Nathan Stouffer

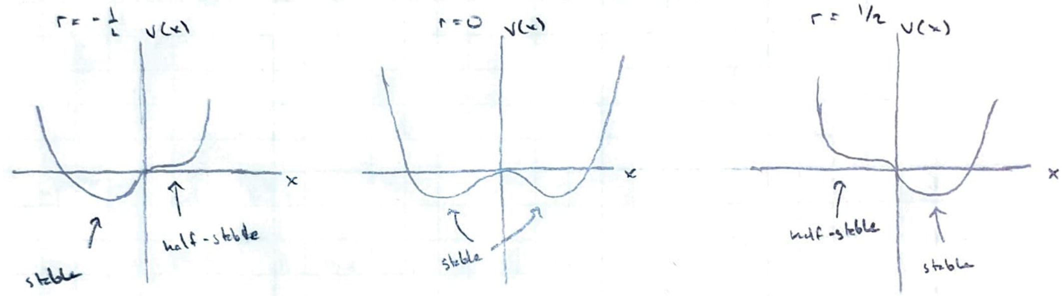
M 454 — Dynamics I

HW 2

2.7.6| Plot the potential $V(x)$ for $\dot{x} = r + x - x^3$. Identify equilibria and their stability. (for different values of r)

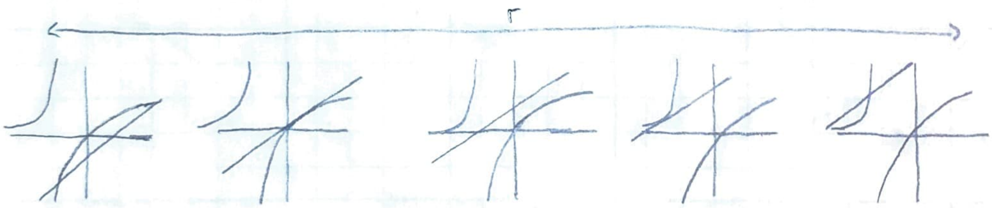
$$V(x) = -\int f(x)\,dx = -\int r + x - x^3\,dx = -\left[rx + \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right] = \tfrac{1}{4}x^4 - \tfrac{1}{2}x^2 - rx$$

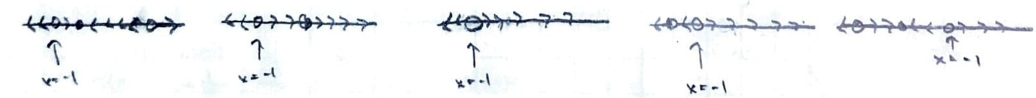
$$= \tfrac{1}{4}x\left(x^3 - 2x - 4r\right)$$



3.1.4| $\dot{x} = r + \frac{1}{2}x - x/(1+x)$

First we sketch all the qualitatively different vector fields for x





We know that a saddle bifurcation occurs for two critical values of r because of the different vector fields shown above. What are the critical values for x and r? We must have $f(x,r) = 0$ and $\frac{d}{dx}f(x,r) = 0$

$$f(x,r) = 0 = r + \tfrac{1}{2}x - \frac{x}{x+1} \Rightarrow r = \frac{x}{x+1} - \tfrac{1}{2}x \Rightarrow r(x+1) = x - \tfrac{1}{2}x(x+1)$$

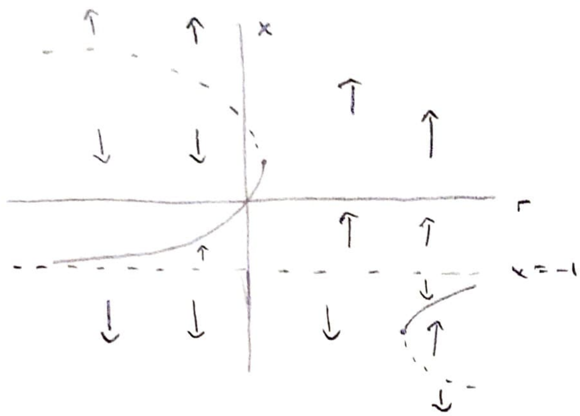
$$\Rightarrow r(x+1) = \tfrac{1}{2}x(1-x) \Rightarrow r = \frac{x(1-x)}{2(x+1)}$$

$$\frac{d}{dx}f(x,r) = 0 = \frac{1}{2} - \frac{(1+x)-x}{(1+x)^2} = \frac{1}{2} - \frac{1}{(1+x)^2} \Rightarrow 2 = (1+x)^2 = x^2 + 2x + 1$$

