Exercises 4.1.5 — Problem 2

Problem. Let A be the set defined by the equations $f_1(x) = 0, f_2(x) = 1, ..., f_n(x) = 0$, where $f_1, ..., f_n$ are continuous functions defined on the whole line. Show that A is closed. Must A be compact?

Proof. We first show that A is closed. Given a function f^k, let $D_k = \{x \mid f^k(x) = 0\}$. Then we can say that $A = \cup_{i=1}^{n} D_i$. So A is union of a finite number of sets. If each D_k is closed, then A (a finite intersection of closed sets) must also be closed.

Let's now verify that every D_k is closed. Pick an arbitrary $D_k = \{x \mid f^k(x) = 0\}$. Now consider the result (from exercise 4.1.5 problem 1) that a function f defined on a closed domain is continuous if and only if the inverse image of every closed set is a closed set. f^k is a continuous function (by hypothesis) defined on the closed set $\mathbb{R}$, so we can equivalently say that the inverse image of every closed set is a closed set. Since $\{0\}$ is a closed set, the inverse image of $\{0\}$ under f^k is closed. But the inverse image is $\{x \mid f^k(x) = 0\}$ which is D_k. Therefore D_k is closed which implies that A is closed.

It is not necessarily the case that A is compact. We already know A is necessarily closed so let's find a counterexample where A is not bounded (since then A will not be compact). Suppose we have f^1 be the constant function $f^1 : x \mapsto 0$. Then $D_1 = \{x \mid f^1(x) = 0\} = \mathbb{R}$ which is not bounded. Therefore A is not necessarily compact.