PROJECTS

Here are an assortment of possible projects for groups of 2, or 3, to work through and present by the last day of class (Friday 6 December).

Expectations. Here are my expectations for these projects.

- I expect each student to volunteer for a project. Largely based on interest in each project, I'll assign groups of 2, or 3; each group will be devoted to a single project. I expect each student to accept their assigned group, and engage with the other members of their group with respect.
- Each project is a narrative, with embedded prompts. I expect each group to work through this narrative, and make notes in response to each prompt. Because each member of each group will be expected to speak about the project, I expect each member of the group to have a large degree of ownership over the narrative and responses to the prompts.
- I'll provide poster-paper at class on Monday 2 December. Each group is expected to sketch a poster that will facilitate a presentation of their project. These presentations will take place on the final day of class (6 December). The format of these presentations will be that one person in each group is stationed at the poster, while the other(s) roam the room to visit other posters; then the roles switch part way through class.
- I expect each group to arrange to meet outside of class to fulfill these expectations. **I encourage each group to share contacts among themselves.** Essentially, the amount of time you'd spend on homework for a 10 day stretch is the amount of time I expect you'll spend on this.

The "participation" component of your final grade will largely be measured by your meeting these expectations.
(Let me know immediately if you have any questions or concerns.)

Date: 24 November 2019.

PROJECT: BROUWER FIXED-POINT THEOREM

This is a celebrated classical result. It articulates a sense in which every a dynamical system on a disk has a fixed point.

Central Question. Does every continuous map $\mathbb{D}^n \to \mathbb{D}^n$ have a fixed point?

- To make sense of this central question, let's first see the definition of a fixed point.

  Definition 0.1. Let X be a set. Let $f\colon X \to X$ a map from X to itself. A *fixed point* for f is an element $x \in X$ for which $f(x) = x$.

- Try to identify all fixed points of
  $$\mathbb{R} \longrightarrow \mathbb{R}~, \qquad t \mapsto 0~,$$
  and of
  $$\mathbb{R} \longrightarrow \mathbb{R}~, \qquad t \mapsto \sin(2\pi t)~,$$
  and of
  $$\mathbb{R} \longrightarrow \mathbb{R}~, \qquad t \mapsto t + \sin(2\pi t)~.$$
- Reflect on the central question in the case that $n = 0$. Well, $\mathbb{D}^0 = \{0\}$ is a singleton. So the answer is trivially true.
- Next, think about the central question in the case that $n = 1$. Experiment with some examples of maps $\mathbb{D}^1 \to \mathbb{D}^1$ (such as linear, or polynomial, or trigonometric functions). Based on your examples, conjecture an answer to the central question (in this case that $n = 1$). Take turns with your project-partner trying to convince them your conjecture is correct; conversely, try to poke holes in your partner's rationale.

- Based on that exercise, see notice what concepts you're referencing to conjure your arguments. See if you can hone in on one or two central themes.

- To me, 'direction' from each point to its image stands out as such a concept.

- Okay, now wuppose, to the contrary, that the answer is "no", there does not exist a fixed point. So, for each $x \in \mathbb{D}^1$, the element $f(x) \in \mathbb{D}^1$ is either strictly to the right of x, or strictly to the left of x. In other words, we have a map
  $$r\colon \mathbb{D}^1 \longrightarrow \mathbb{S}^0~, \qquad x \mapsto \frac{x - f(x)}{|x - f(x)|}~,$$
  which reports -1 if and only if $f(x)$ is to the right of x, and which reports $+1$ if and only if $f(x)$ is strictly to the left of x.
  Argue that r is continuous.
  What must be the value of r on $x = -1 \in \mathbb{S}^0 \subset \mathbb{D}^1$?
  What must be the value of r on $x = +1 \in \mathbb{S}^0 \subset \mathbb{D}^1$?
  Consider the resulting map $\pi_0(r)\colon \pi_0(\mathbb{D}^1) \to \pi_0(\mathbb{S}^0)$. Use your answers to

the above two questions to conclude that $\pi_0(r)$ is surjective. Draw a contradiction from this assertion.

- Think about how you'd improve the above reasoning to the 2-disk. Note that 'left/right' are replaced by 'directions', the set of which is $\mathbb{S}^1$. Because both $\mathbb{D}^2$ and $\mathbb{S}^1$ is connected, literally repeating the above argument doesn't work. But there is still something fundamentally different between $\mathbb{D}^2$ and $\mathbb{S}^1$, isn't there. Namely, in $\mathbb{D}^2$, any two points are connected by a canonical path (straight-line), while that's not the case for $\mathbb{S}^1$. Said another way, there is a "1-dimensional hole" in $\mathbb{S}^1$, but no "1-dimensional" holes in $\mathbb{D}^2$. So there can't be a surjection from the collection of "1-dimensional holes" in $\mathbb{D}^2$ to the collection of "1-dimensional holes" in $\mathbb{S}^1$. This is the spirit of the argument, at least. Let's try to make it a bit more rigorous.

