Exercises 7.4.5 — Problem 2

Problem. If f is analytic in a neighborhood of x_0 and $f(x_0) = 0$, show that $f(x)/(x - x_0)$ is analytic in the same neighborhood.

Proof. So we know that f is analytic in some neighborhood $(x_0 - 1/n, x_0 + 1/n)$. Let y be any fixed point in $(x_0 - 1/n, x_0 + 1/n)$. Since $f(x)$ is analytic at y, there exists a power series expansion $f(x) = \sum_{n=0}^{\infty} a_n(x - y)^n$ about y. By the uniqueness of power series, $\sum_{n=0}^{\infty} a_n(x - y)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$. Then $b_n = \frac{f(x_0)}{0!} = \frac{0}{1} = 0$ and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

We can divide both sides by $x - x_0$, leaving

$$\frac{f(x)}{x - x_0} = \frac{1}{x - x_0} \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Thus $f(x)/(x - x_0)$ has a power series expansion about any point y in the neighborhood so $f(x)$ is analytic.

$\square$