M 431: Assignment 11

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Page 146 — Problem 3

Problem. If $\varphi : R \longrightarrow R'$ is a homomorphism of R onto R' adn R has a unit element, 1, show that $\varphi(1)$ is the unit element of R'.

Proof. To show that $\varphi(1)$ is a unit element of R' we must show that $\varphi(1)r' = r'\varphi(1) = r'$ for all $r' \in R'$. Fix an $r' \in R'$. Since φ is onto, there exists some $r \in R$ such that $\varphi(r) = r'$. Since 1 is a unit element in R we certainly have $1r = r1 = r$. Then we can take φ of each of $1r, r1, r$ and use the fact that φ is a homomorphism to say that

$$\begin{aligned} 1r = r1 &= r \\ \varphi(1r) = \varphi(r1) &= \varphi(r) \\ \varphi(1)\varphi(r) = \varphi(r)\varphi(1) &= \varphi(r) \\ \varphi(1)r' = r'\varphi(1) &= r' \end{aligned}$$

$\square$

Page 146 — Problem 4

Problem. If I, J are ideals of R, definte $I + J$ by $I + J = \{i + j \mid i \in I, j \in J\}$. Prove that $I + J$ is an ideal of R.

Proof. We wish to show that $K = I + J$ is an ideal. First we check that K is nonempty. Since I, J are both nonempty, we can sum elements from them so $I + J = K \neq \emptyset$.

Now we verify that K is an additive subgroup. We use the aesthetic definition so we must show that for any $k, \bar{k} \in K$ that $k - \bar{k} \in K$. Because of their memebership in $K = I + J$, we know that $k = i + j$ and $\bar{k} = \bar{i} + \bar{j}$ for some $i, \bar{i} \in I$ and $j, \bar{j} \in J$. Then $k - \bar{k} = (i + j) - (\bar{i} + \bar{j}) = i + j - \bar{i} - \bar{j} = (i - \bar{i}) + (j - \bar{j}) \in I + J = K$ since I, J are ideals of R.

We know just need to show that $rk, kr \in K$ for all $r \in R$ (we take both I, J to be bi-ideals of R). Consider $rk = r(i + j) = ri + rj = i' + j' \in I + J = K$ and $kr = (i + j)r = ir + ij = i'' + j'' \in I + J = K$ which shows that the multiplicitave requirement of being an ideal is satisfied.

$\square$

Page 146 — Problem 13

Problem. In Example 6, show that $R/I_p \cong H(\mathbb{Z}_p)$.

Proof. Here we have

$$R := \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_i \in \mathbb{Z}\, i = 0,1,2,3\}$$
$$I_p := \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid p \mid \alpha_i\, i = 0,1,2,3\}$$
$$H(\mathbb{Z}_p) := \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_i \in \mathbb{Z}_p\, i = 0,1,2,3\}$$

We will use the first isomorphism theorem to prove the goal. Consider $\varphi : R \longrightarrow H(\mathbb{Z}_p)$ which maps $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mapsto \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k$ where $\mathbb{Z}_p \ni \beta_i = \alpha_i \mod p$. We now investigate whether φ is onto/homomorphism and the kernel φ. That φ is onto is simple: for an element in $H(\mathbb{Z}_p)$ just take the element in R with the exact same coefficients. That φ is a homomorphism holds by the properties of integers modulo p.

Now we check out the kernel of φ. We claim that $\ker\varphi = I_p$. Going to the right, pick $\alpha \in I_p$ then $p \mid \alpha_i \iff \alpha_i \equiv 0 \mod p \implies \varphi(\alpha_i) = 0$ for all $i = 0,1,2,3$ so $\alpha \in \ker\varphi$. Now going to the left, pick any $\alpha \in \ker\varphi$, this implies that $\varphi(\alpha) = 0 \implies \alpha_i \equiv 0 \mod p \implies p \mid \alpha_i$ so α is a member of I_p.

Thus we have provided a suitable epimorphism to apply the first isomorphism thereom, which means that $R/I_p \cong H(\mathbb{Z}_p)$.

$\square$

Baby version of Opt 333

Problem. Let $V = span\left(\begin{bmatrix}1\\1\end{bmatrix}\right) \subset \mathbb{R}^2$. In the ring $R = M_{2\times 2}(\mathbb{R})$, consider $J_V := \{A \in R \mid V \subset N(A)\}$. Show that J_V is a principal ideal, that is, $J_V = (A)$ for some $A \in R$.

Proof. Let $A = \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}$. Let's investigate whether $J_V = (A)$. Going to the left, pick $B \in J_V$ then $V \subset N(B)$ which means that $B\begin{bmatrix}1\\1\end{bmatrix} = 0$. So we must have

$$B\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}b_1 & b_2\\b_3 & b_4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}b_1 + b_2\\b_3 + b_4\end{bmatrix} = 0$$

which means that $b_2 = -b_1$ and $b_4 = -b_3$. Then we can rewrite B as

$$B = \begin{bmatrix}b_1 & -b_1\\b_3 & -b_3\end{bmatrix} = \frac{1}{2}\begin{bmatrix}b_1 & b_1\\b_3 & b_3\end{bmatrix}\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}$$

and since $\frac{1}{2}\begin{bmatrix}b_1 & b_1\\b_3 & b_3\end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ we know that $B \in (A)$.

Now going to the right, fix any $B \in (A)$ then there exists some matrix $D \in M_{2\times 2}(\mathbb{R})$ such that $B = DA$. Let's check if $V \subset N(B)$. This is the case if $B\begin{bmatrix}1\\1\end{bmatrix} = 0$:

$$B\begin{bmatrix}1\\1\end{bmatrix} = DA\begin{bmatrix}1\\1\end{bmatrix} = D\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = D\begin{bmatrix}0\\0\end{bmatrix} = 0$$

So B is also a member of J_V. So we have showed the inclusion both directions and $J_V = (A)$ which means J_V is a principal ideal.

$\square$

Simple version of Opt 383

Problem. Let $R = C([0,1])$ and $x_0 \in [0,1]$. Show that $I_{x_0} := \{f \in R \mid f(x_0) = 0\}$ is not a principal ideal.

Proof. Suppose that I_{x_0} is a principal ideal. Then there is some $f \in I_{x_0}$ such that $I_{x_0} = (f)$. What can we deduce about f? We give a quick proof that $f^{-1}(0) = \{x_0\}$. That $x_0 \in f^{-1}(0)$ is assumed so suppose there is some $y \neq x_0$ in the preimage of 0 under f. Then $g * f(y_0) = 0$ for all g. Yet there exist functions h in I_{x_0} where $h(y_0) \neq 0$ so (f) would not equal I_{x_0}. Thus the preimage is merely the singleton $\{x_0\}$.

Fix any $h \in I_{x_0}$, then since I_{x_0} is a principal ideal there exists some $g \in R$ such that $h = g*f$. Since x_0 is the only input for which f vanishes we must have $g(x) = h(x)/f(x)$ for all $x \neq x_0$. Since $g \in R = C([0,1])$ (which means that $\lim_{x \longrightarrow x_0} g(x)$ exists) we know that $\lim_{x \longrightarrow x_0} \frac{h(x)}{f(x)}$ exists. But this not necessarily the case for any $h(x)$. So we have found a contradiction and we can say that I_{x_0} is not a principal ideal.

$\square$