M 383: Assignment 5

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Exercises 3.2.3 — Problem 1

Problem. Let A be an open set. Show that if a finite number of points are removed from A, the remaining set is still open. Is the same true if a countable number of points are removed?

Proof. Say we have some open set A and a finite set B such that $B \subset A$. We must show that $A \setminus B$ is still open. Since A is open, for every $x \in A$ there must exist an open interval $(a_x, b_x) \subset A$ such that $x \in (a_x, b_x)$. Now remove the elements of B from A, is the remainder still open? For a given $x \in A \setminus B$, if no elements of B were in (a_x, b_x) we satisfy open with (a_x, b_x). If there are elements of B in (a_x, b_x), we must do some work. Let $B_l = \{b \in B \mid b > x\}$ and $B_s = \{b \in B \mid b < x\}$. Choose $a'_x = \max\{a_x, \max\{B_s\}\}$ and $b'_x = \min\{b_x, \min\{B_l\}\}$. Then $A \setminus B$ must contain the open interval (a'_x, b'_x) which contains x. So every $x \in A \setminus B$ is contained in an open interval that is contained in $A \setminus B$ meaning that $A \setminus B$ is open.

The same statement is not true if we remove a countable number of points from an open set. Consider the open set $I = (0, 1)$, which is open because it is an open interval. Then remove (from I) every element in the sequence $x_n = n/(2n + 1)$. Since $x_n \to 1/2 \in I$, every neighborhood of $1/2$ will contain some element x_n. Then there cannot exist an open interval (a, b) containing $1/2$ such that $(a, b) \subset I \setminus \{x_n\}$. Therefore, $I \setminus \{x_n\}$ is not open and we have constructed a non-open set by removing a countable number of points from an open set.

Exercises 3.2.3 — Problem 4

Problem. Let A be a set and x a number. Show that x is a limit-point of A if and only if there exists a sequence $x_1, x_2, ...$ of distinct points in A that converges to x.

Proof. We first suppose that x is a limit-point of A and we must show that there exists a sequence $x_1, x_2, ...$ of distinct points in A that converges to x. Then, by definition of limit-point for a set, for any $1/n$ there exists $y_n \in A$ and $y_n \neq x$ such that $|y_n - x| < 1/n$. Then let $y_1, y_2, ...$ be a sequence of points in A that satisfies $|y_n - x| < 1/n$. By definition of limit, $y_1, y_2, ... \to x$ but each y_j is not necessarily distinct. Each duplicate must appear finitely many times by the Axiom of Archimedes so we can remove all duplicates to produce $x_1, x_2, ...$ a distinct sequence of elements of A that converges to x.

Now we suppose that there exists a sequence $x_1, x_2, ...$ of distinct points in A that converges to x. We must show that x is a limit-point of A. Since the sequence $\{x_k\}$ converges to x, we know that for all n, there exists m such that for all $j \geq m$ we have $|x_j - x| < 1/n$. Equivalently, there are infinitely many terms in the sequence in the neighborhood $(x - 1/n, x + 1/n)$. But how do we know that the set A also contains infinitely many points in every neighborhood of x? We know this because the sequence $x_1, x_2, ...$ is distinct. So we have infinitely many points of A within every nieghborhood of x, meaning that x is a limit-point of A.

Exercises 3.2.3 — Problem 5

Problem. Let A be a closed set, x a point in A, and B be the set A with x removed. Under what conditions is B closed?

Proof. We first show that removing a single point from a set does not change the set of limit-points. We know this because, for a point y to be a limit-point the set C, we must have that every neighborhood around y must contain infinitely many points of C. Removing a single point from C will not change the fact that there are infinitely many points in each neighborhood.

B is closed if and only if x is not a limit-point of A. We prove this in two parts. Going to the right, we must show that B is closed implies that x is not a limit-point of A. We show this with the contrapostive: x is a limit-point of A implies that B is not closed. So we know x to be limit point of A and $B = A \setminus \{x\}$. B has the same limit points as A so x is a limit-point of B. But x is not contained in B so B is not closed.

Now going to the left, we assume that x is not a limit-point of A. Since x is not a limit-point of A, x is also not a limit-point of B (for B has the same limit points as A). Then A contains all of its limits points (for it is closed), and B must contain the same set of limit points since x was not a limit point. So B contains all of its limit points and is closed.