Exercises 6.2.4 — Problem 10

Problem. If f is Riemann integrable on $[a, b]$ and continuous at x_0, prove that $F(x) = \int_a^x f(t)dt$ is differentiable at x_0 and $F'(x_0) = f(x_0)$. Show that if f has a jump discontinuity at x_0, then F is not differentiable at x_0.

Proof. To prove that $F(x)$ is differentiable at x_0, we must show that $\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0}$ exists. Note that $\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{\int_a^x f(t)dt - \int_a^{x_0} f(t)dt}{x-x_0} = \lim_{x\to x_0} \frac{\int_{x_0}^x f(t)dt}{x-x_0}$ by the quick proof in the previous problem. For the limit to exist, we must have the existence of both the right and left limit and the equality $\lim_{x\to x_0^-} \frac{\int_{x_0}^x f(t)dt}{x-x_0} = \lim_{x\to x_0^+} \frac{\int_{x_0}^x f(t)dt}{x-x_0}$.

Let $\Delta x = |x - x_0|$, $m = \inf_{y\in[x_0-\Delta x, x_0]} f(y)$, and $M = \sup_{y\in[x_0,x_0+\Delta x]} f(y)$. Certainly we have $\lim_{x\to x_0^-} m(x - x_0) \leq \lim_{x\to x_0^-} \int_{x_0}^x f(t)dt$ for $x > x_0$. Then, dividing by $x - x_0$ we get $\lim_{x\to x_0^-} m = m \leq \lim_{x\to x_0^-} \frac{\int_{x_0}^x f(t)dt}{x-x_0}$. Then for $x_0 > x$, we must have $\lim_{x\to x_0^+} \int_x^{x_0} f(t)dt \leq \lim_{x\to x_0^+} M(x_0 - x)$. Dividing by $x_0 - x$, we have $\lim_{x\to x_0^+} \frac{\int_x^{x_0} f(t)dt}{x_0-x} = \lim_{x\to x_0^+} \frac{\int_{x_0}^x f(t)dt}{x-x_0} \leq \lim_{x\to x_0^+} M = M$.

So we have $m \leq \lim_{x\to x_0} \frac{\int_{x_0}^x f(t)dt}{x-x_0} \leq M$, but since f is continuous m, M can be made arbitrarily close with sufficiently small Δx so the limit exists because of the squeeze theorm. Further, the value to which m, M converge to is $\Delta x \to 0$ is $f(x_0)$, as desired.

Note that if f has a jump discontinuity at x_0, we cannot use the squeeze theorem. In fact, the left limit would be bounded above by $\sup_{y\in[x_0-\Delta x_0, x_0]} f(y)$ and the right limit would be bounded below by $\inf_{y\in[x_0,x_0+\Delta x_0]} f(y)$. Since f has a jump discontinuity at x_0 we know that $\sup_{y\in[x_0-\Delta x_0,x_0]} f(y) < \inf_{y\in[x_0,x_0+\Delta x_0]} f(y)$. But then the left and right limit cannot by equal, so the limiet does not exist.

$\square$