M 383: Assignment 4

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Exercises 3.1.3 — Problem 1

Problem. Compute the sup, inf, limsup, liminf, and all the limit points of the sequence $x_1, x_2, ...$ where $x_n = 1/n + (-1)^n$.

Proof. First we compute the $\sup$. To this end, we show that $3/2$ is an upper bound for $1/n + (-1)^n$ for all $n \in \mathbb{N}$.

$$3/2 \geq 1/n + (-1)^n \iff 3n/2 \geq 1 + n(-1)^n \geq 1 - n \iff 5n/2 \geq 1 \iff n \geq 2/5$$

which is true for all natural numbers. Now we must show that $3/2$ is the least upper bound for $x_n = 1/n+(-1)^n$. We know this because $3/2$ is an element of the sequence: $x_2 = 1/2+(-1)^2 = 1/2+1 = 3/2$.

Now we compute the $\inf$. First we will show that $-1 \leq x_n$ for all $n \in \mathbb{N}$.

$$-1 \leq 1/n + (-1)^n \iff -n \leq 1 + n(-1)^n \leq 1 + n \iff -n \leq 1 + n \iff n \geq -1/2$$

which is true for all natural numbers. No we must show that -1 is the greatest lower bound for x_n. Suppose there was a lower bound $-1 + 1/k$ for some $k \in \mathbb{N}$. But $-1 + 1/k$ cannot be a lower bound for x_n since x_{2k+1} is certainly less than $-1 + 1/k$:

$$x_{2k+1} = \frac{1}{2k+1} + (-1)^{2k+1} = \frac{1}{2k+1} + -1 < -1 + 1/k \iff \frac{1}{2k+1} < 1/k \iff k < 2k+1 \iff k > -1$$

which is true for all natural numbers. So $-1 + 1/k$ cannot be a lower bound for x_n, which implies that -1 is the greatest lower bound for the sequence.

Before computing $\limsup$ and $\liminf$, we will find all the limit points of $\{x_n\}$. Note that $x_1, x_2, x_3, ... = y_1, z_1, y_2, z_2, y_3, z_3, ...$ where $y_n = 1/(2n-1) + (-1)^{2n-1}$ and $z_n = 1/2n + (-1)^{2n}$. Further $y_n = 1/(2n-1) + (-1)^{2n-1} = 1/(2n-1) + -1$ and $z_n = 1/2n + (-1)^{2n} = 1/2n + 1$. Then we can say that $\lim_{n\to\infty} y_n = -1$ and $\lim_{n\to\infty} z_n = 1$. Since each subsequence converges, $\{y_n\}$ and $\{z_n\}$ each have only one limit point. Additionally, we know that $x_1, x_2, ...$ is just a shuffled sequence of $y_1, y_2, ...$ and $z_1, z_2, ...$ so the limit points of $x_1, x_2, ...$ are the limit points of $\{y_n\}$ and $\{z_n\}$: -1 and 1.

For $\limsup$ and $\liminf$, we know that $\limsup$ is the $\sup$ of the set of limit points and that $\liminf$ is the $\inf$ of the set of the limit points. So the $\limsup$ is 1 and the $\liminf$ is -1.

Exercises 3.1.3 — Problem 2

Problem. If a bounded sequence is the sum of a monotone increasing and a monotone decreasing sequence ($x_n = y_n + z_n$ where $\{y_n\}$ is monotone increasing and $\{z_n\}$ is monotone decreasing), does it follow that the sequence converges? What if $\{y_n\}$ and $\{z_n\}$ are bounded?

Proof. Suppose we have a monotone increasing sequence $\{y_n\}$ and a monotone decreasing sequence $\{z_n\}$. Is their bounded sum $\{x_n\} = \{y_n + z_n\}$ necessarily convergent? No. Consider such sequences $\{y_n\} = 1, 2, 2, 3, 3, 4, 4, ...$ and $\{z_n\} = -1, -1, -2, -2, -3, -3, -4, ...$ with sum $\{x_n\} = 0, 1, 0, 1, 0, 1, 0, ...$ which has no limit.

