Advanced Algorithms, Homework 6

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In any question that you are expected to provide an algorithm, you are expected to provide:

1. Describe the problem in your own words, including describing what the input and output is.

2. Describe, in paragraph form, the algorithm you propose.

3. Provide a nicely formatted algorithm to solve the problem.

4. Use a decrementing function to prove that algorithm terminates OR Give the runtime with justification.

5. Prove partial correctness. In other words, if there is a loop or recursion, what is the loop/recursion invariant? Provide the proof. (Note: you only need to do this for the outer-most loop if there are nested loops).

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CSCI 432 Problem 6-1

Collaborators: *n/a*

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Walk through Kruskal's algorithm, using the graph in Figure 7.7 (left) of the textbook. Label the center vertex a, the other red vertex b, and the remainder c through g in counter-clockwise order. You should use the union-find data structure, with both "heuristics."
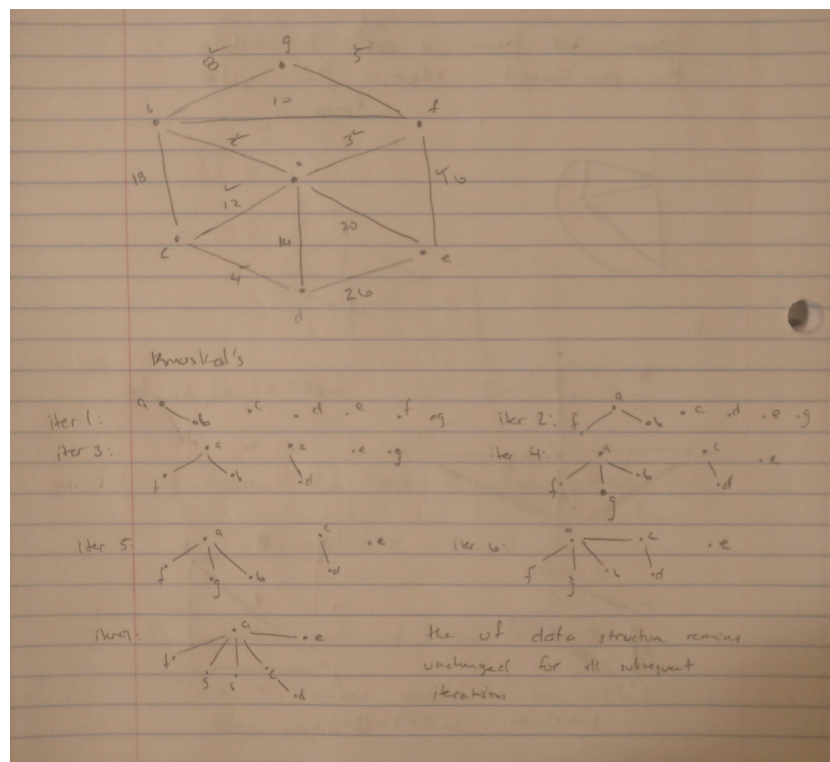
Answer   Here is my walk through:

Figure 1: Walking through Kruskal's algorithm

Chapter 3, Question 9 (Palindromes)

A palindrome is any string that is exactly the same as its reversal.

Any string can be decomposed into a sequence of palindromes. Describe and analyze an efficient algorithm to find the smallest number of palindromes that make up a given input string.

Answer

1. The input to this problem is an array containing a string of characters. The goal is to find the smallest number of palindromes that concatenate to form the input string. For output, we will give an integer (the smallest integer number of palindromes that make the input string).

2. My algorithm works in two parts. First I will describe the recursive portion of the algorithm, and then (to make it efficient) I will discuss how to reach the dynamic programming solution. The recursive algorithm starts at the beginning of the string. If we have a pointer to a letter in the string, that letter must be part of a palindrome which precedes it or it must start a new palindrome. Then we recursively compute the number of palindromes in each case to find the total number of palindromes that compose the input. Then we take the minimum value.

   The algorithm I just described runs in expnonential time since it either chooses to include a letter or not include it. However we would like this algorithm to be efficient. To do so, we employ dynamic programming. If the pointer to the current string is at index f and the beginning of the palindrome containing word$[f]$ is at index s. Then we can say that the decision for s, f depends on the decision for $s, f+1$ and $f+1, f+1$. This induces a dynamic programming algorithm that runs in polynomial time (the size of the table), so we have an efficient algorithm!

