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Dynamics

HW a

7.3.1) Consider $\dot{x} = x - y - x(x^2+5y^2)$
$\dot{y} = x + y - y(x^2+y^2)$

a) classify the fixed point at the origin.

$(0,0)$ is a fixed point since $\dot{\mathbf{x}} = \begin{pmatrix}0\\0\end{pmatrix}$ when $(x,y) = (0,0)$

$$\dot{x} = x - y - x(x^2+5y^2) = x - y - x^3 - 5xy^2$$
$$\dot{y} = x + y - y(x^2+y^2) = x + y - x^2y - y^3$$

$$J = \begin{bmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & 1-x^2-3y^2 \end{bmatrix}$$

$$J_{(0,0)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \tau = 2,\ \Delta = 1 - -1 = 2 \quad \lambda_{1,2} = \frac{2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = 1 \pm i$$

$(0,0)$ is a spiral source

b) Rewrite the system in polar coordinates using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$ (we also use $x = r\cos\theta$ and $y = r\sin\theta$)

$$r\dot{r} = x\dot{x} + y\dot{y} = x(x-y-x(x^2+5y^2)) + y(x+y-y(x^2+y^2))$$
$$= x^2 - xy - x^2(x^2+5y^2) + xy + y^2 - y^2(x^2+y^2)$$
$$= x^2 + y^2 - x^4 - 5x^2y^2 - x^2y^2 - y^4$$
$$r\dot{r} = r - r^4\cos^4\theta - r^4\sin^4\theta - 6r^4\cos^2\theta\sin^2\theta$$
$$\dot{r} = r(1 - r^2\cos^4\theta - r^2\sin^4\theta - 6r^2\cos^2\theta\sin^2\theta)$$
$$\dot{r} = r(1 - r^2\cos^2\theta(1-\sin^2\theta) - r^2\sin^2\theta(1-\cos^2\theta) - 6r^2\cos^2\theta\sin^2\theta)$$
$$\dot{r} = r(1 - r^2\cos^2\theta - r^2\sin^2\theta + r^2\cos^2\theta\sin^2\theta + r^2\cos^2\theta\sin^2\theta - 6r^2\cos^2\theta\sin^2\theta)$$
$$\dot{r} = r(1 - r^2 - 4r^2\cos^2\theta\sin^2\theta)$$
$$\dot{r} = r(1 - r^2 - r^2\sin^2 2\theta)$$

So $\dot{r} = r(1 - r^2 - r^2\sin^2 2\theta)$

$$\dot{\theta} = (x\dot{y} - y\dot{x})/r^2 = (x(x+y-y(x^2+y^2)) - y(x-y-x(x^2+5y^2)))/r^2$$
$$= (x^2 + \cancel{xy} - xy(x^2+y^2) - \cancel{xy} + y^2 + xy(x^2+5y^2))/r^2$$
$$= (r^2 - \cancel{x^3y} - xy^3 + \cancel{x^3y} + 5xy^3)/r^2$$
$$= (r^2 + 4xy^3)/r^2$$
$$= (r^2 + 4r^4\cos\theta\sin^3\theta)/r^2$$
$$\dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta$$

So our system is

$$\dot{r} = r(1 - r^2 - r^2\sin^2 2\theta)$$
$$\dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta$$

c) Determine the circle of maximum radius r_1 centered at the origin such that all trajectories have a radially outward component.

This occurs when $\dot{r} > 0$, since r is greater than 0, $\dot{r} = r(1 - r^2 - r^2\sin^2 2\theta) > 0$ exactly when

$$1 - r^2 - r^2\sin^2 2\theta > 0$$
$$1 > r^2 + r^2\sin^2 2\theta = r^2(1+\sin^2 2\theta)$$
$$1/r^2 > 1 + \sin^2 2\theta$$
$$1/r^2 > 2 = 1 + 1 \geq 1 + \sin^2 2\theta$$

$1/r^2 > 2 \iff 1/2 > r^2 \iff r_1 < 1/\sqrt{2}$. So if we choose r_1 strictly less than $1/\sqrt{2}$ the flow will be radially outward.

d) Determine the circle of minimum radius r_2, centered on the origin such that all trajectories have a radially inward component.

$\dot{r} < 0$ exactly when

$$0 > 1 - r^2 - r^2\sin^2 2\theta$$
$$1 < r^2 + r^2\sin^2 2\theta = r^2(1+\sin^2 2\theta)$$
$$1/r^2 < 1 + \sin^2 2\theta$$
$$1/r^2 < 1 = 1 + 0 \leq 1 + \sin^2 2\theta$$

$1/r^2 < 1 \iff r^2 > 1 \iff r > 1$. So we choose r_2 strictly greater than 1 we will have flow radially inward.

e) Show that the limit cycle in the trapping region $r_1 \leq r \leq r_2$.

