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Infinity modulus and the essential metric ☆

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A R T I C L E I N F O

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A B S T R A C T

We study ∞-modulus on general metric spaces and establish its relation to lengths of paths. This connection was already known for modulus on graphs, but the formulation in metric measure spaces requires more attention to exceptional families. We use this to define a metric that we call the essential metric, and show how this recovers a metric that had already been advanced in the literature by De Cecco and Palmieri.

1. Introduction

The p-modulus of a family of curves is a way to quantify the richness of such a family. This began as an important tool in function theory, because of its conformal invariance in the $p = 2$ case [1,3]. Later, p-modulus was defined and used successfully on n-dimensional Euclidean spaces, and general metric spaces as well [9,10]. The case $p = \infty$ has been studied on general metric spaces by the fourth author and her collaborators [7,6], and on graphs and networks by the first and third authors and their collaborators [2]. In particular, on graphs, ∞-modulus was found to be connected to the shortest path graph distance [2, Theorem 4.1]. This fact gave the impetus for the present paper. Here, we develop the theory of ∞-modulus on general metric spaces from first principles. The presentation is mostly self-contained, so that anyone with some background in analysis can follow easily.

More specifically, we first recall the notion of an ∞-exceptional family of curves, see Definition 3.4 and Lemma 3.6. Then, we introduce the concept of *essential length* for a family of curves Γ, which is denoted by

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$\mathrm{ess}\ell(\Gamma)$. This is the supremum of all the values $a \geq 0$ for which the subfamily of curves in Γ whose length is at most a is ∞-exceptional, see Definition 3.11.

Our first result, Theorem 3.13, can be summarized as follows.

Theorem 1.1. *Let Γ be a family of curves in a metric measure space. Then*

$$\mathrm{Mod}_\infty(\Gamma) = \frac{1}{\mathrm{ess}\ell(\Gamma)}.$$

This notion of essential length gives rise to a new metric that we call the *essential metric*. Basically, the essential distance (or metric) $d_{\mathrm{ess}}(x, y)$ between two points is equal to the essential length of the family of curves connecting x and y, see Definition 4.1. The essential metric is one way to measure the effective shortest path between two points. In the case of a graph, it is indeed associated with the shortest path between two points. In the setting of metric spaces the trajectory of a single curve (or countably many curves) might have measure zero; this does not happen in the case of a graph or network where the underlying space is locally one-dimensional. Thus, in the metric setting the essential metric does not measure the absolute shortest path between the two points. Instead, it can ignore a subfamily of "shortest" paths, if that collection is negligible (that is, has zero modulus). For example, the Sierpinski gasket is a quasiconvex metric space when viewed as a metric subspace of the Euclidean two-dimensional space and equipped with the $\log(3)/\log(2)$-dimensional Hausdorff measure, but the essential distance between distinct pairs of points there is infinite. Hence, in general, it is of interest to know which pairs of points have finite essential distance, and which pairs of points do not.

As an application, we revisit and reinterpret some of the existing results from the literature in this new light. More precisely, in Theorem 4.3, we establish a connection between our essential metric d_{ess} and a notion of intrinsic metric $\hat{d}$, see Definition 4.4, that was first introduced by De Cecco and Palmieri in [5]. The focus of [5] was the setting of Lipschitz manifolds, where (by Rademacher's theorem) the transition maps associated with the Lipschitz charts are Lipschitz continuous and hence are guaranteed to be differentiable almost everywhere. In defining a metric on such a manifold, one needs to ensure that there are curves that avoid (almost everywhere in a 1-dimensional sense) the non-differentiability set for the transition maps. To do so, De Cecco and Palmieri introduced the notion of intrinsic distance $\hat{d}$ on the manifold.

Later, the fourth author and her collaborators, in [7], considered the intrinsic metric $\hat{d}$ in the setting of metric measure spaces, and established connections between the existence of such intrinsic metrics and the fact that the underlying space supports an ∞-Poincaré inequality. Moreover, they also used this notion to establish the uniqueness of ∞-harmonic functions in doubling metric measure spaces which are not necessarily geodesic spaces, but support an ∞-Poincaré inequality.

This paper is structured as follows. In Section 2, we recall the basic tools necessary to compute the length of a curve in a metric space. Then in Section 3, we give two equivalent definitions of ∞-modulus on metric measure spaces, characterize the notion of ∞-exceptional families, introduce the notion of essential length of a family of curves, and show how it relates to ∞-modulus. Finally, in Section 4, we define the essential metric and show that it coincides with a differently defined metric that was first introduced by De Cecco and Palmieri.

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2. Basic tools in metric spaces

2.1. Length in metric spaces

We follow Chapter 5 in [10]. The material in this section can be found in other sources as well, but we collect as much as possible here, for the reader's convenience, so as not to require a lot of background.

A *metric space* (X, d) is a set X equipped with a *metric* d. That is, d is a function $d : X \times X \to \mathbb{R}$ satisfying non-degeneracy: $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$; symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$; and finally, the triangle inequality:

$$d(x, y) \leq d(x, z) + d(z, y) \qquad \forall x, y, z \in X.$$

A *path* γ in X is a continuous map $\gamma : [a, b] \longrightarrow (X, d)$ for some compact interval $[a, b] \subset \mathbb{R}$. Its *length* is the total variation:

$$\operatorname{length}(\gamma) := \sup_{a=t_0 \leq t_1 \cdots \leq t_N = b} \sum_{k=0}^{N-1} d(\gamma(t_k), \gamma(t_{k+1})), \tag{1}$$

where the supremum is taken over all possible partitions with N arbitrary.

The path γ is called *rectifiable* if $\operatorname{length}(\gamma) < \infty$. In this case, we can define the *length function* $s_\gamma : [a, b] \longrightarrow [0, \operatorname{length}(\gamma)]$ as

$$s_\gamma(t) := \operatorname{length}\left(\gamma|_{[a,t]}\right). \tag{2}$$

Clearly, s_γ is increasing (in the weak sense). Also it can be checked using (1) that for any $a \leq t < s \leq b$:

$$d(\gamma(t), \gamma(s)) \leq \operatorname{length}\left(\gamma|_{[t,s]}\right) = s_\gamma(s) - s_\gamma(t). \tag{3}$$

Lemma 2.1. *If γ is a rectifiable path in X, then s_γ is continuous.*

Proof. Suppose s_γ is not left continuous. Then there is $a < t_0 \leq b$ and $\delta > 0$ such that

$$s_\gamma(t_0) > s_\gamma(t) + \delta \qquad \text{for all } a < t < t_0. \tag{4}$$

