M 431: Assignment 1

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Problem. If C is a finite set, let $m(C)$ denote the number of elements in C. If A, B are finite sets, prove that

$$m(A \cup B) = m(A) + m(B) - m(A \cap B)$$

Proof. We take for granted the fact that $m(A \cup B) = m(A) + m(B)$ if A, B are disjoint sets. Additionally, let's eliminate the edge case where one or both of A, B is the empty set. Without loss of generality, let $A = \emptyset$. Then on the left hand side $m(A \cup B) = m(\emptyset \cup B) = m(B)$. Now on the right hand side, $m(A) + m(B) - m(A \cap B) = m(\emptyset) + m(B) - m(\emptyset) \cap B) = 0 + m(B) - m(\emptyset) = m(B) - 0 = m(B)$. Then the LHS equals the RHS. From here on, we assume that A, B are nonempty sets.

Now let's establish some notation. Let $A' = A \setminus (A \cap B)$ and $B' = B \setminus (A \cap B)$. We then have three mutually disjoint sets A', B', and $A \cap B$. Also note that $A = A' \cup (A \cap B)$, $B = B' \cup (A \cap B)$, and $A \cup B = A' \cup B' \cup (A \cap B)$.

We now show, via induction, that for a finite, nonempty collection of mutually disjoint sets $Y = \{Y_1, Y_2, ..., Y_n\}$, we have $m(\bigcup_{i=1}^{n} Y_i) = \sum_{i=1}^{n} m(Y_i)$. Consider the base case where $n = 1$, then $Y = \{Y_1\}$. The left hand side becomes $m(\bigcup_{i=1}^{1} Y_i) = m(Y_1)$ and the right hand side is also $\sum_{i=1}^{1} m(Y_i) = m(Y_1)$ so the base case is verified.

Now suppose that for some $k \in \mathbb{N}$, $m(\bigcup_{i=1}^{k} Y_i) = \sum_{i=1}^{k} m(Y_i)$. Now we prove that the equality holds for $k + 1$. That is, we must show that $m(\bigcup_{i=1}^{k+1} Y_i) = \sum_{i=1}^{k+1} m(Y_i)$. Beginning with the LHS, $m(\bigcup_{i=1}^{k+1} Y_i) = m(Y_{k+1} \cup \bigcup_{i=1}^{k} Y_i)) = m(Y_{k+1}) + m(\bigcup_{i=1}^{k} Y_i)$ since Y_{k+1} and $\bigcup_{i=1}^{k} Y_i$ are disjoint. Then $m(\bigcup_{i=1}^{k} Y_i) = \sum_{i=1}^{k} m(Y_i)$ by our inductive assumption and we have $m(\bigcup_{i=1}^{k+1} Y_i) = m(Y_{k+1}) + \sum_{i=1}^{k} m(Y_i) = \sum_{i=1}^{k+1} m(Y_i)$. So the LHS equals the RHS and the proof by induction is complete.

Now we return to the problem at hand. As noted above, $A \cup B = A' \cup B' \cup (A \cap B)$ so we must have $m(A \cup B) = m(A' \cup B' \cup (A \cap B))$. But A', B', and $A \cap B$ are mutually disjoint sets so $m(A' \cup B' \cup (A \cap B)) = m(A') + m(B') + m(A \cap B)$.

Since $A = A' \cup (A \cap B)$ and $A', A \cap B$ are mutually disjoint, $m(A) = m(A' \cup (A \cap B)) = m(A') + m(A \cap B)$. Then $m(A') = m(A) - m(A \cup B)$. A similar argument shows that $m(B') = m(A) - m(A \cup B)$. Then substituting into $m(A \cup B) = m(A') + m(B') - m(A \cap B)$, we get $m(A \cup B) = m(A) - m(A \cap B) + m(B) - m(A \cap B) + m(A \cap B) = m(A) + m(B) - m(A \cap B)$. So for finite sets A, B it is the case that $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.

□

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Problem. Show, for finite sets A, B, that $m(A \times B) = m(A)m(B)$.

Proof. Again, we take for granted the fact that if A, B are disjoint sets then we have $m(A \cup B) = m(A) + m(B)$. As an edge case, consider if A, B, or both A and B are the empty set. In this case $A \times B = \emptyset$ and $m(A \times B) = m(\emptyset) = 0$. Also $m(A)m(B) = 0$ because one or both of $m(A), m(B)$ will be 0. Since the statement holds when one or both of A, B is the empty set, we assume both sets are nonempty for the remainder of the proof.

