M 431: Assignment 1

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Page 7 — Problem 14

Problem. If C is a finite set, let $m(C)$ denote the number of elements in C. If A, B are finite sets, prove that

$$m(A \cup B) = m(A) + m(B) - m(A \cap B)$$

Proof. We take for granted the fact that $m(A \cup B) = m(A) + m(B)$ if A, B are disjoint sets. Now let's establish some notation. Let $A' = A \setminus (A \cap B)$ and $B' = B \setminus (A \cap B)$.

We now show a quick proof of the fact that $m(A \cap B)$ is finite. By definition of intersection, $A \cap B \subset A$. But then A cannot contain fewer elements than $A \cap B$, so we must have $m(A) \geq m(A \cap B)$. Then since A is finite, $A \cap B$ is also finite.

We also show that $A' \cap B' = \emptyset$. $A' = A \setminus (A \cap B)$ and $B' = B \setminus (A \cap B)$ so

$$\begin{aligned}
A' \cap B' &= \{x \mid x \in A' \text{ and } x \in B'\} \\
&= \{x \mid x \in A \setminus (A \cap B) \text{ and } x \in B \setminus (A \cap B)\} \\
&= \{x \mid (x \in A \text{ and } x \notin A \cap B) \text{ and } (x \in B \text{ and } x \notin A \cap B)\} \\
&= \{x \mid x \in A \text{ and } x \in B \text{ and } x \notin A \cap B\} \\
&= \{x \mid x \in A \cap B \text{ and } x \notin A \cap B\} \\
&= \emptyset
\end{aligned}$$

So $A' \cap B' = \emptyset$.

We now show, via induction, that given $k \in \{0, 1, 2, ..., m(A \cap B)\}$, every subset $C \subset A \cap B$ such that $m(C) = k$ satisfies $m(A' \cup B' \cup C) = m(A' \cup C) + m(B' \cup C) - m(C)$. We will denote a subset of $A \cap B$ containing k elements as C_k. Consider the base case where $k = 0$. Then there is only one C_k and it must be $C_k = \emptyset$. So we must show that $m(A' \cup B' \cup \emptyset) = m(A' \cup \emptyset) + m(B' \cup \emptyset) - m(\emptyset)$. But then

$$\begin{aligned}
m(A' \cup B' \cup \emptyset) &= m(A' \cup \emptyset) + m(B' \cup \emptyset) - m(\emptyset) \\
m(A' \cup B') &= m(A') + m(B') - 0 \\
m(A' \cup B') &= m(A') + m(B')
\end{aligned}$$

which we already know to be true since A' and B' are disjoint sets. So the base case is proved.

Now suppose that for some $i \in \{0, 1, 2, ..., m(A \cap B) - 1\}$, that we have $m(A' \cup B' \cup C_i) = m(A' \cup C_i) + m(B' \cup C_i) - m(C_i)$ for all $C_i \subset A \cap B$. We must now show that $m(A' \cup B' \cup C_{i+1}) = m(A' \cup C_{i+1}) + m(B' \cup C_{i+1}) - m(C_{i+1})$ for all $C_{i+1} \subset A \cap B$.

For every i, we C_{i+1} contains one more element than C_i so we must have $m(C_{i+1}) = m(C_i) + 1 \iff m(C_i) = m(C_{i+1}) - 1$. Furthermore, we constructed A', B' so that for any k, $A' \cap C_k = B' \cap C_k = (A' \cup B') \cap C_k = \emptyset$. Then we can use our fact about disjoint sets to say that

$$\begin{aligned}
m(A' \cup B' \cup C_i) &= m(A' \cup C_i) + m(B' \cup C_i) - m(C_i) \\
m(A' \cup B') + m(C_i) &= m(A') + m(C_i) + m(B') + m(C_i) - m(C_i) \\
m(A' \cup B') + m(C_{i+1}) - 1 &= m(A') + m(C_{i+1}) - 1 + m(B') + m(C_{i+1}) - 1 - (m(C_{i+1}) - 1) \\
m(A' \cup B' \cup C_{i+1}) - 1 &= m(A' \cup C_{i+1}) - 1 + m(B' \cup C_{i+1}) - 1 - m(C_{i+1}) + 1
\end{aligned}$$

Then the ones cancel, leaving $m(A' \cup B' \cup C_{i+1}) = m(A' \cup C_{i+1}) + m(B' \cup C_{i+1}) - m(C_{i+1})$, which completes the induction.

