Exercises 10.1.5 — Problem 3

Problem. If f is differentiable at y, show that $d_u f(y)$ is linear in u, meaning $d_{au+bv} f(y) = a d_u f(y) + b d_v f(y)$ for any scalars a and b.

Proof. Before beginning, note that Theorem 10.1.1 says that if $f : D \to \mathbb{R}^m$ for $D \subset \mathbb{R}^n$ is differentiable at y with differential $df(y)$, then $d_u f(y)$ exists at y for any $u \in \mathbb{R}^n$ and $d_u f(y) = df(y) \cdot u$. Now let $w = au + bv$ for $u, v \in \mathbb{R}^n$ and scalars a, b.

Since $\mathbb{R}^n$ is a vector space, we have that $w \in \mathbb{R}^n$. Thus we have

$$d_{au+bvd} f(y) = d_w f(y) = df(y) \cdot w = df(y) \cdot (au + bv) = adf(y) \cdot u + bdf(y) \cdot v = ad_u f(y) + bd_v f(y)$$

where the crucial steps are justified by Theorem 10.1.1 and because scalars commute with matrices.

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