M 472 – Homework 4
Integrals

Due Monday, March 1, on Gradescope

1. Let a and R be positive real numbers, let C be the positively oriented circle of radius R about the origin, and let $f(z) = z^{a-1} = \exp[(a-1)\operatorname{Log} z]$ be the principal branch of the power function z^{a-1}. Find $\displaystyle\int_C f(z)\,dz$.

   Let's begin by paramatrizing $C : z(t) = Re^{it}$ where $-\pi \le t \le \pi$, then $dz = iRe^{it}\,dt$. Then $\int_C f(z)dz = \int_C z^{a-1}dz = \int_C \exp[(a-1)\operatorname{Log} z]dz = \int_{-\pi}^{\pi} \exp[(a-1)\operatorname{Log}(Re^{it})]iRe^{it}\,dt$. Since $t \in [\pi, \pi]$ we know $\operatorname{Log}(Re^{it}) = \ln R + it$. Now consider the following calculations

$$\begin{aligned}
\int_{-\pi}^{\pi} \exp[(a-1)\operatorname{Log}(Re^{it})]iRe^{it}\,dt &= iR\int_{-\pi}^{\pi} e^{(a-1)(\ln R+it)}e^{it}\,dt \\
&= iR\int_{-\pi}^{\pi} e^{\ln R^{a-1}}e^{i(a-1)t}e^{it}\,dt \\
&= iRR^{a-1}\int_{-\pi}^{\pi} e^{iat}\,dt \\
&= iR^a\left(\frac{1}{ia}e^{iat}\right)\Big|_{-\pi}^{\pi} \\
&= \frac{R^a}{a}\left[e^{ia\pi} - e^{-ia\pi}\right] \\
&= \frac{R^a}{a}\left[(\cos(a\pi) + i\sin(a\pi)) - (\cos(-a\pi) + i\sin(-a\pi))\right] \\
\int_C z^{a-1}dz &= i\frac{2R^a}{a}\sin(a\pi)
\end{aligned}$$

   where the last equality is justified by $e^{ia\pi} - e^{-ia\pi} = (\cos(a\pi) + i\sin(a\pi)) - (\cos(-a\pi) + i\sin(-a\pi)) = \cos(a\pi) + i\sin(a\pi) - \cos(a\pi) + i\sin(a\pi) = i2\sin(a\pi)$. Note that this aligns with the fact that z^a is analytic (its integral is 0 around closed loops) when a is a natural number.

2. Let T be the triangle with vertices 0, 1, and $1+i$.

   (a) Evaluate the integrals $\displaystyle\int_{\partial T} x\,dz$ and $\displaystyle\int_{\partial T} y\,dz$.

   First let's parametrize the boundary of the triangle T. For all parametrizations, let $0 \le t \le 1$. For C_1 we have $z_1(t) = t$ and $dz_1 = 1\,dt$. The curve C_2 is parametrized by $z_2(t) = 1 + it$ which gives $dz_2 = i\,dt$. Finally, C_3 is $z_3(t) = \sqrt{2}(1-t)e^{i\pi/4}$ and $dz_3 = -\sqrt{2}e^{i\pi/4}\,dt$.

   Before going on, note that $\operatorname{Re}\left(\sqrt{2}(1-t)e^{i\pi/4}\right) = \operatorname{Im}\left(\sqrt{2}(1-t)e^{i\pi/4}\right)$ since any z with $\theta = \pi/4$ must satisfy $y = x$. Further, $\operatorname{Re}\left(\sqrt{2}(1-t)e^{i\pi/4}\right) = \sqrt{2}(1-t)\cos(\pi/4) = \sqrt{2}(1-t)\sqrt{2}/2 = 1-t$.

Using the above parametrizations of ∂T, we have $\int_{\partial T} xdz = \int_{C_1} xdz + \int_{C_2} xdz + \int_{C_3} xdz = \int_0^1 t\,dt + \int_0^1 1i\,dt + \int_0^1 \mathrm{Re}\left(\sqrt{2}(1-t)e^{i\pi/4}\right)\left(-\sqrt{2}e^{i\pi/4}\,dt\right)$. We know $\int_0^1 t\,dt = 1/2$ and $\int_0^1 1i\,dt = i$ and

$$\begin{aligned}\int_0^1 \mathrm{Re}\left(\sqrt{2}(1-t)e^{i\pi/4}\right)\left(-\sqrt{2}e^{i\pi/4}\,dt\right) &= \int_0^1 (1-t)\left(-\sqrt{2}e^{i\pi/4}\right)\,dt \\ &= \int_0^1 (t-1)(1+i)\,dt \\ &= \int_0^1 t-1\,dt + i\int_0^1 t-1\,dt \quad = -1/2 - i/2\end{aligned}$$

