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Dynamics
HW 10

8.1.8 Given the equation for

a) Find and classify all bifurcations as ε and γ vary. let $d\phi/d\tau = \phi'$ and $\phi' = \omega$, then

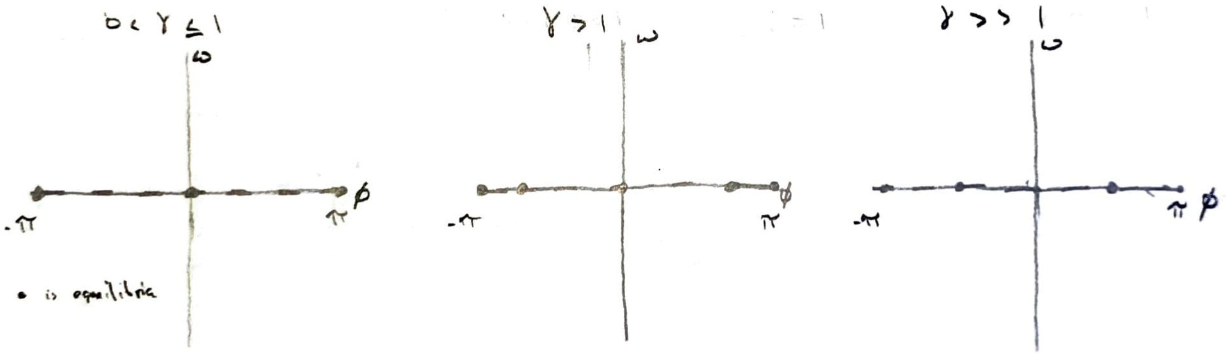
$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma \sin\phi\cos\phi \quad \text{becomes the system}$$

$$\phi' = \omega$$
$$\omega' = 1/\varepsilon\left[-\omega + \sin\phi + \gamma\sin\phi\cos\phi\right]$$

The equilibria must occure when $\phi' = 0 = \omega$ and $\omega' = 0$.
$\omega' = 0$ when
$$0 = 1/\varepsilon\left[-\omega + \sin\phi + \gamma\sin\phi\cos\phi\right]$$
$$0 = 1/\varepsilon\left(\sin\phi + \gamma\sin\phi\cos\phi\right)$$
$$0 = \frac{\sin\phi}{\varepsilon}\left(1 + \gamma\cos\phi\right)$$

So $\omega' = 0$ when $\sin\phi = 0$ or if $\cos\phi = 1/\gamma$ has a solution
$\sin\phi = 0$ when $\phi = 0, \pi$. Then we can plot the null clines
as γ varies: (ϕ: - - -, ω: ——)

I messed up, these are not the null clines for ω, just the intersection points. It is the ϕ-null-cline though!



This tells us that a bifurcation occurs at $\gamma = 1$. But what type?

$$J = \begin{bmatrix} 0 & 1 \\ 1/\varepsilon(\cos\phi + \gamma\cos^2\phi - \gamma\sin^2\phi & -1/\varepsilon \end{bmatrix}$$

$\tau = -1/\varepsilon < 0$ so we can only have sinks, saddles, and centers

Then for each equilibria,

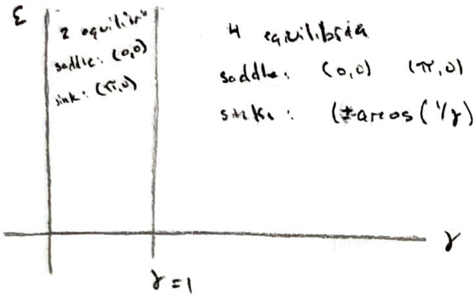
$(0,0)$: $J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \frac{1+\gamma}{\varepsilon} & -1/\varepsilon \end{bmatrix}$ $\tau < 0$ and $\Delta = \frac{-(1+\gamma)}{\varepsilon} < 0$

