Exercises 4.2.4 — Problem 9

Problem. If f and g are uniformly continuous, show that $f+g$ is uniformly continuous.

Proof. So we have to uniformly continuous functions f and g defined on the intersections of their domains. Then for all $1/2m$, there exist $1/n_1, 1/n_2$ such that for all x, x_0 we have $|x-x_0| < 1/n_1$ implies $|f(x)-f(x_0)| < 1/2m$ and $|x-x_0| < 1/n_2$ implies $|g(x)-g(x_0)| < 1/2m$. Is it the case that $f+g$ is also uniformly continuous?

Select $|x-x_0| < 1/n = min(1/n_1, 1/n_2)$, then $|(f+g)(x)-(f+g)(x_0)| = |f(x)+g(x)-f(x_0)-g(x_0)| = |f(x)-f(x_0)+g(x)-g(x_0)|$. By triangle inequality we can say that $|f(x)-f(x_0)+g(x)-g(x_0)| \leq |f(x)-f(x_0)|+|g(x)-g(x_0)| < 1/2m+1/2m = 1/m$. So for any $1/m$, we can choose $1/n = min(1/n_1, 1/n_2)$ to satsify uniform continuity for $f+g$.