M 472 – Homework 1 – Complex numbers

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Due on January 20 on Gradescope

1. Perform the following calculations and express the answer in the form $x + iy$.
   (a) $(3-2i) - i(4+5i)$:
   $(3-2i) - i(4+5i) = (3-2i) + (5-4i) = (3+5) + i(-2-4) = 8 - 6i$

   (b) $(7-2i)(5+3i)$:
   $(7-2i)(5+3i) = (35+6) + i(-10+21) = 41 + 11i$

   (c) $(i-1)^3$:
   $(i-1)^3 = (i-1)(i-1)(i-1) = (-1+1-i-i)(i-1) = -2i(i-1) = 2+2i$

   (d) $\dfrac{1+2i}{3-4i} - \dfrac{4-3i}{2-i}$:
   $\dfrac{1+2i}{3-4i} - \dfrac{4-3i}{2-i} = \dfrac{1+2i}{3-4i}\dfrac{3+4i}{3+4i} - \dfrac{4-3i}{2-i}\dfrac{2+i}{2+i} = \dfrac{3-8+6i+4i}{3^2+4^2} - \dfrac{8+3-6i+4i}{2^2+1^2} =$
   $\dfrac{-5+10i}{25} - \dfrac{11-2i}{5} = \dfrac{-1+2i}{5} + \dfrac{-11+2i}{5} = \dfrac{-12+4i}{5} = -12/5 + i4/5$

2. Find the principal argument $\operatorname{Arg} z$ for
   (a) $z = \dfrac{-3}{4+4i}$:
   $$z = \frac{-3}{4+4i} = \frac{1}{4}\frac{-3}{1+1i}\frac{1-1i}{1-1i} = \frac{1}{4}\frac{-3+3i}{2} = 3/8 - 3/8i$$
   $\operatorname{Arg} z = \arctan((-3/8)/(3/8)) = \arctan(-1) = -\pi/4$

   (b) $z = \left(-\sqrt{3}+i\right)^5$:
   Let $w = -\sqrt{3}+i$, then $w_r = \sqrt{w\bar{w}} = \sqrt{3+1} = 2$ and $\operatorname{Arg} w = \arctan(-1/\sqrt{3}) = 5\pi/6$. Then $w = 2\exp(i5\pi/6)$ and $z = w^5 = 2^5\exp(5 * i5\pi/6) = 32\exp(25\pi/6) = 32\exp(\pi/6)$. So $\operatorname{Arg} z = \pi/6$.

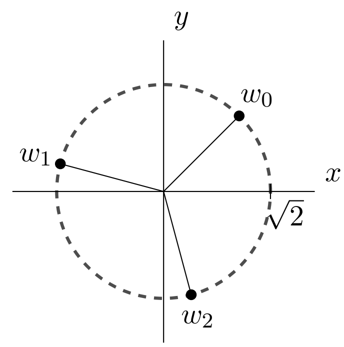
3. Find the square roots of the following and express them in rectangular coordinates.
   (a) $z = 4i$:
   $z = 4i = 4\exp(i\pi/2)$. We want to find $w \in \mathbb{C}$ such that $w^2 = (r\exp(i\theta))^2 = r^2\exp(i2\theta) = 4\exp(i\pi/2)$. Then we must have $r^2 = 4 \implies r = 2$ and $\exp(i2\theta) = \exp(i\pi/2) \implies 2\theta = \pi/2 + 2\pi k$ for $k \in \mathbb{Z}$. Then $\theta_k = \pi/4 + \pi k$. So $\sqrt{z} = w_k = 2\exp(i(\pi/4+\pi k)) = 2\exp(i\pi/4)\exp(i\pi k) = (-1)^k 2\exp(i\pi/4) = \pm 2\exp(i\pi/4) = \pm(\sqrt{2} + i\sqrt{2})$