So now we know that given $k \in \{0, 1, 2, ..., m(A \cap B)\}$, every $C_k \subset A \cap B$ must satisfy $m(A' \cup B' \cup C_k) = m(A' \cup C_k) + m(B' \cup C_k) - m(C_k)$. Take $k = m(A \cap B)$, then there is only one C_k and it must be $C_k = A \cap B$. So we must have $m(A' \cup B' \cup (A \cap B)) = m(A' \cup (A \cap B)) + m(B' \cup (A \cap B)) - m(A \cap B)$. But $A' \cup B' \cup (A \cap B) = A \cup B$, $A' \cup (A \cap B) = A$, and $B' \cup (A \cap B) = B$, so we have $m(A \cup B) = m(A) + m(B) - m(A \cap B)$, which is what needed to be shown.

$\square$

Page 7 — Problem 20

Problem. Show, for finite sets A, B, that $m(A \times B) = m(A)m(B)$.

Proof. Again, we take for granted the fact that if A, B are disjoint sets then we have $m(A \cup B) = m(A) + m(B)$. As an edge case, consider if A, B, or both A and B are the empty set. In this case $A \times B = \emptyset$ and $m(A \times B) = m(\emptyset) = 0$. Also $m(A)m(B) = 0$ because one or both of $m(A), m(B)$ will be 0. Since the statement holds when one or both of A, B is the emptyset, we assume both sets are nonempty for the remainder of the proof.

Now we show that for a singleton $S = \{s\}$ and a nonempty set X, we have $m(S \times X) = m(X)$. We show this equality by establishing a bijection between $S \times X$ and X. Consider $f : S \times X \longrightarrow X$ defined by $f((s, x)) = x$. To show that f is a bijection, we must show it is surjective and injective. The map f is surjective if every element of X is the image of some element of $S \times X$ under f. For any $x' \in X$, just choose $(s, x') \in S \times X$, then $f((s, x')) = x'$. So f is surjective, but what about injective? We can show f is injective by proving that $f((s, x_1)) = f((s, x_2))$ implies $(s, x_1) = (s, x_2)$. Suppose we have $f((s, x_1)) = f((s, x_2))$, then it must also be true that $x_1 = x_2$. Since $s = s$ and $x_1 = x_2$, must have $(s, x_1) = (s, x_2)$. Therefore, f is injective. Since f is both injective and surjective, it is a bijection and $m(S \times X) = m(X)$.

We now show, via induction, that for a finite, nonempty collection of mutually disjoint sets $Y = \{Y_1, Y_2, ..., Y_n\}$, we have $m(\bigcup_{i=1}^{n} Y_i) = \sum_{i=1}^{n} m(Y_i)$. Consider the base case where $n = 1$, then $Y = \{Y_1\}$. The left hand-side becomes $m(\bigcup_{i=1}^{1} Y_i) = m(Y_1)$ and the right hand side is $\sum_{i=1}^{1} m(Y_i) = m(Y_1)$ so the base case is verified.

Now suppose that for some $k \in \mathbb{N}$, $m(\bigcup_{i=1}^{k} Y_i) = \sum_{i=1}^{k} m(Y_i)$. Now we prove that the equality holds for $k + 1$.

We now claim that $A \times B$ is such a finite union of sets and that those sets are mutually disjoint.

$\square$

Page 13 — Problem 6

Problem. If $f : S \longrightarrow T$ is onto and $g : T \longrightarrow U$ and $h : T \longrightarrow U$ are such that $g \circ f = h \circ f$, then $g = h$.

Proof. To show that $g = h$, we must show that $g(t) = h(t)$ for all $t \in T$. Since $g \circ f = h \circ f$, we can say that $(g \circ f)(s) = g(f(s)) = (h \circ f)(s) = h(f(s))$ for all $s \in S$. Then since f is onto, for every $t \in T$, there exists $s_t \in S$ such that $f(s_t) = t$. Then $f(s)$ can be replaced with t to give $g(f(s)) = g(t) = h(f(s)) = h(t)$, which proves that $g = h$.

□

Page 20 — Problem 11

Problem. Can you find a positive integer m such that $f^m = i$ for all $f \in S_4$?

Proof.

$\square$

Page 20 — Problem 13

Problem. Show that there is a positive integer t such that $f^t = i$ for all $f \in S_n$.

Proof.

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