$(\pi, 0)$: $J_{(\pi,0)} = \begin{bmatrix} 0 & 1 \\ \frac{-1+\gamma}{\varepsilon} & -1/\varepsilon \end{bmatrix}$ $\tau < 0$ and $\Delta = \frac{1-\gamma}{\varepsilon}$

At $\gamma = 1$, the determinant of the Jacobian at $(\phi, \omega) = (\pi, 0)$ changes from positive to negative. Since $\tau = -1/k < 0$ this means that $(\pi, 0)$ changes from a sink to a saddle. We also know two more equilibria occur for $\gamma \geq 1$. So the bifurcation is a supercritical bifurcation.

b) Plot the stability diagram in the γ, ε plane (positive quadrant)

The only bifurcation occurs at $\gamma = 1$:



8.1.10) Say we have
$$\dot{S} = r_s S\left(1 - \frac{S}{K_s}\cdot\frac{K_E}{E}\right)$$
$$\dot{E} = r_E E\left(1 - \frac{E}{K_E}\right) - P\frac{B}{S}$$

where B is a constant bug population and $r_E, r_S, K_E, K_S, P > 0$

a) Give a biological interpretation of the system.

We can rewrite $\dot{S}$ as $r_s S\left(1 - \frac{S}{h(E)}\right)$ where $h(E) = K_S E / K_E$ then S' is just a plain old logistic equation with saturating value $h(E)$.

On the other hand, $\dot{E}$ is a logistic equation but we subtract PB/S which is the ratio between the number of worms and the size of the trees.

b) Nondimensionalize the system.

let $s = S/K_S \Rightarrow K_S s = S \Rightarrow K_S \dot{s} = \dot{S}$
$e = E/K_E \Rightarrow K_E e = E \Rightarrow K\dot{e} = \dot{E}$

then the equations become

$$K_S \dot{s} = r_s K_S s(1 - s/e) \Rightarrow \dot{s} = r_s s(1 - s/e)$$
$$K_E \dot{e} = r_E K_E e(1-e) - P\, B/K_S s \Rightarrow \frac{1}{r_E}\dot{e} = e(1-e) - PB/r_E K_E K_S s$$

then let $B' = PB/r_E K_E K_S$ and $\tau = r_E t$

then $\dot{s} = \frac{ds}{dt} = \frac{ds}{d\tau}\frac{d\tau}{dt} = s' r_E$ and $\dot{e} = \frac{de}{dt} = \frac{de}{d\tau}\frac{d\tau}{dt} = e' r_E$

with the time substitutions, we have

$$\begin{aligned} r_e s' &= r_s s(1 - s/e) \\ e' &= e(1-e) - B'/s \end{aligned} \quad \Rightarrow \quad \begin{aligned} s' &= R s(1 - s/e) \\ e' &= e(1-e) - B'/s \end{aligned} \qquad \text{where } R = r_s/r_e$$

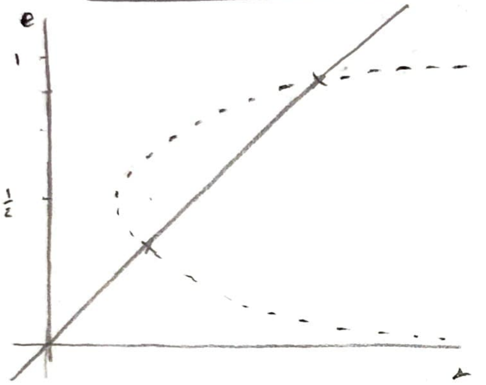
c) Sketch the null clines (show small B ⇒ two fixed points, large B ⇒ no fixed points)

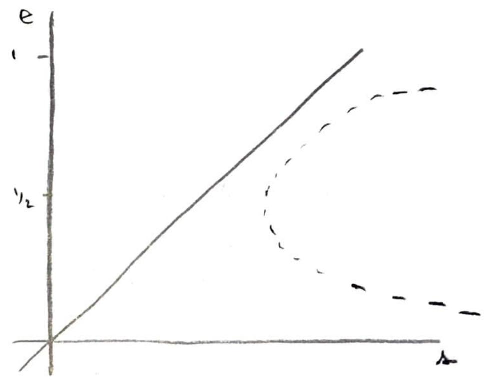
$s' = 0$ when $0 = R s(1 - s/e)$ so $s = 0, e$

