M 383: Assignment 1

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Section 1.1.3 — Exercise 2c

Problem. Begin with the statement "Multiplication of integers is associative." Rewrite the state with explicit quantifiers. Then form the negation of the statement. Finally, recast the negation in a form similar to the original statement.

Proof. The statement "Multiplication of integers is associative" can be rewritten with explicit qunatifiers as

$$\forall a, b, c \in \mathbb{Z}, a * (b * c) = (a * b) * c$$

The negation of the statement is

$$\exists a, b, c \in \mathbb{Z} \text{ such that } a * (b * c) \neq (a * b) * c$$

We can rewrite the negation (in words) as "Multiplication of integers is not always associative."

Section 1.2.3 — Exercise 1

Problem. Prove that every subset of $\mathbb{N}$ is either finite or countable. Conclude from this that there is no infinite set with cardinality less than that of $\mathbb{N}$.

Proof. We must show that every subset of $\mathbb{N}$ is either finite or countable. To do this, we select a subset $A \subset \mathbb{N}$. Then A must be either finite or infinite. If A is finite, there is nothing to show.

If A is infinite, we must show that A is countable. We can show A is countable by listing every element of A:

$$a_1, a_2, a_3, a_4, a_5, \ldots$$

where a_1 is the smallest element of A, a_2 is the next smallest, and so on. Since the above listing includes every element of A without repitition, A must be countable. So we have shown that every subset of the natural numbers must be finite or countable.

Further, every infinite subset of $\mathbb{N}$ has the same cardinality as $\mathbb{N}$. Therefore, there can exist no smaller infinite set than $\mathbb{N}$.

Section 1.2.3 — Exercise 3

Problem. Prove that the rational numbers are countable.

Proof. We must show that the rational numbers are countable. We can do so by listing all the rationals. Before doing so, let's set up some notation. For a given $k \in \mathbb{N}$, let $Q_k = \{\pm j/k \mid j \in \mathbb{N}\}$. We then say that $U = \bigcup_{k=1}^{\infty} Q_k$. Then the set of rational numbers is $\mathbb{Q} = \{0\} \bigcup U$.

Note that each Q_k is countable since we can list all of its elements as

$$\frac{1}{k}, \frac{-1}{k}, \frac{2}{k}, \frac{-2}{k}, \frac{3}{k}, \frac{-3}{k}, \frac{4}{k}, \frac{-4}{k}, \ldots$$

We can then form an infinite table with Q_k as the k^{th} row (we denote the i^{th} element of Q_k as q_{ki}).

$$\begin{array}{cccc} q_{11} & q_{12} & q_{13} & \cdots \\ q_{21} & q_{22} & q_{23} & \cdots \\ q_{31} & q_{32} & q_{33} & \cdots \\ \vdots & & \ddots & \end{array}$$

We can then list all the elements of $\mathbb{Q}$. We start with 0, then we read the above table diagonally and ignore any duplicated elements:

$$L_{\mathbb{Q}} = 0, q_{11}, q_{21}, q_{12}, q_{31}, q_{22}, q_{23}, \cdots$$

Then $\mathbb{N}$ has a one-to-one correspondence with $L_{\mathbb{Q}}$. So, we have shown that $L_{\mathbb{Q}}$ is countable.

Section 1.2.3 — Exercise 4

Problem. Show that if a countable subset is removed from an uncountable set, the remainder is still uncountable.

Proof. We begin by giving the problem some notation. Given an uncountable set A and a countable subset $B \subset A$, we must show that $A \setminus B$ is uncountable.

Towards a contradiction, let's assume that we have an uncountable set A and a countable subset $B \subset A$ such that $A \setminus B$ is countable. Let $C = A \setminus B$. Since B and C are both countable, we can list all of their elements: $b_1, b_2, b_3, ...$ and $c_1, c_2, c_3, ...$ From here, we can also deduce that $B \bigcup C$ is countable because every element of $B \bigcup C$ can be listed: $b_1, c_1, b_2, c_2, b_3, c_3, ...$

Yet, $B \bigcup C = A$ and we know A to be uncountable. So, we have reached a contradiction since A cannot be both countable and uncountable.

Since we reached a contradiction, we must have incorrectly assumed that $A \setminus B$ could be countable. Therefore, $A \setminus B$ must be uncountable and we have shown that removing a countable set from an uncoubtable set must always result in an uncountable set.

Section 1.2.3 — Exercise 5

Problem. Let $A_1, A_2, A_3, ...$ be countable sets, and let their Cartesian product $A_1 \times A_2 \times A_3 \times \cdots$ be defined to be the set of all sequences $(a_1, a_2, ...)$ where a_k is an element of A_k. Prove that the Cartesian product is uncountable. Show that the same conclusion holds if each of the sets $A_1, A_2, ...$ has at least two elements.

Proof. We must show that the Caresian product of a countable number of countable sets is uncountable. Towards a contradiction, let's assume that such a Cartesian product is countalbe. For notation, we assume that $P = A_1 \times A_2 \times A_3 \times \cdots$ is a countable set.

Since P is countable, there exists a bijective map $f : \mathbb{N} \longrightarrow P$. For a given $n \in \mathbb{N}$, we say $f(n) = (a_1^n, a_2^n, a_3^n, ...) \in P$. To get our contradiction, we will construct an element $p = (p_1, p_2, p_3, ...) \in P$ such that $p \notin im(f)$.

When constructing p, we choose p_k such that $p_k \in A_k$ and $p_k \neq a_k^k$. We choose $p_k \in A_k$ so that p is an element of P and we require that $p_k \neq a_k^k$ so that there is no $n \in \mathbb{N}$ such that $f(n) = p$. We know that such a $p_k \in A_k$ exists because A_k has countably many elements. Since there is no n such that $f(n) = p$, the element $p \notin im(f)$. This means that f is not surjective.

Yet we assumed that f is surjective so we have reached the contradiction that f must be both surjective and not surjective. Since we reached a contradiction, we must have incorrectly assumed that P is countable. So we have shown that the Cartesian product of a countable number of countable sets is uncountable.

We can make our result even stronger. Our selection of p_k only requires that there is a single element $a'_k \in A_k$ such that $a'_k \neq a_k^k$. So the proof holds so long as $\{a_k^k, a'_k\} \subset A_k$. Thus, each A_k needs only two elements.