Exercises 5.3.4 — Problem 1

Problem. Define

$$x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Prove that $f(x) = x_+^k$ is continuously differentiable if k is an integer greater than one.

Proof. We must show that $f(x) = x_+^k$ for $k > 1$ is continuously differentiable. That is, we must show $f'(x)$ exists and is continuous on $\mathbb{R}$. First observe that $f(x)$ is certainly continuously differentiable for $x \neq 0$. This is because for every $x \neq 0$, there exists a neighborhood of x such that $f(x) = x^k$ or $f(x) = 0$ which are both continuously differentiable.

All that remains is to show that $f(x)$ is continuously differentiable at $x = 0$. First, $f(x)$ is differentiable at 0 if there exists $f'(0) \in \mathbb{R}$ such that for all $1/m$ there exists a $1/n$ such that we have $|(f(x) - f(0))/(x - 0) - f'(0)| \leq 1/m$ for $x \in (-1/n, 1/n)$ and $x \neq 0$. To find $f'(0)$, we simplify:

$$\left|\frac{f(x) - f(0)}{x - 0} - f'(0)\right| = \left|\frac{f(x)}{x} - f'(0)\right| \leq 1/m$$

Then we split into two cases: we have $x < 0$ and $x > 0$ (we assumed $x \neq 0$). If $x < 0$, then $f(x)/x = 0/x = 0$ and we must have $|f'(0)| \leq 1/m$. The only real number to satisfy this is $f'(0) = 0$. Will $f'(0) = 0$ also satisfy the inequality for $x > 0$? Select $n = m^{1/k-1}$. Then we have $|f(x)/x - f'(0)| = |x^k/x - 0| = |x^{k-1}| \leq 1/n^{k-1} = 1/m$. Then $f'(0) = 0$ satisfies the derivative at 0.

We now must verify that $f'(x)$ is continuous at 0. Note that

$$f'(x) = \begin{cases} kx^{k-1} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

We must show that for all $1/m$, there exists a $1/n$ such that $|f'(x) - f'(0)| = |f'(x)| \leq 1/m$ for $x \in (-1/n, 1/n)$. Again, we split into two cases: $x \leq 0$ and $x > 0$. In the case where $x \leq 0$, we have $f'(x) = 0$ so the inequality is trivially satisfied. Then in the case where $x > 0$, take $n = (km)^{1/(k-1)}$. Then $|f'(x)| < k/n^{k-1} = 1/m$ which shows that $f'(x)$ is continuous. Since $f'(x)$ exists and is continuous, the function $f(x)$ for $k > 1$ is continuously differentiable.