M 431: Assignment 9

Nathan Stouffer

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Problem. Prove that a group of order 35 is cyclic.

Proof. We will prove this by using Theorem 2.8.5 on page 91 of the textbook. Let G be a group of order 35 and $p = 7, q = 5$ then $|G| = 35 = p * q = 7 * 5$ (also $p > q$). Then $5 \nmid (7 - 1)$ since $7 - 1 = 6$ and $5 \nmid 6$. Thus the theorem tells us that G must be cyclic.

$\square$

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Problem. If G is a group with subgroups A, B of orders m, n, respectively, where m and n are relatively prime, prove that the subset of G, $AB = \{ab \mid a \in A, b \in B\}$, has mn distinct elements.

Proof. Note that $A \cap B \leq A, B$ which means that $|A \cap B|$ divides both $|A| = m, |B| = n$ by Lagrange's Theorem. But $(m, n) = 1$ so we must have $|A \cap B| = 1$. This means that AB is an internal direct product (take $H = AB$ and $|A \cap B| = (e)$ so H must be in an internal direct product). Since AB is an internal direct product we have $AB \cong A \times B$.

Since the two groups $AB, A \times B$ are isomorphic, there exists a bijection their underlying sets so we have $|AB| = |A \times B|$. Since A, B are finite, $|A \times B| = mn$ so we have $|AB| = mn$ as well. Thus AB has mn distinct elements.

□

Conjugacy Stabilizers

Problem. Suppose that $K, H \leq G$ and H is normal. Let $[a]$ stand for the $K-$conjugacy class of a: $[a] := \{bab^{-1} \mid b \in K\}$. Introduce the stabilizer of a: $Stab(a) := \{b \in K \mid bab^{-1} = a\}$.

(a) Show that, for any $a' \in [a]$, $Stab(a')$ and $Stab(a)$ are related by conjugation, $\exists k \in K$ where

$$Stab(a') = k\, Stab(a)\, k^{-1}$$

Conclude that $|Stab(a)| = |Stab(a')|$.
(b) Use part (a) to show that following formula for the cardinality of $[a]$:

$$\#[a] = \frac{|K|}{|Stab(a)|}$$

Proof.

(a) Fix a and $a' \in [a]$, this implies that there exists some $\bar{b} \in K$ such that $\bar{b}a\bar{b}^{-1} = a'$. Select $k = \bar{b}$. We now wish to show that $Stab(a') = \bar{b}\, Stab(a)\, \bar{b}^{-1}$. Going to the left, pick any $\bar{b}b\bar{b}^{-1} \in \bar{b}\, Stab(a)\, \bar{b}^{-1}$. Then we know that $bab^{-1} = a$ and $\bar{b}a\bar{b}^{-1} = a'$. Now consider

$$(\bar{b}b\bar{b}^{-1})a'(\bar{b}b\bar{b}^{-1})^{-1} = \bar{b}b\bar{b}^{-1}a'\bar{b}b^{-1}\bar{b}^{-1} = \bar{b}bab^{-1}\bar{b}^{-1} = \bar{b}a\bar{b}^{-1} = a'$$

where the simplifications were made because $bab^{-1} = a$ and $\bar{b}a\bar{b}^{-1} = a'$. So then $\bar{b}b\bar{b}^{-1}$ is also a member of $Stab(a')$.

Now pick any $b' \in Stab(a')$ which means $b'a'(b')^{-1} = a'$. We wish to show that $b' \in \bar{b}\, Stab(a)\bar{b}^{-1}$ which is true of $(\bar{b}^{-1}b'\bar{b})a(\bar{b}^{-1}b'\bar{b}^{-1})^{-1} = a$. So consider

$$(\bar{b}^{-1}b'\bar{b})a(\bar{b}^{-1}b'\bar{b}^{-1})^{-1} = \bar{b}^{-1}b'\bar{b}a\bar{b}^{-1}(b')^{-1}\bar{b} = \bar{b}^{-1}b'a'(b')^{-1}\bar{b} = \bar{b}^{-1}a'\bar{b} = a$$

Since $\bar{b}a\bar{b}^{-1} = a' \iff \bar{b}^{-1}a'\bar{b}$. Thus we have $Stab(a') = \bar{b}\, Stab(a)\, \bar{b}^{-1}$. This gives $|Stab(a)| = |Stab(a')|$ since the map taking $b \in Stab$ to $\bar{b}b\bar{b}^{-1}$ is a bijection (because it has an inverse function).

(b) We will now show that $\#[a] = \frac{|K|}{|Stab(a)|}$ by equivalently showing that $|Stab(a)| * \#[a] = |K|$. From the proof of part (a) we know that every $a' \in [a]$ absorbs $|Stab(a')| = |Stab(a)|$ elements of the form bab^{-1} for $b \in K$. Thus we have $|Stab(a)| + \cdots + |Stab(a)| = |K|$ where $\cdots$ is adding $\#[a]$ times. Thus we have $|Stab(a)| * \#[a] = |K| \iff \#|a| = \frac{|K|}{|Stab(a)|}$.

$\square$

Abelian Classification

Problem. List all abelian isomorphism classes with order 108.

Proof. Let's first find the prime factorization of 108: $108 = 54 * 2 = 27 * 2^2 = 3^3 * 2^2$. There are 3 partitions of 3: $3 = 3, 3 = 2 + 1$, and $3 = 1 + 1 + 1$ and 2 partitions of 2: $2 = 2$ and $2 = 1 + 1$. Thus there are $2 * 3 = 6$ nonisomorphic groups of order 108.

□

Page 101 — Problem 2

Problem. Let G be an abelian group of order p^n, p a prime, and let $a \in G$ have maximal order. Show that $x^{o(a)} = e$ for all $x \in G$.

Proof. By Lagrange's Theorem, any element in G must have order p^j for some $j \in \{0, 1, ..., n\}$. Let $p^k = o(a)$ the order of a. I struggled a lot on this problem so I make the following conjecture which helps me prove the problem (if true). Cosider the map $p^\star : G \longrightarrow G$ defined by taking $x \mapsto x^p$. I conjecture that $p^\star$ is a homormophism onto a subgroup $H \leq G$ with order k. Since $|H| = k$ every $h \in H$ satisfies $h^k = e \in H \subset G$. But note that H consists exactly of the elements $x^p \in G$ so $h^k = (x^p)^k = e$. But x was arbitary in G so we have shown that $x^{o(a)} = e$ for all $x \in G$.

$\square$