Take $r_1 = 1/\sqrt{2} - 0.001$ and $r_2 = 1.001$, by parts c and d, $r_1 \leq r \leq r_2$ is a closed and bounded trapping region. We have a continuously differentiable vector field on R since the functions f and g are compositions of polynomials.

The final thing to verify is that there are no fixed points in R. For this to be case, we must have $\dot{r} = 0$:

$$\dot{r} = 0 = r(1 - r^2 - r^2\sin^2 2\theta) \iff 0 = 1 - r^2 - r^2\sin^2 2\theta$$
$$1 = r^2(1 + \sin^2 2\theta)$$
$$r^2 = 1/(1+\sin^2 2\theta)$$

then we must have $\dot{\theta} = 0$:

$$\dot{\theta} = 0 = 1 + 4r^2\cos\theta\sin^3\theta = 1 + 4\frac{\cos\theta\sin^3\theta}{1+\sin^2 2\theta}$$

$$4\cos\theta\sin^3\theta = -1 - \sin^2 2\theta$$
$$2\sin^2\theta\sin 2\theta = -1 - \sin^2 2\theta$$
$$0 = \sin^2 2\theta + 2\sin^2\theta\sin 2\theta + 1 \qquad \text{then let } z = \sin 2\theta$$
$$0 = z^2 + 2\sin^2\theta\, z + 1$$

which has no real solutions in z since $2\sin^2\theta > 0 \;\forall\theta$. Therefore we cannot have both $\dot{r} = 0$, $\dot{\theta} = 0$ so there are no fixed points in R.

Then we can apply the Poincare-Bendixson theorem to say a closed orbit inside R.

7.3.5 | Show that the system $\dot{x} = -x - y + x(x^2 + 2y^2)$ has at least one periodic solution

$$\dot{y} = x - y + y(x^2 + 2y^2)$$

First note $\dot{x}, \dot{y}$ are continuously differentiably since they are composed of polynomials. let's convert to polar coordinates to find fixed points.

$$r\dot{r} = x\dot{x} + y\dot{y} = x(-x-y+x(x^2+2y^2)) + y(x-y+y(x^2+2y^2))$$
$$= -x^2 - \cancel{xy} + x^2(x^2+2y^2) + \cancel{xy} - y^2 + y^2(x^2+2y^2)$$
$$r\dot{r} = -r^2 + r^2(x^2+2y^2)$$
$$\dot{r} = r(r^2\cos^2\theta + 2r^2\sin^2\theta - 1)$$

$$\dot{\theta} = (x\dot{y} - y\dot{x})/r^2 = (x(x-y+y(x^2+2y^2)) - y(-x-y+x(x^2+2y^2)))/r^2$$
$$= (x^2 - xy + xy(x^2+2y^2) + xy + y^2 - xy(x^2+2y^2))/r^2$$
$$\dot{\theta} = (x^2+y^2)/r^2 = r^2/r^2 = 1$$

Since $\dot{\theta} = 1$, the only fixed point is the origin. Now let's construct our trapping region R.

outward: $\dot{r} > 0$ exactly when $r^2\cos^2\theta + 2r^2\sin^2\theta - 1 > 0$

$$1/r^2 < \cos^2\theta + 2\sin^2\theta$$

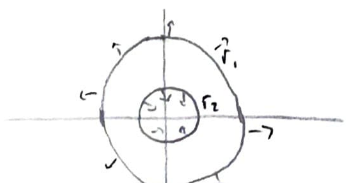
$$1/r^2 < 1 = \cos^2\theta + \sin^2\theta \leq \cos^2\theta + 2\sin^2\theta$$

So we can select $r_1 = 1.001$ for outward flow

inward: $\dot{r} < 0$ exactly when $r^2\cos^2\theta + 2r^2\sin^2\theta - 1 < 0$

$$1/r^2 > \cos^2\theta + 2\sin^2\theta$$

$1/r^2 \geq 3 = 1 + 2 \geq \cos^2\theta + 2\sin^2\theta$ so we choose $r_2 = 1/\sqrt{3}$ for our boundary with inward flow. The we have the following region



The image on the left is not a trapping region but negating $\dot{x}$ and $\dot{y}$ to get the system x' and y' gives a trapping region for x', y'. Therefore, a closed orbit exists in x', y'.