If we require that $\{y_n\}$ and $\{z_n\}$ are bounded, then we can claim that $\{x_n\}$ converges. This is because Theorem 3.1.2 says that any bounded, monotone increasing sequence converges (and there is an analogous result for bounded, monotone decreasing sequences). So $\{y_n\}$ and $\{z_n\}$ converge and $\{x_n\} = \{y_n + z_n\}$ is just the sum of two converging sequences, so $\{x_n\}$ must also converge.

Exercises 3.1.3 — Problem 4

Problem. Prove $\sup(A \cup B) \geq \sup A$ and $\sup(A \cap B) \leq \sup A$.

Proof. We begin by proving that $\sup(A \cup B) \geq \sup A$. To this end, suppose that the opposite were true: that $\sup(A \cup B) < \sup A$. By definition of sup, we know $\sup A = x$ where x is the smallest extended real number satisfying $a \leq x$ for all $a \in A$. We also know $\sup(A \cup B) = z$ where z is the smallest extended real number satisfying $a \leq z$ for all $a \in A$ and $b \leq z$ for all $b \in B$. From this, we conclude that $\sup(A \cup B) < \sup A \iff z < x$. We also know $a \leq z$ for all a, so it must be true that $a \leq z < x$ for all a. But we have just showed that x is not the smallest extended real number satisfying $a \leq x$ for all a, a contradiction! So it must be true that $\sup(A \cup B) \geq \sup A$.

Now we show that $\sup(A \cap B) \leq \sup A$. Suppose that $\sup(A \cap B) > \sup A$. By definition of sup, we know $\sup(A \cap B) = x$ where x is the smallest extended real number satisfying $a \leq x$ for all $a \in A$. We also know that $\sup(A \cap B) = z$ where z is the smallest extended real number satisfying $c \leq z$ for all $c \in C = A \cap B$. Then $\sup(A \cap B) > \sup A \iff z > x$. We know that $x \geq a$ and that every $c \in A$ (for $c \in A \cap B \iff c \in A \wedge c \in B$), so we can conlude that $z > x \geq a \geq c$. But then z is not the smallest extended real number satisfying $z \geq c$ (for x is strictly less than z). so we have reached a contradiction and it must be true that $\sup(A \cap B) \leq \sup A$.

Exercises 3.1.3 — Problem 6

Problem. Is every subsequence of a subsequence of a subsequence also a subsequence of the sequence?

Proof. Given a sequence $\{x_n\}$ with a subsequence $\{x'_n\}$ we must show that any $\{x''_n\}$ (a subsequence of $\{x'_n\}$) is also a subsequence of $\{x_n\}$. We know every element of $\{x'_n\}$ is an element of $\{x_n\}$ since $\{x'_n\}$ is obtained by crossing off elements of $\{x_n\}$. We also know that $\{x''_n\}$ is obtained by crossing off elements of $\{x'_n\}$. Then, we can obtain $\{x''_n\}$ by crossing off every element of $\{x_n\}$ that should not be in $\{x'_n\}$ and that should not be in $\{x''_n\}$. Therefore, $\{x''_n\}$ must also be a subsequence of $\{x_n\}$.

Exercises 3.1.3 — Problem 9

Problem. Can there exist a sequence whose set of limit points is exactly $1, 1/2, 1/3, ...$?

Proof. There is no sequence whose set of limit points is exactly $1, 1/2, 1/3, ...$ We prove this with contradiction. Suppose $x_1, x_2, x_3, ...$ is a sequence whose limit points are exactly $1, 1/2, 1/3, ...$ A limit point y of $x_1, x_2, ...$ must satisfy the following: for all $1/n$ and m there must exist some $j \geq m$ such that $|y - x_j| < 1/n$. For the sake of contradiction, we assumed that any $1/a$ (for $a \in \mathbb{N}$) is a limit point of $x_1, x_2, ...$ Then there must exist an infinite number of terms less than $1/a$ for any a, which is equivalent to saying that 0 is a limit point of $x_1, x_2, ...$ But this is a contradiction for we assumed that the limit points were $1, 1/2, 1/3, ...$, thus there can be no sequence whose limit points are exactly $1, 1/2, 1/3, ...$