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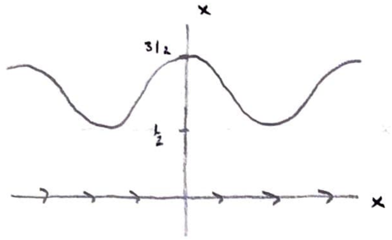
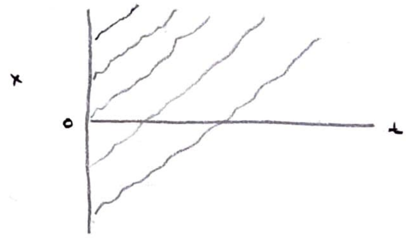
M 454 - Dynamics I

HW 1

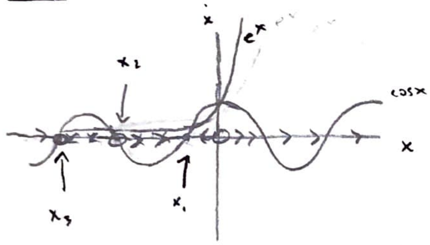
2.2.5 Say we have $\dot{x} = 1 + \frac{1}{2}\cos x$. Analyze the problem graphically

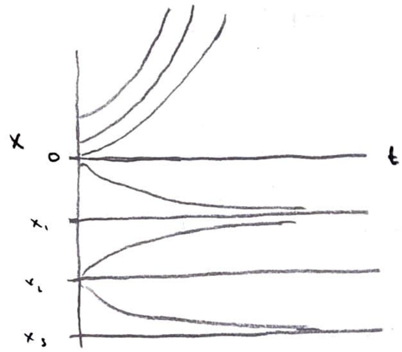




$\dot{x} = 1 + \frac{1}{2}\cos x$ has no fixed points. I tried to solve this analytically but couldn't figure it out.

2.2.7 We now have $\dot{x} = e^x - \cos x$





$x^* = 0$ is an unstable fixed point

$x^* = x_1$ is stable.

If my labeling were to continue x_n is stable when n is odd and unstable when n is even.

I tried to find an analytical solution for this one and struggled as well

2.2.13 Terminal velocity. $v(t)$, the velocity of a skydiver, is governed by $m\dot{v} = mg - kv^2$ where m is mass, g is acceleration due to gravity, and $k > 0$ is a constant dealing with air resistance

(a) Assuming that $v(0) = 0$, solve for $v(t)$ analytically.

$$m\dot{v} = mg - kv^2 = m\frac{dv}{dt}$$

$$(*) \quad \int \frac{m}{mg - kv^2}\,dv = \int 1 \cdot dt$$

We will return to $(*)$ after evaluating $\int \frac{m}{mg - kv^2}\,dv$

$$\int \frac{m}{mg - kv^2}\,dv = \frac{m}{k}\int \frac{1}{\frac{mg}{k} - v^2}\,dv = \frac{m}{k}\int \frac{1}{X^2 - v^2}\,dv \quad \text{where } X = \sqrt{\frac{mg}{k}}$$

Then we can use partial fractions to expand $\frac{1}{x^2-v^2} = \frac{1}{(x-v)(x+v)}$:

$$\frac{1}{(X-v)(X+v)} = \frac{A}{X-v} + \frac{B}{X+v} = \frac{A(X+v) + B(X-v)}{(X-v)(X+v)} = \frac{(A+B)X + (A-B)v}{(X-v)(X+v)}$$

since $X = \sqrt{\frac{mg}{k}}$, we must have

$$A + B = \sqrt{\frac{k}{mg}} \quad \text{and} \quad A - B = 0 \Rightarrow A = B$$

since $A = B$, $2A = \sqrt{\frac{k}{mg}} \Rightarrow A = \frac{1}{2}\sqrt{\frac{k}{mg}} = B$. Using the partial fraction expansion, we can say

$$\int \frac{m}{mg - kv^2}\, dv = \frac{m}{2k}\sqrt{\frac{k}{mg}} \int \frac{1}{X-v} + \frac{1}{X+v}\, dv$$

$$= \sqrt{\frac{m}{4kg}}\left[-\ln|X-v| + \ln|X+v|\right] + c$$

now we can return to $(*)$:

$$\int \frac{m}{mg - kv^2}\, dv = \int 1\, dt$$

$$\sqrt{\frac{m}{4kg}}\left[-\ln|X-v| + \ln|X+v|\right] = t + c$$

$$\ln\left|\frac{X+v}{X-v}\right| = \sqrt{\frac{4kg}{m}}\, t + c$$

$$\frac{X+v}{X-v} = A_0 e^{\sqrt{\frac{4kg}{m}}\, t} \quad \text{where } A_0 = e^c$$

