Exercises 5.2.4 — Problem 1

Problem. Let f and g be continuous functions on $[a, b]$ and differentiable at every point in the interior, with $g(a) \neq g(b)$. Prove that there exists a point x_0 in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

Proof. Let's begin by defining $H(x) = (f(b) - f(a)) * g(x) - (g(b) - g(a)) * f(x)$. Now let $F = f(b) - f(a)$ and $G = g(b) - g(a)$. Then note that

$$H(b) - H(a) = (F * g(b) - G * f(b)) - (F * g(a) - G * f(a)) = F * G - G * F = 0$$

Then since f and g both satisfy the conditions for the mean value theorem, so must $H(x)$ (a linear combination of f and g). So there must exist $x_0 \in (a, b)$ such that

$$H'(x_0) = \frac{H(b) - H(a)}{b - a}$$

But since $H(b) - H(a) = 0$ and $b \neq a$, we must have $H'(x_0) = 0$ for some $x_0 \in (a, b)$. Then $F * g'(x_0) - G * f'(x_0) = (f(b) - f(a)) * g'(x_0) - (g(b) - g(a)) * f'(x_0) = 0$. Then we must have $(f(b) - f(a)) * g'(x_0) = (g(b) - g(a)) * f'(x_0)$. We already know $g(b) - g(a) \neq 0$ and if we assume that $g'(x) \neq 0$ for $x \in (a, b)$, then we get the desired result for some $x_0 \in (a, b)$:

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(x_0)}{g'(x_0)}$$