M 441: Homework 2

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Problem 1

Problem. Use Theorem 2.4 of the notes to prove

$$h - sin(h) = O(h^3) \qquad as\ h \to 0$$

Theorem 2.4 states that for functions $f(h)$ and $g(h)$ that are continuous near $h = 0$ and

$$lim_{h\to 0} \frac{f(h)}{g(h)} = L < \infty$$

then $f = O(g)$ as $h \to 0$.

Answer. To show that $h - sin(h) = O(h^3)$, we will show that $lim_{h\to 0} f(h)/g(h) = L < \infty$ for some $L \in \mathbb{R}$.

$$lim_{h\to 0} \frac{h - sin(h)}{h^3} = \frac{0 - sin(0)}{0^3} = \frac{0}{0}$$

Since the limit is $0/0$, we can apply L'Hospital's rule to say that

$$lim_{h\to 0} \frac{h - sin(h)}{h^3} = \frac{1 - cos(h)}{3h^2} = \frac{1 - cos(0)}{3 * 0^2} = \frac{1 - 1}{0} = \frac{0}{0}$$

By the same reasoning, we apply L'Hospital's rule again

$$lim_{h\to 0} \frac{h - sin(h)}{h^3} = \frac{sin(h)}{6h} = \frac{sin(0)}{6 * 0} = \frac{0}{0}$$

We then apply L'Hospital's rule a final time to say that

$$lim_{h\to 0} \frac{h - sin(h)}{h^3} = \frac{cos(h)}{6} = \frac{cos(0)}{6} = \frac{1}{6}$$

Since $lim_{h\to 0} \frac{h - sin(h)}{h^3} = 1/6 < \infty$, we say that $h - sin(h) = O(h^3)$.

Problem 2

Problem. Using Taylor series of $f(x+h)$ and $f(x-h)$ show

$$f(x+h) - 2f(x) + f(x-h) = f''(x)h^2 + O(h^4)$$

Then, solve for $f''(x)$ to get an approximation for $f''(x)$ and state the order of the truncation error.

Answer. We know the Taylor series expansion of $f(x+h)$ and $f(x-h)$ to be

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(x)h^2 + \frac{1}{3!}f^{(3)}(x)h^3 + O(h^4)$$

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2!}f''(x)h^2 - \frac{1}{3!}f^{(3)}(x)h^3 + O(h^4)$$

Adding these equations together, we get

$$f(x+h) + f(x-h) = 2f(x) + 0 + f''(x)h^2 + 0 + O(h^4)$$

which is equivalent to saying that

$$f(x+h) - 2f(x) + f(x-h) = f''(x)h^2 + O(h^2)$$

which we were required to show. From here, we can rearrange the above equation to say that

