Exercises 3.1.3 — Problem 4

Problem. Prove $\sup(A \cup B) \geq \sup A$ and $\sup(A \cap B) \leq \sup A$.

Proof. We begin by proving that $\sup(A \cup B) \geq \sup A$. To this end, suppose that the opposite were true: that $\sup(A \cup B) < \sup A$. By definition of $\sup$, we know $\sup A = x$ where x is the smallest extended real number satisfying $a \leq x$ for all $a \in A$. We also know $\sup(A \cup B) = z$ where z is the smallest extended real number satisfying $a \leq z$ for all $a \in A$ and $b \leq z$ for all $b \in B$. From this, we conclude that $\sup(A \cup B) < \sup A \iff z < x$. We also know $a \leq z$ for all a, so it must be true that $a \leq z < x$ for all a. But we have just showed that x is not the smallest extended real number satisfying $a \leq x$ for all a, a contradiction! So it must be true that $\sup(A \cup B) \geq \sup A$.

Now we show that $\sup(A \cap B) \leq \sup A$. Suppose that $\sup(A \cap B) > \sup A$. By definition of $\sup$, we know $\sup(A \cap B) = x$ where x is the smallest extended real number satisfying $a \leq x$ for all $a \in A$. We also know that $\sup(A \cap B) = z$ where z is the smallest extended real number satisfying $c \leq z$ for all $c \in C = A \cap B$. Then $\sup(A \cap B) > \sup A \iff z > x$. We know that $x \geq a$ and that every $c \in A$ (for $c \in A \cap B \iff c \in A \land c \in B$), so we can conlude that $z > x \geq a \geq c$. But then z is not the smallest extended real number satisfying $z \geq c$ (for x is strictly less than z). so we have reached a contradiction and it must be true that $\sup(A \cap B) \leq \sup A$.