M 383: Assignment 6

Nathan Stouffer

Exercises 3.3.1 — Problem 1

Problem. Show that compact sets are closed under arbitrary intersections and finite unions.

Proof. First note that a set of real numbers A is compact if and only if A is closed and bounded. Additionally, note that closed sets are closed under arbitrary intersection and finite unions.

We now show that compact sets are closed under arbitrary intersections. Let $\mathbb{B}$ be any collection of compact sets and $B' = \cup_{B \in \mathbb{B}} B$. The set B' is compact if and only if it is closed and bounded. Certainly $\mathbb{B}$ is closed since every $B \in \mathbb{B}$ is closed and their intersection must be closed. We show that B' is bounded by contradiction. Suppose B' is not bounded. Then there exists a sequence $x_1, x_2, ...$ of elements in B' such that for all n, there exists a j such that $x_j > n$ or $x_j < -n$. But sequence $x_1, x_2, ...$ must also be in every $B \in \mathbb{B}$ so we have just shown that every $B \in \mathbb{B}$ is unbounded. But then no $B \in \mathbb{B}$ is compact, which is a contradiction. Therefore B' is both closed and bounded, which implies that B' is compact.

We now show that compact sets are closed under finite unions. Let $C_1, C_2, ..., C_n$ be a finite collection of compact sets and $C' = \cap_{i=1}^{n} C_i$. To show that C' is compact, we will show that it is closed and bounded. C' is closed since it is the intersection of a finite collection of closed sets. Is C' bounded? Suppose that C' is not bounded, then one of two cases must occur. Either there exists a monotonically increasing sequence $x_1, x_2, ...$ of elements in C' such that for all n there exists j such that $x_j > n$ or there exists a monotonically decreasing sequence $y_1, y_2, ...$ of elements in C' such that for all n there exists j such that $y_j < -n$. In either case, we can use the pigeon hole principle to say at least one of B_i contains an infinite number of terms in the sequence (whether it be increasing or decreasing). But then that B_i must be unbounded, which is a contradiction since we assumed every B_i to be compact. Therefore, the union of a finite number of compact sets must be compact.

Exercises 3.3.1 — Problem 4

Problem. If $A \subset B_1 \cup B_2$ where B_1 and B_2 are disjoint open sets and A is compact, show that $A \cap B_1$ is compact. Is the same true if B_1 and B_2 are not disjoint?

Proof. Let's first analyze the case where $B_1 \cap B_2 = \emptyset$. So we must show that $A \cap B_1$ is compact, that is, every sequence that lies entirely in $A \cap B_1$ has a limit-point in $A \cap B_1$. To this end, pick a sequence $x_1, x_2, ...$ that lies entirely in $A \cap B_1$. Every x_j is a point in A and A is compact so $x_1, x_2, ...$ certainly has a limit-point x in A. But is x a member of B_1? Since $A \subset B_1 \cup B_2$ and $x \in A$, it must be true that x is in B_1 or B_2. Further, we know that $B_1 \cap B_2 = \emptyset$ so x must be in only one of B_1 and B_2.

Suppose that x is a point in B_2. Since B_2 is open, there must exist an open interval $(a, b) \subset B_2$ such that $x \in (a, b)$. Since $(a, b) \subset B_2$ and $B_1 \cap B_2 = \emptyset$, no elements of B_1 are in the interval (a, b). But every element in the sequence $x_1, x_2, ...$ is a member of B_1, so we have just shown that there exists a neighborhood (a, b) of x such that no element of $x_1, x_2, ...$ is contained (a, b). Then x is not a limit-point of $x_1, x_2, ...$, but this is a contradiction since we know x to be a limit-point of the sequence. So $x \notin B_2$, which implies that $x \in B_1$. Then x (a limit-point of $x_1, x_2, ...$) is a point in both A and B_1, which implies that $x \in A \cap B_1$. Since $x_1, x_2, ...$ is an arbitrary sequence entirely in $A \cap B_1$, we have just shown that every sequence in $A \cap B_1$ has a limit-point in $A \cap B_1$. In other words, we have shown that $A \cap B_1$ is compact.

We now wonder if $A \cap B_1$ is necessarily compact if we instead assume that $B_1 \cap B_2 \neq \emptyset$. The statement does not hold; for evidence, see the following counterexample. Let $B_1 = (0, 2)$ and $B_2 = (1, 3)$ (open sets with a nonempty intersection). Then let $A = [1, 2] \subset B_1 \cup B_2$ (which is compact since A is closed and bounded). Then $A \cap B_1 = [1, 2)$. The set $A \cap B_1$ is not closed since it does not contain its limit-ponit 2, then since $A \cap B_1$ is not closed, it is not compact.

Exercises 3.3.1 — Problem 8

Problem. If A is compact, show that $\sup A$ and $\inf A$ belong to A. Give an example of a non-compact set A such that both $\sup A$ and $\inf A$ belong to A.

Proof. Suppose that we have some compact set A, we must show that $\sup A \in A$ and $\inf A \in A$. First we will show that $\sup A \in A$. Let $y = \sup A$. Then for all n, there exists some $y_n \in A$ such that $y - 1/n < y_n$. Which implies that for all n, there exists $y_n \in A$ such that $y - y_n = |y - y_n| < 1/n$. Let $y_1, y_2, ...$ be a sequence (not necessarily distinct) that satisfies $|y - y_n| < 1/n$. From this, we can gather that $\{y_n\}$ converges to y (which means that y is the only limit point of the sequence). Then since every $y_n \in A$ and A is compact, some limit-point of $y_1, y_2, ...$ is a point in A. But we already know that y is the only limit-point of $y_1, y_2, ...$, so we must have $y \in A$. And $y = \sup A$ so it must be true that $\sup A \in A$.

Similarly, we now show that $\inf A \in A$. Let $z = \inf A$. Then for all n, there exists some $z_n \in A$ such that $z + 1/n > z_n$. Which implies that for all n, there exists $z_n \in A$ such that $z_n - z = |z_n - z| = |z - z_n| < 1/n$. Let $z_1, z_2, ...$ be a sequence (not necessarily distinct) that satisfies $|z - z_n| < 1/n$. From this, we can gather that $\{z_n\}$ converges to z (which means that z is the only limit point of the sequence). Then since every $z_n \in A$ and A is compact, some limit-point of $z_1, z_2, ...$ is a point in A. But we already know that z is the only limit-point of $z_1, z_2, ...$, so we must have $z \in A$. And $z = \inf A$ so it must be true that $\inf A \in A$.

We now give an example of a non-compact set A such that $\sup A$ and $\inf A$ belong to A. Let $A = [0, 1) \cup (1, 2]$. The set A is not closed since it does not contain its limit-point 1, which means A is not compact. However, $\sup A = 2 \in A$ and $\inf A = 0 \in A$.