M 476 - Homework 5

Nathan Stouffer

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Problem 1

Consider the two subsets $X := \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ and $Z := \{(w, z) \in \mathbb{C}^2 \mid wz = 0\}$.

Part A

Problem: Show that both X and Z are path-connected.

Solution: We first consider X. We will build a continuous map from an arbitrary point in X to the origin of $\mathbb{R}^2$. For an element $(x, y) \in X$, we know, by the definition of X, that either x or y must be 0. Without loss of generality, take $x = 0$. Then, we must map an arbitrary y to 0 on the real number line. This is done by $f : [0, 1] \to \mathbb{R}$, specifically, $t \mapsto ty$. The map f must be continuous by obs (prod). Furthermore, $0 \mapsto 0$ and $1 \mapsto y$. Therefore, the set X must be path-connected.

We now show, through a similar argument, that Z is path-connected. We will build a continuous map from an arbitrary point in Z to the origin of the complex plane. For an element $(w, z) \in Z$, we know, by the definition of Z, that either w or z must be the complex number 0. Without loss of generality, take $w = 0$. Then, we must map an arbitrary z to the origin of the complex plane. This is done by $f : [0, 1] \to \mathbb{C}$, specifically, $t \mapsto tz$. The map f must be continuous by obs (prod). Furthermore, $0 \mapsto 0$ and $1 \mapsto z$. Thus the set Z is path-connected.

Part B

Problem: For each finite cardinality r, identify the subsets $Cut_r(X) \subset X$ and $Cut_r(Z) \subset Z$.

We begin with the set X.
In the case $r = 0$: $Cut_0(X) = \emptyset$
In the case $r = 1$: $Cut_1(X) = \emptyset$
In the case $r = 2$: $Cut_2(X) = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$
In the case $r = 3$: $Cut_3(X) = \emptyset$
In the case $r = 4$: $Cut_4(X) = \{(0, 0)\}$
In the case $r > 5$: Since no two sets of cut points in X will intersect and all elements of X are accounted for in the above cases, any $r > 5$ will produce an empty set.

We now move to the set Z.
In the case $r = 0$: $Cut_0(Z) = \emptyset$
In the case $r = 1$: $Cut_1(Z) = \{(w, z) \in \mathbb{C}^2 \mid (w, z) \neq (0, 0)\}$
In the case $r = 2$: $Cut_2(Z) = \{(0, 0)\}$
In the case $r > 3$: Since no two sets of cut points in Z will intersect and all elements of Z are accounted for in the above cases, any $r > 3$ will produce an empty set.

Problem 2

Problem: Let $n > 0$. Consider the subset

$$U_n := \{U \in GL_n(\mathbb{R}) \mid U \text{ is upper triangular}\} \subset GL_n(\mathbb{R}) \subset Mat_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$$

For each $n > 0$, identify the cardinality of $|\pi_o(U_n(\mathbb{R}))|$.

Solution: We begin with an arbitrary $n \times n$ matrix A and note that A is really just $n \times n$ slots for real numbers. Thus the slots must be filled with $\mathbb{R}^{n^2}$. Now, an element $U' \in U_n$ has constraints on the input. The matrix U' must be upper triangular and invertible. For such a U', the values in the diagonal of U' must all be nonzero. Thus, the slots of U' must be filled with $\{0\}^{\times\binom{n}{2}} \times (\mathbb{R} \setminus O)^n \times \mathbb{R}^{\times\binom{n}{2}}$.

We now apply the proposition that, given two sets X and Y, $\pi_o(X \times Y)$ is bijective with $\pi_o(X) \times \pi_o(Y)$. We also note that bijective sets must have equal cardinalities. Using this proposition, since $|\pi_o(\{0\})| = 1$, it must be true that $|\pi_o(\{0\}^{\times\binom{n}{2}})| = 1$. Likewise, since $|\pi_o(\{\mathbb{R}\})| = 1$, it must be true that $|\pi_o(\{\mathbb{R}\}^{\times\binom{n}{2}})| = 1$. The proposition can be used again to show that $\pi_o(U_n(\mathbb{R}))$ is bijective with $\pi_o(\mathbb{R} \setminus O)^n$. Since the two sets are bijective, their cardinality must be equal. Thus, $|\pi_o(U_n(\mathbb{R}))| = |\pi_o(\mathbb{R} \setminus O)^n|$.

We know that $|\pi_o(\mathbb{R} \setminus O)| = 2$. Therefore, by the same proposition as before, $|\pi_o(\mathbb{R} \setminus O)^n| = 2^n$. Thus, for $n > 0$, $|\pi_o(U_n(\mathbb{R}))| = 2^n$.

Problem 3

Problem: For $n = 2$ and $n = 3$, prove that P_n is finite and identify the cardinality $|P_n|$.

