M 384: Assignment 6

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Exercises 7.5.5 — Problem 7

Problem. If f is C^1 on $[a,b]$ prove that there exists a cubic polynomial P such that $f - P$ and its first derivative vanish at the endpoints of the interval.

Proof. Note that we are able to use the fact that there exists a polynomial of degree $2n-1$ that satisfies $f(x_k) = a_k$ and $f'(x_k) = b_k$ for $k = 1, \ldots, n$. Let $n = 2$ and take $x_1 = a$ and $x_2 = b$. Let $a_1 = f(a)$, $b_1 = f'(a)$ and $a_2 = f(b), b_2 = f'(b)$. We apply our fact and say that there is some polynomial P of degree $2n - 1 = 3$ that satisfies $P(a) = a_1 = f(a)$, $P'(a) = b_1 = f'(a)$, $P(b) = a_2 = f(b)$, and $P'(b) = b_2 = f'(b)$.

Then we have the following four equations:

$$\begin{aligned}
(f-P)(a) &= f(a) - P(a) = f(a) - f(a) = 0 \\
(f-P)'(a) &= f'(a) - P'(a) = f'(a) - f'(a) = 0 \\
(f-P)(b) &= f(b) - P(b) = f(b) - f(b) = 0 \\
(f-P)'(b) &= f'(b) - P'(b) = f'(b) - f'(b) = 0
\end{aligned}$$

Thus we have the desired equalities for a cubic polynomial P.

□

Exercises 7.5.5 — Problem 9

Problem. If $f(c) = 0$ for some point c in (a, b), prove that the polynomials approximating f on $[a, b]$ may be taken to vanish at c.

Proof. By the WAT, we gain a sequence of polynomials f_n that converges to f uniformly. Since uniform convergence implies pointwise convergence, we know $f_n(c) \to 0$ as $n \to \infty$, so the polynomials f_n vanish at c.

$\square$

Exercises 7.5.5 — Problem 14

Problem.

a. For $c_m = \int_{-1}^{1}(1-x^2)^m dx$, obtain the identity $c_m = c_{m-1} - (1/2m)c_m$ by integration by parts.

b. Show that

$$c_m = 2\frac{2*4*6*\cdots*(2m)}{3*5*7*\cdots*(2m+1)} = \frac{2(2^m m!)^2}{(2m+1)!}$$

Proof.

a. We wish to compute $\int_{-1}^{1}(1-x^2)^m dx$. Let $u = (1-x^2)^m$ and $dv = dx$. This implies that $v = x$ and $du = m*(1-x^2)^{m-1}*(-2x) = -2mx(1-x^2)^{m-1}$. Then with integration by parts, we have

$$\begin{aligned}
c_m &= \int_{-1}^{1}(1-x^2)^m dx \\
&= (1-x^2)^m * x\Big|_{-1}^{1} - \int_{-1}^{1} -2mx^2(1-x^2)^{m-1}dx \\
&= 0 + 2m\int_{-1}^{1}(1-1+x^2)(1-x^2)^{m-1}dx \\
&= 2m\left[\int_{-1}^{1}(1-x^2)^{m-1}dx + \int_{-1}^{1}(-1+x^2)(1-x^2)^{m-1}dx\right] \\
&= 2m\left[c_{m-1} - \int_{-1}^{1}(1-x^2)^m dx\right] \\
c_m &= 2m\left[c_{m-1} - c_m\right] \\
(1/2m)c_m &= c_{m-1} - c_m \\
c_m &= c_{m-1} - (1/2m)c_m
\end{aligned}$$

So have shown $c_m = c_{m-1} - (1/2m)c_m$. Note that this can be rearranged to say that $c_m = \frac{2m*c_{m-1}}{2m+1}$, which will be useful in part a.

b. We now show $c_m = \frac{2(2^m m!)^2}{(2m+1)!}$ by induction. Consider the base case $m = 1$: $c_1 = \int_{-1}^{1}(1-x^2)^1 dx = \left(x - (1/3)x^3\right)\Big|_{-1}^{1} = (1-1/3) - (-1+1/3) = 4/3$. Also $\frac{2(2^1*1!)^2}{(2*1+1)!} = 8/3! = 8/6 = 4/3$ so the base case holds.

Now suppose for some natural number m we have $c_m = \frac{2(2^m m!)^2}{(2m+1)!}$. What is c_{m+1}? Recall the rearranged formula from part a, then $c_{m+1} = \frac{2(m+1)c_m}{2(m+1)+1}$. Then we use the induction hypothesis and perfrom some algebra:

$$
\begin{aligned}
c_{m+1} &= \frac{2(m+1)}{2(m+1)+1} c_m \\
&= \frac{2(m+1)}{2(m+1)+1} \frac{2(2^m m!)^2}{(2m+1)!} \\
&= \frac{2(m+1)}{2(m+1)} \frac{2(m+1)}{2(m+1)+1} \frac{2(2^m m!)^2}{(2m+1)!} \\
&= 2 \frac{2^2(m+1)^2(2^m m!)^2}{(2(m+1)+1)(2m+2)(2m+1)!} \\
&= \frac{2(2 * 2^m * (m+1) m!)^2}{(2(m+1)+1)(2m+2)!} \\
&= \frac{2(2^{m+1}(m+1)!)^2}{(2(m+1)+1)(2(m+1))!} \\
c_{m+1} &= \frac{2(2^{m+1}(m+1)!)^2}{(2(m+1)+1)!}
\end{aligned}
$$

Thus the induction step holds and $c_m = \frac{2(2^m m!)^2}{(2m+1)!}$

$\square$