3. Here is the psuedocode of the recursive algorithm.

PALINSEQ(word, s, f)
1. if (f > len(word))
2.   if (ISPALIN(word, s, f-1))
3.   return 1
4.   else
5.   return $+\infty$
6. cur = PALINSEQ(word, s, f+1)
7. nxt = $+\infty$
8. if (ISPALIN(word, s, f))
9.   nxt = 1 + PALINSEQ(word, f+1, f+1)
10.   return min (cur, next)

From here, we can use the following dynamic programming table to build our efficient solution (arrows drawn because of the dependency on $T[s, f + 1]$ and $T[f + 1, f + 1]$):
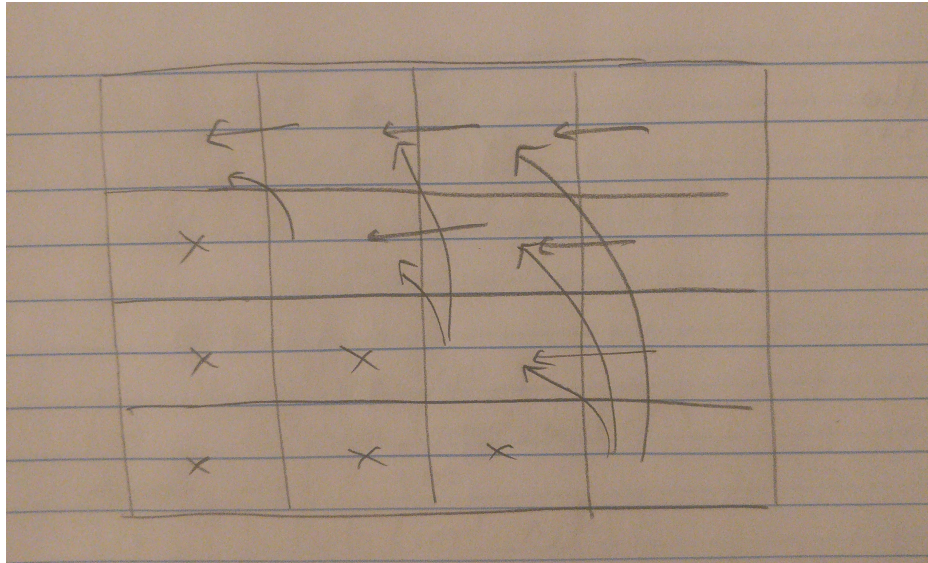
Figure 2: Dependency Table for $n = 4$

4. We will use a decrementing function to show that the recursive version terminates. This implies termination for the dynamic programming solution. Let our state space be letters in the variable word. Our map to the natural numbers is the number of letters following the index f+1. The function is strictly decreasing because becuase each iteration sends in $f_1 + 1$ for f_{i+1}. This implies termination because our stopping condition is that $f > len(word)$.

5. Now suppose that our algorithm terminates, why should it be correct? Again, we need only show correctness of the recursive solution for the dynamic program to also be correct. The recursive solution is correct because we test every possibility for sequences of palindromes and take the minimum value. Then the dynamic program evaluates the program in an intelligent way instead of testing all possibilities.

Chapter 7, Question 1 (Shortest and Longest Edges in Cycle)

Let $G = (V, E)$ be an arbitrary connected graph with weighted edges.

(a) Prove that for any cycle in G, the minimum spanning tree of G excludes the maximum-weight edge in that cycle.

(b) Prove or disprove: The minimum spanning tree of G includes the minimum-weight edge in every cycle in G.

Answer

(a) We will prove that given a cycle in G, the minimum spanning tree of G does not include the maximum-weight edge in that cycle. To this end, pick a cycle C and label the maximum-weight edge e_{max}. Now suppose we have some spanning tree T that includes e_{max}. Now we show that T is not the minimum spanning tree for G.

Since C is a cycle, there exists some edge $e \in C$ that is not contained in T (otherwise T would not be a tree) and that connects the same components that e_{max} connects. We know that $e \neq e_{max}$ since e is not contained in T. Then, by definition of e_{max}, we must have $weight(e) < weight(e_{max})$. Now let $T' = T - e_{max} + e$. Certainly the total weight of T' is less than the total weight of T. If T' is a spanning tree, then T is certainly not the minimum spanning tree. T' must be a spanning tree because e connects the same components that e_{max} connects. Therefore, T cannot be a minimum spanning tree since another spanning tree of G exists with less weight.

