

# CSCI 338: Assignment 1

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## Problem 1

Prove that  $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(4n^2 - 1)$ .

**Proof:** We begin by defining the property  $P(n) := 1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2$  for any  $n \in \mathbb{Z}^+$ . Our task is to show that  $P(n) = \frac{1}{3}n(4n^2 - 1)$  for all  $n \in \mathbb{Z}^+$ . We proceed by induction.

We first must prove the base case of  $n = 1$ . Take the LHS:  $P(n) = 1^2 = 1$ . And now the RHS:  $\frac{1}{3}n(4n^2 - 1) = \frac{1}{3}(1)(4(1)^2 - 1) = 1$ . Since the LHS and the RHS for both 1 for  $n = 1$ , our base case is proved.

We now state the inductive hypothesis. That is, for some  $k \in \mathbb{Z}^+$ , we assume  $P(k) = \frac{1}{3}k(4k^2 - 1)$  to be true.

To complete our proof, we must now show that  $P(k+1) = \frac{1}{3}(k+1)(4(k+1)^2 - 1)$  holds. First note that

$$P(k+1) = 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2(k+1)-1)^2 = P(k) + (2(k+1)-1)^2$$

We now proceed as follows:

$$\begin{aligned} P(k+1) &= P(k) + (2(k+1)-1)^2 \\ &= \frac{1}{3}k(4k^2 - 1) + (2(k+1)-1)^2, \text{ by our inductive hypothesis} \\ &= \frac{1}{3}[4k^3 - k + 3(4k^2 + 4k + 1)] \\ &= \frac{1}{3}[4k^3 + 12k^2 + 11k + 3] \\ &= \frac{1}{3}(k+1)[4k^2 + 8k + 3] \\ &= \frac{1}{3}(k+1)[4(k^2 + 2k) + 3] \\ &= \frac{1}{3}(k+1)[4(k+1)^2 + 3 - 4], \text{ by completing the square} \\ P(k+1) &= \frac{1}{3}(k+1)(4(k+1)^2 - 1) \end{aligned}$$

So we have now shown that  $P(k+1) = \frac{1}{3}(k+1)(4(k+1)^2 - 1)$  holds. This completes our proof by induction. So it must be true that  $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$  for all  $n \in \mathbb{Z}^+$ .

□

## Problem 2

Given a planar graph  $P = (V, E)$ , we have Euler's formula:  $|V| + |F| - |E| = 2$ , where  $F$  is the set of faces of  $P$  and  $E$  is the set of edges of  $P$ . Let  $|V| = n$ , where  $V$  is the set of vertices of  $P$ . Prove that  $|F|$  is at most  $2n$ .

**Proof:** We want to show that, for a planar graph  $P$  with  $n$  vertices, it must be true that  $|F| \leq 2n$ .

If  $P$  is a forest or tree, then there is only one face. So  $|F| = 1$  and  $1 < 2n \forall n \in \mathbb{Z}^+$ . So  $|F| < 2n$  if  $P$  is a forest or tree.

We now consider all other cases. If you were to count the number of edges from the perspective of each face in  $P$ , you would reach exactly  $2|E|$ . Additionally, since we require at least three edges to define a face, it must be true that  $2|E| \geq 3|F|$ . Equivalently,  $|E| \geq \frac{3}{2}|F|$ . Recall that Euler's formula holds for all planar graphs. So we have:

$$\begin{aligned} |V| + |F| - |E| = 2 &\iff n + |F| - |E| = 2 \\ &\iff n + |F| - \frac{3}{2}|F| \geq 2 \\ &\iff -\frac{1}{2}|F| \geq 2 - n \\ &\iff |F| \leq 2n - 4 \\ |V| + |F| - |E| = 2 &\iff |F| \leq 2n \end{aligned}$$

Since Euler's formula must be true, it must also be true that  $|F| \leq 2n$ .

So, it has been shown in all cases of planar graphs with  $n$  vertices that  $|F| \leq 2n$

□

## Problem 3

Prove that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

**Proof:** We want to show that any simple graph has the property that there is a path from any vertex of odd degree to some other vertex of odd degree. Recall that a simple graph is a graph where no pair of vertices  $a, b$  has more than one edge connecting  $a$  and  $b$ .

We now take a simple graph  $G = (V, E)$ . There are two cases. Either  $G$  is connected or not. A graph that is not connected is the union of connected graphs, so we need only prove the connected case.

We now take the connected case of  $G$ . If  $\text{degree}(v)$  is even for every vertex  $v \in V$ , our statement is true since there are no vertices with odd degree. So we need only prove the case where there exists some  $u \in V$  such that  $\text{degree}(v)$  is odd.

