M 383: Assignment 1

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Section 1.1.3 — Exercise 2c

Problem. Begin with the statement "Multiplication of integers is associative." Rewrite the state with explicit quantifiers. Then form the negation of the statement. Finally, recast the negation in a form similar to the original statement.

Proof. The statement "Multiplication of integers is associative" can be rewritten with explicit qunatifiers as

$$\forall a, b, c \in \mathbb{Z}, a * (b * c) = (a * b) * c$$

The negation of the statement is

$$\exists a, b, c \in \mathbb{Z} \text{ such that } a * (b * c) \neq (a * b) * c$$

We can rewrite the negation (in words) as "Multiplication of integers is not always associative."

Section 1.2.3 — Exercise 1

Problem. Prove that every subset of $\mathbb{N}$ is either finite or countable. Conclude from this that there is no infinite set with cardinality less than that of $\mathbb{N}$.

Proof.

Section 1.2.3 — Exercise 3

Problem. Prove that the rational numbers are countable.

Proof. We must show that the rational numbers are countable. We can do so by listing all the rationals. Before doing so, let's set up some notation. For a given $k \in \mathbb{N}$, let $Q_k = \{\pm j/k \mid j \in \mathbb{N}\}$. We then say that $U = \bigcup_{k=1}^{\infty} Q_k$. Then the set of rational numbers is $\mathbb{Q} = \{0\} \bigcup U$.

Note that each Q_k is countable since we can list all of its elements as

$$\frac{1}{k}, \frac{-1}{k}, \frac{2}{k}, \frac{-2}{k}, \frac{3}{k}, \frac{-3}{k}, \frac{4}{k}, \frac{-4}{k}, \cdots$$

We can then form an infinite table with Q_k as the k^{th} row (we denote the i^{th} element of Q_k as q_{ki}).

$$\begin{array}{cccc} q_{11} & q_{12} & q_{13} & \cdots \\ q_{21} & q_{22} & q_{23} & \cdots \\ q_{31} & q_{32} & q_{33} & \cdots \\ \vdots & & \ddots & \end{array}$$

We can then list all the elements of $\mathbb{Q}$ by reading the above table diagonally and ignoring any duplicate elements (seperately, we must remember 0 at the beginning):

$$L_{\mathbb{Q}} = 0, q_{11}, q_{21}, q_{12}, q_{31}, q_{22}, q_{23}, \cdots$$

As a final step, all the duplicates in $L_{\mathbb{Q}}$ must be removed. Then $\mathbb{N}$ has a one-to-one correspondence with $L_{\mathbb{Q}}$. So, we have shown that $L_{\mathbb{Q}}$ is countable.

Section 1.2.3 — Exercise 4

Problem. Show that if a countable subset is removed from an uncountable set, the remainder is still uncountable.

Proof. We begin by giving the problem some notation. Given an uncountable set A and a countable subset $B \subset A$, we must show that $A \setminus B$ is uncountable.

Towards a contradiction, let's assume that we have an uncountable set A and a countable subset $B \subset A$ such that $A \setminus B$ is countable. Let $C = A \setminus B$. Since B and C are both countable, we can list all of their elements: $b_1, b_2, b_3, \ldots$ and $c_1, c_2, c_3, \ldots$. From here, we can also deduce that $B \bigcup C$ is countable because every element of $B \bigcup C$ can be listed: $b_1, c_1, b_2, c_2, b_3, c_3, \ldots$

Yet, $B \bigcup C = A$ and we know A to be uncountable. So, we have reached a contradiction (A cannot be both countable and uncountable).

Since we reached a contradiction, we must have incorrectly assumed that $A \setminus B$ could be countable. Therefore, $A \setminus B$ must be uncountable and we have shown that removing a countable set from an uncoubtable set must always result in an uncountable set.

Section 1.2.3 — Exercise 5

Problem. Let $A_1, A_2, A_3, ...$ be countable sets, and let their Cartesian product $A_1 \times A_2 \times A_3 \times \cdots$ be defined to be the set of all sequences $(a_1, a_2, ...)$ where a_k is an element of A_k. Prove that the Cartesian product is uncountable. Show that the same conclusion holds if each of the sets $A_1, A_2, ...$ has at least two elements.

Proof. We must show that the Caresian product of a countable number of countable sets is uncountable. Towards a contradiction, let's assume not. That is, let's assume that $P = A_1 \times A_2 \times A_3 \times \cdots$ is a countable set.

Since P is countable, there exists a bijective map $f : \mathbb{N} \longrightarrow P$. For a given $n \in \mathbb{N}$, we say $f(n) = (a_1^n, a_2^n, a_3^n, ...)$. To get our contradiction, we will construct an element $p = (p_1, p_2, p_3, ...) \in P$ such that $p \notin im(f)$. We choose p_k such that $p_k \in A_k$ and $p_k \neq a_k^k$.