Exercises 3.3.1 — Problem 1

Problem. Show that compact sets are closed under arbitrary intersections and finite unions.

Proof. First note that a set of real numbers A is compact if and only if A is closed and bounded. Additionally, note that closed sets are closed under arbitrary intersection and finite unions.

We now show that compact sets are closed under arbitrary intersections. Let $\mathbb{B}$ be any collection of compact sets and $B' = \cup_{B \in \mathbb{B}} B$. The set B' is compact if and only if it is closed and bounded. Certainly $\mathbb{B}$ is closed since every $B \in \mathbb{B}$ is closed and their intersection must be closed. We show that B' is bounded by contradiction. Suppose B' is not bounded. Then there exists a sequence $x_1, x_2, ...$ of elements in B' such that for all n, there exists a j such that $x_j > n$ or $x_j < -n$. But sequence $x_1, x_2, ...$ must also be in every $B \in \mathbb{B}$ so we have just shown that every $B \in \mathbb{B}$ is unbounded. But then no $B \in \mathbb{B}$ is compact, which is a contradiction. Therefore B' is both closed and bounded, which implies that B' is compact.

We now show that compact sets are closed under finite unions. Let $C_1, C_2, ..., C_n$ be a finite collection of compact sets and $C' = \cap_{i=1}^{n} C_i$. To show that C' is compact, we will show that it is closed and bounded. C' is closed since it is the intersection of a finite collection of closed sets. Is C' bounded? Suppose that C' is not bounded, then one of two cases must occur. Either there exists a monotonically increasing sequence $x_1, x_2, ...$ of elements in C' such that for all n there exists j such that $x_j > n$ or there exists a monotonically decreasing sequence $y_1, y_2, ...$ of elements in C' such that for all n there exists j such that $y_j < -n$. In either case, we can use the pigeon hole principle to say at least one of B_i contains an infinite number of terms in the sequence (whether it be increasing or decreasing). But then that B_i must be unbounded, which is a contradiction since we assumed every B_i to be compact. Therefore, the union of a finite number of compact sets must be compact.