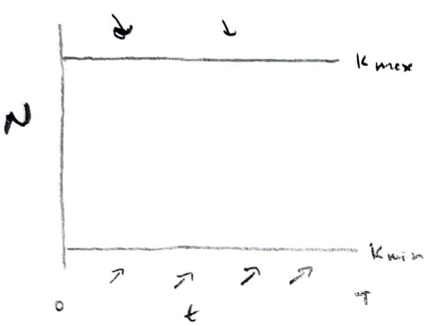
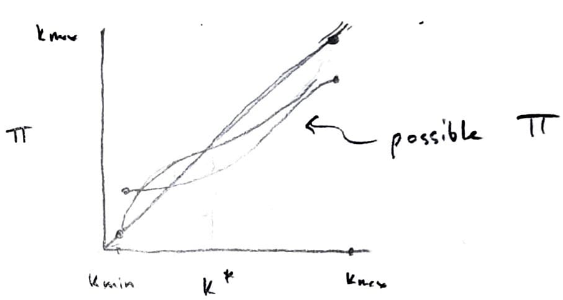
Nathan Stouffer
Dynamics
HW 12

8.5.3 | Consider the logistic equation $\dot{N} = rN(1 - N/K(t))$ where $K(t)$ is positive, smooth, and T-periodic in t.

a) Show that the system has at least one stable limit cycle of period T contained in the strip $K_{min} \le N \le K_{max}$

Well, for $N > K_{max}$, we must have $\dot{N} < 0$ and for $N < K_{min}$ we must have $\dot{N} > 0$. Shown in the following diagram, we must have $\Pi(K_{max}) \le K_{max}$ and $\Pi(K_{min}) \ge K_{min}$ where Π is the Poincare map. This also induces the continuous Poincare map





Then the map Π must cross the diagonal, in wich case $\Pi(K^*) = K^*$ which corresponds to a closed orbit. Now we must show that this closed orbit is stable. Suppose there were no stable closed orbits in $[K_{min}, K_{max}]$, then the flow must move away from all closed orbits, but the flow must also stay inside the region, so a stable orbit must exist.

b) Is the cycle necessarily unique?

My intuition says that the cycle is unique but I have no proof. At first, I looked for a $K(t)$ which caused two or more cycles but I did not find one. I also struggled down several methods of showing the cycle was unique. Oh well!

8.6.1 | In a paper on systems of neural oscillators, the following model was introduced

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1 \cos\theta_2$$
$$\dot{\theta}_2 = \omega_2 + \sin\theta_2 \cos\theta_1 \qquad \text{with} \qquad \omega_1, \omega_2 \ge 0$$

a) sketch all the qualitatively different phase portraits as $\omega_{1,2}$ vary.

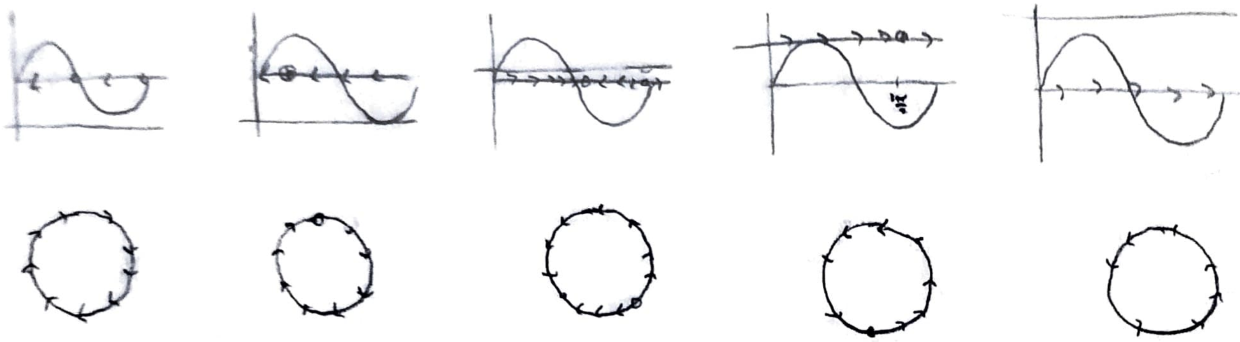
First consider the related system $\left.\begin{aligned}\alpha &= \theta_1 + \theta_2 \\ \beta &= \theta_1 - \theta_2\end{aligned}\right\} \Rightarrow \begin{aligned}\dot{\alpha} &= \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\beta} &= \dot{\theta}_1 - \dot{\theta}_2\end{aligned}$

then

$$\dot{\alpha} = \omega_1 + \omega_2 + \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 = \omega_1 + \omega_2 + \sin\alpha = \dot{\alpha}$$
$$\dot{\beta} = \omega_1 - \omega_2 + \sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 = \omega_1 - \omega_2 + \sin\beta = \dot{\beta}$$