Since γ is continuous at t_0, there is $a < t_1 < t_0$ so that

$$d(\gamma(t), \gamma(t_0)) < \delta/2 \qquad \text{for every } t_1 < t < t_0. \tag{5}$$

By (4), $\operatorname{length}\left(\gamma|_{[t_1,t_0]}\right) = s_\gamma(t_0) - s_\gamma(t_1) > \delta$. So, there is a partition $t_1 = s_0 < s_1 < \cdots < s_{N-1} < s_N = t_0$ of $[t_1, t_0]$, so that

$$\sum_{j=0}^{N-1} d(\gamma(s_j), \gamma(s_{j+1})) > \delta. \tag{6}$$

Using (5) with $t = s_{N-1}$, we get that

$$\sum_{j=0}^{N-2} d(\gamma(s_j), \gamma(s_{j+1})) > \delta/2. \tag{7}$$

Let $t_2 := s_{N-1}$ and repeat the steps above with t_2 in place of t_1, noting that both (4) and (5) remain valid. This process can be iterated indefinitely and it gives rise to a sequence $t_k < t_0$, such that $\operatorname{length}\left(\gamma|_{[t_1,t_k]}\right) \to \infty$. This contradicts the rectifiability of γ.

The proof that s_γ is right continuous is similar. $\square$

The length function s_γ is continuous and increasing, but not necessarily strictly increasing. Still, we can define a right-inverse $s_\gamma^{-1} : [0, \operatorname{length}(\gamma)] \to [a, b]$ as follows:

$$s_\gamma^{-1}(t) = \max\{s : s_\gamma(s) = t\} \qquad \forall t \in [0, \mathrm{length}(\gamma)] \tag{8}$$

Then s_γ^{-1} is increasing, right-continuous, and $s_\gamma(s_\gamma^{-1}(t)) = t$.

Definition 2.2. The *arc-length parametrization* of a rectifiable path $\gamma : [0,1] \to X$ is the curve $\gamma_s : [0, \mathrm{length}(\gamma)] \to X$ defined by

$$\gamma_s(t) := \gamma(s_\gamma^{-1}(t)).$$

In particular, $\gamma(u) = \gamma_s(s_\gamma(u))$, and

$$\mathrm{length}\left(\gamma_s|_{[t,u]}\right) = \mathrm{length}\left(\gamma|_{[s_\gamma^{-1}(t), s_\gamma^{-1}(u)]}\right) = s_\gamma(s_\gamma^{-1}(t)) - s_\gamma(s_\gamma^{-1}(u)) = t - u. \tag{9}$$

Definition 2.3. A path $\gamma : [0,1] \to X$ is *absolutely continuous* if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that whenever $\{(a_i, b_i)\}_{i=1}^N$ are disjoint intervals in $[0,1]$:

$$\sum_{i=1}^N |b_i - a_i| < \delta \Longrightarrow \sum_{i=1}^N d(\gamma(a_i), \gamma(b_i)) < \epsilon.$$

Proposition 2.4. *Suppose $\gamma : [0,1] \to X$ is rectifiable. Then γ is absolutely continuous if and only if s_γ is absolutely continuous.*

Proof. If s_γ is absolutely continuous, then by (3), the same is true of γ.

Conversely, assume that γ is absolutely continuous. Given $\epsilon > 0$, find $\delta = \delta(\epsilon) > 0$ as in Definition 2.3. Let $\{(a_i, b_i)\}_{i=1}^N$ be disjoint intervals in $[0,1]$ with

$$\sum_{i=1}^N |b_i - a_i| < \delta.$$

Then, as we have seen, $s_\gamma(b_i) - s_\gamma(a_i) = \mathrm{length}\left(\gamma\,|_{[a_i,b_i]}\right) < \infty$. By (1), there are disjoint intervals $\{(a_i^j, b_i^j)\}_{j=1}^{N_i}$ contained in (a_i, b_i) such that

$$\sum_{j=1}^{N_i} d(\gamma(a_i^j), \gamma(b_i^j)) \geq s_\gamma(b_i) - s_\gamma(a_i) - \frac{\epsilon}{N}.$$

By absolute continuity of γ, since the disjoint intervals $\{(a_i^j, b_i^j)\}_{i,j}$ also have length adding up to less than δ, we get

$$\sum_{i=1}^N s_\gamma(b_i) - s_\gamma(a_i) \leq \sum_{i=1}^N \sum_{j=1}^{N_i} d(\gamma(a_i^j), \gamma(b_i^j)) + \epsilon \leq 2\epsilon. \qquad \square$$

Next we recall Proposition 5.1.8 of [10].

Proposition 2.5 *([10]). Let $\gamma : [0,1] \to X$ be a compact rectifiable path. Then, its arc-length parametrization γ_s is absolutely continuous. Indeed, γ_s is 1-Lipschitz and*

$$\lim_{u \to t, u \neq t} \frac{d(\gamma_s(t), \gamma_s(u))}{|t-u|} = 1 \qquad \textit{for a.e. } t.$$

Definition 2.6. Suppose $\gamma : [0,1] \to X$ is rectifiable and $\rho : X \to [0,\infty]$ is Borel. Then the *line integral* of ρ along γ is

$$\int_\gamma \rho ds := \int_0^{\mathrm{length}(\gamma)} \rho(\gamma_s(t))dt. \tag{10}$$

Also, if $F \subset X$ is a Borel set, we say that γ *has positive length in* F if

$$\int_{\gamma\cap F} ds := \int_\gamma \mathbb{1}_F\, ds = \int_0^{\mathrm{length}(\gamma)} \mathbb{1}_{\gamma_s^{-1}(F)}(t)dt = m_1\left(\gamma_s^{-1}(F)\right) > 0, \tag{11}$$

where m_1 is the Lebesgue measure on $\mathbb{R}$. We write Γ_F^ℓ for the family of all curves that have positive length in F.

The key observation for (10) is that the composition of the Borel function ρ and the continuous function γ_s, is a measurable function.

In this paper, a *curve* will denote a non-constant path, defined on a possibly infinite interval $[a,b]$, that is locally rectifiable, meaning that every $t \in (a,b)$ has a neighborhood where γ is rectifiable. Unless otherwise stated, all the families that will be considered will be families of such curves.

Definition 2.7. Suppose $\gamma : [0,1] \to X$ is a curve and $F \subset X$ is a Borel set. We say that γ *spends positive time* in F if

$$\int_0^1 \mathbb{1}_{\gamma^{-1}(F)}(t)dt = m_1\left(\gamma^{-1}(F)\right) > 0.$$

We write Γ_F^τ for the family of all curves that spend positive time in F.

