Exercises 4.1.5 — Problem 10

Problem. Show that a function that satisfies a Lipschitz condition is uniformly continuous.

Proof. Suppose we have some function f such that $|x - x_0| < 1/Mm$ implies that $|f(x) - f(x_0)| < 1/m$ for some constant M, is f uniformly continuous? The function f is uniformly continuous if for all m, there exists a n such that $|x - x_0| < 1/n$ implies that $|f(x) - f(x_0)| < 1/m$ for all $x, x_0 \in D$ satisfying $|x - x_0| < 1/n$. From the fact that f satisfies a Lipschitz condition, we can gather that $|f(x) - f(x_0)| \leq M|x - x_0|$. Then we can just take $n = Mm$ to satisfy uniform continuity: $|f(x) - f(x_0)| < M|x - x_0| = M * 1/Mm = 1/m$. Therefore f must also be uniformly continuous.