Exercises 4.2.4 — Problem 3

Problem. If the domain of a continuous function is an interval, show that the image is an interval. Give examples where the image is an open interval.

Proof. To show this, we assume not. That is, we assume that there exists some continuous function f defined on an interval I such that the image is not an interval. Since the image is not an interval, then there exist two nonempty, distinct sets A and B such that $f(I) = A \cup B$ and $A \cup B$ is not an interval. Let C be the open interval $(\inf A \cup B, \sup A \cup B)$. Then since $A \cup B$ is not an interval, there exists some number $c \in C$ that is not a member of $A \cup B$. Then there exist $x_1 \neq x_2 \in I$ such that $f(x_1) < c < f(x_2)$. But this directly contradicts the intermediate value theorem since f is a continuous function containing the interval $[x_1, x_2]$ but does not take on every value betweeen $[f(x_1), f(x_2)]$. So we have reached a contradiction and the image must be an interval.

Here are two examples where the image is an open interval: Consider $f(x) = x$ with domain $\mathbb{R}$ and $g(x) = x + 1$ with domain $(0, 1)$. Both images are open sets.