Exercises 3.3.1 — Problem 8

Problem. If A is compact, show that $\sup A$ and $\inf A$ belong to A. Give an example of a non-compact set A such that both $\sup A$ and $\inf A$ belong to A.

Proof. Suppose that we have some compact set A, we must show that $\sup A \in A$ and $\inf A \in A$. First we will show that $\sup A \in A$. Let $y = \sup A$. Then for all n, there exists some $y_n \in A$ such that $y - 1/n < y_n$. Which implies that for all n, there exists $y_n \in A$ such that $y - y_n = |y - y_n| < 1/n$. Let $y_1, y_2, ...$ be a sequence (not necessarily distinct) that satisfies $|y - y_n| < 1/n$. From this, we can gather that $\{y_n\}$ converges to y (which means that y is the only limit point of the sequence). Then since every $y_n \in A$ and A is compact, some limit-point of $y_1, y_2, ...$ is a point in A. But we already know that y is the only limit-point of $y_1, y_2, ...$, so we must have $y \in A$. And $y = \sup A$ so it must be true that $\sup A \in A$.

Similarly, we now show that $\inf A \in A$. Let $z = \inf A$. Then for all n, there exists some $z_n \in A$ such that $z + 1/n > z_n$. Which implies that for all n, there exists $z_n \in A$ such that $z_n - z = |z_n - z| = |z - z_n| < 1/n$. Let $z_1, z_2, ...$ be a sequence (not necessarily distinct) that satisfies $|z - z_n| < 1/n$. From this, we can gather that $\{z_n\}$ converges to z (which means that z is the only limit point of the sequence). Then since every $z_n \in A$ and A is compact, some limit-point of $z_1, z_2, ...$ is a point in A. But we already know that z is the only limit-point of $z_1, z_2, ...$, so we must have $z \in A$. And $z = \inf A$ so it must be true that $\inf A \in A$.

We now give an example of a non-compact set A such that $\sup A$ and $\inf A$ belong to A. Let $A = [0, 1) \cup (1, 2]$. The set A is not closed since it does not contain its limit-point 1, which means A is not compact. However, $\sup A = 2 \in A$ and $\inf A = 0 \in A$.