WEEK 3

CLASS

We continued to focused our attention on *continuity*, by completing/asserting the list below of examples of continuous maps. We then moved on to the first (non-trivial) topological property: *path-connectedness* and *path-components*.

Here are some important notions we covered in class:

- We showed that the following classes of maps are continuous. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be subsets. Let $f : X \to Y$ be a map between sets.
  - **Constant maps.** Let $y_0 \in Y$ be an element. The *constant map (at $y_0 \in Y$)* is the map

    $$\mathsf{const}_{y_0} : X \longrightarrow Y \ , \qquad x \mapsto y_0 \ .$$

    This map is continuous: ... take δ to be any positive number ... .
  - **Idenity maps.** The identity map

    $$\mathsf{id}_X : X \xrightarrow{=} X \ , \qquad x \mapsto x \ ,$$

    is continuous: ... take $\delta = \varepsilon$ ... .
  - **Inclusions.** Suppose $n = k$ and $X \subset Y$. The inclusion

    $$\mathsf{inc} : X \hookrightarrow Y \ , \qquad x \mapsto x \ ,$$

    is continuous.
  - **Projections.** For each $1 \leq i \leq n$, the projection map

    $$\mathsf{pr}_i : \mathbb{R}^n \longrightarrow \mathbb{R} \ , \qquad (x_1, \ldots, x_n) \mapsto x_i \ ,$$

    is continuous: ... take $\delta = \varepsilon$ ... .
  - **Compositions.** Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two maps. If f and g are continuous then their composition $g \circ f$ is continuous.
    In class we drew pictures to convince ourselves of this. I encourage you to think of this statement in terms of such pictures. Here is a more rigorous argument.
    Indeed, let $x \in X$ and let $\varepsilon > 0$. Because g is continuous, there is a $\gamma > 0$ for which $\mathsf{dist}_Y(y', f(x)) < \gamma$ implies $\mathsf{dist}_Z(g(y'), g \circ f(x)) < \varepsilon$. Because f is continuous, there is a $\delta > 0$ for which $\mathsf{dist}_X(x', x) < \delta$ implies $\mathsf{dist}_Y(f(x'), f(x)) < \gamma$, which, byrom the choice of γ, implies $\mathsf{dist}_Z(g \circ f(x'), g \circ f(x)) < \varepsilon$.
  - **Products.** Let $Z \subset \mathbb{R}^\ell$ be another subset. Let

    $$F : X \longrightarrow Y \times Z \ , \qquad x \mapsto F(x) \ ,$$

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be a map between sets. This map F is continuous if and only if each of the projections

$$\mathrm{pr}_Y \circ F\colon X \xrightarrow{F} Y \times Z \xrightarrow{\mathrm{pr}_Y} Y \qquad \text{and} \qquad \mathrm{pr}_Z \circ F\colon X \xrightarrow{F} Y \times Z \xrightarrow{pr_Z} Z$$

is continuous.
(Said another way, a map $F\colon X \to Y \times Z$ is nothing other than a pair of maps $F_Y\colon X \to Y$ and $F_Z\colon X \to Z$:

$$F(x) = \big(F_Y(x), F_Z(y)\big) \ .$$

As so, this condition says that F is continuous if and only if F_Y and F_Z are both continuous.)
Indeed, suppose F_Y and F_Z are both continuous. Let's show F is continuous. Let $x \in X$ and let $\varepsilon > 0$. Because F_Y is continuous, there is a $\delta_Y > 0$ for which $\mathsf{dist}_X(x', x) < \delta_Y$ implies $\mathsf{dist}_Y(F_Y(x'), F_Y(x)) < \varepsilon/\sqrt{2}$. Because F_Z is continuous, there is a $\delta_Z > 0$ for which $\mathsf{dist}_X(x', x) < \delta_Z$ implies $\mathsf{dist}_Z(F_Z(x'), F_Z(x)) < \varepsilon/\sqrt{2}$. Take $\delta = \mathsf{Min}\{\delta_Y, \delta_Z\}$. Then $\mathsf{dist}_X(x', x) < \delta$ implies

$$\mathsf{dist}_{Y\times Z}(F(x'), F(x)) = \sqrt{\mathsf{dist}_Y(F_Y(x'), F_Y(x))^2 + \mathsf{dist}_Z(F_Z(x'), F_Z(x))^2} < \sqrt{(\varepsilon/\sqrt{2})^2 + (\varepsilon/\sqrt{2})^2} = \varepsilon \ .$$

