M 431: Assignment 7

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Page 74 — Problem 15

Problem. If G is any group, $N \trianglelefteq G$ and $\varphi : G \longrightarrow G'$ a homomorphism of G onto G', prove that the image, $\varphi(N)$, of N is a normal subgroup of G'.

Proof. We already know that the image $\varphi(N)$ is a subgroup of G' since the image of a subgroup under a homomorphism is a subgroup. So we only need to show that $\varphi(N)$ is normal in G'. Fix $\overline{g} \in G'$ and $\overline{n} \in \varphi(N) \subset G'$. Since φ is onto, we can pick $g \in G$ and $n \in N$ such that $\varphi(g) = \overline{g}$ and $\varphi(n) = \overline{n}$.

Now consider $\varphi(g^{-1})\varphi(n)\varphi(g) \in G'$. On the one hand, $\varphi(g^{-1})\varphi(n)\varphi(g) = \varphi(g)^{-1}\varphi(n)\varphi(g) = \overline{g}^{-1}\overline{ng}$. But also $\varphi(g^{-1})\varphi(n)\varphi(g) = \varphi(g^{-1}ng) = \varphi(n_1) \in \varphi(N)$ since φ is a homomorphism. Then our chain of equalities show that $\overline{g}^{-1}\overline{ng} \in \varphi(N)$. Since $\overline{g}, \overline{n}$ were selected arbitrarily from G' and $\varphi(N)$ respectively, we just showed that $\overline{g}^{-1}\varphi(N)\overline{g} \subset \varphi(N)$. Which is to say that $\varphi(N)$ is a normal subgroup of G'.

$\square$

Page 74 — Problem 16

Problem. If $N \trianglelefteq G$ and $M \trianglelefteq G$ and $MN = \{mn \mid m \in M, n \in \mathbb{N}\}$, prove that MN is a subgroup of G and that $MN \trianglelefteq G$.

Proof. First let's show that MN is a subgroup of G. We will use the aesthetic definition. We know $MN \neq \emptyset$ because $e \in M, N$ so $ee = e \in MN$. Now for arbitrary $x, y \in MN$ we show that $xy^{-1} \in MN$. Let $x = mn \in MN$ and $y = \bar{m}\bar{n} \in MN$. Then $xy^{-1} = (mn)(\bar{m}\bar{n})^{-1} = mn\bar{n}^{-1}\bar{m}^{-1}$. Taking $n_1^{-1} = n\bar{n}^{-1}$, we get $mn\bar{n}^{-1}\bar{m}^{-1} = mn_1^{-1}m^{-1}n_1n_1^{-1}$ Then since M is normal in G we know $n_1^{-1}m^{-1}n_1 = m_1 \in M$ so we have $xy = mm_1n_1^{-1} = m_2n_2 \in MN$ where $m_2 = mm_1$ and $n_2 = n_1^{-1}$. Thus $MN \leq G$.

We now prove that MN is normal in G. We must show that $g^{-1}MNg \subset MN$. Fix any $g \in G$ and $mn \in MN$. Then $g^{-1}mng = g^{-1}mgg^{-1}ng$ but M, N are each normal so let $m_1 = g^{-1}mg \in M$ and $n_1 = g^{-1}ng \in N$ and we have $g^{-1}mng = m_1n_1 \in MN$. Since we selected the values arbitrarily, $g^{-1}MNg \subset MN$ and $MN \trianglelefteq G$.

□

Page 83 — Problem 3

Problem. If G is a group and $N \trianglelefteq G$, show that if $\overline{M}$ is a subgroup of G/N and $M = \{a \in G \mid Na \in \overline{M}\}$, then M is a subgroup of G and $N \subset M$.

