M 476 - Homework 4

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Problem 2

Problem: For which $n \geq 1$ is $SO(n)$ path-connected? Justify your answer.

Solution: We begin by defining $SO(n)$:

$$SO(n) := \{A_{n\times n \text{ matrix}} \mid A^T A = I_{n\times n} \text{ and } det(A) > 0\}$$

The set $SO(n)$ can also be thought of as the set of all rotations of $\mathbb{R}^n$ about the origin.

We first show that $n = 1$ is path connected. $SO(1)$ has only one element: $[1]$. Therefore, $SO(1)$ is path connected.

We now shows that, for $n \geq 2$, $SO(n)$ is path-connected. Let $V = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be an orthonormal basis (that respects orientation) for $\mathbb{R}^n$. The basis V must be an element of $SO(n)$ since the elements of V could be the columns of a $n \times n$ matrix that is both orthonormal and whose determinant is greater than 0 (since V respects orientation). Since V is an arbitrary element of $SO(n)$, if we can find a continuous map from V to another arbitrary element $E = \{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$ then we will know that $SO(n)$ is path-connected.

Take an arbitrary natural number $k \geq 2$ for $SO(k)$. Then, $V = \{\vec{v}_1, \vec{v}_2, ...\vec{v}_k\}$ and $E = \{\vec{e}_1, \vec{e}_2, ...\vec{e}_k\}$. We will prove that V and E are connected by inductively rotating each element of V to match the position of E.

We begin with the base case of $\vec{v}_1$. We will now construct a path from $\vec{v}_1$ to $\vec{e}_1$. Let $\vec{p}_1$ be a unit vector such that $\vec{p}_1 \perp \vec{e}_1$ and $v_1 \in span\{\vec{p}_1, \vec{e}_1\}$. Then, $\exists t_1 \in [0, 2\pi]$ such that $\vec{v}_1 = \cos(t) * \vec{e}_1 + \sin(t) * \vec{p}_1$. Then, $[0, t_1] \to SO(k)$ is defined by $t_1 \mapsto (\vec{v}_1, \vec{v}_2, ..., \vec{v}_k)$ and $0 \mapsto (\vec{e}_1, \vec{v\prime}_2, ..., \vec{v\prime}_k)$. Since $\cos(t)\vec{e}_1 + \sin(t)\vec{p}_1$ is made up of continuous functions, the arbitrary elements $\vec{v}_1$ and $\vec{e}_1$ must be path-connected.

Our inductive assumption is as follows: for an arbitrary $s \geq 2$, let $\vec{v}_s \in V$ and assume $\forall \vec{v}_t$ such that $t \leq s$ that $\vec{v}_t = \vec{e}_t$.

We will now show that there exists a path from $\vec{v}_{s+1}$ to $\vec{e}_{s+1}$. Let $V\prime \in SO(k)$ be such a vector as described in the inductive assumption. Then, $V\prime = \{\vec{e}_1, \vec{e}_2, ..., \vec{e}_s, \vec{v\prime}_{s+1}, ...\vec{v\prime}_k\}$. Now let $\vec{p\prime}$ be a vector perpendicular to $\vec{e}_{s+1}$ such that $\vec{v}_{s+1} \in span\{\vec{p\prime}, \vec{e}_{s+1}\}$. We know that such a vector $p\prime$ exists because of Graham-Schmitt orthogonolization. Then $\exists t_2 \in [0, 2\pi]$ such that $\vec{v}_{s+1} = cos(t) * \vec{e}_{s+1} + sin(t) * \vec{p\prime}$. Now, we hold all vectors $\vec{v\prime}_r \in V'$ such that $\vec{v\prime}_r = \vec{e}_r$ constant and rotate all remaining vectors in V' according to $cos(t) * \vec{e}_{s+1} + sin(t) * \vec{p\prime}$. Specifically, $t_2 \mapsto (\vec{e}_1, \vec{e}_2, ..., \vec{e}_s, \vec{e}_{s+1}, ..., \vec{v\prime\prime}_k)$ and $0 \mapsto (\vec{e}_1, \vec{e}_2, ..., \vec{e}_s, \vec{v\prime}_{s+1}, ...\vec{v\prime}_k)$. The path must be continuous for it is made up of continuous input and operations.

