Exercises 6.2.4 — Problem 6

Problem. Prove that if f is Riemann integrable on $[a, b]$ and $g(x) = f(x)$ for every x except for a finite number, then g is Riemann integrable.

Proof. We must show that g is Riemann integrable. The function g is Riemann integrable if there exists sequence of partitions P_j and a real number $\int_b^a g(x)dx$ for which $S(g, P_j) \to \int_b^a g(x)dx$ as $j \to \infty$ for every choice of Cauchy Sums $S(g, P_j)$. We already know that this condition holds for the function f. We will just show that the difference between any Cauchy Sum $S(f, P_j)$ and $S(g, P_j)$ becomes arbitrarily small as $j \to \infty$. Then the same conditions will hold for g.

Given a partition P_j, let $|P_j|$ denote the maximum interval length in P_j. Now, since the number of difference between f and g is finite, label the values $x_1, x_2, ..., x_n$ such that $g(x_i) \neq f(x_i)$. Then we must have $|S(f, P_j) - S(g, P_j)| \leq \sum_{i=1}^n |f(x_i) - g(x_i)||P_j|$ (the differences between the function times the maximum interval length). Now let $D = \max_i |f(x_i) - g(x_i)|$, then $|S(f, P_j) - S(g, P_j)| \leq \sum_{i=1}^n D|P_j| = nD|P_j|$. Since nD is finite, as $j \to \infty$, $|P_j| \to 0$ so the magnitude of the difference is 0 as $j \to \infty$. Then $\int_b^a f(x)dx = \int_b^a g(x)dx$ which implies that g is Riemann integrable.

$\square$