Conversely, if F is continuous, then both of the composite maps $F_Y = \mathrm{pr}_Y \circ F$ and $F_Z = \mathrm{pr}_Z \circ F$ are continuous.

– The **addition map**

$$\mathsf{add}\colon \mathbb{R}^2 \longrightarrow \mathbb{R} \ , \qquad (x,y) \mapsto x + y \ ,$$

is continuous. Indeed, let $(x,y) \in \mathbb{R}^2$. Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{2}$. We now check: Let $(x', y') \in \mathbb{R}^2$ with $\mathsf{dist}_{\mathbb{R}^2}\big((x',y'),(x,y)\big) < \delta$. Then

$$\begin{aligned} \mathsf{dist}_{\mathbb{R}}\big(\mathsf{add}(x',y'), \mathsf{add}(x,y)\big) \ &= \ |(x'+y') - (x+y)| \\ = \ |(x'-x) + (y'-y)| \ &\underset{\text{triag ineq}}{\leq} \ |x'-x| + |y'-y| \end{aligned}$$

$$\leq \ \sqrt{(x'-x)^2 + (y'-y)^2} + \sqrt{(x'-x)^2 + (y'-y)^2} \ < \ \delta \ + \ \delta = \varepsilon \ .$$

– The **multiplication map**

$$\mathsf{mult}\colon \mathbb{R}^2 \longrightarrow \mathbb{R} \ , \qquad (x,y) \mapsto xy \ ,$$

is continuous. Indeed, let $(x,y) \in \mathbb{R}^2$. Let $\varepsilon > 0$. Let $R \geq \mathsf{Max}\big\{|x|, |y|, 1\big\}$. Take $\delta = \mathsf{Min}\big\{\frac{\varepsilon}{2R}, R\big\}$. Indeed, let $(x',y') \in \mathbb{R}^2$ with $\mathsf{dist}_{\mathbb{R}^2}\big((x',y'),(x,y)\big) < \delta$. Then

$$\begin{aligned} \mathsf{dist}_{\mathbb{R}}\big(\mathsf{mult}(x',y'), \mathsf{mult}(x,y)\big) \ &= \ |(x'y') - (xy)| \\ = \ |x'y' - xy' + xy' - xy| \ &= \ |(x'-x)y' - x(y'-y)| \\ &\underset{\text{triag ineq}}{\leq} \ |x'-x||y'| + |x'||y'-y| \end{aligned}$$

$$\leq \ |y'|\sqrt{(x'-x)^2 + (y'-y)^2} + |x|\sqrt{(x'-x)^2 + (y'-y)^2} \ < \ |y'|\delta \ + \ |x|\delta$$

$$\leq \ (|y| + \delta)\delta + |x|\delta \ \leq \ (R+R)\delta + R\delta \ \leq \ 3R\delta \ = \ \varepsilon \ .$$

- So the *diagonal* map

$$\mathsf{diag}\colon X \longrightarrow X \times X \ , \qquad x \mapsto (x,x) \ ,$$

is continuous because it is coordinate-wise the identity map.
- So the *squaring* map

$$\mathsf{sq}\colon \mathbb{R} \longrightarrow \mathbb{R} \ , \qquad x \mapsto x^2 \ ,$$

is continuous because it is a composition:

$$\mathsf{sq}\colon \mathbb{R} \xrightarrow{\mathsf{diag}} \mathbb{R} \times \mathbb{R} \xrightarrow{\mathsf{mult}} \mathbb{R} \ .$$

