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Dynamics

HW 7

**6.5.2** Consider the system $\ddot{x} = x - x^2$

a) Find the equilibria. let $y = \dot{x}$, then we have $\dot{x} = y$

$\dot{y} = x - x^2$

We must have
$\dot{x} = 0 \Rightarrow y = 0$
$\dot{y} = 0 = x - x^2 \Rightarrow 0 = x(1-x)$ $x = 0, 1$

$$J = \begin{bmatrix} 0 & 1 \\ 1-2x & 0 \end{bmatrix}$$

Case $(0,0)$:

$$J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \tau = 0, \Delta = -1 < 0$$

So $(0,0)$ is a saddle!

Case $(1,0)$:

$$J_{(1,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \tau = 0, \Delta = 1 > 0$$

so $(1,0)$ is a linear center

However, we now show the system is conservative which means that $(1,0)$ is a non-linear center.

$$\ddot{x} = x - x^2 \Rightarrow \ddot{x} + x^2 - x = 0 \Rightarrow \ddot{x}\dot{x} + x^2\dot{x} - x\dot{x} = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}\dot{x}^2 + \frac{1}{3}x^3 - \frac{1}{2}x^2\right] = 0$$

so we have the conserved quantity $E(x,y) = \frac{1}{2}y^2 + \frac{1}{3}x^3 - \frac{1}{2}x^2 = C$ along trajectories.

b) Sketch the phase portrait.

let's determine the stable/unstable manifolds of the saddle

For $(0,0)$, our eigenvalues are $\lambda_{1,2} = \frac{\pm\sqrt{0-4(-1)}}{2} = \pm 1$

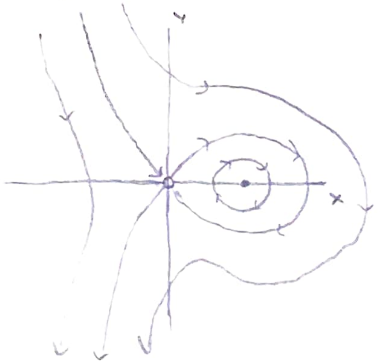
$\lambda_1 = 1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1$

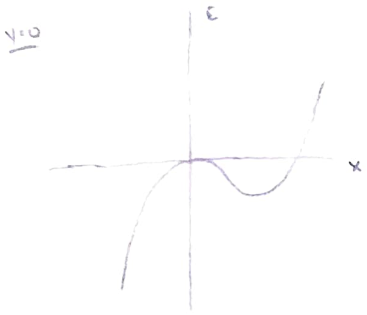
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

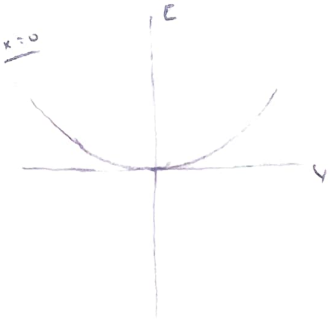
Then we have the phase portrait



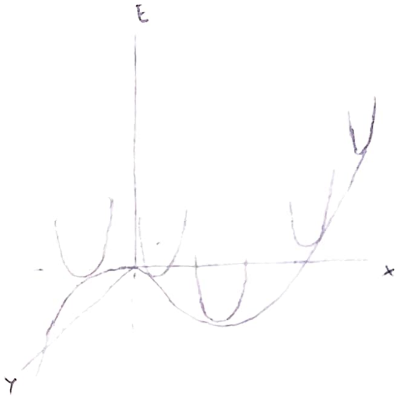
c) Find the equation for the homoclinic orbit.

From part a, we know that $E(x,y) = \frac{1}{2}y^2 + 1/3\,x^3 - 1/2\,x^2 = C$.





So our energy surface is



The homoclinic orbit occurs at the saddle of E, which is (0,0) which implies that the energy is 0. So the equation for the homoclinic orbit is

$$1/2\,y^2 = 1/2\,x^2 - 1/3\,x^3$$

**6.5.8** For a simple harmonic oscillator of mass m, spring constant k, displacement x, and momentum p the Hamiltonian is $H(x,p) = p^2/2m + kx^2/2$. Write out Hamilton's equations explicitly. Show that one equation gives momentum and the other is equivalent to F=ma. Verify that H is the total energy.

