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Dynamics

HW 11

8.2.17 Consider the new binocular rivalry model

$$\dot{x}_1 = -x_1 + F(I - bx_2 - gy_1)$$
$$\dot{y}_1 = (-y_1 + x_1)/T$$
$$\dot{x}_2 = -x_2 + F(I - bx_1 - gy_2)$$
$$\dot{y}_2 = (-y_2 + x_2)/T$$

where $F(z) = 1/(1+e^{-z})$

a) Show that $x_1^* = y_1^* = x_2^* = y_2^* = u$ is a fixed point for all choices of parameters and that u is uniquely defined.

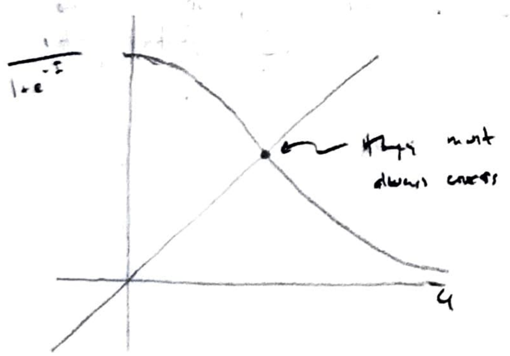
let's take for granted that such a u exists (we will show it later). Since $x_1^* = y_1^*$ and $x_2^* = y_2^*$, we have $\dot{y}_1 = 0$, $\dot{y}_2 = 0$. Additionally we can say that

$$0 = \dot{x}_1 = -x_1^* + F(I - bx_2^* - gy_1^*) = -x_1^* + F(I - bx_1^* - gx_1^*) = -x_2^* + F(I - bx_1^* - gy_2^*) = \dot{x}_2$$

So now we come to the point where there is the question of whether u can exist. Such a u must solve

$$0 = -u + F(I - bu - gu)$$
$$u = F(I - bu - gu) \Rightarrow u = \frac{1}{1 + e^{(b+g)u - I}} \quad (*)$$

To verify there is a solution, we will graph the equations



b) Show that the stability matrix has the form

$$\begin{bmatrix} -c_1 & -c_2 & -c_3 & 0 \\ d_1 & -d_1 & 0 & 0 \\ -c_3 & 0 & -c_1 & -c_2 \\ 0 & 0 & d_1 & -d_1 \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

Show that the e-vals of the 4x4 equal the e-vals of $A-B$ and $A+B$

First we compute the stability matrix:

let
$$\dot{x}_1 = F = -x_1 + (1+\exp(bx_2+gy_1-I))^{-1}$$
$$\dot{y}_1 = H = (-y_1+x_1)/T$$
$$\dot{x}_2 = G = -x_2 + (1+\exp(bx_1+gy_2-I))^{-1}$$
$$\dot{y}_2 = L = (-y_2+x_2)/T$$

$F_{x_1} = -1$ $\quad F_{y_1} = -g(1+\exp(bx_2+gy_1-I))^{-2}$ $\quad F_{x_2} = -b(1+\exp(bx_2+gy_1-I))^{-2}$ $\quad F_{y_2} = 0$

$H_{x_1} = 1/T$ $\quad H_{y_1} = -1/T$ $\quad H_{x_2} = 0$ $\quad H_{y_2} = 0$

$G_{x_1} = -b(1+\exp(bx_1+gy_2-I))^{-2}$ $\quad G_{y_1} = 0$ $\quad G_{x_2} = -1$ $\quad G_{y_2} = -g(1+\exp(bx_1+gy_2-I))^{-2}$

$L_{x_1} = 0$ $\quad L_{y_1} = 0$ $\quad L_{x_2} = 1/T$ $\quad L_{y_2} = -1/T$

then let $c_1 = 1$, $c_2 = g(1+\exp(bu+gu-I))^{-2}$, $c_3 = b(1+\exp(bu+gu-I))^{-2}$, $d = 1/T$
and we can have the stability matrix at (u,u,u,u) be

$$\begin{bmatrix} -c_1 & -c_2 & -c_3 & 0 \\ d_1 & -d_1 & 0 & 0 \\ -c_3 & 0 & -c_1 & -c_2 \\ 0 & 0 & d_1 & -d_1 \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} -c_1 & -c_2 \\ d_1 & -d_1 \end{bmatrix} \quad B = \begin{bmatrix} -c_3 & 0 \\ 0 & 0 \end{bmatrix}$$

