M 472 – Homework 4

Integrals

Due Monday, March 1, on Gradescope

1. Let a and R be positive real numbers, let C be the positively oriented circle of radius R about the origin, and let $f(z) = z^{a-1} = \exp[(a-1)\operatorname{Log} z]$ be the principal branch of the power function z^{a-1}. Find $\displaystyle\int_C f(z)\,dz$.

   Let's begin by paramatrizing $C : z(t) = Re^{it}$ where $-\pi \leq t \leq \pi$, then $dz = iRe^{it}dt$. Then $\int_C f(z)dz = \int_C z^{a-1}dz = \int_C \exp[(a-1)\operatorname{Log} z]dz = \int_{-\pi}^{\pi} \exp[(a-1)\operatorname{Log}(Re^{it})]iRe^{it}dt$. Since $t \in [\pi, \pi]$ we know $\operatorname{Log}(Re^{it}) = \ln R + it$. Now consider the following calculations

$$\begin{aligned}
\int_{-\pi}^{\pi} \exp[(a-1)\operatorname{Log}(Re^{it})]iRe^{it}dt &= iR\int_{-\pi}^{\pi} e^{(a-1)(\ln R + it)}e^{it}dt \\
&= iR\int_{-\pi}^{\pi} e^{\ln R^{a-1}}e^{i(a-1)t}e^{it}dt \\
&= iRR^{a-1}\int_{-\pi}^{\pi} e^{iat}dt \\
&= iR^a\left(\frac{1}{ia}e^{iat}\right)\Big|_{-\pi}^{\pi} \\
&= \frac{R^a}{a}\left[e^{ia\pi} - e^{-ia\pi}\right] \\
\int_C z^{a-1}dz &= i\frac{2R^a}{a}\sin(a\pi)
\end{aligned}$$

   where the last equality is justified by $e^{ia\pi} - e^{-ia\pi} = (\cos(a\pi) + i\sin(a\pi)) - (\cos(-a\pi) + i\sin(-a\pi)) = \cos(a\pi) + i\sin(a\pi) - \cos(a\pi) + i\sin(a\pi) = i2\sin(a\pi)$. Note that this aligns with the fact that z^a is analytic (its integral is 0 around closed loops) when a is a natural number.

2. Let T be the triangle with vertices 0, 1, and $1+i$.
   (a) Evaluate the integrals $\displaystyle\int_{\partial T} x\,dz$ and $\displaystyle\int_{\partial T} y\,dz$.
   (b) Use the results from part (a) to find $\displaystyle\int_{\partial T} z\,dz$ and $\displaystyle\int_{\partial T} \bar{z}\,dz$.
   (c) One of the integrals in part (b) can be evaluated without explicit integration, using instead a theorem from class/textbook. Explain and use this to double-check your result.

3. For $R > 1$, let C_R be the positively oriented circle of radius R about the origin.
   (a) Show that $\displaystyle\left|\int_{C_R} \frac{\operatorname{Log} z}{z^2}\,dz\right| \leq 2\pi\left(\frac{\pi + \ln R}{R}\right)$.
   (b) Conclude that $\displaystyle\lim_{R\to\infty}\int_{C_R} \frac{\operatorname{Log} z}{z^2}\,dz = 0$.