Exercises 3.2.3 — Problem 1

Problem. Let A be an open set. Show that if a finite number of points are removed from A, the remaining set is still open. Is the same true if a countable number of points are removed?

Proof. Say we have some open set A and a finite set B such that $B \subset A$. We must show that $A \setminus B$ is still open. Since A is open, for every $x \in A$ there must exist an open interval $(a_x, b_x) \subset A$ such that $x \in (a_x, b_x)$. Now remove the elements of B from A, is the remainder still open? For a given $x \in A \setminus B$, if no elements of B were in (a_x, b_x) we satisfy open with (a_x, b_x). If there are elements of B in (a_x, b_x), we must do some work. Let $B_l = \{b \in B \mid b > x\}$ and $B_s = \{b \in B \mid b < x\}$. Choose $a'_x = \max\{a_x, \max\{B_s\}\}$ and $b'_x = \min\{b_x, \min\{B_l\}\}$. Then $A \setminus B$ must contain the open interval (a'_x, b'_x) which contains x. So every $x \in A \setminus B$ is contained in an open interval that is contained in $A \setminus B$ meaning that $A \setminus B$ is open.

The same statement is not true if we remove a countable number of points from an open set. Consider the open set $I = (0, 1)$, which is open because it is an open interval. Then remove (from I) every element in the sequence $x_n = n/(2n + 1)$. Since $x_n \to 1/2 \in I$, every neighborhood of $1/2$ will contain some element x_n. Then there cannot exist an open interval (a, b) containing $1/2$ such that $(a, b) \subset I \setminus \{x_n\}$. Therefore, $I \setminus \{x_n\}$ is not open and we have constructed a non-open set by removing a countable number of points from an open set.