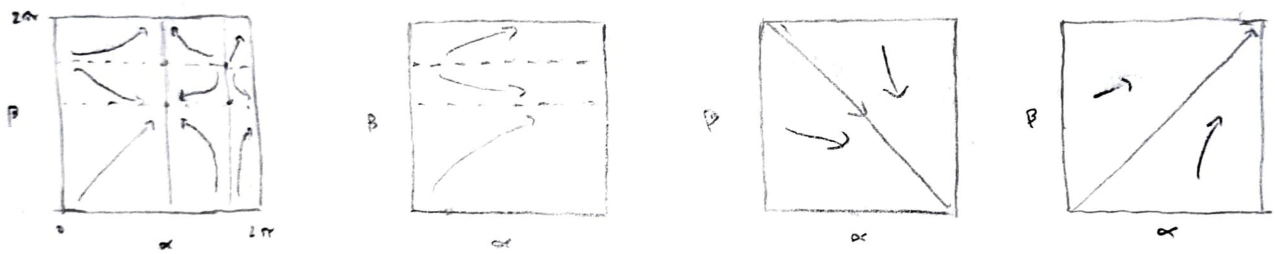
Both $\dot{\alpha}, \dot{\beta}$ are of the form $\dot{\phi} = W + \sin\phi$ so let's analyze this system:

$W < -1$ $\quad$ $W = -1$ $\quad$ $W \in (-1, 1)$ $\quad$ $W = 1$ $\quad$ $W > 1$



Note that for $\dot{\alpha}$, $\omega_1 + \omega_2$ is constrained to non-negative values whereas in $\dot{\beta}$, $\omega_1 - \omega_2$ can be any real. This constrains the bifurcations that can occur.

For $\omega_1 + \omega_2 < 1$ $\quad$ for $\omega_1 - 1 < \omega_2 < \omega_1 + 1$ $\quad$ for $\omega_2 > \omega_1 - 1$, $\quad$ for $\omega_2 > \omega_1 + 1$



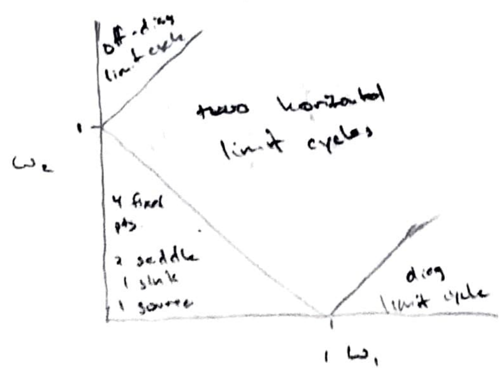
b) Find the curves in ω_1, ω_2 parameter space occur and classify the various bifurcations.

From the results in part a), we can conclude that bifurcations occur along $\omega_2 = 1 - \omega_1$, $\omega_2 = \omega_1 - 1$, and $\omega_2 = \omega_1 + 1$.

At $\omega_2 = 1 - \omega_1$ a Hopf bifurcation occurs since two fixed orbits appear.

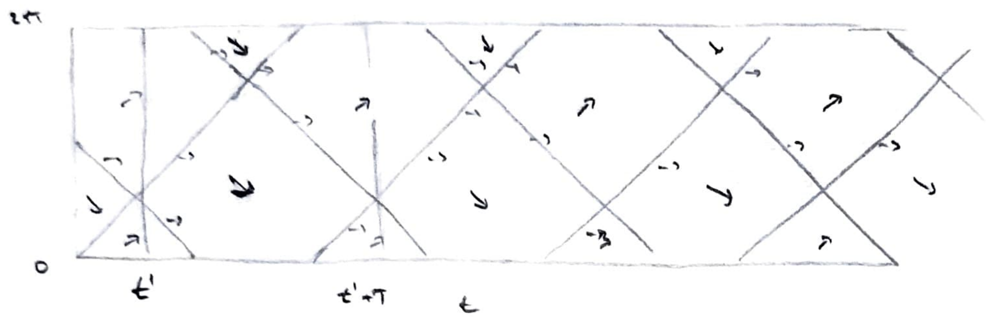
At $\omega_2 = \omega_1 - 1$ and $\omega_2 = \omega_1 + 1$, a different limit cycle appears on the diagonal (or off-diagonal)

c) Plot the stability diagram in ω_1, ω_2 space



8.7.5 Show $\dot{\theta} + \sin\theta = \sin t$ has at least two periodic solutions. Comment on their stability

Consider $\dot{\theta} = \sin t - \sin\theta$, $\dot{t} = 1$ on the cylinder.



The $\dot{\theta}$ null-clines are $t = \theta$, $\pi - \theta$. Now consider the Poincare map Π. Note that $\Pi(t' = \pi/2) < \pi/2$. This means that the trajectory crossing $(\pi/2, \pi/2)$ must move "down" on the cylinder.

Similarly, the trajectory crossing $(3\pi/2, 3\pi/2)$ is $\Pi(3\pi/2) > \frac{3\pi}{2}$ so this trajectory must go "up" the cylinder. From this, we can gather that the trajectories are approaching a <u>stable</u> limit cycle (since they cannot intersect).

But for those two trajectories to move towards each other along one side of the cylinder implies (by continuity of the flow/Π) that another, <u>unstable</u> limit cycle exists in $(\pi/2, 3\pi/2)$.

8.7.6 Give a mechanical interpretation of $\dot{\theta} + \sin\theta = \sin t$ given in the previous exercise

The first thing to note is that the system is lacking an acceleration term, this means that we could be looking at some mechanical system in a viscous fluid.

Let's choose a pendulum in honey. Then we don't just have the pendulum eqn because we have $F = \sin t$ being applied. This represents some oscillation force being applied externally.

Here is a picture.