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Dynamics
HW 6

6.1.3 For the following system, find the fixed points. Then sketch the null clines, vector field, and a plausible phase portrait.

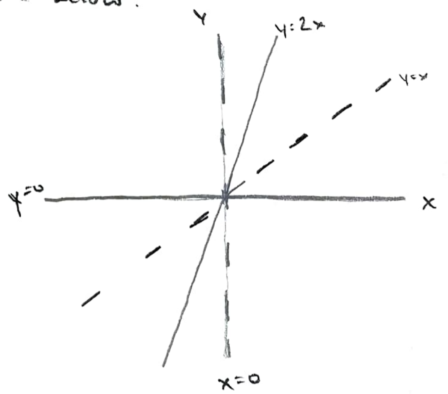
$$\dot{x} = x(x-y)$$
$$\dot{y} = y(2x-y)$$

The fixed points occur when $(\dot{x}, \dot{y}) = (0,0)$: $\dot{x} = x(x-y) = 0$, $\dot{y} = y(2x-y) = 0$

$x(x-y) = 0$ when $x = 0, y$
$y(2x-y) = 0$ when $y = 0, 2x$

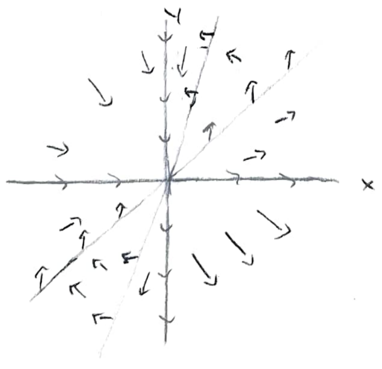
Certainly $(x^*, y^*) = (0,0)$ is a fixed point. Are there others? They would occur when $x = 2x \iff 0 = x$ so the origin is the only fixed point.

We also already found the null clines: $x = 0, y$ and $y = 0, 2x$. We depict them below:

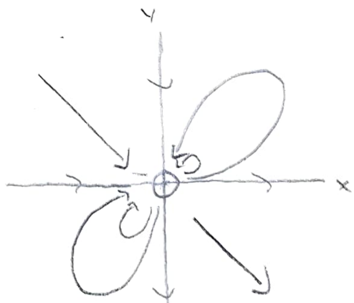


x cline: – – –
y cline: ———

We can also add the general direction of the vector field in each region which allows us to draw a plausible phase portrait.



Vector field



phase portrait

6.2.2 Consider the system $\dot{x} = y$

$$\dot{y} = -x + (1 - x^2 - y^2)\,y$$

a) Let D be the open disk $x^2 + y^2 < 4$. Verify that the system satisfies the hypothesis of existence and uniqueness theorem throughout D.

We must show that the vector valued function $\dot{\mathbf{x}} = f(\mathbf{x})$ is continuous and that each $\partial f_i / \partial x_j$ $i,j = 1,2$ (since $D \subseteq \mathbb{R}^2$) is continous.

$f_1 = y$ and $f_2 = -x + (1 - x^2 - y^2)y$ are continuous so $f(\mathbf{x})$ must be continuous. We can compute

$\partial f_1 / \partial x_1 = 0$ $\qquad$ $\partial f_1 / \partial x_2 = 1$ $\qquad$ (since $x_1 = x$, $x_2 = y$)

$\partial f_2 / \partial x_1 = -1 - 2xy$ $\qquad$ $\partial f_2 / \partial x_2 = 1 - x^2 - 3y^2$

which are all continuous on D. So we know our solutions to exist and to be unique

b) By substitution, show that $x(t) = \sin t$ and $y(t) = \cos t$ is an exact solution of the system

if $x(t) = \sin t \Rightarrow \dot{x} = \cos t$
if $y(t) = \cos t \Rightarrow \dot{y} = -\sin t$

then $\dot{x}$ and $\dot{y}$ can be rewritten as

$\dot{x} = y \longrightarrow \underline{\cos t = \cos t}$ true $\forall t$

and $\dot{y} = -x + (1 - x^2 - y^2)y$

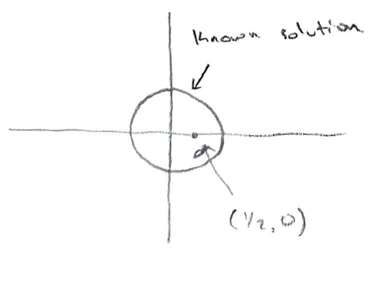
$\downarrow$

$$-\sin t = -\sin t + (1 - (\sin^2 t + \cos^2 t)) \cos t$$
$$-\sin t = -\sin t + (1 - 1)\cos(t) = -\sin t$$
$$-\sin t = -\sin t \;] \text{ true } \forall t.$$