  Let $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ be a continuous map. Suppose, for sake of contradiction, that f has no fixed points. Consider the resulting map
  $$r\colon \mathbb{D}^2 \xrightarrow{x \mapsto (f(x),x))} \mathsf{Conf}_2(\mathbb{D}^2) \xrightarrow{g} \mathsf{Conf}_2(\mathbb{S}^1) \xrightarrow{\mathrm{pr}\colon (y_1,y_2)\mapsto y_2} \mathbb{S}^1 ,$$
  where $g(x_1,x_2)$ is the unique pair $(y_1,y_2) \in \mathsf{Conf}_2(\mathbb{S}^1)$ as in the MidTerm problem.
  Argue that r is continuous.
  What is the value of r on an element $x \in \mathbb{S}^1 \subset \mathbb{D}^2$?
  Conclude that the composite map
  $$\mathbb{S}^1 \xrightarrow{\text{inclusion}} \mathbb{D}^2 \xrightarrow{r} \mathbb{S}^1$$
  is the identity map.
  Use the least you think you have to in order to deduce that the composition of linear maps
  $$\mathsf{H}^1(\mathbb{S}^1) \xrightarrow{\mathsf{H}^0(r)} \mathsf{H}^1(\mathbb{D}^2) \xrightarrow{\mathsf{H}^0(\text{inclusion})} \mathsf{H}^1(\mathbb{S}^1)$$
  is the identity map. Conclude an inequality involving dimensions:
  $$\dim(\mathsf{H}^1(\mathbb{D}^2)) \;\geq\; \dim(\mathsf{H}^1(\mathbb{S}^1)) \;.$$
  Argue that this is impossible, since $\mathbb{D}^2$ has no "1-dimensional holes". Therefore, there is no continuous map $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ without fixed points!

- Now generalize and abstract the above discussion.

  Definition 0.2. Let $A \subset X$ be a subspace of a topological space. A *retraction (of X onto A)* is a continuous map
  $$r\colon X \longrightarrow A$$
  for which $r(a) = a$ for each $a \in A$. We say X *retracts onto* A if such a retraction exists.

- Show that $\mathbb{D}^1$ retracts onto $\mathbb{D}^0$.
  Show that $\mathbb{S}^1 \times \mathbb{S}^1$ retracts onto $\mathbb{S}^1 \times \{1\}$.

- Let $A \subset X$ be a subspace of a topological space. Let $r\colon X \to A$ be a retraction. Denote the inclusion $i\colon A \to X$. Prove that the composite map among 0th cohomologies

$$\mathsf{H}^0(A) \xrightarrow{\mathsf{H}^0(i)} \mathsf{H}^0(X) \xrightarrow{\mathsf{H}^0(r)} \mathsf{H}^0(A)$$

is the identity map (on $\mathsf{H}^0(A)$). Conclude that $\mathsf{H}^0(r)$ is injective, and that $\mathsf{H}^0(i)$ is surjective. Deduce an inequality

$$\dim\big(\mathsf{H}^0(X)\big) \;\geq\; \dim\big(\mathsf{H}^0(A)\big)\ .$$

- Deduce from the above point that, for $A \subset X$ a subspace of a topological space, if

$$\dim\big(\mathsf{H}^0(X)\big) \;<\; \dim\big(\mathsf{H}^0(A)\big)\ ,$$

then there is no retraction of X onto A.

- Conclude that $\mathbb{D}^1$ does *not* retract onto $\mathbb{S}^0$.

- Convince yourselves that $\mathbb{S}^{n-1}$ has an $(n-1)$-dimensional hole, whereas $\mathbb{D}^n$ does not. This intuition is instantiated as $\dim(\mathsf{H}^{n-1}(\mathbb{S}^{n-1})) = 1$ whereas $\dim(\mathsf{H}^{n-1}(\mathbb{D}^n)) = 0$.
Use basic features of $\mathsf{H}^{n-1}(-)$, to likewise conclude that $\mathbb{D}^n$ does not retract onto $\mathbb{S}^{n-1}$. This conclusion is the foundation of the following theorem; check it out.

- Understand the following statement (for $n \leq 1$), and proof (for $n \leq 1$ and $k = 0$), of the following result.

Theorem 0.3 (Brouwer fixed-point Theorem)**.** *Let $f\colon \mathbb{D}^n \to \mathbb{D}^n$ be a map. If f is continuous, then f has a fixed point, which is to say there is an element $x \in \mathbb{D}^n$ for which $f(x) = x$.*

Proof. Assume f is continuous. Suppose, for sake of contradiction, that f does not have a fixed point. Consider the map

$$r\colon \mathbb{D}^n \longrightarrow \mathbb{S}^{n-1}$$

whose value on $x \in \mathbb{D}^n$ is the unique $y \in \mathbb{S}^{n-1}$ for which $\{y, x, f(x)\}$ are colinear and x is either equal to y or x is between y and $f(x)$. Argue that r is continuous (this follows similar logic as in a problem on the MidTerm exam! One way to do this is to supply an explicit formula for the values of r in terms of x and algebraic operations, then argue that any such expression depends continuously on x). Argue that r is a retraction. Therefore, the map composite linear map

$$\mathsf{H}^k(\mathbb{S}^{n-1}) \xrightarrow{\mathsf{H}^0(r)} \mathsf{H}^k(\mathbb{D}^n) \xrightarrow{\mathsf{H}^0(i)} \mathsf{H}^k(\mathbb{S}^{n-1})$$

is the identity map. So $\dim(\mathsf{H}^k(\mathbb{S}^{n-1})) \leq \dim(\mathsf{H}^k(\mathbb{D}^n))$, for each $k \geq 0$. But for $n \leq 1$, this is *not* true for $k = 0$. So we have a contradiction, therefore the assumption that f does not have a fixed point is incorrect. [1]

$\square$

- There are many applications of this result. Here are a couple to consider.

