Nathan Stouffer
Dynamics
HW8

6.8.7 Use index theory to show that the system $\dot{x} = x(4 - y - x^2)$, $\dot{y} = y(x-1)$ has no closed orbits.

First let's look at some null clines!

$$\left.\begin{aligned} \dot{x} &= 0 \text{ when } x = 0 \text{ or } y = 4 - x^2 \\ \dot{y} &= 0 \text{ when } y = 0 \text{ or } x = 1 \end{aligned}\right\} \Rightarrow \text{ our equilibria occur at } (0,0),\ (1,3),\ (\pm 2, 0)$$

$$f(x,y) = 4x - xy - x^3 \Rightarrow f_x = 4 - y - 3x^2 \text{ and } f_y = -x$$

$$g(x,y) = y(x-1) \Rightarrow g_x = y \text{ and } g_y = x - 1$$

this gives our jacobian matrix
$$J = \begin{bmatrix} 4 - y - 3x^2 & -x \\ y & x-1 \end{bmatrix}$$

then analyzing each equilibria:

$(0,0)$:

$$J_{(0,0)} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \quad \lambda_1 = 4,\ \lambda_2 = -1 \Rightarrow \text{"saddle" with } I_{(0,0)} = -1$$

$(1,3)$

$$J_{(1,3)} = \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix} \quad \tau = -2,\ \Delta = 4 \quad \lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \cdot 4}}{2} = -1 \pm \sqrt{3}i \Rightarrow \text{spiral sink } I_{(1,3)} = 1$$

$(-2,0)$

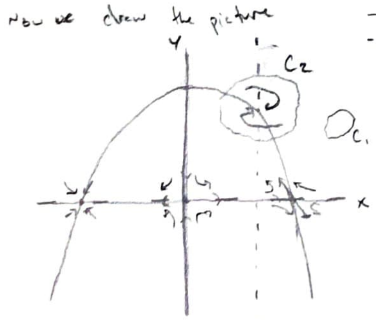
$$J_{(-2,0)} = \begin{bmatrix} -8 & 2 \\ 0 & -3 \end{bmatrix} \quad \tau = -11,\ \Delta = 24 \quad \lambda_{1,2} = \frac{-11 \pm \sqrt{121 - 4 \cdot 24}}{2} = \frac{-11 \pm \sqrt{25}}{2} = -8, -3 \Rightarrow \text{sink } I_{(-2,0)} = 1$$

$(2,0)$

$$J_{(2,0)} = \begin{bmatrix} -8 & -2 \\ 0 & 1 \end{bmatrix} \quad \lambda_{1,2} = -8, 1 \Rightarrow \text{Saddle w/ } I_{(2,0)} = -1$$

Now we draw the picture

— x cline
--- y cline



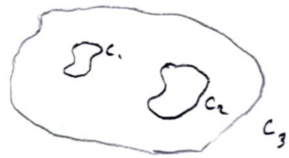
There cannot be a closed orbit around any of $(-2,0)$, $(0,0)$, or $(2,0)$ because $y=0$ means the solution stays at $y=0$ for all time and we must have a unique solution.

There cannot be a closed orbit not enclosing a fixed point (eliminates C_1)

C_2 might exist, as we discussed in class. Or it might not.

6.8.6 A smooth vector field on the phase plane is known to have exactly three closed orbits. Two of the cycles, say C_1 and C_2, lie inside the third cycle C_3. However, C_1 does not lie inside C_2 (nor vice-versa).

a) Sketch the arrangement of the three cycles:



b) Show that there must be at least one fixed point in the region bounded by C_1, C_2, C_3. Let this region be I_{C_*}

we know $I_{C_3} = 1$ and $I_{C_3} = I_{C_2} + I_{C_1} + I_{C_*}$

$1 = 1 + 1 + I_{C_*}$

$I_{C_*} = -1 \Rightarrow C_*$ contains at least one fixed point

7.2.5 Let $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ be a smooth vector field defined on the plane

a) Show that if $\dot{\mathbf{x}} = h(\mathbf{x})$ is a gradient system then $\partial f/\partial y = \partial g/\partial x$

so $\dot{\mathbf{x}} = h(\mathbf{x})$ is a gradient system $\Rightarrow$ there exists some $V(x)$ such that

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla V(x,y) = \begin{bmatrix} -\partial V/\partial x \\ -\partial V/\partial y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

then $\frac{\partial f}{\partial y} = -V_{xy}$ and $\frac{\partial g}{\partial x} = -V_{yx}$. Since the vector field is smooth, $-V_{xy} = -V_{yx}$ and $\partial f/\partial y = \partial g/\partial x$

b) Is condition a) sufficient?

Now suppose we have a smooth vector field defined on the plane such that $\partial f/\partial y = \partial g/\partial x$. Does this mean the system is gradient? The system will be gradient if some $U(x,y)$ exists at $f = V_x$ and $g = V_y$.

Consider $V(x,y) = V(0,0) + \int_\gamma \langle \dot{x}(t), \dot{y}(t) \rangle \, d\gamma$ where γ is some path on the plane. Is $V(x,y)$ which satisfies $f = V_x, g = V_y$ well-defined (γ is arbitrary)?

