M 472 – Homework 2 – Complex functions and differentiation

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Due Monday, February 1, on Gradescope

1. Find a domain in the z-plane whose image under the transformation $w = z^2$ is the square domain in the w-plane bounded by the lines $u = 1$, $u = 2$, $v = 1$, and $v = 2$. (See Section 14 in the textbook.)

2. Show that the function $f(z) = \left(\frac{z}{\bar{z}}\right)^2$ has the value 1 at all nonzero points on the real and imaginary axes, but that it has the value -1 at all nonzero points on the line $x = y$. Conclude that the limit of $f(z)$ as z tends to 0 does not exist.

   Consider $z = x \neq 0$. Then we have $f(z) = \left(\frac{z}{\bar{z}}\right)^2 = \left(\frac{x}{x}\right)^2 = 1^2 = 1$. Along the imaginary axis, we have $z = iy \neq 0 \implies f(z) = \left(\frac{iy}{-iy}\right)^2 = \left(\frac{-y}{y}\right)^2 = (-1)^2 = 1$. Then we consider $z = x + ix \neq 0$ which gives $f(z) = \left(\frac{x+ix}{x-ix}\right)^2 = \left(\frac{x+ix}{x-ix} * \frac{x+ix}{x+ix}\right)^2 = \left(\frac{x^2-x^2+2ix^2}{x^2+x^2}\right)^2 = i^2 = -1$. But then the limit of $f(z)$ cannot exist as z tends to 0 since there exist approaches to 0 that tend to different values.

3. Let f be the function defined by

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

   Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz-plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point on the line $\Delta x = \Delta y$. Conclude that $f'(0)$ does not exist. Note that to obtain this result it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz-plane.

   For $z = 0$, we then have $\Delta w = f(0 + \Delta z) - f(0) = f(\Delta z)$. We consider three cases for Δz. First, we have $\Delta z = \Delta x \neq 0$. Here $\Delta w/\Delta z = f(\Delta x)/\Delta x = \frac{\overline{\Delta x}^2}{\Delta x^2} = \frac{\Delta x^2}{\Delta x^2} = 1$. Then consider $\Delta z = i\Delta y \neq 0$. Then $\Delta w/\Delta z = \frac{\overline{i\Delta y}^2/i\Delta y}{i\Delta y} = \frac{(-i\Delta y)^2}{(i\Delta y)^2} = \frac{-\Delta y^2}{-\Delta y^2} = 1$. In the final case, we have $\Delta x + i\Delta x \neq 0$. Then $\Delta w/\Delta z = \frac{(\overline{\Delta x + i\Delta x})^2/(\Delta x + i\Delta x)}{\Delta x + i\Delta x} = \frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)^2} = \frac{\Delta x^2 - \Delta x^2 - 2i\Delta x^2}{\Delta x^2 - \Delta x^2 + 2i\Delta x^2} = -1$. Since we have two approaches that tend to different limits, we know that $f'(0)$ does not exist.

4. Show from the definition of complex derivatives that the functions $f(z) = \operatorname{Re} z$ and $g(z) = \operatorname{Im} z$ are not complex differentiable at any point in the plane.

   For the complex derivative to exist at a point, we must have the limit of the difference quotient converge as $\Delta z \to 0$. We will show that this cannot hold for either of f, g at any point $z \in \mathbb{C}$. Let's begin with $f(z) = \operatorname{Re} z$. Take $\Delta z = \Delta x$, then $\lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} = \lim_{\Delta x \to 0} \Delta x/\Delta x = 1$. But then for $\Delta z = i\Delta y$ we have $\lim_{\Delta y \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y} = \lim_{\Delta y \to 0} 0/i\Delta y = 0$. So the limit of f cannot exist at any point $z \in C$.

Now consider $g(z) = \operatorname{Im} z$. Again take $\Delta z = \Delta x$ then $\lim_{\Delta x \to 0} \frac{g(z+\Delta x)-g(z)}{\Delta x} = \lim_{\Delta x \to 0} 0/\Delta x = 0$. And then for $\Delta z = i\Delta y$ we have $\lim_{\Delta y \to 0} \frac{g(z+i\Delta y)-g(z)}{i\Delta y} = \lim_{\Delta y \to 0} \frac{i\Delta y}{i\Delta y} = 1$. So, again, the limit g does not exist for any complex number z.

5. Using the definitions, rules for derivatives, or the Cauchy-Riemann equations, determine where the following functions are complex differentiable and find $f'(z)$ where it exists.

   (a) $f(z) = 1/(z^2+1)$

   (b) $f(z) = x^2 + iy^2$

   (c) $f(z) = \sin x \cosh y - i \cos x \sinh y$

   (d) $f(z) = \sin x \cosh y + i \cos x \sinh y$

   We consider the following functions.

   (a) For $f(z) = 1/(z^2+1)$, the function f is the quotient of two complex differentiable functions so it is complex differentiable for $z^2+1 \neq 0$ $(z \neq i)$. The derivative is $f'(z) = -2z/(z^2+1)^2$.

   (b) Take $f(z) = x^2 + iy^2$, then $u = x^2$ and $v = y^2$. And we must have $u_x = 2x = v_y = 2y$ and $v_x = 0 = -u_y = 0$. This holds for $y = x$. Further, the partials are continuous so the derivative exists for z such that $y = x$. The derivative is $f'(z) = 2x$.

   (c) $f(z) = \sin x \cosh y - i \cos x \sinh y$ gives $u = \sin x \cosh y$ and $v = -\cos x \sinh y$. Then we have $u_x = \cos x \cosh y, u_y = \sin x \sinh y, v_x = \sin x \sinh y, v_y = -\cos x \cosh y$. We must have $\cos x \cosh y = -\cos x \cosh y \iff \cos x \cosh y = 0$ and $\sin x \sinh y = -\sin x \sinh y \iff \sin x \sinh y = 0$. But $\cos x$ and $\sin x$ are not 0 for the same x so we must have $\cosh y = \sinh y = 0$. But this is also never the case for any y so $f(z)$ is not complex differntiable anywhere.

   (d) For $f(z) = \sin x \cosh y + i \cos x \sinh y$ we have almost the same equations as above. u_x, u_y are the same but we have $v_x = -\sin x \sinh y$ and $v_y = -\cos x \cosh y$. These satisfy the Cauchy Riemann equations for all x, y and are continuous partials, therefore $f'(z)$ exists for any complex number. It's formula is $f'(z) = \cos x \cosh y - i \sin x \sinh y$.