# M 431: Assignment 6

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## Homomorphisms between clocks

*Problem.* Find all homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{15}$.

*Proof.*

$\square$

## Page 74 — Problem 13

*Problem.* If $G$ is a finite abelian group of order $n$ and $\varphi : G \longrightarrow G$ is defined by $\varphi(a) = a^m$ for all $a \in G$, find the necessary and sufficient condition that $\varphi$ be an isomorphism of $G$ onto itself.

*Proof.*

$\square$

## Page 75 — Problem 26

*Problem.* If $G$ is a group and $a \in G$, define $\sigma_a(g) = aga^{-1}$. We saw in Example 9 of this section that $\sigma_a$ is an isomorphism of $G$ onto itself, so $\sigma_a \in A(G)$, the group of all $1-1$ mappings of $G$ (as a set) onto itself. Define $\psi : G \longrightarrow A(G)$ by $\psi(a) = \sigma_a$ for all $a \in G$. Prove that

**(a)** $\psi$ is a homomorphism of $G$ into $A(G)$.

**(b)** $\ker\psi = Z(G)$ the center of $G$.

*Proof.*

**(a)** We wish to show that $\psi$ is a homomorphism. That is, we must show that $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in G$. On the LHS, we have $\psi(ab) = \sigma_{ab}$ and on the RHS we have $\psi(a)\psi(b) = \sigma_a \circ \sigma_b$. For the LHS to equal the RHS we must have $\sigma_{ab}(g) = (\sigma_a \circ \sigma_b)(g)$ for all $g \in G$. Fix any $g \in G$, then $\sigma_{ab}(g) = (ab)g(ab)^{-1} = abgb^{-1}a^{-1} = \sigma_a(bgb^{-1}) = \sigma_a(\sigma_b(g)) = (\sigma_a \circ \sigma_b)(g)$. Since $g$ was arbitrary in $G$, we have shown the equality between maps $\sigma_{ab} = \sigma_a \circ \sigma_b$ which means $\psi$ is a homomorphism.

**(b)** Now we want to show that $\ker\psi = Z(G)$. Let's write their definitions. We know $\ker\psi := \{a \in G \mid \psi(a) = id \in A(G)\} \subset G$ and that $Z(G) := \{a \in G \mid ag = ga \text{ for all } g \in G\}$. We will show equality by showing inclusion in both directions. Going to the left, pick any $a \in \ker\psi$, then $\psi(a) = id \in A(G)$. Also, by definition, $\psi(a) = \sigma_a$ this gives the equality of maps $\sigma_a = id$. Then we have $\sigma_a(g) = id(g) = g$ for all $g \in G$. By definition, $\sigma_a(g) = aga^{-1}$ so we also have $aga^{-1} = g$ for all $g \in G$. Left multiplying by $a$ gives $ag = ga$ for all $g$, which is exactly the condition for membership in $Z(G)$. Therefore, $a \in Z(G)$ and since $a$ was arbitray in $\ker\psi$ we have the inclusion $\ker\psi \subset Z(G)$.

We now show the other inclusion $Z(G) \subset \ker\psi$. Fix any $b \in Z(G)$ then $bg = gb$ for every $b \in G$. Left multplying by $b^{-1}$ gives the equality $bgb^{-1} = g$ for all $b \in G$. The LHS is exactly $\sigma_b(g)$ so we have just shown that $\sigma_b(g) = g$ for every $b$. But then $\sigma_b$ performs the same actions as the identity map $id \in A(G)$ so we have $\sigma_b = id$. Further, $\sigma_b = \psi(b)$ so we also have $\psi(b) = id \in A(G)$, which is the condition for membership in $\ker\psi$. By arbitrariness of $b$, we have shown $Z(G) \subset \ker\psi$.

□

## Heisenberg to plane

*Problem.* Find an epimorphism the Heisenberg group $\mathbb{H}_3(\mathbb{R})$ onto $\mathbb{R}^2$.

*Proof.* Consider the function $\varphi : \mathbb{H}_3(\mathbb{R}) \longrightarrow \mathbb{R}^2$ which maps a matrix $A = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{H}_3(\mathbb{R})$ to the point $(x, y) \in \mathbb{R}^2$. For $\varphi$ to be an epimorphism, we must verify that $\varphi$ is both surjective and a homomorphism. For $\varphi$ to be surjective, for every point in $\mathbb{R}^2$ we must have a matrix in $\mathbb{H}_3(\mathbb{R})$ such that $\varphi$ maps the matrix to the point. For the point $(a, b) \in \mathbb{R}^2$, choose the matrix $A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$. Then $\varphi(A) = (a, b)$ as desired. Now let's verify that $\varphi$ is a homomorphism. We must check that $\varphi(AA') = \varphi(A)\varphi(A')$ for all $A, A' \in \mathbb{H}_3(\mathbb{R})$.

$$AA' = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' + x & z' + y'x + z \\ 0 & 1 & y' + y \\ 0 & 0 & 1 \end{bmatrix}$$

On the LHS, $\varphi(AA') = (x' + x, y' + x)$. Then on the RHS, we have $\varphi(A)\varphi(A') = (x, y) + (x', y') = (x + x', y + y') = (x' + x, y' + y)$. So the LHS equals the RHS and $\varphi$ is a homomorphism.

$\square$

## Klein group

*Problem.* Show that the group $Sym(R)$ where $R$ is a rectangle that is not a square is isomorphic to the product $\mathbb{Z}_2 \times \mathbb{Z}_2$.

*Proof.*

$\square$