WEEK 9

Class

There is a take-home midterm across the period Saturday 26 October through Friday 1 November, due at 3:10pm.

Really, the notion of a *continuous map* is what justifies the notion of an abstract topological space.

Definition 0.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y$ be topological spaces. Let $f\colon X \to Y$ be a map between their underlying sets. We say f is *continuous* if, for each member $V \in \mathcal{T}_Y$ of the topology on Y, the preimage $f^{-1}(V) := \{x \in X \mid f(x) \in V\} \in \mathcal{T}_X$ is a member of the topology on X.

Observation 0.2. Let X be a topological space. The identity map $\mathsf{id}_X\colon X \to X$ is continuous.

Observation 0.3. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be continuous maps between topological spaces. The composition $X \xrightarrow{g\circ f} Z$ is also continuous.

We discussed how this definition abstracts the definition of continuity between subspaces of Euclidean spaces. Here is the formal assertion, and justification.

Lemma 0.4. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be subsets of Euclidean spaces. Let $f\colon X \to Y$ be a map between sets. Then f is continuous in the ε-δ sense if and only if f is continuous with respect to the subspace topologies on both X and Y.*

Proof. Suppose f is continuous with respect to the subspace topologies on both X and Y (in the sense of the abstract definition of *continuous*). Let's show that f is continuous in the ε-δ definition. So let $x \in X$ and $\varepsilon > 0$. By definition of the topology on $\mathbb{R}^n$, the subset $\mathsf{B}_\varepsilon(f(x)) \subset \mathbb{R}^k$ is a member of $\mathcal{T}_{\mathbb{R}^n}$. By definition of the subspace topology on Y, the subset $\mathsf{B}_\varepsilon(f(x)) \cap Y \subset Y$ is a member of the subspace topology $\mathcal{T}_Y$ on Y. By definition of continuity of f, the preimage

$$f^{-1}\big(\mathsf{B}_\varepsilon(f(x)) \cap Y\big) \ \subset\ X$$

is a member of the subspace topology $\mathcal{T}_X$ on X. By definition of the subspace topology \$$cT_X$ on X from the topology on $\mathbb{R}^n$, because $x \in f^{-1}\big(\mathsf{B}_\varepsilon(f(x)) \cap Y\big)$, there is a $\delta > 0$ for which

$$\mathsf{B}_\delta(x) \ \subset\ f^{-1}\big(\mathsf{B}_\varepsilon(f(x)) \cap Y\big) \ .$$

By definition of this preimage, this inclusion implies, for each $x' \in \mathsf{B}_\delta(x)$, that the value $f(x') \in \mathsf{B}_\varepsilon(f(x))$. By definition of balls in Euclidean spaces, this is to say, $\mathsf{dist}(x', x) < \delta$ implies $\mathsf{dist}(f(x'), f(x)) < \varepsilon$. We have verified the ε-δ criteria, thereby concluding one of the implications of the statement of the lemma.

Now suppose f is continuous with respect to the ε-δ definition of continuity. Let's show that f is continuous with respect to the subspace topologies on both X and Y

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(in the sense of the abstract definition of *continuous*). So let $V \subset Y$ be a member of the subspace topology $\mathcal{T}_Y$. We must show the preimage $f^{-1}(V) \subset X$ is a member of the subspace topology $\mathcal{T}_X$. By definition of this subspace topology $\mathcal{T}_X$ on X, this is to show there is a member $\widetilde{U} \in \mathcal{T}_{\mathbb{R}^n}$ of the topology on $\mathbb{R}^n$ for which $\widetilde{U} \cap X = f^{-1}(V)$. By definition of the topology on $\mathbb{R}^n$, this is to show, for each $x \in f^{-1}(V)$, there is a $\delta > 0$ for which $\mathsf{B}_\delta(x) \cap X \subset f^{-1}(V)$. So let $x \in f^{-1}(V)$. By definition of the subspace topology $\mathcal{T}_Y$ on Y, there is a member $\widetilde{V}$ of the topology on $\mathbb{R}^k$ for which $\widetilde{V} \cap Y = V$. By definition of the topology on $\mathbb{R}^k$, there is an $\varepsilon > 0$ for which $\mathsf{B}_\varepsilon(f(x)) \cap Y \subset V$. Now, by the assumption that f is continuous in the ε-δ sense, there is a $\delta > 0$ for which $\mathsf{dist}(x',x) < \delta$ implies $\mathsf{dist}(f(x'), f(x)) < \varepsilon$. In other words, for this $\delta > 0$, there is an inclusion between subsets

