Exercises 7.6.3 — Problem 6

Problem. Prove that the family of all polynomials of degree $\leq N$ with coefficients in $[-1, 1]$ is uniformly bounded and uniformly equicontinuous on any compact interval.

Proof. Let $p(x)$ be a such a polynomial on a compact interval $[a, b]$. That is, $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_i x^i$ for $x \in [a, b]$ a compact interval, $c_i \in [-1, 1]$, and $0 \leq i \leq N$.

First we will bound all such polynomials. Choose $m = \max(|a|, |b|)$ then any $c_j x^j \leq |c_j x^j| = |c_j||x^j| \leq 1m^j = m^j$ and take $M = \sum_{i=0}^{N} m^i$. Then, term for term, we have $p(x) \leq M$ for any of the polynomials in the family. Thus, the family is uniformly bounded.

Now we show that the family is uniformly equicontinuous. For any such $p(x)$, consider $p'(x) = c_1 + 2c^2 x + 3c_3 x^2 + \cdots + Nc_N x^{N-1}$. Then $p'(x) \leq M' = \sum_{i=0}^{N-1} N * m^i$ since $N * m^i$ is larger term for term. Thus by the MVT we have the following for some $z \in (x, y)$: $|p(x) - p(y)| \leq |p'(z)||x - y| \leq M'|x - y|$. Then for a given $1/m$ we can choose $1/n = 1/(Mm)$ to satisfy equicontinuity.

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