M 441: Homework 4

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Problem 1

Problem. Compute $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$ where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

For $\|A\|_2$ you may use the fact that the characteristic polynomial of A^TA is

$$P(\lambda) = \det(A^TA - \lambda I) = (\lambda - 2)(\lambda - 8)(\lambda - 16)$$

Use the fact that $\|A\|_2 = \sqrt{\lambda_{max}}$ where λ_{max} is the largest eigenvalue of A^TA.

Answer. First we compute $\|A\|_1$:

$$\|A\|_1 = \max_j \sum_{i=1}^{2} |a_{ij}| = \max\{4, 4, 4\} = 4$$

Then we compute $\|A\|_2$ (note that λ_{max} is the largest eigenvalue of A^TA):

$$\|A\|_2 = \sqrt{\lambda_{max}} = \sqrt{16} = 4$$

Finally, we compute $\|A\|_\infty$:

$$\|A\|_\infty = \max_i \sum_{j=1}^{2} |a_{ij}| = \max\{5, 2, 5\} = 5$$

Problem 2

Problem. Recall that the condition number $\kappa(A)$ of a matrix A is defined by

$$\kappa(A) = \|A\| \ \|A^{-1}\|$$

The main theoerm conclusion related to the condition number is that

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa(A) \frac{\|b - \hat{b}\|}{\|b\|}$$

where x and $\hat{x}$ are solutions to $Ax = b$ and $A\hat{x} = \hat{b}$. When $\kappa(A)$ is large, the system is said to be ill-conditioned and even small relative errors in b can result in large relative errors in the solution x. In this problem

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6.01 \end{bmatrix} \qquad b = \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} \qquad \hat{b} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

and all norms are the ∞-norm as in $\|A\|_\infty$.

a) Compute A^{-1} exactly.

b) Compute x and $\hat{x}$ exactly.

c) Find the relative errors below. Do they differ by a lot?

$$e_x = \frac{\|x - \hat{x}\|}{\|x\|} \qquad e_b = \frac{\|b - \hat{b}\|}{\|b\|}$$

d) Compute the condition number

$$\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

Answer.

a) We compute the exact inverse of A. Note that $\det A = 1/100$. Then the exact inverse of A is

$$A^{-1} = \frac{1}{100} \begin{bmatrix} 6.01 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 601 & -200 \\ -300 & 100 \end{bmatrix}$$

b) We now compute x and $\hat{x}$ exactly.

$$x = A^{-1}b = \begin{bmatrix} 601 & -200 \\ -300 & 100 \end{bmatrix} \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} = \begin{pmatrix} 1803 - 1802 \\ -900 + 901 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{x} = A^{-1}\hat{b} = \begin{bmatrix} 601 & -200 \\ -300 & 100 \end{bmatrix} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1803 - 1800 \\ -900 + 90 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

c) We now compute and compare the relative errors e_x and e_b.

$$e_x = \frac{\|x - \hat{x}\|}{\|x\|} = \frac{\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \|}{\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \|} = \frac{\| \begin{pmatrix} -2 \\ 1 \end{pmatrix} \|}{\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \|} = \frac{2}{1} = 2$$

$$e_b = \frac{\|b - \hat{b}\|}{\|b\|} = \frac{\| \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \end{pmatrix} \|}{\| \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} \|} = \frac{\| \begin{pmatrix} 0 \\ 0.01 \end{pmatrix} \|}{\| \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} \|} = \frac{0.01}{9.01} = \frac{1}{901}$$

The relative errors differ by a large amount. The relative error in our input variable is $e_b = 1/901 \approx 0.11\%$ which is rather small. However, our relative output error is $e_x = 2 = 200\%$, which is much larger than our input error. This suggests that our system is highly sensitive to small changes in initial conditions.

d) We now compute $\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$. Note that $\|A\|_\infty = \max\{3, 9.01\} = 9.01$ and $\|A^{-1}\|_\infty = \max\{801, 400\} = 801$. Then we can compute the condition number:

$$\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty = 9.01 * 801 = 7217.01$$

