Exercises 3.2.3 — Problem 5

Problem. Let A be a closed set, x a point in A, and B be the set A with x removed. Under what conditions is B closed?

Proof. We first show that removing a single point from a set does not change the set of limit-points. We know this because, for a point y to be a limit-point the set C, we must have that every neighborhood around y must contain infinitely many points of C. Removing a single point from C will not change the fact that there are infinitely many points in each neighborhood.

B is closed if and only if x is not a limit-point of A. We prove this in two parts. Going to the right, we must show that B is closed implies that x is not a limit-point of A. We show this with the contrapostive: x is a limit-point of A implies that B is not closed. So we know x to be limit point of A and $B = A \setminus \{x\}$. B has the same limit points as A so x is a limit-point of B. But x is not contained in B so B is not closed.

Now going to the left, we assume that x is not a limit-point of A. Since x is not a limit-point of A, x is also not a limit-point of B (for B has the same limit points as A). Then A contains all of its limits points (for it is closed), and B must contain the same set of limit points since x was not a limit point. So B contains all of its limit points and is closed.