$$\Rightarrow 0 = x^2 + 2x - 1$$

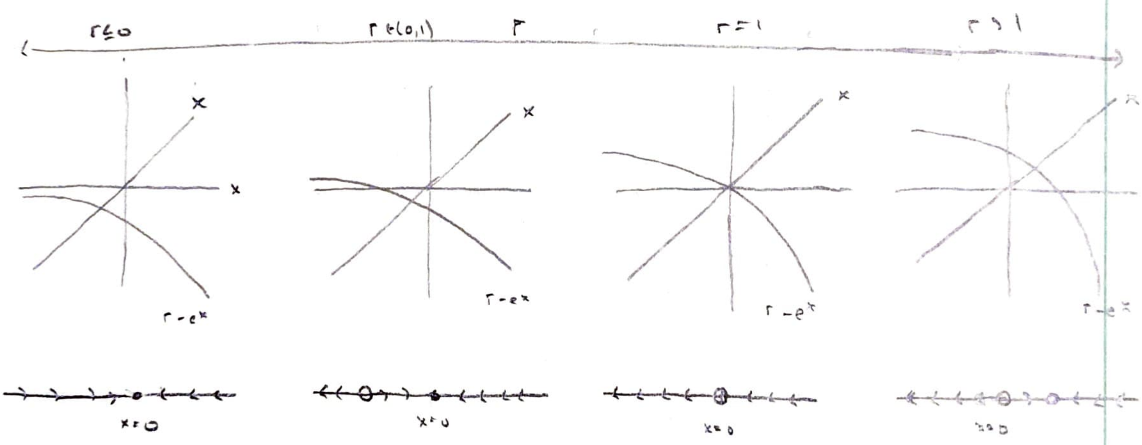
by Quadratic Formula $x = -1 \pm \sqrt{2}$ and the points that are saddle nodes are $\left(-1-\sqrt{2}, \frac{3+2\sqrt{2}}{2}\right)$ and $\left(-1+\sqrt{2}, \frac{3-2\sqrt{2}}{2}\right)$

We now sketch the bifurcation diagram:



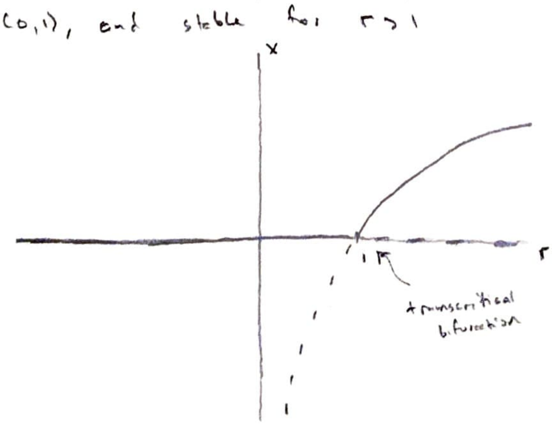
3.2.4) $\dot{x} = x(r - e^x)$

First we sketch all the qualitatively different vector fields as r varies



A transcritical bifurcation occurs at $x = 0$ when $r = 1$ (by looking at the vector fields for $r \in (0,1)$, $r = 1$, and $r > 1$)

To sketch the bifurcation diagram, we know $x = 0$ is a stable equilibria for $r < 1$ and unstable for $r > 1$. The other equilibria satisfies $r = e^x \Rightarrow x = \ln(r)$ which is nonexistent for $r \leq 0$, unstable for $r \in (0,1)$, and stable for $r > 1$



3.2.5| We have the following chemical reactions

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X \qquad B + X \xrightarrow{k_2} C$$

a) Assume that A and B are both held constant, show that $\dot{x} = c_1 x - c_2 x^2$ for some constants c_1, c_2.

We know that $A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$ induces $\dot{x} = k_1 a x - k_{-1} x^2$, including the second reaction produces

$$\dot{x} = k_1 a x - k_2 b x - k_{-1} x^2$$

$$\dot{x} = (k_1 a - k_2 b) x - k_{-1} x^2$$

$$\dot{x} = c_1 x - c_2 x^2 \quad \text{for } c_1 = k_1 a - k_2 b, \; c_2 = k_{-1}$$

b) Show that $x^* = 0$ is stable for $k_2 b > k_1 a$ and explain why this makes sense chemically.

First note that $x^* = 0$ is always an equilibria, then $x^* = 0$ is stable when $f'(0) < 0$

$$f'(x) = c_1 - 2c_2 x$$
$$f'(0) = c_1$$

$$c_1 < 0 \iff k_1 a - k_2 b < 0 \iff k_1 a < k_2 b \iff k_2 b > k_1 a$$

Chemically, this makes sense for two reasons. First, $k_2 b > k_1 a$ ensures that $B + X \xrightarrow{k_2} C$ has a chance to react, otherwise its effect is negligible. Second, if the second reaction is not negligible the concentration of X tends to 0 because there is no return from the second reaction. (while the first just oscillates)