So $x(t) = \sin t$ and $y(t) = \cos t$ satisfy $\dot{x}, \dot{y}$ and are an exact solution.

c) Now consider a different solution with $x(0) = 1/2$, $y(0) = 0$. Show $x(t)^2 + y(t)^2 < 1$ for all $t < \infty$

We already have shown existence and uniqueness and we know $x_1(t) = \sin t$, $y_1(t) = \cos t$ is a solution (as t varies, this is the unit circle)



By existence/uniqueness, our new solution can not leave the unit circle (or be on it for that matter).

6.3.6 For the following system, find the fixed points, classify them, sketch the neighboring trajectories, and try to fill in the rest of the phase portrait.

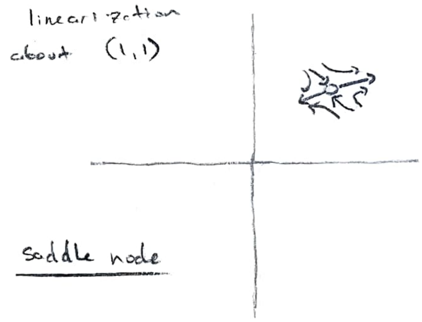
$$\dot{x} = xy - 1$$
$$\dot{y} = x - y^3$$

We want both $\dot{x} = 0$ and $\dot{y} = 0$ for a fixed point, so we must have $xy = 1$ and $y^3 = x$, which means that $y^4 = 1$ so $y = \pm 1$. This gives $x = \pm 1$ and fixed points $(1,1)$ and $(-1,-1)$

To classify the fixed points, we linearize the system at each point. The jacobian is

$$J = \begin{bmatrix} y & x \\ 1 & -3y^2 \end{bmatrix}$$

case $(x^*, y^*) = (1,1)$:



$$J_{(1,1)} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \qquad \tau = -2, \ \Delta = -4$$

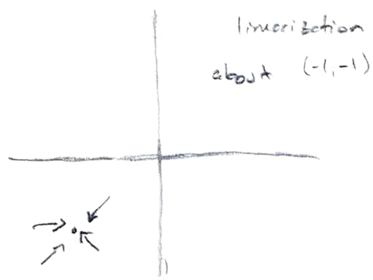
saddle node

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4(-4)}}{2} = -1 \pm \sqrt{5}$$

for $\lambda_1 = -1 + \sqrt{5} > 0$: $\begin{bmatrix} 1+1-\sqrt{5} & 1 \\ 1 & -3+1-\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} -1 & 2+\sqrt{5} \\ 1 & -2-\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} -1 & 2+\sqrt{5} \\ 0 & 0 \end{bmatrix}$ so $v_1 = \begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix}$

for $\lambda_2 = -1 - \sqrt{5} < 0$: $\begin{bmatrix} 1+1+\sqrt{5} & 1 \\ 1 & -3+1+\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} -1 & 2-\sqrt{5} \\ 1 & -2+\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} -1 & 2-\sqrt{5} \\ 0 & 0 \end{bmatrix}$ so $v_2 = \begin{pmatrix} 2-\sqrt{5} \\ 1 \end{pmatrix}$

case $(x^*, y^*) = (-1,-1)$:



$$J_{(-1,-1)} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \qquad \tau = -4, \ \Delta = 4$$

attractor

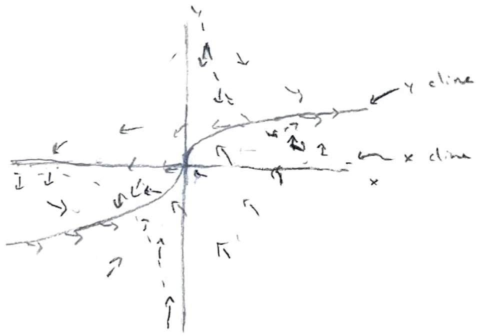
$$\lambda_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4(4)}}{2} = -2$$

for $\lambda_1 = \lambda_2 = -2$: $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so $v_1 = v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