   (b) $z = -1 + \sqrt{3}i$:
   $z = -1+\sqrt{3}i = 2\exp(i2\pi/3)$. Let $\sqrt{z} = w$, then $w^2 = r^2\exp(i2\theta) = 2\exp(i2\pi/3)$. So we must have $r^2 = 2 \implies r = \sqrt{2}$ and $\exp(i2\theta) = \exp(i2\pi/3) \implies 2\theta = 2\pi/3 + 2\pi k$ for

$k \in \mathbb{Z}$. Then $\theta = \pi/3 + \pi k$. So $\sqrt{z} = w_k = \sqrt{2}\exp(i(\pi/3+\pi k)) = \sqrt{2}\exp(i\pi/3)\exp(i\pi k) = (-1)^k\sqrt{2}\exp(i\pi/3) = \pm\sqrt{2}\exp(i\pi/3) = \pm(\sqrt{2}/2 + i\sqrt{6}/2)$.

4. Find and sketch the three cube roots of $z = -2 + 2i$. (Hint: They should form the vertices of an equilateral triangle centered at zero.)

   $z = -2 + 2i = \sqrt{8}\exp(i3\pi/4)$. We seek $\sqrt[3]{z} = w$. So $w^3 = r^3\exp(i3\theta) = \sqrt{8}\exp(i3\pi/4)$. Then we must have $r^3 = \sqrt{8} \implies r = \sqrt{2}$ and $(i3\theta) = \exp(i3\pi/4) \implies 3\theta = 3\pi/4 + 2\pi k$ for $k \in \mathbb{Z}$. This gives $\theta = \pi/4 + 2\pi/3k$. So $\sqrt{z} = w_k = \sqrt{2}\exp(i\pi/4)\exp(i2\pi/3k)$. On a graph, these are:



5. Assuming that $|z_3| \neq |z_4|$, show that

$$\frac{\mathrm{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

   To show the above inequality, we show that the numerator on the RHS is greater than the numerator on the LHS and the denominator on the RHS is less than the denominator on the LHS. We first show that $\mathrm{Re}(z_1 + z_2) \leq |z_1| + |z_2|$. Consider $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(x_1 + iy_1) + (x_2 + iy_2) = \mathrm{Re}(x_1 + x_2 + i(y_1 + y_2)) = x_1 + x_2$. Then $x_1 + x_2 \leq \sqrt{x_1^2} + \sqrt{x_2^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + t_2^2} = |z_1| + |z_2|$. Then we can show that $|z_3 + z_4| \geq ||z_3| - |z_4||$. Beginning with $|z_3 + z_4| = |z_3 - -z_4| \geq ||z_3| - |-z_4|| = ||z_3| - |z_4||$. Since $|z_3| \neq |z_4|$, the denomonitor on the RHS is not 0, so the expression is defined. Therefore,

$$\frac{\mathrm{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

6. Use de Moivre's formula to derive the trigonometric identities

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\,\sin^2\theta,$$
$$\sin(3\theta) = 3\cos^2\theta\,\sin\theta - \sin^3\theta.$$

   We begin with $\exp(i\theta) = \cos\theta + i\sin\theta$. Then $(\exp(i\theta))^3 = (\cos\theta + i\sin\theta)^3$. Expanding the RHS,

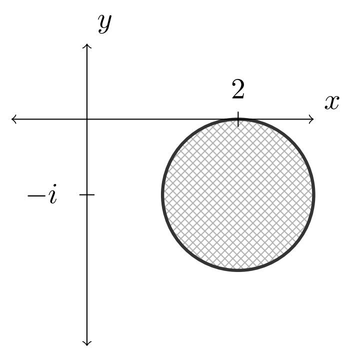
$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3\\ &= \cos^3\theta + i3\cos^2\theta\sin\theta + i^2 3\sin^2\theta\cos\theta + i^3\sin^3\theta\\ &= \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

Then the LHS is $(\exp(i\theta))^3 = \exp(i3\theta) = \cos(3\theta) + i\sin(3\theta)$. Since the LHS must equal the RHS, the real and imaginary parts of each side must match. This gives us