- So the map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \ , \qquad (x,y) \mapsto (xy, y+2) \ ,$$

is continuous because the first coordinate-map is mult and the secont coordimate map is a composition

$$\mathbb{R}^2 \xrightarrow{\mathsf{pr}_2\colon (x,y)\mapsto y} \mathbb{R} \xrightarrow{(\mathsf{id}_Y, \mathsf{const}_2)\colon y \mapsto (y,2)} \mathbb{R}^2 \xrightarrow{\mathsf{add}} \mathbb{R} \ .$$

- We gave a pretty thorough sketch for the following assertion.

Theorem 0.1 (Polynomial maps are continuous)**.** *Let $p_1, \dots, p_k$ be k polynomials each in n variables. Consider the map*

$$P\colon \mathbb{R}^n \longrightarrow \mathbb{R}^k \ , \qquad (x_1,\dots,x_n) \mapsto \big(p_1(x_1,\dots,x_n), p_2(x_1,\dots,x_n), \dots, p_k(x_1,\dots,x_n)\big) \ .$$

The map P is continuous.

The above theorem is stated pretty generally. I strongly encourage you to think of specific instances of it (such as those of the above points). In doing so, you might appreciate how broad the class of continuous maps you now know!

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Theorem 0.2 (Power series maps are continuous, where they converge)**.** *Let $a_0, a_1, \dots$ be a sequence of real numbers. Consider the power series,*

$$a_0 + a_1 x + a_2 x^2 + \dots \ .$$

Let $0 \leq R \leq \infty$ be the radius of convergence of this power series. The map

$$(-R,R) \longrightarrow \mathbb{R} \ , \qquad x \mapsto a_0 + a_1 x + a_2 x^2 + \dots \ ,$$

is continuous.

As examples, we see that $x \mapsto \cos(x)$ and $x \mapsto \sin(x)$ and $x \mapsto e^x$ are each continuous maps from $\mathbb{R}$ to $\mathbb{R}$. So the class of maps that you know are continuous is pretty large now! Even so, reflect on some natural maps that are not manifestly described in terms of any particular formula (ray-tracing, etc); it's not obvious that such maps are continuous.

- We next moved on to path-connectedness, and path-components. As we'll see, path-connectedness is a topological property; and the set of path-components is a topological invariant.

- We initiated this by considering if $\mathbb{R} \setminus \{0\}$ is path-connected, and if $\mathbb{R}^2 \setminus \{0\}$ is path-connected. We noted that *showing* a space is path-connected is not obvious without direct pictures, and is, in principle, requires constructing uncountable data; likewise, *showing* a space is *not* path-connecte also seems to require some tools. So we embark on developing some formalism around path-components so we can more easily identify the set of path-components of various spaces.
- **Definition 0.3.** Let X be a set.
  - An *equivalence relation (on X)* is a subset $E \subset X \times X$ of the product satisfying the following three conditions. For convenience, let's denote $x \sim x'$ in place of $(x,x') \in E$; and $\sim$ in place of E.
    * $x \sim x$ for all $x \in X$.
    * $x \sim x'$ implies $x' \sim x$.
    * $x \sim x'$ and $x' \sim x''$ implies $x \sim x''$.