Before explicitly solving for v, we will find A using $v(0) = 0$,

$$\frac{X+0}{X-0} = A_0 e^{\sqrt{\frac{4kg}{m}}\, 0} \;:\; A_0 = \frac{X}{X} = 1$$

then we have

$$\frac{X+v}{X-v} = e^{\sqrt{\frac{4kg}{m}}\, t} \Rightarrow X + v = X e^{\sqrt{\frac{4kg}{m}}\, t} - v e^{\sqrt{\frac{4kg}{m}}\, t}$$

$$v\left(1 + e^{\sqrt{\frac{4kg}{m}}\, t}\right) = X\left(-1 + e^{\sqrt{\frac{4kg}{m}}\, t}\right)$$

$$v = \sqrt{\frac{mg}{k}}\; \frac{e^{\sqrt{\frac{4kg}{m}}\, t} - 1}{e^{\sqrt{\frac{4kg}{m}}\, t} + 1}$$

Thus our explicit solution for $v(t) = \sqrt{\frac{mg}{k}}\; \dfrac{e^{\sqrt{\frac{4gk}{m}}\, t} - 1}{e^{\sqrt{\frac{4gk}{m}}\, t} + 1}$

(b) we now want to compute $\lim_{t\to\infty} v(t)$:

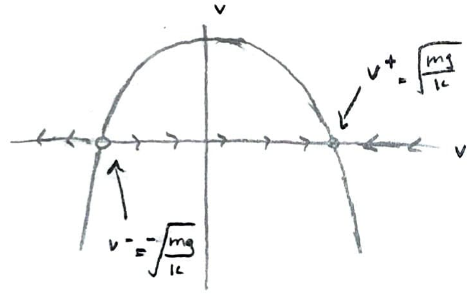
$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \sqrt{\frac{mg}{k}}\,\frac{e^{\sqrt{4gk/m}\,t} - 1}{e^{\sqrt{4gk/m}\,t} + 1} \underset{\text{by dominating terms}}{=} \lim_{t\to\infty} \sqrt{\frac{mg}{k}} \cdot 1 = \sqrt{\frac{mg}{k}}$$

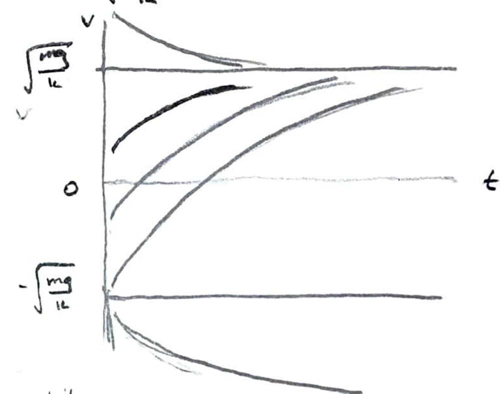
so our terminal velocity is $\sqrt{\frac{mg}{k}}$

(c) We now solve $m\dot{v} = mg - kv^2$ graphically.

equivalently, $\dot{v} = g - \frac{k}{m}v^2 = \left(\sqrt{g} + \sqrt{\frac{k}{m}}\,v\right)\left(\sqrt{g} - \sqrt{\frac{k}{m}}\,v\right)$

So we have equilibria at $v^* = \pm\sqrt{\frac{mg}{k}}$





So $v^+ = \sqrt{\frac{mg}{k}}$ is a stable equilibria while $v^- = -\sqrt{\frac{mg}{k}}$ is an unstable equilibria. From our graph in time, we can see that most valid initial conditions (objects beginning freefall) will tend toward a terminal velocity of v^+ in time.

