M 441: Homework 3

Nathan Stouffer

Problem 1

Problem. The Secant method for finding roots of $f(x) = 0$ is defined by

$$x_{n+2} = x_{n+1} - \frac{f(x_{n+1})f(x_{n+1} - x_n)}{f(x_{n+1} - f(x_n))}$$

where x_1 and x_2 are specified initial guesses. Write matlab code to solve the following problem using the Secant method:

$$f(x) = exp(x) - ln(x+4) \qquad\qquad x_1 = 1 \quad x_2 = 0.5$$

Since the root is not known, we can't compute the exact error. Instead, we shall use the difference in successive x_n values as a measure of convergence

$$E_n = |x_n - x_{n-1}|$$

Include the code and an output of R showing convergence. As before, R has $n, x(n), E(n)$ as the n^{th} row.

Answer. I wrote some matlab code that implements the Secant method for finding roots. Using $f(x) = \exp(x) - \ln(x+4)$ with $x_1 = 1$ and $x_2 = 0.5$, I produced the following table:

n	x_n	$E_n = \lvert x_n - x_{n-1}\rvert$
2	0.500000000000000	0.500000000000000
3	0.424992808749178	0.075007191250822
4	0.393980247342822	0.031012561406356
5	0.391919950214853	0.002060297127969
6	0.391877859305723	0.000042090909130
7	0.391877805146257	0.000000054159467
8	0.391877805144861	0.000000000001396

The matlab code used to produce the above table is on the next page.

```
%  A "for loop" implementing the Secant method for finding f(x)=0
%
%  1) the function f defines f(x)
%  2) xexact is not known, so the error is computed
%     as the difference of consecutive terms
%  3) tolerance is the error we require to stop
%     iterating the sequence
%
%  The output matrix is R with n-th row R(n,:)
%  The columns of R are
%
%          R(n,:) = [ n , x(n) , E_n ]
%
%  where E_n is difference between consecutive terms in the sequence.
%
clear x;
clear R;
format long;
x(1)=1;                      % initial guesses for the roots
x(2)=0.5;
tolerance=1e-08;             % maximum error to stop iterating
n=2;                         % set current iteration value
E(n)=abs(x(n)-x(n-1));       % compute initial "error"
R(1,:)=[n,x(n),E(n)];
while E(n) > tolerance
    x(n+1)=x(n) - f(x(n))*(x(n)-x(n-1))/(f(x(n))-f(x(n-1)));
    E(n+1)=abs(x(n+1)-x(n));
    R(n,:)=[n+1,x(n+1),E(n+1)];
    n=n+1;                   % update number of iterations
end;
R

function val = f(x)
    val = exp(x) - log(x+4);
end
```

Problem 2

Problem. We illustrate by way of simple systems how solutions of $A\mathbf{x} = \mathbf{b}$ can change dramatically when we approximate $\mathbf{b}$. Below M is some very large number.

(a) Use Gauss elimination to find the exact solution of $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 2M \\ 1 & M+1 \end{bmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2+6M \\ 4+3M \end{pmatrix}$$

Note: simplify your answers. If done correctly, the solutions $\mathbf{x}$ do not depend on M.

(b) When M is large $\mathbf{b} \approx \mathbf{b}_{new} = M \begin{pmatrix} 6 \\ 3 \end{pmatrix}$. Resolve the system with this new appproximate $\mathbf{b}_{new}$.
Are the solutions in (a) and (b) close?

Answer.

(a) First we setup the augmented matrix $[A \mid \mathbf{b}]$:

$$\begin{bmatrix} 2 & 2M & | & 2+6M \\ 1 & M+1 & | & 4+3M \end{bmatrix} \sim \begin{bmatrix} 2 & 2M & | & 2+6M \\ 0 & 1 & | & 3 \end{bmatrix}$$

So $x_2 = 3$ and $2x_1 + 2Mx_2 = 2x_1 + 6M = 2 + 6M \implies x_1 = 1$. So the exact solution to $[A \mid \mathbf{b}]$ is $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(b) For large M, we can approximate $\mathbf{b} = \mathbf{b}_{new} = M \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and then solve $A\mathbf{x} = \mathbf{b}_{new}$. To do this, we set up the augmented matrix $[A \mid \mathbf{b}_{new}]$:

$$\begin{bmatrix} 2 & 2M & | & 6M \\ 1 & M+1 & | & 3M \end{bmatrix} \sim \begin{bmatrix} 2 & 2M & | & 6M \\ 0 & 1 & | & 0 \end{bmatrix}$$

Then $x_2 = 0$ and $2x_1 + 2Mx_2 = 2x_1 = 6M \implies x_1 = 3M$. So the exact solution to $[A \mid \mathbf{b}_{new}]$ is $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3M \\ 0 \end{pmatrix}$.

