

CSCI 338: Assignment 3

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Problem 1

Design context-free grammars for the following languages

- 1.1 $A = \{a^n b^m | n \neq 2m\}.$
- 1.2 $B = \{a^i b^j c^k | i, j, k \geq 0 \text{ and either } i = j \text{ or } j = k\}.$
- 1.3 $C = \{a^n b^m | n = 3m\}.$
- 1.4 $D = \{a^n b^m | n \leq m + 3\}.$

Proof: We now give the context grammars for each of the above languages.

$$\begin{aligned} 1.1 \quad S &\longrightarrow aaSb \mid aB \mid A \mid B \\ A &\longrightarrow Aa \mid a \\ B &\longrightarrow Bb \mid b \end{aligned}$$

$$\begin{aligned} 1.2 \quad S &\longrightarrow IC \mid AK \\ I &\longrightarrow aIb \mid \epsilon \\ K &\longrightarrow bKc \mid \epsilon \\ C &\longrightarrow Cc \mid \epsilon \\ A &\longrightarrow Aa \mid \epsilon \end{aligned}$$

$$1.3 \quad S \longrightarrow \epsilon \mid aaaSb$$

$$\begin{aligned} 1.4 \quad S &\longrightarrow X \mid aX \mid aaX \mid aaaX \\ X &\longrightarrow aXb \mid B \\ B &\longrightarrow Bb \mid \epsilon \end{aligned}$$

□

Problem 2

Decide whether the following grammar is ambiguous.

$$\begin{aligned}S &\rightarrow AB|aaB \\A &\rightarrow a|Aa \\B &\rightarrow b\end{aligned}$$

Proof: To show that the above grammar is ambiguous, we must show that there exist two leftmost derivations for some string in the grammar. Consider the following two derivations for aab .

$$\begin{aligned}\text{Derivation 1: } S &\xrightarrow{*} AB \xrightarrow{*} AaB \xrightarrow{*} aaB \xrightarrow{*} aab \\ \text{Derivation 2: } S &\xrightarrow{*} aaB \xrightarrow{*} aab\end{aligned}$$

The above derivations are both leftmost and distinct, yet they derive the same string. So the given grammar must be ambiguous.

□

Problem 3

Convert the following CFG G to an equivalent PDA.

$$R \rightarrow XRX|S$$

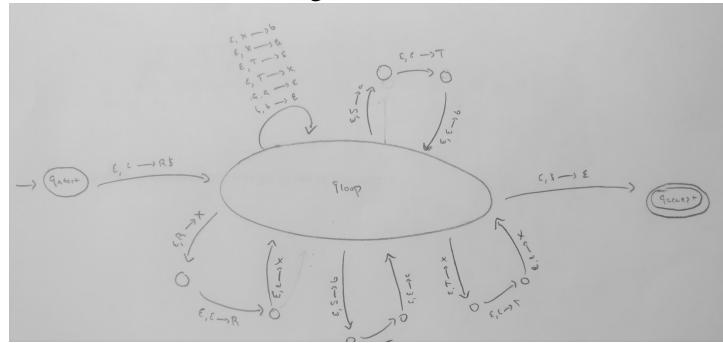
$$S \rightarrow aTb|bTa$$

$$T \rightarrow XTX|X|\epsilon$$

$$X \rightarrow a|b$$

Proof: We show the equivalent PDA below.

Figure 1: PDA



□

Problem 4

Let $G = (V, \Sigma, R, S)$ be the following grammar. $V = \{S, T, U\}$; $\Sigma = \{0, \#\}$; and R is the set of rules:

$$\begin{aligned} S &\rightarrow TT|U \\ T &\rightarrow 0T|T0|\# \\ U &\rightarrow 0U00|\# \end{aligned}$$

4.1 Describe $L(G)$ in English.

Proof: In English, $L(G)$ consists of the union of two sets. The first is the set of all strings that include two $\#$ symbols with any number of zeros on either side and in between the $\#$ symbols. The second is the set consisting of all strings beginning with n zeros, then a $\#$ symbol, and then $2n$ zeros.

□

4.2 Prove that $L(G)$ is not regular.

Proof: We will prove that $L(G)$ is not regular by contradiction. That is, we will assume that $L(G)$ is regular. Since $L(G)$ is regular, we can apply the pumping lemma for regular languages. Take p to be the pumping length of $L(G)$. Then, for any string $s \in L(G)$ where $|s| \geq p$ (we know s exists since p is finite and we construct an element of $L(G)$ to be as large as needed), the following must be true:

1. $xy^i z \in L(G) \forall i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$

We now choose $s = 0^p \# 0^{2p} \in L(G)$. Since $L(G)$ is regular, we can decompose $s = xyz$. Because of condition 3, it must be the case that xy consists of no more than the first p letters of s , meaning that $y = 0^a$ where $1 \leq a \leq p$. We then choose $i = 0$ and wonder if it is true that $xz = 0^{p-a} \# 0^{2p} \in L(G)$.

This rests on $2(p - a) = 2p \iff 2p - 2a = 2p \iff -2a = 0 \iff a = 0$, which is certainly not true since a cannot be 0. So $xz \notin L(G)$ and s cannot be pumped for $i = 0$.

Since s cannot be pumped for $i = 0$, the pumping lemma does not apply to $L(G)$. Since the pumping lemma does not apply to $L(G)$, it must be the case that $L(G)$ is non-regular.

