Exercises 10.1.5 — Problem 10

Problem. Let $g : [a, b] \to \mathbb{R}^n$ be differentiable. If $f : \mathbb{R}^n \to \mathbb{R}^1$ is differentiable, what is the derivative $(d/dt)f(g(t))$.

Proof. Here we have $g(t) = (g_1(t), \ldots, g_n(t))$ for $t \in [a, b]$ and $f(z) = f(z_1, \ldots, z_n)$ where $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$. Then $f \circ g = f(g(t)) = f(g_1(t), \ldots, g_n(t)) \in \mathbb{R}$. Consider $f \circ g$ and let $z_k = g_k(t)$. The textbook provides this formula for partial derivatives:

$$\frac{\partial f}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}\left(\frac{\partial z_k}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial f}{\partial z_1} & \cdots & \frac{\partial f}{\partial z_n} \end{bmatrix} \begin{bmatrix} \frac{\partial z_1}{\partial x_j} \\ \vdots \\ \frac{\partial z_n}{\partial x_j} \end{bmatrix} = \nabla f \cdot \begin{bmatrix} \frac{dg_1}{dt} \\ \vdots \\ \frac{dz_n}{dt} \end{bmatrix} = \nabla f \cdot \frac{dg}{dt}$$

where $\frac{dg}{dt}$ is a vector valued function. In this case, the domain is $[0, 1]$ so there is no need for x_j, which is why it is replaced with t.

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