(d) Given that the average weight of a person in free fall is 261.2 lb (with gear) and $g = 32.2$ ft/s, use the fact that some one fell 29300 ft in 116 s (and reached terminal velocity) to compute the average velocity V_{avg}.

$$V_{avg} = d/t = 29300\text{ ft}/116\text{ s} \approx 252.586\text{ ft/s}$$

(e) Now estimate the terminal velocity and the drag constant k

Consider our analytic solution for $v(t)$ rewritten as

$$v(t) = V\,\frac{e^{2Dt} - 1}{e^{2Dt} + 1} \quad \text{where} \quad V = \sqrt{\frac{mg}{k}},\ D = \sqrt{gk/m}$$

$$v(t) = V\,\frac{e^{2Dt} - 1}{e^{2Dt} + 1}\,\frac{e^{-Dt}}{e^{-Dt}} = V\,\frac{e^{Dt} - e^{-Dt}}{e^{Dt} + e^{-Dt}} = V\,\frac{\sinh(Dt)}{\cosh(Dt)}$$

then we can compute $s(t) = \int v(t)\, dt$

$$s(t) = \int v(t)\,dt = \int V \frac{\sinh(Dt)}{\cosh(Dt)}\,dt = V\frac{1}{D}\ln(\cosh(Dt)) + C_0$$

since $s(0) = 0 \Rightarrow 0 = V\frac{1}{D}\ln(\cosh(D0)) + C_0 = V\frac{1}{D}\ln(1) + C_0 = C_0 = 0$

so we can write

$$s(t) = V\frac{1}{D}\ln(\cosh(Dt)) = \frac{m}{k}\ln(\cosh(Dt))$$

since we are interested in large t, consider the following approximation for $\ln(\cosh(x))$ when x is large $(x \gg 1)$

$$\ln(\cosh(x)) = \ln\left(\frac{e^x + e^{-x}}{2}\right) \approx \ln\left(\frac{e^x}{2}\right) = \ln e^x - \ln 2 = x - \ln 2$$

then

$$s(t) \approx \frac{m}{k}(Dt - \ln 2) = \frac{m}{k}\left(\sqrt{\frac{gk}{m}}\,t - \ln 2\right) = \sqrt{\frac{mg}{k}}\,t - \frac{m}{k}\ln 2$$

then I plugged in all the known values to $\begin{pmatrix} m = 118.5\ \text{lb}, & t = 116\ \text{s} \\ g = 32.2\ \text{ft/s}^2 & \end{pmatrix}$

$$s(116) = 29300 = \sqrt{\frac{mg}{k}}\,t - \frac{m}{k}\ln 2$$

and solved for k on a calculator, $k \approx 0.054055$

plugging this into $V = \sqrt{\frac{mg}{k}}$, we get a terminal velocity of

$$V = 265.686\ \text{ft/s}$$

2.3.4) Allee Growth

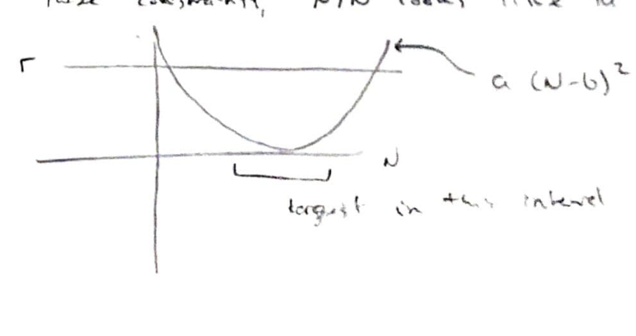
(a) show that $\dot{N}/N = r - a(N-b)^2$ is an example of Allee Growth if r, a, and b satisfy certain conditions