(1) Suppose we're interested in the health of Yellowstone's ecosystem. We've identified 30 indicator species for assessing this, each whose population is practical to count. So we record the 'health' of Yellowstone's ecosystem simply as a sequence of 30 numbers: the ith number in this sequence is the population of the ith species. Suppose that, based on modeling, or something, we happen to know that each species population is bounded above by some number N (the "carrying capacity"). So we're recording the 'health' of Yellowstone's ecosystem as a point in

$$[0,N]^{\times 30} \subset \mathbb{R}^{30}$$

a 30-dimensional cube.
Explain this paragraph in your own words.

Now, based on modeling, we abstract the year-by-year change of these populations as a map

$$T\colon [0,N]^{\times 30} \longrightarrow [0,N]^{\times 30} .$$

So, the value of f on a point is the following year's population-state, which is a function of the given year's population-state. Explain this map in your own words.

Suppose our model suggests that T is continuous. Argue that there is a homeomorphism $[0,N]^{\times 30} \cong \mathbb{D}^{30}$ with the 30-dimensional disk. Apply Brouwer's fixed-point Theorem to conclude that there is some population-state $x \in [0,N]^{\times 30}$ for which $T(x) = x$.
Explain such an x as an *equilibrium-state*.

(2) Undergo an analogous discussion but replacing each species with a company in a (closed) economic system; each population of a species with the value produced a company; the map T by the month-by-month change of value.
Explain how to use Brouwer's fixed-point Theorem to conclude the existence of an equilibrium-state to the economic system.

(3) Think of another, conceptually distinct, application of Brouwer's fixed-point Theorem.

[1] In fact, for $n > 1$, then $\dim(\mathsf{H}^{n-1}(\mathbb{S}^{n-1})) = 1$ whereas $\dim(\mathsf{H}^{n-1}(\mathbb{D}^n)) = 0$. So this argument still lends to a contradiction even in the case that $n > 1$ ... we just have to know what $\mathsf{H}^k(-)$ is for $k > 0$ in order to accept this more general $n > 1$ argument.

PROJECT: 15-PUZZLE

This project uses the concept of path-components to assess solvability of the "15-puzzle". Recall, or look up, the "15-puzzle".

Central Question. Given a state of the 15-puzzle, what are the chances that it's solvable?

- Entertain the existence of a topological space in which each state of the 15-puzzle is an element in it, while each simple move relating two states implements a (continuous) path in it.
  Argue that the universal/minimal such topological space is a graph, X_{15}, in which a vertex is a state of the 15-puzzle and an edge is given by a simple move.
  Argue that every state of the 15-puzzle can be solved if and only if this graph is path-connected. More generally, consider the path-component containing the "solved" puzzle, denote it simply $[\text{solved}] \in \pi_0(X_{15})$. Argue that the ratio
  $$\frac{|[\text{solved}]|}{|\pi_0(X_{15})|}$$
  can be interpreted as the probability of a random state being solvable.

- More generally, let $k > 0$ and consider the likewise graph X_n, where $n = k^2 - 1$, that codifies the "n-puzzle". Argue that not every state of the "3-puzzle" is solvable. Specifically, argue that X_3 has at least two path-components. Conceptualize this as a surjective continuous map
  $$X_3 \longrightarrow \{\pm 1\} = \mathbb{S}^0 , \tag{1}$$
  whose value on a state of the "3-puzzle" is +1 if it is connected via a path to a state of the "3-puzzle" in which 1 is placed in the top left corner and 2 is placed next to it, then 3 is in the bottom row; and whose value is -1 otherwise. Let's phrase this map is a less ad-hoc way.

  First, argue that a state of the "3-puzzle" is precisely the same data as a permutation σ of 4 letters:
  $$\{1,2,3,4\} \xrightarrow[\text{bijection}]{\sigma} \{1,2,3,4\} .$$
  Specifically, order the 4 spots in the way reading is ordered: from top left, rightward, then down a notch and rightward. Now, given a state of the "3-puzzle", declare $\sigma(i)$ to be the number appearing in the ith place (where the blank number is understood as "4"). Next, consider the permutation matrix P_σ associated to σ. Then consider the map from the set of vertices of X_3 as
  $$\sigma \mapsto (-1)^{\sigma^{-1}(4)} \det(P_\sigma) .$$
  Argue that if σ and σ' are related by a simple move, then these two values agree:
  $$(-1)^{\sigma^{-1}(4)} \det(P_\sigma) \;=\; (-1)^{(\sigma')^{-1}(4)} \det(P_{\sigma'}) .$$

Conclude that there is a continuous map (1). [2]

- Now, return to the case of the "15-puzzle". Construct a finite, abstract, directed graph $\Gamma_{15} = (V, E, s, t)$ in which V is the set of all possible states of the "15-puzzle";
$$E \;:=\; \Big\{ (u,v) \mid u \text{ and } v \text{ are related by a simple move} \Big\} \;\subset\; V \times V \;;$$
$$s \colon E \xrightarrow{(u,v) \mapsto u} V \qquad \text{and} \qquad t \colon E \xrightarrow{(u,v) \mapsto v} V \;.$$
Name a natural bijection between V and the set of permutations of the set of 16 letters, $\{1, \ldots, 16\}$. Through this bijection, we'll just think of elements in V as permutations of $\{1, \ldots, 16\}$.

