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Dynamics I

HW 3

3.4.6 Discuss $\dot{x} = rx - \frac{x}{1+x}$ and draw the bifurcation diagram

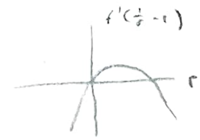
$\dot{x} = 0 = x(r - 1/(1+x))$ $\qquad 0 = r - 1/(1+x) \Rightarrow \frac{1}{1+x} = r \Rightarrow x+1 = \frac{1}{r}$

$x = \frac{1}{r} - 1$

$f(x) = rx - \frac{x}{1+x} \qquad f'(x) = r - \frac{1+x-x}{(1+x)^2} = r - \frac{1}{(1+x)^2}$

$x^* = 0$: $f'(0) = r - \frac{1}{(1+0)^2} = r - 1 < 0 \Rightarrow x^* = 0$ is stable for $r > 1$

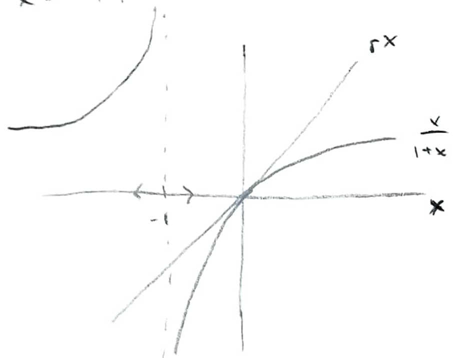
$x^* = \frac{1}{r} - 1$: $f(\frac{1}{r} - 1) = r - 1/(1 + \frac{1}{r} - 1)^2 = r - r^2 = r(1-r) \Rightarrow$

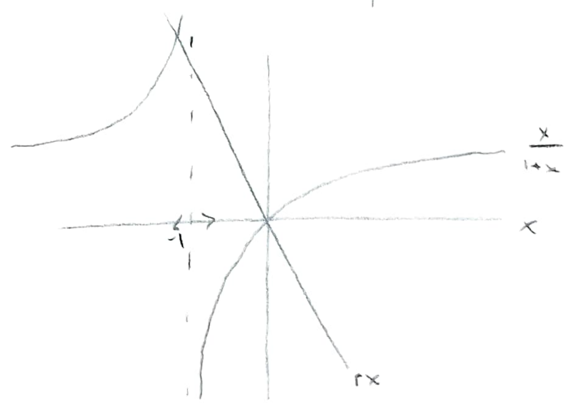


Case $r < 0 \Rightarrow$ stable $\qquad$ Case $r \in (0,1) \Rightarrow$ unstable

Case $r > 1 \Rightarrow$ stable

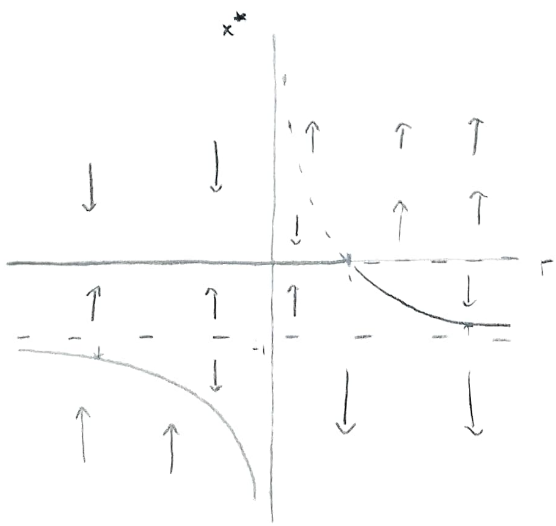
Seperately from r, we must also have an unstable equilibria at $x^* = -1$:





There must always exist a small unstable window about $x = -1$ since r must be finite and the limits on each side of $x = -1$ approach $\mp\infty$ respectively.