By definition, $A \times B = \{(a, b) \mid a \in A, b \in B\}$. Since A is finite, let $m(A) = n \in \mathbb{N}$ and we can enumerate its elements $A = \{a_1, a_2, ..., a_n\}$. Now let $A_k = \{a_k\} \times B$. Then $\bigcup_{i=1}^{n} A_i = A \times B$. Further, the sets $A_1, A_2, ..., A_n$ are mutually disjoint because the pair (a_k, b) can only be a member of the set A_k. Then, using the inductive result from the previous problem, $m(A \times B) = m(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} m(A_i)$.

Now we show that for any singleton $S = \{s\}$ and a nonempty set X, we have $m(S \times X) = m(X)$. We show this equality by establishing a bijection between $S \times X$ and X. Consider $f : S \times X \longrightarrow X$ defined by $f((s, x)) = x$. To show that f is a bijection, we must show it is surjective and injective. The map f is surjective if every element of X is the image of some element of $S \times X$ under f. For any $x' \in X$, just choose $(s, x') \in S \times X$, then the image of (s, x') under f is x', as desired. Is f injective? We can show f is injective by proving that $f((s, x_1)) = f((s, x_2))$ implies $(s, x_1) = (s, x_2)$. Suppose we have $f((s, x_1)) = f((s, x_2))$, evaluating f gives us $x_1 = x_2$. Since $s = s$ and $x_1 = x_2$, must have $(s, x_1) = (s, x_2)$. Therefore, f is injective. Since $f : S \times X \longrightarrow X$ is both injective and surjective, it is a bijection and $m(S \times X) = m(X)$.

Recall that $A_i = \{a_i\} \times B$. Since $\{a_i\}$ is a singleton, we can use the result just shown and say that $m(A_i) = m(B)$. So now we have $m(A \times B) = \sum_{i=1}^{n} m(B) = n * m(B)$. But $n = m(A)$ so we really just showed that $m(A \times B) = m(A)m(B)$, which was the goal.

$\square$

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Problem. If $f : S \longrightarrow T$ is onto and $g : T \longrightarrow U$ and $h : T \longrightarrow U$ are such that $g \circ f = h \circ f$, then $g = h$.

Proof. To show that $g = h$, we must show that $g(t) = h(t)$ for all $t \in T$. Since $g \circ f = h \circ f$, we can say that $(g \circ f)(s) = g(f(s)) = (h \circ f)(s) = h(f(s))$ for all $s \in S$. Then since f is onto, for every $t \in T$, there exists $s_t \in S$ such that $f(s_t) = t$. Then $f(s)$ can be replaced with t to give $g(f(s)) = g(t) = h(f(s)) = h(t)$, which proves that $g = h$.

$\square$

Page 20 — Problem 11

Problem. Can you find a positive integer m such that $f^m = i$ for all $f \in S_4$?

Proof. In this proof, we verify that $m = 4! = 24$ satisfies $f^m = f^{4!} = i$ for all $f \in S_4$. Let f be any map in S_4. We already know, from class lecture three, that any $f \in S_n$ can be uniquely decomposed into disjoint cycles. Further, the minimum cycle length is 1 and the maximum cycle length is n (since there are only n elements to permute). Since $n = 4$ in our case, f can be decomposed into cycles and every cycle in f must have length 1, 2, 3, or 4. Then, the image of s_0 (an element in a cycle of length l) under $f^{4!}$ is $f^{4!}(s_0) = (f^l(s_0))^{4!/l} = (i(s_0))^{4!/l} = i(s_0) = s_0$ for cycles of lengths 1, 2, 3, and 4. Since these are the only cycle lengths that can occur, $f^{4!}$ acts identically to i, which implies $f^{4!} = i$. Then since our choice of f was arbitrary, $m = 4!$ satisifes $f^m = i$ for every $f \in S_4$.

$\square$

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Problem. Show that there is a positive integer t such that $f^t = i$ for all $f \in S_n$.

Proof. For this question, we do not give a value for t, we only show its existence. Note that $|S_n| = n! < +\infty$. Then enumerate the elements of $S_n = \{f_1, f_2, ..., f_{n!-1}, f_{n!}\}$. We know from class lecture three that for every $f_k \in S_n$, there exists some $p_k \in \mathbb{N}$ such that $f_k^{p_k} = i$.

We show for any map $f : S \longrightarrow S$ and natural numbers r, s we have the equality $f^{rs} = (f^r)^s$. On the LHS, we have $f^{rs} = f^{\sum_{i=1}^{s} r} = f^r f^r \cdots f^r$ s times.

Then take $t = \prod_{k=1}^{n!} p_k$. For any $f_k \in S_4$, we have $f_k^t = (f_k^{p_k})^{t/p_k}$. Also note that t/p_k is a natural number because p_k is a factor in the construction of t. But we already know $f_k^{p_k} = i$, so we really have $f_k^t = i^{t/p_k} = i$. The identity map repeated any number of times is still the identity map, so we have $f_k^t = i$ as desired.

$\square$