M 383: Assignment 3

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Exercises 2.2.4 — Problem 3

Problem. If x is a real number, show that there exists a Cauchy sequence of rationals $x_1, x_2, ...$ representing x such that $x_n < x$ for all n.

Proof. Pick $y_1, y_2, ...$ a Cauchy sequence of rationals with $lim_{k \to \infty} y_k = x$, we know that $\forall n \in \mathbb{N}, \exists m(n) \in \mathbb{N}$ such that $|y - y_k| < 1/n$ for all $k \geq m(n)$. We will then edit the sequence $\{y_k\}$ such that every $y_k < y$ but we will not change the limit x.

First, choose some rational number $q \in \mathbb{Q}$ such that $q < y$. Then change the first $m(1)$ elements of $\{y_k\}$ to have the value q. Since the sequence is Cauchy, $m(1)$ must be finite so we have just changed a finite number of terms of $\{y_k\}$. Thus, the $lim_{k \to \infty} y_k$ did not change must still be x.

Next we construct a sequence $\{m_k\}$ where the first $m(1)$ elements are 0, the next $m(2)$ elements are 1, the next $m(3)$ elements are $1/2$, the next $m(4)$ elements are $1/3$, and so on:

$$0, 0, ..., 0, 1, 1, ..., 1, 1/2, 1/2, ..., 1/2, 1/3, 1/3, ...1/3, 1/4, ...$$

We can see that $\{m_k\}$ is Cauchy and that $lim_{k \to \infty} m_k = 0$. This implies that $lim_{k \to \infty}(y_k - m_k) = x - 0 = x$. However, for each k, we have constructed m_k such that $y_k - m_k < x$. Then let $x_k = y_k - m_k$ and we have a Cauchy sequence of rationals $x_1, x_2, ...$ that represents x such that $x_k < x$ for all $k \in \mathbb{N}$.

Exercises 2.2.4 — Problem 7

Problem. Prove that $|x - y| \geq |x| - |y|$ for any real numbers x and y.

Proof. We must show that $|x - y| \geq |x| - |y|$ holds for all real numbers x and y. We take $|x + y| \leq |x| + |y|$ (the triangle inequality) to be true. Then let x and y be any real numbers. Certainly it is true that $|x| = |x - y + y|$ since $\mathbb{R}$ is an ordered field. Also, by the triangle inequality, $|x - y + y| \leq |x - y| + |y|$. Since $|x - y + y| = |x|$, we can say that $|x - y| + |y| \geq |x|$ which we can reorder to be $|x - y| \geq |x| - |y|$. So we have shown the desired inequality for all real numbers x and y.

Exercises 2.3.3 — Problem 1

Problem. Write out a proof that $lim_{k\to\infty}(x_k + y_k) = x + y$ if $lim_{k\to\infty}x_k = x$ and $lim_{k\to\infty}y_k = y$ for sequences of real numbers.

Proof. We want to show (for sequences of real numbers) that the limit of a sum is the same as the sum of the limits. Given that $lim_{k\to\infty}x_k = x$ and $lim_{k\to\infty}y_k = y$, we want to show that $\{x_k + y_k\}$ satisfies the Cauchy criterion and converges to $x + y$.

Saying that $\{x_k\}$ and $\{y_k\}$ have limits is equivalent to saying $\{x_k\}$ and $\{y_k\}$ are Cauchy. We also know the sum of two Cauchy sequences to be Cauchy, so certainly $\{x_k + y_k\}$ is Cauchy.

But what is the limit of $\{x_k + y_k\}$? Let's begin by noting that $lim_{k\to\infty}x_k = x \implies \forall n, \exists m_a$ such that $|x_a - x| \leq 1/2n$ for all $a \geq m_a$. Similarly $lim_{k\to\infty}y_k = y \implies \forall n, \exists m_b$ such that $|y_b - y| \leq 1/2n$ for all $b \geq m_b$. Choose $k = max\{a, b\}$. Then $|(x_k+y_k)-(x+y)| = |(x_k-x)+(y_k-y)| \leq 1/2n+1/2n = 1/n$. So $x + y$ satisfies the definition of a limit for the sequence $\{x_k + y_k\}$. Thus the limit of a sum is equal the sum of the limits.

Exercises 2.3.3 — Problem 3

Problem. Let $x_1, x_2, ...$ be a sequence of real numbers such that $|x_n| \leq 1/2^n$, and set $y_n = x_1 + x_2 + \cdots + x_n$. Show that the sequence $y_1, y_2, ...$ converges.

Proof. We must show that $y_1, y_2, ...$ converges. To this end, we introduce the following result for any natural number n: $1/2^n = \sum_{k=n+1}^{\infty} 1/2^k$. Let's now prove this result. First note that for some natural number a, $1/2^{a+1} = 1/2 * 1/2^a$. Then for any a, we have

$$1/2^a + 1/2^{a+1} + 1/2^{a+2} + \cdots = 1/2^a + 1/2 * 1/2^a + 1/2 * 1/2^{a+1} + \cdots = 1/2^a + 1/2 * (1/2^a + 1/2^{a+1} + \cdots)$$

which implies that

$$1/2^a = 1/2 * (1/2^a + 1/2^{a+1} + \cdots) = 1/2 * 1/2^a + 1/2 * 1/2^{a+1} + \cdots = 1/2^{a+1} + 1/2^{a+2} + \cdots = \sum_{k=a+1}^{\infty} 1/2^k$$

Now to show that $y_1, y_2, ...$ converges, we will show that $y_1, y_2, ...$ is Cauchy. So, given a natural number n, we must show the existence of an index m such that $|y_j - y_k| \leq 1/n$ for all $j, k \geq m$. For any index m, we can provide an upper bound for $|y_j - y_k|$ by choosing the largest possible y_j and the smallest possible y_k. Because $1/2^n = \sum_{k=n+1}^{\infty} 1/2^k$ and $x_n \leq 1/2^n$, we can say that the largest y_j could be is $y_m + 1/2^m$ and the smallest y_k could be is $y_m - 1/2^m$. Then

$$|y_j - y_k| \leq |(y_m + 1/2^m) - (y_k - 1/2^m)| = |2/2^m| = 1/2^{m-1}$$

Choosing $m = n$, we can say that $1/2^{m-1} = 1/2^{n-1} \leq 1/n$ for all n. So there exists an index m such that $|y_j - y_k| \leq 1/n$ for all $j, k \geq m$. Thus $y_1, y_2, ...$ satisfies the Cauchy criterion and must also converge.