M 431: Assignment 7

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Page 74 — Problem 15

Problem. If G is any group, $N \trianglelefteq G$ and $\varphi : G \longrightarrow G'$ a homomorphism of G onto G', prove that the image, $\varphi(N)$, of N is a normal subgroup of G'.

Proof. We already know that the image $\varphi(N)$ is a subgroup of G' since the image of a subgroup under a homomorphism is a subgroup. So we only need to show that $\varphi(N)$ is normal in G'. Fix $\overline{g} \in G'$ and $\overline{n} \in \varphi(N) \subset G'$. Since φ is onto, we can pick $g \in G$ and $n \in N$ such that $\varphi(g) = \overline{g}$ and $\varphi(n) = \overline{n}$.

Now consider $\varphi(g^{-1})\varphi(n)\varphi(g) \in G'$. On the one hand, $\varphi(g^{-1})\varphi(n)\varphi(g) = \varphi(g)^{-1}\varphi(n)\varphi(g) = \overline{g}^{-1}\overline{ng}$. But also $\varphi(g^{-1})\varphi(n)\varphi(g) = \varphi(g^{-1}ng) = \varphi(n_1) \in \varphi(N)$ since φ is a homomorphism. Then our chain of equalities show that $\overline{g}^{-1}\overline{ng} \in \varphi(N)$. Since $\overline{g}, \overline{n}$ were selected arbitrarily from G' and $\varphi(N)$ respectively, we just showed that $\overline{g}^{-1}\varphi(N)\overline{g} \subset \varphi(N)$. Which is to say that $\varphi(N)$ is a normal subgroup of G'.

$\square$

Page 74 — Problem 16

Problem. If $N \trianglelefteq G$ and $M \trianglelefteq G$ and $MN = \{mn \mid m \in M, n \in \mathbb{N}\}$, prove that MN is a subgroup of G and that $MN \trianglelefteq G$.

Proof. First let's show that MN is a subgroup of G. We will use the aesthetic definition. We know $MN \neq \emptyset$ because $e \in M, N$ so $ee = e \in MN$. Now for arbitrary $x, y \in MN$ we show that $xy^{-1} \in MN$. Let $x = mn \in MN$ and $y = \bar{m}\bar{n} \in MN$. Then $xy^{-1} = (mn)(\bar{m}\bar{n})^{-1} = mn\bar{n}^{-1}\bar{m}^{-1}$. Taking $n_1^{-1} = n\bar{n}^{-1}$, we get $mn\bar{n}^{-1}\bar{m}^{-1} = mn_1^{-1}m^{-1}n_1n_1^{-1}$ Then since M is normal in G we know $n_1^{-1}m^{-1}n_1 = m_1 \in M$ so we have $xy = mm_1n_1^{-1} = m_2n_2 \in MN$ where $m_2 = mm_1$ and $n_2 = n_1^{-1}$. Thus $MN \leq G$.

We now prove that MN is normal in G. We must show that $g^{-1}MNg \subset MN$. Fix any $g \in G$ and $mn \in MN$. Then $g^{-1}mng = g^{-1}mgg^{-1}ng$ but M, N are each normal so let $m_1 = g^{-1}mg \in M$ and $n_1 = g^{-1}ng \in N$ and we have $g^{-1}mng = m_1n_1 \in MN$. Since we selected the values arbitrarily, $g^{-1}MNg \subset MN$ and $MN \trianglelefteq G$.

□

Page 83 — Problem 3

Problem. If G is a group and $N \trianglelefteq G$, show that if $\overline{M}$ is a subgroup of G/N and $M = \{a \in G \mid Na \in \overline{M}\}$, then M is a subgroup of G and $N \subset M$.

Proof. First let's show that $M \leq G$. Let's use the aesthetic definition again; we know the identity element $e \in M$ since $Ne = N$ the identity element of G/N. Since $\overline{M}$ is a subgroup of G/N, we know $N \in \overline{M}$. Now fix any $x, y \in M$ and we show $xy^{-1} \in M$. Which is equivalent to showing that $Nxy^{-1} \in \overline{M}$. Since $x, y \in M$ we know $Nx, Ny \in \overline{M}$. But $\overline{M}$ is a subgroup of G/N so the aesthetic condition holds in $\overline{M}$: $(Nx)(Ny)^{-1} \in \overline{M}$. We know $(Ny)^{-1} = Ny^{-1}$ so $(Nx)(Ny)^{-1} = NxNy^{-1}$. Since N is normal in G, the left coset xN equals the right coset Nx. So $NxNy^{-1} = NNxy^{-1} = Nxy^{-1} \in \overline{M}$ the exact condition we wanted to show.

Now we show that $N \subset M$. Pick any $x \in N$ and consider the right coset Nx. Since $x \in N$, we know $Nx = N$. We already worked out that $N \in \overline{M}$. So $Nx \in \overline{M}$ which means $x \in M$. Therefore $N \subset M$.

□

Page 83 — Problem 4

Problem. If $\overline{M}$ in the previous problem is normal in G/N, show that the M defined is normal in G.

Proof. Since $\overline{M} \trianglelefteq G/N$ we know that $(Ng)^{-1}\overline{M}(Ng) \subset \overline{M}$ for arbitary $g \in G$. That is to say, for any $Nm \in \overline{M}$ (where $m \in M$) there exists some $Nm' \in \overline{M}$ (where $m' \in M$) such that $Ng^{-1}NmNg = Nm'$. Then since left cosets of N equal right cosets of N in G and $NN = N$, the LHS can be rearranged to say that $Ng^{-1}mg = Nm' \in \overline{M}$. Since $Ng^{-1}mg \in \overline{M}$, we must have $g^{-1}mg \in M$. But then g was arbitary in G and m was arbitrary in M so $g^{-1}Mg \subset M$ and $M \trianglelefteq G$.

□

Page 87 — Problem 3

Problem. Let G be the group of nonzero real numbers under multiplication and let $N = \{1, -1\}$. Prove that $G/N \cong$ positive real numbers under multiplication.

Proof.

$\square$

Page 88 — Problem 6

Problem. If G is a group and $N \trianglelefteq G$, show that if $a \in G$ has finite order $o(a)$, then Na in G/N has finite order m where $m \mid o(a)$.

Proof.

$\square$