## Exercises 2.2.4 — Problem 3

*Problem.* If $x$ is a real number, show that there exists a Cauchy sequence of rationals $x_1, x_2, ...$ representing $x$ such that $x_n < x$ for all $n$.

*Proof.* Pick $y_1, y_2, ...$ a Cauchy sequence of rationals with $lim_{k\to\infty} y_k = x$, we know that $\forall n \in \mathbb{N}, \exists m(n) \in \mathbb{N}$ such that $|y - y_k| < 1/n$ for all $k \geq m(n)$. We will then edit the sequence $\{y_k\}$ such that every $y_k < y$ but we will not change the limit $x$.

First, choose some rational number $q \in \mathbb{Q}$ such that $q < y$. Then change the first $m(1)$ elements of $\{y_k\}$ to have the value $q$. Since the sequence is Cauchy, $m(1)$ must be finite so we have just changed a finite number of terms of $\{y_k\}$. Thus, the $lim_{k\to\infty} y_k$ did not change must still be $x$.

Next we construct a sequence $\{m_k\}$ where the first $m(1)$ elements are 0, the next $m(2)$ elements are 1, the next $m(3)$ elements are $1/2$, the next $m(4)$ elements are $1/3$, and so on:

$$0, 0, ..., 0, 1, 1, ..., 1, 1/2, 1/2, ..., 1/2, 1/3, 1/3, ...1/3, 1/4, ...$$

We can see that $\{m_k\}$ is Cauchy and that $lim_{k\to\infty} m_k = 0$. This implies that $lim_{k\to\infty}(y_k - m_k) = x - 0 = x$. However, for each $k$, we have constructed $m_k$ such that $y_k - m_k < x$. Then let $x_k = y_k - m_k$ and we have a Cauchy sequence of rationals $x_1, x_2, ...$ that represents $x$ such that $x_k < x$ for all $k \in \mathbb{N}$.