Exercises 10.1.5 — Problem 15

Problem. If $f : D \to \mathbb{R}$ is C^1 with $D \subset \mathbb{R}^n$ and D contains the line segment joining x and y, show that $f(y) = f(x) + \nabla f(z) \cdot (y - x)$ for some point z on the line segment. Explain why this is an n-dimensional analog of the mean value theorem.

Proof. For fixed vectors x and y, let $g : [0,1] \to \mathbb{R}^n$ be defined by taking $t \mapsto x + t(y - x)$ and define $h(t) = (f \circ g)(t) = f(g(t))$ a map from $[0,1] \to \mathbb{R}$. Since x, y are connected by a line segment contained in D we have that the image of $[0,1]$ under g is a subset of D, which is where f is differentiable. Then since f is differentiable, we can apply the formula from two problems ago and say that

$$\frac{d}{dt} f(g(t)) = \nabla f \cdot \begin{bmatrix} \frac{d}{dt}(x_1 + t(y_1 - x_1)) \\ \vdots \\ \frac{d}{dt}(x_n + t(y_n - x_n)) \end{bmatrix} = \nabla f \cdot \begin{bmatrix} y_1 - x_1 \\ \vdots \\ y_n - x_n \end{bmatrix} = \nabla f \cdot (y - x)$$

Then since $h : [0,1] \to \mathbb{R}$ is C^1 we can apply the MVT and say that there exists some $\lambda \in (0,1)$ such that $h(\lambda) = h(1) - h(0) = f(g(1)) - f(g(0)) = f(y) - f(x)$. Taking $z = g(\lambda)$, we have that

$$f(y) - f(x) = \nabla f(z) \cdot (y - x) \iff f(y) = f(x) + \nabla f(z) \cdot (y - x)$$

This is an n-dimensional analogue of the MVT since it provides with only the existence of some point "in-between" two selected points for which the MVT condition holds.

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