Hamilton's equations ($\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$) give

$$\dot{x} = \frac{\partial H}{\partial p} = p/m \Leftrightarrow p = m\dot{x} \quad (\text{momentum})$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx \Leftrightarrow \dot{p} = m\ddot{x} = -kx \quad (\text{which Newton's law for a spring})$$

Now we verify that $H(x,p)$ is the total energy along trajectories

$$H(x,p) = p^2/2m + kx^2/2 = (m\dot{x})^2/2m + kx^2/2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Newton's law: $m\ddot{x} = -kx \Rightarrow m\ddot{x} + kx = 0 \Rightarrow m\ddot{x}\dot{x} + kx\dot{x} = 0$

$$\frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right] = 0$$

So $E(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = C$
which equals $H(x,p)$!

**6.5.9** Show that for any Hamiltonian system, show that $H(x,p)$ is a conserved quantity.

For $H(x,p)$ to be conserved along trajectories, we must have

$\frac{d}{dt}H(x,p) = 0$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p}$$ but we know that the system is Hamiltonian so $\dot{x} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial x}$ $\Rightarrow$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial x} = 0$$

since the time derivative of the Hamiltonian function $H(x,p)$ it must be a conserved quantity.

**6.5.14** Consider a glider flying at speed $v$ at an angle $\theta$ to the horizontal. Its motion is governed approximately by the dimensionless equations

$$\dot{v} = -\sin\theta - Dv^2$$
$$v\dot{\theta} = -\cos\theta + v^2$$

where the trigonometric terms represent the effects of gravity and the $v^2$ terms represent the effects of drag and lift

a) Suppose there is no drag ($D=0$). Show that $v^3 - 3v\cos\theta$ is a conserved quantity. Sketch the phase portrait in this case. Interpret your results physically.

Since $D=0$, we have
$$\dot{v} = -\sin\theta$$
$$v\dot{\theta} = -\cos\theta + v^2$$

Now to show that $v^3 - 3v\cos\theta$ is conserved, we take the time derivative:

$$\frac{d}{dt}\left[v^3 - 3v\cos\theta\right] = 3v^2\dot{v} - (3\dot{v}\cos\theta - 3v\sin\theta\,\dot{\theta})$$
$$= 3v^2\dot{v} - 3\dot{v}\cos\theta + 3v\dot{\theta}\sin\theta$$
$$= -3v^2\sin\theta + 3\sin\theta\cos\theta + 3\sin\theta(-\cos\theta + v^2)$$
$$= -3v^2\sin\theta + 3v^2\sin\theta + 3\sin\theta\cos\theta - 3\sin\theta\cos\theta$$
$$= 0$$

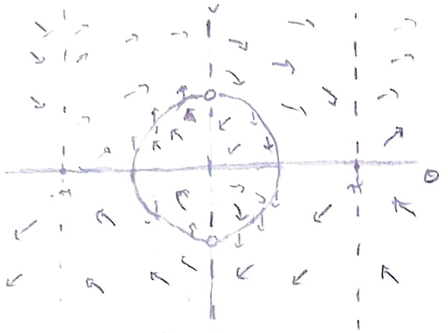
So $v^3 - 3v\cos\theta$ is conserved when $D=0$

Now we sketch the phase portrait. Note that null clines occur when

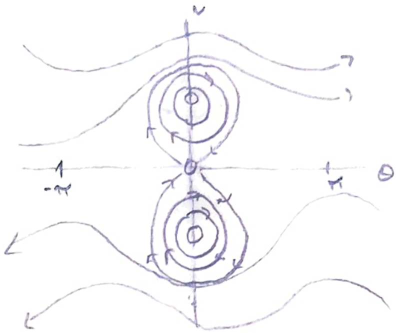
$\dot{v} = -\sin\theta = 0$ when $\theta = 0, \pi, -\pi$ (- - - )

$v\dot{\theta} = v^2 - \cos\theta = 0$ when $v^2 = \cos\theta$ (but $\dot{\theta}$ has a discontinuity @ $v=0$) (—)

Then we can sketch the null clines and phase portrait



null clines



phase portrait

Physically this means that if the glider starts in an area close to the centers, it will oscillate like a sine wave. Otherwise, the glider will do loop-dee-loops!

b) Analyze the case with positive drag $(D > 0)$.