- As in the case of the "3-puzzle", consider the map
$$V \longrightarrow \{\pm 1\} = \mathbb{S}^0 \;, \qquad \sigma \mapsto (-1)^{\sigma^{-1}(16)} \det(P_\sigma) \;,$$
where P_σ is the 16×16 permutation matrix associated to σ. Argue further that this map extends (as being constant on each edge) as a continuous map from the geometric realization:
$$|\Gamma_{15}| \longrightarrow \{\pm 1\} = \mathbb{S}^0 \;.$$

- Conclude that
$$\Big| \pi_0\big(|\Gamma_{15}|\big) \Big| \geq 2 \;. \tag{2}$$
Interpret this as the statement that not every state of the "15-puzzle" is solvable.

- See if you can argue that, in fact, the relation (2) is an equality.

- See what other numbers n, besides 3 and 15, you can generalize these methods to concerning the "n-puzzle".

[2] In other words, conclude that assigning to each element in an edge of X_3 the value, $(-1)^{\sigma^{-1}(4)} \det(P_\sigma)$, of either of its vertices, σ, is well-defined and continuous.

PROJECT: PERSISTENT COHOMOLOGY

This concept is outlined in the Week 13 notes, as an example concerning marmot chirps. This is a computational method for identifying native clusters of a data set.

Central Problem. Given a data set $S \subset \mathbb{R}^N$, figure out how it's "clustered".

- Justify, through at least 3 conceptually distinct examples, the following definition. [3] [4]

  Definition 0.4. A *data set* is a finite subset
  $$S \subset \mathbb{R}^N$$
  of a Euclidean space.

  From here on, fix a data set $S \subset \mathbb{R}^N$, which we'll assume has more than 1 element in it: $|S| > 1$.

- Outline some intuition behind "clustering" for S; appeal to examples of data sets for this. [5] Try to come up with your own definition of "clustering"; test any such definition against some (reasonably robust) examples of data sets; reflect on how your definition lends to computation (for instance, how might you write an algorithm for a machine to identify clustering of S?).

- For me, and many others, a reasonable notion of "clustering" would be path-components of this data set. But, literally, this is nonsensical: $\pi_0(S) = S$. [6] So a new idea is needed for regarding "clusters" of S in terms of path-components.

  One approach is that to say, for $\varepsilon \geq 0$, that an "ε-cluster" of S is a path-component of the (abstract, directed) graph $\Gamma_\varepsilon = (S, E_\varepsilon, s_\varepsilon, t_\varepsilon)$ defined as follows. Its set of vertices is $V = S$, the given data set. Its set of directed edges
  $$E_\varepsilon \ := \ \Big\{(u,v) \in V \times V \mid \|u - v\| \leq \varepsilon\Big\}$$
  consists of those pairs of data points (ie, elements in S) that are within ε of each other. Its source map and its target map are projections onto each coordinate of E_ε:
  $$s_\varepsilon \colon E_\varepsilon \xrightarrow{(u,v)\mapsto u} S \qquad \text{and} \qquad t_\varepsilon \colon E_\varepsilon \xrightarrow{(u,v)\mapsto v} S \ .$$

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[3] For example, a collection of audio files, or a collection of digital photos, etc.

[4] Remember, like all mathematical definitions, they are as useful in as much as they balance a tension: on the one hand, it's easy to find examples of this definition 'in the wild'. On the other hand, it's easy to prove stuff about this definition. So this prompt is to articulate your reflection about how this definition indeed balances this tension.

[5] For instance, if your data set consists of a collection of images, you might expect for "cat photos" to be clustered away from "night sky" photos.

[6] Here, as always, we're regarding a subset of a Euclidean space (such as $S \subset \mathbb{R}^N$) as a topological space via the *subspace* construction.

Describe this (abstract, directed) graph Γ_ε in your own words, as if you're speaking to a student in this class of a different project.
Explain why this definition of an "ε-cluster" is a reasonable one; [7] if you don't think it's a reasonable one, then why not?

- Point out that, for $\varepsilon \geq 0$ really small, then $\pi_0\big(|\Gamma_\varepsilon|\big) = S$. Point out that, for $\varepsilon \geq 0$ really large, then $\pi_0\big(|\Gamma_\varepsilon|\big)$ is a singleton. Reflect on how the function

$$\mathbb{R}_{\geq 0} \longrightarrow \mathbb{N}~, \qquad \varepsilon \mapsto \Big|\pi_0\big(|\Gamma_\varepsilon|\big)\Big|~, \tag{3}$$

  is non-increasing (from $|S|$ to 1).
  Reflect on how to figure out for which ε the set $\pi_0\big(|\Gamma_\varepsilon|\big)$ is the native clustering of S. Undergo such reflections through examples, even of just dots you mark on a piece of paper (ie, a finite subset $S \subset \mathbb{R}^2$).

