M 383: Assignment 7

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Exercises 4.1.5 — Problem 2

Problem. Let A be the set defined by the equations $f_1(x) = 0, f_2(x) = 1, ..., f_n(x) = 0$, where $f_1, ..., f_n$ are continuous functions defined on the whole line. Show that A is closed. Must A be compact?

Proof. We first show that A is closed. Given a function f^k, let $D_k = \{x \mid f^k(x) = 0\}$. Then we can say that $A = \cup_{i=1}^n D_i$. So A is union of a finite number of sets. If each D_k is closed, then A (a finite union of closed sets) must also be closed.

Let's now verify that every D_k is closed. Pick an arbitrary $D_k = \{x \mid f^k(x) = 0\}$. Now consider the result (from exercise 4.1.5 problem 1) that a function f defined on a closed domain is continuous if and only if the inverse image of every closed set is a closed set. f^k is a continuous function (by hypothesis) defined on the closed set $\mathbb{R}$, so we can equivalently say that the inverse image of every closed set is a closed set. Since $\{0\}$ is a closed set, the inverse image of $\{0\}$ under f^k is closed. But the inverse image is $\{x \mid f^k(x) = 0\}$ which is D_k. Therefore D_k is closed which implies that A is closed.

It is not necessarily the case that A is compact. We already know A is necessarily closed so let's find a counterexample where A is not bounded (since then A will not be compact). Suppose we have f^1 be the constant function $f^1 : x \mapsto 0$. Then $D_1 = \{x \mid f^1(x) = 0\} = \mathbb{R}$ which is not bounded. Therefore A is not necessarily compact.

Exercises 4.1.5 — Problem 4

Problem. Give a definition of $\lim_{x\to\infty} f(x) = y$. Show that this is true if and only if for every sequence $x_1, x_2, ...$ of points in the domain of f such that $\lim_{n\to\infty} x_n = +\infty$, we have $\lim_{n\to\infty} f(x_n) = y$.

Proof. We define $\lim_{x\to\infty} f(x)$ (for functions with unbounded positive domain) to be $y \in \mathbb{R}$ such that for all m there exists n such that for all $x > n$ ($x \in \mathbb{R}$) we have $|y - f(x)| < 1/m$. We now show that this is the case if and only if every sequence of points in the domain with a limit of $+\infty$ has $\lim_{n\to\infty} f(x_n) = y$.

First suppose that y is a real number such that for every m there exists n such that for all $x > n$ we have $|y - f(x)| < 1/m$. Then we would like to show every sequence $x_1, x_2, ...$ in the domain of f with $+\infty$ as a limit has $\lim_{n\to\infty} f(x_n) = y$. The sequence $x_1, x_2, ... \to +\infty$ so for every $a \in \mathbb{N}$ there exists an index b such that for all $j \geq b$ we have $x_j > a$. But then taking $a = n$ we have $|y - f(x_j)| < 1/m$ for all $j \geq n$. In the limit $j \to \infty$, we have $|y - f(x)| \leq 1/m$ which satisfies $\lim_{n\to\infty} f(x_n) = y$.

Now suppose that every sequence of points in the domain of f such that $\lim_{x\to\infty} x_n = +\infty$, we have $\lim_{n\to\infty} f(x_n) = y$. We want to show that $\lim_{x\to\infty} f(x) = y$. By Theorem 4.1.1, what we just said is equivalent to claiming that $\lim_{x\to\infty} f(x)$ exists. Further, the theorem states that every sequence $f(x_1), f(x_2), ...$ has a common limit and that common limit the limit of the function. Therefore $\lim_{x\to\infty} f(x) = y$.

Exercises 4.1.5 — Problem 7

Problem. Give an example of a continuous function with domain $\mathbb{R}$ such that the inverse image of a compact set is not compact.

Proof. Consider the example I gave in Problem 2 where f is the constant function that takes every real number to 0. Certainly f is continuous and the compact set 0 has $\mathbb{R}$ for an inverse image under f. Although $\mathbb{R}$ is closed, it is not bounded so $\mathbb{R}$ is not compact.

Exercises 4.1.5 — Problem 10

Problem. Show that a function that satisfies a Lipschitz condition is uniformly continuous.

Proof. Suppose we have some function f such that $|x - x_0| < 1/Mm$ implies that $|f(x) - f(x_0)| < 1/m$ for some constant M, is f uniformly continuous? The function f is uniformly continuous if for all m, there exists a n such that $|x - x_0| < 1/n$ implies that $|f(x) - f(x_0)| < 1/m$ for all $x, x_0 \in D$ satisfying $|x - x_0| < 1/n$. From the fact that f satisfies a Lipschitz condition, we can gather that $|f(x) - f(x_0)| \leq M|x - x_0|$. Then we can just take $n = Mm$ to satisfy uniform continuity: $|f(x) - f(x_0)| < M|x - x_0| = M * 1/Mm = 1/m$. Therefore f must also be uniformly continuous.