Exercises 7.3.4 — Problem 7

Problem. If $|f_n(x)| \leq a_n$ for all x and $\sum_{n=1}^{\infty} a_n$ convreges, prove that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Proof. First define $F_k(x) = \sum_{n=1}^{k} f_n(x)$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly if and only if the sequence $F_1(x), F_2(x), ...$ has a uniform limit $F(x)$. By Theorem 7.3.1, the sequence (F_k) has a uniform limit if and only if (F_k) satisfies the Cauchy criterion: for every $1/m$ there exists some $N \in \mathbb{N}$ such that for all $k, j \geq N$ we have $|f_k(x) - f_j(x)| \leq 1/m$ for all x in the common domain.

Let's now show this. We define $A_k = \sum_{n=1}^{k} a_n$. Since the series $\sum_{n=1}^{\infty} a_n$ converes, we know the sequence $A_1, A_2, ...$ satisfies the Cauchy criterion for sequences of real numbers. Then for any $1/m$ choose N such that for all $k, j \geq N$ we have $|A_k - A_j| \leq 1/m$. Without loss of generality, select $k \geq j$. Then we have

$$\begin{aligned}
1/m \geq |A_k - A_j| &= |(a_1 + a_2 + \cdots + a_j + \cdots a_k) - (a_1 + a_2 + \cdots + a_k)| \\
&= |a_{j+1} + \cdots + a_k| \\
&\geq |\,|f_{j+1}(x)| + \cdots + |f_k(x)|\,| \\
&\geq |f_{j+1}(x) + \cdots + f_k(x)| \\
&= |(f_1(x) + \cdots + f_j(x) + \cdots + f_k(x)) - (f_1(x) + \cdots + f_j(x)| \\
&= |F_k(x) - F_j(x)|
\end{aligned}$$

This is exactly the Cauchy criterion for sequences of functions. So we have shown that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

□