Exercises 3.1.3 — Problem 1

Problem. Compute the sup, inf, limsup, liminf, and all the limit points of the sequence $x_1, x_2, ...$ where $x_n = 1/n + (-1)^n$.

Proof. First we compute the $\sup$. To this end, we show that $3/2$ is an upper bound for $1/n + (-1)^n$ for all $n \in \mathbb{N}$.

$$3/2 \geq 1/n + (-1)^n \iff 3n/2 \geq 1 + n(-1)^n \geq 1 - n \iff 5n/2 \geq 1 \iff n \geq 2/5$$

which is true for all natural numbers. Now we must show that $3/2$ is the least upper bound for $x_n = 1/n+(-1)^n$. We know this because $3/2$ is an element of the sequence: $x_2 = 1/2+(-1)^2 = 1/2+1 = 3/2$.

Now we compute the $\inf$. First we will show that $-1 \leq x_n$ for all $n \in \mathbb{N}$.

$$-1 \leq 1/n + (-1)^n \iff -n \leq 1 + n(-1)^n \leq 1 + n \iff -n \leq 1 + n \iff n \geq -1/2$$

which is true for all natural numbers. No we must show that -1 is the greatest lower bound for x_n. Suppose there was a lower bound $-1 + 1/k$ for some $k \in \mathbb{N}$. But $-1 + 1/k$ cannot be a lower bound for x_n since x_{2k+1} is certainly less than $-1 + 1/k$:

$$x_{2k+1} = \frac{1}{2k+1}+(-1)^{2k+1} = \frac{1}{2k+1}+-1 < -1+1/k \iff \frac{1}{2k+1} < 1/k \iff k < 2k+1 \iff k > -1$$

which is true for all natural numbers. So $-1 + 1/k$ cannot be a lower bound for x_n, which implies that -1 is the greatest lower bound for the sequence.

Before computing $\limsup$ and $\liminf$, we will find all the limit points of $\{x_n\}$. Note that $x_1, x_2, x_3, ... = y_1, z_1, y_2, z_2, y_3, z_3, ...$ where $y_n = 1/(2n-1) + (-1)^{2n-1}$ and $z_n = 1/2n + (-1)^{2n}$. Further $y_n = 1/(2n-1) + (-1)^{2n-1} = 1/(2n-1) + -1$ and $z_n = 1/2n + (-1)^{2n} = 1/2n + 1$. Then we can say that $\lim_{n\to\infty} y_n = -1$ and $\lim_{n\to\infty} z_n = 1$. Since each subsequence converges, $\{y_n\}$ and $\{z_n\}$ each have only one limit point. Additionally, we know that $x_1, x_2, ...$ is just a shuffled sequence of $y_1, y_2, ...$ and $z_1, z_2, ...$ so the limit points of $x_1, x_2, ...$ are the limit points of $\{y_n\}$ and $\{z_n\}$: -1 and 1.

For $\limsup$ and $\liminf$, we know that $\limsup$ is the $\sup$ of the set of limit points and that $\liminf$ is the $\inf$ of the set of the limit points. So the $\limsup$ is 1 and the $\liminf$ is -1.