CSCI 338: Assignment 1

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Problem 1

Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$.

Proof: We begin by defining the property $P(n) := 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$ for any $n \in \mathbb{Z}^+$. Our task is to show that $P(n) = \frac{1}{3}n(4n^2 - 1)$ for all $n \in \mathbb{Z}^+$. We proceed by induction.

We first must prove the base case of $n = 1$. Take the LHS: $P(n) = 1^2 = 1$. And now the RHS: $\frac{1}{3}n(4n^2 - 1) = \frac{1}{3}(1)(4(1)^2 - 1) = 1$. Since the LHS and the RHS for both 1 for $n = 1$, our base case is proved.

We now state the inductive hypothesis. That is, for some $k \in \mathbb{Z}^+$, we assume $P(k) = \frac{1}{3}k(4k^2 - 1)$ to be true.

To complete our proof, we must now show that $P(k+1) = \frac{1}{3}(k+1)(4(k+1)^2 - 1)$ holds. First note that

$$P(k+1) = 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2(k+1)-1)^2 = P(k) + (2(k+1)-1)^2$$

We now proceed as follows:

$$\begin{aligned}
P(k+1) &= P(k) + (2(k+1)-1)^2 \\
&= \frac{1}{3}k(4k^2 - 1) + (2(k+1)-1)^2, \text{ by our inductive hypothesis} \\
&= \frac{1}{3}[4k^3 - k + 3(4k^2 + 4k + 1)] \\
&= \frac{1}{3}[4k^3 + 12k^2 + 11k + 3] \\
&= \frac{1}{3}(k+1)[4k^2 + 8k + 3] \\
&= \frac{1}{3}(k+1)[4(k^2 + 2k) + 3] \\
&= \frac{1}{3}(k+1)[4(k+1)^2 + 3 - 4], \text{ by completing the square} \\
P(k+1) &= \frac{1}{3}(k+1)(4(k+1)^2 - 1)
\end{aligned}$$

So we have now shown that $P(k+1) = \dfrac{1}{3}(k+1)(4(k+1)^2 - 1)$ holds. This completes our proof by induction. So it must be true that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$ for all $n \in \mathbb{Z}^+$.

$\square$

Problem 2

Given a planar graph $P = (V, E)$, we have Euler's formula: $|V| + |F| - |E| = 2$, where F is the set of faces of P and E is the set of edges of P. Let $|V| = n$, where V is the set of vertices of P. Prove that $|F|$ is at most $2n$.

Proof: We want to show that, for a planar graph P with n vertices, it must be true that $|F| \leq 2n$.

If P is a forest or tree, then there is only one face. So $|F| = 1$ and $1 < 2n$ $\forall n \in \mathbb{Z}^+$. So $|F| < 2n$ if P is a forest or tree.

We now consider all other cases. If you were to count the number of edges from the perspective of each face in P, you would reach exactly $2|E|$. Additionally, since we require at least three edges to define a face, it must be true that $2|E| \geq 3|F|$. Equivalently, $|E| \geq \frac{3}{2}|F|$. Recall that Euler's formula holds for all planar graphs. So we have:

$$\begin{aligned}
|V| + |F| - |E| = 2 &\iff n + |F| - |E| = 2 \\
&\iff n + |F| - \frac{3}{2}|F| \geq 2 \\
&\iff -\frac{1}{2}|F| \geq 2 - n \\
&\iff |F| \leq 2n - 4 \\
|V| + |F| - |E| = 2 &\iff |F| \leq 2n
\end{aligned}$$

Since Euler's formula must be true, it must also be true that $|F| \leq 2n$.

So, it has been shown in all cases of planar graphs with n vertices that $|F| \leq 2n$

$\square$

Problem 3

Prove that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

Proof: We want to show that any simple graph has the property that there is a path from any vertex of odd degree to some other vertex of odd degree. Recall that a simple graph is a graph where no pair of vertices a, b has more than one edge connecting a and b.

We now take a simple graph $G = (V, E)$. There are two cases. Either G is connected or not. A graph that is not connected is the union of connected graphs, so we need only prove the connected case.

We now take the connected case of G. If $degree(v)$ is even for every vertex $v \in V$, our statement is true since there are no vertices with odd degree. So we need only prove the case where there exists some $u \in V$ such that $degree(v)$ is odd.