$e' = 0$ when $0 = e(1-e) - B'/s$ then we must have $s = \dfrac{B'}{e(1-e)}$

Then we have the following null cline portraits

small B (which implies small B') | large B





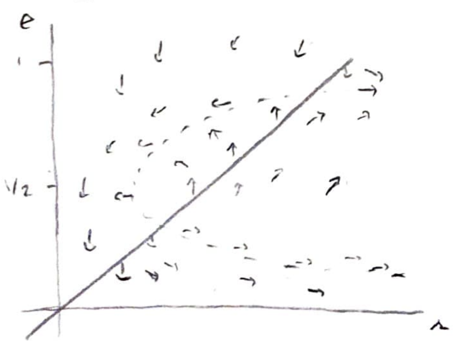
s cline: ———

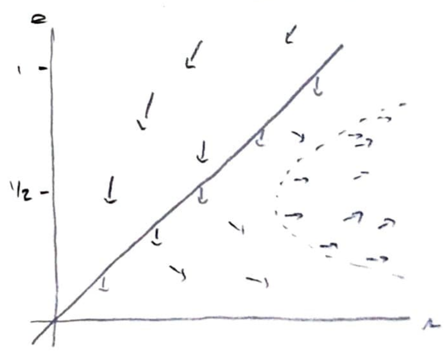
e cline: - - -

For small B, there are two intersections of the null clines and for large B there are no intersections. Therefore at some critical B-value, a saddle node bifurcation occurs

d) Sketch the phase portrait for both small and large values of B.

small B | large B





given s, we can find an e small enough st the direction is ↘

same applies here

8.1.13 Consider the laser model $\dot{n} = GnN - kn$, $\dot{N} = -GnN - fN + p$ where $G, k, f > 0$ and p can be + or −

$n(t)$ denotes the number of photons
$N(t)$ denotes the number of excited atoms
G is the gain coefficient
k is the decay rate from mirror transmission
f is the decay rate from spontaneous emission
p is the pump strength

a) Nondimensionalize the system

Note that GN has the same units as k and Gn has the same units as f. Then let

$$x = \tfrac{G}{f} n \Rightarrow n = \tfrac{f}{G} x \Rightarrow \dot{n} = \tfrac{f}{G}\dot{x}$$
$$y = \tfrac{G}{k} N \Rightarrow N = \tfrac{k}{G} y \Rightarrow \dot{N} = \tfrac{k}{G}\dot{y}$$

and our system can be rewritten as

$$\tfrac{f}{G}\dot{x} = G\left(\tfrac{f}{G}x\right)\left(\tfrac{k}{G}y\right) - k\tfrac{f}{G}x \qquad \Rightarrow \qquad \dot{x} = kxy - kx = kx(y-1)$$
$$\tfrac{k}{G}\dot{y} = -G\left(\tfrac{f}{G}x\right)\left(\tfrac{k}{G}y\right) - f\tfrac{k}{G}y + p \qquad \quad \tfrac{1}{f}\dot{y} = -xy - y + p\tfrac{G}{kf}$$

then let $\tau = t$

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = x' f \qquad \dot{y} = \frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} = y' f$$

and we can write the system as $x' = Fx(y-1)$ where $F = fk$
$y' = -xy - y + P$ where $P = pG/kf$

b) Find and classify all the fixed points.

