Exercises 7.4.5 — Problem 2

Problem. If f is analytic in a neighborhood of x_0 and $f(x_0) = 0$, show that $f(x)/(x - x_0)$ is analytic in the same neighborhood.

Proof. So we know that f is analytic in some neighborhood $(x_0 - 1/n, x_0 + 1/n)$. Let y be any fixed point in $(x_0 - 1/n, x_0 + 1/n)$. Since $f(x)$ is analytic at y, there exists a power series expansion $f(x) = \sum_{n=0}^{\infty} a_n(x - y)^n$ about y. By the uniqueness of power series, $\sum_{n=0}^{\infty} a_n(x - y)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$. Then $b_n = \frac{f(x_0)}{0!} = \frac{0}{1} = 0$ and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

We can divide both sides by $x - x_0$, leaving

$$\frac{f(x)}{x - x_0} = \frac{1}{x - x_0} \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^{n-1}$$

Thus $f(x)/(x - x_0)$ has a power series expansion P about x_0. Let's now check that the power series P has the same radius of convergence as the power series expansion of $f(x)$ about x_0. The power series expansion of $f(x)$ about x_0 has radius of convergence $1/R$. For P, the radius of convergence is $\limsup_{n\to\infty} \left(\frac{f^{(n)}(x_0)}{n!}\right)^{1/n}$, but this is the exact same expression as the expansion of $f(x)$ so we know its value is also $1/R$. Therefore, the two expansions have the same radius of convergence which implies that that $f(x)/(x - x_0)$ is analytic in the same neighborhood.

$\square$