    Let $\sim$ be an equivalence class on X. An *equivalence class (with respect to $\sim$)* is a non-empty subset
    $$\emptyset \neq S \ \subset \ X$$
    for which, for $x \sim x'$, then $x \in S$ if and only if $x' \in S$.
  - For $\sim$ an equivalence relation on X, the *quotient* is the set of equivalence classes:
    $$X_{/\sim} \ := \ \Big\{ S \subset X \mid S \text{ is an equivalence class} \Big\} .$$
    Note the canonical map between sets
    $$X \longrightarrow \pi_0(X) , \qquad x \mapsto \Big\{ x' \in X \mid x \sim x' \Big\} \ =: \ [x] ,$$
    which assigns to x the equivalence class containing x, which is denoted as $[x]$.
- **Definition 0.4** (Path-components)**.** Let $X \subset \mathbb{R}^n$.
  (1) Let $x, x' \in X$. Declare $x \underset{\text{path}}{\sim} x'$ to mean there is a continuous map $\gamma \colon [0,1] \to X$ for which $\gamma(0) = x$ and $\gamma(1) = x'$.
  (2) A *path-component (of X)* is a subset
  $$S \ \subset \ X$$
  satisfying the following three properties:
    - $S \neq \emptyset$.
    - Let $x, x' \in X$ for which $x \underset{\text{path}}{\sim} x'$. Then $x \in S$ if and only if $x' \in S$.

  (3) The *set of path-components (of X)* is the quotient set
  $$\pi_0(X) \ := \ X_{/\underset{\text{path}}{\sim}} .$$
- We verified that $\underset{\text{path}}{\sim}$ is indeed an equivalence relation on X. The hardest part is transitivity (the third property of an equivalence relation). For that, we used the following result, which we proved in class.

Lemma 0.5. *Let $a < c < b$ be real numbers. Let $\alpha\colon [a,c] \to X$ be a map; let $\beta\colon [c,b] \to X$ be a map. Suppose $\alpha(c) = \beta(c)$. Consider the map*

$$\alpha * \beta\colon [a,b] \longrightarrow \mathbb{R}\ , t \mapsto \alpha(t)\ (\ \mathit{if}\ t \in [a,c])\ , \qquad t \mapsto \beta(t)\ (\ \mathit{if}\ t \in [c,b])\ .$$

If α and β are both continuous, then $\alpha \star \beta$ is continuous.

Reading

By Wednesday 18 September. §3. Be especially familiar with these aspects.

- The definition of a *homeomorphism*, and the definition of *homeomorphic*.
- The notion of a topological property.
- The definition of path-connected, togetherness, and the set of path-components.
- The definition of cut-points.

Exercises

These are due by **5pm on Friday 20 September**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) Consider the map

$$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2\ , \qquad (x,y,z) \mapsto \big(3xy^2 + 1, 2z^2 + x\big)\ .$$

Deduce that f is continuous by writing it a composition of maps in the **bolded** classes of maps in the first bullet point above.

(2) Let

$$X\ :=\ \Big\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } y < 1\Big\}\ \subset\ \mathbb{R}^2\ .$$

Let

$$Z\ :=\ \Big\{(x,y) \in \mathbb{R}^2 \mid y = 0\Big\}\ .$$

Consider the map

$$f\colon X \longrightarrow Z$$

whose value on (x,y) is the unique $(x',0) \in Z$ for which the triple of points in $\mathbb{R}^2$

$$\{\ (0,1)\ ,\ (x,y)\ ,\ (x',0)\ \}$$

are colinear. Prove that there is indeed a unique such value $f(x,y)$ for each $(x,y) \in X$; prove that f is continuous.
Hint: try 'building' f out of maps you know are continuous (such as multiplication, division, addition, etcetera).

(3) (I adjusted the numerology.) For $n \geq 0$, denote

$$\mathbb{S}^n\ :=\ \Big\{(x_0, x_1, \dots, x_n) \in \mathbb{R}^n \mid x_0^2 + x_1^2 + \cdots + x_n^2 = 1\Big\}\ \subset\ \mathbb{R}^{n+1}\ .$$

Prove that $\mathbb{S}^n$ is connected if and only if $n > 0$.
(Hint: For $p, q \in \mathbb{S}^n$ orthogonal (i.e., their dot product $p \cdot q = 0$), consider the great circle

$$\mathbb{R} \longrightarrow \mathbb{S}^n\ , \qquad t \mapsto \cos(t)p + \sin(t)q\ .$$

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(4) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$. Prove that $X \times Y$ is path-connected if and only if both X and Y are path-connected.
Deduce from this and the previous problem that the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is path-connected.