(b) The minimum spanning tree of G does not necessarily have to include the minimum-weight ege in every cycle of G. Consider the following counterexample:
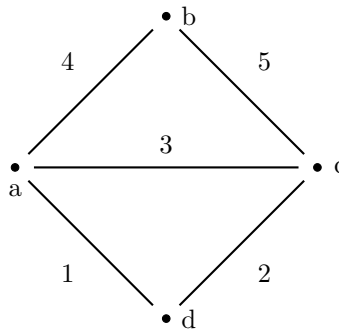
Consider the cycle $a \to b \to c \to a$ which has minimum-weight edge ac. But the above graph has minimum spanning tree with edges ad, dc, ab. The minimum spanning tree does not contain ac so we have found a counterexample.

Chapter 8, Question 3 (Weights on Vertices)

Suppose we are given an undirected graph G in which every vertex has a positive weight.

(a) Describe and analyze an algorithm to find a spanning tree of G with minimum total weight.

(b) Describe and analyze an algorithm to find a path in G from one given vertex s to another given vertex t with minimum total weight.

Answer

(a) We first describe and analyze an algorithm to find a spanning tree of G with minimum total weight.

a) The input to the problem is a connected, undirected graph $G = (V, E)$ paired with a weight function $w : V \rightarrow \mathbb{R}^+$. We define the weight of a graph G to be $weight(G) = \sum_{v \in V} w(v)$.

The output will be a tree with minimum weight $T = (V_T, E_T) \subset G$ such that every vertex of V is in T and T is connected. In this problem, every vertex must be contained in T so the weight of every spanning tree is the same value.

b) Since every spanning tree of G will have the same weight, we need only provide an algorithm that computes a spanning tree T of G. To do this, we will use a union-find data structure. First, we will initialize the data structure with every vertex in its own component. Then we will iterate over every edge. For each edge $e = (s, t)$, we will test if s and t are in the same component. If not, union them in the data structure and add e to T. After iterating through the edges, T will be a miminum spanning tree for G.

c) Here is the algorithm:

$\mathrm{MST}(G = (V, E))$
1. $T = (V, \emptyset)$
2. $\mathrm{uf} = \mathrm{init}(V)$
3.   for $i = 1..|E|$
4. $(s_i, t_i) = E[i]$
5.   if $(\mathrm{uf.find}(s_i) \neq \mathrm{uf.find}(t_i))$
6. $\mathrm{uf.union}(s_i, t_i)$
7. $T = T \cup (\emptyset, (s_i, t_i))$ // add edge to tree
8.   return T

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d) We prove that MST terminates by giving a run time analysis. Line 1 takes constant time and line 2 takes $\Theta(V)$ time. Line 3 runs exactly $|E|$ times and lines 4, 7, and 8 all run in constant time. This leaves only lines 5 and 6 unaccounted for. If we use the heuristic version of union find, then both lines run in constant time. Therefore the run time of the algorithm is a composition of polynomials, which is finite and the algorithm must terminate.

e) We now prove partial correctness. Suppose that MST terminates, will it terminate in a correct state? We will define a loop invariant for the loop beginning on line 3. At iteration i, the loop invariant L_i is that T_i is a forest of the connected components of G after processing the first i edges. Prior to the for loop, we have not processed any edges so T should just contain every vertex in its own connected component, which is the case. As we are processing the for loop, an edge is added to T only if the vertices are found to be in different components, thus connecting the components and preserving the forest structure of T. After the for loop terminates (which we know will occur because we proved termination), we will have processed every edge. Then T is not a forest, but a single tree since the graph G is assumed to be connected. Additionally, T is a spanning tree because every vertex G must be in T. Thus, we return a spanning tree from the MST which must be a minimum spanning tree since every spanning tree has the same total weight.

(b) Describe and analyze an algorithm to find a path in G from one given vertex s to another given vertex t with minimum total weight.

a) The input to the problem is the same as part a). That is, a connected, undirected graph $G = (V, E)$ paired with a weight function $w : V \to \mathbb{R}^+$. Again, we define the weight of a graph G to be $weight(G) = \sum_{v \in V} w(v)$. We say the weight of a simple path P of vertices in G is $weight(P) = \sum_{v \in P} w(v)$. We need only consider simple paths because the vertex weights are all positive so no shortest path will have a cycle.

