Exercises 2.3.3 — Problem 3

Problem. Let $x_1, x_2, ...$ be a sequence of real numbers such that $|x_n| \leq 1/2^n$, and set $y_n = x_1 + x_2 + \cdots + x_n$. Show that the sequence $y_1, y_2, ...$ converges.

Proof. We must show that $y_1, y_2, ...$ converges. To this end, we introduce the following result for any natural number n: $1/2^n = \sum_{k=n+1}^{\infty} 1/2^k$. Let's now prove this result. First note that for some natural number a, $1/2^{a+1} = 1/2 * 1/2^a$. Then for any a, we have

$$1/2^a + 1/2^{a+1} + 1/2^{a+2} + \cdots = 1/2^a + 1/2 * 1/2^a + 1/2 * 1/2^{a+1} + \cdots = 1/2^a + 1/2 * (1/2^a + 1/2^{a+1} + \cdots)$$

which implies that

$$1/2^a = 1/2 * (1/2^a + 1/2^{a+1} + \cdots) = 1/2 * 1/2^a + 1/2 * 1/2^{a+1} + \cdots = 1/2^{a+1} + 1/2^{a+2} + \cdots = \sum_{k=a+1}^{\infty} 1/2^k$$

Now to show that $y_1, y_2, ...$ converges, we will show that $y_1, y_2, ...$ is Cauchy. So, given a natural number n, we must show the existence of an index m such that $|y_j - y_k| \leq 1/n$ for all $j, k \geq m$. For any index m, we can provide an upper bound for $|y_j - y_k|$ by choosing the largest possible y_j and the smallest possible y_k. Because $1/2^n = \sum_{k=n+1}^{\infty} 1/2^k$ and $x_n \leq 1/2^n$, we can say that the largest y_j could be is $y_m + 1/2^m$ and the smallest y_k could be is $y_m - 1/2^m$. Then

$$|y_j - y_k| \leq |(y_m + 1/2^m) - (y_k - 1/2^m)| = |2/2^m| = 1/2^{m-1}$$

Choosing $m = n$, we can say that $1/2^{m-1} = 1/2^{n-1} \leq 1/n$ for all n. So there exists an index m such that $|y_j - y_k| \leq 1/n$ for all $j, k \geq m$. Thus $y_1, y_2, ...$ satisfies the Cauchy criterion and must also converge.