Exercises 2.3.3 — Problem 1

Problem. Write out a proof that $lim_{k\to\infty}(x_k + y_k) = x + y$ if $lim_{k\to\infty}x_k = x$ and $lim_{k\to\infty}y_k = y$ for sequences of real numbers.

Proof. We want to show (for sequences of real numbers) that the limit of a sum is the same as the sum of the limits. Given that $lim_{k\to\infty}x_k = x$ and $lim_{k\to\infty}y_k = y$, we want to show that $\{x_k + y_k\}$ satisfies the Cauchy criterion and converges to $x + y$.

Saying that $\{x_k\}$ and $\{y_k\}$ have limits is equivalent to saying $\{x_k\}$ and $\{y_k\}$ are Cauchy. We also know the sum of two Cauchy sequences to be Cauchy, so certainly $\{x_k + y_k\}$ is Cauchy.

But what is the limit of $\{x_k + y_k\}$? Let's begin by noting that $lim_{k\to\infty}x_k = x \implies \forall n, \exists m_a$ such that $|x_a - x| \leq 1/2n$ for all $a \geq m_a$. Similarly $lim_{k\to\infty}y_k = y \implies \forall n, \exists m_b$ such that $|y_b - y| \leq 1/2n$ for all $b \geq m_b$. Choose $k = max\{a, b\}$. Then $|(x_k+y_k)-(x+y)| = |(x_k-x)+(y_k-y)| \leq 1/2n+1/2n = 1/n$. So $x + y$ satisfies the definition of a limit for the sequence $\{x_k + y_k\}$. Thus the limit of a sum is equal the sum of the limits.