M 384: Assignment 9

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Problem 1

Problem. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Show that $\partial f/\partial x$, $\partial f/\partial y$ exist for all $(x,y) \in \mathbb{R}^2$

(b) Show that both $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ exist but $\frac{\partial^2 f(0,0)}{\partial x \partial y} \neq \frac{\partial^2 f(0,0)}{\partial y \partial x}$

Proof.

(a) For $(x,y) \neq (0,0)$ we can just compute the partial derivatives. Consider

$$\frac{\partial f}{\partial x} = \frac{[y(x^2-y^2)+2xyx](x^2+y^2) - xy(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{(3x^2y-y^3)(x^2+y^2)+2x^2y^3-2x^4y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{[x(x^2-y^2)-2yxy](x^2+y^2) - xy(x^2-y^2)2y}{(x^2+y^2)^2} = \frac{(x^3-2xy^2)(x^2+y^2)+2xy^4-2x^3y^2}{(x^2+y^2)^2}$$

Then for each partial evaluated at 0, we take a limit. First consider $\frac{\partial f}{\partial x}(0,0)$:

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} \\ &= \lim_{x\to 0} \frac{\frac{x*0(x^2-0^2)}{x^2+0^2}}{x} \\ &= \lim_{x\to 0} \frac{0}{x^2}\frac{1}{x} = \lim_{x\to 0} \frac{0}{x^3} = \lim_{x\to 0} \frac{0}{3!} = \lim_{x\to 0} 0 = 0\end{aligned}$$

where we used L'Hopitals rule three times. Now consider $\frac{\partial f}{\partial y}(0,0)$

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} \\ &= \lim_{y\to 0} \frac{\frac{0*y(0^2-y^2)}{0^2+y^2}}{y} \\ &= \lim_{y\to 0} \frac{0}{y^2}\frac{1}{y} = \lim_{y\to 0} \frac{0}{y^3} = \lim_{y\to 0} \frac{0}{3!} = \lim_{y\to 0} 0 = 0\end{aligned}$$

(b) Now we must compute the mixed partials and show their nonequality.

$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{x\to 0} \frac{\frac{\partial f(x,0)}{\partial y} - \frac{\partial f(0,0)}{\partial y}}{x-0} = \lim_{x\to 0} \frac{x^5/x^4}{x} = \lim_{x\to 0} \frac{x^5}{x^5} = \lim_{x\to 0} 1 = 1$$

$$\frac{\partial^2 f(0,0)}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{y\to 0} \frac{\frac{\partial f(0,y)}{\partial x} - \frac{\partial f(0,0)}{\partial x}}{y-0} = \lim_{y\to 0} \frac{-y^5/y^4}{y} = \lim_{y\to 0} \frac{-y^5}{y^5} = \lim_{y\to 0} -1 = -1$$

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Problem 2

Problem. For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, prove that $|x^\alpha| \leq |x|^{|\alpha|}$ where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$

Proof. With the defined operations, we have (crucially) that $x_i \leq \sqrt{x_i^2} \leq \sqrt{x_1^2 + \cdots + x_i^2 + \cdots + x_n^2} = |x|$. This implies that, for any x_i, we have $|x_i^{\alpha_i}| \leq ||x|^{\alpha_i}| = |x|^{\alpha_i}$. This allows us to deduce the following:

$$|x^\alpha| = |x_1^{\alpha_1} \cdots x_n^{\alpha_n}| = |x_1^{\alpha_1}| \cdots |x_n^{\alpha_n}| \leq |x|^{\alpha_1} \cdots |x|^{\alpha_n} = |x|^{\alpha_1 + \cdots + \alpha_n} = |x|^{|\alpha|}$$

which was the goal!

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