The solutions from (a) and (b) are not close at all, this is a result of our approximation of $\mathbf{b}$.

Problem 3

Problem. Consider the non-symmetric matrix

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 9 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

L, U, D are lower unit triangular, upper unit triangular, and diagonal matrices respectively. By convention, a tilde indicates that a matrix need not have ones on the diagonal.

(a) $A = L\tilde{U}$

(b) $A = LDU$

(c) Note that $det(AB) = det(A)det(B)$ and the determinant of triangular matrices equals the product of its diagonal elements. Use these face to compute $det(A) = det(L)det(\tilde{U})$.

Answer.

(a) We wish to find matrices $L, \tilde{U}$ such that $A = L\tilde{U}$. To this end, we row reduce A via the elementary matrices E_1 and E_2:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 4 & 9 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 3 \end{bmatrix} = \tilde{U}$$

Now let $L = E_1^{-1} E_2^{-1}$:

$$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

To retain equality with A, we can just left multiply $\tilde{U}$ by L: $L\tilde{U} = E_1^{-1} E_2^{-1} E_2 E_1 A = A$. So, we have the desired factorization:

$$A = L\tilde{U} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) We now want to find matrices L, D, U such that $A = LDU$. To do this, we will use the result from part (a). Let L and $\tilde{U}$ be defined as in (a). Then let D be the diagonal of $\tilde{U}$ and D^{-1} be the inverse of D:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/7 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Since $DD^{-1} = I$, we can say that $\tilde{U} = DD^{-1}\tilde{U} = DU$ where $U = D^{-1}\tilde{U}$. Note that $U = D^{-1}U$ is an upper unit triangular matrix:

$$U = D^{-1}\tilde{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/7 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 3/7 \\ 0 & 0 & 1 \end{bmatrix}$$

Since L is unit lower triangular, D is diagonal, and U is upper unit triangular, we have the desired factorization $A = LDU$:

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 3/7 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Now we use the fact that $det(AB) = det(A)det(B)$ (for matrices A, B) to compute $det(A)$. We know $det(A) = det(L)det(\tilde{U})$. Since L and $\tilde{U}$ are both triangular matrices, their determinants are the products of the diagonals: $det(L) = 1$ and $det(\tilde{U}) = 1 * -7 * 3 = -21$. So $det(A) = 1 * -21 = -21$.

Problem 4

Problem. Below is a symmetric positive definite matrix:

$$A = A^T = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

Find the Cholesky factorization $A = \tilde{L}^T \tilde{L}$

Answer. Towards finding the Cholesky factorization of $A = A^T$, we first find the $L\tilde{U}$ factorization of A. As in Problem 3, we will introduce elementary matrices and preserve equality with A by mulitiplying by the inverses:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$E_1^{-1} E_2^{-1} E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

By combining $E_2^{-1} E_1^{-1} = L$ and $E_2 E_1 A = \tilde{U}$, we get

$$A = L\tilde{U} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now let D be the diagonal of $\tilde{U}$ and D^{-1} be the inverse of D:

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$DD^{-1} = I$ so we can say that $A = L\tilde{U} = LDD^{-1}\tilde{U} = LDU_1$ where $U_1 = D^{-1}\tilde{U}$:

$$D^{-1}\tilde{U} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

But then $U_1 = L^T$, so we really have the matrix factorization $A = LDL^T$. This is close to the desired factorization $A = \tilde{L}\tilde{L}^T$. To get the desired factorization, let $D^{1/2}$ be the matrix where each element is the square root of the corresponding element in D:

$$D^{1/2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $D^{1/2} D^{1/2} = D$ so we can write the factorization as $A = LD^{1/2} D^{1/2} L^T$. Let us now compute $LD^{1/2}$:

$$LD^{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

We can just compute $D^{1/2}L^T$ as $(LD^{1/2})^T$ because $(LD^{1/2})^T = D^{1/2T}L^T = D^{1/2}L^T$ since $D^{1/2}$ is diagonal. But then $A = LD^{1/2}D^{1/2}L^T = (LD^{1/2})(LD^{1/2})^T$ so setting $\tilde{L} = LD^{1/2}$ gives us the desired factorization $A = \tilde{L}\tilde{L}^T$:

$$A = \tilde{L}\tilde{L}^T = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$