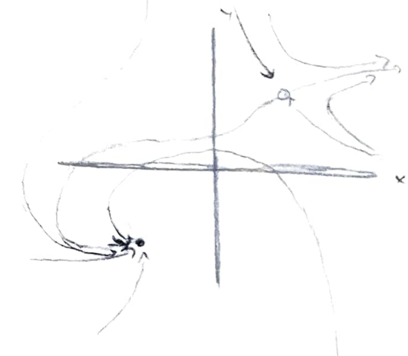
then we can fill in the rest of the phase portrait using null clines.

x clines: $\dot{x} = 0$ when $x = 1/4$ (- - - - -)
y clines: $\dot{y} = 0$ when $y = (x)^{1/3}$ (———)

null clines



phase portrait



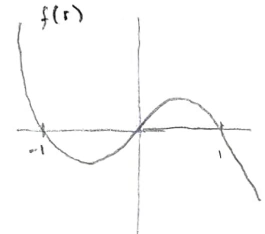
6.3.15 | Consider the system $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1 - \cos\theta$ where r, θ are polar coordinates

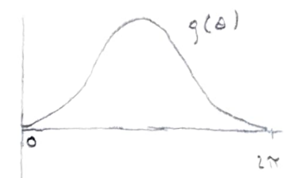
Sketch the phase portrait and show that the fixed point $r^* = 1$, $\theta^* = 0$ is attracting but not Liapunov stable.

Although we are in polar coordinates, we can plot the trajectories on the regular phase plane. Also note that as equations are entirely uncoupled.

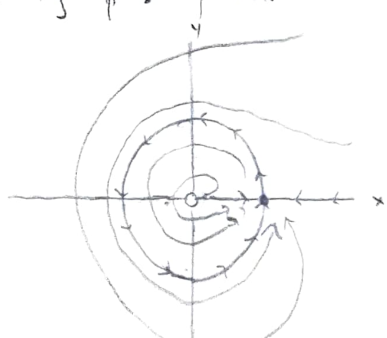
$\dot{r} = 0 = r(1 - r^2)$ which is the origin and the unit circle
$\dot{\theta} = 0 = 1 - \cos\theta$ for $\theta = 0$

We can also plot the graphs $f(r) = r(1 - r^2)$ and $g(\theta) = 1 - \cos\theta$





This gives us the following phase portrait:



From this phase portrait, we can say that $(r^*, \theta^*) = (1, 0)$ is an attractor but not Liapunov stable because of small positive perturbation from $\theta^* = 0$ sends the trajectory around the circle.

<u>6.4.3</u> | For the following competition problems where $x, y \geq 0$, find the fixed points, investigate their stability, draw the null clines, and sketch plausible phase portraits. Indicate basins of attraction.

$$\dot{x} = x(3 - 2x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

$\dot{x} = 0 = x(3 - 2x - 2y)$ when $x = 0, \ \frac{1}{2}(3 - 2y)$

$\dot{y} = 0 = y(2 - x - y)$ when $y = 0, \ 2 - x$

This gives the three equilibria $(0,0)$, $(0,2)$, and $(3/2, 0)$
(since $x = \frac{1}{2}(3-2y) = 1/2\,(3 - 2(2-x)) = 1/2\,(3 - 4 + 2x) = -1/2 + x$ has no solns).

Since $f(x,y) = 3x - 2x^2 - 2xy$ and $g(x,y) = 2y - xy - y^2$, we have the Jacobian

$$J = \begin{bmatrix} 3 - 4x - 2y & -2x \\ -y & 2 - x - 2y \end{bmatrix}$$

Then we can consider the case of each fixed point.

