Exercises 7.2.4 — Problem 4

Problem. Prove the ratio test. What does this tell you if $\lim_{n\to\infty} |x_{n+1}/x_n|$ exists?

Proof. The ratio test claims that if $|x_{n+1}/x_n| < r$ for all sufficiently large n and some $r < 1$, then $\sum x_n$ converges absolutely while if $|x_{n+1}/x_n| \geq 1$ for all sufficiently large n, then $\sum x_n$ diverges.

First we prove the absolutely convergent result. Consider some infinite series that satisfies $|x_{n+1}/x_n| < r$ for all sufficiently large n. Then there exists a natural number K such that for all $n > K$ we have $|x_{n+1}/x_n| < r$. Then the infinite series $\sum x_n = \sum_{n=1}^{K} x_n + \sum_{n=K+1}^{\infty} x_n$. Letting $y_n = x_{n+K}$, the convergence of $\sum x_n$ rests on $\sum_{n=K+1}^{\infty} x_n = \sum_{n=1}^{\infty} y_n$.

By supposition, we have (for all $n > K$) $|x_{n+1}/x_n| < r$. Because of how we defined y_n, this means we have $|y_{n+1}/y_n| < r$ for all n. This implies that $|y_{n+1}| < r|y_n|$, which, in turn, means that $|y_n| < r^{n-1}|y_1|$ for $n \geq 2$. Then let $z_n = r^{n-1}y_1$. We know $\sum_{n=1}^{\infty} z_n$ converges because $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} r^{n-1}y_1 = y_1 \sum_{n=1}^{\infty} r^{n-1}$ converges since $|r| < 1$. Now we can use the comparison test for $\sum z_n$ and $\sum y_n$ to say that $\sum y_n$ converges absolutely, which implies the absolute convergence of $\sum x_n$.

We now show that if $|x_{n+1}/x_n| \geq 1$ for all sufficiently large n then $\sum x_n$ diverges. Suppose we have such a summation $\sum x_n$. Then there exists some K such that $|x_{n+1}/x_n| \geq 1$ for all $n > K$. But then we have $|x_{n+1}| \geq |x_n|$ for all $n > K$. This certainly cannot satisfy $\lim_{n\to\infty} x_n = 0$, a necessary condition for convergence. Therefore, $\sum x_n$ diverges.

If we know that $\lim_{n\to\infty} |x_{n+1}/x_n|$ exists, then I'm not sure what we can say. It seems like the series may converge or diverge.

$\square$