M 383: Assignment 10

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Exercises 5.3.4 — Problem 1

Problem. Define

$$x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Prove that $f(x) = x_+^k$ is continuously differentiable if k is an integer greater than one.

Proof. We must show that $f(x) = x_+^k$ for $k > 1$ is continuously differentiable. That is, we must show $f'(x)$ exists and is continuous on $\mathbb{R}$. First observe that $f(x)$ is certainly continuously differentiable for $x \neq 0$. This is because for every $x \neq 0$, there exists a neighborhood of x such that $f(x) = x^k$ or $f(x) = 0$ which are both continuously differentiable.

All that remains is to show that $f(x)$ is continuously differentiable at $x = 0$. First, $f(x)$ is differentiable at 0 if there exists $f'(0) \in \mathbb{R}$ such that for all $1/m$ there exists a $1/n$ such that we have $|(f(x) - f(0))/(x - 0) - f'(0)| \leq 1/m$ for $x \in (-1/n, 1/n)$ and $x \neq 0$. To find $f'(0)$, we simplify:

$$\left| \frac{f(x) - f(0)}{x - 0} - f'(0) \right| = \left| \frac{f(x)}{x} - f'(0) \right| \leq 1/m$$

Then we split into two cases: we have $x < 0$ and $x > 0$ (we assumed $x \neq 0$). If $x < 0$, then $f(x)/x = 0/x = 0$ and we must have $|f'(0)| \leq 1/m$. The only real number to satisfy this is $f'(0) = 0$. Will $f'(0) = 0$ also satisfy the inequality for $x > 0$? Select $n = m^{1/k-1}$. Then we have $|f(x)/x - f'(0)| = |x^k/x - 0| = |x^{k-1}| \leq 1/n^{k-1} = 1/m$. Then $f'(0) = 0$ satisfies the derivative at 0.

We now must verify that $f'(x)$ is continuous at 0. Note that

$$f'(x) = \begin{cases} kx^{k-1} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

We must show that for all $1/m$, there exists a $1/n$ such that $|f'(x) - f'(0)| = |f'(x)| \leq 1/m$ for $x \in (-1/n, 1/n)$. Again, we split into two cases: $x \leq 0$ and $x > 0$. In the case where $x \leq 0$, we have $f'(x) = 0$ so the inequality is trivially satisfied. Then in the case where $x > 0$, take $n = (km)^{1/(k-1)}$. Then $|f'(x)| < k/n^{k-1} = 1/m$ which shows that $f'(x)$ is continuous. Since $f'(x)$ exists and is continuous, the function $f(x)$ for $k > 1$ is continuously differentiable.

Exercises 5.2.4 — Problem 1

Problem. Suppose $f'(x_0) = 0, f''(x_0) = 0, ... f^{n-1}(x_0) = 0$ and $f^{(n)}(x_0) > 0$ for a C^n function f. Prove that f has a local minimum at x_0 if n is even and that x_0 is neither a local maximum nor a local minimum if n is odd.

Proof. As in the textbook, let

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n$$

But since $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, we have $T_n(x) = f(x_0) + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n$. Then, by Taylor's Theorem, we must have

$$f - T_n = o(|x - x_0|^n) \iff \lim_{x \to x_0} \frac{f - T_n}{|x - x_0|^n} = 0$$

which is equivalent to

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - 1/n! * f^{(n)}(x_0)(x - x_0)^n}{|x - x_0|^n} = 0$$

and to

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{|x - x_0|^n} = \lim_{x \to x_0} \frac{1/n! * f^{(n)}(x_0)(x - x_0)^n}{|x - x_0|^n}$$

The LHS looks like the derivative at x_0, note that it is slightly different because of the absolute value in the denominator. We now consider the two cases where n is even and n is odd. If n is even then $|x - x_0|^n = (x - x_0)^n$ and we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{|x - x_0|^n} = 1/n! * f^{(n)}(x_0) > 0$$

But then since $|x - x_0|^n$ is strictly positive, we must have $f(x) > f(x_0)$ to satisfy the equality as $x \to x_0$ and $x \neq x_0$. Then there must exist a neighborhood of x_0 such that $f(x) > f(x_0)$ for all $x \neq x_0$ in the neighborhood. But then f has a strict local minimum at x_0 as desired.

Now consider the case where n is odd. Then $|x - x_0|^n = (x - x_0)^n$ for $x > x_0$ and $|x - x_0|^n = -(x - x_0)^n$ fro $x < x_0$. So for $x < x_0$, we must have

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{|x - x_0|^n} = -1/n! * f^{(n)}(x_0) < 0$$

which means $f(x) < f(x_0)$ for $x < x_0$ in a neighborhood of x_0. And for $x > x_0$, we can use the logic from the even case of n to show that $f(x) > f(x_0)$ in a neighborhood of x_0.

So we have just shown that, for some neighborhood of x_0, $f(x)$ is strictly less than $f(x_0)$ when $x < x_0$ and $f(x)$ is strictly greater than $f(x_0)$ when $x > x_0$. Therefore, x_0 cannot be either of a local minimum or a local maximum.