Section 1.2.3 — Exercise 4

Problem. Show that if a countable subset is removed from an uncountable set, the remainder is still uncountable.

Proof. We begin by giving the problem some notation. Given an uncountable set A and a countable subset $B \subset A$, we must show that $A \setminus B$ is uncountable.

Towards a contradiction, let's assume that we have an uncountable set A and a countable subset $B \subset A$ such that $A \setminus B$ is countable. Let $C = A \setminus B$. Since B and C are both countable, we can list all of their elements: $b_1, b_2, b_3, ...$ and $c_1, c_2, c_3, ...$. From here, we can also deduce that $B \bigcup C$ is countable because every element of $B \bigcup C$ can be listed: $b_1, c_1, b_2, c_2, b_3, c_3, ...$

Yet, $B \bigcup C = A$ and we know A to be uncountable. So, we have reached a contradiction since A cannot be both countable and uncountable.

Since we reached a contradiction, we must have incorrectly assumed that $A \setminus B$ could be countable. Therefore, $A \setminus B$ must be uncountable and we have shown that removing a countable set from an uncoubtable set must always result in an uncountable set.