M 384: Assignment 4

Nathan Stouffer

Exercises 7.3.4 — Problem 2

Problem. Suppose $f_n \to f$ and the function f_n all satisfy the Lipschitz condition $|f_n(x) - f_n(y)| \leq M|x - y|$ for some constant M independant of n. Prove that f also satisfies the same Lipschitz condition.

Proof. For fixed x, y in the common domain, we want to show that $|f(x) - f(y)| \leq M|x - y|$. Let's begin with $|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$. Then by the triangle inequality applied twice, we get the following for any $n \in \mathbb{N}$

$$|f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

By supposition, $|f_n(x) - f_n(y)| \leq M|x - y|$ so we have

$$|f(x) - f(y)| \leq M|x - y| + |f(x) - f_n(x)| + |f(y) - f_n(y)|$$

for any n. But this is just a comparison of real numbers so we can just pass to the limit $n \to \infty$ and the non-strict inequality will hold. But the limit of the $|f - f_n| = 0$ so we have $|f(x) - f(y)| \leq M|x - y|$ as desired.

$\square$

Exercises 7.3.4 — Problem 5

Problem. If $\lim_{n\to\infty} f_n = f$ and the functions f_n are all monotone increasing, must f be monotone increasing? What happens if f_n are all strictly increasing?

Proof. In this problem, the limit f must be monotone increasing. To see this, fix x, y in the common domain D such that $x > y$. We know that $f_n(x) \geq f_n(y)$ for all $n \in \mathbb{N}$ with co-domain $\mathbb{R}$ so we really have a sequence of numbers satisfying

$$\lim_{n\to\infty} f_n(x) \geq \lim_{n\to\infty} f_n(y)$$

When passing to the limit the non-strict inequality holds. By pointwise convergence of $\lim_{n\to\infty} f_n$ to f we have $f(x) \geq f(y)$. But we just showed exactly that f is monotone increasing.

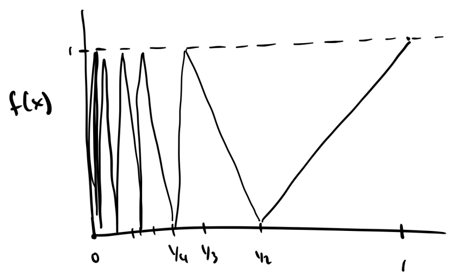
The same is not true for strictly increasing functions. Consider a function $f_n(x) = \frac{1}{n}x$ with domain $D = [0, 1]$. Any f_n is strictly increasing but the limit of this sequence is the function $f(x) = 0$, which is only monotone increasing.

$\square$

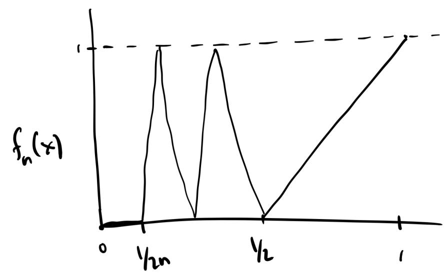
Exercises 7.3.4 — Problem 6

Problem. Give an example of a sequence of continuous functions converging pointwise to a function with a discontinuity of the second kind.

Proof. We begin by defining a function with domain $[0, 1]$ and a discontinuity of the second kind at $x = 0$. For $x = 1/2k$, define $f(x) = 0$ and for $x = 1/(2k+1)$ define $f(x) = 1$. For values in between an "even" and "odd" pair, linearly interpolate between 0 and 1. Finally define $f(0) = 0$. The left figure depicts the limit (as far as I drew it). Then on the right we have a depiction of an element of the sequence. For a



Limit function $f(x)$



Element of the sequence

function f_n in the sequence, we define $f_n(x) = 0$ for $x < 1/2n$ and defined as the limit for $x \geq 1/2n$. Certainly the sequence $f_1, f_2, ...$ converges pointwise to f on their common domain $[0, 1]$. Further, each f_n is continuous but the limit has a discontinuity of the second kind.

□

Exercises 7.3.4 — Problem 7

Problem. If $|f_n(x)| \leq a_n$ for all x and $\sum_{n=1}^{\infty} a_n$ convreges, prove that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Proof. First define $F_k(x) = \sum_{n=1}^{k} f_n(x)$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly if and only if the sequence $F_1(x), F_2(x), ...$ has a uniform limit $F(x)$. By Theorem 7.3.1, the sequence (F_k) has a uniform limit if and only if (F_k) satisfies the Cauchy criterion: for every $1/m$ there exists some $N \in \mathbb{N}$ such that for all $k, j \geq N$ we have $|f_k(x) - f_j(x)| \leq 1/m$ for all x in the common domain.

Let's now show this. We define $A_k = \sum_{n=1}^{k} a_n$. Since the series $\sum_{n=1}^{\infty} a_n$ converes, we know the sequence $A_1, A_2, ...$ satisfies the Cauchy criterion for sequences of real numbers. Then for any $1/m$ choose N such that for all $k, j \geq N$ we have $|A_k - A_j| \leq 1/m$. Without loss of generality, select $k \geq j$. Then we have

$$\begin{aligned}
1/m \geq |A_k - A_j| &= |(a_1 + a_2 + \cdots + a_j + \cdots a_k) - (a_1 + a_2 + \cdots + a_k)| \\
&= |a_{j+1} + \cdots + a_k| \\
&\geq |\,|f_{j+1}(x)| + \cdots + |f_k(x)|\,| \\
&\geq |f_{j+1}(x) + \cdots + f_k(x)| \\
&= |(f_1(x) + \cdots + f_j(x) + \cdots + f_k(x)) - (f_1(x) + \cdots + f_j(x)| \\
&= |F_k(x) - F_j(x)|
\end{aligned}$$

This is exactly the Cauchy criterion for sequences of functions. So we have shown that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

□