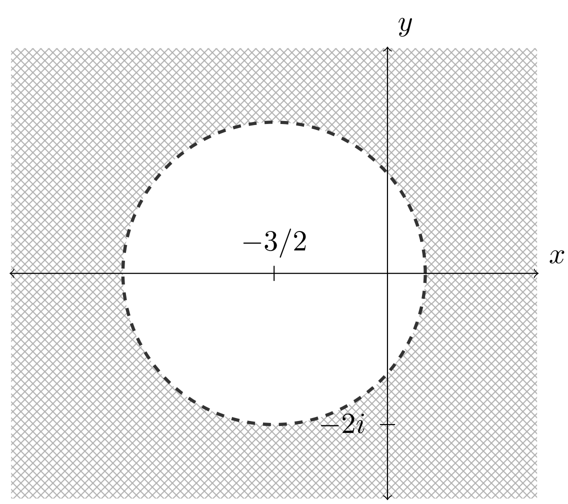
$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\,\sin^2\theta,$$
$$\sin(3\theta) = 3\cos^2\theta\,\sin\theta - \sin^3\theta.$$

7. In each case, sketch the set of points determined by the given condition.

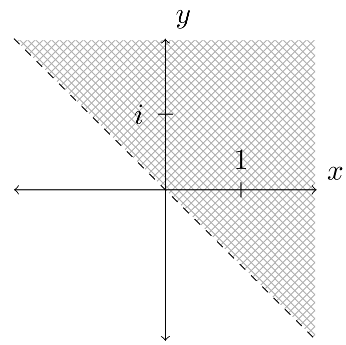
   (a) $|z - 2 + i| \leq 1$
   $|z - 2 + i| = |z - (2 - i)| \leq 1$:



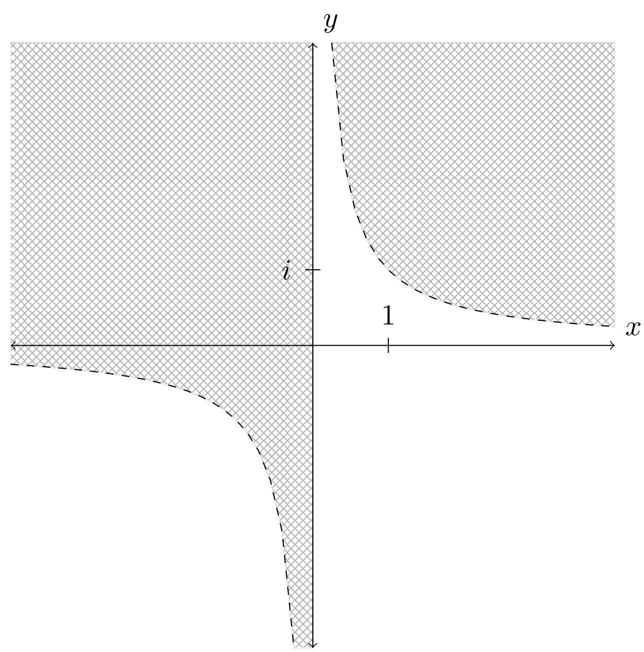
   (b) $|2z + 3| > 4$
   $|2z + 3| = |2(z + 3/2)| = 2|z - -3/2| > 4 \iff |z - -3/2| > 2$:



   (c) $|z - 1| < |z + i|$
   $|z-1| < |z+i| \iff |x-1+iy| < |x+i(y+1)| \iff \sqrt{(x-1)^2 + y^2} < \sqrt{x^2 + (y+1)} \iff$
   $x^2 - 2x + 1 + y^2 < x^2 + y^2 + 1 \iff -2x < 2y \iff y > -x$



(d) $(\operatorname{Re} z)(\operatorname{Im} z) > 1$
$(\operatorname{Re} z)(\operatorname{Im} z) > 1 \iff xy > 1$



(e) $0 < \arg z < \pi/4$