- Motivate that the native clusters of S is the set $\pi_0\big(|\Gamma_\varepsilon|\big)$ whose cardinality is unchanged (yet greater than 1) for the largest interval of εs. [8] Supply a compelling argument for why, if no such large interval exists, then we can interpret there being one cluster.

- Finally, reflect on the computability of this method. For instance, how might you instruct a machine to implement this (for a given data set $S \subset \mathbb{R}^N$)? For instance, you might consider only evaluating the function (3) on elements $\varepsilon = \frac{n}{100} \in \mathbb{R}_{\geq 0}$, with $n = 0, 1, 2, \dots, R$, for some very large natural number R. But, still, for a given such $\varepsilon = \frac{n}{100}$, how would a machine identify the cardinality of $\pi_0\big(|\Gamma_\varepsilon|\big)$? This is where the Week 12 HW problem is useful: identify the cardinality of the set of path-components as the dimension of the kernel of a the linear map:

$$\Big|\pi_0\big(|\Gamma_\varepsilon|\big)\Big| ~=~ \mathsf{dim}\Big(\mathsf{Ker}\big(\mathbb{R}^S \xrightarrow{t_\varepsilon{}^* - s_\varepsilon{}^*} \mathbb{R}^{E_\varepsilon}\big)\Big)~.$$

  To figure out this dimension, we choose bases for $\mathbb{R}^S$ and $\mathbb{R}^{E_\varepsilon}$, with respect to these bases write this linear map as a matrix, then compute the nullity of this matrix.

  Here is an easy implementation of this. Enumerating $S = \{v_1, v_2, \dots, v_k\}$, then endowing $E_\varepsilon \subset S \times S$ with the dictionary enumeration therefrom: $E_\varepsilon = \{a_1, \dots, a_\ell\}$. [9] Recall that such bases determine an $\ell \times k$-matrix A_ε implementing the linear map $(t_\varepsilon^* - s_\varepsilon^*)$. So each entry of A_ε is either 0 or ± 1. In fact, argue that, for $u, v, w \in S$, then
  - the $\big((u,v),w\big)$-entry is 1 if and only if $u \neq w = v$ (and u is within ε of v, as it would have to be in order for $(u,v) \in E_\varepsilon$);

[7] For instance, appeal to the reflections generated by the previous paragraph.
[8] In other words, the largest interval $[\varepsilon, \varepsilon'] \subset \mathbb{R}_{\geq 0}$ for which $\Big|\pi_0\big(|\Gamma_\varepsilon|\big)\Big| = \Big|\pi_0\big(|\Gamma_{\varepsilon'}|\big)\Big|$.
[9] Note that $\ell \leq k^2$ since $E_\varepsilon \subset S \times S$.

– the $((u,v),w)$-entry is -1 if and only if $u = w \neq v$ (and u is within ε of v, as it would have to be in order for $(u,v) \in E_\varepsilon$);
– the $((u,v),w)$-entry is 0 otherwise (though u is within ε of v, as it would have to be in order for $(u,v) \in E_\varepsilon$).

Conclude, in particular, that A_ε is a symmetric matrix, with all diagonal entries being 0. In any case, this is a very simple type of matrix!
Write this matrix out for various εs in the case that S has three elements in it.

In summary, we're then seeking the largest interval for which the following function (which is the computable version of (3)),

$$\{0,1,2,3,\ldots,R\} \longrightarrow \mathsf{nullity}(A_\varepsilon)\ , \tag{4}$$

is constant (and greater than 1). Finally, computing nullities of matrices is what computers do for breakfast!

- **Example.** Undergo this method for the following data set:

$$S := \Big\{\Big(\ 100\cos(\frac{2\pi p}{400})\ ,\ 100\sin(\frac{2\pi p}{400})\ ,\ 0\ \Big)\ |\ p=0,\ldots,399\Big\}$$
$$\bigcup$$
$$\Big\{\Big(\ \cos(\frac{2\pi p}{40})\sin(\frac{\pi q}{20})\ ,\ \sin(\frac{2\pi p}{40})\sin(\frac{\pi q}{20})\ ,\ \cos(\frac{\pi q}{20})\ \Big)\ |\ p=0,\ldots,39\ ,\ q=0,\ldots,19\Big\}$$
$$\subset\ \mathbb{R}^3\ .$$

Specifically,
(1) Sketch a graph of the function (4);
(2) Identify the largest interval of ns in $\{1,\ldots,R\}$ on which the values of the function (4) are constant yet greater than 1;
(3) Identify the native clusters of S.

PROJECT: EULER CHARACTERISTIC

This project outlines what Euler characteristic is, and explains that it satisfies an inclusion-exclusion principle (even though $|\pi_0(-)|$ does not).
(Note, much of this material is essentially repeated from the last set of notes.)

Central Question. Can the failure of an "inclusion-exclusion principle" for $|\pi_0(-)|$, as it pertains to open covers of a topological space, be fixed? Specifically, is there a different way to assign a number $\chi(-)$ to each topological space so that $\chi(-)$ possesses an "inclusion-exclusion principle" pertaining to open covers of a topological space?

