# M 383: Assignment 6

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# Exercises 3.2.3 — Problem 1

*Problem.* Show that compact sets are closed under arbitrary intersections and finite unions.

*Proof.*

# Exercises 3.3.1 — Problem 4

*Problem.* If $A \subset B_1 \cup B_2$ where $B_1$ and $B_2$ are disjoint open sets and $A$ is compact, show that $A \cap B_1$ is compact. Is the same true if $B_1$ and $B_2$ are not disjoint?

*Proof.* Let's first analyze the case where $B_1 \cap B_2 = \emptyset$. So we must show that $A \cap B_1$ is compact, that is, every sequence that lies entirely in $A \cap B_1$ has a limit-point in $A \cap B_1$. To this end, pick a sequence $x_1, x_2, \dots$ that lies entirely in $A \cap B_1$. Every $x_j$ is a point in $A$ and $A$ is compact so $x_1, x_2, \dots$ certainly has a limit-point $x$ in $A$. But is $x$ a member of $B_1$? Since $A \subset B_1 \cup B_2$ and $x \in A$, it must be true that $x$ is in $B_1$ or $B_2$. Further, we know that $B_1 \cap B_2 = \emptyset$ so $x$ must be in only one of $B_1$ and $B_2$.

Suppose that $x$ is a point in $B_2$. Since $B_2$ is open, there must exist an open interval $(a, b) \subset B_2$ such that $x \in (a, b)$. Since $(a, b) \subset B_2$ and $B_1 \cap B_2 = \emptyset$, no elements of $B_1$ are in the interval $(a, b)$. But every element in the sequence $x_1, x_2, \dots$ is a member of $B_1$, so we have just shown that there exists a neighborhood $(a, b)$ of $x$ such that no element of $x_1, x_2, \dots$ is contained $(a, b)$. Then $x$ is not a limit-point of $x_1, x_2, \dots$, but this is a contradiction since we know $x$ to be a limit-point of the sequence. So $x \notin B_2$, which implies that $x \in B_1$. Then $x$ (a limit-point of $x_1, x_2, \dots$) is a point in both $A$ and $B_1$, which implies that $x \in A \cap B_1$. Since $x_1, x_2, \dots$ is an arbitrary sequence entirely in $A \cap B_1$, we have just shown that every sequence in $A \cap B_1$ has a limit-point in $A \cap B_1$. In other words, we have shown that $A \cap B_1$ is compact.

We now wonder if $A \cap B_1$ is necessarily compact if we instead assume that $B_1 \cap B_2 \neq \emptyset$. The statement does not hold; for evidence, see the following counterexample. Let $B_1 = (0, 2)$ and $B_2 = (1, 3)$ (open sets with a nonempty intersection). Then let $A = [1, 2] \subset B_1 \cup B_2$ (which is compact since $A$ is closed and bounded). Then $A \cap B_1 = [1, 2)$. The set $A \cap B_1$ is not closed since it does not contain its limit-ponit 2, then since $A \cap B_1$ is not closed, it is not compact.

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## Exercises 3.3.1 — Problem 8

*Problem.* If $A$ is compact, show that $\sup A$ and $\inf A$ belong to $A$. Give an example of a non-compact set $A$ such that both $\sup A$ and $\inf A$ belong to $A$.

*Proof.* Suppose that we have some compact set $A$, we must show that $\sup A \in A$ and $\inf A \in A$. First we will show that $\sup A \in A$. Let $y = \sup A$. Then for all $n$, there exists some $y_n \in A$ such that $y - 1/n < y_n$. Which implies that for all $n$, there exists $y_n \in A$ such that $y - y_n = |y - y_n| < 1/n$. Let $y_1, y_2, ...$ be a sequence (not necessarily distinct) that satisfies $|y - y_n| < 1/n$. From this, we can gather that $\{y_n\}$ converges to $y$ (which means that $y$ is the only limit point of the sequence). Then since every $y_n \in A$ and $A$ is compact, some limit-point of $y_1, y_2, ...$ is a point in $A$. But we already know that $y$ is the only limit-point of $y_1, y_2, ...$, so we must have $y \in A$. And $y = \sup A$ so it must be true that $\sup A \in A$.

Similarly, we now show that $\inf A \in A$. Let $z = \inf A$. Then for all $n$, there exists some $z_n \in A$ such that $z + 1/n > z_n$. Which implies that for all $n$, there exists $z_n \in A$ such that $z_n - z = |z_n - z| = |z - z_n| < 1/n$. Let $z_1, z_2, ...$ be a sequence (not necessarily distinct) that satisfies $|z - z_n| < 1/n$. From this, we can gather that $\{z_n\}$ converges to $z$ (which means that $z$ is the only limit point of the sequence). Then since every $z_n \in A$ and $A$ is compact, some limit-point of $z_1, z_2, ...$ is a point in $A$. But we already know that $z$ is the only limit-point of $z_1, z_2, ...$, so we must have $z \in A$. And $z = \inf A$ so it must be true that $\inf A \in A$.

We now give an example of a non-compact set $A$ such that $\sup A$ and $\inf A$ belong to $A$. Let $A = [0, 1) \cup (1, 2]$. The set $A$ is not closed since it does not contain its limit-point 1, which means $A$ is not compact. However, $\sup A = 2 \in A$ and $\inf A = 0 \in A$.