M 384: Assignment 7

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Exercises 7.6.3 — Problem 2

Problem. If $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$ for some fixed M and $\alpha > 0$ and all x, y in a compact interval, show that $\{f_n\}$ is uniformly equicontinuous.

Proof. A sequence is uniformly equicontinuous if for all $1/m$ there exists a $1/k$ such that $|x-y| < 1/k$ implies $|f_n(x) - f_n(y)| < 1/m$ for every f_n in the sequence. If we are given $1/m$, choose $1/k = (1/Mm)^{1/\alpha}$, then we have $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha \leq M1/k^\alpha = M((1/Mm)^{1/\alpha})^\alpha = M/Mm = 1/m$ for $|x-y| < 1/k$ as desired.

□

Exercises 7.6.3 — Problem 5

Problem. Give an example of a sequence that is uniformly equicontinuous but not uniformly bounded.

Proof. Consider the sequence $\{f_n(x)\}$ where we take $f_n(x) \equiv n$. Then $|f_n(x) - f_n(y)| = |n - n| = 0$ for all x, y yet there is no bound on the sequence.

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Exercises 7.6.3 — Problem 6

Problem. Prove that the family of all polynomials of degree $\leq N$ with coefficients in $[-1, 1]$ is uniformly bounded and uniformly equicontinuous on any compact interval.

Proof. Let $p(x)$ be a such a polynomial on a compact interval $[a, b]$. That is, $p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_ix^i$ for $x \in [a, b]$ a compact interval, $c_i \in [-1, 1]$, and $0 \leq i \leq N$.

First we will bound all such polynomials. Choose $m = \max(|a|, |b|)$ then any $c_jx^j \leq |c_jx^j| = |c_j||x^j| \leq 1m^j = m^j$ and take $M = \sum_{i=0}^{N} m^i$. Then, term for term, we have $p(x) \leq M$ for any of the polynomials in the family. Thus, the family is uniformly bounded.

Now we show that the family is uniformly equicontinuous. For any such $p(x)$, consider $p'(x) = c_1 + 2c^2x + 3c_3x^2 + \cdots + Nc_Nx^{N-1}$. Then $p'(x) \leq M' = \sum_{i=0}^{N-1} N * m^i$ since $N * m^i$ is larger term for term. Thus by the MVT we have the following for some $z \in (x, y)$: $|p(x) - p(y)| \leq |p'(z)||x - y| \leq M'|x - y|$. Then for a given $1/m$ we can choose $1/n = 1/(Mm)$ to satisfy equicontinuity.

□

Exercises 7.6.3 — Problem 9

Problem. Give an example of a uniformly bounded and uniformly equicontinuous sequence of functions on the whole line that does not have any uniformly convergent subsequences.

Proof. Consider the function

$$f_n(x) = \begin{cases} 0 & x \leq n-1 \\ x-(n-1) & n-1 < x \leq n \\ 1 & x > n \end{cases}$$

Any f_n is bounded by taking $M = 2$ so the sequence is uniformly bounded. Also the sequence is uniformly continuous by taking $1/n = 1/2m$ for a given $1/m$ (also the derivative is bounded above by 1). Now we show that the sequence has no uniformly converging subsequences. To have a converging subsequence, there must be a subsequence that satisfies the Cauchy criterion. We show that no pair of distinct elements even satisfies the Cauchy criterion, so there is no chance of a subsequence existing since every element of the sequence is distinct. Pick any $k > j$ and consider $|f_k(j) - f_j(j)| = |0 - 1| = 1$. Thus there is no uniformly converging subsequence of f_n.

$\square$