We now proceed with a proof by contradiction, that is, assume there is no path p connecting vertices u and w where $w \in V$ and $degree(w)$ is odd. Recall that we are in the case where G is connected. So for p to not exist, it must also be true that there is no $w \in V$ such that $degree(w)$ is odd. This implies that u is the only vertex in G with odd degree.

We can now partition V into two subsets: $\{u\}$ and $C = \{c \in V \mid c \neq u\}$. Note that for $c \in C$, $degree(c)$ must be even. Denote m and n as the sum of the degrees of the vertices in C and V respectively. Since m is the sum of even numbers, m must be even. Now we know that $degree(u)$ is odd. Since $n = degree(u) + m$, n must be odd. Yet, the sum of degrees of a graph is always $2 * |E|$, which is even. Since n cannot be both even and odd, a contradiction is found. So it must be true that the path p exists.

Since p exists, it must be true that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

$\square$

Problem 4

A fully binary tree T is a tree such that all internal nodes have two children. Prove that a fully binary tree with n internal nodes in total has $2n + 1$ nodes.

Proof: Our task is to show that the fully binary tree T with n internal nodes has a total of $2n + 1$ nodes. We will define T_n to be the number of nodes in such a tree. So we must show that $T_n = 2n + 1$ for arbitrary $n \in \{0\} \cup \mathbb{Z}^+$. We will proceed by induction.

Consider the base case of T_0, that is, a tree with no internal nodes. Since there are no internal nodes, there is only a root node. So surely, $T_0 = 1$. Now, does $T_n = 2n + 1$ hold? Certainly: $2n + 1 = 2 * 0 + 1 = 1$. So the base case holds.

We now state the inductive hypothesis. We assume, for some $k \in \{0\} \cup \mathbb{Z}^+$, that $T_n = 2n + 1$ holds for all $n \leq k$.

We now show that $T_{k+1} = 2(k + 1) + 1$. Let M and N be trees with T_k and T_{k+1} internal nodes respectively. By definition of T_n, it must be true that N contains one more internal node than M, we name this internal node i. Since i must have two child nodes, it follows that

$$\begin{aligned}
T_{k+1} &= T_k + 2 \\
&= (2k + 1) + 2, \text{ by inductive hypothesis} \\
&= 2k + 2 + 1 \\
T_{k+1} &= 2(k + 1) + 1
\end{aligned}$$

Since $T_{k+1} = 2(k + 1) + 1$ holds, our proof by induction is complete. So, it must be true that a fully binary tree T with n internal nodes has a total of $2n + 1$ nodes.

$\square$

Problem 5

Given an undirected graph $G = (V, E)$, the breadth-first-search starting at $v \in V$ ($bfs(v)$ for short) is to generate a shortest path tree starting at vertex $v \in V$. The diameter of G is the longest of all shortest paths $\delta(u, v), u, v \in V$.

When G is a tree, the following algorithm is proposed to compute the diameter of G.

1. Run $bfs(w), w \in V$, and compute the vertex $x \in V$ furthest from w.
2. Run $bfs(x)$ and compute the vertex $y \in V$ furthest from x.
3. Return $\delta(x, y)$ as the diameter of G.

Prove that this algorithm is correct; i.e., $\delta(x, y)$ is in fact the longest among all the shortest paths between $u, v \in V$.

Proof: Take the end points of the longest shortest path to be a and b. We know that y is certainly the farthest vertex from x. So whether or not this algorithm is correct relies on whether $bfs(w)$ yields $x = a$.

We will prove this by contradiction. That is, we assume there exists some vertex $m \in V$ such that $bfs(m)$ yields a furthest vertex n such that $n \neq a$.

Without loss of generality, take $\delta(m, a) \geq \delta(m, b)$. Since $bfs(m)$ yields n, it must be true that $\delta(m, n) \geq \delta(m, a)$. Recall that $\delta(a, b)$ is the diameter yet we can form a longer diameter by using n and a as endpoints. So we have found a contradiction. This means that $bfs(w)$ must yield the end point a on the longest shortest path.

Since an arbitrary point yields an end point of the longest shortest path, it must be the case that the above algorithm computes the diameter.

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