constraints: $b > 0$ so that $a(N-b)^2$ is shifted right

$a > 0$ so that $a(N-b)^2$ opens down

$r > a(N-b)^2$ so that $\dot{N}/N$ is not negative for intermediate values

If we satisfy these constraints, $\dot{N}/N$ looks like the difference of

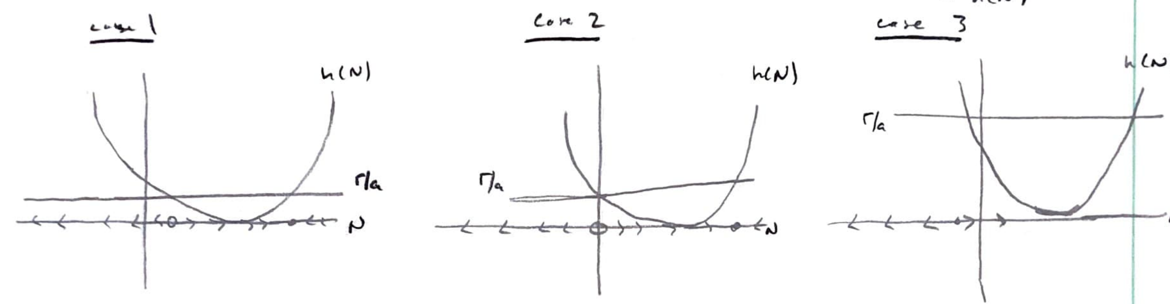


(b) Find all the fixed points of $\dot{N} = N(r - a(N-b)^2)$ and classify their stability.

Certainly $N^* = 0$ will always be an equilibria, but its classification is not always the same. we have 3 cases.

$$0 = r - a(N-b)^2 \iff r = a(N-b)^2 \iff \frac{r}{a} = \underbrace{(N-b)^2}_{h(N)}$$

only for $\frac{r}{a} = (N-b)^2$ which excludes $N=0$ as equilibria. I do include $N^*=0$ here



Our equilibria are always $N^* = 0,\ b \pm \sqrt{r/a}$. let $N_\circ^* = 0$, $N^- = b - \sqrt{r/a}$ and $N^+ = b + \sqrt{r/a}$

Case 1: $r/a < b^2$

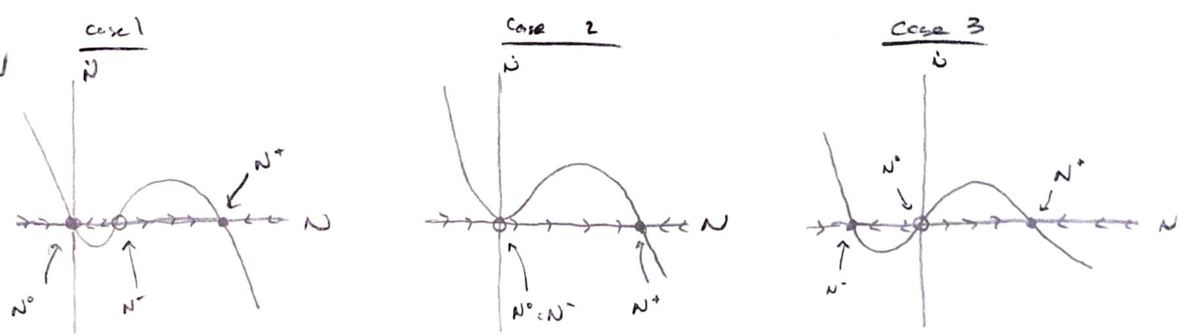
N^0 is stable, N^- is unstable, N^+ is stable

Case 2: $r/a = b^2$

$N^0 = N^-$ is unstable and N^+ is stable

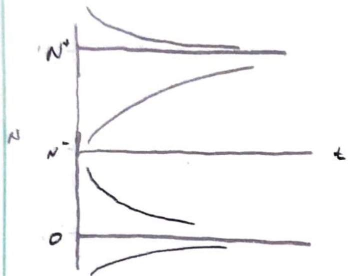
Case 3: $r/a > b^2$

N^0 is unstable, N^- has no context, and N^+ is stable

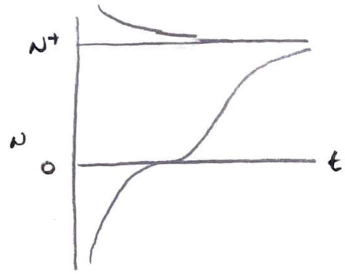


(c) On the next page, we will sketch solutions for $N(t)$ in each of these three cases

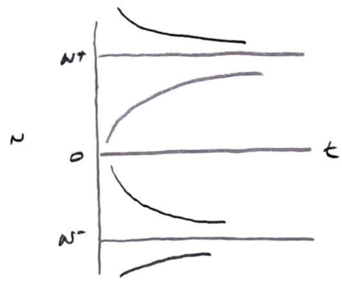
Case 1



Case 2



Case 3



(d) We now compare the solutions for $N(t)$ to the logistic equation. First, note that any $N<0$ makes no physical sense so we will only compare $N>0$ with the logistic eqn.

In this context, cases 2 and 3 are qualitatively identical to the logistic eqn. There is a stable equilibria at the carrying capacity and an unstable equilibria at $N=0$.

