Exercises 6.3.2 — Problem 1

Problem. For which values of a and b does the improper integral $\int_0^{1/2} x^a |\log x|^b dx$ exist?

Proof. We consider three cases for the value of a: $a < -1, a = -1, a > -1$. Beginning with $a > -1$ then we know there exists some $\epsilon > 0$ such that $a - \epsilon > -1$. Then $\int_0^{1/2} x^a |\log x|^b dx = \int_0^{1/2} x^{a-\epsilon} x^{\epsilon} |\log x|^b dx$. Then $\lim_{x \to 0^+} x^{\epsilon} |\log x|^b = 0$ so we need only worry about the convergence of $\int_0^{1/2} x^{a-\epsilon}$. But $a - \epsilon > -1$ so we know this integral to converge.

Now consider $a < -1$ then there exists some $\epsilon > 0$ sucht that $a + \epsilon < -1$. Then $\int_0^{1/2} x^a |\log x|^b dx = \int_0^{1/2} x^{a+\epsilon} x^{-\epsilon} |\log x|^b dx$. Then $\lim_{x \to 0^+} x^{-\epsilon} |\log x|^b = \infty$ so we are toast in this case.

When $a = -1$, we have $\int_0^{1/2} x^a |\log x|^b dx = \int_0^{1/2} \frac{|\log x|^b}{x} dx$. Taking $\log$ to be the natural logarithm, we have $\int_0^{1/2} \frac{|\ln x|^b}{x} dx = \frac{|\ln x|^{b+1}}{b+1} \Big|_0^{1/2}$ (even when $b = 0$) which diverges to $-\infty$ when 0 is plugged in.

$\square$