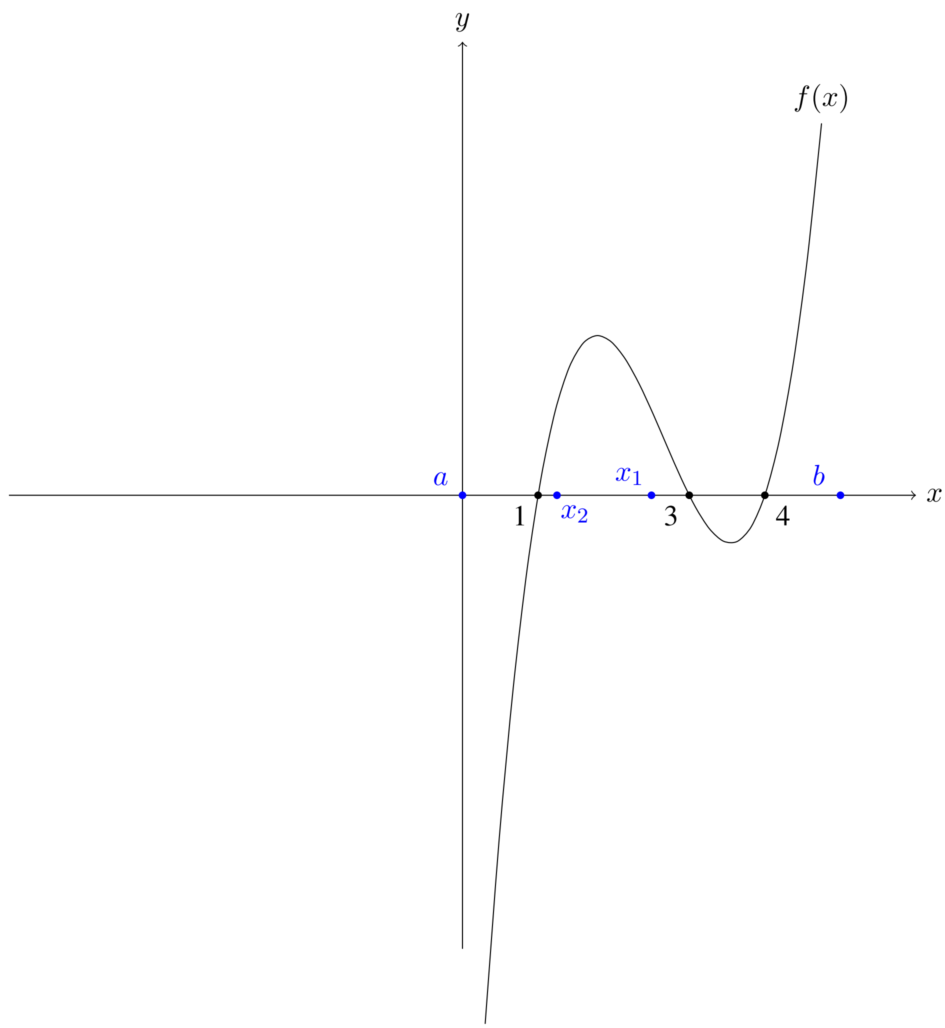
$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + \frac{O(h^4)}{h^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

Thus we can approximate $f''(x) \approx \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$ with $O(h^2)$ error.

Problem 3

Problem. Let $f(x) = (x-1)(x-3)(x-4)$. What root of $f(x)$ does the Bisection Method converge to on the interval $[a,b] = [0,5]$. Sketch $f(x)$ and label the first two midpoint values x_1, x_2.

Answer. Based on the graph below, we can say that the Bisection Method converges to $x^* = 0$ on the interval $[a,b] = [0,5]$. We now display the graph of $f(x) = (x-1)(x-3)(x-4)$ and label the first two midpoints x_1 and x_2.



Problem 4

Problem. For the following three problems use modified versions of the posted $Newton.m$, $f.m$, and $df.m$ matlab files to find an approximation of a root of $f(x)$ using the indicated starting guess x_1. For each case, print output in the three columns

$$n \qquad x_n \qquad E_n \qquad\qquad (1 \leq n \leq 10)$$

Lastly, state if the convergence rate is linear, superlinear, or quadratic.

a) Newton's Method: $x_1 = 4, f(x) = (x-1)^2 - 2$

b) Newton's Method: $x_1 = 4, f(x) = (x-1)^2$

c) Accelerated Newton's Method: $x_1 = 4, f(x) = (x-1)^2$

Answer.

Problem 5

Problem. Steffensen method for solving $f(x) = 0$ is defined by:

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)} = x_n - F(x_n)$$

where

$$g(x) = \frac{f(x + f(x)) - f(x)}{f(x)}$$

For simple roots $\bar{x}$ where $f(\bar{x}) \neq 0$, one can show the method has quadratic convergence. Write a Matlab file Steffensen.m whose output is

$$n \qquad x_n \qquad E_n$$

for $1 \leq n \leq 10$ for the case $f(x) = x^2 - 4$ and $x_0 = 1$. Include your code for Steffensen.m and output.

Answer.