Exercises 4.2.4 — Problem 1

Problem. If f is monotone increasing on an interval and has a jump discontinuity at x_0 in the interior of the domain, show that the jump is bounded above by $f(x_2) - f(x_1)$ for any two points x_1, x_2 of the domain surrounding x_0, $x_1 < x_0 < x_2$.

Proof. For some monotone increasing function f defined on an interval I with a jump at x_0 in the interior of I. Let the height of the jump be $h = \lim_{x\to x_0^+} f(x) - \lim_{x\to x_0^-} f(x)$. Now suppose we have two points $x_1, x_2 \in I$ such that $x_1 < x_0 < x_2$. We want to show that $h \leq f(x_2) - f(x_1)$.

Since f increases monotonically, we know that $f(x_1) \leq \lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x) \leq f(x_2)$, which imply that $f(x_1) + \lim_{x\to x_0^+} f(x) \leq \lim_{x\to x_0^-} f(x) + f(x_2)$. Equvalently, $f(x_2) - f(x_1) \geq \lim_{x\to x_0^-} f(x) - \lim_{x\to x_0^+} f(x) = h$. Therefore, the jump is bounded above by $f(x_2) - f(x_1)$.