# M 431: Assignment 8

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## Page 87 — Problem 2

*Problem.* Let $G$ be the group of all real-valued functions on the unit interval $[0,1]$, where we define, for $f,g \in G$, addition by $(f+g)(x) = f(x)+g(x)$ for every $x \in [0,1]$. If $N = \{f \in G \mid f(1/4) = 0\}$, prove that $G/N \cong$ the real numbers under $+$.

*Proof.* Note that our ambient group $G$ is abelian so any subgroup is a normal subgroup. We first claim that $N \leq G$ by showin the aesthetic definition holds for $N$. Certainly $id \in N$ for $id(1/4) = 0$. Now pick $f, g \in N$ and recall that $g^{-1} = -g$. Then $f + g^{-1} = f + (-g)$ and now evaluating at $x = 1/4$: $(f + (-g))(1/4) = f(1/4) + (-g(1/4)) = 0 + -0 = 0$ so $f + g^{-1} \in N$.

Now the quotient group $G/N = \{g + N \mid g \in G\}$ and $(g+N)(1/4) = g(1/4) + N(1/4) = g(1/4) + 0 = g(1/4)$. Let's define the mapping $\psi : G/N \longrightarrow \mathbb{R}$ which takes $g+N$ to $g(1/4)$. We must verify four things for $\psi$ to be an isomorphism: well-defined, homomorphism, onto, and 1-1.

Well-defined: pick $g + N = \bar{g} + N$. Then there exists some $f \in N$ such that $g = \bar{g} + f$ which implies that $\bar{g} = g + f$. Now consider $\psi(\bar{g} + N) = \psi(g + f + N) = (g + f)(1/4) = g(1/4) + f(1/4) = g(1/4) = \psi(g + N)$ where we know $f(1/4) = 0$ since $f \in N$.

Homomorphism: fix $g_1+N, g_2+N \in G/N$. Certainly we have $\psi(g_1+N)+\psi(g_2+N) = g_1(1/4)+g_2(1/4)$. But we also have $\psi(g_1 + N + g_2 + N) = \psi(g_1 + g_2 + N) = (g_1 + g_2)(1/4) = g_1(1/4) + g_2(1/4)$ so $\psi$ is a homomorphism.

Onto: this is verified quite easily. If you give me an $x^\star \in \mathbb{R}$, I will give you the function $f(x) = x^\star$ for $x \in [0,1]$. Certainly $\psi(f + N) = f(1/4) = x^\star$.

1-1: fix any $f_1, f_2 \in G$ such that $\psi(f_1 + N) = \psi(f_2 + N)$. We wish to show that $f_1 + N = f_2 + N$ which is true if there exists some $n \in N$ such that $f_1 = f_2 + n$. We will provide such a function $n(x)$. Note that $\psi(f_1 + N) = \psi(f_2 + N) \implies f_1(1/4) = f_2(1/4)$. Here is our function defined on $[0,1]$:

$$n(x) = \begin{cases} 0 & x = 1/4 \\ f_1(x) - f_2(x) & x \neq 1/4 \end{cases}$$

Now consider $f_2(x) + n(x)$. If $x = 1/4$ then $f_2(1/4) + n(1/4) = f_2(1/4) + 0 = f_1(1/4)$. Then if $x \neq 1/4$ we have $f_2(x) + n(x) = f_2(x) + f_1(x) - f_2(x) = f_1(x)$. Thus $f_1(x) = f_2(x) + n(x)$ for all $x \in [0,1]$ and we have $f_1 = f_2 + n$ which implies that the cosets $f_1 + N$ and $f_2 + N$ are equal. Thus $\psi$ is 1-1.

We have check everything we need to for $\psi$ to be an isomorphism so the two groups are isomorphic!

$\square$

## 3rd Iso Thm Example

*Problem.* Identify and illustrate with pictures the three quotient groups in the 3rd isomorphism theorem instantiated for $G = \mathbb{R} \times \mathbb{Z}_2$, $N = \mathbb{Z} \times \mathbb{Z}_2$, and $K = 2\mathbb{Z} \times \{0\}$.

*Proof.*

$\square$

## 2rd Iso Thm Example

*Problem.* Identify and illustrate with pictures all the groups involved in the 2nd isomorhpism theorem isntantiated for $G = \mathbb{R} \times \mathbb{R}$, $N = \mathbb{Z} \times \mathbb{R}$, $H = \mathbb{R} \times \{0\}$. In particular, draw the cosets making up the quotient groups and recognize the groups as familiar concrete groups.

*Proof.*

$\square$

## Page 96 — Problem 5

*Problem.* Let $G$ be a finite group, $N_1, N_2, ..., N_k$ normal subgroups of $G$ such that $G = N_1 N_2 \cdots N_k$ and $|G| = |N_1||N_2| \cdots |N_k|$. Prove that $G$ is the direct product of $N_1, N_2, \ldots, N_k$.

*Proof.*

$\square$

## Page 96 — Problem 6

*Problem.* Let $G$ be a group, $N_1, N_2, \ldots, N_k$ normal subgroups of $G$ such that:
(1) $G = N_1 N_2 \cdots N_k$
(2) For each $i$, $N_i \cap (N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_k) = (e)$
Prove that $G$ is the direct product of $N_1, N_2, \ldots, N_k$

*Proof.* We begin with a lemma that is a consequence of property (2). Given a fixed $N_i$, let $H_i$ be the product of any selection of $\bar{N}_i = N_1, ..., N_{i-1}, N_{i+1}, ..., N_k$. We claim that $N_i \cap H_i = (e)$. Here is the proof. Certainly $e \in H_i$ and $e \in N_i$ so $e \in N_i \cap H_i$. From property (2), we know that $N$ intersect all the other normal subgroups $N_j$ is $(e)$. Further, we know $H_i$ must be a subset of $N_1, ..., N_{i-1}, N_{i+1}, ..., N_k$ for every element of $H_i$ can be written as an element of $\bar{N}_i$ with $e$ chosen as the element for sets that were not selected for $H_i$. Thus we are just constricting one of the sets involved in an intersection (and we certainly keep the elements that are already in the intersection) so $N_i \cap H_i = (e)$.

Now let's proceed by induction. For a base case, take $K_2 = N_1 N_2$. By the lemma just proven, we have $N_1 \cap N_2 = (e)$ and the corollary on page 95 of the textbook tells us that $K_2$ is the internal product of $N_1$ and $N_2$. Futhermore, $K_2 \trianglelefteq G$ since $N_1, N_2 \trianglelefteq G$. Now supppose for some $j \in \{2, 3, ..., k-1\}$ that $K_j$ is the direct product $K_j = N_1 N_2 \cdots N_j$. We will show that $K_{j+1} = N_1 N_2 \cdots N_j N_{j+1}$ is also a direct product. The lemma tells us that $K_j \cap N_{j+1} = (e)$ and we can apply the corollary again to say that $K_{j+1}$ is the direct product of $N_1 N_2 \cdots N_j N_{j+1}$.

Thus, by induction, every $K_j$ for $j \in \{2, 3, ..., k\}$ is the direct product of $N_1 N_2 \cdots N_j$. This implies that $G = N_1 N_2 \cdots N_k$ is a direct product as well!

□