Thus our bifurcation diagram is:



So we have a <u>transcritical</u> bifurcation at $r = 1$

3.4.8 | Discuss $\dot{x} = rx - \frac{x}{1+x^2}$ and draw a bifurcation diagram

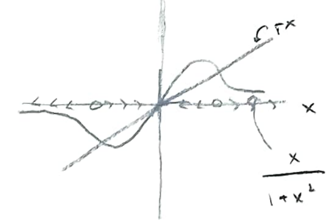
$\dot{x} = 0 : x\left(r - \frac{1}{1+x^2}\right)$ Solving equilibria $0 = r - \frac{1}{1+x^2} \Rightarrow r = \frac{1}{1+x^2} \Rightarrow x^2+1 = \frac{1}{r}$

$x = \pm\sqrt{\frac{1}{r} - 1}$

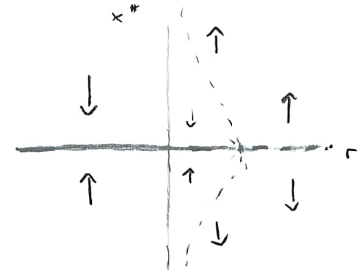
$f(x) = rx - \frac{x}{1+x^2}$ $f'(x) = r - \frac{1+x^2-2x^2}{1+x^2} = r - \frac{1-x^2}{1+x^2}$

Case $x^* = 0$: $f'(0) = r - 1 < 0$ when $r < 1$

Case $x^* = \pm\sqrt{1/r - 1}$ only exists when $r \in (0,1)$.
the picture shows that $x^* = \pm\sqrt{1/r - 1}$ are both unstable

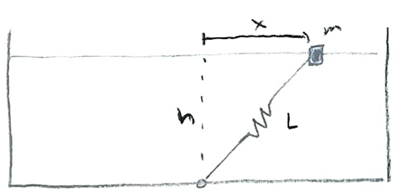


Thus we have the bifurcation diagram:



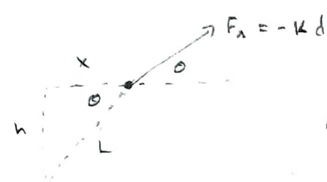
A transcritical bifurcation!
(we also discussed a bifurcation from infinity in class)

3.5.4 | Bead on a horizontal wire



A bead w/ mass m is constrained along a horizontal wire. A spring of relaxed length L_0 and spring constant k. Additionally there is viscous damping force $b\dot{x}$

(a) Free body diagram of relevant forces for bead:



where $d = L - L_0$
$= \sqrt{x^2+h^2} - L_0$

$F_s = -k\left(\sqrt{x^2+h^2} - L_0\right) = k\left(L_0 - \sqrt{x^2+h}\right)$ but we must project F_s onto the direction of the wire: $F_s \cos\theta = F_s\, x/L = k\left(\frac{L_0}{L} - 1\right)x$

So Newton's law says $m\ddot{x} = -b\dot{x} + k\left(\frac{L_0}{\sqrt{x^2+h^2}} - 1\right)x$

(b) Find all equilibria. To do this, we set $\ddot{x} = \dot{x} = 0$. Then

$0 = k\left(\frac{L_0}{\sqrt{x^2+h^2}} - 1\right)x$ (certainly $x^* = 0$ is an equilibria)

$0 = \frac{L_0}{\sqrt{x^2+h^2}} - 1 \Rightarrow 1 = \frac{L_0}{\sqrt{x^2+h^2}} \Rightarrow \sqrt{x^2+h^2} = L_0 \Rightarrow x^2 = L_0^2 - h^2$

$x = \pm\sqrt{L_0^2 - h^2}$

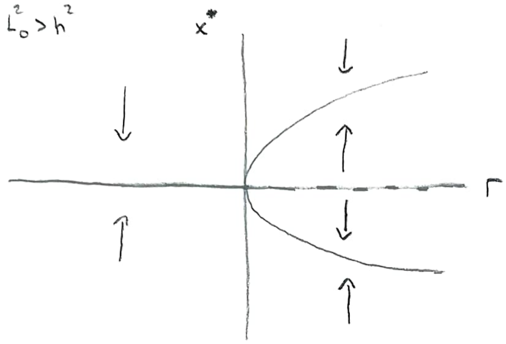
So we have equilibria at $x^* = 0, \pm\sqrt{L_0^2 - h^2}$

(c) Now suppose $m=0$, Classify all the fixed points and draw a bifurcation diagram.