Since a closed orbit exists in the negated system x', y', a closed orbit must exist (in the same location but opposite direction) in the original system $\dot{x}, \dot{y}$.

7.4.1| Show that the equation $\ddot{x} + \mu(x^2-1)\dot{x} + \tanh x = 0$ for $\mu > 0$ has exactly one periodic solution and classify its stability.

If we satisfy the assumptions for Lienard's theorem, the equation will have a unique, stable limit cycle. Let $f(x) = \mu(x^2-1)$ and $g(x) = \tanh x$. We must verify the following assumptions

(1) f, g are continuously differentiable $\forall x$.

f is a polynomial so we are good. $g(x) = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

which is cont. diff.

(2) $g(x)$ is odd: $g(-x) = \dfrac{e^{-x} - e^{x}}{e^{-x}+e^{x}} = -\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = -g(x)$

(3) $g(x) > 0$ with $x > 0$

$g(x) > 0$ when $e^x - e^{-x} > 0 \iff e^x > e^{-x} \iff x > -x$
which is true since $x > 0$.

(4) $f(x)$ is even:
$f(-x) = \mu((-x)^2 - 1) = \mu(x^2-1) = f(x)$

(5) $F(x) = \int_0^x f(u)\,du$ has one positive zero a and $F(x) < 0$ for $0 < x < a$, positive and non-decreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$

$$F(x) = \int_0^x f(u)\,du = \int_0^x \mu(u^2-1)\,du = \mu\left(\tfrac{1}{3}u^3 - u\right)\Big|_0^x = \mu\left(\tfrac{1}{3}x^3 - x\right)$$
$$= \tfrac{1}{3}\mu x(x^2-3)$$

$F(x)$ has one positive 0 at $x = \sqrt{3} = a$
$F(1) = 1/3\,\mu(1-3) = -2/3\,\mu < 0$ so $F(x) < 0$ on $0 < x < a$.

For $x > a$, $F'(x) = f(x) > 0$ so $F(x)$ is non-decreasing and must be positive since $F(3) = \frac{1}{3}\mu 3(3^2-3) > 0$.

$\lim_{x \to \infty} F(x) = +\infty$

Thus we have verified all conditions for the Lienard theorem and $\ddot{x} + \mu(x^2-1)\dot{x} + \tanh x = 0$ must have a unique stable limit cycle.

7.5.7| Consider
$$\dot{u} = b(v-u)(\alpha + u^2) - u \qquad b \gg 1,\ \alpha \ll 1$$
$$\dot{v} = c - u \qquad 8\alpha b < 1$$

a) Sketch the null clines. The $\dot{v}$ null cline is $u = c$!

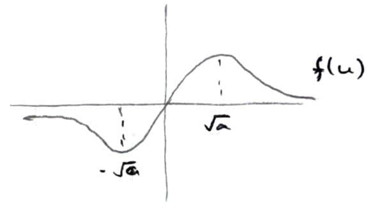
The u-cline is harder. $\dot{u} = 0 = b(v-u)(\alpha+u^2) - u$

$$u = b(v-u)(\alpha+u^2)$$

$$v-u = \frac{u/b}{\alpha+u^2} \iff v = u + \frac{u/b}{\alpha+u^2}$$

let $f(u) = \dfrac{u/b}{\alpha+u^2}$

$$f'(u) = \frac{1}{b}\,\frac{\alpha+u^2-2u^2}{(\alpha+u^2)^2} = \frac{1}{b}\,\frac{\alpha-u^2}{(\alpha+u^2)^2}$$



zeros at $u = \pm\sqrt{\alpha}$ and positive for $u>0$, odd, with $\lim_{u\to\infty} f(u) = 0$.

Then we just add $f(u)$ to u to get our null cline

$g(u) = u + \frac{1}{b}\,\frac{u}{\alpha+u^2}$, what does this look like?

