Problem 1

Problem. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Show that $\partial f/\partial x$, $\partial f/\partial y$ exist for all $(x,y) \in \mathbb{R}^2$

(b) Show that both $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ exist but $\frac{\partial^2 f(0,0)}{\partial x \partial y} \neq \frac{\partial^2 f(0,0)}{\partial y \partial x}$

Proof.

(a) For $(x,y) \neq (0,0)$ we can just compute the partial derivatives. Consider

$$\frac{\partial f}{\partial x} = \frac{[y(x^2-y^2)+2xyx](x^2+y^2) - xy(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{(3x^2y-y^3)(x^2+y^2)+2x^2y^3-2x^4y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{[x(x^2-y^2)-2yxy](x^2+y^2) - xy(x^2-y^2)2y}{(x^2+y^2)^2} = \frac{(x^3-2xy^2)(x^2+y^2)+2xy^4-2x^3y^2}{(x^2+y^2)^2}$$

Then for each partial evaluated at 0, we take a limit. First consider $\frac{\partial f}{\partial x}(0,0)$:

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} \\ &= \lim_{x\to 0} \frac{\frac{x*0(x^2-0^2)}{x^2+0^2}}{x} \\ &= \lim_{x\to 0} \frac{0}{x^2}\frac{1}{x} = \lim_{x\to 0}\frac{0}{x^3} = \lim_{x\to 0}\frac{0}{3!} = \lim_{x\to 0} 0 = 0 \end{aligned}$$

where we used L'Hopitals rule three times. Now consider $\frac{\partial f}{\partial y}(0,0)$

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} \\ &= \lim_{y\to 0} \frac{\frac{0*y(0^2-y^2)}{0^2+y^2}}{y} \\ &= \lim_{y\to 0} \frac{0}{y^2}\frac{1}{y} = \lim_{y\to 0}\frac{0}{y^3} = \lim_{y\to 0}\frac{0}{3!} = \lim_{y\to 0} 0 = 0 \end{aligned}$$

(b) Now we must compute the mixed partials and show their nonequality.

$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{x\to 0} \frac{\frac{\partial f(x,0)}{\partial y} - \frac{\partial f(0,0)}{\partial y}}{x-0} = \lim_{x\to 0}\frac{x^5/x^4}{x} = \lim_{x\to 0}\frac{x^5}{x^5} = \lim_{x\to 0} 1 = 1$$

$$\frac{\partial^2 f(0,0)}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{y\to 0} \frac{\frac{\partial f(0,y)}{\partial x} - \frac{\partial f(0,0)}{\partial x}}{y-0} = \lim_{y\to 0}\frac{-y^5/y^4}{y} = \lim_{y\to 0}\frac{-y^5}{y^5} = \lim_{y\to 0} -1 = -1$$

$\square$