M 383: Assignment 1

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Section 1.1.3 — Exercise 2c

Problem. Begin with the statement "Multiplication of integers is associative." Rewrite the state with explicit quantifiers. Then form the negation of the statement. Finally, recast the negation in a form similar to the original statement.

Proof. The statement "Multiplication of integers is associative" can be rewritten with explicit qunatifiers as

$$\forall a, b, c \in \mathbb{Z}, a * (b * c) = (a * b) * c$$

The negation of the statement is

$$\exists a, b, c \in \mathbb{Z} \text{ such that } a * (b * c) \neq (a * b) * c$$

We can rewrite the negation (in words) as "Multiplication of integers is not always associative."

Section 1.2.3 — Exercise 1

Problem. Prove that every subset of $\mathbb{N}$ is either finite or countable. Conclude from this that there is no infinite set with cardinality less than that of $\mathbb{N}$.

Proof.

Section 1.2.3 — Exercise 3

Problem. Prove that the rational numbers are countable.

Proof. We can prove that the rational numbers are countable by providing a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{Q}$. Before showing that $\mathbb{Q}$ is countable, we give some notation. For a given $k \in \mathbb{N}$, let $Q_k = \{\pm j/k \mid j \in \mathbb{N}\}$. We then say that $U = \bigcup_{k=1}^{\infty} Q_k$. Then we can write the set of rational numbers as $\mathbb{Q} = \{0\} \bigcup U$.

Note that each Q_k is countable.

Section 1.2.3 — Exercise 4

Problem. Show that if a countable subset is removed from an uncountable set, the remainder is still uncountable.

Proof.

Section 1.2.3 — Exercise 5

Problem. Let $A_1, A_2, A_3, \ldots$ be countable sets, and let their Cartesian product $A_1 \times A_2 \times A_3 \times \cdots$ be defined to be the set of all sequences $(a_1, a_2, \ldots)$ where a_k is an element of A_k. Prove that the Cartesian product is uncountable. Show that the same conclusion holds if each of the sets $A_1, A_2, \ldots$ has at least two elements.

Proof.