Exercises 2.1.3 — Problem 3

Problem. What kinds of real numbers are representable by Cauchy sequences of integers?

Proof. We must say what types of real numbers are representable by Cauchy sequences of integers. Let $x_n = x_1, x_2, ...$ be a Cauchy sequence of integers. Since x_n is Cauchy, it must be true that $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$ such that $\forall j, k \geq m \ |x_j - x_k| \leq 1/n$.

Choose $n = 2$, then $1/n = 1/2$. We must also have x_j (an integer) and $|x_j - x_k| \leq 1/2$, but the only integer value of x_k that satisifes $|x_j - x_k| \leq 1/2$ is $x_k = x_j$. So, the sequence must be have a constant integer value x_j beyond the m^{th} term. Then $x_n \to x_j$. Since x_j is an integer, a Cauchy sequence of integers can only represent an integer.