## Exercises 7.3.4 — Problem 5

*Problem.* If $\lim_{n\to\infty} f_n = f$ and the functions $f_n$ are all monotone increasing, must $f$ be monotone increasing? What happens if $f_n$ are all strictly increasing?

*Proof.* In this problem, the limit $f$ must be monotone increasing. To see this, fix $x, y$ in the common domain $D$ such that $x > y$. We know that $f_n(x) \geq f_n(y)$ for all $n \in \mathbb{N}$ with co-domain $\mathbb{R}$ so we really have a sequence of numbers satisfying

$$\lim_{n\to\infty} f_n(x) \geq \lim_{n\to\infty} f_n(y)$$

When passing to the limit the non-strict inequality holds. By pointwise convergence of $\lim_{n\to\infty} f_n$ to $f$ we have $f(x) \geq f(y)$. But we just showed exactly that $f$ is monotone increasing.

The same is not true for strictly increasing functions. Consider a function $f_n(x) = \frac{1}{n}x$ with domain $D = [0, 1]$. Any $f_n$ is strictly increasing but the limit of this sequence is the function $f(x) = 0$, which is only monotone increasing.

$\square$