M 472 – Homework 7

Series

Due Friday, April 23, on Gradescope

1. In each case, write the principal part of the function at its isolated singular point (i.e., the part of the Laurent series with negative exponents), and determine whether that point is a removable singularity, a pole, or an essential singularity.

   (a) $z^2 \sin \dfrac{1}{z}$

   (b) $\dfrac{z^2}{1+z}$

   (c) $\dfrac{1 - \cos z}{z^2}$

   (d) $\dfrac{1 - \cos z}{z^3}$

2. Evaluate the following residues.

   (a) $\operatorname*{Res}_{z=2i} \left[\dfrac{1}{z^2+4} \right]$

   (b) $\operatorname*{Res}_{z=2i} \left[\dfrac{e^{ix}}{(z^2+1)(z^2+4)} \right]$

   (c) $\operatorname*{Res}_{z=0} \left[z^2 \sin \dfrac{1}{z} \right]$

   (d) $\operatorname*{Res}_{z=0} \left[\dfrac{\sin z}{1 - \cos z} \right]$

3. Use residues to find the following integrals:

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$

   (c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\, dx$

   (d) $\displaystyle\int_{0}^{2\pi} \frac{\cos \theta}{5 + 4\cos\theta}\, d\theta$

4. Consider the function $f(z) = \dfrac{\csc(\pi z)}{z^2} = \dfrac{1}{z^2 \sin(\pi z)}$. Following the example in class used to evaluate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, we define D_N to be the square region in the complex plane with vertices $\pm(N + 1/2) \pm i(N + 1/2)$. Again, just as in class, the ML-estimate, combined with explicit formulas for $f(z)$ on the boundary ∂D_N, can be used to show that $\displaystyle\lim_{N\to\infty} \int_{\partial D_N} f(z)\, dz = 0$. (You can take this limit for granted, no need to show this.) Applying the Residue Theorem, as we did in the similar example in class, you will get the explicit value of some infinite series. Which infinite series is this, and what is its sum?