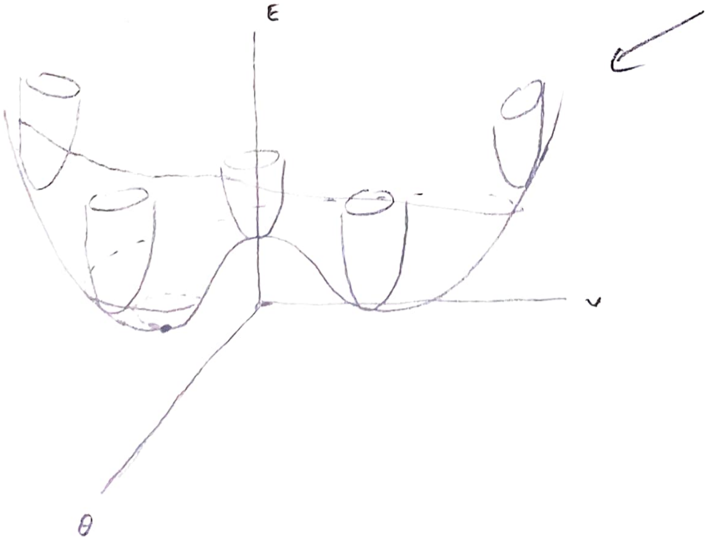
In part a, we found that $v^3 - 3v\cos\theta$ was conserved, But we now have

$\dot{v} = -\sin\theta - Dv^2$

$\dot{\theta} = -\cos\theta + v^2$

Because of the conserve quantity, we were able to draw level curves on our phase portrait. Without the conserved quantity, we can just move down the energy curve into one of the centers which are now sinks:

An example trajectory



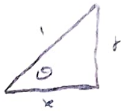
6.7.2] The equation $\ddot{\theta} + \sin\theta = \gamma$ describes the dynamics of an undamped pendulum driven by constant torque.

a) Find all equilibrium points and classify them as $\gamma$ varies.

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \gamma - \sin\theta$$

For equilibria, we must have $\omega = 0$ and $\sin\theta = \gamma$. If $|\gamma| > 1$, there are no equilibria. But if $\gamma \in [-1, 1]$ we can linearize the system.

$$J = \begin{bmatrix} 0 & 1 \\ -\cos\theta & 0 \end{bmatrix}$$ but $\sin\theta = \gamma \Rightarrow$



$\Rightarrow x = \sqrt{1 - \gamma^2}$

so $\cos\theta = \pm\sqrt{1-\gamma^2}$

From here, ~~we~~ can conclude if $\gamma = \pm 1$, then $\cos\theta = 0$ and

$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\tau = 0,\ \Delta = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$ degenerate case

Otherwise, $\gamma \in (-1, 1)$ and there are two equilibria with their respective Jacobians:

$J_- = \begin{bmatrix} 0 & 1 \\ \sqrt{1-\gamma^2} & 0 \end{bmatrix}$ $\tau = 0,\ \Delta = -\sqrt{1-\gamma^2}$ $\Rightarrow$ saddle

$J_+ = \begin{bmatrix} 0 & 1 \\ -\sqrt{1-\gamma^2} & 0 \end{bmatrix}$ $\tau = 0,\ \Delta = \sqrt{1-\gamma^2}$ $\Rightarrow$ linear center

However we can show that the system is conservative:

$$\ddot{\theta} + \sin\theta = \gamma \Rightarrow \ddot{\theta} + \sin\theta - \gamma = 0 \Rightarrow \ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta - \gamma\dot{\theta} = 0$$

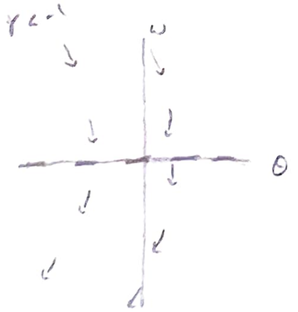
$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \cos\theta - \gamma\theta\right] = 0$$

The time derivative of $E(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \cos\theta - \gamma\theta = C$ is constant so our linear center is also a non-linear center

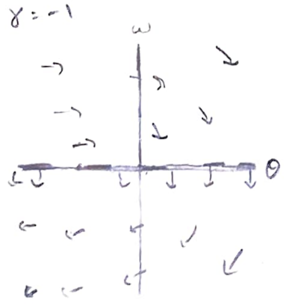
b) Sketch the null clines and the vector field.

Certainly ~~we~~ must have $\omega = 0$ and $\gamma = \sin\theta$ for null clines. This gives a few cases:

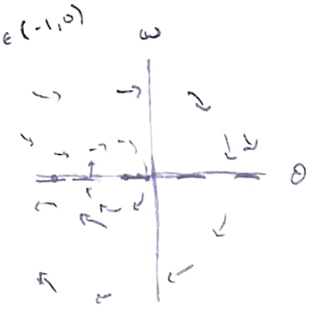
$\gamma < -1$



$\gamma = -1$

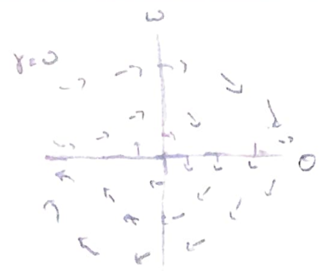


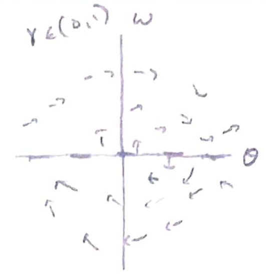
$\gamma \in (-1, 0)$

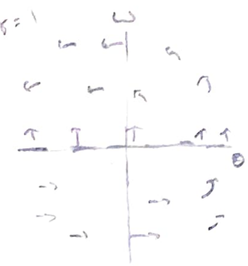


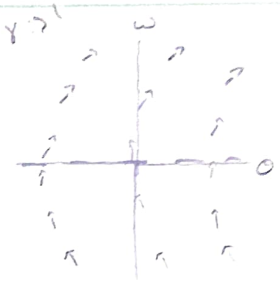
$2\pi$ periodic, this is on $[-\pi, \pi]$

2π periodic









c) The system is conservative, which I showed in part a. We conserve $\varepsilon(\theta, \omega) = \frac{1}{2}\omega^2 - \cos\theta - \gamma\theta = C$. How about reversible?

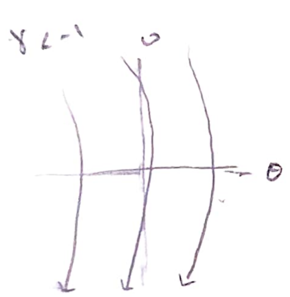
also see phase portraits for qualitative evidence

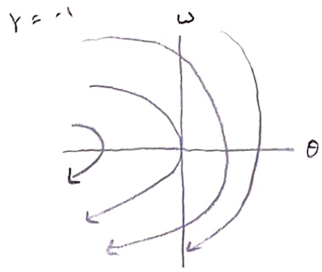
$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \gamma - \sin\theta \end{cases} \xrightarrow[\omega \to -\omega]{t \to -t} \underbrace{\begin{aligned} -\dot{\theta} &= -\omega \\ \dot{\omega} &= \gamma - \sin\theta \end{aligned}} \iff \begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \gamma - \sin\theta \end{aligned}$$

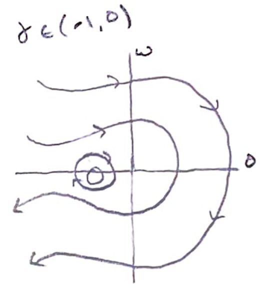
we have $\dot{\omega}$ since $\frac{d}{dt}(-\omega) = -\dot{\omega}$ but we scale by $-1$ w/ our new time $t \to -t$ $(\tau = -t)$

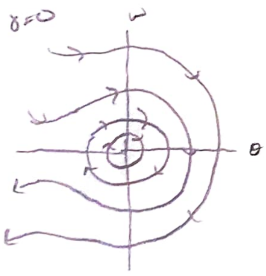
So the system is reversible!

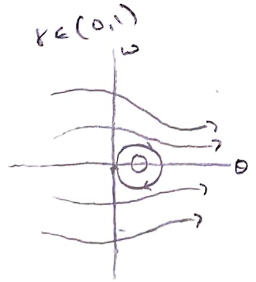
d) Sketch the phase portrait as $\gamma$ varies.

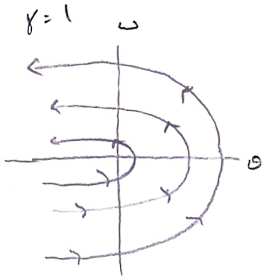


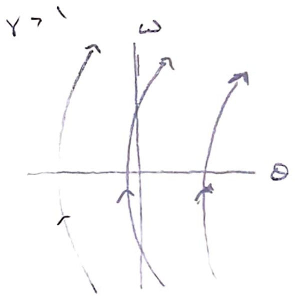












e) Find the approximate frequency of small oscillations about any centers in the phase portrait.

Since the system is conservative, we can use our linear approximation about the center to estimate frequency.

$$J_+ = \begin{bmatrix} 0 & 1 \\ -\sqrt{1-\gamma^2} & 0 \end{bmatrix} \qquad \lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{\pm\sqrt{-\sqrt{1-\gamma^2}}}{2} = \pm \frac{(1-\gamma^2)^{1/4}}{2} i = \pm \beta i$$

solutions are $\theta(t) = A \sin\beta t + B \cos\beta t$ So our frequency is $\frac{T}{\beta} = 2\pi \Rightarrow T = 2\beta\pi = (1-\gamma^2)^{1/4}\pi$