I also did some algebra on the last page of the hw

After some googling, I found that $\det\left(\begin{bmatrix} A-\lambda I & B \\ B & A-\lambda I \end{bmatrix}\right) = \det(A - I\lambda - B) \cdot \det(A - \lambda I + B)$

$\det(A-B-\lambda I) \cdot \det(A+B-\lambda I) = 0$ so we can just solve for the eigenvalues of $A-B$ and $A+B$

c) Show that the eigenvalues of $A+B$ are all negative

$$A+B = \begin{bmatrix} -(c_1+c_3) & -c_2 \\ d_1 & -d_1 \end{bmatrix}$$

$\tau = -(c_1+c_3+d_1)$

$\Delta = d_1(c_1+c_3) + c_2 d_1 = d_1(c_1+c_2+c_3)$

$$\lambda_{1,2} = \frac{-(c_1+c_3+d_1) \pm \sqrt{(c_1+c_3+d_1)^2 - 4\Delta}}{2}$$

$\Delta > 0$ so the square root will be $< |\tau|$
$\Rightarrow \lambda_{1,2} < 0$

d) Show that, depending on the sizes of g and T, the matrix can have either a negative determinant (pitchfork bifurcation) or a positive trace (Hopf bifurcation)

$$A - B = \begin{bmatrix} -c_1 + c_3 & -c_2 \\ d_1 & -d_1 \end{bmatrix} \qquad \tau = -c_1 + c_3 - d_1$$
$$\Delta = d_1(c_1 - c_3) + c_2 d_1 = d_1(c_1 + c_2 - c_3)$$

T does not affect the sign of the determinant but for sufficiently small g we will have $\Delta < 0$ since c_3 will be fairly large while $c_2 \to 0$ with small g. Additionally τ will be negative with sufficiently small T. This will cause a pitchfork bifurcation

Then for large g and T, we have $\tau > 0$ since $\tau = -c_1 + c_3 - d_1 > 0 \Updownarrow -c_1 + c_3 > d_1$ with large T, this is easily acheived. ($d_1 = 1/T$) and $\Delta > 0$ since c_2 will grow with g. Then a Hopf bifurcation occurs

8.3.1 Consider the reaction equation
$$\dot{x} = 1 - (b+1)x + ax^2y$$
$$\dot{y} = bx - ax^2y$$
where $a, b > 0$ are dimensionless parameters and $x, y \geq 0$ are dimensionless concentrations.

a) Find all fixed points and use the Jacobian to classify them.

For a fixed point to occur, we must have

$\dot{x} = 0 = 1 - (b+1)x + ax^2y \Rightarrow 0 = 1 - (b+1)x + bx \Rightarrow 0 = 1 - \cancel{bx} - x + \cancel{bx} \Rightarrow x = 1$

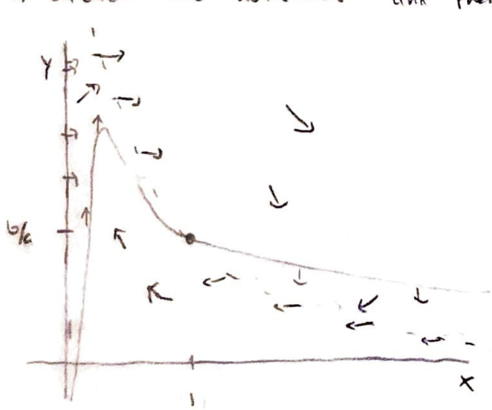
$\dot{y} = 0 = bx - ax^2y \Rightarrow ax^2y = bx$

since $x = 1$, $y = b/a$ at the fixed point. The general Jacobian is
$$J = \begin{bmatrix} -(b+1) + 2axy & ax^2 \\ b - 2axy & -ax^2 \end{bmatrix}$$

At the equilibria, the Jacobian evaluates to
$$J_{(1, b/a)} = \begin{bmatrix} b-1 & a \\ -b & -a \end{bmatrix} \qquad \tau = b - a - 1, \quad \Delta = -a(b-1) + ab = a$$

