Exercises 5.1.3 — Problem 1

Problem. Show that $f(x) = O(|x - x_0|^2)$ as $x \to x_0$ implies $f(x) = o(|x - x_0|)$ as $x \to x_0$, but give an example to show that the converse is not true.

Proof. So we know that $f(x) = O(|x - x_0|^2)$ as $x \to x_0$ and we want to show that $f(x) = o(|x - x_0|)$ as $x \to x_0$. Equivalently, we could show that $\lim_{x \to x_0} f(x)/|x - x_0| = 0$. From the definition of big-O, we can say that as $x \to x_0$, there exists $1/n$ and a positive constant c such that $|x - x_0| < 1/n$ implies that $|f(x)| \leq c * |x - x_0|^2$. Then for the same neighborhood $(x_0 - 1/n, x_0 + 1/n)$, we must have $|f(x)|/|x - x_0| \leq c * |x - x_0|$. And since non-strict inequalities are preserved by limits, $\lim_{x \to x_0} |f(x)|/|x - x_0| \leq \lim_{x \to x_0} c * |x - x_0| = 0$ for $x \neq x_0$. Since $\lim_{x \to x_0} |f(x)|/|x - x_0|$ cannot be negative, it must equal 0. Then $f(x) = o(|x - x_0|)$ as $x \to x_0$, which was the goal.

A counterexample of the converse is $f(x) = |x - x_0|^{3/2}$. Certainly $|x - x_0|^{3/2} = o(|x - x_0|)$ since

$$\lim_{x \to x_0} \frac{|x - x_0|^{3/2}}{|x - x_0|} = \lim_{x \to x_0} |x - x_0|^{1/2} = 0$$

But, we do not have $|x - x_0|^{3/2} = O(|x - x_0|^2)$. To verify this, let's unpack the definition of big-O. $|x - x_0|^{3/2} = O(|x - x_0|^2)$ only if for all $1/n$ there exists a positive constant c such that $|x - x_0| < 1/n$ implies that $|x - x_0|^{3/2} \leq c|x - x_0|^2$. Then for $x \neq x_0$ (which is trivially true), we must have $1 \leq c|x - x_0|^{1/2}$. However, this condition fails if $|x - x_0|^{1/2} < 1/c$. But since we are considering a neighborhood of x_0, the value of $|x - x_0|^{1/2}$ can be made arbitrarily small with an appropriate choice of x. So no c exists to staisfy the definition of big-O and we have found a counterexample to the converse.