Exercises 5.2.4 — Problem 1

Problem. Suppose $f'(x_0) = 0, f''(x_0) = 0, ...f^{n-1}(x_0) = 0$ and $f^{(n)}(x_0) > 0$ for a C^n function f. Prove that f has a local minimum at x_0 if n is even and that x_0 is neither a local maximum nor a local minimum if n is odd.

Proof. As in the textbook, let

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n$$

But since $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, we have $T_n(x) = f(x_0) + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n$. Then, by Taylor's Theorem, we must have

$$f - T_n = o(|x - x_0|^n) \iff \lim_{x \to x_0} \frac{f - T_n}{|x - x_0|^n} = 0$$

which is equivalent to

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - 1/n! * f^{(n)}(x_0)(x - x_0)^n}{|x - x_0|^n} = 0$$

and to

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{|x - x_0|^n} = \lim_{x \to x_0} \frac{1/n! * f^{(n)}(x_0)(x - x_0)^n}{|x - x_0|^n}$$

The LHS looks like the derivative at x_0, note that it is slightly different because of the absolute value in the denominator. We now consider the two cases where n is even and n is odd. If n is even then $|x - x_0|^n = (x - x_0)^n$ and we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{|x - x_0|^n} = 1/n! * f^{(n)}(x_0) > 0$$

But then since $|x - x_0|^n$ is strictly positive, we must have $f(x) > f(x_0)$ to satisfy the equality as $x \to x_0$ and $x \neq x_0$. Then there must exist a neighborhood of x_0 such that $f(x) > f(x_0)$ for all $x \neq x_0$ in the neighborhood. But then f has a strict local minimum at x_0 as desired.

Now consider the case where n is odd. Then $|x - x_0|^n = (x - x_0)^n$ for $x > x_0$ and $|x - x_0|^n = -(x - x_0)^n$ fro $x < x_0$. So for $x < x_0$, we must have

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{|x - x_0|^n} = -1/n! * f^{(n)}(x_0) < 0$$

which means $f(x) < f(x_0)$ for $x < x_0$ in a neighborhood of x_0. And for $x > x_0$, we can use the logic from the even case of n to show that $f(x) > f(x_0)$ in a neighborhood of x_0.

So we have just shown that, for some neighborhood of x_0, $f(x)$ is strictly less than $f(x_0)$ when $x < x_0$ and $f(x)$ is strictly greater than $f(x_0)$ when $x > x_0$. Therefore, x_0 cannot be either of a local minimum or a local maximum.