M 431: Assignment 1

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Page 7 — Problem 14

Problem. If C is a finite set, let $m(C)$ denote the number of elements in C. If A, B are finite sets, prove that

$$m(A \cup B) = m(A) + m(B) - m(A \cap B)$$

Proof. We take for granted the fact that $m(A \cup B) = m(A) + m(B)$ if A, B are disjoint sets. Additionally, let's eliminate the edge case where one or both of A, B is the empty set. Without loss of generality, let $A = \emptyset$. Then on the left hand side $m(A \cup B) = m(\emptyset \cup B) = m(B)$. Now on the right hand side, $m(A) + m(B) - m(A \cap B) = m(\emptyset) + m(B) - m(\emptyset) \cap B) = 0 + m(B) - m(\emptyset) = m(B) - 0 = m(B)$. Then the LHS equals the RHS. From here on, we assume that A, B are nonempty sets.

Now let's establish some notation. Let $A' = A \setminus (A \cap B)$ and $B' = B \setminus (A \cap B)$. We then have three mutually disjoint sets A', B', and $A \cap B$. Also note that $A = A' \cup (A \cap B)$, $B = B' \cup (A \cap B)$, and $A \cup B = A' \cup B' \cup (A \cap B)$.

We now show, via induction, that for a finite, nonempty collection of mutually disjoint sets $Y = \{Y_1, Y_2, ..., Y_n\}$, we have $m(\bigcup_{i=1}^n Y_i) = \sum_{i=1}^n m(Y_i)$. Consider the base case where $n = 1$, then $Y = \{Y_1\}$. The left hand side becomes $m(\bigcup_{i=1}^1 Y_i) = m(Y_1)$ and the right hand side is also $\sum_{i=1}^1 m(Y_i) = m(Y_1)$ so the base case is verified.

Now suppose that for some $k \in \mathbb{N}$, $m(\bigcup_{i=1}^k Y_i) = \sum_{i=1}^k m(Y_i)$. Now we prove that the equality holds for $k + 1$. That is, we must show that $m(\bigcup_{i=1}^{k+1} Y_i) = \sum_{i=1}^{k+1} m(Y_i)$. Beginning with the LHS, $m(\bigcup_{i=1}^{k+1} Y_i) = m(Y_{k+1} \cup \bigcup_{i=1}^k Y_i)) = m(Y_{k+1}) + m(\bigcup_{i=1}^k Y_i)$ since Y_{k+1} and $\bigcup_{i=1}^k Y_i$ are disjoint. Then $m(\bigcup_{i=1}^k Y_i) = \sum_{i=1}^k m(Y_i)$ by our inductive assumption and we have $m(\bigcup_{i=1}^{k+1} Y_i) = m(Y_{k+1}) + \sum_{i=1}^k m(Y_i) = \sum_{i=1}^{k+1} m(Y_i)$. So the LHS equals the RHS and the proof by induction is complete.

Now we return to the problem at hand. As noted above, $A \cup B = A' \cup B' \cup (A \cap B)$ so we must have $m(A \cup B) = m(A' \cup B' \cup (A \cap B))$. But A', B', and $A \cap B$ are mutually disjoint sets so $m(A' \cup B' \cup (A \cap B)) = m(A') + m(B') + m(A \cap B)$.

$\square$

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Problem. Show, for finite sets A, B, that $m(A \times B) = m(A)m(B)$.

Proof. Again, we take for granted the fact that if A, B are disjoint sets then we have $m(A \cup B) = m(A) + m(B)$. As an edge case, consider if A, B, or both A and B are the empty set. In this case $A \times B = \emptyset$ and $m(A \times B) = m(\emptyset) = 0$. Also $m(A)m(B) = 0$ because one or both of $m(A), m(B)$ will be 0. Since the statement holds when one or both of A, B is the empty set, we assume both sets are nonempty for the remainder of the proof.

Now we show that for a singleton $S = \{s\}$ and a nonempty set X, we have $m(S \times X) = m(X)$. We show this equality by establishing a bijection between $S \times X$ and X. Consider $f : S \times X \longrightarrow X$ defined by $f((s, x)) = x$. To show that f is a bijection, we must show it is surjective and injective. The map f is surjective if every element of X is the image of some element of $S \times X$ under f. For any $x' \in X$, just choose $(s, x') \in S \times X$, then $f((s, x')) = x'$. So f is surjective, but what about injective? We can show f is injective by proving that $f((s, x_1)) = f((s, x_2))$ implies $(s, x_1) = (s, x_2)$. Suppose we have $f((s, x_1)) = f((s, x_2))$, then it must also be true that $x_1 = x_2$. Since $s = s$ and $x_1 = x_2$, must have $(s, x_1) = (s, x_2)$. Therefore, f is injective. Since f is both injective and surjective, it is a bijection and $m(S \times X) = m(X)$.

We now show, via induction, that for a finite, nonempty collection of mutually disjoint sets $Y = \{Y_1, Y_2, ..., Y_n\}$, we have $m(\bigcup_{i=1}^{n} Y_i) = \sum_{i=1}^{n} m(Y_i)$. Consider the base case where $n = 1$, then $Y = \{Y_1\}$. The left hand-side becomes $m(\bigcup_{i=1}^{1} Y_i) = m(Y_1)$ and the right hand side is $\sum_{i=1}^{1} m(Y_i) = m(Y_1)$ so the base case is verified.

Now suppose that for some $k \in \mathbb{N}$, $m(\bigcup_{i=1}^{k} Y_i) = \sum_{i=1}^{k} m(Y_i)$. Now we prove that the equality holds for $k + 1$.

We now claim that $A \times B$ is such a finite union of sets and that those sets are mutually disjoint.

$\square$

Page 13 — Problem 6

Problem. If $f : S \longrightarrow T$ is onto and $g : T \longrightarrow U$ and $h : T \longrightarrow U$ are such that $g \circ f = h \circ f$, then $g = h$.

Proof. To show that $g = h$, we must show that $g(t) = h(t)$ for all $t \in T$. Since $g \circ f = h \circ f$, we can say that $(g \circ f)(s) = g(f(s)) = (h \circ f)(s) = h(f(s))$ for all $s \in S$. Then since f is onto, for every $t \in T$, there exists $s_t \in S$ such that $f(s_t) = t$. Then $f(s)$ can be replaced with t to give $g(f(s)) = g(t) = h(f(s)) = h(t)$, which proves that $g = h$.

□

Page 20 — Problem 11

Problem. Can you find a positive integer m such that $f^m = i$ for all $f \in S_4$?

Proof.

$\square$

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Problem. Show that there is a positive integer t such that $f^t = i$ for all $f \in S_n$.

Proof.

□