Section 1.2.3 — Exercise 1

Problem. Prove that every subset of $\mathbb{N}$ is either finite or countable. Conclude from this that there is no infinite set with cardinality less than that of $\mathbb{N}$.

Proof. We must show that every subset of $\mathbb{N}$ is either finite or countable. To do this, we select a subset $A \subset \mathbb{N}$. Then A must be either finite or infinite. If A is finite, there is nothing to show.

If A is infinite, we must show that A is countable. We can show A is countable by listing every element of A:

$$a_1, a_2, a_3, a_4, a_5, ...$$

where a_1 is the smallest element of A, a_2 is the next smallest, and so on. Since the above listing includes every element of A without repitition, A must be countable. So we have shown that every subset of the natural numbers must be finite or countable.

Further, every infinite subset of $\mathbb{N}$ has the same cardinality as $\mathbb{N}$. Therefore, there can exist no smaller infinite set than $\mathbb{N}$.