M 472 – Homework 7

Series

Due Friday, April 23, on Gradescope

1. In each case, write the principal part of the function at its isolated singular point (i.e., the part of the Laurent series with negative exponents), and determine whether that point is a removable singularity, a pole, or an essential singularity.

   (a) $z^2 \sin\frac{1}{z} = z^2 * \left(1/z - \frac{(1/z)^3}{3!} + \frac{(1/z)^5}{5!} - \cdots\right)$
   $= z - \frac{1}{z * 3!} + \frac{1}{z^3 * 5!} - \cdots$
   So we have an essential singularity at $z = 0$.

   (b) $\frac{z^2}{1+z} = \frac{z^2}{z * (1 + 1/z)} = \frac{z}{1 - (-1/z)}$
   $= z * \left(1 - 1/z + 1/z^2 - 1/z^3 + \cdots\right)$
   $= z - 1 + 1/z - 1/z^2 + \cdots$.
   So we have another essential singularity at $z = 0$.

   (c) $\frac{1 - \cos z}{z^2} = \frac{1 - (1 - z^2/2 + z^4/4! - \cdots)}{z^2}$
   $= \frac{z^2/2 - z^4/4! + z^6/6! - \cdots}{z^2}$
   $= 1/2 - z^2/4! + z^4/6! - \cdots$
   So we have a removable singularity here.

   (d) $\frac{1 - \cos z}{z^3} = \frac{1 - (1 - z^2/2 + z^4/4! - \cdots)}{z^3}$
   $= \frac{z^2/2 - z^4/4! + z^6/6! - \cdots}{z^3}$
   $= 1/2z - z/4! + z^3/6! - \cdots$
   So we have pole at $z = 0$.

2. Evaluate the following residues.

   (a) $\operatorname*{Res}_{z=2i}\left[\frac{1}{z^2+4}\right]$

   (b) $\operatorname*{Res}_{z=2i}\left[\frac{e^{iz}}{(z^2+1)(z^2+4)}\right]$

   (c) $\operatorname*{Res}_{z=0}\left[z^2 \sin\frac{1}{z}\right]$

   (d) $\operatorname*{Res}_{z=0}\left[\frac{\sin z}{1-\cos z}\right]$

   Here are my answers.

   (a) $\operatorname*{Res}_{z=2i}\left[\frac{1}{z^2+4}\right] = \operatorname*{Res}_{z=2i}\frac{1}{(z+2i)(z-2i)} = \lim_{z\to 2i}\frac{1}{z+2i} = -i/4.$

   (b) $\operatorname*{Res}_{z=2i}\left[\frac{e^{iz}}{(z^2+1)(z^2+4)}\right] = \operatorname*{Res}_{z=2i}\frac{e^{iz}}{(z^2+1)(z+2i)(z-2i)} = \lim_{z\to 2i}\frac{e^{iz}}{(z^2+1)(z+2i)} = i/12e^2$

   (c) $\operatorname*{Res}_{z=0}\left[z^2 \sin\frac{1}{z}\right] = -1/3! = -1/6$ since from the Laurent series in the previous problem.

   (d) $\operatorname*{Res}_{z=0}\left[\frac{\sin z}{1-\cos z}\right] = \operatorname*{Res}_{z=0}\frac{z - z^3/3! + z^5/5! - \cdots}{1 - (1 - z^2/2! + z^4/4! - \cdots)} = \operatorname*{Res}_{z=0}\frac{z - z^3/3! + z^5/5! - \cdots}{-z^2/2! + z^4/4! - \cdots)}$
   $= \lim_{z\to 0}\frac{1 - z^2/3! + z^4/5! - \cdots}{1/2! - z^2/4! + z^4/6! - \cdots} = 2$

