Exercises 5.2.4 — Problem 2

Problem. If f is a function satisfying

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

for all x and y and some fixed point M and $\alpha > 1$, prove that f is constant.

Proof. Note that the domain of f is the real number line. Then let $y = x_0$ be a real number and take $x \neq x_0$ (every function satisfies the inequality for $x = x_0$). To show that f is constant, we will show that $f'(x_0)$ exists and equals 0. We know that for some fixed M and another fixed $\alpha > 1$ that $|f(x) - f(x_0)| \leq M|x - x_0|^\alpha$. Since $x \neq x_0$ and $\alpha - 1 > 0$, we must also have

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} = \left|\frac{f(x) - f(x_0)}{x - x_0}\right| = \left|\frac{f(x) - f(x_0)}{x - x_0} - 0\right| \leq M|x - x_0|^{\alpha - 1}$$

Then we just choose $|x - x_0| < 1/(Mm)^{1/(\alpha-1)}$ and we have exactly that $f'(x_0) = 0$. Since the derivative at an arbitrary x_0 is 0, the derivative is 0 everywhere and the function f must be constant.