Exercises 2.1.3 — Problem 8

Problem. Can a Cauchy sequence of positive rational numbers be equivalent to a Cauchy sequence of negative rational numbers?

Proof. We will show that a Cauchy sequence of positive rational numbers can be equivalent to a Cauchy sequence of negative numbers by providing an example of two such sequences. We define the two sequences x_n^+ and x_n^- where the l^{th} terms of x_n^+ and x_n^- are $1/l$ and $-1/l$ respectively.

Before showing the equivalence of x_n^+ and x_n^-, we must show that both sequences are Cauchy. A sequence x is Cauchy if $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$ such that $\forall j, k \geq m \ \ |x_j - x_k| \leq 1/n$. For both x_n^+ and x_n^-, select $m = n$ to satisfy the Cauchy criterion.

The sequences x_n^+ and x_n^- are equivalent if $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$ such that $\forall k \geq m \ \ |x_k^+ - x_k^-| \leq 1/n$. Given an n, take $m = 2n$. Then $\forall k \geq m$,

$$|x_k^+ - x_k^-| = |x_k^+ - -x_k^+| = |x_k^+ + x_k^+| = 2x_k^+ \leq 2/(2n) = 1/n$$

Since there exists an m such that the difference between terms in the sequence is bounded by $1/n$, the sequences x_n^+ and x_n^- are equivalent. Since the two sequences are equivalent, we have shown the existence of a Cauchy sequence of positive rational numbers that is equivalent to a Cauchy sequence of negative rational numbers.