M 431: Assignment 8

Nathan Stouffer

Page 87 — Problem 2

Problem. Let G be the group of all real-valued functions on the unit interval $[0, 1]$, where we define, for $f, g \in G$, addition by $(f + g)(x) = f(x) + g(x)$ for every $x \in [0, 1]$. If $N = \{f \in G \mid f(1/4) = 0\}$, prove that $G/N \cong$ the real numbers under $+$.

Proof. Note that our ambient group G is abelian so any subgroup is a normal subgroup. We first claim that $N \leq G$ by showin the aesthetic definition holds for N. Certainly $id \in N$ for $id(1/4) = 0$. Now pick $f, g \in N$ and recall that $g^{-1} = -g$. Then $f + g^{-1} = f + (-g)$ and now evaluating at $x = 1/4$: $(f + (-g))(1/4) = f(1/4) + (-g(1/4)) = 0 + -0 = 0$ so $f + g^{-1} \in N$.

Now the quotient group $G/N = \{g + N \mid g \in G\}$ and $(g + N)(1/4) = g(1/4) + N(1/4) = g(1/4) + 0 = g(1/4)$. Let's define the mapping $\psi : G/N \longrightarrow \mathbb{R}$ which takes $g + N$ to $g(1/4)$. We must verify four things for ψ to be an isomorphism: well-defined, homomorphism, onto, and 1-1.

Well-defined: pick $g + N = \bar{g} + N$. Then there exists some $f \in N$ such that $g = \bar{g} + f$ which implies that $\bar{g} = g + f$. Now consider $\psi(\bar{g} + N) = \psi(g + f + N) = (g + f)(1/4) = g(1/4) + f(1/4) = g(1/4) = \psi(g + N)$ where we know $f(1/4) = 0$ since $f \in N$.

Homomorphism: fix $g_1+N, g_2+N \in G/N$. Certainly we have $\psi(g_1+N)+\psi(g_2+N) = g_1(1/4)+g_2(1/4)$. But we also have $\psi(g_1 + N + g_2 + N) = \psi(g_1 + g_2 + N) = (g_1 + g_2)(1/4) = g_1(1/4) + g_2(1/4)$ so ψ is a homomorphism.

Onto: this is verified quite easily. If you give me an $x^\star \in \mathbb{R}$, I will give you the function $f(x) = x^\star$ for $x \in [0, 1]$. Certainly $\psi(f + N) = f(1/4) = x^\star$.

1-1: fix any $f_1, f_2 \in G$ such that $\psi(f_1 + N) = \psi(f_2 + N)$. We wish to show that $f_1 + N = f_2 + N$ which is true if there exists some $n \in N$ such that $f_1 = f_2 + n$. We will provide such a function $n(x)$. Note that $\psi(f_1 + N) = \psi(f_2 + N) \implies f_1(1/4) = f_2(1/4)$. Here is our function defined on $[0, 1]$:

$$n(x) = \begin{cases} 0 & x = 1/4 \\ f_1(x) - f_2(x) & x \neq 1/4 \end{cases}$$

Now consider $f_2(x) + n(x)$. If $x = 1/4$ then $f_2(1/4) + n(1/4) = f_2(1/4) + 0 = f_1(1/4)$. Then if $x \neq 1/4$ we have $f_2(x) + n(x) = f_2(x) + f_1(x) - f_2(x) = f_1(x)$. Thus $f_1(x) = f_2(x) + n(x)$ for all $x \in [0, 1]$ and we have $f_1 = f_2+n$ which implies that the cosets f_1+N and f_2+N are equal. Thus ψ is 1-1.

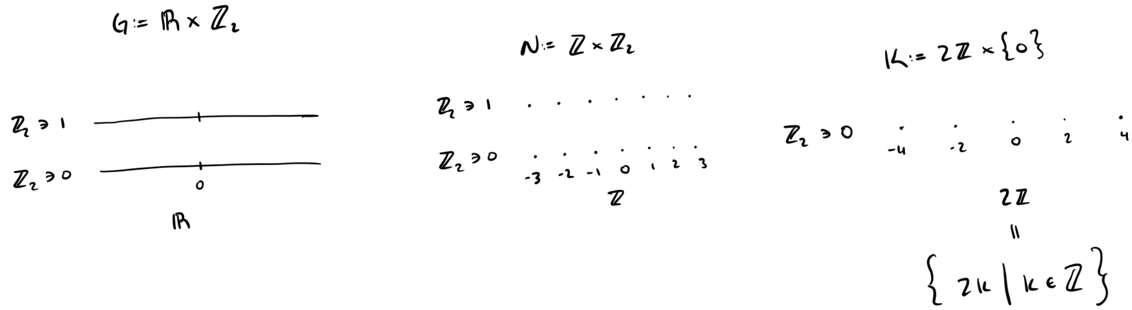
We have check everything we need to for ψ to be an isomorphism so the two groups are isomorphic!

