M 441: Homework 4

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Problem 1

Problem. Compute $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$ where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

For $\|A\|_2$ you may use the fact that the characteristic polynomial of A^TA is

$$P(\lambda) = \det(A^TA - \lambda I) = (\lambda - 2)(\lambda - 8)(\lambda - 16)$$

Use the fact that $\|A\|_2 = \sqrt{\lambda_{max}}$ where λ_{max} is the largest eigenvalue of A^TA.

Answer. First we compute $\|A\|_1$:

$$\|A\|_1 = \max_j \sum_{i=1}^{2} |a_{ij}| = \max\{4, 4, 4\} = 4$$

Then we compute $\|A\|_2$ (note that λ_{max} is the largest eigenvalue of A^TA):

$$\|A\|_2 = \sqrt{\lambda_{max}} = \sqrt{16} = 4$$

Finally, we compute $\|A\|_\infty$:

$$\|A\|_\infty = \max_i \sum_{j=1}^{2} |a_{ij}| = \max\{5, 2, 5\} = 5$$

Problem 2

Problem. Recall that the condition number $\kappa(A)$ of a matrix A is defined by

$$\kappa(A) = \|A\| \ \|A^{-1}\|$$

The main theoerm conclusion related to the condition number is that

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa(A) \frac{\|b - \|}{\|b\|}$$

where x and $\hat{x}$ are solutions to $Ax = b$ and $A\hat{x} = .$ When $\kappa(A)$ is large, the system is said to be ill-conditioned and even small relative errors in b can result in large relative errors in the solution x. In this problem

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6.01 \end{bmatrix} \qquad b = \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} \qquad = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

and all norms are the ∞-norm as in $\|A\|_\infty$.

a) Compute A^{-1} exactly.

b) Compute x and $\hat{x}$ exactly.

c) Find the relative errors below. Do they differ by a lot?

$$e_x = \frac{\|x - \hat{x}\|}{\|x\|} \qquad e_b = \frac{\|b - \|}{\|b\|}$$

d) Compute the condition number

$$\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

Answer.

a) We compute the exact inverse of A. Note that $\det A = 1/100$. Then the exact inverse of A is

$$A^{-1} = \frac{1}{100} \begin{bmatrix} 6.01 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 601 & -200 \\ -300 & 100 \end{bmatrix}$$

b) We now compute x and $\hat{x}$ exactly.

$$x = A^{-1}b = \begin{bmatrix} 601 & -200 \\ -300 & 100 \end{bmatrix} \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} = \begin{pmatrix} 1803 - 1802 \\ -900 + 901 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{x} = A^{-1} = \begin{bmatrix} 601 & -200 \\ -300 & 100 \end{bmatrix} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1803 - 1800 \\ -900 + 90 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

c) We now compute and compare the relative errors e_x and e_b.

$$e_x = \frac{\|x - \hat{x}\|}{\|x\|} = \frac{\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \|}{\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \|} = \frac{\| \begin{pmatrix} -2 \\ 1 \end{pmatrix} \|}{\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \|} = \frac{2}{1} = 2$$

$$e_b = \frac{\|b - \|}{\|b\|} = \frac{\| \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \end{pmatrix} \|}{\| \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} \|} = \frac{\| \begin{pmatrix} 0 \\ 0.01 \end{pmatrix} \|}{\| \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} \|} = \frac{0.01}{9.01} = \frac{1}{901}$$

The relative errors differ by a large amount. The relative error in our input variable is $e_b = 1/901 \approx 0.11\%$ which is rather small. However, our relative output error is $e_x = 2 = 200\%$, which is much larger than our input error. This suggests that our system is highly sensitive to small changes in initial conditions.

d) We now compute $\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$. Note that $\|A\|_\infty = \max\{3, 9.01\} = 9.01$ and $\|A^{-1}\|_\infty = \max\{801, 400\} = 801$. Then we can compute the condition number:

$$\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty = 9.01 * 801 = 7217.01$$

Problem 3

Problem. Recall the general iterative technique for solving $Ax = b$ has the split equation

$$Qx_{n+1} = (Q - A)x_n + b$$

which may be written as

$$x_{n+1} = Kx_n + c \qquad K = I - Q^{-1}A \qquad c = Q^{-1}b$$

In all of the following questions, we have

$$A = \begin{bmatrix} 20 & 1 \\ -1/2 & 2 \end{bmatrix} \qquad b = \begin{pmatrix} 120 \\ 159 \end{pmatrix} \qquad x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

a) Use the matlab function "Iterate.m" to approximat ethe solution of $Ax = b$ using the Gauss-Seidel, Jacobi, and Richardson iteration techniques for $N = 10$ iterates each. For each method, print out the iteration matrix x and state if the method converges.

b) For the Gauss-Seidel technique, $\|K\|_1 = 1/16$. Use this and the fact that $\|e_{n+1}\| \le \|K\|^n \|e_0\|$ to find the minimum value for n that ensures the following relative error tolerance:

$$\frac{\|x_{n+1} - x\|_1}{\|x\|_1} < 10^{-12}$$

Answer.