Exercises 7.5.5 — Problem 7

Problem. If f is C^1 on $[a, b]$ prove that there exists a cubic polynomial P such that $f - P$ and its first derivative vanish at the endpoints of the interval.

Proof. Note that we are able to use the fact that there exists a polynomial of degree $2n - 1$ that satisfies $f(x_k) = a_k$ and $f'(x_k) = b_k$ for $k = 1, \ldots, n$. Let $n = 2$ and take $x_1 = a$ and $x_2 = b$. Let $a_1 = f(a)$, $b_1 = f'(a)$ and $a_2 = f(b), b_2 = f'(b)$. We apply our fact and say that there is some polynomial P of degree $2n - 1 = 3$ that satisfies $P(a) = a_1 = f(a)$, $P'(a) = b_1 = f'(a)$, $P(b) = a_2 = f(b)$, and $P'(b) = b_2 = f'(b)$.

Then we have the following four equations:

$$
\begin{aligned}
(f - P)(a) &= f(a) - P(a) = f(a) - f(a) = 0 \\
(f - P)'(a) &= f'(a) - P'(a) = f'(a) - f'(a) = 0 \\
(f - P)(b) &= f(b) - P(b) = f(b) - f(b) = 0 \\
(f - P)'(b) &= f'(b) - P'(b) = f'(b) - f'(b) = 0
\end{aligned}
$$

Thus we have the desired equalities for a cubic polynomial P.

□