□

3rd Iso Thm Example

Problem. Identify and illustrate with pictures the three quotient groups in the 3rd isomorphism theorem instantiated for $G = \mathbb{R} \times \mathbb{Z}_2$, $N = \mathbb{Z} \times \mathbb{Z}_2$, and $K = 2\mathbb{Z} \times \{0\}$.

Proof. For this problem and the next, I ended up drawing a lot of pictures so all my work is in the figures below.



$$G/N := \{(x,i) + \mathbb{Z}\times\mathbb{Z}_2 \mid (x,i) \in \mathbb{R}\times\mathbb{Z}_2\}$$
$$= \{(x+\mathbb{Z},\ i+\mathbb{Z}_2) \mid (x,i) \in \mathbb{R}\times\mathbb{Z}_2\}$$

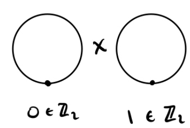
circle grp

just $\mathbb{Z}_2$ pf/

$i=0$: $i+\mathbb{Z}_2 = \{0+0, 0+1\} = \mathbb{Z}_2$

$i=1$: $i+\mathbb{Z}_2 = \{1+0, 1+1\} = \mathbb{Z}_2$

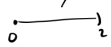
Thus G/N is the Torus



$\cong S^1 \times S^1 = T^2$

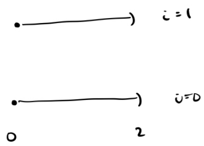
$$G/K := \{ (x,i) + 2\mathbb{Z}\times\{0\} \mid (x,i) \in \mathbb{R}\times\mathbb{Z}_2 \}$$

$$= \{ (x+2\mathbb{Z}, i+0) \mid (x,i) \in \mathbb{R}\times\mathbb{Z}_2 \}$$



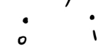
the "fat" torus described in lecture

$G/K =$

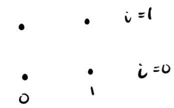


$$N/K := \{ (j,i) + 2\mathbb{Z}\times\{0\} \mid (j,i) \in \mathbb{Z}\times\mathbb{Z}_2 \}$$

$$= \{ (j+2\mathbb{Z}, i+0) \mid (j,i) \in \mathbb{Z}\times\mathbb{Z}_2 \}$$



$N/K =$



now consider $(G/K)/(N/K)$. if pts differ by 1, we identify them

which gives $T^2 = S^1 \times S^1$ just like G/N!

Thus we have depicted the groups!

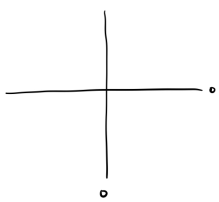
$\square$

2rd Iso Thm Example

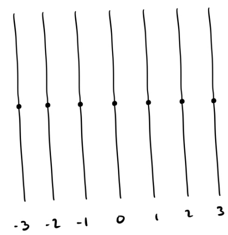
Problem. Identify and illustrate with pictures all the groups involved in the 2nd isomorhpism theorem isntantiated for $G = \mathbb{R} \times \mathbb{R}$, $N = \mathbb{Z} \times \mathbb{R}$, $H = \mathbb{R} \times \{0\}$. In particular, draw the cosets making up the quotient groups and recognize the groups as familiar concrete groups.

Proof.

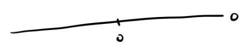
$G = \mathbb{R} \times \mathbb{R}$



$N = \mathbb{Z} \times \mathbb{R}$



$H = \mathbb{R} \times \{0\}$



$$HN := \{ (x,0) + (k, \bar{x}) \mid x, \bar{x} \in \mathbb{R} \quad k \in \mathbb{Z} \}$$
$$= \{ (x+k, \bar{x}) \mid x, \bar{x} \in \mathbb{R} \quad k \in \mathbb{Z} \}$$

but x, k are arbitrary in their respective sets so $x + k = x'$ any element of $\mathbb{R}$, so

$HN = \mathbb{R} \times \mathbb{R}$

$$H \cap N = \{ (a,b) \mid (a,b) \in \underset{\mathbb{R}\times\{0\}}{H}, \underset{\mathbb{Z}\times\mathbb{R}}{N} \} = \{ (a,b) \mid a \in \mathbb{R} \cap \mathbb{Z},\ b \in \{0\} \cap \mathbb{R} \} = \{ (a,0) \mid a \in \mathbb{Z} \} = \mathbb{Z} \times \{0\}$$

... -3 -2 -1 0 1 2 3 ...

concretely, $H \cap N = \{ (a,0) \mid a \in \mathbb{Z} \} \underset{\text{iso}}{\cong} \mathbb{Z}$ via $\psi : (a,0) \longmapsto a$

$$H/_{(H\cap N)} := \left\{ (x,0) + H\cap N \mid x\in\mathbb{R} \right\} = \left\{ (x,0) + \mathbb{Z}\times\{0\} \mid x\in\mathbb{R} \right\}$$

$$= \left\{ (x+\mathbb{Z}, 0) \mid x\in\mathbb{R} \right\} \underset{\text{iso}}{\cong}$$

"circle grp"

This gives us the circle group again!

