M 472 – Homework 3

Cauchy-Riemann Equations, Elementary Functions

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Due Wednesday, February 17, on Gradescope

1. Assume that a function f is analytic in a domain D, and that f is real-valued. Show that f must be a constant function. (Hint: Use the Cauchy-Riemann equations.)

   A function f is analytic on an open set D if f is complex differentiable at all $z_0 \in D$. Since f is complex differentiable on D, it satisfies the Cauchy-Riemann equations. Namely, for $f = u + iv$ we must have $u_x = v_y$ and $v_x = -u_y$. But since f is real-valued, $v_x = v_y = 0$ so we konw $u_x = u_y = 0$. But then u must be constant, for if it were not, then one the partial derivatives would be non-zero.

2. Find all values of z such that

   (a) $e^z = -1 + i$
   $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) = e^x \cos y + ie^x \sin y = -1 + i$. Equivalently, we must have $e^x \cos y = -1$ and $e^x \sin y = 1$. Then $-e^x \cos y = e^x \sin y$, but since $e^x \neq 0$ for all $x \in \mathbb{R}$ we must satisfy $-\cos y = \sin y \implies y = 3\pi/4 + 2\pi n$ for $n \in \mathbb{Z}$ (since we are in the second quadrant). But we must also have $e^x \cos y = -1$. Since y is $3\pi/4$ plus and integer multiple of 2π, $\cos y = -1/\sqrt{2}$ so we must have $e^{=}\sqrt{2}$, equivalently, $x = \ln \sqrt{2}$. Therefore, $z = \ln \sqrt{2} + i(3\pi/4 + 2\pi n)$ for any integer n.

   (b) e^z is purely imaginary
   We must have $e^z = e^x \cos y + ie^x \sin y = ib$. This is true if and only if $e^x \cos y = 0$ but we already noted $e^x \neq 0$ for all $x \in \mathbb{R}$ so the solutions are equivalent to that of $\cos y = 0$ which we know to be $\pi/2 + n\pi$ for any integer n. Thus the solutions are $z = x + i(\pi/2 + 2\pi n)$ for $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

   (c) $|e^z| < e^2$
   Note that $|e^z| = |e^x||\cos y + i \sin y| = |e^x| * 1 = e^x$. So we must have $z = x + iy$ where $x < 2$.

3. Find $\text{Log}(i^3)$ and $\text{Log}\, i$, and show that $\text{Log}(i^3) \neq 3\,\text{Log}\, i$.

   Let's begin with $\text{Log}(i^3) = \text{Log}(-i) = \ln|-i| + i\,\text{Arg}(-i) = \ln 1 - i\pi/2 = -i\pi/2$. Then $\text{Log}\, i = \ln|i| + i\,\text{Arg}\, i = \ln 1 + i\pi/2 = i\pi/2$. Certainly $-i\pi/2 \neq i\pi/2$ so $\text{Log}(i^3) \neq 3\,\text{Log}\, i$.

4. (a) Find a complex number z such that $\text{Log}(e^z) \neq z$.
   Choose $z = 1 + i3\pi/2$. Then $\text{Log}(e^z) = \ln|e^z| + i\,\text{Arg}(e^z) = \ln(e^1) + i\,\text{Arg}(e^{3\pi/x}) = 1 - i\pi/2 \neq z$.

   (b) For which complex numbers z does the equality $\text{Log}(e^z) = z$ hold?
   $\text{Log}(e^z) = \ln|e^z| + i\,\text{Arg}(e^z) = \ln e^x + i\,\text{Arg}\, e^{iy} = x + i\theta$ where $\theta \in (-\pi, \pi]$ and $y = \theta + 2\pi k$ for some $k \in Z$. Thus for $z \in \{x + iy \mid y \in (-\pi, \pi]\} \subset \mathbb{C}$ we have $\text{Log}(e^z) = z$.

5. (a) Find the principal value of $(-1+i)^i$
   Principal value: $(-1+i)^i = \exp(i\,\mathrm{Log}(-1+i)) = \exp(i[\ln|-1+i| + i\,\mathrm{Arg}(-1+i)]) = \exp(i[\ln\sqrt{2} + i3\pi/4]) = \exp(-3\pi/4 + i\ln\sqrt{2})$.

   (b) Find all values of $(-1+i)^i$
   To find all values of $(-1+i)^i$, we just use arg instead of Arg in the above sub-problem. This gives $\arg(-1+i) = 3\pi/4 + 2\pi n$ for $n \in Z$ and $(-1+i)^i = \exp(i[\ln\sqrt{2} + i(3\pi/4 + 2\pi n)]) = \exp(-(3\pi/4 + 2\pi n) + i\ln\sqrt{2})$.

6. Find all values z such that

   (a) $\sin z = 2$
   To solve for z, we have $w = 2$ in the formula $z = -i\log(iw + \sqrt{1-w^2}) = -i\log(i(2+\sqrt{3})) = -i[\ln|i(2+\sqrt{3})| + i\arg(i(2+\sqrt{3}))] = -i[\ln(2+\sqrt{3}) + i(\pi/2 + 2\pi n)] = \pi/2 + 2\pi n - i\ln(2+\sqrt{3})$.

   (b) $\sin z = 2i$
   Similarly to the previous subproblem, we have $w = 2i$ in $z = -i\log(iw + \sqrt{1-w^2}) = -i\log(-2+\sqrt{5}) = -i[\ln|-2+\sqrt{5}| + i\arg(-2+\sqrt{5})] = -i[\ln(-2+\sqrt{5}) + i2\pi n] = 2\pi n - i\ln(-2+\sqrt{5})$