We now proceed with a proof by contradiction, that is, assume there is no path  $p$  connecting vertices  $u$  and  $w$  where  $w \in V$  and  $\text{degree}(w)$  is odd. Recall that we are in the case where  $G$  is connected. So for  $p$  to not exist, it must also be true that there is no  $w \in V$  such that  $\text{degree}(w)$  is odd. This implies that  $u$  is the only vertex in  $G$  with odd degree.

We can now partition  $V$  into two subsets:  $\{u\}$  and  $C = \{c \in V \mid c \neq u\}$ . Note that for  $c \in C$ ,  $\text{degree}(c)$  must be even. Denote  $m$  and  $n$  as the sum of the degrees of the vertices in  $C$  and  $V$  respectively. Since  $m$  is the sum of even numbers,  $m$  must be even. Now we know that  $\text{degree}(u)$  is odd. Since  $n = \text{degree}(u) + m$ ,  $n$  must be odd. Yet, the sum of degrees of a graph is always  $2 * |E|$ , which is even. Since  $n$  cannot be both even and odd, a contradiction is found. So it must be true that the path  $p$  exists.

Since  $p$  exists, it must be true that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

□

## Problem 4

A fully binary tree  $T$  is a tree such that all internal nodes have two children. Prove that a fully binary tree with  $n$  internal nodes in total has  $2n + 1$  nodes.

**Proof:** Our task is to show that the fully binary tree  $T$  with  $n$  internal nodes has a total of  $2n + 1$  nodes. We will define  $T_n$  to be the number of nodes in such a tree. So we must show that  $T_n = 2n + 1$  for arbitrary  $n \in \{0\} \cup \mathbb{Z}^+$ . We will proceed by induction.

Consider the base case of  $T_0$ , that is, a tree with no internal nodes. Since there are no internal nodes, there is only a root node. So surely,  $T_0 = 1$ . Now, does  $T_n = 2n + 1$  hold? Certainly:  $2n + 1 = 2 * 0 + 1 = 1$ . So the base case holds.

We now state the inductive hypothesis. We assume, for some  $k \in \{0\} \cup \mathbb{Z}^+$ , that  $T_n = 2n + 1$  holds for all  $n \leq k$ .

We now show that  $T_{k+1} = 2(k + 1) + 1$ . Let  $M$  and  $N$  be trees with  $T_k$  and  $T_{k+1}$  internal nodes respectively. By definition of  $T_n$ , it must be true that  $N$  contains one more internal node than  $M$ , we name this internal node  $i$ . Since  $i$  must have two child nodes, it follows that

$$\begin{aligned} T_{k+1} &= T_k + 2 \\ &= (2k + 1) + 2, \text{ by inductive hypothesis} \\ &= 2k + 2 + 1 \\ T_{k+1} &= 2(k + 1) + 1 \end{aligned}$$

Since  $T_{k+1} = 2(k + 1) + 1$  holds, our proof by induction is complete. So, it must be true that a fully binary tree  $T$  with  $n$  internal nodes has a total of  $2n + 1$  nodes.  $\square$

## Problem 5

Given an undirected graph  $G = (V, E)$ , the breadth-first-search starting at  $v \in V$  ( $bfs(v)$  for short) is to generate a shortest path tree starting at vertex  $v \in V$ . The diameter of  $G$  is the longest of all shortest paths  $\delta(u, v), u, v \in V$ .

When  $G$  is a tree, the following algorithm is proposed to compute the diameter of  $G$ .

1. Run  $bfs(w), w \in V$ , and compute the vertex  $x \in V$  furthest from  $w$ .
2. Run  $bfs(x)$  and compute the vertex  $y \in V$  furthest from  $x$ .
3. Return  $\delta(x, y)$  as the diameter of  $G$ .

Prove that this algorithm is correct; i.e.,  $\delta(x, y)$  is in fact the longest among all the shortest paths between  $u, v \in V$ .

**Proof:** Take the end points of the longest shortest path to be  $a$  and  $b$ . We know that  $y$  is certainly the farthest vertex from  $x$ . So whether or not this algorithm is correct relies on whether  $bfs(w)$  yields  $x = a$ .

We will prove this by contradiction. That is, we assume there exists some vertex  $m \in V$  such that  $bfs(m)$  yields a furthest vertex  $n$  such that  $n \neq a$ .

Without loss of generality, take  $\delta(m, a) \geq \delta(m, b)$ . Since  $bfs(m)$  yields  $n$ , it must be true that  $\delta(m, n) \geq \delta(m, a)$ . Recall that  $\delta(a, b)$  is the diameter yet we can form a longer diameter by using  $n$  and  $a$  as endpoints. So we have found a contradiction. This means that  $bfs(w)$  must yield the end point  $a$  on the longest shortest path.

Since an arbitrary point yields an end point of the longest shortest path, it must be the case that the above algorithm computes the diameter.

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