Exercises 7.3.4 — Problem 2

Problem. Suppose $f_n \to f$ and the function f_n all satisfy the Lipschitz condition $|f_n(x) - f_n(y)| \leq M|x-y|$ for some constant M independant of n. Prove that f also satisfies the same Lipschitz condition.

Proof. For fixed x, y in the common domain, we want to show that $|f(x) - f(y)| \leq M|x - y|$. Let's begin with $|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$. Then by the triangle inequality applied twice, we get the following for any $n \in \mathbb{N}$

$$|f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

By supposition, $|f_n(x) - f_n(y)| \leq M|x - y|$ so we have

$$|f(x) - f(y)| \leq M|x - y| + |f(x) - f_n(x)| + |f(y) - f_n(y)|$$

for any n. But this is just a comparison of real numbers so we can just pass to the limit $n \to \infty$ and the non-strict inequality will hold. But the limit of the $|f - f_n| = 0$ so we have $|f(x) - f(y)| \leq M|x - y|$ as desired.

$\square$