Exercises 1 — Problem

Problem. Prove that between any two distinct raional numbers there are infinitely many other rationals.

Proof. We must show that between any two distinct rationals there exist infinitely many other rationals. We will prove this by contradiction, assume that there exist two distinct rational numbers a, b such that there are finitely many rationals between a and b. Without loss of generality, assume $a < b$.

We define the set $\mathbb{Q}_{ab} := \{q \in \mathbb{Q} \mid a < q \leq b\}$. Since there are finitely many rationals between a and b, $\mathbb{Q}_{ab}$ has a finite number of elements. Any finite set of rationals has a minimum, so pick p'/q' to be an expression of the rational number $min\{\mathbb{Q}_{ab}\}$. Since p'/q' is the smallest element of $\mathbb{Q}_{ab}$, there are no rationals between a and p'/q'. We can also conclude that for p/q (some expression of a), $p/q < p'/q' \iff pq' < p'q$.

From here, we construct

$$x = \frac{pq' + p'q}{2qq'}$$

Certainly x must be rational by closure of integers under addition and multiplication. Since x is rational, it cannot be between p/q and p'/q' so x must satisfy one of the following cases.

Case $x \leq p/q$:

$$\frac{pq' + p'q}{2qq'} \leq p/q \iff pq' + p'q \leq 2pq' \iff p'q \leq pq' \iff pq' \geq p'q$$

which is a contradiction since $pq' < p'q$.

Case $x \geq p'/q'$:

$$\frac{pq' + p'q}{2qq'} \geq p'/q' \iff pq' + p'q \geq 2p'q \iff pq' \geq p'q$$

which is a contradiction since $pq' < p'q$.

Since both cases for x lead to a contradiction, we must have incorrectly assumed that there exist distinct rationals with finitely many rationals between them. Therefore, it must be true that between any two distinct rationals there are infinitely many other rationals.