

# M 431: Assignment 4

Nathan Stouffer

## Page 55 — Problem 21

*Problem.* If  $A, B$  are subgroups of  $G$  such that  $b^{-1}Ab \subset A$  for all  $b \in B$ , show that  $AB$  is a subgroup of  $G$ .

*Proof.* Our task is to show that  $AB \leq G$ . Note that  $AB := \{ab \mid a \in A, b \in B\}$ . We are also given the conditions that  $A, B \leq G$  and  $b^{-1}Ab = \{b^{-1}ab \mid a \in A\} \subset A$  for all  $b \in B$ . We get associativity for free and we can quickly deduce that  $e \in AB$  since  $e \in A, B$  and  $e = ee \in AB$ .

We now show that  $AB$  is closed under the group operation in  $G$ . Pick any  $x, y \in AB$  then  $x = a_1b_1$  and  $y = a_2b_2$  for some  $a_1, a_2 \in A$  and  $b_1, b_2 \in B$ . Then  $xy = a_1b_1a_2b_2 = b_1b_1^{-1}a_1b_1a_2b_2$  where the existence of  $b_1^{-1}$  is guaranteed since  $G$  is a group. But then  $b_1^{-1}a_1b_1 \in A$  so let  $b_1^{-1}a_1b_1 = a_3 \in A$ . Then we have  $xy = b_1a_3a_2b_2$ . Setting  $a_4 = a_3a_2$  ( $a_4 \in A$  by closedness), we get  $xy = b_1a_4b_2$ . Again since  $G$  is a group, we can say that  $b_1a_4b_2 = b_1a_4b_1^{-1}b_1b_2 = (b_1^{-1})^{-1}a_4b_1^{-1}b_1b_2$ . Then let  $a = (b_1^{-1})^{-1}a_4b_1^{-1}$  (a member of  $A$  by our assumed property) and  $b = b_1b_2$  (a member of  $B$  by closedness) and then we have  $xy = ab$  where  $a \in A, b \in B$ . Thus  $cd \in AB$ .

As the final step towards proving  $AB \leq G$ , we must verify that every element has an inverse in  $AB$ . Fix any  $x = ab \in AB$ , we wonder if  $x^{-1} = (ab)^{-1} \in AB$ . Certainly  $(ab)^{-1} \in G$  and  $(ab)^{-1} = b^{-1}a^{-1} = b^{-1}a^{-1}bb^{-1}$  by properties of  $G$ . But then we know  $b^{-1}a^{-1}b \in A$  (since  $a^{-1}$  is an element of  $A$  so the property applies). Letting  $b^{-1}a^{-1}b = a' \in A$  we have  $x^{-1} = a'b^{-1}$  which must be a member of  $AB$  since  $a' \in A$  and  $b^{-1} \in B$  (since  $B$  is subgroup). Therefore,  $AB$  is a subgroup of  $G$ .

□

## Page 65 — Problem 19

*Problem.* Find all the distinct conjugacy classes of  $S_3$ .

*Proof.* Our goal is to find all the distinct conjugacy classes of  $S_3$ . Recall that  $S_3 = \{id, (12), (13), (23), (123), (321)\}$  and note that  $(123)^{-1} = (321)$  (see Figure 1 for proof). Given  $f, g \in S_3$  we have  $f \sim g$  ( $f$  and  $g$  are conjugate) if there exists some  $h \in G$  such that  $g = h^{-1}fh$ . The textbook also notes that  $\sim$  is an equivalence relation.

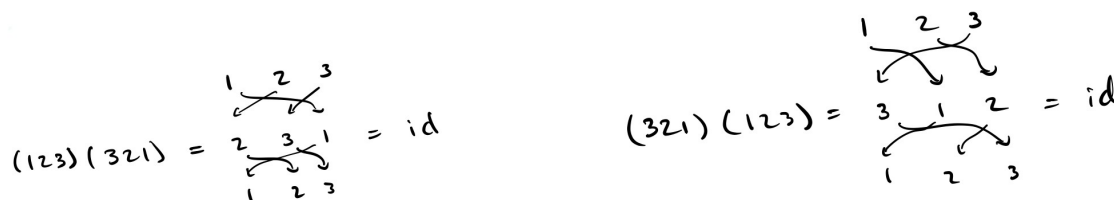


Figure 1: Showing  $(123)^{-1} = (321)$

First, we show that  $\{id\}$  is a conjugacy class. Since  $\sim$  is an equivalence relation, we necessarily have  $id \sim id$ . Now pick  $f$  a member of the conjugacy class of  $id$ . Then  $id = g^{-1}fg$  for some  $g \in S_3$ . Left multiplying by  $g$  and right multiplying by  $g^{-1}$ , we obtain  $f = id$ , therefore the conjugacy class of  $id$  is the singleton  $\{id\}$ .

With the definition for  $\sim$ , certainly  $(123)$  and  $(321)$  are in the same conjugacy class (Figure 2 and the fact that  $\sim$  is symmetric). Additionally, the maps  $(12)$ ,  $(13)$ , and  $(23)$  are in the same conjugacy class (Figure 3 and the symmetric/transitive properties of equivalence relations).