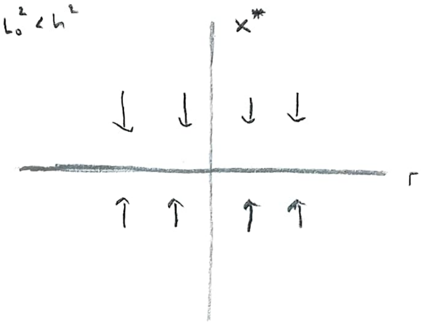
Since $m=0$, $\quad b\dot{x} = k\left(\frac{L_0}{\sqrt{x^2+h^2}} - 1\right)x \Rightarrow \dot{x} = \frac{k}{b}\left(\frac{L_0}{\sqrt{x^2+h^2}} - 1\right)x$

We already know that equilibria are $x^* = 0, \pm\sqrt{L_0^2 - h^2}$. Let $r = L_0^2 - h^2$. Then we can sketch the bifurcation diagram for two cases:

case $L_0^2 > h^2$



Case $L_0^2 < h^2$



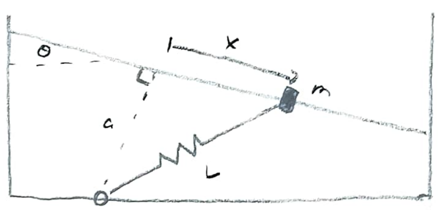
$x^* = 0$ is stable for $r<0$ and unstable for $r>0$.
$x^* = \pm\sqrt{L_0^2 - h^2}$ do not exist for $r<0$ but are stable for $r>0$.

(d) If $m \neq 0$, how small must m be to be considered negligible?

$m\ddot{x} = -b\dot{x} + k\left(\frac{L_0}{\sqrt{x^2+h^2}} - 1\right)x \Rightarrow m\ddot{x} + b\dot{x} - k\left(\frac{L_0}{\sqrt{x^2+c^2}} - 1\right)x$

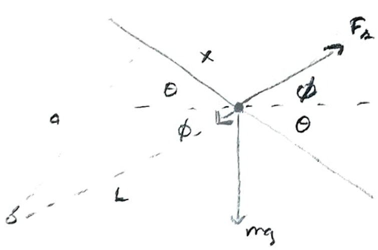
$\Rightarrow \frac{m}{b}\ddot{x} + \dot{x} - \frac{k}{b}\left(\frac{L_0}{\sqrt{x^2+h^2}} - 1\right)x$. Our apprx. is valid for $m \ll b$

<u>3.6.5</u> We now consider a bead on a tilted wire



spring constant k, spring rest length L_0,

(a) First we construct Newton's law: $\quad m\ddot{x} = F_g + F_s\}$ subject to projections



To project gravity onto the wire, multiply by $\cos(\pi/2 - \theta)$

$F_g = mg\cos(\pi/2 - \theta) = mg\sin\theta$

We must project the spring onto the wire with $F_s \cos(\phi+\theta) = F_s\, x/L = F_s\, x/(\sqrt{a^2+x^2} - L_0)$

Further, $F_s = -k\left(\sqrt{a^2+x^2} - L_0\right)$

So we have our equation $\quad m\ddot{x} = mg\sin\theta + k\left(\frac{L_0}{\sqrt{a^2+x^2}} - 1\right)x$

To find equilibria, we set $\ddot{x} = 0$:

$$0 = mg\sin\theta + k\left(\frac{L_0}{\sqrt{a^2+x^2}} - 1\right)x \Rightarrow mg\sin\theta = k\left(1 - \frac{L_0}{\sqrt{a^2+x^2}}\right)x$$

which is what ~~we~~ needed to show.

