Exercises 3.2.3 — Problem 4

Problem. Let A be a set and x a number. Show that x is a limit-point of A if and only if there exists a sequence $x_1, x_2, ...$ of distinct points in A that converges to x.

Proof. We first suppose that x is a limit-point of A and we must show that there exists a sequence $x_1, x_2, ...$ of distinct points in A that converges to x. Then, by definition of limit-point for a set, for any $1/n$ there exists $y_n \in A$ and $y_n \neq x$ such that $|y_n - x| < 1/n$. Then let $y_1, y_2, ...$ be a sequence of points in A that satisfies $|y_n - x| < 1/n$. By definition of limit, $y_1, y_2, ... \to x$ but each y_j is not necessarily distinct. Each duplicate must appear finitely many times by the Axiom of Archimedes so we can remove all duplicates to produce $x_1, x_2, ...$ a distinct sequence of elements of A that converges to x.

Now we suppose that there exists a sequence $x_1, x_2, ...$ of distinct points in A that converges to x. We must show that x is a limit-point of A. Since the sequence $\{x_k\}$ converges to x, we know that for all n, there exists m such that for all $j \geq m$ we have $|x_j - x| < 1/n$. Equivalently, there are infinitely many terms in the sequence in the neighborhood $(x - 1/n, x + 1/n)$. But how do we know that the set A also contains infinitely many points in every neighborhood of x? We know this because the sequence $x_1, x_2, ...$ is distinct. So we have infinitely many points of A within every nieghborhood of x, meaning that x is a limit-point of A.