Exercises 4.1.5 — Problem 4

Problem. Give a definition of $\lim_{x\to\infty} f(x) = y$. Show that this is true if and only if for every sequence $x_1, x_2, ...$ of points in the domain of f such that $\lim_{n\to\infty} x_n = +\infty$, we have $\lim_{n\to\infty} f(x_n) = y$.

Proof. We define $\lim_{x\to\infty} f(x)$ (for functions with unbounded positive domain) to be $y \in \mathbb{R}$ such that for all m there exists n such that for all $x > n$ ($x \in \mathbb{R}$) we have $|y - f(x)| < 1/m$. We now show that this is the case if and only if every sequence of points in the domain with a limit of $+\infty$ has $\lim_{n\to\infty} f(x_n) = y$.

First suppose that y is a real number such that for every m there exists n such that for all $x > n$ we have $|y - f(x)| < 1/m$. Then we would like to show every sequence $x_1, x_2, ...$ in the domain of f with $+\infty$ as a limit has $\lim_{n\to\infty} f(x_n) = y$. The sequence $x_1, x_2, ... \to +\infty$ so for every $a \in \mathbb{N}$ there exists an index b such that for all $j \geq b$ we have $x_j > a$. But then taking $a = n$ we have $|y - f(x_j)| < 1/m$ for all $j \geq n$. In the limit $j \to \infty$, we have $|y - f(x)| \leq 1/m$ which satisfies $\lim_{n\to\infty} f(x_n) = y$.

Now suppose that every sequence of points in the domain of f such that $\lim_{x\to\infty} x_n = +\infty$, we have $\lim_{n\to\infty} f(x_n) = y$. We want to show that $\lim_{x\to\infty} f(x) = y$. By Theorem 4.1.1, what we just said is equivalent to claiming that $\lim_{x\to\infty} f(x)$ exists. Further, the theorem states that every sequence $f(x_1), f(x_2), ...$ has a common limit and that common limit the limit of the function. Therefore $\lim_{x\to\infty} f(x) = y$.