Exercises 3.3.1 — Problem 4

Problem. If $A \subset B_1 \cup B_2$ where B_1 and B_2 are disjoint open sets and A is compact, show that $A \cap B_1$ is compact. Is the same true if B_1 and B_2 are not disjoint?

Proof. Let's first analyze the case where $B_1 \cap B_2 = \emptyset$. So we must show that $A \cap B_1$ is compact, that is, every sequence that lies entirely in $A \cap B_1$ has a limit-point in $A \cap B_1$. To this end, pick a sequence $x_1, x_2, ...$ that lies entirely in $A \cap B_1$. Every x_j is a point in A and A is compact so $x_1, x_2, ...$ certainly has a limit-point x in A. But is x a member of B_1? Since $A \subset B_1 \cup B_2$ and $x \in A$, it must be true that x is in B_1 or B_2. Further, we know that $B_1 \cap B_2 = \emptyset$ so x must be in only one of B_1 and B_2.

Suppose that x is a point in B_2. Since B_2 is open, there must exist an open interval $(a, b) \subset B_2$ such that $x \in (a, b)$. Since $(a, b) \subset B_2$ and $B_1 \cap B_2 = \emptyset$, no elements of B_1 are in the interval (a, b). But every element in the sequence $x_1, x_2, ...$ is a member of B_1, so we have just shown that there exists a neighborhood (a, b) of x such that no element of $x_1, x_2, ...$ is contained (a, b). Then x is not a limit-point of $x_1, x_2, ...$, but this is a contradiction since we know x to be a limit-point of the sequence. So $x \notin B_2$, which implies that $x \in B_1$. Then x (a limit-point of $x_1, x_2, ...$) is a point in both A and B_1, which implies that $x \in A \cap B_1$. Since $x_1, x_2, ...$ is an arbitrary sequence entirely in $A \cap B_1$, we have just shown that every sequence in $A \cap B_1$ has a limit-point in $A \cap B_1$. In other words, we have shown that $A \cap B_1$ is compact.

We now wonder if $A \cap B_1$ is necessarily compact if we instead assume that $B_1 \cap B_2 \neq \emptyset$. The statement does not hold; for evidence, see the following counterexample. Let $B_1 = (0, 2)$ and $B_2 = (1, 3)$ (open sets with a nonempty intersection). Then let $A = [1, 2] \subset B_1 \cup B_2$ (which is compact since A is closed and bounded). Then $A \cap B_1 = [1, 2)$. The set $A \cap B_1$ is not closed since it does not contain its limit-ponit 2, then since $A \cap B_1$ is not closed, it is not compact.