$(1, b/a)$ is a sink for $b - a - 1 < 0$ and a source for $b - a - 1 > 0$

b) Sketch the nullclines and thereby construct a trapping region for the flow



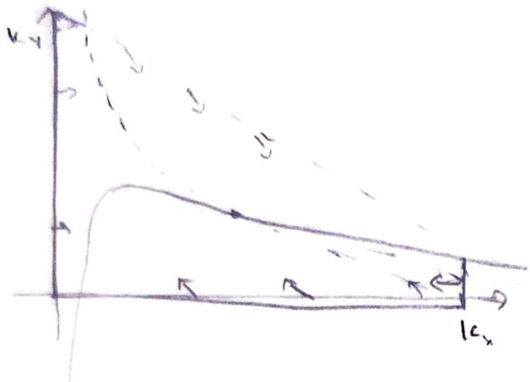
x cline: $y = \frac{(b+1)x - 1}{ax^2}$ ——

y cline: $y = \frac{b/a}{x}$ - - -

$x = 0$

large $x, y \Rightarrow$

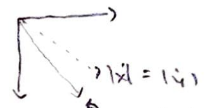
Now we must construct a trapping region. Certainly $x=0$, $y=0$ are good boundaries.



then some vertical line $x = k_x$ will have inward flow, and a line $y = k_y$ will also have inward flow } up to the null

We can make the need k_y as large as needed. Then, with massive x, y we must have $|\dot{x}| < |\dot{y}|$ (since there is $-(b+1)x$ vs bx)

=>



then some line with slope -1 and a large y-intercept will complete the trapping region.

c) Show that a Hopf bifurcation appears for some $b = b_c$

A Hopf bifurcation will appear when the fixed point $(1, b/a)$ turns from a sink to a source. This occurs at $b_c = a+1$

d) The limit cycle must exist for $b > b_c$ since $b > b_c \Rightarrow \tau = b - a - 1$
$> a + 1 - a - 1$
$= 0$

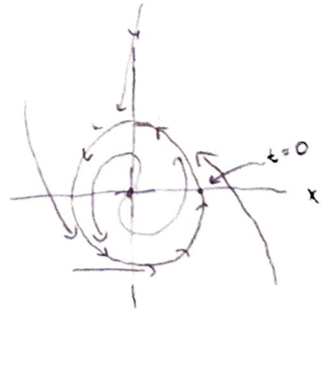
and since $\tau > 0$ (and $\Delta > 0$) the equilibria is a source. Then we have a punctured trapping region containing no fixed points in the plane. Then the Poincare-Bendixson theorem claims there must be a limit cycle in the region

e) Find an approximate period of the limit cycle for $b \gtrsim b_c$

From the table on page 267, a supercritical hopf bifurcation has a period $O(1)$. More precisely, $\omega \approx \Delta^{1/2} = \sqrt{a}$ and $T = 2\pi/\omega = 2\pi/\sqrt{a}$

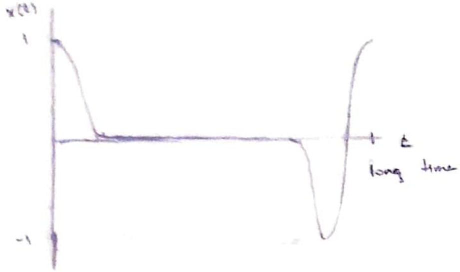
$\boxed{8.4.1}$ Consider the system $\dot{r} = r(1-r^2)$ for μ slightly greater than 1.
let $x = r\cos\theta$ $\quad\dot{\theta} = \mu - \sin\theta$
$y = r\sin\theta$
and sketch the wave forms $x(t), y(t)$.