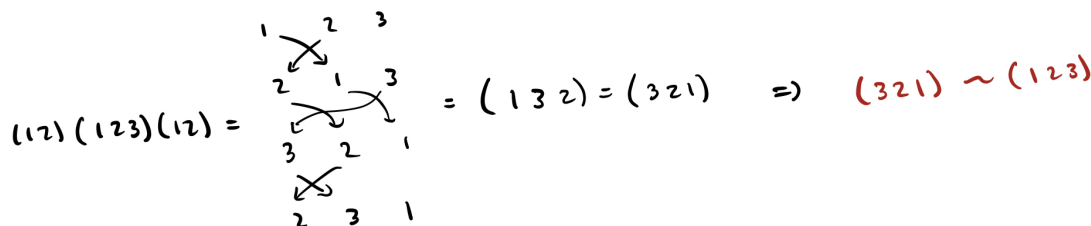


Figure 2: Showing  $(123) \sim (321)$

$$(123)(12)(321) = \begin{array}{c} 1 \quad 2 \quad 3 \\ \swarrow \quad \searrow \quad \swarrow \\ 2 \quad 3 \quad 1 \\ \swarrow \quad \searrow \quad \swarrow \\ 3 \quad 2 \quad 1 \\ \swarrow \quad \searrow \quad \swarrow \\ 1 \quad 3 \quad 2 \end{array} = (23) \Rightarrow (23) \sim (12)$$

$$(123)(13)(321) = \begin{array}{c} 1 \quad 2 \quad 3 \\ \swarrow \quad \searrow \quad \swarrow \\ 2 \quad 3 \quad 1 \\ \swarrow \quad \searrow \quad \swarrow \\ 1 \quad 3 \quad 2 \\ \swarrow \quad \searrow \quad \swarrow \\ 2 \quad 1 \quad 3 \end{array} = (12) \Rightarrow (12) \sim (13)$$

Figure 3: Showing  $(12) \sim (13) \sim (23)$

We now show that the conjugacy classes of  $S_3$  are  $A_1 = \{id\}$ ,  $A_2 = \{(12), (13), (23)\}$ , and  $A_3 = \{(123), (321)\}$ . We already know  $\{id\}$  to be its own conjugacy class. So we only need show that  $A_2, A_3$  are not part of the same conjugacy class. Since  $\sim$  is an equivalence relation, it suffices to show that a single pair of elements from  $A_2, A_3$  are not in the same conjugacy class. Can we show  $(12)$  and  $(123)$  are not in the same conjugacy class? Suppose  $(12) \sim (123)$ , then there exists  $h \in G$  such that  $(12) = h^{-1}(123)h$ . Then it must also be true that  $(12)^3 = (h^{-1}(123)h)^3$ . Simplifying,  $(12)^3 = (12)$  and  $(h^{-1}(123)h)^3 = h^{-1}(123)^3h = h^{-1}idh = h^{-1}h = id$ . But  $(12) \neq id$  so we have reached a contradiction and the elements  $(12)$  and  $(123)$  cannot be part of the same conjugacy class. Then their conjugacy classes must be distinct (since  $\sim$  is an equivalence relation). Therefore the conjugacy classes of  $S_3$  are  $A_1 = \{id\}$ ,  $A_2 = \{(12), (13), (23)\}$ , and  $A_3 = \{(123), (321)\}$ .

□

## Page 65 — Problem 21

*Problem.* Let  $G$  be the dihedral group of order 8. Find the conjugacy classes in  $G$ .

*Proof.*

□

## Heisenberg group problem

*Problem.* Find the center of our new friend, the Heisenberg group,

$$\mathbb{H}_3(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

*Proof.* We wish to find the center of  $\mathbb{H}_3(\mathbb{R})$ :  $Z(\mathbb{H}_3(\mathbb{R})) = \{A \in \mathbb{H}_3(\mathbb{R}) \mid AB = BA \text{ for all } B \in \mathbb{H}_3(\mathbb{R})\}$ . Note that  $I_3 \in \mathbb{H}_3(\mathbb{R})$  so  $\mathbb{H}_3(\mathbb{R})$  is nonempty. We first give a necessary condition for  $A \in Z(\mathbb{H}_3(\mathbb{R}))$ . Fix  $A \in Z(\mathbb{H}_3(\mathbb{R}))$  and let  $B \in \mathbb{H}_3(\mathbb{R})$ , then we have  $AB = BA$ :

$$AB = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' + x & z' + xy' + z \\ 0 & 1 & y' + y \\ 0 & 0 & 1 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x + x' & z + x'y + z' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{bmatrix}$$

Since  $AB = BA$ , their elements must match. From this, we gather that  $xy' = x'y$  for all  $x', y' \in \mathbb{R}$ . This can only be true if  $x = y = 0$  for  $A \in Z(\mathbb{H}_3(\mathbb{R}))$ . We now show that  $x = y = 0$  is a sufficient condition for membership in

□

## Crayon the clock

*Problem.* Let  $G$  be equal to  $\mathbb{Z}_{15} := \{0, 1, 2, \dots, 14\}$ . Draw it some way. Find a subgroup  $H$  of order  $|H| = 5$  and then color differently all the different subsets of  $\mathbb{Z}_{15}$  of the form  $aH$ . (How many are there?) If you have more crayons, do another drawing for an  $H$  with  $|H| = 3$ .

*Proof.*

□