Thus $\int_{\partial T} xdz = 1/2 + i - 1/2 - i/2 = i/2$

Now let's consider $\int_{\partial T} ydz = \int_{C_1} ydz + \int_{C_2} ydz = \int_{C_3} ydz$. With our parametrizations, we have $\int_{C_1} ydz = \int_0^1 0\,dt = 0$, $\int_{C_2} ydz = \int_0^1 ti\,dt = i/2$, and

$$\int_{C_3} ydz = \int_0^1 \mathrm{Im}\left(\sqrt{2}(1-t)e^{i\pi/4}\right)\left(-\sqrt{2}e^{i\pi 4}\,dt\right) = -1/2 - i/2$$

since the real part equals the imaginary part as noted before. Thus $\int_{\partial T} y\,dz = 0 + i/2 + -1/2 - i/2 = -1/2$.

(b) Use the results from part (a) to find $\displaystyle\int_{\partial T} z\,dz$ and $\displaystyle\int_{\partial T} \bar{z}\,dz$.

Let's begin with $\int_{\partial T} z\,dz = \int_{\partial T} x + iy\,dz = \int_{\partial T} xdz + i\int_{\partial T} ydz = i/2 + i(-1/2) = 0$. Then for $\int_{\partial T} \bar{z}\,dz = \int_{partialT} x - iy\,dz = \int_{\partial T} x\,dz - i\int_{\partial T} y\,dz = i/2 - i(-1/2) = i$.

(c) One of the integrals in part (b) can be evaluated without explicit integration, using instead a theorem from class/textbook. Explain and use this to double-check your result.

We know that the integral on a closed loop of an analytic function (on the interior of the loop) is 0. $f(z) = z$ is analytic everywhere and ∂T is a closed loop so the integral must be 0.

3. For $R > 1$, let C_R be the positively oriented circle of radius R about the origin.

(a) Show that $\displaystyle\left|\int_{C_R} \frac{\mathrm{Log}\, z}{z^2}\,dz\right| \leq 2\pi\left(\frac{\pi + \ln R}{R}\right)$.

Recall that $\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$. Then

$$\left|\int_{C_R} \frac{\mathrm{Log}\, z}{z^2}\,dz\right| \leq \int_{C_R} \left|\frac{\mathrm{Log}\, z}{z^2}\right|\,dz = \int_{C_R} \frac{|\ln|z| + i\,\mathrm{Arg}\, z|}{|z^2|}\,dz = \int_{C_R} \frac{|\ln R + i\,\mathrm{Arg}\, z|}{R^2}\,dz$$

where the final equality is justified since $|z| = R$ on C_R. Now we want to choose the maximum value for the value of the integrand along the path C_R. Then $|\ln R + iArgz| = \sqrt{(\ln R)^2 + (\mathrm{Arg}\, z)^2} \leq \sqrt{(\ln R)^2 + \pi^2}$ since largest magnitude of the function Arg is a member of $\{\pi, -\pi\}$. By the triangle inequality for real numbers, $\sqrt{(\ln R)^2 + \pi^2} \leq \ln R + \pi$. Then we have

$$\left|\int_{C_R} \frac{\mathrm{Log}\, z}{z^2}\,dz\right| \leq \int_{C_R} \frac{\pi + \ln R}{R^2}\,dz = 2\pi R\left(\frac{\pi + \ln R}{R^2}\right) = 2\pi\left(\frac{\pi + \ln R}{R}\right)$$

(b) Conclude that $\lim_{R\to\infty} \int_{C_R} \frac{\text{Log}\, z}{z^2}\, dz = 0$.

To deduce this, take the limit on both sides of the inequality proved in part (a). The RHS is

$$\lim_{R\to\infty} 2\pi \left(\frac{\pi + \ln R}{R} \right) = \lim_{R\to\infty} 2\pi/R = 0$$

by L'Hopital's rule (which we can use since we are only dealing with a real valued limit). So we have just shown that the magnitude of the integral is less than or equal 0 as $R \to \infty$, which implies the value of the integral is 0.