M 383: Assignment 9

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Exercises 5.1.3 — Problem 1

Problem. Show that $f(x) = O(|x - x_0|^2)$ as $x \to x_0$ implies $f(x) = o(|x - x_0|)$ as $x \to x_0$, but give an example to show that the converse is not true.

Proof. So we know that $f(x) = O(|x - x_0|^2)$ as $x \to x_0$ and we want to show that $f(x) = o(|x - x_0|)$ as $x \to x_0$. Equivalently, we could show that $\lim_{x \to x_0} f(x)/|x - x_0| = 0$. From the definition of big-O, we can say that as $x \to x_0$, there exists $1/n$ and a positive constant c such that $|x - x_0| < 1/n$ implies that $|f(x)| \leq c * |x - x_0|^2$. Then for the same neighborhood $(x_0 - 1/n, x_0 + 1/n)$, we must have $|f(x)|/|x - x_0| \leq c * |x - x_0|$. And since non-strict inequalities are preserved by limits, $\lim_{x \to x_0} |f(x)|/|x - x_0| \leq \lim_{x \to x_0} c * |x - x_0| = 0$ for $x \neq x_0$. Since $\lim_{x \to x_0} |f(x)|/|x - x_0|$ cannot be negative, it must equal 0. Then $f(x) = o(|x - x_0|)$ as $x \to x_0$, which was the goal.

A counterexample of the converse is $f(x) = |x - x_0|^{3/2}$. Certainly $|x - x_0|^{3/2} = o(|x - x_0|)$ since

$$\lim_{x \to x_0} \frac{|x - x_0|^{3/2}}{|x - x_0|} = \lim_{x \to x_0} |x - x_0|^{1/2} = 0$$

But, we do not have $|x - x_0|^{3/2} = O(|x - x_0|^2)$. To verify this, let's unpack the definition of big-O. $|x - x_0|^{3/2} = O(|x - x_0|^2)$ only if for all $1/n$ there exists a positive constant c such that $|x - x_0| < 1/n$ implies that $|x - x_0|^{3/2} \leq c|x - x_0|^2$. Then for $x \neq x_0$ (which is trivially true), we must have $1 \leq c|x - x_0|^{1/2}$. However, this condition fails if $|x - x_0|^{1/2} < 1/c$. But since we are considering a neighborhood of x_0, the value of $|x - x_0|^{1/2}$ can be made arbitrarily small with an appropriate choice of x. So no c exists to staisfy the definition of big-O and we have found a counterexample to the converse.

Exercises 5.2.4 — Problem 1

Problem. Let f and g be continuous functions on $[a, b]$ and differentiable at every point in the interior, with $g(a) \neq g(b)$. Prove that there exists a point x_0 in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

Proof. Let's begin by defining $H(x) = (f(b)-f(a))*g(x)-(g(b)-g(a))*f(x)$. Now let $F = f(b)-f(a)$ and $G = g(b) - g(a)$. Then note that

$$H(b) - H(a) = (F * g(b) - G * f(b)) - (F * g(a) - G * f(a)) = F * G - G * F = 0$$

Then since f and g both satisfy the conditions for the mean value theorem, so must $H(x)$ (a linear combination of f and g). So there must exist $x_0 \in (a, b)$ such that

$$H'(x_0) = \frac{H(b) - H(a)}{b - a}$$

But since $H(b) - H(a) = 0$ and $b \neq a$, we must have $H'(x_0) = 0$ for some $x_0 \in (a, b)$. Then $F * g'(x_0) - G*f'(x_0) = (f(b)-f(a))*g'(x_0)-(g(b)-g(a))*f'(x_0) = 0$. Then we must have $(f(b)-f(a))*g'(x_0) = (g(b) - g(a)) * f'(x_0)$. We already know $g(b) - g(a) \neq 0$ and if we assume that $g'(x) \neq 0$ for $x \in (a, b)$, then we get the desired result for some $x_0 \in (a, b)$:

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(x_0)}{g'(x_0)}$$

Exercises 5.2.4 — Problem 2

Problem. If f is a function satisfying

$$|f(x) - f(y)| \leq M|x - y|^{\alpha}$$

for all x and y and some fixed point M and $\alpha > 1$, prove that f is constant.

Proof. Note that the domain of f is the real number line. Then let $y = x_0$ be a real number and take $x \neq x_0$ (every function satisfies the inequality for $x = x_0$). To show that f is constant, we will show that $f'(x_0)$ exists and equals 0. We know that for some fixed M and another fixed $\alpha > 1$ that $|f(x) - f(x_0)| \leq M|x - x_0|^{\alpha}$. Since $x \neq x_0$ and $\alpha - 1 > 0$, we must also have

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} = \left|\frac{f(x) - f(x_0)}{x - x_0}\right| = \left|\frac{f(x) - f(x_0)}{x - x_0} - 0\right| \leq M|x - x_0|^{\alpha - 1}$$

Then we just choose $|x - x_0| < 1/(Mm)^{1/(\alpha-1)}$ and we have exactly that $f'(x_0) = 0$. Since the derivative at an arbitrary x_0 is 0, the derivative is 0 everywhere and the function f must be constant.

Exercises 5.2.4 — Problem 3

Problem. Is the converse of the mean value theorem ture, in the sense that if f is continuous on $[a, b]$ and differentiable on (a, b), given a point x_0 in (a, b) must there exists points x_1, x_2 in $[a, b]$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0)$$

Proof. The converse of the mean value theorem is not true. Consider the counterexample $f(x) = x^3$ defined on $[-1, 1]$ and take $x_0 = 0$. Then we must show that the exist no $x_1, x_2 \in [a, b]$ that satisfy

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

Without loss of generality, take $x_2 > x_1$. Then the above equality is only true if $f(x_2) = f(x_1)$ but this can never occur since $f(x) = x^3$ increases monotonically so $f(x_2) > f(x_1)$. So we have found a counterexample to the converse of the mean value theorem.