In contrast to part a), the output will be the path (of vertices) P in G beginning at s and terminating at t with minimum weight.

b) The algorithm is just a modified version of Dijkstra's. Like Dijkstra's, we begin at vertex s and keep a priority queue of the shortest known paths to the other vertices. While the priority queue contains vertices, we greedily select the shortest known path until and see if we should relax any edges leaving the final vertex in the path. Instead of using the weight of the outgoing edge as the weight we add when testing relaxation, we should add the weight of the other vertex. Then we iterate until the priority queue is empty. This will give us a shortest path tree and we can select the shortest path via the predecessor chain.

c) Here is the psuedocode for the algorithm.

MODIFIEDDIJKSTRAS($G = (V, E, w)$, s, t)
1.   pred, dist, pq $\leftarrow$ INIT(V, w, s)
2.   while pq is not empty
3. $u \leftarrow$ pq.extractMin()
4.   for all edges (u, v) and (v, u)
5.   if (dist[u] + w[v] < dist[v])
6.   pred[v] = u
7.   dist[v] = dist[u] + w[v]
8.   pq.decreaseKey(v, dist[v])
9.   return PATH($pred$, s, t)

INIT($V[1..n]$, w, s)
1.   pred $\leftarrow$ [null, null, ..., null] // array of length $|V|$
2.   dist $\leftarrow$ [+∞, +∞, ..., +∞] // array of length $|V|$
3.   pq $\leftarrow$ empty priority queue
4.   pred[s] = s
5.   dist[s] = w[s]
6.   for i in 1..$|V|$
7.   pq.insert(v, dist[v])
8.   return (pred, dist, pq)

PATH(pred[$1..n$], s, t)
1.   rev, $i \leftarrow$ [], t
2.   while ($i \neq s$)
3.   rev.append(i)
4. $i \leftarrow$ pred[i]
5.   rev.append(s)
6.   path $\leftarrow$ reverse.reverse()
7.   return path

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d) The subroutines INIT and PATH terminate so long as MODIFIEDDIJKSTRAS runs correctly. So we must show that the while loop in MODIFIEDDIJKSTRAS terminates. We do so with a decrementing function. Let our state space be the contents of the priority queue pq. Our map to the natural numbers is the number of elements in pq. Then we say $f : PQ \to \mathbb{N}$ and $f(i)$ returns the natural number on the ith iteration. For our loop to terminate, we must show that $f(i + 1) < f(i)$. This must be true since the first line of the while loop removes an element from pq and no element is ever added to pq. So $f(i + 1) = f(i) - 1 < f(i)$. Therefore the algorithm terminates.

e) Now suppose that MODIFIEDDIJKSTRAS terminates, is it correct? Let the loop invariant $L_i =$ pred contains the shortest path tree for G_i (the graph with the i processed vertices). Certainly if we have G_i and its shortest path tree, we obtain G_{i+1} by adding the new vertex to the graph. Then we can update the shortest path tree by testing if the new vertex gains us any new shortest paths and updating the current tree as needed. Then when the algorithm terminates, we will have the shortest path tree for all $|V|$ vertices in the graph. Then pred contains the shortest path from s to t and we can just return this path.

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CSCI 432 Problem 6-5

Collaborators: *n/a*

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Describe a "real-life" problem that can be modeled as:

1. An undirected graph.

2. A directed, weighted graph.

3. A tree.

4. A forest.

Answer

1. A an example of an undirected graph is who *can* reach who in a cell tower communication system. Each tower has a fixed radius of broadcast, so if cell tower a can reach b then b can reach a (it is symmetric). Because of this symmetry, there is not need for directed edges and we can just represent the connectivity with an undirected graph.

2. An example of a real-life problem that can be modeled as a directed, weighted graph is a road network. Cities are vertices and roads are edges. The weights on the edges could be determined by distance or time taken to travel the road in that direction.

3. A nice way to think of a tree in a real world application is a file structure system with directories and subdirectories and so on. There can be no cycles in such a file system.

4. An example of a forest in a real-world application is a cloud computing network. Each warehouse is a tree of computers that run in parrallel to complete the tasks thrown at them. And the entire computing network is the union of those trees.