Exercises 3.1.3 — Problem 9

Problem. Can there exist a sequence whose set of limit points is exactly $1, 1/2, 1/3, ...$?

Proof. There is no sequence whose set of limit points is exactly $1, 1/2, 1/3, ...$ We prove this with contradiction. Suppose $x_1, x_2, x_3, ...$ is a sequence whose limit points are exactly $1, 1/2, 1/3, ...$ A limit point y of $x_1, x_2, ...$ must satisfy the following: for all $1/n$ and m there must exist some $j \geq m$ such that $|y - x_j| < 1/n$. For the sake of contradiction, we assumed that any $1/a$ (for $a \in \mathbb{N}$) is a limit point of $x_1, x_2, ...$ Then there must exist an infinite number of terms less than $1/a$ for any a, which is equivalent to saying that 0 is a limit point of $x_1, x_2, ...$ But this is a contradiction for we assumed that the limit points were $1, 1/2, 1/3, ...$, thus there can be no sequence whose limit points are exactly $1, 1/2, 1/3, ...$