(b) Show that equilibria can be written in dimensionless form as $1 - h/u = R/\sqrt{1+u^2}$

First note the units: $m \to kg$ $\quad g \to N/kg$ $\quad k \to N/m$
$a \to m$ $\quad L_0 \to m$ $\quad x \to m$

Then

$$\frac{mg}{k}\sin\theta = \left(1 - \frac{L_0}{\sqrt{a^2+x^2}}\right)x = \left(1 - \frac{L_0}{a\sqrt{1+(\frac{x}{a})^2}}\right)x$$

let $u = \frac{x}{a} \Rightarrow x = au$ and $R = L_0/a$

$$\frac{mg}{k}\sin\theta = \left(1 - \frac{R}{\sqrt{1+u^2}}\right)au \Rightarrow \frac{mg}{kau}\sin\theta = 1 - \frac{R}{\sqrt{1+u^2}}$$

let $h = \frac{mg}{ka}\sin\theta$, we then have

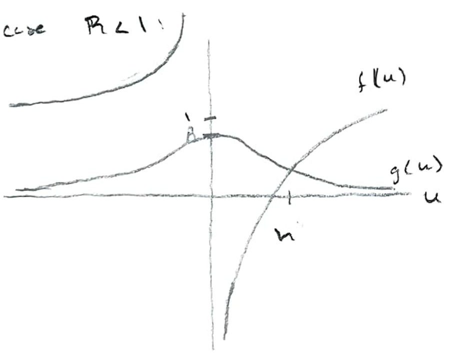
$$\frac{h}{u} = 1 - \frac{R}{\sqrt{1+u^2}} \Rightarrow 1 - h/u = \frac{R}{\sqrt{1+u^2}}$$

also note that $h = \frac{mg}{ka}\sin\theta$, $R = L_0/a$, and $u = \frac{x}{a}$ are all dimensionless

(c) Give a graphical analysis of the dimensionless equation for $R<1$ and $R>1$

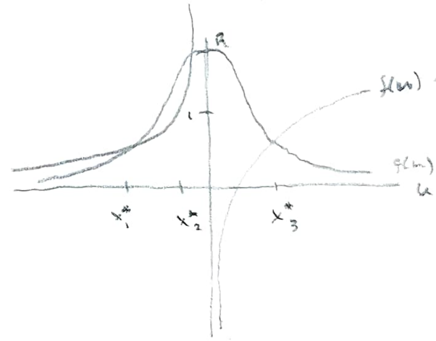
let $f(u) = 1 - h/u$ and $g(u) = \frac{R}{\sqrt{1+u^2}}$

case $R<1$:



A single equilibria

case $R>1$: (w/ sufficiently small h)



3 equilibria

(d) Let $r = R-1$. Show that the equilibrium equation reduces to $h + ru - \frac{1}{2}u^3 \approx 0$ for small r, h, u

From $1 - h/u = R/\sqrt{1+u^2}$ we get $1 - h/u = r-1/\sqrt{1+u^2} \Rightarrow \sqrt{1+u^2}(u-h) = ur - u$

the Taylor expansion for $\sqrt{1+u^2}$ is $1 + \frac{u^2}{2} + O(u^4)$

so $\sqrt{1+u^2}(u-h) \approx (1+\frac{u^2}{2})(u-h) = u - \frac{hu^2}{2} + \frac{u^3}{2} - h$

Then we can say that $-h - u + \frac{u^3}{2} - \frac{hu^2}{2} \approx ur - u \Rightarrow \frac{-hu^2}{2} \approx h + ur - \frac{u^3}{2}$

But $-hu^2/2 \approx 0$ so $0 \approx h + ur - \frac{u^3}{2}$

(e) Find an approximate formula for the saddle-node bifurcation curves in the limit of small r, h, and u.

$f(u) = h + ur - \frac{1}{2}u^3$ $\quad f'(u) = r - \frac{3}{2}u^2$

So $r = \frac{3}{2}u^2$. This implies that $0 = h + u\left(\frac{3}{2}u^2\right) - \frac{1}{2}u^3 = h + u^3$
$\Rightarrow h = -u^3$

Our approximate bifurcation curves are $h = -u^3$ and $r = \frac{3}{2}u^2$

(f) Find the exact equations for the bifurcation curves

We have $1 - h/u = R/\sqrt{1+u^2}$ which we can implicitly derive to obtain another eqn. for a saddle node bifurcation

$$\frac{d}{du}\left(1 - h/u\right) = \frac{d}{du}\left(R/\sqrt{1+u^2}\right) \rightarrow hu^{-2} = -uR\,(1+u^2)^{-3/2}$$