$$g'(u) = 1 + \frac{1}{b}\,\frac{\alpha-u^2}{(\alpha+u^2)^2} = \frac{(\alpha+u^2)^2 + 1/b\,(\alpha-u^2)}{(\alpha+u^2)^2} = 0$$

$$b(\alpha+u^2)^2 + (\alpha-u^2) = 0$$

let $z = u^2$, then we have

$$\begin{aligned} 0 &= b(\alpha+z)^2 + \alpha - z \\ &= b(\alpha^2+2\alpha z+z^2)+\alpha - z \\ &= bz^2 + 2\alpha b z + b\alpha^2 + \alpha - z \\ 0 &= bz^2 + (2\alpha b-1)z + \alpha + b\alpha^2 \end{aligned}$$

$$z = \frac{1-2\alpha b \pm\sqrt{(2\alpha b-1)^2 - 4b(\alpha+b\alpha^2)}}{2b}$$

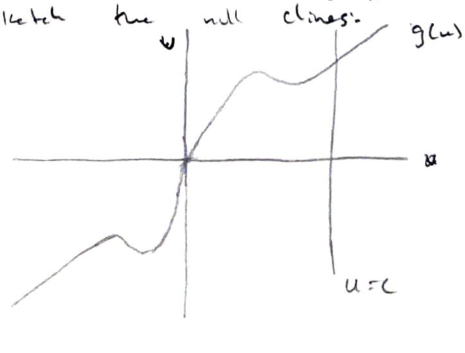
we must have $1-2\alpha b > 0$ and $(2\alpha b-1)^2 - 4b(\alpha+b\alpha^2) > 0$

$1-2\alpha b > 0 \iff 1 > 2\alpha b$ but $1 > 8\alpha b$ so certainly $1 > 2\alpha b$

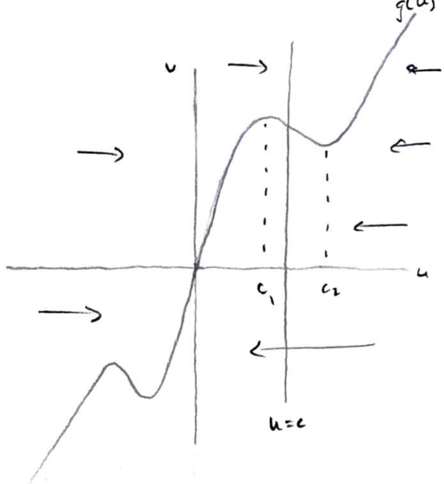
$$\begin{aligned} &(2\alpha b-1)^2 - 4b(\alpha+b\alpha^2) > 0 \\ &4\alpha^2 b^2 - 4\alpha b + 1 - 4\alpha b - 4\alpha^2 b^2 > 0 \\ &1 - 8\alpha b > 0 \\ &1 > 8\alpha b \end{aligned}$$

which we assumed to be true.

Then $g(u)$ must have 4 zeros and $g(u)$ is not monotonic and we can sketch the null clines.



b) Show that the system exhibits relaxation oscillations for $c_1 < c < c_2$ for approximate c_1, c_2.



let's redraw the null clines

c_1, c_2 on the graph would be the bounds if we can show that relaxation oscillations occur.

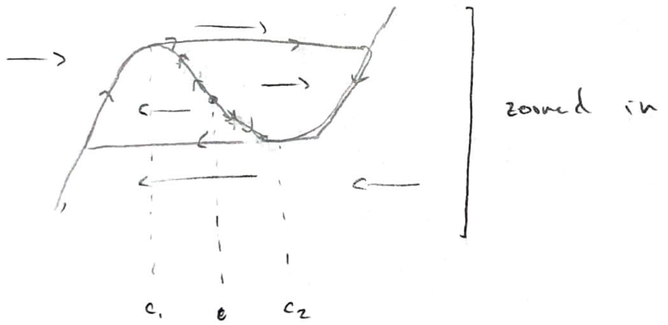
note if $v \neq g(u)$, then $\|\dot{u}\| \gg \|\dot{v}\|$

since

$\dot{u} \approx b\, v u^2 - u$

for left of $g(u)$, $\dot{u} > 0$ since the point $(u,v) = (0,1)$ gives $\dot{u} = b(1-\alpha)(\alpha + 0) + 0$
$= b\alpha(1-\alpha) > 0$

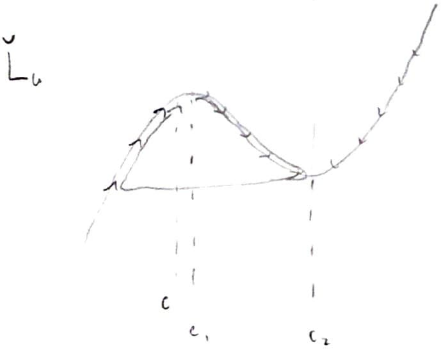
the particle the travels along $g(u)$ until rapidly moving to another part of the null cline.



The picture is a relaxation oscillation!

c) Show that the system is excitable of c is slightly less than c_1

We just draw the zoomed in image with a new cacline.



for perturbations to the left, we are stable but a perturbation to the right will take that triangle shaped path I drew. Therefore the system is excitable.