Exercises 6.2.4 — Problem 9

Problem. If f is a Riemann integrable function on $[a, b]$ prove that $F(x) = \int_a^x f(t)dt$ satisfies a Lipschitz condition.

Proof. The function $F(x)$ is Lipschitz if there exists a natural number M such that $|F(x) - F(x_0)| \leq M|x - x_0|$ for all $x, x_0 \in [a, b]$. Plugging in x, we must have $| \int_a^x f(t)dt - \int_a^{x_0} | \leq M|x - x_0|$ for all $x, x_0 \in [a, b]$. Choose $M = \sup_{y \in [a,b]} f(y)$.

Now we will show that $\int_a^x f(t)dt - \int_a^{x_0} f(t)dt = \int_{x_0}^x f(t)dt$. We consider three cases. Case $x = x_0$: then $\int_a^x f(t)dt - \int_a^{x_0} f(t)dt = 0 = \int_{x_0}^x f(t)dt$. Case $x < x_0$: then $\int_a^x f(t)dt - \int_a^{x_0} f(t)dt = \int_a^x f(t)dt - [\int_a^x f(t)dt + \int_x^{x_0} f(t)dt] = -\int_x^{x_0} f(t)dt = \int_{x_0}^x f(t)dt$. Case $x > x_0$: then $\int_a^x f(t)dt - \int_a^{x_0} f(t)dt = [\int_a^{x_0} f(t)dt + \int_{x_0}^x f(t)dt] - \int_a^{x_0} f(t)dt = \int_{x_0}^x f(t)dt$.

So now have $| \int_a^x f(t)dt - \int_a^{x_0} f(t)dt| = | \int_{x_0}^x f(t)dt|$. Then we immediately obtain $| \int_{x_0}^x f(t)dt| \leq M_0|x - x_0|$ where $M_0 = \sup f(x)$ on $[x_0, x]$ (or $[x, x_0]$ if $x < x_0$). Then $M_0|x - x_0| \leq \sup_{x \in [a,b]} f(x)|x - x_0| = M|x - x_0|$. So we have just shown that $|F(x) - F(x_0)| \leq M|x - x_0|$ for our chosen M. Therefore $F(x)$ is Lipshitz continuous.

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