Exercises 4.2.4 — Problem 11

Problem. If f is a continuous function on a compact set, show that either f has a zero or f is bounded away from zero ($|f(x)| > 1/n$ for all x in the domain, for some $1/n$).

Proof. So we know that f is a continuous function on a compact set D. We must show that $f(D)$ contains 0 or is bounded away from 0. We know that continuous functions map compact sets to compact sets. Therefore $f(D)$ is a compact set. If $0 \in f(D)$ then we are done. If $0 \notin f(D)$, then suppose that f is not bounded away from 0. Then there exists a sequence $f(x_1), f(x_2), ...$ such that for all $1/n$ there exists m such that for $j \geq m$ we have $|f(x_j) - 0| < 1/n$. Then $f(x_1), f(x_2), ...$ converges to 0. And $f(D)$ is compact so $f(D)$ must contain 0 (since it is the only limit point of the sequence). But now we have reached a contradiction since we assumed $0 \notin f(D)$. Therefore, if $0 \notin f(D)$ then f is bounded away from 0.