$x' = 0$ when $0 = Fx(y-1)$ so $x = 0,\ y = 1$
$y' = 0$ when $0 = -xy - y + P$
$xy + y = P$
$y = P/(x+1)$

null clines

these are the fixed points but they might not have a physical interpretation

Then we have the fixed points $(0, P)$ and $(P-1, 1)$. To classify their stability, we compute the Jacobian

$$J = \begin{bmatrix} F(y-1) & Fx \\ -y & -x-1 \end{bmatrix} \qquad J_{(0,P)} = \begin{bmatrix} F(P-1) & 0 \\ -P & -1 \end{bmatrix} \qquad \begin{aligned}\tau &= F(P-1) - 1 \\ \Delta &= -F(P-1)\end{aligned}$$

for $P < 1$: sink $(\Delta > 0,\ \tau < 0)$
for $P > 1$: saddle $(\Delta < 0)$

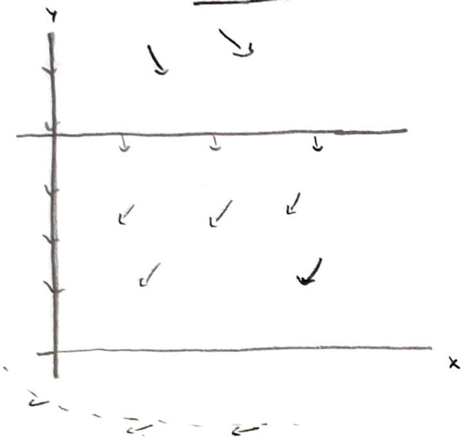
$$J_{(P-1,1)} = \begin{bmatrix} 0 & F(P-1) \\ -1 & -P \end{bmatrix} \qquad \begin{aligned}\tau &= -P \\ \Delta &= F(P-1)\end{aligned}$$

for $P < 1$: saddle $(\Delta < 0)$
for $P > 1$: sink $(\Delta > 0,\ \tau < 0)$

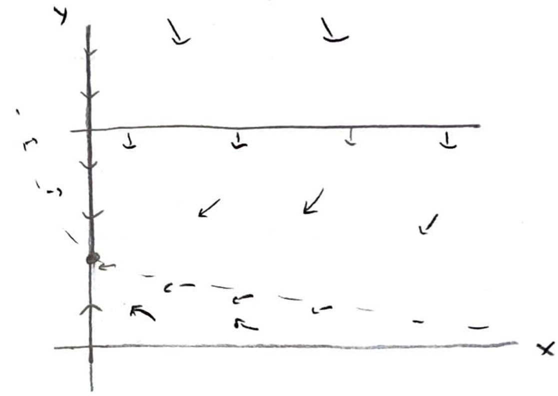
c) Sketch all the qualitatively different phase portraits that occur as the dimensionless parameters are varied.

