M 431: Assignment 4

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Problem. If A, B are subgroups of G such that $b^{-1}Ab \subset A$ for all $b \in B$, show that AB is a subgroup of G.

Proof. Our task is to show that $AB \leq G$. Note that $AB := \{ab \mid a \in A, b \in B\}$. We are also given the conditions that $A, B \leq G$ and $b^{-1}Ab = \{b^{-1}ab \mid a \in A\} \subset A$ for all $b \in B$. We get associativity for free and we can quickly deduce that $e \in AB$ since $e \in A, B$ and $e = ee \in AB$.

We now show that AB is closed under the group operation in G. Pick any $x, y \in AB$ then $x = a_1b_1$ and $y = a_2b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $xy = a_1b_1a_2b_2 = b_1b_1^{-1}a_1b_1a_2b_2$ where the existence of b_1^{-1} is guaranteed since G is a group. But then $b_1^{-1}a_1b_1 \in A$ so let $b_1^{-1}a_1b_1 = a_3 \in A$. Then we have $xy = b_1a_3a_2b_2$. Setting $a_4 = a_3a_2$ ($a_4 \in A$ by closedness), we get $xy = b_1a_4b_2$. Again since G is a group, we can say that $b_1a_4b_2 = b_1a_4b_1^{-1}b_1b_2 = (b_1^{-1})^{-1}a_4b_1^{-1}b_1b_2$. Then let $a = (b_1^{-1})^{-1}a_4b_1^{-1}$ (a member of A by our assumed property) and $b = b_1b_2$ (a member of B by closedness) and then we have $xy = ab$ where $a \in A, b \in B$. Thus $cd \in AB$.

As the final step towards proving $AB \leq G$, we must verify that every element has an inverse in AB. Fix any $x = ab \in AB$, we wonder if $x^{-1} = (ab)^{-1} \in AB$. Certainly $(ab)^{-1} \in G$ and $(ab)^{-1} = b^{-1}a^{-1} = b^{-1}a^{-1}bb^{-1}$ by properties of G. But then we know $b^{-1}a^{-1}b \in A$ (since a^{-1} is an element of A so the property applies). Letting $b^{-1}a^{-1}b = a' \in A$ we have $x^{-1} = a'b^{-1}$ which must be a member of AB since $a' \in A$ and $b^{-1} \in B$ (since B is subgroup). Therefore, AB is a subgroup of G.

$\square$

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Problem. Find all the distinct conjugacy classes of S_3.

Proof. Our goal is to find all the distinct conjugacy classes of S_3. Recall that $S_3 = \{id, (12), (13), (23), (123), (321)\}$ and note that $(123)^{-1} = (321)$ (see Figure 1 for proof). Given $f, g \in S_3$ we have $f \sim g$ (f and g are conjugate) if there exists some $h \in G$ such that $g = h^{-1}fh$. The textbook also notes that $\sim$ is an equivalence relation.

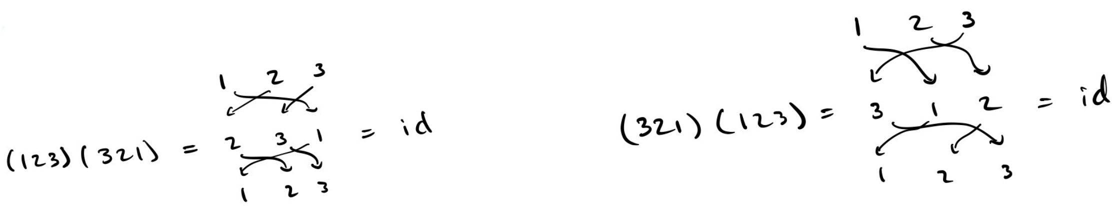

Figure 1: Showing $(123)^{-1} = (321)$

First, we show that $\{id\}$ is a conjugacy class. Since $\sim$ is an equivalence relation, we necessarily have $id \sim id$. Now pick f a member of the conjugacy class of id. Then $id = g^{-1}fg$ for some $g \in S_3$. Left multiplying by g and right multiplying by g^{-1}, we obtain $f = id$, therefore the conjugacy class of id is the singleton $\{id\}$.

With the definition for $\sim$, certainly (123) and (321) are in the same conjugacy class (Figure 2 and the fact that $\sim$ is symmetric). Additionally, the maps $(12), (13)$, and (23) are in the same conjugacy class (Figure 3 and the symmetric/transitive properties of equivalence relations).