However, case 1 has a threshold (greater than 0) that the population must reach for $N(t)$ to reach carrying capacity. This makes $N^*=0$ a stable equilibria (in addition to N^+).

2.4.2 | Use linear stability analysis to classify the fixed points of $\dot{x} = x(1-x)(2-x)$ w/ solns $x=0,1,2$

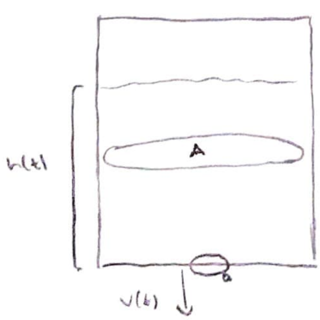
let $f(x) = x(1-x)(2-x) = x(2-3x+x^2) = x^3-3x^2+2x$

$f'(x) = 3x^2 - 6x + 2 = 0$ w/ solns $x = 1 \pm 1/\sqrt{3}$

$f'(x) = (x-(1-1/\sqrt{3}))(x-(1+1/\sqrt{3})) = (x-1+1/\sqrt{3})(x-1-1/\sqrt{3})$

$x=0$: $f'(0) = (0-1+1/\sqrt{3})(0-1-1/\sqrt{3}) = (-1+1/\sqrt{3})(-1-1/\sqrt{3}) > 0 \Rightarrow$ unstable
$x=1$: $f'(1) = (1-1+1/\sqrt{3})(1-1-1/\sqrt{3}) = 1/\sqrt{3} \cdot -1/\sqrt{3} < 0 \Rightarrow$ stable
$x=2$: $f'(2) = (2-1+1/\sqrt{3})(2-1-1/\sqrt{3}) = (1+1/\sqrt{3})(1-1/\sqrt{3}) > 0 \Rightarrow$ unstable

2.5.6 | Consider the following setup of a bucket with a hole in it



(a) Show that $a\,v(t) = A\,\dot{h}(t)$ for our system.

If we use conservation of volume, we know that the amount lost from the top $(A\dot{h}(t))$ at any instant t must be equal to the amount flowing through the hole $(a\,v(t))$.

(b) To relate v and h, we use conservation of energy. First

$$\Delta PE = mgh - mg(h - \Delta h) \quad \text{for some small } \Delta h$$

Since the bucket has constant cross-sectional area A, $m = \rho A h$. So

$$\Delta PE \quad \rho g A h^2 - \rho g A (h - \Delta h)^2$$

By conservation of energy, what is lost in potential must be found in kinetic energy: $KE = \frac{1}{2} m v^2 = \frac{1}{2}\rho A \Delta h\, v^2$. Setting $\Delta PE = KE$, we can solve for v^2:

$$\rho g A h^2 - \rho g A (h - \Delta h)^2 = \frac{1}{2}\rho A \Delta h\, v^2$$

$$g(h^2 - (h^2 - 2h\Delta h + \Delta h^2)) = \frac{1}{2}\Delta h\, v^2$$

$$2gh\Delta h + g\Delta h^2 = \frac{1}{2}\Delta h\, v^2$$

$$v^2 = 4gh + 2g\Delta h$$

But we let $\Delta h \to 0$ and $v^2 = 4gh$

(c) Now show $\dot{h} = -c\sqrt{h}$ where $c = \sqrt{2g}\,\frac{a}{A}$

$$A\dot{h}(t) = a\,v(t) = a^{\pm}\sqrt{4gh} = 2a^{\pm}\sqrt{gh}$$

$$\dot{h}(t) = \frac{\pm 2a\sqrt{g}}{A}\sqrt{h}$$

since the height is decreasing, we choose the negative sign!

$$\dot{h}(t) = -\frac{2a\sqrt{g}}{A}\sqrt{h} = -c\sqrt{h} \quad \text{where} \quad c = \frac{2a\sqrt{g}}{A}$$

(d) Given $h(0) = 0$, show that $h(t)$ is non-unique in backwards time.

We know that $h(t)$ is not unique at $t=0$ since

$$\ddot{h}(t) = -\frac{c}{2}\frac{1}{\sqrt{h}}\dot{h} \quad \text{is not continuous at } t=0$$

$h(0) = 0$ is not continuous

so we can apply the existence and uniqueness theorem!