Proof. First let's show that $M \leq G$. Let's use the aesthetic definition again; we know the identity element $e \in M$ since $Ne = N$ the identity element of G/N. Since $\overline{M}$ is a subgroup of G/N, we know $N \in \overline{M}$. Now fix any $x, y \in M$ and we want to show $xy^{-1} \in M$. But $xy^{-1} \in M$ if $Nxy^{-1} \in \overline{M}$. Since $x, y \in M$ we know $Nx, Ny \in \overline{M}$. But $\overline{M}$ is a subgroup of G/N so the aesthetic condition holds in $\overline{M}$: $(Nx)(Ny)^{-1} \in \overline{M}$. We know $(Ny)^{-1} = Ny^{-1}$ so $(Nx)(Ny)^{-1} = NxNy^{-1}$. Then by the operation defined in the subgroup $\overline{M} \leq G/N$, we have $NxNy^{-1} = Nxy^{-1} \in \overline{M}$ the exact condition we wanted to show.

Now we show that $N \subset M$. Pick any $x \in N$ and consider the right coset Nx. Since $x \in N$, we know $Nx = N$. We already worked out that $N \in \overline{M}$. So $Nx \in \overline{M}$ which means $x \in M$. Therefore $N \subset M$.

$\square$

Page 83 — Problem 4

Problem. If $\overline{M}$ in the previous problem is normal in G/N, show that the M defined is normal in G.

Proof. Since $\overline{M} \trianglelefteq G/N$ we know that $(Ng)^{-1}\overline{M}(Ng) \subset \overline{M}$ for arbitary $g \in G$. That is to say, for any $Nm \in \overline{M}$ (where $m \in M$) there exists some $Nm' \in \overline{M}$ (where $m' \in M$) such that $Ng^{-1}NmNg = Nm'$. Then using properties of the subgroup $\overline{M} \leq G/N$, the LHS can be rearranged to say that $Ng^{-1}mg = Nm' \in \overline{M}$. Since $Ng^{-1}mg \in \overline{M}$, we must have $g^{-1}mg \in M$. But then g was arbitary in G and m was arbitrary in M so $g^{-1}Mg \subset M$ and $M \trianglelefteq G$.

□

Page 87 — Problem 3

Problem. Let G be the group of nonzero real numbers under multiplication and let $N = \{1, -1\}$. Prove that $G/N \cong$ positive real numbers under multiplication.

Proof. First let's explore the group G/N and show that the concept of G mod N even makes sense. We must have $N \trianglelefteq G$, although since $G = \mathbb{R} \setminus \{0\}$ with multiplication is abelian we only need to show that $N \leq G$. The identity $1 \in G$ is a member of N. We inherit associativity since $N \subset G$. As far as inverses go, the only element not equal to 1 is -1, which is its own inverse. The only remaining property to prove is closure: $1 * -1 = -1 \in N$ (the remaining combinations are shown by abelian or inverse properties). So $N \leq G$ and since G is abelian, $N \trianglelefteq G$. Because of this, it makes sense to contemplate G/N.

Consider the group $G/N = \{Nx \mid x \in G\}$ where $Nx = \{nx \mid n \in N\} = \{x, -x\}$. Thus $G/N = \{\{x, -x\} \mid x \in \mathbb{R} \setminus \{0\}\}$. Let's now consider the function $\varphi : G/N \longrightarrow \mathbb{R}^+$ defined by $\varphi(Nx) = \varphi(\{x, -x\}) = |x| \in \mathbb{R}^+$. The function φ is well defined since $|x| = |-x| > 0$ and $|x| > 0$ when $x \neq 0$, so $|x|$ is in the co-domain and choosing x or $-x$ results in the same output.

We now prove that φ is a homomorphism and a bijection, thus an isomorphism. For any $Nx, Ny \in G/N$, we have $\varphi(Nx)\varphi(Ny) = \varphi(\{x, -x\})\varphi(\{y, -y\}) = |x||y| = |xy|$. Also we have $\varphi(NxNy) = \varphi(Nxy) = |xy|$ by the group operation in G/N. So $\varphi(NxNy) = \varphi(Nx)\varphi(Ny)$ and φ is a homomorphism. Now we show that φ is 1-1. Pick $Nx, Ny \in G/N$ such that $\varphi(Nx) = \varphi(Ny) \implies |x| = |y|$ which means $y = x$ or $y = -x$. Consider $Nx = \{x, -x\}$. If $y = x$, then certainly $Nx = Ny$ and if $y = -x$ then $Ny = N(-x) = \{-x, x\} = Nx$ so we have $Nx = Ny$ in either case. Thus φ is an injection. What about a surjection? This is quite easy, for $x \in \mathbb{R}^+$ choose $Nx \in G/N$. Then $\varphi(Nx) = |x| = x$ since $x > 0$. Thus φ is a homomorphism and a bijection so φ is an isomorphism.