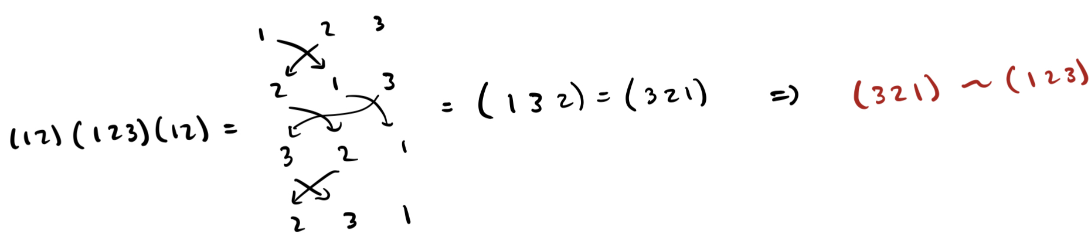

Figure 2: Showing $(123) \sim (321)$

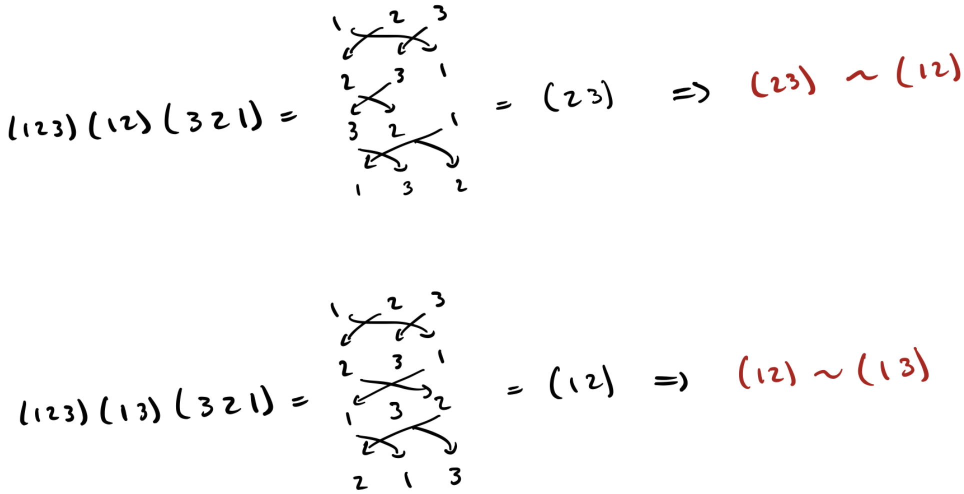

Figure 3: Showing $(12) \sim (13) \sim (23)$

We now show that the conjugacy classes of S_3 are $A_1 = \{id\}$, $A_2 = \{(12), (13), (23)\}$, and $A_3 = \{(123), (321)\}$. We already know $\{id\}$ to be its own conjugacy class. So we only need show that A_2, A_3 are not part of the same conjugacy class. Since $\sim$ is an equivalence relation, it suffices to show that a single pair of elements from A_2, A_3 are not in the same conjugacy class. Can we show (12) and (123) are not in the same conjugacy class? Suppose $(12) \sim (123)$, then there exists $h \in G$ such that $(12) = h^{-1}(123)h$. Then it must also be true that $(12)^3 = (h^{-1}(123)h)^3$. Simplifying, $(12)^3 = (12)$ and $(h^{-1}(123)h)^3 = h^{-1}(123)^3 h = h^{-1} id h = h^{-1}h = id$. But $(12) \neq id$ so we have reached a contradiction and the elements (12) and (123) cannot be part of the same conjugacy class. Then their conjugacy classes must be distinct (since $\sim$ is an equivalence relation). Therefore the conjugacy classes o S_3 are $A_1 = \{id\}$, $A_2 = \{(12), (13), (23)\}$, and $A_3 = \{(123), (321)\}$.

□

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Problem. Let G be the dihedral group of order 8. Find the conjugacy classes in G.

Proof.