Solution: We begin by writing the definition of P_n.

$$P_n := \{(z_1, z_2, ...z_n) \in \mathbb{C}^{\times n} \mid z_1 = 0 \text{ and } z_2 = 1 \text{ and } dist(z_i, z_{i+1}) = 1 \text{ and } dist(z_n, z_1) = 1\} \subset \mathbb{C}^{\times n}$$

We now examine the case $n = 2$. The set $P_2 = \{(0, 1)\}$. Since P_2 has only one element, P_2 must be finite and $|P_2| = 1$.

We now shift our focus to the case $n = 3$. The set $P_3 = \{(0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i), (0, 1, \frac{1}{2} - \frac{\sqrt{3}}{2}i)\}$ Since P_3 has two discrete elements, P_3 must be finite and $|P_3| = 2$.

Problem 4

Problem: For each $n \geq 2$, identify the cardinality $|\pi_o(P_n)|$.

Solution: For cases $n = 2, 3$, we have already shown in Problem 3 that $|\pi_o(P_2) = 1|$ and $|\pi_o(P_3) = 2|$.

We now consider the case $n \geq 4$. We will show that $|\pi_o(P_n)| = 1$ where $n \geq 4$ by induction. Additionally, we will use the following statement: $|\pi_o(P_n)| = 1$ iff P_n is path-connected.

Consider the base case of $n = 4$. The set P_4 is path-connected if we can build a continuous map from an arbitrary element in P_4 to the element $(0, 1, 0, 1)$. Take $(0, 1, a_3, a_4)$ as an arbitrary element of P_4. While staying in P_4, we can rotate the position of a_3 to 0. Given that we must stay in P_4, this will map a_4 to a point a_4', giving $(0, 1, 0, a_4')$. We can now rotate the point a_4' to the point 1, which gives $(0, 1, 0, 1)$. Thus the base case is proved.

We now state the inductive assumption: Assume there exists $k \geq 4$ for which $P_{n=k}$ is path-connected. That is, assume, for an arbitrary $p = (0, 1, a_3, a_4, ..., a_k) \in P_{n=k}$, there exists a continuous path in $P_{n=k}$ that maps p to $(0, 1, 0, 1, ..., a_k') \in P_{n=k}$.

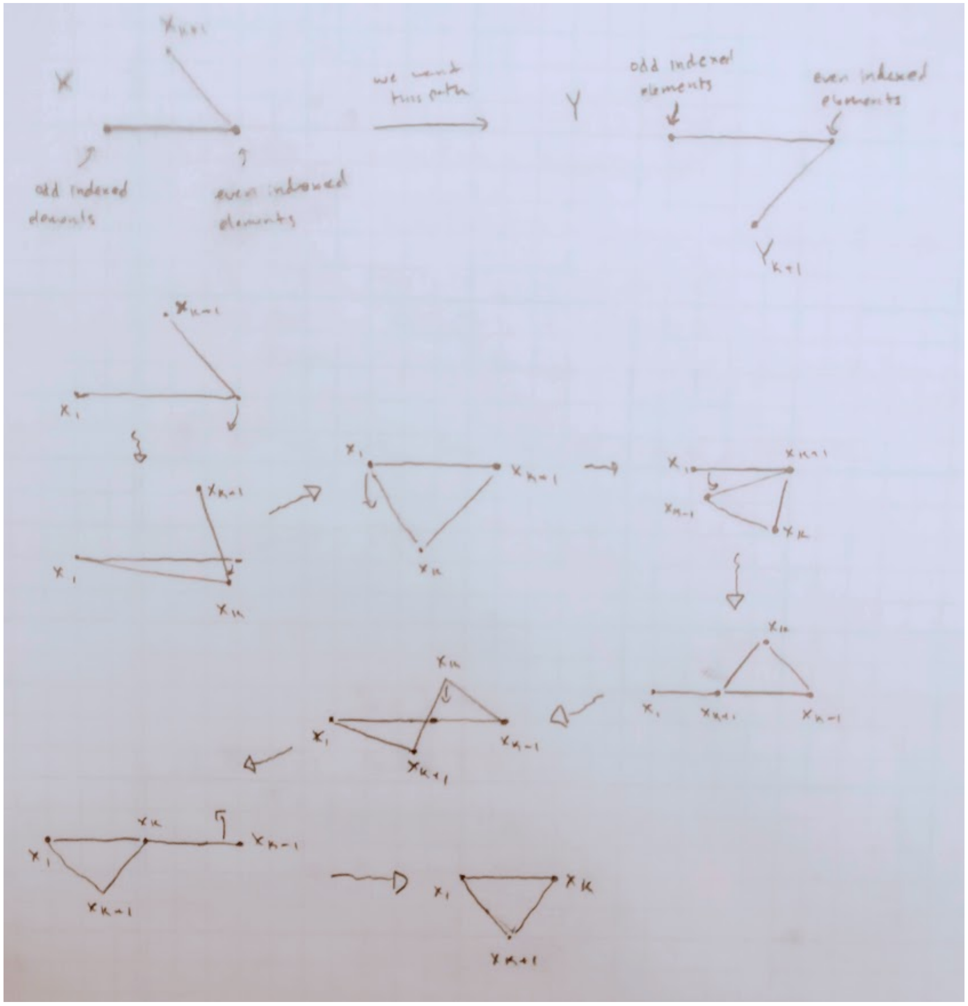
We will now show that $P_{n=k+1}$ must also be path-connected. Take $q = (0, 1, a_3, a_4, ..., a_k, a_{k+1})$ as an arbitrary element of $P_{n=k+1}$. By inductive assumption, we know that q maps to $q' = (0, 1, 0, 1, ..., a_k', a_{k+1}')$.

