M 472 – Homework 2 – Complex functions and differentiation

Due Monday, February 1, on Gradescope

1. Find a domain in the z-plane whose image under the transformation $w = z^2$ is the square domain in the w-plane bounded by the lines $u = 1$, $u = 2$, $v = 1$, and $v = 2$. (See Section 14 in the textbook.)

2. Show that the function $f(z) = \left(\frac{z}{\bar{z}}\right)^2$ has the value 1 at all nonzero points on the real and imaginary axes, but that it has the value -1 at all nonzero points on the line $x = y$. Conclude that the limit of $f(z)$ as z tends to 0 does not exist.

3. Let f be the function defined by

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

   Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz-plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point on the line $\Delta x = \Delta y$. Conclude that $f'(0)$ does not exist. Note that to obtain this result it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz-plane.

4. Show from the definition of complex derivatives that the functions $f(z) = \operatorname{Re} z$ and $g(z) = \operatorname{Im} z$ are not complex differentiable at any point in the plane.

5. Using the definitions, rules for derivatives, or the Cauchy-Riemann equations, determine where the following functions are complex differentiable and find $f'(z)$ where it exists.
   (a) $f(z) = 1/(z^2 + 1)$,
   (b) $f(z) = x^2 + iy^2$,
   (c) $f(z) = \sin x \cosh y - i \cos x \sinh y$,
   (d) $f(z) = \sin x \cosh y + i \cos x \sinh y$.