3. Use residues to find the following integrals:

   (a) $\int_{-\infty}^{\infty}\frac{dx}{x^2+4}$

   (b) $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)}$

   (c) $\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+1)(x^2+4)}\,dx$

   (d) $\int_{0}^{2\pi}\frac{\cos\theta}{5+4\cos\theta}\,d\theta$

(a) For this one consider the complex integral over the boundary of D_R, the disk of radius R only in the upper half-plane. $\int_{\partial D_R} \frac{1}{z^2+4}\,dz$. By the residue theorem, this integral equals $2\pi i * \text{Res}_{z=2i} \frac{1}{z^2+4} = 2\pi i \frac{-i}{4} = \pi/2$. Now we show that integral over the arc is 0 (meaning that $\int_{\infty}^{\infty} \frac{1}{x^2+4}\,dx = \pi/2$). For the remainder of problem 3, let C_1 be the arc of D_R parameterized by $z(t) = Re^{it}$ for $0 \le t \le \pi$. Then, by the ML-estimate $\left|\int_{C_1} \frac{1}{z^2+4}\,dz\right| \le \int_{C_1} \left|\frac{1}{z^2+4}\right|\,dz \le \int_{C_1} \left|\frac{1}{z^2}\right|\,dz = \int_{C_1} \frac{1}{R^2}\,dz = piR/R^2 = \pi/R$ which goes to 0 as $R \to \infty$.

(b) Let $f(z) = \dfrac{1}{(z^2+1)(z^2+4)}$. Then $\displaystyle\int_{\partial D_R} \frac{1}{(z^2+1)(z^2+4)}\,dz = 2\pi i \left[\operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=i} f(z)\right] = 2\pi i\,[1/(-3*4i) + 1/(2i*3)] = \pi/6$. Then we have the ML estimate $\left|\int_{C_1} f(z)\,dz\right| \le \int_{C_1} |f(z)|\,dz \le \int_{C_1} 1/z^4\,dz = \pi R/R^4 = \pi/R^3$ which goes to 0 as R goes to ∞.

(c) Let $g(z) = \dfrac{e^{iz}}{(z^2+1)(z^2+4)}$. Then $\displaystyle\int_{\partial D_R} \frac{e^{iz}}{(z^2+1)(z^2+4)}\,dz = 2\pi i \left[\operatorname*{Res}_{z=2i} g(z) + \operatorname*{Res}_{z=i} g(z)\right] = \frac{\pi}{6e^2}(2e-1)$. This is entirely real so there is no need to take the real part of this to get our actual answer. Also, we need the ML estimate: $\displaystyle\left|\int_{C_1} g(z)\,dz\right| \le \int_{C_1} |g(z)|\,dz \le \int_{C_1} 1/z^4\,dz = \pi R/R^4 = \pi/R^3$ which goes to 0 and $R \to \infty$.

(d) I struggled on this one and I am already a day late so I didn't type it up. Sorry!

4. Consider the function $f(z) = \dfrac{\csc(\pi z)}{z^2} = \dfrac{1}{z^2 \sin(\pi z)}$. Following the example in class used to evaluate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, we define D_N to be the square region in the complex plane with vertices $\pm(N+1/2) \pm i(N+1/2)$. Again, just as in class, the ML-estimate, combined with explicit formulas for $f(z)$ on the boundary ∂D_N, can be used to show that $\displaystyle\lim_{N\to\infty} \int_{\partial D_N} f(z)\,dz = 0$. (You can take this limit for granted, no need to show this.) Applying the Residue Theorem, as we did in the similar example in class, you will get the explicit value of some infinite series. Which infinite series is this, and what is its sum?

First note that $f(z)$ has singularities at $z = 0, \pm 1, \pm 2, \cdots, \pm N$ in the domain D_N. From mathematica, we know that $\text{Res}_{z=0} f(z) = \pi/6$. For some $k \in \mathbb{Z} \setminus \{0\}$ we have $\displaystyle\operatorname*{Res}_{z=k} \frac{\csc(\pi z)}{z^2} = \operatorname*{Res}_{z=k} \frac{1/z^2}{\sin(\pi z)} = \frac{1/k^2}{\pi \cos(\pi k)} = \frac{(-1)^k}{\pi k^2}$. By the residue theorem, we have $\displaystyle\frac{1}{2\pi i}\int_{\partial D_N} f(z)\,dz = \pi/6 + \sum_{k=1}^{N}\left(\frac{(-1)^k}{\pi k^2} + \frac{(-1)^{-k}}{\pi k^2}\right) = \pi/6 + \sum_{k=1}^{N} \frac{2(-1)^k}{\pi k^2}$. Then taking $N \to \infty$ gives $\displaystyle 0 = \pi/6 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{\pi k^2}$ which means that $\displaystyle\pi^2/12 = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - 1/4 + 1/9 - 1/16 + \cdots$. This is the alternating series version of the Basel problem.