$V(x,y)$ is well-defined if $\int_\alpha \langle \dot{x}(t), \dot{y}(t) \rangle \, d\alpha = 0$ for every closed loop α.

$$\int_\alpha \langle \dot{x}(t), \dot{y}(t) \rangle \, d\alpha = \int_\alpha \langle V_x, V_y \rangle \cdot d\alpha = \int_\alpha V_x \, dx + V_y \, dy$$

$$= \iint_A -\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \, dx \, dy$$

note:

$$= \iint_A -\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} \, dx \, dy$$

but $-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} = 0$ so the path integral is 0 and $V(x,y)$ is well-defined!

7.2.9 For each of the following systems, decide whether it is a gradient system. If so, find V and sketch the phase portrait. On a seperate graph, sketch the equipotentials V = constant.

a) $\dot{x} = y + x^2 y \longrightarrow f_y = 1 + x^2$

$\dot{y} = -x + 2xy \longrightarrow g_x = -1 + 2y$ $\Rightarrow$ not a gradient system

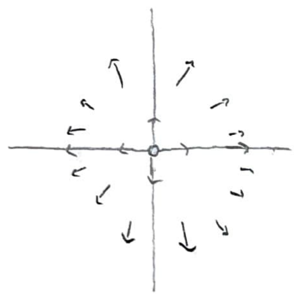
b) $\dot{x} = 2x \longrightarrow f_y = 0$

$\dot{y} = 8y \longrightarrow g_x = 0$ $\Rightarrow$ gradient system

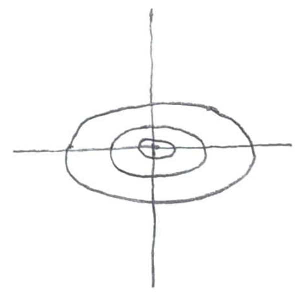
let $V(x,y) = -\int f(x,y)\,dx - \int g(x,y)\,dy = -\int 2x\,dx - \int 8y\,dy = -x^2 - 4y^2$

verification: $-\nabla V(x,y) = \begin{bmatrix} -V_x \\ -V_y \end{bmatrix} = \begin{bmatrix} 2x \\ 8y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$

phase portrait



equipotential

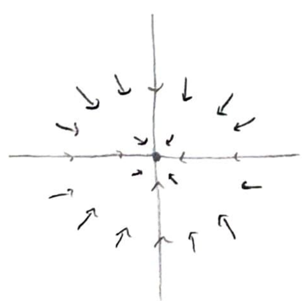


c) $\dot{x} = -2x e^{x^2+y^2} \longrightarrow f_y = -4xy\, e^{x^2+y^2}$

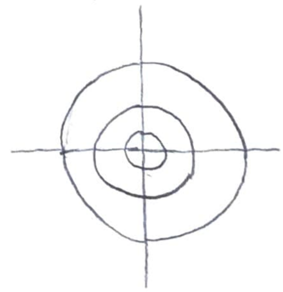
$\dot{y} = -2y e^{x^2+y^2} \longrightarrow g_x = -4xy\, e^{x^2+y^2}$ $\Rightarrow$ gradient system

let $V(x,y) = e^{x^2+y^2}$ then $-\nabla V(x,y) = \begin{bmatrix} -V_x \\ -V_y \end{bmatrix} = \begin{bmatrix} -2x e^{x^2+y^2} \\ -2y e^{x^2+y^2} \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$

phase portrait



equipotentials



7.2.14 Consider $\dot{x} = x^2 - y - 1$

$\dot{y} = y(x-2)$

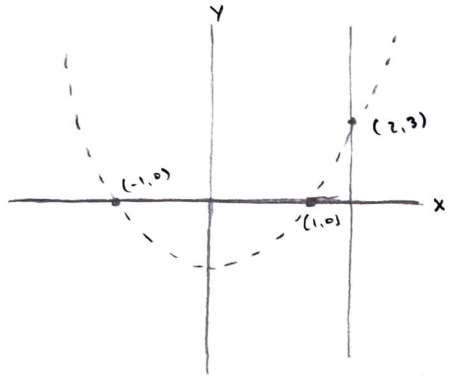
a) Show that there are three fixed points and classify them.

$\dot{x} = 0$ when $y = x^2 - 1$

$\dot{y} = 0$ when $y = 0$ or $x = 2$

lets sketch the null clines now

— y cline
---- x cline



So we have three equilibria: $(\pm 1, 0)$ and $(2,3)$

let's classify them with the Jacobian

$$J = \begin{bmatrix} 2x & -1 \\ y & x-2 \end{bmatrix}$$

$(-1,0)$:

$$J_{(-1,0)} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix} \quad \lambda_{1,2} = -2, -3 \Rightarrow \text{"sink"}$$

$(1,0)$:

$$J_{(1,0)} = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} \quad \lambda_{1,2} = -1, 2 \Rightarrow \text{"saddle"}$$

$(2,3)$:

$$J_{(2,3)} = \begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix} \quad \tau = 4, \ \Delta = 4 \quad \lambda_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4}}{2} = -2 \Rightarrow \text{"degenerate case"}$$

b) Show that there are no closed orbits.

Note $I_{(-1,0)} = 1$, $I_{(1,0)} = -1$, and $I_{(2,3)} = 1$ and also note the x-axis is a solution for all time so no closed orbit can cross it. No closed orbit exists below the x-axis since there are no equilibria the x-axis. This leaves the only possibility above the x axis. If a closed orbit existed, it would have to surround $(2,3)$ but this cannot occur since $(2,3)$ sends flow to $(1,0)$ along the stable manifold.

c) sketch the phase portrait.