M 472 – Homework 6

Series

Due Monday, March 29, on Gradescope

1. (a) Find the Maclaurin series (Taylor series centered at $z = 0$) for the function $f(z) = e^{-z^2}$.

   We know the Maclaurin series for $e^w = 1 + w + w^2/2! + w^3/3! + \cdots$. Here we have $w = -z^2$ so the Maclaurin series turns into $1 - z^2 + z^4/2! - z^6/3! + z^8/4! + \cdots$.

   (b) Find the Maclaurin series for the function $F(z) = \int_0^z e^{-\zeta^2}\, d\zeta$.
   (Since the integrand is an entire function, the integral is independent of path, i.e., $F(z)$ can be evaluated by integrating along any piecewise smooth path from 0 to z.)

   Since the integral is independent of the path, we can just compute the antiderivative and evaluate $F(z) - F(0)$. The antiderivate can be computed term by term (as long as we have absolute convergence):

$$\begin{aligned} F(z) &= \int_0^z e^{-\zeta^2} d\zeta \\ &= \int_0^z \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^{2n}}{n!} d\zeta \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^z \zeta^{2n} d\zeta \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} \zeta^{2n+1} \Big|_0^z \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} z^{2n+1} \end{aligned}$$

2. Find the Laurent series of $f(z) = \dfrac{z}{(z-1)(z+2)}$ in the annulus $\{1 < |z| < 2\}$. (Hint: Your first step should be a partial fraction decomposition.)

   We have $f(z) = z\dfrac{1}{(z-1)(z+2)} = z\left(\dfrac{1/3}{z-1} + \dfrac{-1/3}{z+2}\right) = \dfrac{z}{3}\left(\dfrac{1}{z-1} + \dfrac{1}{z+2}\right)$ by partial fraction decomposition. Then $\dfrac{1}{z-1} = \dfrac{1}{z}\dfrac{1}{1-1/z}$ and recalling that $1 < |z|$ allows us to expand into a power series:

$$\frac{1}{z}\left(1 + 1/z + 1/z^2 + 1/z^3 + \cdots\right) = 1/z \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \sum_{n=-\infty}^{-1} z^n$$

On the other hand, $\dfrac{1}{z+2} = \dfrac{1}{2}\dfrac{1}{1-(-z/2)}$ and since $|z| < 2$ we expand to get

$$\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-z}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^{n+1}}$$

This gives us the final Laurent series as

$$f(z) = z\left(\sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^{n+1}}\right) = \sum_{n=-\infty}^{-1} z^{n+1} + \sum_{n=0}^{\infty}\frac{(-1)^n z^{n+1}}{2^{n+1}}$$

3. (a) Find the Laurent series of $f(z) = \dfrac{\sin z}{z}$, $g(z) = \dfrac{\cos z}{z}$, and $h(z) = \dfrac{\cos z}{z^2}$ in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

$$f(z) = \frac{\sin z}{z} = \frac{z - z^3/3! + z^5/5! + \cdots}{z} = 1 - z^2/3! + z^4/5! - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}$$

$$g(z) = \frac{\cos z}{z} = \frac{1 - z^2/2! + z^4/4! - \cdots}{z} = 1/z - z/2! + z^3/4! - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n-1}$$

$$h(z) = \frac{\cos z}{z^z} = \frac{1 - z^2/2! + z^4/4! - \cdots}{z^2} = 1/z^2 - 1/2! + z^2/4! - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n-2}$$

(b) Which of the functions in (a) are entire?

The only function from part (a) that is entire is $f(z)$. Both $g(z)$ and $h(z)$ have singularities at 0 so they are not entire. However, the power series expansion of $f(z)$ is entire, so $f(z)$ can be adapted to be entire.

(c) For each of the functions in (a), find the Laurent series of their antiderivative in $\mathbb{C}^*$ or explain why no such antiderivative exists.

The antiderivative of $f(z)$ is $F(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+1)!}z^{2n+1}$ and the antiderivate of $h(z)$ is $H(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-1)(2n)!}z^{2n-1}$. The function $g(z)$ has no antiderivate since the integral of $1/z$ has a branch cut.

4. Assume that $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} a_n z^n$ is a Laurent series which converges in some annulus containing the circle $\{|z| = r\}$. Find $\displaystyle\int_{|z|=r} f(z)\,dz$ and $\displaystyle\int_{|z|=r} \overline{f(z)}\,dz$ in terms of the coefficients (a_n) and the radius r.
(Hint for the second integral: On the circle $|z| = r$ we have that $z\bar{z} = r^2$, so $\bar{z} = \frac{r^2}{z}$.)

We wish to compute $\int_{|z|=r} f(z)dz$ given that $f(z)$ has a Laurent series.

$$\int_{|z|=r} f(z)dz = \int_{|z|=r}\sum_{n=-\infty}^{\infty} a_n z^n dz = \sum_{n=-\infty}^{\infty} a_n \int_{|z|=r} z^n dz = 2\pi i a_{-1}$$

where the final step is justified since $\int_{|z|=r} z^n dz = 0$ when $n \neq -1$ and $\int_{|z|=r}^n dz = 2\pi i$ when $n = -1$. Now for $\overline{f(z)}$ we use the hint that $\bar{z} = r^2/z$:

$$\int_{|z|=r} \overline{f(z)} dz = \int_{|z|=r} \sum_{n=-\infty}^{\infty} \overline{a_n z^n} dz = \int_{|z|=r} \sum_{n=-\infty}^{\infty} \overline{a_n} (\overline{z})^n dz = \sum_{n=-\infty}^{\infty} \overline{a_n} \int_{|z|=r} \left(\frac{r^2}{z}\right)^n dz$$

$$= \sum_{n=-\infty}^{\infty} \overline{a_n} r^{2n} \int_{|z|=r} z^{-n} dz = 2\pi i r^2 \overline{a_1}$$

where the final simplification is justified similarly to the previous integral.