From 6, we found that the critical parameter is P and F does not matter so much

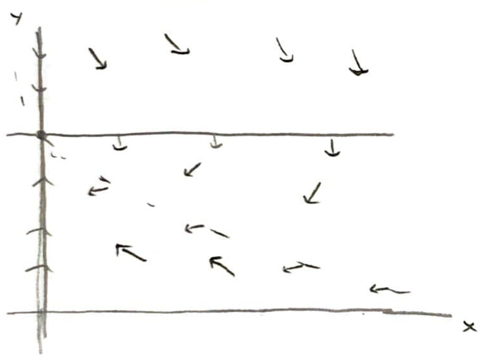
$P \leq 0$



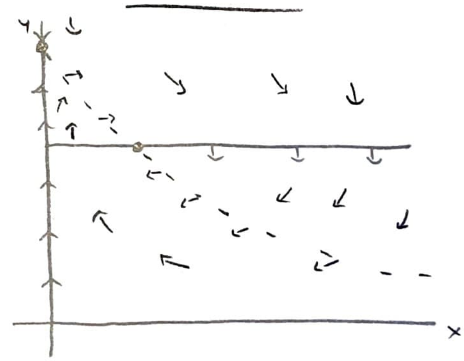
$0 < P < 1$



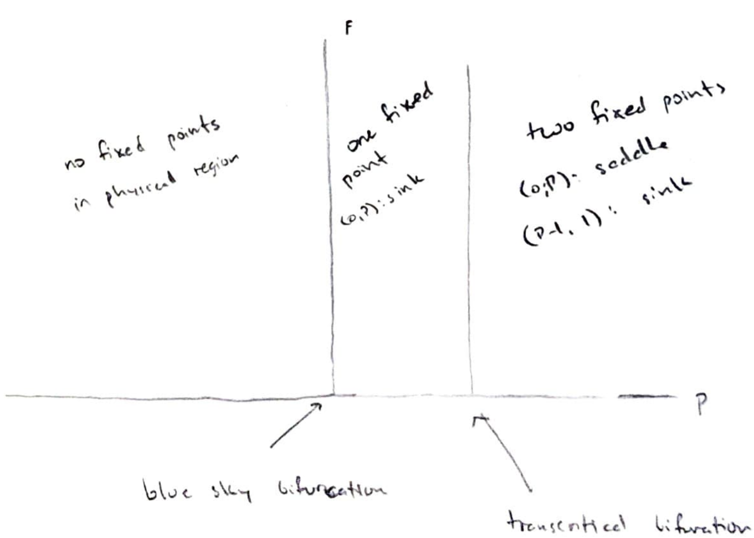
$P = 1$



$P > 1$



d) Plot the stability diagram for the system. What types of bifurcations occur



8.1.14 | Consider

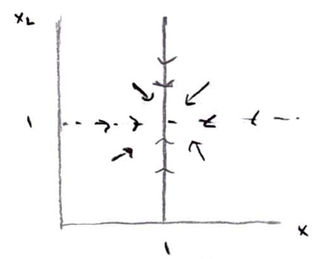
$$\dot{x}_1 = -x_1 + F(I - b x_2) \qquad \text{where} \qquad F(y) = 1/1+e^{-y}$$
$$\dot{x}_2 = -x_2 + F(I - b x_1)$$

I is the strength of the input stimulus
b is the strength of the mutual antagonism

a) sketch the phase plane for various values of I and b (both positive).

case $b \ll 1$, $I \gg 1$

x_1 cline: $0 = -x_1 + 1/1+e^{bx_2 - I} \approx -x_1 + 1 \Rightarrow x_1 = 1$ —
x_2 cline: $0 = -x_2 + 1/1+e^{bx_1 - I} \approx -x_2 + 1 \Rightarrow x_2 = 1$ - - -



case $b \gg 1$, $I \ll 1$

x_1 cline: $0 = -x_1 + 1/1+e^{bx_2 - I} \approx -x_1 \Rightarrow x_1 = 0$ —
x_2 cline: $0 = -x_2 + 1/1+e^{bx_1 - I} \approx -x_2 \Rightarrow x_2 = 0$ - -

b) Show that the symmetric fixed point $x_1^* = x_1^* = x^*$ is always a solution.

We must show the existence of some x^* s.t. $x^* = x_1^* = x_2^*$ is a fixed point $\forall I, b$. Since $x^* = x^*_1 = x^*_2$ we have

$$\dot{x}_1 = -x^* + 1/1+e^{bx^* - I}$$
$$\dot{x}_2 = -x^* + 1/1+e^{bx^* - I}$$

Certainly $\dot{x}_1 = \dot{x}_2$ for $x_1^* = x_2^* = x^*$, but do there exist I, b s.t. $(\dot{x}_1, \dot{x}_2) = (0,0)$? This will be the case iff $x^* = 1/1+e^{bx^* - I} \iff 1 + e^{bx^* - I} = 1/x^*$. if $x^* \in (0,1)$ the range of the LHS is $(1, \infty)$ in the b, I space and any value can be achieved by choosing inputs carefully

c) Show that a sufficiently large b the symmetric loses stability at a pitchfork bifurcation

when b is super large, small values of $x_1 x_2$, will make the null clines go to
x_1 : $x_1 = 1/2$ (instead of $x_1 = 0$, $x_2 = 0$)
x_2 : $x_2 = 1/2$

Since this creates a sort of - - -) ——— in the b, I parameter space, this is a subcritical pitchfork