1,1) = \varphi(fr^2)& \\
\varphi(f) + \varphi(fr^2) = (1,0) + (1,1) = (0,1) = \varphi(r^2) = \varphi(ffr^2)& \\
\varphi(fr^2) + \varphi(r^2) = (1,1) + (0,1) = (1,0) = \varphi(f) = \varphi(fr^4)&
\end{aligned}
$$

$\square$