So we have (1) $1 - h/u = R/\sqrt{1+u^2}$ and (2) $hu^{-2} = -uR(1+u^2)^{-3/2}$

From (2), we gather that $h = -u^3 R\,(1+u^2)^{-3/2}$, Then using (1) we find that

$$1 + u^2 R\left(1+u^2\right)^{-3/2} = R\,(1+u^2)^{-1/2} \Rightarrow (1+u^2)^{3/2} + u^2 R = R\,(1+u^2)$$
$$\Rightarrow (1+u^2)^{3/2} = R + Ru^2 - Ru^2 = R$$

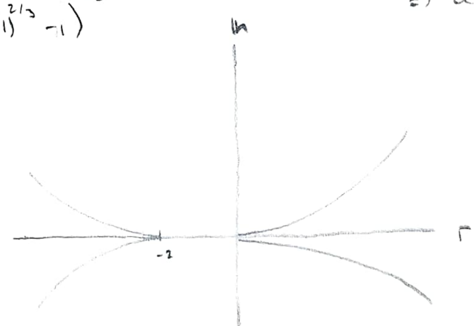
So $R = (1+u^2)^{3/2}$ and $h = -u^3 R R^{-1} = -u^3$

(g) Give a numerically accurate plot of the bifurcation curves in the (r, h) plane

$R = r+1 = (1+u^2)^{3/2} \Rightarrow r(u) = (1+u^2)^{3/2} - 1$ and $h(u) = -u^3$

To graph this, we can solve for u: $r+1 = (1+u^2)^{3/2}$
$\Rightarrow (r+1)^{2/3} = 1 + u^2$
$\Rightarrow u^2 = (r+1)^{2/3} - 1$
$\Rightarrow u = \pm\sqrt{(r+1)^{2/3} - 1}$

then $h(u) = -u^3 = \pm\left((r+1)^{2/3} - 1\right)^{3/2}$

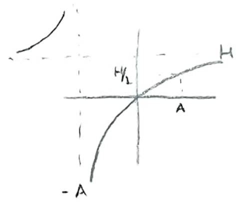


(h) In terms of the original problem, as the distance the spring is stretched to reach the wire increases, the gravitational force must be stronger to reach a saddle bifurcation

3.7.4 Consider the model for the population of fish N in a fishery

$$\dot{N} = rN(1 - N/k) - H\frac{N}{A+N} \qquad \text{where} \qquad H, A > 0$$

(a) Biologically, A is the half-saturation value for catching fish:



let $x(N) = \frac{HN}{A+N}$, then $x(A) = \frac{HA}{A+A} = \frac{HA}{2A} = \frac{H}{2}$

(b) Show that the system can be rewritten in dimensionless form

$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x} \qquad \text{for suitable } \tau, x, h, a$$

Beginning with

$$\dot{N} = rN(1 - N/k) - H\frac{N}{A+N}$$

we divide by r:

$$\frac{1}{r}\dot{N} = N(1 - N/k) - \frac{H}{r}\frac{N}{A+N} \qquad \text{let } x = \frac{N}{k} \Rightarrow N = kx$$

$$\dot{x} = \frac{\dot{N}}{k} \Rightarrow \dot{N} = k\dot{x}$$

$$\frac{k}{r}\dot{x} = kx(1-x) - \frac{H}{r}\frac{kx}{A+kx} = kx(1-x) - \frac{H}{r}\frac{\cancel{k}}{\cancel{k}}\frac{x}{\frac{A}{k}+x}$$

$$\frac{1}{r}\dot{x} = x(1-x) - \frac{H}{rk}\frac{x}{A/k+x} \qquad \text{let } h = H/rk, \ a = A/k$$

$$(*) \qquad \frac{1}{r}\dot{x} = x(1-x) - h\frac{x}{a+x} \qquad \text{let } \tau = rt \Rightarrow t = \tau/r$$

