M 383: Assignment 2

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Exercises 1 — Problem

Problem. Prove that between any two distinct raional numbers there are infinitely many other rationals.

Proof. We must show that between any two distinct rationals there exist infinitely many other rationals. We will prove this by contradiction, assume that there exist two distinct rational numbers a, b such that there are finitely many rationals between a and b. Without loss of generality, assume $a < b$.

We define the set $\mathbb{Q}_{ab} := \{q \in \mathbb{Q} \mid a < q \leq b\}$. Since there are finitely many rationals between a and b, $\mathbb{Q}_{ab}$ has a finite number of elements. Any finite set of rationals has a minimum, so pick p'/q' to be an expression of the rational number $min\{\mathbb{Q}_{ab}\}$. Since p'/q' is the smallest element of $\mathbb{Q}_{ab}$, there are no rationals between a and p'/q'. We can also conclude that for p/q (some expression of a), $p/q < p'/q' \iff pq' < p'q$.

From here, we construct

$$x = \frac{pq' + p'q}{2qq'}$$

Certainly x must be rational by closure of integers under addition and multiplication. Since x is rational, it cannot be between p/q and p'/q' so x must satisfy one of the following cases.

Case $x \leq p/q$:

$$\frac{pq' + p'q}{2qq'} \leq p/q \iff pq' + p'q \leq 2pq' \iff p'q \leq pq' \iff pq' \geq p'q$$

which is a contradiction since $pq' < p'q$.

Case $x \geq p'/q'$:

$$\frac{pq' + p'q}{2qq'} \geq p'/q' \iff pq' + p'q \geq 2p'q \iff pq' \geq p'q$$

which is a contradiction since $pq' < p'q$.

Since both cases for x lead to a contradiction, we must have incorrectly assumed that there exist distinct rationals with finitely many rationals between them. Therefore, it must be true that between any two distinct rationals there are infinitely many other rationals.

Exercises 2.1.3 — Problem 3

Problem. What kinds of real numbers are representable by Cauchy sequences of integers?

Proof. We must say what types of real numbers are representable by Cauchy sequences of integers. Let $x_n = x_1, x_2, ...$ be a Cauchy sequence of integers. Since x_n is Cauchy, it must be true that $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$ such that $\forall j, k \geq m \ |x_j - x_k| \leq 1/n$.

Choose $n = 2$, then $1/n = 1/2$. We must also have x_j (an integer) and $|x_j - x_k| \leq 1/2$, but the only integer value of x_k that satisifes $|x_j - x_k| \leq 1/2$ is $x_k = x_j$. So, the sequence must be have a constant integer value x_j beyond the m^{th} term. Then $x_n \to x_j$. Since x_j is an integer, a Cauchy sequence of integers can only represent an integer.

Exercises 2.1.3 — Problem 5

Problem. Prove that if a Cauchy sequence $x_1, x_2, ...$ of rationals is modified by changing a finite number of terms, the result is an equivalent Cauchy sequence.

Proof. We must show that modifying a finite number of terms in a Cauchy sequence of rationals results in an equivalent Cauchy sequence. Suppose we have a Cauchy sequence of rationals $x_n = x_1, x_2, ...$ and we modify a finite number of terms to produce another sequence $x'_n = x'_1, x'_2, ...$ Is x'_n Cauchy?

Since we modified a finite number of terms, let x'_l be the last term in x'_n such that $x_l \neq x'_l$. Let $m(n)$ be the index of the term in x_n that satisfies the Cauchy criterion for error $1/n$. For x'_n, we say that $m'(n) = max\{m(n), l+1\}$ to provide an index that satisfies the Cauchy criterion for error $1/n$. So we know that x'_n is Cauchy.

But are x_n and x'_n equivalent? The sequences x_n and x'_n are equivalent if $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$ such that $\forall j \geq m \ |x_j - x'_j| \leq 1/n$. We have already identified that x'_l is the last term in x'_n that differs from x_n. So, regardless of what n is chosen, we select $m = l + 1$. Then, $x_j = x'_j \implies |x_j - x'_j| = |x_j - x'_j| = |0| = 0$ which is certainly less than $1/n$. So the sequences x_n and x'_n are equivalent.

Therefore, modifying a finite number of terms in a Cauchy sequence of rationals results in an equivalent Cauchy sequence.

Exercises 2.1.3 — Problem 8

Problem. Can a Cauchy sequence of positive rational numbers be equivalent to a Cauchy sequence of negative rational numbers?

Proof.

Exercises 2.1.3 — Problem 9

Problem. Show that if $x_1, x_2, ...$ is a Cuachy sequence of rational numbers there exists a positive integer N such that $x_j \leq N$ for all j.

Proof.