$$\mathsf{B}_\delta(x) \cap X \ \subset \ f^{-1}(\mathsf{B}_\varepsilon(f(x)) \cap Y) \ .$$

The inclusion $\mathsf{B}_\varepsilon(f(x)) \cap Y \subset V$ implies the inclusion $f^{-1}(\mathsf{B}_\varepsilon(f(x)) \cap Y) \subset f^{-1}(V)$, which grants the desired inclusion

$$\mathsf{B}_\delta(x) \cap X \ \subset \ f^{-1}(V) \ .$$

$\square$

The advantage to this abstract notion of a *topological space* is that interesting examples, such as $\mathbb{RP}^n$ and $\mathsf{Gr}_k(n)$ and $|\Gamma|$, are now in our realm of rigorous examination. One advantage of this abstract notion of *continuous map* is that we can now speak of 'continuous paths' in an abstract topological space. **I strongly suggest this to tie your intuition to the abstract definition of a topological space**. In other words, a topology $\mathcal{T}$ on a set X determines which maps $[0,1] \to X$ are 'continuous'; and continuous maps $[0,1] \to X$ implement wiggling points in X around. So a topology on X determines how points in X wiggle around.

Definition 0.5 (Path-components)**.** Let $X = (X, \mathcal{T}_X)$ be a topological space.

(1) Declare $x \underset{\text{path}}{\sim} x'$ to mean there is a continuous map $\gamma \colon [0,1] \to X$ for which $\gamma(0) = x$ and $\gamma(1) = x'$.

(2) A *path-component (of X)* is a subset

$$S \ \subset \ X$$

satisfying the following three properties:
- $S \neq \emptyset$.
- Let $x, x' \in X$ for which $x \underset{\text{path}}{\sim} x'$. Then $x \in S$ if and only if $x' \in S$.

(3) The *set of path-components (of X)* is the quotient set

$$\pi_0(X) \ := \ X_{/\underset{\text{path}}{\sim}} \ .$$

We verified that $\underset{\text{path}}{\sim}$ is indeed an equivalence relation on X. The hardest part is transitivity (the third property of an equivalence relation). For that, we used the same Lemma (from Week3 Notes) we used when considering subsets of Euclidean spaces.

The following observations amount to unpacking the above definitions (of π_0 and of *continuity* and of *topology*).

- Let $f\colon X \longrightarrow Y$ be a continuous map. There results a map
$$\pi_0(f)\colon \pi_0(X) \longrightarrow \pi_0(Y)~, \qquad [x] \mapsto [f(x)]~.$$
- The resulting diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"] & \pi_0(Y) \end{tikzcd}$$
commutes.
- For $X \xrightarrow{f} Y \xrightarrow{g} Z$ continuous maps, then $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.
- $\pi_0(\mathsf{id}_X) = \mathsf{id}_{\pi_0(X)}$.
- For $f\colon X \xrightarrow{\cong} Y$ a homeomorphism, the map $\pi_0(f)\colon \pi_0(X) \xrightarrow{\cong} \pi_0(Y)$ is a bijection.
In particular, if the cardinalities $\big|\pi_0(X)\big| \neq \big|\pi_0(Y)\big|$ don't agree, then X and Y are not homeomorphic.
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Theorem 0.6. *Let $f\colon X \to Y$ be a continuous map between topological spaces. If f is surjective, then $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is surjective; in particular, there is a bound between cardinalities:*
$$\big|\pi_0(X)\big| \geq \big|\pi_0(Y)\big|~.$$

- In particular, for $f\colon X \to Y$ a continuous surjection between topological spaces, if X is path-connected then so is Y.