$$\frac{dt}{d\tau} = 1/r$$

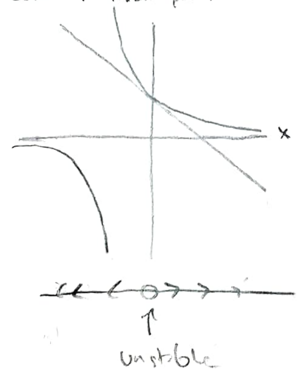
then let $x' = \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \dot{x}r$.
substituting $x' = \dot{x}r$ into $(*)$, we get

$$\frac{dx}{d\tau} = x' = x(1-x) - h\frac{x}{a+x}$$

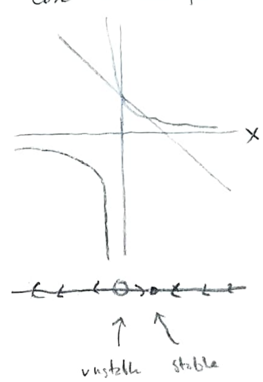
(c) Show that the system can have 1, 2, or 3 fixed points (depending on a, h).
Classify their stability

$$\frac{dx}{d\tau} = x' = 0 = x(1-x) - h\frac{x}{a+x} = x\left(1 - x - \frac{h}{a+x}\right)$$

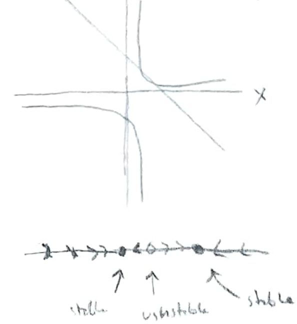
Case 1 fixed point



Case 2 fixed points



Case 3 fixed points
(recall $x^* = 0$)



(d) Analyze the dynamics near $x=0$, show a bifurcation occurs when $h=a$

let $f(x) = x(1-x) - \frac{hx}{a+x}$ $\quad f'(x) = 1-x-x - \frac{(a+x)h - hx}{(a+x)^2} = 1-2x - \frac{ha}{(a+x)^2}$

We know $x^* = 0$ to be a fixed point, when is it stable?

$f'(0) = 1 - \frac{ha}{a^2} = 1 - \frac{h}{a} < 0$ when $a < h$. when $a = h$, $f'(0) = 0$
which implies a saddle bifurcation!

(e) Show that another bifurcation occurs when $h = \frac{1}{4}(a+1)^2$ for $a < a_c$.
Classify this bifurcation

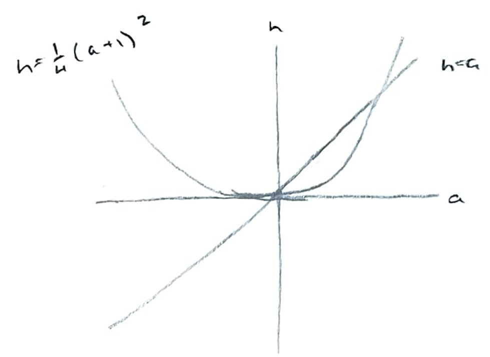
Spose $h = \frac{1}{4}(a+1)^2$, then $0 = x(1-x) - \frac{1}{4}(a+1)^2 \frac{x}{a+x} = x\left(1-x-\frac{(a+1)^2}{4(a+x)}\right)$

$0 = 1-x-\frac{(a+1)^2}{4(a+x)} \Rightarrow (a+1)^2 = 4(a+x)(1-x) \Rightarrow a^2+2a+1 = 4(a+x-ax-x^2)$
$a^2+2a+1 = -4x^2 - 4ax + 4x + 4a$

$0 = a^2 - 2a + 4ax + 1 - 4x + 4x^2 = a^2 - 2a(1-x) + 1 - 4x + 4x^2$ $\quad (\star)$

$(\star)$ can be solved for a with the quadratic formula.
This will give a pitchfork bifurcation because of $\pm$.