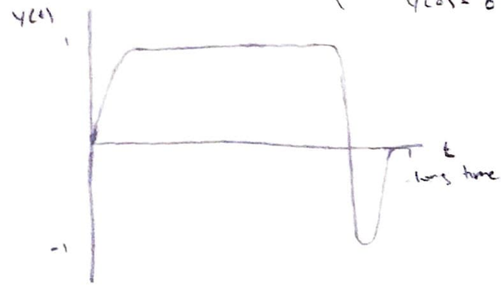
Here is a phase portrait in x, y:



We should note that for $\theta \approx \frac{\pi}{2}$ the flow is mostly in negative $\dot{y}$ and $\theta = 3\pi/2$ the flow is mostly in positive x

Then we can sketch the wave forms for $x(t)$ and $y(t)$ $\left(\text{let } \begin{array}{l} x(0)=1 \\ y(0)=0 \end{array}\right)$



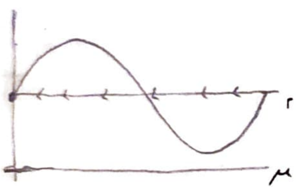


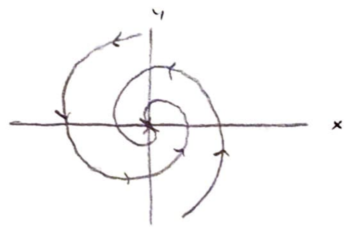
<u>8.4.2</u> | Discuss the bifurcations of the system $\dot{r} = r(\mu - \sin r)$, $\dot{\theta} = 1$ as μ varies

Note that the system is uncoupled so we can perform our analysis separately. $\dot{\theta} = 1$ is constant for all values of μ.

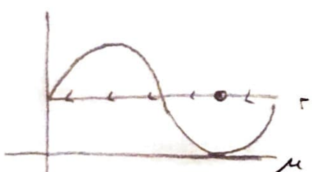
For $\dot{r} = r(\mu - \sin r)$ there is always the fixed point $r=0$.

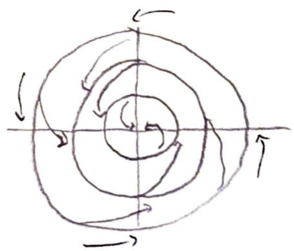
<u>Case $\mu < -1$</u>





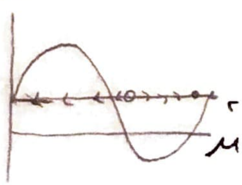
<u>Case $\mu = -1$</u>

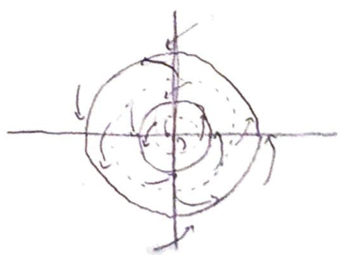




At $\mu = -1$, a half stable limit cycle appears at $r = \frac{3\pi}{2} + 4\pi n$ $\forall n \in \{0,1,2,\dots\}$
This is an infinite period bifurcation.

<u>Case $\mu \in (-1, 0)$</u>



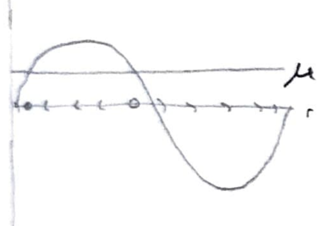


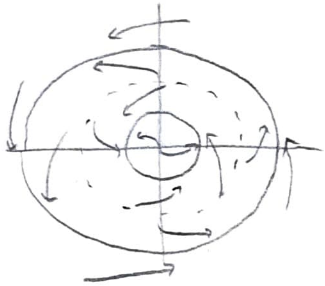
Then there is a splitting of the limit cycles, analogous to a transcritical bifurcation.

case $\mu = 0$:

case $\mu \in (0,1)$:

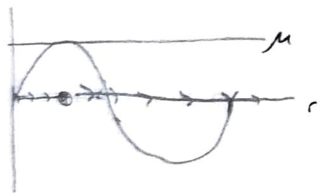
"Zoomed in"

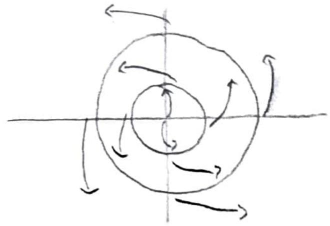




The origin switches stability at $\mu = 0$ and is a supercritical Hopf bifurcation

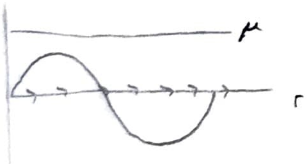
case $\mu = 1$:

