M 383: Assignment 1

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Section 1.1.3 — Exercise 2c

Problem. Begin with the statement "Multiplication of integers is associative." Rewrite the state with explicit quantifiers. Then form the negation of the statement. Finally, recast the negation in a form similar to the original statement.

Proof. The statement "Multiplication of integers is associative" can be rewritten with explicit qunatifiers as

$$\forall a, b, c \in \mathbb{Z}, a * (b * c) = (a * b) * c$$

The negation of the statement is

$$\exists a, b, c \in \mathbb{Z} \text{ such that } a * (b * c) \neq (a * b) * c$$

We can rewrite the negation (in words) as "Multiplication of integers is not always associative."

Section 1.2.3 — Exercise 1

Problem. Prove that every subset of $\mathbb{N}$ is either finite or countable. Conclude from this that there is no infinite set with cardinality less than that of $\mathbb{N}$.

Proof.

Section 1.2.3 — Exercise 3

Problem. Prove that the rational numbers are countable.

Proof. We can prove that the rational numbers are countable by listing all the rationals. Before doing so, let's set up some notation. For a given $k \in \mathbb{N}$, let $Q_k = \{\pm j/k \mid j \in \mathbb{N}\}$. We then say that $U = \bigcup_{k=1}^{\infty} Q_k$. Then the set of rational numbers is $\mathbb{Q} = \{0\} \bigcup U$.

Note that each Q_k is countable since we can list all of its elements as

$$\frac{1}{k}, \frac{-1}{k}, \frac{2}{k}, \frac{-2}{k}, \frac{3}{k}, \frac{-3}{k}, \frac{4}{k}, \frac{-4}{k}, \cdots$$

We can then form an infinite table with Q_k as the k^{th} row (we denote the i^{th} element of Q_k as q_{ki}).

$$\begin{array}{cccc} q_{11} & q_{12} & q_{13} & \cdots \\ q_{21} & q_{22} & q_{23} & \cdots \\ q_{31} & q_{32} & q_{33} & \cdots \\ \vdots & & \ddots & \end{array}$$

We can then list all the elements of $\mathbb{Q}$ by reading the above table diagonally and ignoring any duplicate elements (seperately, we must remember 0 at the beginning):

$$L_{\mathbb{Q}} = 0, q_{11}, q_{21}, q_{12}, q_{31}, q_{22}, q_{23}, \cdots$$

As a final step, all the duplicates in $L_{\mathbb{Q}}$ must be removed. Then $\mathbb{N}$ has a one-to-one correspondence with $L_{\mathbb{Q}}$. So, we have shown that $L_{\mathbb{Q}}$ is countable.

Section 1.2.3 — Exercise 4

Problem. Show that if a countable subset is removed from an uncountable set, the remainder is still uncountable.

Proof.

Section 1.2.3 — Exercise 5

Problem. Let $A_1, A_2, A_3, \ldots$ be countable sets, and let their Cartesian product $A_1 \times A_2 \times A_3 \times \cdots$ be defined to be the set of all sequences $(a_1, a_2, \ldots)$ where a_k is an element of A_k. Prove that the Cartesian product is uncountable. Show that the same conclusion holds if each of the sets $A_1, A_2, \ldots$ has at least two elements.

Proof.