(f) we now plot stability in the (a,h) plane



3.7.6 We model an epidemic where

$x(t)$ = number of healthy people
$y(t)$ = number of sick people
$z(t)$ = number of dead people

$\dot{x} = -kxy$
$\dot{y} = kxy - ly$
$\dot{z} = ly$

$k, l \in \mathbb{R}^+$

(a) Show that $x+y+z$ is constant.

$x+y+z = \int \dot{x} + \dot{y} + \dot{z}\, dt = \int -kxy + kxy - ly + ly\, dt = \int 0\, dt = N$ a constant

(b) Use $\dot{x}$ and $\dot{z}$ to show $x(t) = x_0 \exp(-k z(t)/l)$ where $x_0 = x(0)$

$\dot{x} = -kxy$ and $\dot{z} = ly \Rightarrow y = \dot{z}/l$

$x(0) = e^{0+c} = x_0$

$\dot{x} = -kx\dot{z}/l \Rightarrow \int \frac{1}{x}\, dx = \int -\frac{k}{l}\dot{z}\, dt \Rightarrow \ln x = -\frac{k}{l} z + C \Rightarrow x = e^{-\frac{k}{l}z + c}$

$\Rightarrow x(t) = x_0 \exp\left(-k z(t)/l\right)$

(c) Show z satisfies $\dot{z} = l(N - z - x_0 \exp(-kz/l))$

$\dot{z} = ly$ but $y = N - z - x = N - z - x_0 \exp(-kz/l)$

$\dot{z} = l\left(N - z - x_0 \exp(-kz/l)\right)$

(d) Show $\dot{z} = l\left[N - z - x_0 \exp(-kz/l)\right]$ can be non-dimensionalized to

$\frac{dz}{d\tau} = a - bu - e^{-u}$

we begin with $\dot{z} = l\left[N - z - x_0 \exp(-kz/l)\right]$

$\frac{1}{lx_0}\dot{z} = N/x_0 - \frac{1}{x_0} z - \exp(-kz/l)$ let $u = \frac{kz}{l} \Rightarrow z = \frac{lu}{k}$

$\dot{u} = \frac{k}{l}\dot{z} \Rightarrow \dot{z} = \frac{l\dot{u}}{k}$

$\frac{1}{kx_0}\dot{u} = N/x_0 - \frac{l}{kx_0} u - e^{-u}$

now let $a = N/x_0$ and $b = l/kx_0$; $\frac{1}{kx_0}\dot{u} = a - bu - e^{-u}$

also let $\tau = \frac{1}{kx_0} t \Rightarrow t = kx_0\tau \Rightarrow \frac{dt}{d\tau} = kx_0$

$\frac{du}{dt} = \frac{du}{d\tau}\frac{d\tau}{dt} = \frac{du}{dt} kx_0 = kx_0 u'$ (where $u' = \frac{du}{d\tau}$)

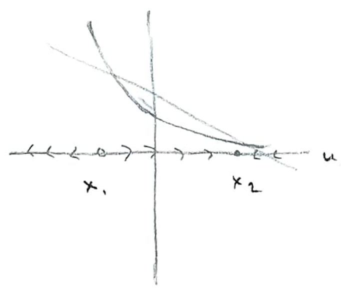
We then have $u' = a - bu - e^{-u}$

(e) Show that $a \geq 1$ and $b > 0$

$a = N/x_0$: We know that $x, y, z \geq 0$ and $x + y + z = N$ $\forall t \in \mathbb{R}^+$
So it must be true that $x_0 \leq N \Rightarrow N/x_0 \geq 1 \Rightarrow a \geq 1$

$b = l/kx_0$: We know $l, k > 0$ and $x_0 > 0$ since we assume everyone (except initial patients) to be healthy. So $b > 0$

(f) Determine the number of fixed points and classify their stability

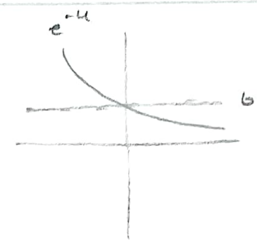


Since $a \geq 1$ and $b > 0$, there must be two fixed points x_1 and x_2 ($x_1 < x_2$). x_1 is unstable and x_2 is stable

