Exercises 6.1.5 — Problem 4

Problem. Prove the integral mean value theorem: if f is continuous on $[a, b]$ then there exists y in (a, b) such that $\int_a^b f(x)dx = (b-a)f(y)$.

Proof. Suppose we have some continuous function f defined on $[a, b]$. Then let $F(x) = \int_a^x f(t)dt$, we know $F(x) \in C^1[a, b]$ and $F' = f$ by the differentiation of the integral theorem. Since $F(x)$ is differentiable on $[a, b]$, the mean value theorem tells us there exists $y \in (a, b)$ such that

$$F'(y) = f(y) = \frac{F(b) - F(a)}{b-a} = \frac{\int_a^b f(x)dx - \int_a^a f(x)dx}{b-a} = \frac{\int_a^b f(x)dx - 0}{b-a} = \frac{\int_a^b f(x)dx}{b-a}$$

So we have $f(y) = \frac{\int_a^b f(x)dx}{b-a}$ and we can multiply both sides by $b-a$ to obtain $\int_a^b f(x)dx = (b-a)f(y)$ for some $y \in (a, b)$ as desired.

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