Exercises 1.2.3 — Problem 5

Problem. Let $A_1, A_2, A_3, ...$ be countable sets, and let their Cartesian product $A_1 \times A_2 \times A_3 \times \cdots$ be defined to be the set of all sequences $(a_1, a_2, ...)$ where a_k is an element of A_k. Prove that the Cartesian product is uncountable. Show that the same conclusion holds if each of the sets $A_1, A_2, ...$ has at least two elements.

Proof. We must show that the Caresian product of a countable number of countable sets is uncountable. Towards a contradiction, let's assume that such a Cartesian product is countalbe. For notation, we assume that $P = A_1 \times A_2 \times A_3 \times \cdots$ is a countable set.

Since P is countable, there exists a bijective map $f : \mathbb{N} \longrightarrow P$. For a given $n \in \mathbb{N}$, we say $f(n) = (a_1^n, a_2^n, a_3^n, ...) \in P$. To get our contradiction, we will construct an element $p = (p_1, p_2, p_3, ...) \in P$ such that $p \notin im(f)$.

When constructing p, we choose p_k such that $p_k \in A_k$ and $p_k \neq a_k^k$. We choose $p_k \in A_k$ so that p is an element of P and we require that $p_k \neq a_k^k$ so that there is no $n \in \mathbb{N}$ such that $f(n) = p$. We know that such a $p_k \in A_k$ exists because A_k has countably many elements. Since there is no n such that $f(n) = p$, the element $p \notin im(f)$. This means that f is not surjective.

Yet we assumed that f is surjective so we have reached the contradiction that f must be both surjective and not surjective. Since we reached a contradiction, we must have incorrectly assumed that P is countable. So we have shown that the Cartesian product of a countable number of countable sets is uncountable.

We can make our result even stronger. Our selection of p_k only requires that there is a single element $a_k' \in A_k$ such that $a_k' \neq a_k^k$. So the proof holds so long as $\{a_k^k, a_k'\} \subset A_k$. Thus, each A_k needs only two elements.