M 384: Assignment 9

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Problem 1

Problem. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-x^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Show that $\partial f/\partial x$, $\partial f/\partial y$ exist for all $(x,y) \in \mathbb{R}^2$

(b) Show that both $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ exist but $\frac{\partial^2 f(0,0)}{\partial x \partial y} \neq \frac{\partial^2 f(0,0)}{\partial y \partial x}$

Proof.

Problem 2

Problem. For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, prove that $|x^\alpha| \leq |x|^{|\alpha|}$ where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$

Proof. With the defined operations, we have (crucially) that $x_i \leq \sqrt{x_i^2} \leq \sqrt{x_1^2 + \cdots + x_i^2 + \cdots + x_n^2} = |x|$. This implies that, for any x_i, we have $|x_i^{\alpha_i}| \leq ||x|^{\alpha_i}| = |x|^{\alpha_i}$. This allows us to deduce the following:

$$|x^\alpha| = |x_1^{\alpha_1} \cdots x_n^{\alpha_n}| = |x_1^{\alpha_1}| \cdots |x_n^{\alpha_n}| \leq |x|^{\alpha_1} \cdots |x|^{\alpha_n} = |x|^{\alpha_1 + \cdots + \alpha_n} = |x|^{|\alpha|}$$

which was the goal!