Exercises 6.1.5 — Problem 3

Problem. Derive the integration of the derivative theorem from the differentiation of the integral theorem.

Proof. So we must show that $\frac{d}{dx}\int_a^x g(t)dt = g(x)$ for all continuous functions g on $[a,b]$ with $a \leq b$ implies that $\int_a^b f'(x)dx = f(b) - f(a)$ for all $f \in C^1$ on $[a,b]$. Take $F(x) \in C^1[a,b]$, then F has a continuous derivative $F' = f$ defined on $[a,b]$. By the differentiation of the integral theorem, we have $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. Then $f(x) = F'(x) = \frac{d}{dx}F(x)$ so we must have $F(x) = \int_a^x F'(t)dt$ an arbitrary C^1 function on $[a,b]$.

Then we can compute the value of $F(b) - F(a) = \int_a^b F'(x)dx - \int_a^a F'(x)dx = \int_a^b F'(x)dx + 0$. So we have shown that $F(b) - F(a) = \int_a^b F'(x)dx$ holds for any C^1 function defined on $[a,b]$, which is the integration of the derivative theorem.

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