- We've seen that 0th cohomology, $\mathsf{H}^0(-)$, is an assignment of a vector space to each topological space. There is a name for such an assignment:

  Definition 0.5. A *functor from the category of topological spaces and continuous maps among them to the category of vector spaces and linear maps among them* is
  - an assignment of a vector space to each topological space:
  $$F \ : \qquad \underset{\text{a top space}}{X} \qquad \mapsto \qquad \underset{\text{a vect space}}{F(X)} \quad ;$$
  - an assignment of a linear map to each continuous map:
  $$F \ : \qquad X \xrightarrow[\text{continuous}]{f} Y \qquad \mapsto \qquad F(X) \xleftarrow[\text{linear}]{F(f)} F(Y) \ ;$$

  satisfying the following conditions:
  - for each pair $X \xrightarrow{f} Y \xrightarrow{Z}$ of composable continuous maps among topological spaces, the assignment of their composition is the composition of their assignments: the two linear maps from $F(Z)$ to $F(X)$,
  $$\mathsf{H}^0(g \circ f) \ = \ \mathsf{H}^0(f) \circ \mathsf{H}^0(g) \ ,$$
  agree;
  - for each topological space X, the linear map associated to the identity map on X is the identity map on $F(X)$:
  $$F(\mathrm{id}_X) \ = \ \mathrm{id}_{F(X)} \ .$$

  Verify that, indeed, H^0 is a such a functor.

- For F such a functor, it follows from the above that $F(f)$ is an isomorphism between vector spaces whenever f is a homeomorphism between topological spaces!

- The Mayer-Vietoris Theorem reveals a sense in which $\mathsf{H}^0(-)$ satisfies a sort of inclusion-exclusion principle. This is phrased as an *exact sequence (of vector spaces)*. Inspecting the statement of that Theorem, one might ask if the named exact sequence of vector spaces extends to a longer exact sequence of vector spaces. Namely, is there is a vector space V together with a linear map $\partial\colon \mathsf{H}^0(A\cap B)\to V$ for which the resulting sequence

$$0 \xrightarrow{\zeta} \mathsf{H}^0(X) \xrightarrow{\Phi} \mathsf{H}^0(A)\oplus\mathsf{H}^0(B) \xrightarrow{\Psi} \mathsf{H}^0(A\cap B) \xrightarrow{\partial} V$$

is exact. In fact, why stop there; let's ask if this sequence can continue as an exact sequence of linear maps among vector spaces:

$$0 \xrightarrow{\zeta} \mathsf{H}^0(X) \xrightarrow{\Phi} \mathsf{H}^0(A)\oplus\mathsf{H}^0(B) \xrightarrow{\Psi} \mathsf{H}^0(A\cap B) \xrightarrow{\partial} V \to U \to U' \to U'' \to \dots\ .$$

In fact, it does: for each $k\geq 0$, *kth cohomology* is a *functor* (as above):

$$\mathsf{H}^k \text{ is the assignments : } \qquad Z\mapsto \mathsf{H}^k(Z) \qquad \text{and} \qquad (W\xrightarrow{f} Z)\mapsto \big(\mathsf{H}^k(W)\xleftarrow{\mathsf{H}^k(f)}\mathsf{H}^k(Z)\big)\ .$$ [10]

- These functors H^k have the following properties, which I'll state as theorems (since, once H^k is defined, are celebrated results in algebraic topology). [11]

Theorem 0.6 ("Homotopy Invariance"). *Let $C\subset\mathbb{R}^n$ a star-shaped subset. Let Z be a topological space. Consider the product topological space $Z\times C$. For each $k\geq 0$, the projection map $Z\times C\to Z$ implements an isomorphism between vector spaces:*

$$\mathsf{H}^k(Z)\xrightarrow{\cong}\mathsf{H}^k(Z\times C)\ .$$

Theorem 0.7 ("Mayer-Vietorus"). *Let X be a topological space. Let $A,B\subset X$ be subsets. Suppose $A,B\in\mathcal{T}_X$ and $A\cup B=X$. Then, for each $k>0$, there is a linear map*

$$\partial^k_{k-1}\colon \mathsf{H}^{k-1}(A\cap B)\longrightarrow\mathsf{H}^k(X)$$

for which the sequence of linear maps among vector spaces, [12]

$$\begin{aligned} 0 \xrightarrow{\partial^0_{-1}} \mathsf{H}^0(X) \xrightarrow{\Phi^0} \mathsf{H}^0(A)\oplus\mathsf{H}^0(B) \xrightarrow{\Psi^0} \mathsf{H}^0(A\cap B) \quad {}^{13} \\ \xrightarrow{\partial^1_0} \mathsf{H}^1(X) \xrightarrow{\Phi^1} \mathsf{H}^1(A)\oplus\mathsf{H}^1(B) \xrightarrow{\Psi^1} \mathsf{H}^1(A\cap B) \\ \xrightarrow{\partial^2_1} \mathsf{H}^2(X) \xrightarrow{\Phi^2} \mathsf{H}^2(A)\oplus\mathsf{H}^2(B) \xrightarrow{\Psi^2} \mathsf{H}^2(A\cap B) \\ \xrightarrow{\partial^3_2} \cdots \end{aligned} \tag{5}$$

is **exact**.