The two concepts of having positive length in F and spending positive time in F are in general unrelated. For instance, suppose γ is a curve traveling from left to right on $\mathbb{R}$ at constant speed, and suppose γ stops at the origin for one unit of time. Then, if $F = \{0\}$ is the singleton containing the origin, γ spends positive time in F, but γ does not have positive length in F. Conversely, consider the curve

$$\gamma(t) = (t, C(t)), \qquad t \in [0,1],$$

where $C(t)$ is the usual Cantor step-function. Let C be the middle-third Cantor set, D be the dyadic rationals in $[0,1]$, and $F = [0,1] \times ([0,1]\backslash D)$. Then $\gamma^{-1}(F) = C$, and so $m_1\left(\gamma^{-1}(F)\right) = 0$. Intuitively, the curve γ has infinite speed on C, and therefore spends zero time on $\gamma(C)$. Now, letting $E = [0,1] \times D$, we have

$$\sqrt{2} \le \mathrm{length}(\gamma) = m_1\left(\gamma_s^{-1}([0,1]\times[0,1])\right) = m_1\left(\gamma_s^{-1}(F)\right) + m_1\left(\gamma_s^{-1}(E)\right)$$

A simple computation shows $m_1\left(\gamma_s^{-1}(E)\right) = 1$, and hence, $m_1\left(\gamma_s^{-1}(F)\right) \ge \sqrt{2} - 1 > 0$. Therefore, γ has positive length in F. Note also that F can be intersected with $\gamma([0,1])$ so as to get an example where F has area measure zero.

On the other hand, if γ is absolutely continuous, then for every Borel set $F \subset X$ we have

$$m_1\left(\gamma^{-1}(F)\right) = 0 \Longrightarrow m_1\left(\gamma_s^{-1}(F)\right) = 0,$$

meaning that, in this case, if γ has positive length in F, then it also spends positive time in F. To see this, let $f = s_\gamma^{-1}$ and $g = \gamma$. Then we have $f^{-1}(A) \subset s_\gamma(A)$, $\forall A \subset [0,1]$ and also,

$$\gamma_s^{-1}(F) = (g \circ f)^{-1}(F) = f^{-1}\left(g^{-1}(F)\right).$$

By Proposition 2.4, s_γ is absolutely continuous, and hence,

$$m_1\left(\gamma^{-1}(F)\right) = 0 \Rightarrow m_1\left(s_\gamma\left(\gamma^{-1}(F)\right)\right) = 0 \Rightarrow m_1\left(\gamma_s^{-1}(F)\right) = 0.$$

Observe that, by Proposition 5.1.8 of [10] (see Proposition 2.5), an arc-length parametrized curve γ_s is necessarily absolutely continuous. Hence, for such a curve, the notion of spending positive time in F and the notion of having positive length in F coincide.

We end this section by defining admissible densities for a family of curves in a metric space.

Definition 2.8. Let Γ be a family of locally rectifiable curves in the metric space (X, d). A *density*, that is, a non-negative Borel function $\rho : X \to [0, \infty]$ is *admissible* for Γ if

$$\ell_\rho(\gamma) := \int_\gamma \rho ds \geq 1 \qquad \forall \gamma \in \Gamma.$$

We write $\mathrm{Adm}(\Gamma)$ for the set of all admissible densities for Γ, and note that this is purely a metric concept.

2.2. The supremum-modulus on metric spaces

On finite graphs, ∞-modulus is connected to shortest paths. Here, we extend this connection to the setting of metric spaces.

Given a family Γ of locally rectifiable curves in the metric space (X, d), the *supremum-modulus* of Γ is

$$\mathrm{Mod}_{\sup}(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \sup_X(\rho),$$

where $\sup_X(\rho) = \sup\{\rho(x) : x \in X\}$.

Proposition 2.9. *Let Γ be a non-empty family of locally rectifiable curves in a metric space (X, d). Assume that*

$$0 < \ell(\Gamma) := \inf_{\gamma \in \Gamma} \mathrm{length}(\gamma) < \infty.$$

Then

$$\mathrm{Mod}_{\sup}(\Gamma) = \frac{1}{\ell(\Gamma)}. \tag{12}$$

Proof. Since $\ell(\Gamma) > 0$, the density $\rho_0 \equiv \frac{1}{\ell(\Gamma)}$ is well-defined. Note that for all $\gamma \in \Gamma$:

$$\ell_{\rho_0}(\gamma) = \int_\gamma \rho_0 ds = \rho_0 \,\mathrm{length}(\gamma) = \frac{\mathrm{length}(\gamma)}{\ell(\Gamma)} \geq 1.$$

Therefore, $\rho_0 \in \mathrm{Adm}(\Gamma)$, hence $\mathrm{Adm}(\Gamma) \neq \emptyset$ and

$$\mathrm{Mod}_{\sup}(\Gamma) \leq \sup_X(\rho_0) = \ell(\Gamma)^{-1} < \infty.$$

Conversely, suppose $\rho \in \mathrm{Adm}(\Gamma)$. Then, given $\gamma \in \Gamma$,

$$1 \leq \int_\gamma \rho ds = \int_0^{\mathrm{length}(\gamma)} \rho(\gamma_s(t))dt \leq \sup_X(\rho)\,\mathrm{length}(\gamma).$$

Since $\gamma \in \Gamma$ and $\rho \in \mathrm{Adm}(\Gamma)$ are arbitrary and both Γ and $\mathrm{Adm}(\Gamma)$ are non-empty, we can take the infimum and get

$$\ell(\Gamma)\,\mathrm{Mod}_{\sup}(\Gamma) \geq 1. \qquad \square$$

Remark 2.10. Note that (12) makes sense also in limiting cases. For instance, if $\ell(\Gamma) = \infty$, then any constant $\rho > 0$ is admissible, hence $\mathrm{Mod}_{\sup}(\Gamma) = 0$. At the other extreme, if $\ell(\Gamma) = 0$, then there are arbitrarily short curves in Γ, and hence for each $\rho \in \mathrm{Adm}(\Gamma)$ we must have that $\sup_X(\rho) = \infty$ and hence $\mathrm{Mod}_{\sup}(\Gamma) = \infty$.

3. Infinity modulus on metric measure spaces

Geometric function theory grew out of complex analysis and real analysis on Euclidean spaces. It is therefore common to assume that the metric space X is equipped with a regular Borel measure μ. The triple (X, d, μ) is referred to as a *metric measure space*. For example, the measure μ can be obtained from the metric d as a Hausdorff measure for an appropriate dimension, but it doesn't have to be. It is customary to require that (X, d) is separable, and that for every point $x \in X$ there is a radius $r > 0$ such the corresponding metric ball has positive and finite measure, i.e., $0 < \mu(B(x, r)) < \infty$.

