Exercises 2.1.3 — Problem 5

Problem. Prove that if a Cauchy sequence $x_1, x_2, ...$ of rationals is modified by changing a finite number of terms, the result is an equivalent Cauchy sequence.

Proof. We must show that modifying a finite number of terms in a Cauchy sequence of rationals results in an equivalent Cauchy sequence. Suppose we have a Cauchy sequence of rationals $x_n = x_1, x_2, ...$ and we modify a finite number of terms to produce another sequence $x'_n = x'_1, x'_2, ...$ Is x'_n Cauchy?

Since we modified a finite number of terms, let x'_l be the last term in x'_n such that $x_l \neq x'_l$. Let $m(n)$ be the index of the term in x_n that satisfies the Cauchy criterion for error $1/n$. For x'_n, we say that $m'(n) = max\{m(n), l+1\}$ to provide an index that satisfies the Cauchy criterion for error $1/n$. So we know that x'_n is Cauchy.

But are x_n and x'_n equivalent? The sequences x_n and x'_n are equivalent if $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$ such that $\forall j \geq m \ |x_j - x'_j| \leq 1/n$. We have already identified that x'_l is the last term in x'_n that differs from x_n. So, regardless of what n is chosen, we select $m = l + 1$. Then, $x_j = x'_j \implies |x_j - x'_j| = |x_j - x'_j| = |0| = 0$ which is certainly less than $1/n$. So the sequences x_n and x'_n are equivalent.

Therefore, modifying a finite number of terms in a Cauchy sequence of rationals results in an equivalent Cauchy sequence.