Since there is an isomorphism from G/N onto $\mathbb{R}^+$ (both with $*$), the two groups are isomorphic.

$\square$

Page 88 — Problem 6

Problem. If G is a group and $N \trianglelefteq G$, show that if $a \in G$ has finite order $o(a)$, then Na in G/N has finite order m where $m \mid o(a)$.

Proof. We begin with a general lemma: if H is a finite, cyclic subgroup of G and $\sigma : G \longrightarrow K$ is a homomorphism then $\sigma(H)$ is a finite, cyclic subgroup of K. Here is the proof. We already know that the image of a subgroup under a homomorphism is a subgroup so we only need show the finite and cyclic properties. $\sigma(H)$ is finite since σ is a function and H (the set of inputs) is finite. Further, a group is cyclic if there exists some element x such that any member of the group y can be written as $y = x^j$ for some integer j. We know this to be true for H. Let $|H| = n$, then $H = \{e, h, h^2, ..., h^{n-1}\}$. The image of any element of H is $\sigma(h^k) = \sigma(h)^k$ (apply induction for the proof) so $\sigma(h)$ can be taken as the element that generates $\sigma(H)$.

Now let's return to the problem at hand. Fix $a \in G$ such that $o(a)$ is finite and let $o(a) = n$. Then let $A = \{e, a, a^2, ..., a^{n-1}\}$ a finite, cyclic subgroup of G. Let φ be the natural homomorphism from G onto G/N and $\psi = \varphi\mid_A$, the restriction of φ to A. Since ψ is a restriction of a homomorphism to G/N, the map $\psi : A \longrightarrow G/N$ is a homomorphism. Thus $\ker\psi = \{x \in A \mid \psi(x) = N \in G/N\}$ is a subgroup of A (it is also normal but we do not need this). Since $\ker\psi \leq A$, Lagrange's theorem tells us that $|\ker\psi| \mid |A|$. Let $k = |\ker\psi|$ then we must have $k \mid n$.

Let's now apply the lemma, so $\psi(A)$ is a finite, cyclic subgroup of G/N. Let $|\psi(A)| = m \leq n$ and $\psi(A) = \{N, Na, Na^2, ..., Na^{m-1}\}$. Note that the right cosets listed in $\psi(A)$ are all mutually disjoint since $\psi(A)$ is of order m. For any $0 \leq j < m$, pick any $Na^j \in \psi(A)$. Its pre-image $\psi^{-1}(Na^j) = \{x \in A \mid \psi(x) = Na^j\} = (\ker\psi)a^j$ by Lemma 2.5.4 from the text. Also note that the set of pre-images is mutually disjoint because $0 \leq j < m$ and the elements listed in $\psi(A)$ are mutually disjoint.

We can also deduce that $|\ker\psi| = |(\ker\psi)a^j|$ since the map $R_{a^j} : \ker\psi \longrightarrow (\ker\psi)a^j$ defined by right multiplying by a^j is a bijection between the two sets (it has an inverse). Then since $|\ker\psi| = k$ we have $|(\ker\psi)a^j| = k$ for all $0 \leq j < m$. Note that $A = \cup_{j=0}^{m-1}\psi^{-1}(Na^j)$ and the pre-images are mutually disjoint (this is important in the chain of equalities below). Taking the cardinality of both sides, we must have

$$|A| = n = \left|\bigcup_{j=0}^{m-1}\psi^{-1}(Na^j)\right| = \bigcup_{j=0}^{m-1}\left|\psi^{-1}(Na^j)\right| = \bigcup_{j=0}^{m-1}|(\ker\psi)a^j| = \bigcup_{j=0}^{m-1} k = m * k$$

Thus $o(a) = n = m * k$ where $k \in \mathbb{N} \subset \mathbb{Z}$ which is equivalent to saying that $m \mid o(a)$, as desired!

$\square$