3.1. Infinity modulus and ∞-exceptional families

On a metric measure space (X, d, μ), it makes sense to talk about the *essential supremum*:

$$\|\rho\|_\infty := \inf\{a \geq 0 : \mu(\{x : \rho(x) > a\}) = 0\}.$$

Definition 3.1. Given a curve family Γ, let

$$\mathrm{Mod}_\infty(\Gamma) := \inf_{\rho\in\mathrm{Adm}(\Gamma)} \|\rho\|_\infty.$$

Next, we establish some standard modulus properties for Mod_∞.

Lemma 3.2. $\mathrm{Mod}_\infty(\Gamma)$ *has the following properties:*

(i) If $\Gamma_1 \subset \Gamma_2$, *then* $\mathrm{Mod}_\infty(\Gamma_1) \leq \mathrm{Mod}_\infty(\Gamma_2)$ **(monotone)***;*
(ii) Given $\{\Gamma_i\}_{i=1}^\infty$, $\mathrm{Mod}_\infty(\cup_{i=1}^\infty \Gamma_i) \leq \sup_i \mathrm{Mod}_\infty(\Gamma_i)$ **(ultra-subadditive)***;*
(iii) If for every $\gamma \in \Gamma_1$ *there is a subcurve* $\sigma \subset \gamma$ *such that* $\sigma \in \Gamma_2$ *(denoted* $\Gamma_2 \preceq \Gamma_1$*), then* $\mathrm{Mod}_\infty(\Gamma_1) \leq \mathrm{Mod}_\infty(\Gamma_2)$ **(shorter walks)***.*

Remark 3.3. Note that properties (i) and (ii) imply the following properties that are usually stated for p-modulus in the case $p < \infty$:

(ii′) $\mathrm{Mod}_\infty(\bigcup_{j\in\mathbb{N}}\Gamma_j) \le \sum_{j=1}^\infty \mathrm{Mod}_\infty(\Gamma_j)$ **(subadditive)**;
(ii″) If $\Gamma_1 \subset \Gamma_2 \subset \cdots$ and $\Gamma = \bigcup_{j=1}^\infty \Gamma_j$, then $\lim_{j\to\infty} \mathrm{Mod}_\infty(\Gamma_j) = \mathrm{Mod}_\infty(\Gamma)$ **(continuous from below)**.

Proof. (i) Suppose that $\Gamma_1 \subset \Gamma_2$. Then $\mathrm{Adm}(\Gamma_2) \subset \mathrm{Adm}(\Gamma_1)$. Thus, $\mathrm{Mod}_\infty(\Gamma_1) \le \mathrm{Mod}_\infty(\Gamma_2)$.

(ii) Without loss of generality, assume $\sup \mathrm{Mod}_\infty(\Gamma_i) < \infty$. Fix $\epsilon > 0$ and let $\rho_i \in \mathrm{Adm}(\Gamma_i)$ satisfy $\|\rho_i\|_\infty < \mathrm{Mod}_\infty(\Gamma_i) + \epsilon$. Let $Z_i = \{x \in X : \rho_i(x) > \|\rho_i\|_\infty\}$ and $Z = \cup_{i=1}^\infty Z_i$. Since $\mu(Z_i) = 0$, we also have $\mu(Z) = 0$. Define $\rho(x) := \sup_i \rho_i(x)$, for $x \in X$. Then, for any i, we have

$$\int_\gamma \rho ds \ge \int_\gamma \rho_i ds \ge 1$$

for every $\gamma \in \Gamma_i$. Hence, $\rho \in \mathrm{Adm}(\cup_{i=1}^\infty \Gamma_i)$.

Moreover, let $a = \sup_i \|\rho_i\|_\infty$ and let $Y = \{x \in X : \rho(x) > a\}$. By definition of ρ, if $x \in Y$, then there exists i so that $\rho_i(x) > a \ge \|\rho_i\|_\infty$, i.e. $x \in Z_i$. Therefore, $Y \subset Z$, which implies that $\|\rho\|_\infty \le a$. Thus, we have

$$\|\rho\|_\infty \le \sup_i \|\rho_i\|_\infty \le \sup_i \mathrm{Mod}_\infty(\Gamma_i) + \epsilon$$

Letting $\epsilon \to 0$, we obtain the result.

(iii) Note that $\mathrm{Adm}(\Gamma_2) \subset \mathrm{Adm}(\Gamma_1)$, so the infimum is taken over a larger set. □

Definition 3.4. A curve family Γ is said to be ∞-*exceptional*, if $\mathrm{Mod}_\infty(\Gamma) = 0$.

Example 3.5. Let Γ_∞ be the collection of all locally rectifiable curves that are not rectifiable. Then Γ_∞ is ∞-exceptional. Indeed, every constant density $\rho = a > 0$ is admissible for Γ_∞. Therefore, $\mathrm{Mod}_\infty(\Gamma_\infty) = 0$.

As a consequence of Example 3.5, from now on we will always assume, without loss of generality, that our curve families consist solely of rectifiable curves.

In the next lemma we combine Lemma 5.7 and Lemma 5.8 of [6], and add the direction (c) ⇒ (d). We rewrite the whole proof here for completeness.

Lemma 3.6. *Let Γ be a family of rectifiable curves in the metric measure space (X, d, μ). Then, the following are equivalent:*

(a) Γ is ∞-exceptional.
(b) There exists a Borel function $\rho : X \to [0, \infty]$ such that $\|\rho\|_\infty < \infty$ and $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$.
(c) There exists a Borel function $\rho : X \to [0, \infty]$ such that $\|\rho\|_\infty = 0$ and $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$.
(d) There is a Borel set F with $\mu(F) = 0$ such that $\Gamma \subset \Gamma_F^\ell$. In words, there is a Borel set of measure zero such that every curve in the family has positive length in that set.

Proof. (a) ⇒ (b): Assume $\mathrm{Mod}_\infty(\Gamma) = 0$. Then for $k = 1, 2, \dots$ there is $\rho_k \in \mathrm{Adm}(\Gamma)$ such that $\|\rho_k\|_\infty \le 2^{-k}$. Set $\rho := \sum_{k=1}^\infty \rho_k$. Then, $\|\rho\|_\infty \le 1$ and $\ell_\rho(\gamma) = \sum_{k=1}^\infty \ell_{\rho_k}(\gamma) = \infty$ for all $\gamma \in \Gamma$.

