Section 1.2.3 — Exercise 3

Problem. Prove that the rational numbers are countable.

Proof. We must show that the rational numbers are countable. We can do so by listing all the rationals. Before doing so, let's set up some notation. For a given $k \in \mathbb{N}$, let $Q_k = \{\pm j/k \mid j \in \mathbb{N}\}$. We then say that $U = \bigcup_{k=1}^{\infty} Q_k$. Then the set of rational numbers is $\mathbb{Q} = \{0\} \bigcup U$.

Note that each Q_k is countable since we can list all of its elements as

$$\frac{1}{k}, \frac{-1}{k}, \frac{2}{k}, \frac{-2}{k}, \frac{3}{k}, \frac{-3}{k}, \frac{4}{k}, \frac{-4}{k}, \ldots$$

We can then form an infinite table with Q_k as the k^{th} row (we denote the i^{th} element of Q_k as q_{ki}).

$$\begin{array}{cccc} q_{11} & q_{12} & q_{13} & \cdots \\ q_{21} & q_{22} & q_{23} & \cdots \\ q_{31} & q_{32} & q_{33} & \cdots \\ \vdots & & \ddots & \end{array}$$

We can then list all the elements of $\mathbb{Q}$. We start with 0, then we read the above table diagonally and ignore any duplicated elements:

$$L_{\mathbb{Q}} = 0, q_{11}, q_{21}, q_{12}, q_{31}, q_{22}, q_{23}, \cdots$$

Then $\mathbb{N}$ has a one-to-one correspondence with $L_{\mathbb{Q}}$. So, we have shown that $L_{\mathbb{Q}}$ is countable.