Case $(x^*, y^*) = (0,0)$:

$$J_{(0,0)} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$\lambda_1 = 3 > 0$ and $\lambda_2 = 2 > 0 \Rightarrow$ unstable

Case $(x^*, y^*) = (0,2)$:

$$J_{(0,2)} = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$$

$\lambda_1 = -1 < 0$ and $\lambda_2 = -2 < 0 \Rightarrow$ stable

Case $(x^*, y^*) = (3/2, 0)$:

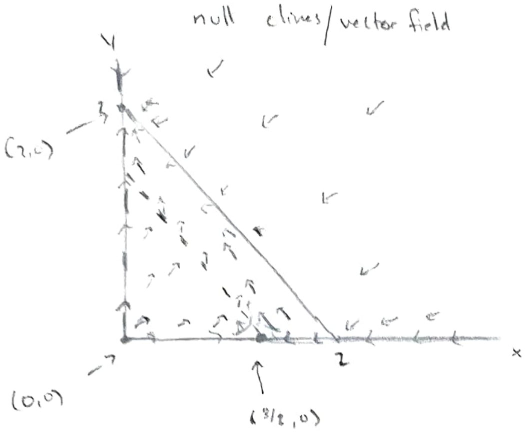
$$J_{(3/2,0)} = \begin{bmatrix} -6 & 3 \\ 0 & 1/2 \end{bmatrix}$$

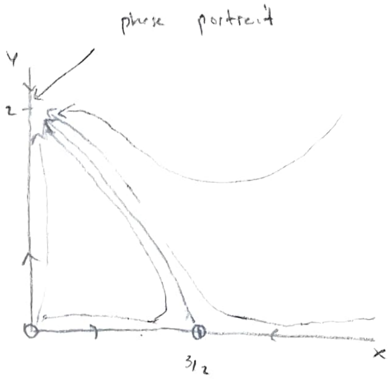
$\lambda_1 = 1/2 > 0$ and $\lambda_2 = -6 < 0 \Rightarrow$ saddle

Then we can sketch the null clines:

x cline: $x = 0$ and $y = 3/2 - x$ (- - - - -)

y cline: $y = 0$ and $y = 2 - x$ (———)





The basin of attraction for $(0,2)$ is all $x, y > 0$

6.4.6 Given
$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$$
$$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$$
for parameters $r_1, r_2, K_1, K_2, b_1, b_2$

a) Non-dimensionalize the model. How many dimensionless groups are needed?

For our dimensionless groups, we choose $x = \frac{N_1}{K_1} \Rightarrow \dot{x} = \frac{\dot{N}_1}{K_1}$

$y = N_2/K_2 \Rightarrow \dot{y} = \dot{N}_2/K_2$

This gives the new equations

$$K_1 \dot{x} = K_1 r_1 x(1-x) - b_1 K_1 K_2 xy \qquad \Rightarrow \qquad \dot{x} = r_1 x(1-x) - b_1 K_2 xy$$
$$K_2 \dot{y} = K_2 r_2 y(1-y) - b_2 K_1 K_2 xy \qquad \qquad \dot{y} = r_2 y(1-y) - b_2 K_1 xy$$

then let $\tau = r_1 t \Rightarrow \frac{d}{dt}(x(\tau)) = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau} r_1 = r_1 x'$

$$\frac{d}{dt}(y(\tau)) = \frac{dy}{d\tau}\frac{d\tau}{dt} = \frac{dy}{d\tau} r_1 = r_1 y'$$

then we have

$$r_1 x' = r_1 x(1-x) - b_1 K_2 xy \qquad \Rightarrow \qquad x' = x(1-x) - B_1 xy$$
$$r_1 y' = r_2 y(1-y) - b_2 K_1 xy \qquad \qquad y' = R y(1-y) - B_2 xy$$

where $B_1 = b_1 K_2 / r_1$, $R = r_2/r_1$, and $B_2 = b_2 K_1 / r_1$ (all positive)
Thus the non-dimensionalized equation is

$$x' = x(1-x) - B_1 xy$$
$$y' = R y(1-y) - B_2 xy$$

b) Show that there are four qualitatively different phase portraits as far as long term behavior is concerned.

