WEEK 6

CLASS

This past week's classes were devoted to examining P_4 (see the Week 5 notes for the definition of P_n, for general n). I'll summarize a few techniques we implemented in this examination – these techniques are the stronger lesson here.

- We chopped P_4 into smaller, more comprehensible, parts. In the case at hand, each of these parts was a point (a vertex) or homeomorphic with an open interval (the interior of an edge). A bit of thinking yielded how these pieces fit together in P_4. So the space P_4 can be reconstructed from these pieces, by 'gluing' the edges together along common vertices.

- The way we chopped up P_4 was natural to its very definition. Namely, we conceive of an element in P_4 as an equallateral 4-gon in the plane, up to translation and rotation of the plane. The 'up to' clause allows us to fix the "12-stick" in such a 4-gon to be the interval in the (complex) plane from the origin to the complex number 1. Now, we 'chopped up' P_4 into chambers, each chamber is characterized by non-horizontality of the "sticks" of 4-gons; the interfaces of these chambers are characterized as where "sticks" in 4-gons are horizontal.
  Note that not much is particular to $n = 4$.

- It just so happened that, in the case of P_4, each 'chamber' has 1 degree of freedom, and each 'interface' has 0 degrees of freedom. Some thought reveals that there are 3 such interfaces, and there are 6 such chambers. Some thought reveals that there are 4 ways to exit each 'interface' into each 'chamber'.
  So P_4 is a *graph*, with 3 vertices and 6 edges and each vertex has valence 4. Up to homeomorphism, tere aren't many such graphs. Some more thought reveals that there are no edges connecting a vertex to itself; and with this stipulation, there is only 1 such graph (up to homeomorphism).

We talked about variations of P_4. Namely, consider

$$P_4^{\mathsf{UnSpec}} := \Big\{3 \text{ unit-sticks in the plane, connected end-to-end}\Big\}_{/\text{trans \& rot}}$$

$$= \Big\{(z_3, z_4) \mid \mathsf{dist}(1, z_3) = 1 \ , \ \mathsf{dist}(z_4, 0) = 1\Big\} \ \subset \ \mathbb{C}^{\times 2} \ .$$

For $0 \leq \ell$ a real number, consider the subset

$$P_4^{\ell} := \Big\{3 \text{ unit-sticks in the plane, connected end-to-end, with the extremal ends distance } \ell\Big\}_{/\text{trans \& rot}}$$

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$$= \left\{ (z_3, z_4) \mid \mathsf{dist}(1, z_3) = 1\ ,\ \mathsf{dist}(z_4, 0) = 1\ ,\ \mathsf{dist}(z_3, z_4) = \ell \right\} \subset \mathbb{C}^{\times 2}\ .$$

So, in the case that $\ell = 1$, we recover the original $P_4 = P_4^1$.

Reading

By Friday 11 October. §3, just the material about *graphs* (pages 44-45). §5, through the definition of a polyhedron (pages 69-73). Be especially familiar with these aspects.

- The definition of a *graph*, and of an *isomorphism* between graphs.
- The definition of a *polyhedron*, and of a *symmetry* between polyhedrons.

Exercises

These are due by **5pm on Friday 11 October**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) (a) Identify the space P_4^{UnSpec} as a familiar one. More precisely, name a familiar subset $T \subset \mathbb{R}^n$, for some n, and a homeomorphism $T \cong P_4^{\mathsf{UnSpec}}$. Explain why T is *not* a graph.

(b) For each ℓ, identify the space P_4^ℓ as a familiar one. More precisely, name a familiar subset $X_\ell \subset \mathbb{R}^n$, for some n, and a homeomorphism $X_\ell \cong P_4^\ell$. Explain why each such X_ℓ is a graph.

(c) Consider the equivalence relation on $\mathbb{R}_{\geq 0}$ declaring $\ell \sim \ell'$ to mean, for each $0 \leq t \leq 1$, there is a homeomorphism $P_4^{t\ell + (1-t)\ell'} \cong P_4^\ell$. For this equivalence relation, describe the equivalence classes of $\mathbb{R}_{\geq 0}$.
(Your answer will be a *partition* of $\mathbb{R}_{\geq 0}$, which is a list of subsets of $\mathbb{R}_{\geq 0}$ that are mutually disjoint and whose union is all of $\mathbb{R}_{\geq 0}$. Note: the definition of this equivalence relation is construed so that each partition will be a *connected* subset of $\mathbb{R}$.)

(d) For a choice of representative ℓ of *each* equivalence class of the above equivalence relation on $\mathbb{R}_{\geq 0}$, describe $X_\ell \cong P_4^\ell$ as a subset of $T \cong P_4^{\mathsf{UnSpec}}$.
(Your answer can be a drawing, or series of drawings, with thorough annotation and descriptions.)
(You might find it helpful to chop up both X_ℓ and T according to where sticks are *horizontal*, then make such a drawing only in each chamber and interface at a time, then patch these drawings together for the full answer.)

(2) Consider the set

$$\mathbb{RP}^2 \ := \ \left\{ V \subset \mathbb{R}^3 \mid V \text{ is a 1-dimensional subvector space} \right\}$$

of all 1-dimensional subvector spaces of $\mathbb{R}^3$. Chop up $\mathbb{RP}^2$ according to how 1-dimensional vector spaces include, or not, into coordinate planes. Specifically, consider the following subsets of $\mathbb{RP}^2$:

$$\mathbb{RP}^2_{xyz} \ := \ \left\{ V \in \mathbb{RP}^2 \mid V \text{ is not contained in the } xy\text{- or } yz\text{- or } zx\text{-planes} \right\}\ ;$$

$$\mathbb{RP}^2_{xy} := \left\{ V \in \mathbb{RP}^2 \mid V \subset xy\text{-plane but not the } x\text{-axis nor the } y\text{-axis} \right\};$$
$$\mathbb{RP}^2_{yz} := \left\{ V \in \mathbb{RP}^2 \mid V \subset yz\text{-plane but not the } y\text{-axis nor the } z\text{-axis} \right\};$$
$$\mathbb{RP}^2_{zx} := \left\{ V \in \mathbb{RP}^2 \mid V \subset zx\text{-plane but not the } z\text{-axis nor the } x\text{-axis} \right\};$$
$$\mathbb{RP}^2_{x} := \left\{ V \in \mathbb{RP}^2 \mid V \subset x\text{-axis} \right\};$$
$$\mathbb{RP}^2_{y} := \left\{ V \in \mathbb{RP}^2 \mid V \subset y\text{-axis} \right\};$$
$$\mathbb{RP}^2_{z} := \left\{ V \in \mathbb{RP}^2 \mid V \subset z\text{-axis} \right\}.$$

(a) For each non-empty subset $S \subset \{x, y, z\}$, identify the above set $\mathbb{RP}^2_S$ as a familiar space.
(These are the "chambers" and "interfaces" of this 'chopping-up' of $\mathbb{RP}^2$.)

(b) Show that the union
$$\mathbb{RP}^2_x \cup \mathbb{RP}^2_y \cup \mathbb{RP}^2_z \cup \mathbb{RP}^2_{xy} \cup \mathbb{RP}^2_{yz} \cup \mathbb{RP}^2_{xz} \cup \mathbb{RP}^2_{xyz} = \mathbb{RP}^2$$
is entier; show also that the intersection of any two of these terms is empty.

(c) Describe, informally, how each the subsets $\mathbb{RP}^2_S \subset \mathbb{RP}^2$ assemble together to reconstruct $\mathbb{RP}^2$.
(Your answer can be an annotated and fully explained picture or series of pictures.)