Exercises 2.1.3 — Problem 9

Problem. Show that if $x_1, x_2, ...$ is a Cauchy sequence of rational numbers there exists a positive integer N such that $x_j \leq N$ for all j.

Proof. We must show that for every Cauchy sequence of rational numbers, there exists a natural number larger than every term in the sequence. Take the Cauchy sequence $x_1, x_2, ...$ Then, for every $n \in \mathbb{N}$ there must exist $m \in \mathbb{N}$ such that $\forall j, k \geq m \ \ |x_j - x_k| \leq 1/n$.

Pick $n = 1$ and let m be an index that satisfies the Cauchy criterion for x_n. Then define the finite set $L := \{x_k \mid k < m\}$. That is, the set of sequence elements prior to x_m. Sequence elements following (and including) x_m must be smaller than $\lceil x_m + 1 \rceil$. Then we can take $N = max\{L, \lceil x_m + 1 \rceil\}$, which must be larger than every term in the sequence.

So we have shown the existence of a natural number larger than every term in a Cauchy sequence.