Theorem 0.8 ("Point Value"). *There are canonical isomorphisms between vector spaces*

$$\mathsf{H}^0(*)\cong\mathbb{R} \qquad \textit{and} \qquad \mathsf{H}^k(*)=\{0\}=:0 \qquad (\textit{for } k>0)\ .$$

[10] In the case that $W\to Z$ is the inclusion of a subspace, we'll denote the induced linear map simply as

$$\mathsf{H}^k(Z)\longrightarrow\mathsf{H}^k(W)\ , \qquad \omega\mapsto\omega_{|W}\ ,$$

suggesting "restriction of ω to $W\subset Z$".

[11] It can be fruitful to convince yourself that one might expect these theorems to be true, even just through the intuition of "k-dimensional holes".

[12] Here, each instance of the maps Φ and Ψ is given in the same way as it is concerning 0th cohomology Namely, $\Phi^k(\omega):=(\omega_{|A},\omega_{|B})$ and $\Psi^k(\alpha,\beta):=\alpha_{|A\cap B}-\beta_{|A\cap B}$.

[13] Here, $\Phi^0:=\Phi$ and $\Psi^0:=\Psi$.

- These three theorems, along, allow one to compute a the vector spaces $\mathsf{H}^k(X)$ for a broad class of topological spaces X. This is amazing, since, evidently, one doesn't even need to know the definition of $\mathsf{H}^k(X)$ in order to compute it (at least for X in this class).

- Use the "Point Value" and "Homotopy Invariance" to conclude that the unique map $Z \to *$ implements an isomorphism
$$\mathsf{H}^k(*) \cong \mathsf{H}^k(Z)$$
for every $k \geq 0$ for each topological space Z that is homeomorphic with a star-shaped subset of a Euclidean space.

- Another consequence of the above is that
$$\mathsf{H}^k(\emptyset) = 0$$
for every $k \geq 0$. As a consequence of this, for $X = A \sqcup B$, then
$$\Phi\colon \mathsf{H}^k(X) \xrightarrow{\cong} \mathsf{H}^k(A) \oplus \mathsf{H}^k(B)$$
is an isomorphism.

- In fact, there is a remarkable theorem of Eilenberg-Steenrod that states that there is a *unique*! collection of functors $\{\mathsf{H}^k\}_{k\geq 0}$ satisfying the above three theorems! In other words, the above two theorems *define* $\mathsf{H}^k(-)$!

- So, you can imagine that to compute $\mathsf{H}^k(X)$ is an inductive exercise involving linear algebra and judicious choices of open covers A, B of X. Implement such an inductive argument, through clever choices of $A, B \subset \mathbb{S}^n$, to compute, for $n > 0$:
$$\dim(\mathsf{H}^k(\mathbb{S}^n)) = 0 \qquad \text{if } k \neq 0, n \qquad \text{and} \qquad \dim(\mathsf{H}^k(\mathbb{S}^n)) = 1 \qquad \text{if } k = 0, n~.$$
Even just doing this for $n = 1, 2$, and possibly $n = 3$ would be enlightening.

- The Mayer-Vietorus Theorem supplies an intuition behind $\mathsf{H}^k(Z)$: it is spanned by the "k-dimensional holes in Z".
Test this intuition on a few simple examples, such as simple graphs. In other words, convince yourself that H^1 of some choice graphs (which you can access using the Mayer-Vietorus Theorem) indeed have the dimension of the intuitive number of "1-dimensional holes" in the graphs.

- Finally, here is the definition of the Euler characteristic. **(This is the main part of this project.)**

  Definition 0.9 (Euler characteristic)**.** For X a topological space, its *Euler characteristic* is the alternating sum of dimensions of its cohomology:
$$\chi(X) ~:=~ \dim(\mathsf{H}^0(X)) - \dim(\mathsf{H}^1(X)) + \dim(\mathsf{H}^2(X)) - \dots$$

$$= \sum_{k\geq 0}(-1)^k \dim(\mathsf{H}^k(X))\ .$$[14]

- So the Euler characteristic is an assignment of an integer to each topological space
$$\chi\colon X \mapsto \chi(X)\ \in \mathbb{Z}\ .$$
Note that a term in $\chi(X)$ is $|\pi_0(X)|$, but there are, in general, more terms of course.

- For X is a finite set with its discrete topology, use the features of H^k discussed above to deduce that its Euler characteristic agrees with its cardinality:
$$|X|\ =\ \chi(X)\ .$$

- A consequence of the Mayer-Vietorus theorem is that **Euler characteristic** (as opposed to $|\pi_0(-)|$) satisfies an inclusion-exclusion principle:
$$\chi(X)\ =\ \chi(A)+\chi(B)-\chi(A\cap B)\ \ ! \tag{6}$$
Verify this identity in a few non-trivial cases of $A,B\subset X$ where X is a graph (such as a graph with 2 vertices, and $n+1$ edges between'em).

- Try to deduce the inclusion-exclusion principle for χ (6) from the Mayer-Vietorus Theorem. **(This is the main conceptual part of this project.)** Specifically, try to prove the following lemma.