(b) ⇒ (c): We may assume without loss of generality that $\|\rho\|_\infty \le 1$ and $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$. For $a \ge 0$, consider the level sets

$$S_a(\rho) = \{x : \rho(x) > a\}.$$

Since $\|\rho\|_\infty \le 1$, we have $\mu(F) = 0$, where $F = S_1(\rho)$. Define $\tilde{\rho} := \rho\, \mathbb{1}_F$. Then, $\|\tilde{\rho}\|_\infty = 0$. We are left to show that $\ell_{\tilde{\rho}}(\gamma) = \infty$ for all $\gamma \in \Gamma$. Using (b) and the rectifiability of γ,

$$\infty = \int_\gamma \rho ds - \ell(\gamma) = \int_{\gamma\cap F} (\rho - 1)ds + \int_{\gamma\setminus F} (\rho - 1)ds$$
$$\leq \int_{\gamma\cap F} \rho ds = \int_\gamma \tilde{\rho} ds.$$

(c) $\Rightarrow$ (d): Assume $\|\rho\|_\infty = 0$ and $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$. Then $\mu(S_a(\rho)) = 0$ for every $a > 0$. Set $F = S_0(\rho) = \bigcup_{k\in\mathbb{N}} S_{1/k}(\rho)$. Then, by the subadditivity of measures, $\mu(F) = 0$. However, by the Cavalieri principle, for all $\gamma \in \Gamma$,

$$\infty = \int_\gamma \rho ds = \int_0^{\text{length}(\gamma)} \rho(\gamma_s(t))dt = \int_0^\infty m_1\left(\gamma_s^{-1}(S_a(\rho))\right) da.$$

In particular, there is $a_0 > 0$, such that $m_1\left(\gamma_s^{-1}(S_{a_0}(\rho))\right) > 0$. Therefore, choosing a positive integer k such that $a_0 > 1/k$, we have

$$\int_\gamma \mathbb{1}_{\gamma_s^{-1}(F)}\, du \geq \int_0^{\text{length}(\gamma)} \mathbb{1}_{\gamma_s^{-1}(S_{1/k}(\rho))}(t)dt > 0.$$

So $\Gamma \subset \Gamma_F^\ell$.

(d) $\Rightarrow$ (a): Assume F is a Borel set with $\mu(F) = 0$ and $\Gamma \subset \Gamma_F^\ell$. Then, $\rho := \infty\chi_F$ is a non-negative Borel function on X such that for each $\gamma \in \Gamma_F^\ell$ (and hence for each $\gamma \in \Gamma$) we have $\int_\gamma \rho\, ds = \infty$. Observe that $\|\rho\|_\infty = 0$, and so Γ is ∞-exceptional.

Hence, we have shown that (a), (b), (c), (d) are all equivalent. $\square$

Definition 3.7. We say that a property holds for ∞-*almost every curve*, if it fails only for an ∞-exceptional set of curves.

For instance, Lemma 3.6 says that if F is a Borel set with $\mu(F) = 0$, then ∞-almost every curve has no length in F.

3.2. Weakly admissible densities

In addition to $\mathrm{Mod}_{\sup}$ and Mod_∞ here we consider a third notion of infinity-modulus, which we will show coincides with Mod_∞.

Definition 3.8. We say that a Borel function $\rho : X \to [0, \infty]$ is *weakly admissible* and write $\rho \in \text{w-Adm}(\Gamma)$, if

$$\int_\gamma \rho ds \geq 1 \qquad \text{for } \infty\text{-a.e. } \gamma \in \Gamma.$$

Then, the *weak ∞-modulus* of a family Γ is

$$\mathrm{Mod}_\infty^*(\Gamma) := \inf_{\rho\in\text{w-Adm}(\Gamma)} \|\rho\|_\infty.$$

Remark 3.9. Since it is easier to be weakly admissible than admissible,

$$\mathrm{Mod}_\infty^*(\Gamma) \le \mathrm{Mod}_\infty(\Gamma).$$

In particular, an ∞-exceptional family has weak ∞-modulus zero.

Lemma 3.10. *We have*

$$\mathrm{Mod}_\infty^*(\Gamma) = \mathrm{Mod}_\infty(\Gamma).$$

Proof. In light of Remark 3.9, it suffices to show that $\mathrm{Mod}_\infty^*(\Gamma) \ge \mathrm{Mod}_\infty(\Gamma)$. To this end, let ρ be weakly admissible for Γ. Then there is a family $\Gamma_0 \subset \Gamma$ with $\mathrm{Mod}_\infty(\Gamma_0) = 0$ such that whenever $\gamma \in \Gamma \setminus \Gamma_0$ we have $\int_\gamma \rho\, ds \ge 1$. By Lemma 3.6 (c), there is a Borel function $\rho_0 : X \to [0, \infty]$ such that $\|\rho_0\|_\infty = 0$ and such that for each $\gamma \in \Gamma_0$ we have $\int_\gamma \rho_0\, ds = \infty$. Note then that $h := \rho + \rho_0$ belongs to $\mathrm{Adm}(\Gamma)$. Thus

$$\mathrm{Mod}_\infty(\Gamma) \le \|h\|_\infty = \|\rho + \rho_0\|_\infty \le \|\rho\|_\infty + \|\rho_0\|_\infty = \|\rho\|_\infty.$$

Taking the infimum over all ρ that are weakly admissible for Γ yields that

$$\mathrm{Mod}_\infty(\Gamma) \le \mathrm{Mod}_\infty^*(\Gamma),$$

as desired. □

3.3. Essential length

Definition 3.11. Let Γ be a family of curves. For every $a \ge 0$, let

$$\Gamma(a) := \{\gamma \in \Gamma : \ell(\gamma) < a\}. \tag{13}$$

The *essential length* of Γ is

$$\mathrm{ess}\ell(\Gamma) := \sup\{a \ge 0 : \Gamma(a) \text{ is } \infty\text{-exceptional}\}.$$

Note that, by definition, we always have

$$\ell(\Gamma) \le \mathrm{ess}\ell(\Gamma).$$

Remark 3.12. For all $a < \mathrm{ess}\ell(\Gamma)$, the family $\Gamma(a)$ from (13) is ∞-exceptional. Writing $a_0 := \mathrm{ess}\ell(\Gamma)$, and using the ultra-subadditivity of infinity modulus, Lemma 3.2 (ii), we get that $\Gamma(a_0)$ is ∞-exceptional.

Theorem 3.13. *Let (X, d, μ) be a metric measure space with μ Borel. Let Γ be a family of rectifiable curves in X.*

(a) If $\mathrm{ess}\ell(\Gamma) \in (0, \infty)$, then

$$\mathrm{Mod}_\infty(\Gamma) = \frac{1}{\mathrm{ess}\ell(\Gamma)} \in (0, \infty),$$

and $\rho_0 \equiv \mathrm{ess}\ell(\Gamma)^{-1}$ is an extremal weakly admissible density.

