Exercises 7.5.5 — Problem 14

Problem.

a. For $c_m = \int_{-1}^{1}(1-x^2)^m dx$, obtain the identity $c_m = c_{m-1} - (1/2m)c_m$ by integration by parts.

b. Show that

$$c_m = 2\frac{2 * 4 * 6 * \cdots * (2m)}{3 * 5 * 7 * \cdots * (2m+1)} = \frac{2(2^m m!)^2}{(2m+1)!}$$

Proof.

a. We wish to compute $\int_{-1}^{1}(1-x^2)^m dx$. Let $u = (1-x^2)^m$ and $dv = dx$. This implies that $v = x$ and $du = m * (1-x^2)^{m-1} * (-2x) = -2mx(1-x^2)^{m-1}$. Then with integration by parts, we have

$$\begin{aligned}
c_m &= \int_{-1}^{1}(1-x^2)^m dx \\
&= (1-x^2)^m * x\Big|_{-1}^{1} - \int_{-1}^{1} -2mx^2(1-x^2)^{m-1}dx \\
&= 0 + 2m\int_{-1}^{1}(1-1+x^2)(1-x^2)^{m-1}dx \\
&= 2m\left[\int_{-1}^{1}(1-x^2)^{m-1}dx + \int_{-1}^{1}(-1+x^2)(1-x^2)^{m-1}dx\right] \\
&= 2m\left[c_{m-1} - \int_{-1}^{1}(1-x^2)^m dx\right] \\
c_m &= 2m\left[c_{m-1} - c_m\right] \\
(1/2m)c_m &= c_{m-1} - c_m \\
c_m &= c_{m-1} - (1/2m)c_m
\end{aligned}$$

So have shown $c_m = c_{m-1} - (1/2m)c_m$. Note that this can be rearranged to say that $c_m = \frac{2m * c_{m-1}}{2m+1}$, which will be useful in part a.

b. We now show $c_m = \frac{2(2^m m!)^2}{(2m+1)!}$ by induction. Consider the base case $m = 1$: $c_1 = \int_{-1}^{1}(1-x^2)^1 dx = \left(x - (1/3)x^3\right)\Big|_{-1}^{1} = (1-1/3) - (-1+1/3) = 4/3$. Also $\frac{2(2^1 * 1!)^2}{(2*1+1)!} = 8/3! = 8/6 = 4/3$ so the base case holds.

Now suppose for some natural number m we have $c_m = \frac{2(2^m m!)^2}{(2m+1)!}$. What is c_{m+1}? Recall the rearranged formula from part a, then $c_{m+1} = \frac{2(m+1)c_m}{2(m+1)+1}$. Then we use the induction hypothesis and perfrom some algebra:

$$
\begin{aligned}
c_{m+1} &= \frac{2(m+1)}{2(m+1)+1} c_m \\
&= \frac{2(m+1)}{2(m+1)+1} \frac{2(2^m m!)^2}{(2m+1)!} \\
&= \frac{2(m+1)}{2(m+1)} \frac{2(m+1)}{2(m+1)+1} \frac{2(2^m m!)^2}{(2m+1)!} \\
&= 2\frac{2^2(m+1)^2(2^m m!)^2}{(2(m+1)+1)(2m+2)(2m+1)!} \\
&= \frac{2(2 * 2^m * (m+1)m!)^2}{(2(m+1)+1)(2m+2)!} \\
&= \frac{2(2^{m+1}(m+1)!)^2}{(2(m+1)+1)(2(m+1))!} \\
c_{m+1} &= \frac{2(2^{m+1}(m+1)!)^2}{(2(m+1)+1)!}
\end{aligned}
$$

Thus the induction step holds and $c_m = \frac{2(2^m m!)^2}{(2m+1)!}$

$\square$