WEEK 5

CLASS

We finished a couple formal results about path-components, and computed the cardinalities of path-components in a few examples.

Here are some important notions we covered in class:

- We observed that, for $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ subsets, and for $f\colon X \to Y$ a continuous map, the diagram among sets,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\text{canonical}"'] & Y \arrow[d, "\text{canonical}"] \\ \pi_0(X) \arrow[r, "\pi_0(f)"] & \pi_0(Y), \end{tikzcd}$$

  commutes. Observing this amounted to unwinding the definition of all terms involved, in particular $\pi_0(f)$.
- **Surjections satisfy a 2-out-of-3 property.**

  Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of maps between sets. If the composition $g \circ f$ is surjective, then g is surjective.
- Because both of the "canonical" maps are surjective, and because surjective maps satisfiy a "2-out-of-3" property, then if the continuous map f is surjective, so is $\pi_0(f)$.
- A consequence of this is that the continuous image of a path-connected space is path-connected. Indeed, a set (such as $\pi_0(Y)$) is a singleton if it receives a surjection (such as $\pi_0(f)$) from a singleton (such as $\pi_0(X)$).
- Let $k > 0$. We considered the set

$$C_k \ := \ \Big\{(t_1,\dots,t_k) \in (0,1) \mid t_i = t_j \iff i = j\Big\} \ \subset \ (0,1)^{\times k} \ \subset \ \mathbb{R}^{\times k} \ = \ \mathbb{R}^k \ .$$

  We discussed how to think of an element in this set: it's k ordered distinct elements in the open intervatl $(0,1)$. We talked about how such an element moves about among other such elements: it's a movie of such (never witnessing two such elements collide).
- Consider the set Σ_k of self-bijections of the set $\{1,\dots,k\}$. Another way to think of Σ_k is as the set of linear orders on the set $\{1,\dots,k\}$. The cardinality of Σ_k is

$$\big|\Sigma_k\big| \ = \ k! \ .$$

- We constructed a map

$$C_k \longrightarrow \Sigma_k \ , \qquad (t_1,\dots,t_k) \mapsto \text{ the unique } \sigma \in \Sigma_k \text{ for which } t_{\sigma(1)} < t_{\sigma(2)} < \cdots < t_{\sigma(k)} \ .$$

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This map is continuous (check!), so it induces a map between sets of connected components:

$$\pi_0(C_k) \longrightarrow \pi_0(\Sigma_k) \ \cong\ \Sigma_k \ ,$$

where the last bijection is the canonical map $\Sigma_k \to \pi_0(\Sigma_k)$, which is a bijection because Σ_k is finite (remember, we talked about this). This map is evidently surjective (check this!). We showed this map is injective, and therefore a bijection. Indeed, suppose this map evaluates identically, as σ, on two elements $(t_1,\dots,t_k)$ and $(t'_1,\dots,t'_k)$ in C_k. We must then construct a path in C_k between these two elements. Here is such a path:

$$\gamma\colon\ [0,1] \longrightarrow C_k\ , \qquad s \mapsto \Big((1-s)t_1 + st'_1\ ,\ (1-s)t_2 + st'_2\ ,\ \dots\ ,\ (1-s)t_k + st'_k\Big)\ .$$

(Check that this is indeed valued in C_k!)

- Finally, the bijection $\pi_0(C_k) \xrightarrow{\text{bijection}} \Sigma_k$ implies that the cardinality

$$\Big|\pi_0(C_k)\Big| \ \cong\ k!\ .$$

- We connected the above calculation to problem (3) on HW4.
- Let $n \geq 2$. At the end of class, we introduced the following set:

$$\widetilde{P}_n \ :=\ \Big\{(z_1,z_2,\dots,z_n) \in (\mathbb{R}^2)^{\times n} \mid \|z_1 - z_2\| = \|z_2 - z_3\| = \cdots = \|z_{n-1} - z_n\| = \|z_n - z_1\| = 1\Big\}$$

$$\subset\ (\mathbb{R}^2)^{\times n} \ =\ \mathbb{R}^{2n}\ .$$

So, an element in $\widetilde{P}_n$ is a sequence of n points in the plane for which the distance between adjascently ordered points is 1 (including the first and final point). Let's call such data an *equallateral n-gon (in the plane)*; so $\widetilde{P}_n$ is the set of equallateral n-gons in the plane.

Next, lets declare two elements $p,q \in \widetilde{P}_n$ to be *equivalent*,

$$p \underset{\text{trans \& rot}}{\sim} q\ ,$$

if there is an *affine symmetry of the plane* relating p and q. This is to say that the equallteral n-gon p can be translated and rotated to result in the equallateral n-gon q.

$$P_n \ :=\ (\widetilde{P}_n)_{/\underset{\text{trans \& rot}}{\sim}}\ .$$

So an element in P_n is an equallateral n-gon up to translation and rotation.