(b) If $\mathrm{ess}\ell(\Gamma) = 0$, then $\mathrm{Mod}_\infty(\Gamma) = \infty$ and no extremal weakly admissible density exists in $L^\infty(X)$.

(c) If $\mathrm{ess}\ell(\Gamma) = \infty$, then $\mathrm{Mod}_\infty(\Gamma) = 0$.

Proof. For part (a), write $a_0 := \mathrm{ess}\ell(\Gamma) \in (0, \infty)$. Suppose that $\rho \in \mathrm{Adm}(\Gamma)$. Set $F := \{x : \rho(x) > \|\rho\|_\infty\}$. Then F is a Borel set with $\mu(F) = 0$, so, by Lemma 3.6, Γ_F^ℓ is ∞-exceptional. On the other hand, Definition 3.11 and the fact that $\mathrm{ess}\ell(\Gamma) < \infty$ imply that, for every $\epsilon > 0$, $\Gamma(a_0 + \epsilon)$ is not ∞-exceptional. So, there is at least one curve $\gamma \in \Gamma(a_0 + \epsilon) \setminus \Gamma_F^\ell$. In particular, by the definition of Γ_F^ℓ given in (11),

$$\int_{\gamma \cap F} \rho ds = 0,$$

and by the admissibility of ρ,

$$1 \le \int_\gamma \rho ds = \int_{\gamma \cap F} \rho ds + \int_{\gamma \setminus F} \rho ds = \int_{\gamma \setminus F} \rho ds \le \|\rho\|_\infty \ell(\gamma) \le \|\rho\|_\infty (\mathrm{ess}\ell(\Gamma) + \epsilon).$$

Letting $\epsilon \to 0$, we find that

$$1 \le \|\rho\|_\infty \mathrm{ess}\ell(\Gamma).$$

Since $\rho \in \mathrm{Adm}(\Gamma)$ was arbitrary, we obtain that

$$\mathrm{Mod}_\infty(\Gamma) \ge \frac{1}{\mathrm{ess}\ell(\Gamma)}. \tag{14}$$

Conversely, let $\rho_0 \equiv \mathrm{ess}\ell(\Gamma)^{-1}$. Then, for every $\gamma \in \Gamma \setminus \Gamma(a_0)$, we have $\ell(\gamma) \ge \mathrm{ess}\ell(\Gamma)$, hence,

$$\int_\gamma \rho_0 ds = \frac{\ell(\gamma)}{\mathrm{ess}\ell(\Gamma)} \ge 1.$$

By Remark 3.12, $\Gamma(a_0)$ is ∞-exceptional. So ρ_0 is weakly admissible and therefore,

$$\mathrm{Mod}_\infty(\Gamma) = \mathrm{Mod}_\infty^*(\Gamma) \le \|\rho_0\|_\infty = \frac{1}{\mathrm{ess}\ell(\Gamma)}. \tag{15}$$

Thus, combining (14) with (15) and Lemma 3.10, part (a) is proved.

For (b), assume that $\mathrm{ess}\ell(\Gamma) = 0$. We shall show that no weakly admissible density ρ has finite essential norm; by convention, the infimum of an empty set is ∞, so it will follow that $\mathrm{Mod}_\infty(\Gamma) = \infty$. Let ρ be a density such that $\|\rho\|_\infty < \infty$ and, as before, define $F := \{z : \rho(z) > \|\rho\|_\infty\}$. By Definitions 3.4 and 3.11 and Lemma 3.10, $\mathrm{ess}\ell(\Gamma) = 0$ implies that $\mathrm{Mod}_\infty(\Gamma(a)) > 0$ for any $a > 0$. Moreover, since Γ_F^ℓ is ∞-exceptional, it follows that $\mathrm{Mod}_\infty(\Gamma(a) \setminus \Gamma_F^\ell) > 0$. But, for any $\gamma \in \Gamma(a) \setminus \Gamma_F^\ell$,

$$\int_\gamma \rho\, ds \le \|\rho\|_\infty \ell(\gamma) < a\|\rho\|_\infty.$$

For sufficiently small a,

$$\Gamma(a) \setminus \Gamma_F^\ell \subset \left\{ \gamma \in \Gamma : \int_\gamma \rho\, ds < 1 \right\}.$$

Since the former set has positive ∞-modulus, $\rho \notin \text{w-Adm}(\Gamma)$.

Finally for (c), assume that $\mathrm{ess}\ell(\Gamma) = \infty$. Then, $\Gamma(n)$ is ∞-exceptional for every $n \in \mathbb{N}$. Also, $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma(n)$. Therefore, by ultra-subadditivity of modulus, Lemma 3.2 (ii), we have that $\mathrm{Mod}_\infty(\Gamma) = 0$. $\square$

4. The essential metric

Definition 4.1. Let (X, d, μ) be a metric measure space, with μ Borel. Given two points $x \neq y \in X$, let

$$d_{\text{ess}}(x, y) := \text{ess}\ell(\Gamma(x, y)) = \text{Mod}_\infty(\Gamma(x, y))^{-1}, \tag{16}$$

where $\Gamma(x, y)$ is the "connecting family" consisting of all curves connecting x and y.

Note that $d_{\text{ess}}(x, y)$ could be infinite for some $x, y \in X$, e.g., if $\Gamma(x, y) = \emptyset$.

Example 4.2. If X is the Sierpinski gasket in the plane, equipped with the Euclidean metric and the natural Hausdorff measure, then from the results of [8] (see also [4]), the collection of all non-constant rectifiable curves in X is ∞-exceptional; thus in this case as well, even though $\Gamma(x, y)$ is non-empty for each pair of points $x, y \in X$, we have that $d_{\text{ess}}(x, y) = \infty$ when $x \neq y$.

Theorem 4.3. *The function $d_{\text{ess}} : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined in (16) for $x \neq y$, and defined to be zero on the diagonal of $X \times X$, is an extended metric on X. Moreover, if for each $x, y \in X$ with $x \neq y$ we have $\text{Mod}_\infty(\Gamma(x, y)) > 0$, then d_{ess} is a metric on X, with the property that $d_{\text{ess}} \geq d$.*

Proof. Let $x, y \in X$ with $x \neq y$. Since every curve can be reversed, the symmetry axiom holds: $d_{\text{ess}}(x, y) = d_{\text{ess}}(y, x)$.

