M 384: Assignment 2

Nathan Stouffer

Exercises 6.2.4 — Problem 6

Problem. Prove that if f is Riemann integrable on $[a, b]$ and $g(x) = f(x)$ for every x except for a finite number, then g is Riemann integrable.

Proof. We must show that g is Riemann integrable. The function g is Riemann integrable if there exists sequence of partitions P_j and a real number $\int_b^a g(x)dx$ for which $S(g, P_j) \to \int_b^a g(x)dx$ as $j \to \infty$ for every choice of Cauchy Sums $S(g, P_j)$. We already know that this condition holds for the function f. We will just show that the difference between any Cauchy Sum $S(f, P_j)$ and $S(g, P_j)$ becomes arbitrarily small as $j \to \infty$. Then the same conditions will hold for g.

Given a partition P_j, let $|P_j|$ denote the maximum interval length in P_j. Now, since the number of difference between f and g is finite, label the values $x_1, x_2, ..., x_n$ such that $g(x_i) \neq f(x_i)$. Then we must have $|S(f, P_j) - S(g, P_j)| \leq \sum_{i=1}^n |f(x_i) - g(x_i)||P_j|$ (the differences between the function times the maximum interval length). Now let $D = \max_i |f(x_i) - g(x_i)|$, then $|S(f, P_j) - S(g, P_j)| \leq \sum_{i=1}^n D|P_j| = nD|P_j|$. Since nD is finite, as $j \to \infty$, $|P_j| \to 0$ so the magnitude of the difference is 0 as $j \to \infty$. Then $\int_b^a f(x)dx = \int_b^a g(x)dx$ which implies that g is Riemann integrable.

$\square$

Exercises 6.2.4 — Problem 9

Problem. If f is a Riemann integrable function on $[a, b]$ prove that $F(x) = \int_a^x f(t)dt$ satisfies a Lipschitz condition.

Proof. The function $F(x)$ is Lipschitz if there exists a natural number M such that $|F(x) - F(x_0)| \leq M|x - x_0|$ for all $x, x_0 \in [a, b]$. Plugging in x, we must have $|\int_a^x f(t)dt - \int_a^{x_0}| \leq M|x - x_0|$ for all $x, x_0 \in [a, b]$. Choose $M = \sup_{y \in [a,b]} f(y)$.

Now we will show that $\int_a^x f(t)dt - \int_a^{x_0} f(t)dt = \int_{x_0}^x f(t)dt$. We consider three cases. Case $x = x_0$: then $\int_a^x f(t)dt - \int_a^{x_0} f(t)dt = 0 = \int_{x_0}^x f(t)dt$. Case $x < x_0$: then $\int_a^x f(t)dt - \int_a^{x_0} f(t)dt = \int_a^x f(t)dt - [\int_a^x f(t)dt + \int_x^{x_0} f(t)dt] = -\int_x^{x_0} f(t)dt = \int_{x_0}^x f(t)dt$. Case $x > x_0$: then $\int_a^x f(t)dt - \int_a^{x_0} f(t)dt = [\int_a^{x_0} f(t)dt + \int_{x_0}^x f(t)dt] - \int_a^{x_0} f(t)dt = \int_{x_0}^x f(t)dt$.

So now have $|\int_a^x f(t)dt - \int_a^{x_0} f(t)dt| = |\int_{x_0}^x f(t)dt|$. Then we immediately obtain $|\int_{x_0}^x f(t)dt| \leq M_0|x - x_0|$ where $M_0 = \sup f(x)$ on $[x_0, x]$ (or $[x, x_0]$ if $x < x_0$). Then $M_0|x - x_0| \leq \sup_{x \in [a,b]} f(x)|x - x_0| = M|x - x_0|$. So we have just shown that $|F(x) - F(x_0)| \leq M|x - x_0|$ for our chosen M. Therefore $F(x)$ is Lipshitz continuous.

$\square$

Exercises 6.2.4 — Problem 10

Problem. If f is Riemann integrable on $[a, b]$ and continuous at x_0, prove that $F(x) = \int_a^x f(t)dt$ is differentiable at x_0 and $F'(x_0) = f(x_0)$. Show that if f has a jump discontinuity at x_0, then F is not differentiable at x_0.

Proof. To prove that $F(x)$ is differentiable at x_0, we must show that $\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0}$ exists. Note that $\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{\int_a^x f(t)dt-\int_a^{x_0} f(t)dt}{x-x_0} = \lim_{x\to x_0} \frac{\int_{x_0}^x f(t)dt}{x-x_0}$ by the quick proof in the previous problem. For the limit to exist, we must have the existence of both the right and left limit and the equality $\lim_{x\to x_0^-} \frac{\int_{x_0}^x f(t)dt}{x-x_0} = \lim_{x\to x_0^+} \frac{\int_{x_0}^x f(t)dt}{x-x_0}$.

Let $\Delta x = |x - x_0|$, $m = \inf_{y\in[x_0-\Delta x,x_0]} f(y)$, and $M = \sup_{y\in[x_0,x_0+\Delta x]} f(y)$. Certainly we have $\lim_{x\to x_0^-} m(x - x_0) \le \lim_{x\to x_0^-} \int_{x_0}^x f(t)dt$ for $x > x_0$. Then, dividing by $x - x_0$ we get $\lim_{x\to x_0^-} m = m \le \lim_{x\to x_0^-} \frac{\int_{x_0}^x f(t)dt}{x-x_0}$. Then for $x_0 > x$, we must have $\lim_{x\to x_0^+} \int_x^{x_0} f(t)dt \le \lim_{x\to x_0^+} M(x_0 - x)$. Dividing by $x_0 - x$, we have $\lim_{x\to x_0^+} \frac{\int_x^{x_0} f(t)dt}{x_0-x} = \lim_{x\to x_0^+} \frac{\int_{x_0}^x f(t)dt}{x-x_0} \le \lim_{x\to x_0^+} M = M$.

So we have $m \le \lim_{x\to x_0} \frac{\int_{x_0}^x f(t)dt}{x-x_0} \le M$, but since f is continuous m, M can be made arbitrarily close with sufficiently small Δx so the limit exists because of the squeeze theorm. Further, the value to which m, M converge to is $\Delta x \to 0$ is $f(x_0)$, as desired.

Note that if f has a jump discontinuity at x_0, we cannot use the squeeze theorem. In fact, the left limit would be bounded above by $\sup_{y\in[x_0-\Delta x_0,x_0]} f(y)$ and the right limit would be bounded below by $\inf_{y\in[x_0,x_0+\Delta x_0]} f(y)$. Since f has a jump discontinuity at x_0 we know that $\sup_{y\in[x_0-\Delta x_0,x_0]} f(y) < \inf_{y\in[x_0,x_0+\Delta x_0]} f(y)$. But then the left and right limit cannot by equal, so the limiet does not exist.

$\square$