M 441: Homework 5

Nathan Stouffer

Problem 1

Problem. Recall the Power Method algorithm is defined by:

$$\mathbf{y}_n = A\mathbf{x}_n$$

$$\lambda_n = \phi(\mathbf{y}_n)/\phi(\mathbf{x}_n)$$

$$\mathbf{x}_{n+1} = \mathbf{y}_n/\|\mathbf{y}_n\|_2$$

where $\phi(\mathbf{z}) = z_1 + z_2 + \cdots + z_n$ and $\mathbf{z} = (z_1, z_2, ..., z_n)$. For most (but not all) initial guesses x_0, λ_n will approach the dominant eigenvalue λ of A with x_n being an approximation of the associated (unit) eigenvector.

a) Let

$$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Using hand calculations (no decimal approximations) only, compute in sequence $\mathbf{y}_0, \lambda_0, \mathbf{x}_1, \mathbf{y}_1, \lambda_1, \mathbf{x}_2$. You should find $\mathbf{x}_1 = -\mathbf{x}_2$. Is the method converging to the dominat eigenvalue? Give some reasonable explanation.

b) Let

$$A = \begin{bmatrix} 16 & 2 & 4 \\ 1 & 40 & -3 \\ 0 & 3 & 5 \end{bmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Use the posted *Power.m* and *phi.m* matlab scripts to approximate the dominatn eigenvalue using $N = 15$ iterates. Include an extra column in the output matrix R as follows.

$$>> R = [x', lambda(k), residual(k), abs(r - edom)]$$

where $edom$ is the dominant eigenvalue. You may find this value using the matlab statement

$$>> evals = eig(A)$$

where $evals$ is a vector of all the eigenvalues of A.

Answer.

a) Now we show the computations for $y_0, \lambda_0, x_1, y_1, \lambda_1, x_2$:

$$y_0 = Ax_0 = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 7-9 \\ 3+3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

$$\lambda_0 = \phi(y_0)/\phi(x_0) = \frac{-2+6}{1+-3} = \frac{4}{-2} = -2$$

$$x_1 = \frac{y_0}{\|y_0\|_2} = \frac{1}{\sqrt{2^2+6^2}} \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \frac{1}{2\sqrt{10}} \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$$

$$y_1 = Ax_1 = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \begin{pmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix} = \begin{pmatrix} -7/\sqrt{10} + 9/\sqrt{10} \\ -3/\sqrt{10} - 3/\sqrt{10} \end{pmatrix} = \begin{pmatrix} 2/\sqrt{10} \\ -6/\sqrt{10} \end{pmatrix}$$

$$\lambda_1 = \phi(y_1)/\phi(x_1) = \frac{2/\sqrt{10} - 6\sqrt{10}}{-1/\sqrt{10} + 3/\sqrt{10}} = \frac{-4/\sqrt{10}}{2/\sqrt{10}} = -2$$

$$x_2 = \frac{y_1}{\|y_1\|_2} = \frac{1}{\sqrt{(4/\sqrt{10})^2 + (6/\sqrt{10})^2}} \begin{pmatrix} 2/\sqrt{10} \\ -6/\sqrt{10} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2/\sqrt{10} \\ -6/\sqrt{10} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{pmatrix}$$

TODO: explanation of why same vector stuff

Problem 2

Problem. Recall that the steepest descent algorithm for approximating the solution of $Ax = b$ (when $A = A^T$ is positive definite) is

$$\mathbf{r}_n = A\mathbf{x} - n - \mathbf{b}$$

$$\alpha_n = \frac{\mathbf{r}_n^T \mathbf{r}_n}{\mathbf{r}_n^T A \mathbf{r}_n}$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \alpha \mathbf{r}_n$$

where $\mathbf{r}_n$ is the residual vector and $\mathbf{x}_n$ is the approximation of the solution $Ax = b$.

a) Let

$$A = \begin{bmatrix} 2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

First prove A is postive definite adn then use hadn calculations (no decimal approximations) to compute in sequence $\mathbf{r}_0, \alpha_0, \mathbf{x}_1, \mathbf{r}_1, \alpha_1, \mathbf{x}_2$.

b) Use the code $steepest.m$ to approximate the solution of $Ax = b$ where $A \in \mathbb{R}^{50\times 50}$ is a tri-diagonal matrix. The iterat output in $steepest.m$ is stored in the matrix R.

$$R(n, :) = [n, xn', norm(en, 2)];$$

The last comlumn of R is the 2-norm of the absolute error. Plot this absolute error versus iteration number for $N = 3000$ iterates. Use matlab to do this.

Answer.

a)