Math 441 (2020) – Homework 4 : 25 points

Due: Thursday, October 15, 2020, 11:59pm (HARD Deadline).

1. [6pts] Compute $\| A \|_1$, $\| A \|_2$ and $\| A \|_\infty$ where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

For $\| A \|_2$ you may use the fact that the characteristic polynomial of $A^T A$ is:

$$P(\lambda) = det(A^T A - \lambda I) = (\lambda - 2)(\lambda - 8)(\lambda - 16)$$

Use the result on pg 9 of Lecture03 to compute $\| A \|_2$.

2. [9pts] Recall that the <u>condition number</u> $\kappa(A)$ of a matrix A is defined by

$$\kappa(A) = \| A \| \ \ \| A^{-1} \|$$

The main theorem conclusion related to the condition number is that

$$\frac{\| x - \hat{x} \|}{\| x \|} \leq \kappa(A) \ \frac{\| b - \hat{b} \|}{\| b \|}$$

where x and $\hat{x}$ are solutions to $Ax = b$ and $A\hat{x} = \hat{b}$. When $\kappa(A)$ is large, the system is said to be ill-conditioned and even small relative errors in b can result in large relative errors in the solution x. In this problem

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6.01 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 3 \\ 9.01 \end{bmatrix} \quad , \quad \hat{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

and all norms are the ∞-norm as in $\| A \|_\infty$.

a) Compute A^{-1} exactly.

b) Compute x and $\hat{x}$ exactly.

c) Find the relative errors below. Do they differ by a lot?

$$e_x = \frac{\| x - \hat{x} \|}{\| x \|} \qquad e_b = \frac{\| b - \hat{b} \|}{\| b \|}$$

d) Compute the condition number

$$\kappa(A) = \| A \|_\infty \ \ \| A^{-1} \|_\infty$$

3. [10pts] Recall the general iterative technique for solving $Ax = b$ has the split equation

$$Qx_{n+1} = (Q - A)x_n + b$$

which may be written

$$x_{n+1} = Kx_n + c \qquad , \qquad K = I - Q^{-1}A$$

In all of the following questions

$$A = \begin{bmatrix} 20 & 1 \\ -\frac{1}{2} & 2 \end{bmatrix} \quad , \quad b = \begin{pmatrix} 120 \\ 159 \end{pmatrix} \quad , \quad x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

a) Use the matlab function in "Iterate.m" (in D2L Matlab contents) to approximate the solution of $Ax = b$ using the Gauss-Seidel, Jacobi and Richardson iteration techniques for $N = 10$ iterates each. For example, define $x0, b, A$ in the matlab command window and then execute

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[x, NORM_K, eig_K]=Iterate(A,b,10,'richardson',x0,1)
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for Richardson's method. For each method, print out the iteration matrix x and state if the method converges.

b) For the Gauss-Seidel technique, $\| K \|_1 = \frac{1}{16}$. (K defined above) Use this and the theory presented in equation (3) of page 11 of Lecture04 to find the minimum n value that assures the following relative error tolerance:

$$\frac{\| x_{n+1} - x \|_1}{\| x \|_1} < 10^{-12}$$