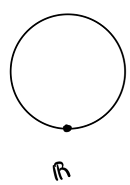
$$\underbrace{(HN)}_{=\mathbb{R}^2}/_{N} := \left\{ (x_1,x_2) + \mathbb{Z}\times\mathbb{R} \mid x_1,x_2\in\mathbb{R} \right\} = \left\{ (x_1+\mathbb{Z}, x_2+\mathbb{R}) \mid x_1,x_2\in\mathbb{R} \right\}$$

$\wr$

our friend the circle grp

note that $x_2+\mathbb{R} = \mathbb{R}$ a single coset

This gives the following picture



$\underset{\text{iso}}{\cong}$

the familiar circle group

Thus we have depicted the groups!

$\square$

Page 96 — Problem 5

Problem. Let G be a finite group, $N_1, N_2, ..., N_k$ normal subgroups of G such that $G = N_1 N_2 \cdots N_k$ and $|G| = |N_1||N_2| \cdots |N_k|$. Prove that G is the direct product of $N_1, N_2, \ldots, N_k$.

Proof. This one threw me for a loop, I could not come up with a rigorous proof of the statement. Here is what I got. G is the direct product of $N_1, N_2, \ldots N_k$ if and only if $N_1 \times N_2 \times \cdots \times N_k$ is isomorphic to G. Consider the function $\psi : N_1 \times N_2 \times \cdots \times N_k \longrightarrow G$ defined by taking $(n_1, n_2, \ldots, n_k)$ to $n_1 n_2 \cdots n_k$. Since $G = N_1 N_2 \cdots N_k$ we know ψ to be a surjection. Furthermore, since $|G| = |N_1||N_2| \cdots |N_k| = |N_1 \times N_2 \times \cdots \times N_k|$ and each cardinality is finite, we can deduce that ψ is also an injection.

It only remains to show that ψ is a homomorphism. That is, we must show that

$$\psi((n_1, n_2, \ldots, n_k))\psi((\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_k)) = \psi((n_1, \ldots, n_k)(\bar{n}_1, \ldots, \bar{n}_k))$$

Cetainly the LHS is $n_1 \cdots n_k \bar{n}_1 \cdots \bar{n}_k$ and the RHS is $\psi(n_1\bar{n}_1, \ldots, n_k\bar{n}_k) = n_1\bar{n}_1 \cdots n_k\bar{n}_k$. However, I could not not figure out a way to show that the RHS equals the LHS.

□

Page 96 — Problem 6

Problem. Let G be a group, $N_1, N_2, \ldots, N_k$ normal subgroups of G such that:

(1) $G = N_1 N_2 \cdots N_k$

(2) For each i, $N_i \cap (N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_k) = (e)$

Prove that G is the direct product of $N_1, N_2, \ldots, N_k$

Proof. We begin with a lemma that is a consequence of property (2). Given a fixed N_i, let H_i be the product of any selection of $\bar{N}_i = N_1, ..., N_{i-1}, N_{i+1}, ..., N_k$. We claim that $N_i \cap H_i = (e)$. Here is the proof. Certainly $e \in H_i$ and $e \in N_i$ so $e \in N_i \cap H_i$. From property (2), we know that N intersect all the other normal subgroups N_j is (e). Further, we know H_i must be a subset of $N_1, ..., N_{i-1}, N_{i+1}, ..., N_k$ for every element of H_i can be written as an element of $\bar{N}_i$ with e chosen as the element for sets that were not selected for H_i. Thus we are just constricting one of the sets involved in an intersection (and we certainly keep the elements that are already in the intersection) so $N_i \cap H_i = (e)$.

Now let's proceed by induction. For a base case, take $K_2 = N_1 N_2$. By the lemma just proven, we have $N_1 \cap N_2 = (e)$ and the corollary on page 95 of the textbook tells us that K_2 is the internal product of N_1 and N_2. Futhermore, $K_2 \trianglelefteq G$ since $N_1, N_2 \trianglelefteq G$. Now supppose for some $j \in \{2, 3, ..., k-1\}$ that K_j is the direct product $K_j = N_1 N_2 \cdots N_j$. We will show that $K_{j+1} = N_1 N_2 \cdots N_j N_{j+1}$ is also a direct product. The lemma tells us that $K_j \cap N_{j+1} = (e)$ and we can apply the corollary again to say that K_{j+1} is the direct product of $N_1 N_2 \cdots N_j N_{j+1}$.

Thus, by induction, every K_j for $j \in \{2, 3, ..., k\}$ is the direct product of $N_1 N_2 \cdots N_j$. This implies that $G = N_1 N_2 \cdots N_k$ is a direct product as well!

$\square$