Next we show that if $x \neq y$, then $d_{\text{ess}}(x, y) \geq d(x, y) > 0$. Since X is a metric space, we have $d(x, y) > 0$. By definition of length, every curve γ connecting x to y satisfies $\ell(\gamma) \geq d(x, y) > 0$. Therefore, for any $a < d(x, y)$, the family $\Gamma(x, y)(a)$, using the notation from (13), is empty, and thus ∞-exceptional. This shows that $d_{\text{ess}}(x, y) \geq d(x, y) > 0$.

Next, we verify the triangle inequality. Fix three distinct points $a, b, c \in X$. For simplicity, let $\Gamma_1 = \Gamma(a, c)$, $\Gamma_2 = \Gamma(c, b)$ and $\Gamma_0 = \Gamma(a, b)$. Also assume that $\delta_0 := d_{\text{ess}}(a, b)$, $\delta_1 := d_{\text{ess}}(a, c)$, $\delta_2 := d_{\text{ess}}(c, b)$, are all positive. Let $\Gamma = \Gamma(a, b; c)$ be the family of all curves that start at a, end at b, and go through c. Then $\Gamma \subset \Gamma_0$, and so $\delta_0 = \text{ess}\ell(\Gamma_0) \leq \text{ess}\ell(\Gamma)$.

If $\delta_1 + \delta_2 = \infty$, then clearly $\delta_0 \leq \delta_1 + \delta_2$. So we can assume that both δ_1 and δ_2 are finite. Fix $\epsilon > 0$ and let $\lambda := \delta_1 + \delta_2 + 2\epsilon$. We want to show that

$$\text{Mod}_\infty(\Gamma(\lambda)) > 0, \tag{17}$$

because, by monotonicity, that implies that $\text{Mod}_\infty(\Gamma_0(\lambda)) > 0$, and thus, by definition of essential length, $\delta_0 \leq \lambda = \delta_1 + \delta_2 + 2\epsilon$. Then letting ϵ tend to zero, yields the conclusion.

To that end, fix a density $\rho \in L^\infty(X)$ which is admissible for $\Gamma(\lambda)$. Such a ρ exists, because we have assumed that $\delta_0 > 0$, so $\text{Mod}_\infty(\Gamma_0) < \infty$. By Definition 3.11,

$$\text{Mod}_\infty(\Gamma_j(\delta_j + \epsilon/2)) > 0, \qquad \text{for } j = 1, 2. \tag{18}$$

Hence $\Gamma_j(\delta_j + \epsilon/2)$, $j = 1, 2$, are non-empty families. Fix two arbitrary curves $\alpha \in \Gamma_1(\delta_1 + \epsilon/2)$ and $\beta \in \Gamma_2(\delta_2 + \epsilon/2)$ and set $\gamma := \alpha + \beta$ to be the concatenation of α and β. Then

$$\ell(\gamma) = \ell(\alpha) + \ell(\beta) < (\delta_1 + \epsilon/2) + (\delta_2 + \epsilon/2).$$

So $\gamma \in \Gamma(\lambda)$. Therefore, by admissibility, $\int_\gamma \rho\, ds \geq 1$. In particular, if, for $j = 1, 2$, we set

$$\ell_\rho(\Gamma_j(\delta_j + \epsilon/2)) := \inf_{\gamma' \in \Gamma_j(\delta_j + \epsilon/2)} \int_{\gamma'} \rho\, ds,$$

then, since α and β were arbitrary,

$$\ell_\rho(\Gamma_1(\delta_1+\epsilon/2))+\ell_\rho(\Gamma_2(\delta_2+\epsilon/2))\geq 1. \tag{19}$$

If $\ell_\rho(\Gamma_j(\delta_j+\epsilon/2))=\infty$ for some $j\in\{1,2\}$, then for each $\gamma_0\in\Gamma_j(\delta_j+\epsilon/2)$ we would have $\int_{\gamma_0}\rho\,ds=\infty$. By Lemma 3.6, this implies that $\mathrm{Mod}_\infty(\Gamma_j(\delta_j+\epsilon/2))=0$, which violates (18) above. Hence $\ell_\rho(\Gamma_j(\delta_j+\epsilon/2))$ is finite for $j=1,2$.

Moreover, we claim that

$$\ell_\rho(\Gamma_j(\delta_j+\epsilon/2))\leq\frac{\|\rho\|_\infty}{\mathrm{Mod}_\infty(\Gamma_j(\delta_j+\epsilon/2))},\qquad\text{for } j=1,2. \tag{20}$$

To prove this, assume first that $\ell_\rho(\Gamma_j(\delta_j+\epsilon/2))>0$, for both $j=1,2$. Then,

$$\frac{\rho}{\ell_\rho(\Gamma_j(\delta_j+\epsilon/2))}\in\mathrm{Adm}(\Gamma_j(\delta_j+\epsilon/2)),\qquad\text{for } j=1,2. \tag{21}$$

In particular,

$$0<\mathrm{Mod}_\infty(\Gamma_j(\delta_j+\epsilon/2))\leq\frac{\|\rho\|_\infty}{\ell_\rho(\Gamma_j(\delta_j+\epsilon/2))},\qquad\text{for } j=1,2,$$

which implies (20). Lastly, if, say $\ell_\rho(\Gamma_1(\delta_1+\epsilon/2))=0$, then (20) holds trivially for $j=1$, and (19) implies that $\ell_\rho(\Gamma_2(\delta_2+\epsilon/2))\geq 1$, so the same admissibility argument in (21) works for $j=2$.

Combining (19) and (20), we get that

$$1\leq\|\rho\|_\infty\left[\frac{1}{\mathrm{Mod}_\infty(\Gamma_1(\delta_1+\epsilon/2))}+\frac{1}{\mathrm{Mod}_\infty(\Gamma_2(\delta_2+\epsilon/2))}\right],$$

and taking the infimum over all such admissible $\rho\in\mathrm{Adm}(\Gamma(\lambda))$ yields

$$\frac{1}{\mathrm{Mod}_\infty(\Gamma(\lambda))}\leq\left[\frac{1}{\mathrm{Mod}_\infty(\Gamma_1(\delta_1+\epsilon/2))}+\frac{1}{\mathrm{Mod}_\infty(\Gamma_2(\delta_2+\epsilon/2))}\right]<\infty,$$

where the finiteness follows from (18). This shows that $\mathrm{Mod}_\infty(\Gamma(\lambda))>0$, hence (17) is established, and the triangle inequality is proved.