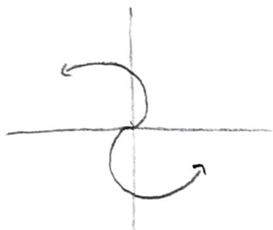




Then we have another "transcritical" when the two limit cycles merge into one limit cycle.

case $\mu > 1$:





Near $\mu = 1$, we have an infinite period situation but as μ grows beyond 1, the origin is really just a source

Here is some algebra for why the e-vals of the 4x4 block matrix $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ is the same as the evals of $A+B$ and $A-B$

let $A = \begin{bmatrix} -c_1 & -c_2 \\ d_1 & -d_1 \end{bmatrix}$ and $B = \begin{bmatrix} -c_3 & 0 \\ 0 & 0 \end{bmatrix}$ and $A+B = \begin{bmatrix} -c_1-c_3-\lambda & -c_2 \\ d_1 & -d_1-\lambda \end{bmatrix}$

$A-B = \begin{bmatrix} -c_1+c_3-\lambda & -c_2 \\ d_1 & -d_1-\lambda \end{bmatrix}$

then $\det(A+B) = (-c_1-c_3-\lambda)(-d_1-\lambda) + c_2 d_1$
$= (c_1+c_3+\lambda)(d_1+\lambda) + c_2 d_1$

$\det(A-B) = (-c_1+c_3-\lambda)(-d_1-\lambda) + c_2 d_1$
$= (c_1-c_3+\lambda)(d_1+\lambda) + c_2 d_1$ then let $C = c_1+\lambda,\ D = d_1+\lambda$

and our determinants become

$\det(A+B) = D(C+c_3) + c_2 d_1$ $\det(A-B) = D(C-c_3) + c_2 d_1$

now lets see if the e-values of the 4x4 satisfy $0 = \det(A+B)\cdot\det(A-B)$

$$\begin{vmatrix} -c_1-\lambda & -c_2 & -c_3 & 0 \\ d_1 & -d_1-\lambda & 0 & 0 \\ -c_3 & 0 & -c_1-\lambda & -c_2 \\ 0 & 0 & d_1 & -d_1-\lambda \end{vmatrix} = \begin{vmatrix} -C & -c_2 & -c_3 & 0 \\ d_1 & -D & 0 & 0 \\ -c_3 & 0 & -C & -c_2 \\ 0 & 0 & d_1 & -D \end{vmatrix}$$

$$= -d_1 \begin{vmatrix} -C & -c_2 & 0 \\ d_1 & -D & 0 \\ -c_3 & 0 & -c_2 \end{vmatrix} - D \begin{vmatrix} -C & -c_2 & -c_3 \\ d_1 & -D & 0 \\ -c_3 & 0 & -C \end{vmatrix}$$

(A note $\begin{bmatrix} -C & -c_2 \\ d_1 & -D \end{bmatrix} = A - \lambda I$)

$= c_2 d_1 \det(A-\lambda I) - D\left[-c_3 \begin{vmatrix} -c_2 & -c_3 \\ -D & 0 \end{vmatrix} - C\det(A-\lambda I)\right]$ $\det(A-\lambda I) = CD + c_2 d_1$

$= c_2 d_1(CD + c_2 d_1) - D(c_3^2 D - C(CD + c_2 d_1))$ $= c_2 d_1 CD + c_2^2 d_1^2 + CD(CD + c_2 d_1) - c_3^2 D^2$

$= C^2 D^2 - c_3^2 D^2 + c_2 d_1 CD + c_2 d_1 CD + c_2^2 d_1^2$ "add 0"

$= D^2(C+c_3)(C-c_3) + c_2 d_1 CD + c_2 c_3 d_1 D + c_2 d_1 CD - c_2 c_3 d_1 D + c_2^2 d_1^2$

$= D^2(C+c_3)(C-c_3) + c_2 d_1 D(C+c_3) + c_2 d_1 D(C-c_3) + c_2^2 d_1^2$

$= D(C+c_3)(D(C-c_3) + c_2 d_1) + c_2 d_1(D(C-c_3) + c_2 d_1)$

$= (D(C+c_3) + c_2 d_1)(D(C-c_3) + c_2 d_1)$

$= \det(A+B)\cdot\det(A-B)$

So then the evals of the big matrix must also satisfy $\det(A+B)\det(A-B)$.