Notation 0.10. Let $V_0,V_1,V_2,\dots$ be a collection of finite-dimensional vector spaces. Suppose there is some $k\geq 0$ for which $V_\ell = 0$ is the zero vector space for $\ell\geq k$. Denote the alternating sum:
$$\chi(V_\bullet)\ :=\ \sum_{k\geq 0}(-1)^k\dim(V_k)\ .$$

Lemma 0.11. *Let*
$$\begin{aligned} 0 \xrightarrow{B_0} U_0 \xrightarrow{F_0} V_0 \xrightarrow{G_0} W_0 \\ \xrightarrow{B_1} U_1 \xrightarrow{F_1} V_1 \xrightarrow{G_1} W_2 \\ \xrightarrow{B_2} U_2 \xrightarrow{F_2} V_2 \xrightarrow{G_2} W_3 \\ \xrightarrow{B_3} \cdots \end{aligned} \tag{7}$$
be an exact sequence (of linear maps among finite-dimensional vector spaces). Suppose there is a $k\geq 0$ for which all vector spaces in this sequence beyond the kth term is the zero vector space. Then
$$\chi(U_\bullet)\ -\chi(V_\bullet)\ +\ \chi(W_\bullet)\ =\ 0\ .$$

[14]Obviously, the Euler characteristic isn't guaranteed to be defined unless each of these dimensions if finite, and there are only finitely many that are not zero.

Proof. Try proving this by induction on k. So, for instance, suppose U_0 onwards is the zero vector space. Then all dimensions are zero, and the statement of the lemma is trivially true. Next, suppose V_0 onwards is the zero vector space. Then exactness of this sequence at U_0 ensures that U_0 must also be the zero vector space. The statement of the lemma is then trivially true again. Next, suppose W_0 onwards is the zero vector space. Use exactness to conclude that F_0 is an isomorphism, there therefore $\dim(U_0) = \dim(V_0)$. Deduce the lemma in this case. Next, suppose U_1 onwards is the zero vector space. In this more interesting case, use the definition of exactness to conclude the statement of the lemma as an instance of the **Rank-Nullity Theorem**! Next, suppose V_1 onwards is the zero vector space. Exactness at U_1 states that $\mathsf{Im}(B_1) = \mathsf{Ker}(F_1) = U_1$. Exactness at W_0 states that $\mathsf{Im}(G_0) = \mathsf{Ker}(B_1)$. The **Rank-Nullity Theorem** thusly gives the identities:

$$\begin{gathered}\dim(W_0) \\ = \dim(\mathsf{Ker}(B_1)) + \dim(\mathsf{Im}(B_1)) \\ = \dim(\mathsf{Im}(G_0)) + \dim(U_1)\ .\end{gathered}$$

The key idea now is to notice that the sequence

$$0 \xrightarrow{B_0} U_0 \xrightarrow{F_0} V_0 \xrightarrow{G_0} \mathsf{Im}(G_0) \to 0 \to 0 \to \cdots$$

is exact, and of shorter "length" (ie, one less k). So you know, from the consideration of lesser k above, that

$$\dim(U_0) - \dim(V_0) + \dim(\mathsf{Im}(G_0)) = 0\ .$$

Substituting the previous identity into this identity gives the statement, in this case. I believe this communicates the spirit of a proof of this lemma. Try now to supply a legitimate inductive proof of the lemma.

□

- Explain that the usual inclusion-exclusion principle agrees with the inclusion-exclusion principle (6). Namely, consider (6) in the case that X is a finite set with its discrete topology.

- For $\Gamma = (V, E, s, t)$ a finite (abstract, directed) graph, outline an argument for why
$$\chi(|\Gamma|) = |V| - |E|\ .$$
For this, you might use the inclusion-exclusion principle for χ applied to $A \subset |\Gamma|$ the complement of the center of each edge of the graph, and B to be the (disjoint) union of the interiors of each edge.
- For clever choices of $A, B \subset \mathbb{S}^n$, compute the Euler characteristic
$$\chi(\mathbb{S}^n) \ = \ 1 + (-1)^n\ .$$

- Let X be a finite union of polyhedra, each glued along their faces. Reflect on an inductive approach to computing $\mathsf{H}^*(X)$. Namely, choose a highest-dimensional polygon $P \subset X$, say its dimension is n. Construct an open cover $A, B \subset X$ so that $B \subset P$ is the interior of this polygon, and A is a slight enlargement of the complement $X \smallsetminus P$ of this polygon, and $A \cap B$ is a slight enlargement of the (point-set) boundary ∂P of this polygon. Using the "Homotopy Invariance" property of H^*, argue that there are isomorphisms

$$\mathsf{H}^k(B) \cong \mathsf{H}^k(*) \qquad \text{and} \qquad \mathsf{H}^k(A \cap B) \cong \mathsf{H}^k(\partial P) \cong \mathsf{H}^k(\mathbb{S}^{n-1})$$

for each $k \geq 0$.

- See if you can use the previous point to identify

$$\chi(\mathbb{RP}^n)$$

even/especially only for $n = 2$. Remember our description of $\mathbb{RP}^2$ as a union of 4 triangles. Alternatively, remember our description of $\mathbb{RP}^2$ as a union of a Mobius strip and a 2-disk.