Now, note that there *exists* a *unique* representative of each $\underset{\text{trans\&rot}}{\sim}$-equivalence class $[p] = [(z_1,\dots,z_n)] \in P_n$ for which $z_1 = 0 \in \mathbb{R}^2 = \mathbb{C}$ and $z_2 = e_1 = 1 \in \mathbb{C}$. So another way to define P_n is as the subset

$$P_n \ :=\ \Big\{(z_3,z_4,\dots,z_n) \in \mathbb{C}^{\times n-2} \mid \|1 - z_3\| = \|z_3 - z_4\| = \cdots = \|z_{n-1} - z_n\| = \|z_n - 0\| = 1\Big\}$$

$$\subset\ \mathbb{C}^{\times n-2} \ =\ \mathbb{C}^{n-2}\ .$$

So, an element in P_n is a sequence of $(n-2)$ points in the (complex) plane such that adjascently numbered points are distance 1 from one another, and the last point is distance 1 from $0 \in \mathbb{C}$ while the first point is distance 1 from $1 \in \mathbb{C}$.

- Here is a definition we'll discuss in class.

Definition 0.1.

- A *set of spokes* is a subset $S \subset \mathbb{R}^n$ for which
  - the intersection $S \cap \mathbb{S}^{n-1}$ is finite
  - there is an equality between subsets of $\mathbb{R}^n$:
$$S = \left\{ t \cdot v \mid t \geq 0 \text{ and } v \in S \cap \mathbb{S}^{n-1} \right\} .$$

  The cardinality $\left| S \cap \mathbb{S}^{n-1} \right|$ is the *valence* of such a set of spokes.
- A subset $X \subset \mathbb{R}^n$, together with a discrete subset $V \subset X$, is a *graph* if it is *locally an edge or a set of spokes*, which is to say the following:
  - for each $x \in X \smallsetminus V$, there is an $\varepsilon > 0$ and a homeomorphism $\varphi \colon X \cap \mathsf{B}_\varepsilon(x) \cong \mathbb{R}$;
  - for each $v \in V$, there is an $\varepsilon > 0$, a set S of spokes, and a homeomorphism $\varphi \colon X \cap \mathsf{B}_\varepsilon(v) \cong S$.

  The subset $V \subset X$ is the *set of vertices*; the *valence* of $v \in V$ is the valence of the *set of spokes* of an ε-neighborhood of $v \in X$. Each path-component of the complement $X \smallsetminus V$ is the *interior of an edge*. An *edge* of X is the closure of the interior of an edge; such an edge *connects* $v, v' \in V$ if v, v' are contained in this edge.

Note: it's not obvious that this definition of a *graph* is the same as that in the book. Nevertheness, these two definitions agree, and you're welcome to use this as you wish.

Reading

By Wednesday 2 October. §3, just the material about *graphs* (pages 44-45). §5, through the definition of a polyhedron (pages 69-73). Be especially familiar with these aspects.

- The definition of a *graph*, and of an *isomorphism* between graphs.
- The definition of a *polyhedron*, and of a *symmetry* between polyhedrons.

Exercises

These are due by **5pm on Wednesday 2 October**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) Consider the two subsets
$$X := \left\{ (x,y) \in \mathbb{R}^2 \mid xy = 0 \right\} \subset \mathbb{R}^2 \qquad \text{and} \qquad Z := \left\{ (w,z) \in \mathbb{C}^2 \mid wz = 0 \right\} \subset \mathbb{C}^2 .$$
(Note: I'm referencing multiplication of complex numbers in the definition of Z.)

(a) Show that both X and Z are path-connected.

(b) For each finite cardinality r, identify the subsets
$$\mathsf{Cut}_r(X) \subset X \qquad \text{and} \qquad \mathsf{Cut}_r(Z) \subset Z .$$

(Recall that, for $A \subset \mathbb{R}^n$ a path-connected subset, the subset

$$\mathsf{Cut}_r(A) := \Big\{ a \in A \mid |\pi_0(A \smallsetminus \{a\})| = r \Big\} \subset A .$$

So, an element $a \in \mathsf{Cut}_1(A)$ is not a cut-point.)

(2) Let $n > 0$. Consider the subset

$$\mathsf{U}_n(\mathbb{R}) := \Big\{ U \in \mathsf{GL}_n(\mathbb{R}) \mid U \text{ is upper triangular} \Big\} \subset \mathsf{GL}_n(\mathbb{R}) \subset \mathsf{Mat}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2} .$$

For each $n > 0$, identify the cardinality

$$\big| \pi_0\big(\mathsf{U}_n(\mathbb{R})\big) \big| .$$

(3) For $n = 2$ and $n = 3$, prove that P_n is finite, and identify the cardinality $|P_n|$.

(4) For each $n \geq 2$, identify the cardinality

$$\big| \pi_0(P_n) \big| .$$

(An informal argument is acceptible here, if necessary.)

(5) Identify a graph X for which there is a homeomorphism $X \cong P_4$.
(Hint: First, argue that P_4 is a graph – to do this, identify the discrete subset $V \subset P_4$ of *vertices*. (Recall, V being finite automatically ensures V is discrete.) Next, identify the *valence* of each such vertex in P_4. Next, identify, for each $v, v' \in P_4$, the set of *edges* in P_4 connecting v and v'.)