On Friday, we examined a few different topologies on $\mathbb{S}^1$, and described paths in each.

- The *discrete* topology $\mathcal{T}_{\text{discrete}} := \{U \subset \mathbb{S}^1\}$ consists of *all* subsets of the circle. Let $[0,1] \xrightarrow{\gamma} \mathbb{S}^1$ be a map. If γ is constant, then it is continuous with respect to the subspace topology on $[0,1] \subset \mathbb{R}$ and this discrete topology on $\mathbb{S}^1$ (check this!). We now show that if γ is not constant, then it is not continuous (with respect to these topologies). Toward a contradiction, suppose γ is continuous yet not constant. Let $z_- \neq z_+ \in \mathsf{Image}(\gamma) \subset \mathbb{S}^1$ be two elements in its image. Consider the map $\mathbb{S}^1 \to \mathbb{S}^0$ whose value on z_- is -1 and whose value on every other element is $+1$. Note that this map $\mathbb{S}^1 \to \mathbb{S}^0$ is continuous with respect to the discrete topology on $\mathbb{S}^1$ and the subspace topology on $\mathbb{S}^0 \subset \mathbb{R}$. So the composition $[0,1] \xrightarrow{\gamma}\!\!\to \mathbb{S}^1 \to \mathbb{S}^0$ is continuous. By construction, this composite map is surjective as well. Therefore, there is a bound $1 = |\pi_0([0,1])| \geq |\pi_0(\mathbb{S}^0)| = 2$, which is a contradiction.
- The quotient topology from the surjection
$$[0,2\pi) \xrightarrow{\alpha} \mathbb{S}^1~, \qquad t \mapsto e^{it}~.$$
Let's see that the map
$$[0,1] \xrightarrow{\gamma} \mathbb{S}^1~, \qquad s \mapsto e^{2\pi i s}$$
is *not* continuous with respect to the subspace topology on $[0,1] \subset \mathbb{R}$ and this quotient topology on $\mathbb{S}^1$. To see this, we must show there exists a member

$U \subset \mathbb{S}^1$ of this quotient topology on $\mathbb{S}^1$ for which $\gamma^{-1}(U) \subset [0,1]$ is *not* a member of the subspace topology on $[0,1] \subset \mathbb{R}$. Consider the subset $U := \{e^{i\theta} \mid 0 \leq \theta < \pi\} \subset \mathbb{S}^1$. This is a member of this quotient topology on $\mathbb{S}^1$: indeed, $\alpha^{-1}(U) = [0,\pi) \subset [0,2\pi)$ is a member of the subspace topology on $[0,2\pi) \subset \mathbb{R}$ (since it's the intersection of an open subset of $\mathbb{R}$ with $[0,2\pi)$). The preimage $\gamma^{-1}(U) = \{s \in [0,1] \mid \gamma(s) \in U\} = [0,\pi) \sqcup \{1\} \subset [0,1]$. This preimage is *not* a member of the subspace topology on $[0,1]$, since there is no open subset of $\mathbb{R}$ whose intersection with $[0,1]$ equals this preimage $[0,\pi) \sqcup \{1\}$ (convince yourself of this!).

- The *codiscrete* topology on $\mathbb{S}^1$ consists solely of the two members $\emptyset$ and $\mathbb{S}^1$. Note that *any* map $[0,1] \xrightarrow{\gamma} \mathbb{S}^1$ is continuous with respect to the subspace topology on $[0,1]$ and this codisctete topology on $\mathbb{S}^1$. Indeed, both $\gamma^{-1}(\emptyset) = \emptyset \subset [0,1]$ and $\gamma^{-1}(\mathbb{S}^1) = [0,1] \subset [0,1]$ are members of the subspace topology on $[0,1] \subset \mathbb{R}$.

Exercises

These are due by **5pm on Friday 1 November**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) (Nothn')