Exercises 7.4.5 — Problem 6

Problem. Prove that if $f(x)$ is analytic on (a, b), then $F(x) = \int_c^x f(t)dt$ is also analytic on (a, b), where c is any point in (a, b).

Proof. Fix $c \in (a, b)$. Then since f is analytic on (a, b), we have $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ where $a_n = \frac{f^{(n)}(c)}{n!}$. Then since $F(x) = \int_c^x f(t)dt$,

$$F(x) = \int_c^x \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(t-c)^n dt = \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{(n+1)!}(t-c)^{n+1} \right) \Bigg|_c^x$$

Then evaluating,

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{(n+1)!}(x-c)^{n+1} - \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{(n+1)!}(c-c)^{n+1}$$

But the second term is 0 so we are left with $F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{(n+1)!}(x-c)^{n+1}$. Since c was arbitrary in (a, b), $F(x)$ has a power series expansion about any point in (a, b), thus $F(x)$ is analytic.

□