There are now two cases: either k is odd or k is even.

Case k is odd: Since k is odd, the $k-1$ point in q' must be 1. Then, while staying in $P_{n=k+1}$, we can then rotate a_k' to 0, which gives $(0, 1, 0, 1, ..., 0, a_{k+1}'') \in P_{n=k+1}$. Now, a_{k+1}'' can be rotated to be 1, yielding $(0, 1, 0, 1, ..., 0, 1) \in P_{n=k+1}$. Thus, in the case that k is odd, $P_{n=k+1}$ is path-connected.

Case k is even: Since k is even, the $k-1$ point in q' must be 0. Then, while staying in $P_{n=k+1}$, we can then rotate a_k' to 1, which gives $q'' = (0, 1, 0, 1, ..., 1, a_{k+1}'') \in P_{n=k+1}$. Since $q'' \in P_{k+1}$, it must be the case that $a_{k+1}'' = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ or $a_{k+1}'' = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

We must now show that both cases of q'' are path connected. That is, we must show that there is a continuous map from $X = (0, 1, 0, 1, ..., 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i) \in P_{k+1}$ to $Y = (0, 1, 0, 1, ..., 1, \frac{1}{2} - \frac{\sqrt{3}}{2}i) \in P_{k+1}$ in P_{k+1} This is done in the figure on the following page.



Thus, in the case that k is odd, $P_{n=k+1}$ is path-connected.

Since it has been inductively proven (in both the case that k is odd and that k is even) that $P_{n=k+1}$ is path-connected. It must be true that P_n is path-connected for $n \geq 4$.

Finally, since P_n for $n \geq 4$ is path-connected, it must be true that $|\pi_o(P_n)| = 1$ for $n \geq 4$.

Problem 5

Problem: Identify a graph X for which there is a homeomorphism $X \cong P_4$.

Solution: Consider the following set of elements of P_4: $v = \{(0,1,0,1), (0,1,0,-1), (0,1,2,1)\}$. We now refer to Figure 1.

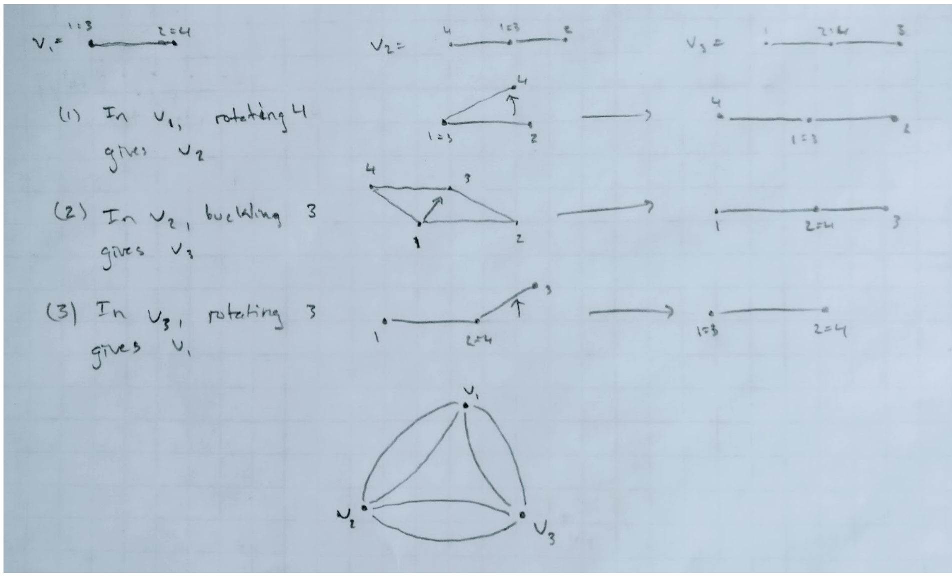


Figure 1: Graph X that is homeomorphic P_4

Consider the distinct operations (1), (2), and (3). Without loss of generality, take an operation o. The operation o can be defined by an angle θ. Additionally, we could choose to define o by the angle $-\theta$. Thus, for each operation, there are two directions to choose from. The set of elements within those operations (excluding the set v) consist of edges in the graph X since there is only one direction of travel and its opposite.

We now examine the elements in v. Such elements have four distinct choices for the direction of travel (two directions and their opposites). Thus the set v is the set of vertices in the graph X.

For clarity, the graph X is pictured in Figure 1.