M 384: Assignment 1

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Exercises 6.1.5 — Problem 3

Problem. Derive the integration of the derivative theorem from the differentiation of the integral theorem.

Proof. So we must show that $\frac{d}{dx}\int_a^x g(t)dt = g(x)$ for all continuous functions g on $[a,b]$ with $a \leq b$ implies that $\int_a^b f'(x)dx = f(b) - f(a)$ for all $f \in C^1$ on $[a,b]$. Take $F(x) \in C^1[a,b]$, then F has a continuous derivative $F' = f$ defined on $[a,b]$. By the differentiation of the integral theorem, we have $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. Then $f(x) = F'(x) = \frac{d}{dx}F(x)$ so we must have $F(x) = \int_a^x F'(t)dt$ an arbitrary C^1 function on $[a,b]$.

Then we can compute the value of $F(b) - F(a) = \int_a^b F'(x)dx - \int_a^a F'(x)dx = \int_a^b F'(x)dx + 0$. So we have shown that $F(b) - F(a) = \int_a^b F'(x)dx$ holds for any C^1 function defined on $[a,b]$, which is the integration of the derivative theorem.

$\square$

Exercises 6.1.5 — Problem 4

Problem. Prove the integral mean value theorem: if f is continuous on $[a, b]$ then there exists y in (a, b) such that $\int_a^b f(x)dx = (b-a)f(y)$.

Proof. Suppose we have some continuous function f defined on $[a, b]$. Then let $F(x) = \int_a^x f(t)dt$, we know $F(x) \in C^1[a, b]$ and $F' = f$ by the differentiation of the integral theorem. Since $F(x)$ is differentiable on $[a, b]$, the mean value theorem tells us there exists $y \in (a, b)$ such that

$$F'(y) = f(y) = \frac{F(b) - F(a)}{b-a} = \frac{\int_a^b f(x)dx - \int_a^a f(x)dx}{b-a} = \frac{\int_a^b f(x)dx - 0}{b-a} = \frac{\int_a^b f(x)dx}{b-a}$$

So we have $f(y) = \frac{\int_a^b f(x)dx}{b-a}$ and we can multiply both sides by $b-a$ to obtain $\int_a^b f(x)dx = (b-a)f(y)$ for some $y \in (a, b)$ as desired.

□

Exercises 6.1.5 — Problem 8

Problem. Let f be a C^1 function on the line, and let $g(x) = \int_0^1 f(xy)y^2 dy$. Prove that g is a C^1 function and establish a formula for $g'(x)$ in terms of f.

Proof. First define $h(x, y) = f(xy)y^2$ then $g(x) = \int_0^1 h(x, y)dy$. Since $f(xy)$ and y^2 are continuous functions, Theorem 6.1.8 tells us that $\int_0^1 h(x, y)dy$ is a continuous function. Further, the fact that each of $f(xy), y^2$ is C^1 means that $g(x)$ is C^1 since C^1 is closed under multpilication. Then the conditions for Theorem 6.1.7 are met (since the constant functions 0 and 1 are certainly C^1) so we can give the derivative formula:

$$g'(x) = \int_0^1 \frac{\partial h}{\partial x}(x, y)dy = \int_0^1 f'(xy)y^3 dy$$

$\square$

Exercises 6.1.5 — Problem 10

Problem. For a continuous, positive function $w(x)$ on $[a, b]$, define the weighted average operator A_w to be

$$A_w(f) = \int_a^b f(x)w(x)dx / \int_a^b w(x)dx$$

for continous functions f. Prove that A_w is linear and lies between the maximum and minimum values of f.

Proof. First we prove that A_w is linear. A_w is linear if $A_w(c_1 f + c_2 g) = c_1 A_w(f) + c_2 A_w(g)$ for $c_1, c_2 \in \mathbb{R}$ and continuous functions f, g on $[a, b]$. Towards proving linearity, let f, g be continuous functions on $[a, b]$ and $c_1, c_2 \in \mathbb{R}$. Then, using properties of integrals and functions,

$$\begin{aligned}
A_w(c_1 f + c_2 g) &= \frac{\int_a^b [c_1 f + c_2 g](x)w(x)}{\int_a^b w(x)dx} \\
&= \frac{\int_a^b c_1 f(x)w(x) + c_2 g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= \frac{\int_a^b c_1 f(x)w(x)dx + \int_a^b c_2 g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= \frac{c_1 \int_a^b f(x)w(x)dx + c_2 \int_a^b g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= \frac{c_1 \int_a^b f(x)w(x)dx}{\int_a^b w(x)dx} + \frac{c_2 \int_a^b g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= c_1 A_w(f) + c_2 A_w(g)
\end{aligned}$$

So A_w is linear. Now we must show that $A_w(f)$ lies in between the maximum and minimum values of f. Let $F^+ = \sup_{x\in[a,b]} f(x)$ and $F^- = \inf_{x\in[a,b]} f(x)$, so we must show that $A_w(f) \in [F^-, F^+]$ which is equivalent to $F^- \leq A_w(f) \leq F^+$. Since $\int_a^b w(x)dx$ is a postive number, let $W = \int_a^b w(x)dx$. If multiply the desired inequality by W, we must equivalently show that $WF^- \leq \int_a^b f(x)w(x)dx \leq WF^+$.

Now it is certainly true that $\int_a^b f(x)w(x)dx \leq \int_a^b F^+ w(x)dx$, but then F^+ is a constant so $\int_a^b F^+ w(x)dx = F^+ \int_a^b w(x)dx = F^+ W$ so $\int_a^b f(x)w(x)dx \leq WF^+$. Similarly, $\int_a^b f(x)w(x)dx \geq \int_a^b F^- w(x)dx = F^- \int_a^b w(x)dx = F^- W$. Then we have $WF^- \leq \int_a^b f(x)w(x)dx \leq WF^+$, which was the goal.

$\square$