□

Problem 5

Convert the following CFG into an equivalent CFG in Chomsky Normal Form

$$\begin{aligned} A &\rightarrow BAB | B | \epsilon \\ B &\rightarrow 00 | \epsilon \end{aligned}$$

Proof: We begin by adding a start rule $S \rightarrow A$. This gives

$$\begin{aligned} S &\rightarrow A \\ A &\rightarrow BAB | B | \epsilon \\ B &\rightarrow 00 | \epsilon \end{aligned}$$

We now remove the rule $B \rightarrow \epsilon$

$$\begin{aligned} S &\rightarrow A \\ A &\rightarrow BAB | BA | AB | A | B | \epsilon \\ B &\rightarrow 00 \end{aligned}$$

And now $A \rightarrow \epsilon$

$$\begin{aligned} S &\rightarrow A | \epsilon \\ A &\rightarrow BAB | BB | BA | AB | A | B \\ B &\rightarrow 00 \end{aligned}$$

The unit rule $A \rightarrow A$ has no consequence so we remove it. In addition, we replace the rule $S \rightarrow A$

$$\begin{aligned} S &\rightarrow BAB | BB | BA | AB | B | \epsilon \\ A &\rightarrow BAB | BB | BA | AB | B \\ B &\rightarrow 00 \end{aligned}$$

We now introduce the new rule $C \rightarrow AB$

$$\begin{aligned} S &\rightarrow BC | BB | BA | AB | B | \epsilon \\ A &\rightarrow BC | BB | BA | AB | B \\ B &\rightarrow 00 \\ C &\rightarrow AB \end{aligned}$$

Finally, we create the new rule $D \rightarrow 0$. This puts us in Chomsky Normal Form.

$$S \rightarrow BC \mid BB \mid BA \mid AB \mid DD \mid \epsilon$$

$$A \rightarrow BC \mid BB \mid BA \mid AB \mid DD$$

$$B \rightarrow DD$$

$$C \rightarrow AB$$

$$D \rightarrow 0$$

□

Problem 6

Using pumping lemma to prove that the following languages are not context-free.

$$6.1 \quad L = \{a^n b^j c^k \mid k = nj\}.$$

Proof: We will prove that L is not context free by contradiction. That is, we will assume that L is context free. Since L is context free, we can apply the pumping lemma. Take p to be the pumping length of L . Then, for any string $s \in L$ where $|s| \geq p$ (we know s exists since p is finite and we could construct an element of L to be as large as needed), the following must be true:

1. $uv^i xy^i z \in L \forall i \geq 0$
2. $|vy| > 0$
3. $|vxy| \leq p$

Choose $s = a^p b^p c^{p^2} \in L$. We can then decompose $s = uvxyz$. We now consider three cases for s .

The first case is when one of v, y contains more than one type of symbol. Pumping up results in a string not of the form $a^n b^j c^k$, since there will be some symbols in between (ie $aabaabbcccc$).

We now consider the case where $v = a^{m_1}$ and $y = b^{m_2}$. Pumping up when $i = 2$ results in a string $a^{p+m_1} b^{j+m_2} c^k$. So is $k = (p + m_1)(p + m_2) = p^2 + (m_1 + m_2)p + m_1 * m_2$? No, since $k = p^2 < p^2 + (m_1 + m_2)p + m_1 * m_2$ and one of m_1, m_2 is greater than 0.

Now we have the case $v = b^{m_1}$ and $y = c^{m_2}$. Pumping up for $i = 2$ results in the string $s = a^p b^{p+m_1} c^{p^2+m_2}$. So is $k = p^2 + m_2 = p(p + m_1) \iff p^2 + m_2 = p^2 + p * m_1 \iff m_2 = p * m_1$. So $s_i \in L$ only when $m_2 = p * m_1$. Can this ever be the case? Well, $m_1 = 0 \implies m_2 = 0$, but that cannot be the case since $|vy| > 0$. So it must be true that $p * m_1 \geq p \implies m_2 \geq p$, yet this also cannot be the case since $|vxy| \leq p$.

Thus a contradiction is found in all cases and L is not context free. □

$$6.2 \quad L = \{a^n b^j \mid n \geq (j-1)^3\}.$$

Proof: We will prove that L is not context free by contradiction. That is, we will assume that L is context free. Since L is context free, we can apply the pumping lemma. Take p to be the pumping length of L . Then, for any string $s \in L$ where $|s| \geq p$ (we know s exists since p is finite and we could construct an element of L to be as large as needed), the following must be true:

- 1. $uv^i xy^i z \in L \forall i \geq 0$
- 2. $|vy| > 0$
- 3. $|vxy| \leq p$

Choose $s = a^{(p-1)^3} b^p \in L$. Then we can decompose $s = uvxyz$. Now consider the following four cases.

If one of v, y contains more than one type of symbol then pumping results in leaving the form $a^n b^j$ for any $n, j \in \mathbb{N}$.

Now consider the case where $vxy = a^m$ for $1 \leq m \leq p$. Then we can pump down for the case $i = 0$ so that $s = a^{(p-1)^3 - m} b^p$. Clearly $(p-1)^3 \neq (p-1)^3 - m$ since $m \neq 0$.

We now view the case where $vxy = b^m$ where $1 \leq m \leq p$. We again pump down ($i = 0$) and now $s = a^{(p-1)^3} b^{p-m}$. Yet it cannot be the case that $(p-1)^3 = (p-m-1)^3$ since $m \neq 0$.

We now consider the final case where $v = a^{m_1}$ and $y = b^{m_2}$ for both $m_1, m_2 \geq 1$. We then pump up for the case $i = 2$ and now $s = a^{(p-1)^3 + m_1} b^{p+m_2}$. This case rests on the fact that $(p-1)^3 + m_1 = (p+m_2-1)^3$. Yet, since $m_2 \geq 1$, it must be true that $(p+m_2-1)^3 \geq p^3$. But since $m_1 < p$, it must also be true that $(p-1)^3 + m_1 < (p-1)^3 + p < p^3$. So $(p-1)^3 + m_1 < (p+m_2-1)^3$.

So we have found a contradiction in all cases and L is not context free. □