M 383: Assignment 8

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Exercises 4.2.4 — Problem 1

Problem. If f is monotone increasing on an interval and has a jump discontinuity at x_0 in the interior of the domain, show that the jump is bounded above by $f(x_2) - f(x_1)$ for any two points x_1, x_2 of the domain surrounding x_0, $x_1 < x_0 < x_2$.

Proof. For some monotone increasing function f defined on an interval I with a jump at x_0 in the interior of I. Let the height of the jump be $h = \lim_{x \to x_0^+} f(x) - \lim_{x \to x_0^-} f(x)$. Now suppose we have two points $x_1, x_2 \in I$ such that $x_1 < x_0 < x_2$. We want to show that $h \leq f(x_2) - f(x_1)$.

Since f increases monotonically, we know that $f(x_1) \leq \lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x) \leq f(x_2)$, which imply that $f(x_1) + \lim_{x \to x_0^+} f(x) \leq \lim_{x \to x_0^-} f(x) + f(x_2)$. Equivalently, $f(x_2) - f(x_1) \geq \lim_{x \to x_0^-} f(x) - \lim_{x \to x_0^+} f(x) = h$. Therefore, the jump is bounded above by $f(x_2) - f(x_1)$.

Exercises 4.2.4 — Problem 3

Problem. If the domain of a continuous function is an interval, show that the image is an interval. Give examples where the image is an open interval.

Proof.

Exercises 4.2.4 — Problem 9

Problem. If f and g are uniformly continuous, show that $f + g$ is uniformly continuous.

Proof.

Exercises 4.2.4 — Problem 11

Problem. If f is a continuous function on a compact set, show that either f has a zero or f is bounded away from zero ($|f(x)| > 1/n$ for all x in the domain, for some $1/n$).

Proof.