Exercises 6.1.5 — Problem 8

Problem. Let f be a C^1 function on the line, and let $g(x) = \int_0^1 f(xy)y^2 dy$. Prove that g is a C^1 function and establish a formula for $g'(x)$ in terms of f.

Proof. First define $h(x, y) = f(xy)y^2$ then $g(x) = \int_0^1 h(x, y)dy$. Since $f(xy)$ and y^2 are continuous functions, Theorem 6.1.8 tells us that $\int_0^1 h(x, y)dy$ is a continuous function. Further, the fact that each of $f(xy), y^2$ is C^1 means that $g(x)$ is C^1 since C^1 is closed under multpilication. Then the conditions for Theorem 6.1.7 are met (since the constant functions 0 and 1 are certainly C^1) so we can give the derivative formula:

$$g'(x) = \int_0^1 \frac{\partial h}{\partial x}(x, y)dy = \int_0^1 f'(xy)y^3 dy$$

□