Finally, suppose that for each $x,y\in X$ with $x\neq y$ we have $\mathrm{Mod}_\infty(\Gamma(x,y))>0$. Then to show that d_{ess} is a metric on X it is enough to show that $d_{\mathrm{ess}}(x,y)<\infty$ for $x,y\in X$. Note that by Remark 3.3 (ii″), $\mathrm{Mod}_\infty(\Gamma(x,y))=\lim_{n\to\infty}\mathrm{Mod}_\infty(\Gamma(x,y)(n))$. Therefore, since $\mathrm{Mod}_\infty(\Gamma(x,y))>0$, we must have $\mathrm{Mod}_\infty(\Gamma(x,y)(n))>0$ for some $n\in\mathbb{N}$. This implies that $d_{\mathrm{ess}}(x,y)\leq n<\infty$, completing the proof. □

The following definition is from [7], and is due to De Cecco and Palmieri [5].

Definition 4.4. Let (X,d,μ) be a metric measure space. Given a set $N\subset X$ with $\mu(N)=0$, and for $x,y\in X$ with $x\neq y$, we set

$$d_N(x,y):=\inf\{\ell(\gamma)\,:\,\gamma\text{ connects } x \text{ to } y,\text{ and } m_1(\gamma_s^{-1}(N))=0\}.$$

Also, define $\widehat{d}:X\times X\to\mathbb{R}$ by $\widehat{d}(x,x)=0$ and for $x\neq y$,

$$\widehat{d}(x,y):=\sup\{d_N(x,y)\,:\,N\subset X\text{ with }\mu(N)=0\}.$$

As in the case of the essential metric d_{ess}, it might happen that $\widehat{d}$ is not finite. It was shown in [7] that if μ is doubling and X is complete, then $\widehat{d}$ is biLipschitz equivalent to the original metric d if and only if X supports an ∞-Poincaré inequality.

Remark 4.5. From the results in [7] it follows that if $\widehat{d}$ is a metric on X such that μ is doubling with respect to this metric, then $(X, \widehat{d}, \mu)$ supports an ∞-Poincaré inequality. Here we say that μ is *doubling* if there is a constant $C \geq 1$ such that whenever $x \in X$ and $r > 0$, we have

$$\mu(B(x, 2r)) \leq C\, \mu(B(x, r)).$$

We say that X supports *an ∞-Poincaré inequality* if there are constants $C > 0$ and $\lambda \geq 1$ such that whenever $u \in L^\infty(X)$ and g is an upper gradient of u

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C\, r\, \|g\chi_{B(x,\lambda r)}\|_\infty \qquad \forall x \in X, r > 0.$$

A non-negative Borel function g on X is said to be an *upper gradient* if for each rectifiable curve γ in X we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds. \tag{22}$$

The notion of upper gradients is due to Heinonen and Koskela, see [10]. See the papers [7,8] for more on the ∞-Poincaré inequality. Heuristically, the ∞-Poincaré inequality gives us a way of controlling the variance of a function on a ball in terms of its ∞-energy on a slightly enlarged ball.

Proposition 4.6. *If (X, d, μ) is a metric measure space, then $\widehat{d} = d_{ess}$.*

Proof. Fix $x, y \in X$ with $x \neq y$. We will first show that $\widehat{d}(x, y) \leq d_{\text{ess}}(x, y)$. To this end, note that if $d_{\text{ess}}(x, y) = \infty$, then the above inequality holds trivially. Therefore, let us assume that $d_{\text{ess}}(x, y) < \infty$. Recall that $\Gamma := \Gamma(x, y)$ denotes the collection of all rectifiable curves in X connecting x to y. For each $\varepsilon > 0$, if $a = [1+\varepsilon]d_{\text{ess}}(x, y)$, then $\operatorname{Mod}_\infty(\Gamma(a)) > 0$. So, by Lemma 3.6 (d), whenever $N \subset X$ with $\mu(N) = 0$ there must exist a curve γ in $\Gamma(a)$ that does not spend positive time in N. In particular, $m_1(\gamma_s^{-1}(N)) = 0$. Hence,

$$d_N(x, y) \leq \ell(\gamma) < [1 + \varepsilon]d_{\text{ess}}(x, y).$$

Taking the supremum over all such nulls sets N gives $\widehat{d}(x, y) \leq [1 + \varepsilon]d_{\text{ess}}(x, y)$. Letting $\varepsilon \to 0$ gives the desired inequality.

We next show that $d_{\text{ess}}(x, y) \leq \widehat{d}(x, y)$. First suppose that $d_{\text{ess}}(x, y) = \infty$. Then, by Theorem 3.13 (c), $\operatorname{Mod}_\infty(\Gamma(x, y)) = 0$. Thus by Lemma 3.6 (d), there is some $N \subset X$ such that $\mu(N) = 0$ and $\Gamma(x, y) \subset \Gamma_N^\ell$. Thus $\widehat{d}(x, y) \geq d_N(x, y) = \infty$.

Now consider the case that $d_{\text{ess}}(x, y) < \infty$. By Theorem 4.3, $d_{\text{ess}}(x, y) \geq d(x, y) > 0$, since $x \neq y$. Therefore, $\operatorname{Mod}_\infty(\Gamma(x, y)(a)) = 0$, whenever $a = [1 - \varepsilon]d_{\text{ess}}(x, y)$ for $0 < \varepsilon < 1$. By Lemma 3.6 (d), there is a Borel set $N \subset X$, with $\mu(N) = 0$, such that every curve in $\Gamma(x, y)(a)$ has positive length in N. In particular, $d_N(x, y) \geq a$. Note that if $\Gamma(x, y)(a)$ is empty, then we can choose N to be the empty set, and by the fact that $\Gamma(x, y)(a)$ is empty we know that *every* curve with end points x and y must have length at least a. Observe that $d_\emptyset(x,y)$ is merely the inner length metric on X. Hence,

$$\widehat{d}(x,y) \geq d_N(x,y) \geq [1-\varepsilon]d_{\mathrm{ess}}(x,y).$$

Letting $\varepsilon \to 0$ gives $\widehat{d}(x,y) \geq d_{\mathrm{ess}}(x,y)$ as desired. $\square$

Remark 4.7. By Theorem 4.3, $d_{\mathrm{ess}} \geq d$. Thus, if $\{x_i\}_i$ is a sequence in X converging with respect to d_{ess} to a point $x \in X$, then this sequence also converges in the original metric d. Therefore, in general, the topology generated by d_{ess} is finer than that of d.

Given a metric measure space, it is of interest to know whether there are a great many curves of controlled length connecting any given pair of disjoint continua. The quantity d_{ess} is a valuable tool in this, as $d_{\mathrm{ess}}(x,y) = \infty$ tells us that there are very few such curves connecting x to y. Therefore d_{ess} is a more sensitive metric on X, and it would be interesting to know when the space $(X, d_{\mathrm{ess}}, \mu)$ supports an ∞-Poincaré inequality.

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