$\square$

Heisenberg group problem

Problem. Find the center of our new friend, the Heisenberg group,

$$\mathbb{H}_3(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

Proof. We wish to find the center of $\mathbb{H}_3(\mathbb{R})$: $Z(\mathbb{H}_3(\mathbb{R})) = \{A \in \mathbb{H}_3(\mathbb{R}) \mid AB = BA \text{ for all } B \in \mathbb{H}_3(\mathbb{R})\}$. Note that $I_3 \in \mathbb{H}_3(\mathbb{R})$ so $\mathbb{H}_3(\mathbb{R})$ is nonempty. We first give a necessary condition for $A \in Z(\mathbb{H}_3(\mathbb{R}))$. Fix $A \in Z(\mathbb{H}_3(\mathbb{R}))$ and let $B \in \mathbb{H}_3(\mathbb{R})$, then we have $AB = BA$:

$$AB = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' + x & z' + xy' + z \\ 0 & 1 & y' + y \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x + x' & z + x'y + z' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{bmatrix}$$

Since $AB = BA$, their elements must match. From this, we gather that $xy' = x'y$ for all $x', y' \in \mathbb{R}$. This can only be true if $x = y = 0$ for $A \in Z(\mathbb{H}_3(\mathbb{R}))$. We now show that $x = y = 0$ is a sufficient condition for membership in $Z(\mathbb{H}_3(\mathbb{R}))$. Pick such a matrix A and then choose $B \in \mathbb{H}_3(\mathbb{R})$. Then we have

$$AB = \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' & z' + z \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = BA$$

Since B was arbitrary in $\mathbb{H}_3(\mathbb{R})$ we must have A in the center of $\mathbb{H}_3(\mathbb{R})$. Therefore $x = y = 0$ is a necessary and sufficient condition for members of $Z(\mathbb{H}_3(\mathbb{R}))$. Thus we have

$$Z(\mathbb{H}_3(\mathbb{R})) = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid z \in \mathbb{R} \right\}$$

$\square$

Crayon the clock

Problem. Let G be equal to $\mathbb{Z}_{15} := \{0, 1, 2, ..., 14\}$. Draw it some way. Find a subgroup H of order $|H| = 5$ and then color differently all the different subsets of $\mathbb{Z}_{15}$ of the form aH. (How many are there?) If you have more crayons, do another drawing for an H with $|H| = 3$.

Proof. Here is a figure with my subgroups.

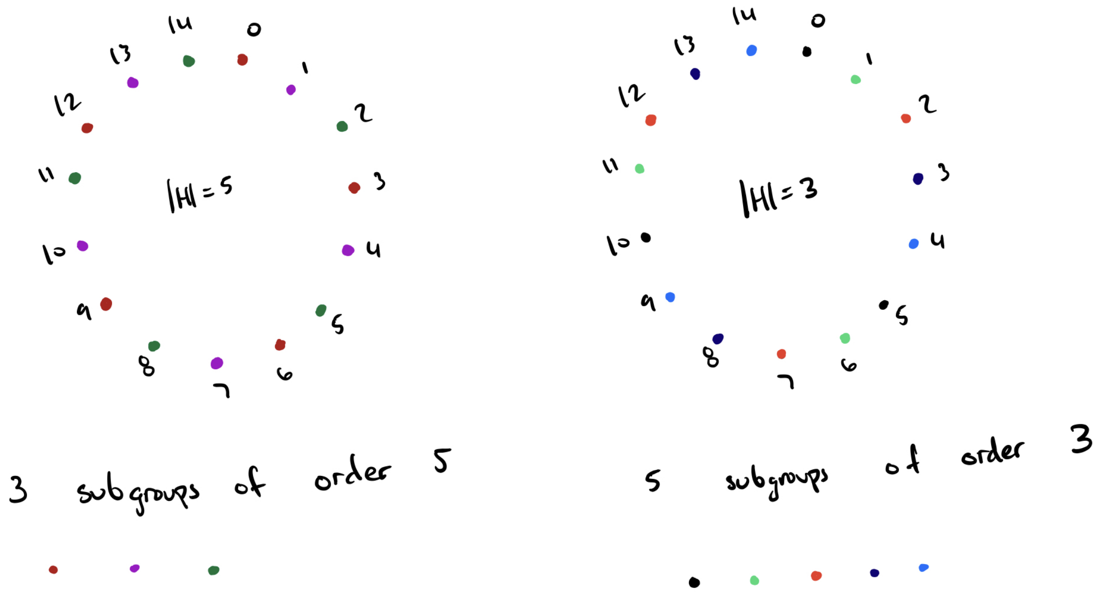

Figure 4: Subgroups of Z_{15}

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