M 383: Assignment 8

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Exercises 4.2.4 — Problem 1

Problem. If f is monotone increasing on an interval and has a jump discontinuity at x_0 in the interior of the domain, show that the jump is bounded above by $f(x_2) - f(x_1)$ for any two points x_1, x_2 of the domain surrounding x_0, $x_1 < x_0 < x_2$.

Proof. For some monotone increasing function f defined on an interval I with a jump at x_0 in the interior of I. Let the height of the jump be $h = \lim_{x \to x_0^+} f(x) - \lim_{x \to x_0^-} f(x)$. Now suppose we have two points $x_1, x_2 \in I$ such that $x_1 < x_0 < x_2$. We want to show that $h \leq f(x_2) - f(x_1)$.

Since f increases monotonically, we know that $f(x_1) \leq \lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x) \leq f(x_2)$, which imply that $f(x_1) + \lim_{x \to x_0^+} f(x) \leq \lim_{x \to x_0^-} f(x) + f(x_2)$. Equvalently, $f(x_2) - f(x_1) \geq \lim_{x \to x_0^-} f(x) - \lim_{x \to x_0^+} f(x) = h$. Therefore, the jump is bounded above by $f(x_2) - f(x_1)$.

Exercises 4.2.4 — Problem 3

Problem. If the domain of a continuous function is an interval, show that the image is an interval. Give examples where the image is an open interval.

Proof. To show this, we assume not. That is, we assume that there exists some continuous function f defined on an interval I such that the image is not an interval. Since the image is not an interval, then there exist two nonempty, distinct sets A and B such that $f(I) = A \cup B$ and $A \cup B$ is not an interval. Let C be the open interval $(\inf A \cup B, \sup A \cup B)$. Then since $A \cup B$ is not an interval, there exists some number $c \in C$ that is not a member of $A \cup B$. Then there exist $x_1 \neq x_2 \in I$ such that $f(x_1) < c < f(x_2)$. But this directly contradicts the intermediate value theorem since f is a continuous function containing the interval $[x_1, x_2]$ but does not take on every value betweeen $[f(x_1), f(x_2)]$. So we have reached a contradiction and the image must be an interval.

Here are two examples where the image is an open interval: Consider $f(x) = x$ with domain $\mathbb{R}$ and $g(x) = x + 1$ with domain $(0, 1)$. Both images are open sets.

Exercises 4.2.4 — Problem 9

Problem. If f and g are uniformly continuous, show that $f + g$ is uniformly continuous.

Proof. So we have to uniformly continuous functions f and g defined on the intersections of their domains. Then for all $1/2m$, there exist $1/n_1, 1/n_2$ such that for all x, x_0 we have $|x - x_0| < 1/n_1$ implies $|f(x) - f(x_0)| < 1/2m$ and $|x - x_0| < 1/n_2$ implies $|g(x) - g(x_0)| < 1/2m$. Is it the case that $f + g$ is also uniformly continuous?

Select $|x-x_0| < 1/n = min(1/n_1, 1/n_2)$, then $|(f+g)(x)-(f+g)(x_0)| = |f(x)+g(x)-f(x_0)-g(x_0)| = |f(x)-f(x_0)+g(x)-g(x_0)|$. By triangle inequality we can say that $|f(x)-f(x_0)+g(x)-g(x_0)| \leq |f(x)-f(x_0)|+|g(x)-g(x_0)| < 1/2m+1/2m = 1/m$. So for any $1/m$, we can choose $1/n = min(1/n_1, 1/n_2)$ to satsify uniform continuity for $f + g$.

Exercises 4.2.4 — Problem 11

Problem. If f is a continuous function on a compact set, show that either f has a zero or f is bounded away from zero ($|f(x)| > 1/n$ for all x in the domain, for some $1/n$).

Proof. So we know that f is a continuous function on a compact set D. We must show that $f(D)$ contains 0 or is bounded away from 0. We know that continuous functions map compact sets to compact sets. Therefore $f(D)$ is a compact set. If $0 \in f(D)$ then we are done. If $0 \notin f(D)$, then suppose that f is not bounded away from 0. Then there exists a sequence $f(x_1), f(x_2), ...$ such that for all $1/n$ there exists m such that for $j \geq m$ we have $|f(x_j) - 0| < 1/n$. Then $f(x_1), f(x_2), ...$ converges to 0. And $f(D)$ is compact so $f(D)$ must contain 0 (since it is the only limit point of the sequence). But now we have reached a contradiction since we assumed $0 \notin f(D)$. Therefore, if $0 \notin f(D)$ then f is bounded away from 0.