Nathan Stouffer
Dynamics I
HW 5

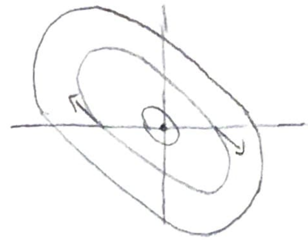
5.2.7 | Given $\dot{x} = 5x + 2y$
$\dot{y} = -17x - 5y$

$$A = \begin{bmatrix} 5 & 2 \\ -17 & -5 \end{bmatrix}$$

$\tau = 0, \quad \Delta = -25 - -34 = 9 > 0$

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \pm \frac{\sqrt{-4\Delta}}{2} = \pm 3i$$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -17 \end{pmatrix} \qquad A\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 17 \end{pmatrix}$$



The origin is a center

5.2.9 | Given $\dot{x} = -3x + 4y$
$\dot{y} = -2x + 3y$

$$A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$$

$\tau = 0, \quad \Delta = -1 < 0$

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \pm \frac{\sqrt{-4\Delta}}{2} = \pm 1$$

case $\lambda_1 = 1$:

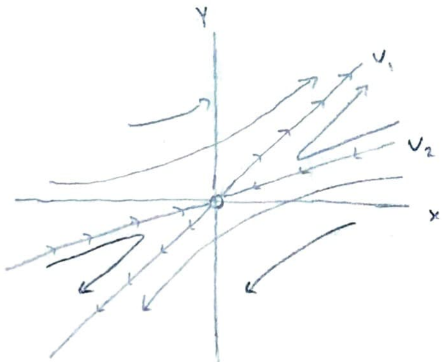
$$\begin{bmatrix} -3-1 & 4 \\ -2 & 3-1 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix}$$

so choose $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

case $\lambda_2 = -1$

$$\begin{bmatrix} -3- -1 & 4 \\ -2 & 3- -1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$$

choose $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



A saddle node!

5.2.9 | Given $\dot{x} = 4x - 3y$
$\dot{y} = 8x - 6y$

$A = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$ $\quad \tau = -2, \quad \Delta = -24 - -24 = 0$

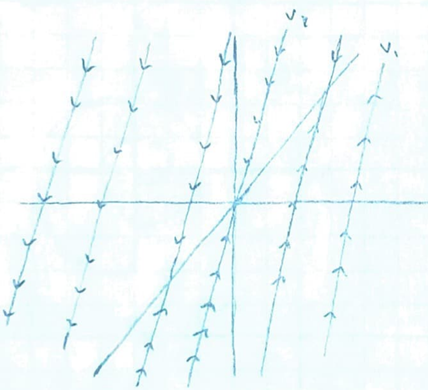
$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{\tau \pm \sqrt{\tau^2}}{2} = \frac{-2 \pm \sqrt{(-2)^2}}{2} = -1 \pm 1 = 0, -2$$

case $\lambda_1 = 0$:

$\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$ so choose $v_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

case $\lambda_2 = -2$:

$\begin{bmatrix} 4 - -2 & -3 \\ 8 & -6 - -2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix}$ so choose $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



5.2.13 | The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$ where $m, b, k > 0$

(a) Rewrite the equation as a 2 dimensional linear system.

let $\dot{x} = v \Rightarrow \dot{v} = \ddot{x}$. Then $m\ddot{x} + b\dot{x} + kx = 0$
$m\dot{v} + bv + kx = 0$
$\dot{v} = -\frac{k}{m}x - \frac{b}{m}v$

so we have our system of equations

$\dot{x} = v$
$\dot{v} = -k/m \, x - b/m \, v$

(b) For all cases, classify the fixed point at the origin and sketch the phase portrait.

$\dot{x} = v$
$\dot{v} = -k/m\, x - b/m\, v$

$A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}$ $\tau = -b/m, \ \Delta = k/m$

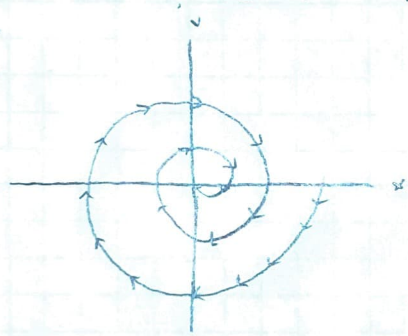
$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = -b/2m \pm \sqrt{b^2/m^2 - 4k/m}\,/2 = -b/2m \pm \sqrt{b^2/m^2 - 4km/m^2}\,/2$$
$$= \frac{-b \pm \sqrt{b^2 - 4km}}{2m}$$

case $b^2 - 4km < 0$:

$$\lambda_{1,2} = \frac{-b}{2m} \pm \frac{\sqrt{b^2-4km}}{2m} = -\alpha \pm \omega i \quad \text{where} \quad \alpha = b/2m > 0, \quad \omega = \sqrt{4km - b^2}\,/2m > 0$$

Since $-\alpha < 0$, we know that the origin is a spiral sink in this case.

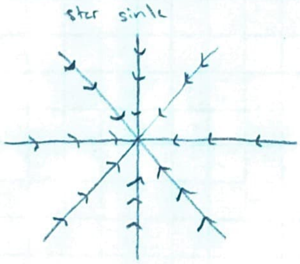
$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -k/m \end{pmatrix}$

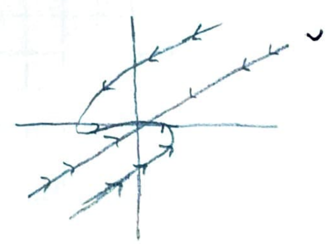


case $b^2 - 4km = 0$:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -b/2m$$ this is a star sink at the origin or a degenerate case:

star sink





case $b^2 - 4km > 0$:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4km}}{2m}$$ let $\lambda_1 = \frac{-b + \sqrt{b^2 - 4km}}{2m}$ $\lambda_2 = \frac{-b - \sqrt{b^2 - 4km}}{2m}$

Since $b, k, m > 0$, we know that $-b < \sqrt{b^2 - 4km}$ so both $\lambda_1, \lambda_2 < 0$

Since λ_1 and λ_2 are less than 0, the origin must be stable and all trajectories approach it (a sink). Here is the general picture:

let v_1 be the e-vector paired with λ_1 and v_2 be the e-vector paired with λ_2. Note that $\lambda_1 > \lambda_2$.

V_1 is the slow direction

V_2 is the fast direction