Problem 3

Problem. Recall the general iterative technique for solving $Ax = b$ has the split equation

$$Qx_{n+1} = (Q - A)x_n + b$$

which may be written as

$$x_{n+1} = Kx_n + c \qquad K = I - Q^{-1}A \qquad c = Q^{-1}b$$

In all of the following questions, we have

$$A = \begin{bmatrix} 20 & 1 \\ -1/2 & 2 \end{bmatrix} \qquad b = \begin{pmatrix} 120 \\ 159 \end{pmatrix} \qquad x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

a) Use the matlab function "Iterate.m" to approximate the solution of $Ax = b$ using the Gauss-Seidel, Jacobi, and Richardson iteration techniques for $N = 10$ iterates each. For each method, print out the iteration matrix x and state if the method converges.

b) For the Gauss-Seidel technique, $\|K\|_1 = 1/16$. Use this and the fact that $\|e_{n+1}\| \leq \|K\|^n \|e_0\| = \|K\|^n \|x\|$ (when $x_0 = \mathbf{0}$) to find the minimum value for n that ensures the following relative error tolerance:

$$\frac{\|x_{n+1} - x\|_1}{\|x\|_1} < 10^{-12}$$

Answer.

a) The iteration matrix for Gauss-Seidel technique. This method converges.

x_1	x_2
0	0
6.000000000000000	81.000000000000000
1.950000000000000	79.987499999999997
2.000625000000000	80.000156250000003
1.999992187499999	79.999998046875007
2.000000097656249	80.000000024414064
1.999999998779296	79.999999999694822
2.000000000015259	80.000000000003823
1.999999999999809	79.999999999999957
2.000000000000002	80.000000000000000

The iteration matrix for Jacobi technique. This method converges.

x_1	x_2
0	0
6.000000000000000	79.500000000000000
2.025000000000000	81.000000000000000
1.950000000000000	80.006249999999994
1.999687500000000	79.987499999999997
2.000625000000000	79.999921874999998
2.000003906250000	80.000156250000003
1.999992187499999	80.000000976562504
1.999999951171874	79.999998046875007
2.000000097656249	79.999999987792975

The iteration matrix for Richardson technique (note that each value is multiplied by 10^{12}). This method does not converge.

x_1	x_2
0	0
0.000000000120000	0.000000000159000
-0.000000002319000	0.000000000060000
0.000000044121000	-0.000000001060500
-0.000000837118500	0.000000023280000
0.000015882091500	-0.000000441680250
-0.000301317938250	0.000008382885000
0.005716658061750	-0.000159041695125
-0.108457461358125	0.003017370885000
2.057674395039375	-0.057246101405062

b) We now want to know the smallest n that satisfies $\dfrac{\|x_{n+1}-x\|_1}{\|x\|_1} < 10^{-12}$. Since we have chosen $x_0 = \mathbf{0}$, we can say that

$$\|x_{n+1}-x\|_1 = \|e_{n+1}\|_1 \leq \|K\|_1^n \|e_0\|_1 = \|K\|_1^n \|x\|_1$$

But then we can simplify:

$$\frac{\|x_{n+1}-x\|_1}{\|x\|_1} \leq \frac{\|K\|_1^n \|x\|_1}{\|x\|_1} = \|K\|_1^n$$

So we need only wonder what is the smallest n that satisfies $\|K\|_1^n < 10^{-12}$:

$$\|K\|_1^n = 1/16^n < 10^{-12} = 1/10^{12} \iff 16^n > 10^{12} \iff n > \lfloor 12 \log_{16}(10) \rfloor$$

But $\lfloor 12 \log_{16}(10) \rfloor = \lfloor 12 \dfrac{\log 10}{\log 16} \rfloor = \lfloor 12/\log 16 \rfloor = 9$ so we must have $n > 9$. Therefore, the smallest n that satisfies $\dfrac{\|x_{n+1}-x\|_1}{\|x\|_1} < 10^{-12}$ is $n = 10$.