(g) Show that the max of $\dot{u}(t)$ occurs at the same time as the max of $\dot{z}(t)$ and $y(t)$

$\dot{u}(t) = f(u) = a - bu - e^{-u} \Rightarrow f'(u) = -b + e^{-u} = 0 \Rightarrow u = -\ln(b)$

but $u = \frac{k}{l} z \Rightarrow u = \frac{k}{l} z = -\ln(b) \Rightarrow z = -\frac{l}{k}\ln(b) = -\frac{l}{k}\ln\left(\frac{l}{kx_0}\right)$



We know $z = -\frac{l}{k}\ln(b)$ gives a maximum because $b>0$ and zeros occur on $0 = e^{-u} - b$

Where do the maxes of $\dot{z}$ and y occur?

$$\dot{z} = l(N - z - x_0 \exp(-kz/l)) = g(u)$$

$$g'(u) = -l + kx_0 \exp(-kz/l) = 0 \Rightarrow kx_0 \exp(-kz/l) = l$$

$$\exp(-kz/l) = l/kx_0$$

$$-\frac{k}{l} z = \ln\left(\frac{l}{kx_0}\right)$$

$$z = -\frac{l}{k}\ln\left(\frac{l}{kx_0}\right)$$

For $\dot{y} = kxy - ly = y(kx - l) = 0$

$$kx - l = kx_0 \exp(-kz/l) - l = 0 \Rightarrow z = -\frac{l}{k}\ln\left(\frac{l}{kx_0}\right)$$

So all three of $\dot{u}, y, \dot{z}$ have critical points at $z = -\frac{l}{k}\ln\left(\frac{l}{kx_0}\right)$. We already have shown that $\dot{u}$ is a maximum. Note that $g'(u)$ appears in the derivations of critical points for $\dot{z}$ and y. Thus consider

$$g'\left(-\frac{l}{k}\ln\left(\frac{l}{kx_0} \pm \varepsilon\right)\right) = kx_0 \cancel{\exp}\left(-\cancel{\frac{k}{l}}\frac{l}{k}\cancel{\ln}\left(\frac{l}{kx_0} \pm \varepsilon\right)\right) - l = l \pm \varepsilon kx_0 - l = \pm \varepsilon kx_0$$

for small ε. Since $k, x_0 > 0$, adding ε (which decreases z from the critical value) means the derivative is positive. This is analogous for subtracting ε. So $z = -\frac{l}{k}\ln\left(\frac{l}{kx_0}\right)$ is a maximum.

(h) Show that if $b<1$, then $\dot{u}(t)$ is increasing at $t=0$, reaches a max at some t_{peak}. Also show $\dot{u}(t)$ goes to 0 eventually

$$\dot{u}(t) = f(u) = a - bu - e^{-u} \Rightarrow f'(u) = -b + e^{-u} > -1 + e^{-u}$$

$$f'(0) = -b + e^{-u} = -b + 1 > -1 - 1 = 0$$

so $\dot{u}$ is increasing at $t=0$. The max occurs @ $u = -\ln(b)$ as previously shown. As $u \to \infty$, $f'(u) = -b < 0$ so $\dot{u}$ decreases as $t \to \infty \Rightarrow \dot{u} = 0$ in the real world.

(i) Show $t_{peak} = 0$ if $b>1$. Recall $f'(u) = -b + e^{-u}$

$$f'(u) = -b + e^{-u} < -1 + e^{-u}, \qquad f'(0) = -b + e^{0} = -b + 1 < -1 + 1 = 0$$

So $\dot{u}$ is decreasing at $t=0$. Additionally $f'(u) = 0$ when $u = -\ln(b)$ but since $b>1 \Rightarrow u<0$. But u can't be negative in the physical context of the problem. So $f'(u) < 0$ on $[0, \infty)$ and $t_{peak} = 0$ for $b>1$

(j) Give a biological interpretation of b.

We can consider b to be a parameter that governs how effective the virus transmits itself. High b values mean the disease struggles to spread but low b values imply a highly contagious disease.

(k) How could this model fit aids?

AIDS is a much slower spreading/acting disease. To better model this, we should include a variable population in our differential equations.