Exercises 7.2.4 — Problem 1

Problem. Give and example of two convergent series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ such that $\sum_{k=1}^{\infty} x_k y_k$ diverges. Can this happen if one of the series is absolutely convergent?

Proof. For the infinite series $\sum x_k, \sum y_k$, let $x_k = (-1)^k/\sqrt{k}$ and $y_k = (-1)^{k+1}/\sqrt{k}$. Note that $-\sum x_k = \sum y_k$ so we only need to show that convergence of one series. Let's show $\sum x_k$. If we set $A_k = 1/\sqrt{k}$ then $\sum x_k = \sum_{k=1}^{\infty}(-1)^k A_k$ where A_k monotonically converges to 0. Then we can apply Theorem 7.2.5 to say that $\sum x_k$ is convergent. So each of $\sum x_k$ and $\sum y_k$ converge, what about their pairwise product?

$$\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty}((-1)^k/\sqrt{k}) * ((-1)^{k+1}/\sqrt{k}) = \sum_{k=1}^{\infty}(-1)^{2k+1}/k = \sum_{k=1}^{\infty} -1/k = -\sum_{k=1}^{\infty} 1/k$$

But this is the negative of the harmonic series, which we know diverges so we have given an example of the desired series.

Now suppose that one of the series is absolutely convergent. Without loss of generality, pick $\sum x_k$. Then $\sum_{k=1}^{\infty} |x_k|$ converges. Additionally, we know $\sum y_k$ converges. Since $\sum y_k$ converges, we know $\lim_{k\to\infty} y_k = 0$. Then for all $1/n$ there exists and m for which $|y_j| < 1/n$ for all $j \geq m$. Choose $n = 1$ and let m_1 be the index that satisfies the Cauchy criterion. Certainly the partial sum $\sum_{k=1}^{m_1-1} x_k y_k$ converges. What about $\sum_{k=m_1}^{\infty} x_k y_k$? Choose any $N > m_1$, then we have $\left|\sum_{k=m_1}^{N} x_k y_k\right| \leq \sum_{k=m_1}^{N} |x_k y_k| = \sum_{k=m_1}^{N} |x_k||y_k|$ by the triangle inequality. But since $|y_j| < 1$ for all $j \geq m_1$ so $\sum_{k=m_1}^{N} |x_k||y_k| \leq \sum_{k=m_1}^{N} |x_k|$. Then we can the limit as $N \to \infty$ and the non-strict inequality is preserved. But we already know $\sum |x_n|$ converges so $\sum x_n y_n$ converges.

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