Exercises 5.2.4 — Problem 3

Problem. Is the converse of the mean value theorem ture, in the sense that if f is continuous on $[a, b]$ and differentiable on (a, b), given a point x_0 in (a, b) must there exists points x_1, x_2 in $[a, b]$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0)$$

Proof. The converse of the mean value theorem is not true. Consider the counterexample $f(x) = x^3$ defined on $[-1, 1]$ and take $x_0 = 0$. Then we must show that the exist no $x_1, x_2 \in [a, b]$ that satisfy

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

Without loss of generality, take $x_2 > x_1$. Then the above equality is only true if $f(x_2) = f(x_1)$ but this can never occur since $f(x) = x^3$ increases monotonically so $f(x_2) > f(x_1)$. So we have found a counterexample to the converse of the mean value theorem.