Exercises 6.1.5 — Problem 10

Problem. For a continuous, positive function $w(x)$ on $[a, b]$, define the weighted average operator A_w to be

$$A_w(f) = \int_a^b f(x)w(x)dx / \int_a^b w(x)dx$$

for continous functions f. Prove that A_w is linear and lies between the maximum and minimum values of f.

Proof. First we prove that A_w is linear. A_w is linear if $A_w(c_1 f + c_2 g) = c_1 A_w(f) + c_2 A_w(g)$ for $c_1, c_2 \in \mathbb{R}$ and continuous functions f, g on $[a, b]$. Towards proving linearity, let f, g be continuous functions on $[a, b]$ and $c_1, c_2 \in \mathbb{R}$. Then, using properties of integrals and functions,

$$\begin{aligned}
A_w(c_1 f + c_2 g) &= \frac{\int_a^b [c_1 f + c_2 g](x)w(x)}{\int_a^b w(x)dx} \\
&= \frac{\int_a^b c_1 f(x)w(x) + c_2 g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= \frac{\int_a^b c_1 f(x)w(x)dx + \int_a^b c_2 g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= \frac{c_1 \int_a^b f(x)w(x)dx + c_2 \int_a^b g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= \frac{c_1 \int_a^b f(x)w(x)dx}{\int_a^b w(x)dx} + \frac{c_2 \int_a^b g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= c_1 A_w(f) + c_2 A_w(g)
\end{aligned}$$

So A_w is linear. Now we must show that $A_w(f)$ lies in between the maximum and minimum values of f. Let $F^+ = \sup_{x \in [a,b]} f(x)$ and $F^- = \inf_{x \in [a,b]} f(x)$, so we must show that $A_w(f) \in [F^-, F^+]$ which is equivalent to $F^- \leq A_w(f) \leq F^+$. Since $\int_a^b w(x)dx$ is a postive number, let $W = \int_a^b w(x)dx$. If multiply the desired inequality by W, we must equivalently show that $WF^- \leq \int_a^b f(x)w(x)dx \leq WF^+$.

Now it is certainly true that $\int_a^b f(x)w(x)dx \leq \int_a^b F^+ w(x)dx$, but then F^+ is a constant so $\int_a^b F^+ w(x)dx = F^+ \int_a^b w(x)dx = F^+ W$ so $\int_a^b f(x)w(x)dx \leq WF^+$. Similarly, $\int_a^b f(x)w(x)dx \geq \int_a^b F^- w(x)dx = F^- \int_a^b w(x)dx = F^- W$. Then we have $WF^- \leq \int_a^b f(x)w(x)dx \leq WF^+$, which was the goal.

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