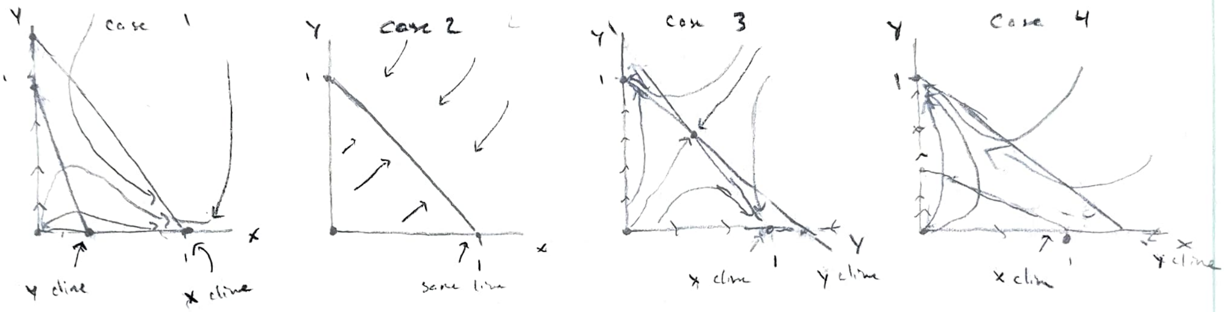
To analyze long term behavior, let's look at the equilibrate points.

$$x' = 0 = x(1-x) - B_1 xy = x(1 - x - B_1 y)$$
$$y' = 0 = Ry(1-y) - B_2 xy = y(R - Ry - B_2 x)$$

Certainly $(0,0)$ is an equilibria. The x null clines are $x=0$ and $y = \frac{1}{B_1}(1-x)$ while the y null clines are $y=0$ and $y = 1 - B_2/R\, x$. So we could have the following cases for the phase portraits.

important! { Note that $y = \frac{1}{B_1}(1-x)$ must intersect the x-axis at $x=1$ and that $y = 1 - B_2/R\, x$ must intersect the y-axis at $y=1$ }

Also note that $(x,y) = (1,1)$ gives $\begin{pmatrix} -B_1 \\ -B_2 \end{pmatrix}$ as the velocity vector



c) Find conditions under which the two species can coexist. Explain the biological meaning of these conditions.

The two species can coexist if there is an equilibria off an axis. This occurs only in case 2 and case 3.

For case 2, we must have $\frac{1}{B_1}(1-x) = 1 - B_2/R\, x \iff 1 - x = B_1 - B_1B_2/R\, x$ which can only be true if $B_1 = 1$ and $B_1B_2/R = \frac{B_2}{R} = 1 \iff B_2 = R$. So in case 2, we must have $B_1 = 1$ and $B_2 = R$.

In case 3, we must have the two lines intersect at exactly one point. We know there is no intersection if $1/B_1 \neq 1$ and $1/B_1 = B_2/R \iff B_1 \neq 1$ and $R = B_1 B_2$. So an intersection exists if either $B_1 = 1$ or $R \neq B_1B_2$. But we must also have $x, y > 0$!

$$1/B_1 (1-x) = 1 - B_2/R\, x \iff 1 - x = B_1 - B_1B_2/R\, x \iff R - RB_1 = (R - B_1B_2)x$$

so $x = R\dfrac{1 - B_1}{R - B_1B_2}$ (so long as $R \neq B_1B_2$). But we must also have $x = R\dfrac{1-B_1}{R - B_1B_2} > 0$ and $y = 1 - B_2 \underbrace{\dfrac{1 - B_1}{R - B_1B_2}}_{\text{obtained from plugging in } x} > 0$

As far as biological interpretation goes, the saddle point seems difficult to give an explanation for. However, for the line of equilibria that occurs when $B_1 = 1$ and $B_2 = R$ means that we must have

$$B_1 = \frac{b_1 K_2}{r_1} = 1 \quad \text{and} \quad B_2 = R = \frac{b_2 K_1}{r_1} = \frac{r_2}{r_1} \iff b_1 K_2 = r_1 \text{ and } b_2 K_1 = r_2$$

From this we can gather that the growth rates of each species must exactly balance the loss from competition for resources.

this equilibria also depends on the initial conditions, which must be very precise.