Thus, by induction, $SO(n)$ is path-connected for $n \geq 2$.

Furthermore, by inspection, we saw that $SO(1)$ is path-connected. Therefore, $SO(n)$ for $n \geq 1$ is path-connected.

Problem 3

Problem: Let $r > 0$. Consider the set

$$X_r := \{(a_1, a_2, ..., a_d) \in (\mathbb{S}^1)^{\times r} \mid a_i = a_j \text{ only if } i = j\} \subset (\mathbb{S}^1)^{\times r} \subset (\mathbb{R}^2)^{\times r} = \mathbb{R}^{2r}$$

What is the cardinality of $\pi_0(X_r)$?

Solution: We note that elements of X_r are sets of r distinct points on the unit circle. Given an arbitrary set of points $x \in X_r$, there is guaranteed to exist a homeomorphism (specifically a rotation) that takes $x_1 \in x$ to the point $(1, 0)$. Now consider the complement of $x \subset \mathbb{S}^1$. This will be the disjointed circle $\mathbb{S}^1 \setminus x$. There now exists a homeomorphism from $\mathbb{S}^1 \setminus x$ to the interval $(0, 2\pi)$.

The points that were initially in x are now distributed as holes across the interval $(0, 2\pi)$. We call this set of holes Y_{r-1}. Since the complements of X_r and Y_{r-1} are homeomorphic, X_r and Y_{r-1} must also be homeomorphic by Theorem 3.2 from the textbook (Equivalent subsets have equivalent complements). Furthermore, since $X_r \cong Y_{r-1}$, it must also be true that $|\pi_0(X_r)| = |\pi_0(Y_{r-1})|$. Now, $|\pi_0(Y_{r-1})|$ is equal to the number of orderings of $r - 1$ numbers on the real line. This is given by $(r-1)!$. Thus, $|\pi_0(X_r)| = (r-1)!$

Problem 4

Problem: Is there a continuous surjection $f : \mathbb{S}^2 \times \mathbb{S}^1 \to O(3)$?

Solution: We will prove this by contradiction. Assume there exists a continuous and surjective $f : \mathbb{S}^2 \times \mathbb{S}^1 \to O(3)$. Then, the map $\gamma : \pi_0(\mathbb{S}^2 \times \mathbb{S}^1) \to \pi_0(O(3))$ must also be surjective. We now disprove this statement.

We first examine the cardinality of $\mathbb{S}^2 \times \mathbb{S}^1$. The following was proved in Homework 3 Problem 3: $\mathbb{S}^n$ is path connected if $n \geq 1$. This implies that both $\mathbb{S}^2$ and $\mathbb{S}^1$ are path connected. Furthermore, it was proved in Homework 3 Problem 4 that $X \times Y$ is path-connected iff X and Y are each path-connected. Thus, $\mathbb{S}^2 \times \mathbb{S}^1$ is path connected. This implies that $|\pi_0(\mathbb{S}^2 \times \mathbb{S}^1)| = 1$.

We now turn to the cardinality of $O(3)$. The set of matrices that compose $O(3)$ must orthonormal. We now choose two specific orthonormal matrices and show that they are in different path components. Take a matrix A and the matrix A' where the columns of A' are equal to the columns of A with the exception of the first column. Set $\vec{a\prime}_1 = -1 * \vec{a}_1$. The matrices A and A' are identical with the exception that one pair of their vectors are scaled by -1. We can then begin rotating $\vec{a\prime}_1$ to $\vec{a}$ while staying within the same path component. However, we cannot completely rotate $\vec{a\prime}_1$ since, to reach $\vec{a}_1$, $\exists \vec{a}_i, \vec{a}_j$ such that $\vec{a\prime}_1 \in span\{\vec{a}_i, \vec{a}_j\}$ at some point. This means that, at some point in the rotation, A' will not be orthonormal. This implies that $\pi_0 O(3) > 1$.

Since $1 \not\leq 1$, know that the number of path components of $\mathbb{S}^2 \times \mathbb{S}^1$ is different than the number of path components of $O(3)$. Thus, a contradiction is found and there does not exist a continuous surjection $f : \mathbb{S}^2 \times \mathbb{S}^1 \to O(3)$.