Exercises 7.6.3 — Problem 9

Problem. Give an example of a uniformly bounded and uniformly equicontinuous sequence of functions on the whole line that does not have any uniformly convergent subsequences.

Proof. Consider the function

$$f_n(x) = \begin{cases} 0 & x \leq n-1 \\ x-(n-1) & n-1 < x \leq n \\ 1 & x > n \end{cases}$$

Any f_n is bounded by taking $M = 2$ so the sequence is uniformly bounded. Also the sequence is uniformly continuous by taking $1/n = 1/2m$ for a given $1/m$ (also the derivative is bounded above by 1). Now we show that the sequence has no uniformly converging subsequences. To have a converging subsequence, there must be a subsequence that satisfies the Cauchy criterion. We show that no pair of distinct elements even satisfies the Cauchy criterion, so there is no chance of a subsequence existing since every element of the sequence is distinct. Pick any $k > j$ and consider $|f_k(j) - f_j(j)| = |0 - 1| = 1$. Thus there is no uniformly converging subsequence of f_n.

$\square$