M 384: Assignment 5

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Exercises 7.4.5 — Problem 2

Problem. If f is analytic in a neighborhood of x_0 and $f(x_0) = 0$, show that $f(x)/(x - x_0)$ is analytic in the same neighborhood.

Proof. So we know that f is analytic in some neighborhood $(x_0 - 1/n, x_0 + 1/n)$. Let y be any fixed point in $(x_0 - 1/n, x_0 + 1/n)$. Since $f(x)$ is analytic at y, there exists a power series expansion $f(x) = \sum_{n=0}^{\infty} a_n(x - y)^n$ about y. By the uniqueness of power series, $\sum_{n=0}^{\infty} a_n(x - y)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$. Then $b_n = \frac{f(x_0)}{0!} = \frac{0}{1} = 0$ and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

We can divide both sides by $x - x_0$, leaving

$$\frac{f(x)}{x - x_0} = \frac{1}{x - x_0} \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^{n-1}$$

Thus $f(x)/(x - x_0)$ has a power series expansion P about x_0. Let's now check that the power series P has the same radius of convergence as the power series expansion of $f(x)$ about x_0. The power series expansion of $f(x)$ about x_0 has radius of convergence $1/R$. For P, the radius of convergence is $\limsup_{n\to\infty} \left(\frac{f^{(n)}(x_0)}{n!}\right)^{1/n}$, but this is the exact same expression as the expansion of $f(x)$ so we know its value is also $1/R$. Therefore, the two expansions have the same radius of convergence which implies that that $f(x)/(x - x_0)$ is analytic in the same neighborhood.

$\square$

Exercises 7.4.5 — Problem 6

Problem. Prove that if $f(x)$ is analytic on (a,b), then $F(x) = \int_c^x f(t)dt$ is also analytic on (a,b), where c is any point in (a,b).

Proof. Fix $c \in (a,b)$. Then since f is analytic on (a,b), we have $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ where $a_n = \frac{f^{(n)}(c)}{n!}$. Then since $F(x) = \int_c^x f(t)dt$,

$$F(x) = \int_c^x \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(t-c)^n dt = \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{(n+1)!}(t-c)^{n+1} \right)\Bigg|_c^x$$

Then evaluating,

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{(n+1)!}(x-c)^{n+1} - \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{(n+1)!}(c-c)^{n+1}$$

But the second term is 0 so we are left with $F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{(n+1)!}(x-c)^{n+1}$. Since c was arbitrary in (a,b), $F(x)$ has a power series expansion about any point in (a,b), thus $F(x)$ is analytic.

□

Exercises 7.4.5 — Problem 7

Problem. Compute the power-series expansion of the $f(x) = x^2/(1-x^2)$ about $x = 0$.

Proof. We wish to find the power series expansion for $x^2/(1-x^2)$ about $x = 0$. Let $u = x^2$. Then

$$\frac{x^2}{1-x^2} = \frac{u}{1-u} = u\frac{1}{1-u}$$

Now we know the power series for $1/(1-u) = \sum_{n=0}^{\infty} u^n$ so

$$u\frac{1}{1-u} = u\sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} u^{n+1}$$

Since, we have $u = x^2$, $\sum_{n=0}^{\infty} u^{n+1} = \sum_{n=0}^{\infty}(x^2)^{n+1} = \sum_{n=0}^{\infty} x^{2n+2}$ so we have the power expansion for $f(x) = x^2/(1-x^2)$ about $x = 0$. Note this is valid only for $|u| = |x^2| = x^2 < 1$.

□

Exercises 7.4.5 — Problem 8

Problem. Compute the radius of convergence of the following power series:

a. $\sum n^4/n!x^n$
b. $\sum \sqrt{n}x^n$
c. $\sum n^2 2^n x^n$

Proof.

a. Here $1/R = \limsup_{n\to\infty}(n^4/n!)^{1/n} = \lim_{n\to\infty}(n^4)^{1/n} \lim_{n\to\infty}(1/n!)^{1/n}$. Since n^4 is a polynomial in n we know $\lim_{n\to\infty}(n^4)^{1/n} = 1$ and it was proven in the textbook that $\lim_{n\to\infty}(1/n!)^{1/n} = 0$. Since the RHS is 0, $R = +\infty$.

b. Again $1/R = \limsup_{n\to\infty}(\sqrt{n})^{1/n} = (\limsup_{n\to\infty} n^{1/n})^{1/2} = 1^{1/2} = 1$. So $R = 1$.

c. As always, $1/R = \limsup_{n\to\infty}(n^2 2^n)^{1/n} = \lim_{n\to\infty}(n^2)^{1/n} \lim_{n\to\infty}(2^n)^{1/n} = 1 \lim_{n\to\infty} 2 = 2$ which means $R = 1/2$.

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