WEEK 13

0TH COHOMOLOGY OF GRAPHS

Let $\Gamma = (V,E,s,t)$ be an abstract directed graph. The cohomology of its geometric realization $|\Gamma|$ can be identified relatively easily purely in terms of (V,E,s,t). Let's assume V and E are finite.

Lemma 0.1. *There is an isomorphism between vector spaces*

$$\begin{aligned} \mathsf{H}^0\big(|\Gamma|\big) \;&\cong\; \mathsf{Ker}\Big(\mathbb{R}^V \xrightarrow{t^*-s^*} \mathbb{R}^E\Big) \\ &= \Big\{(V \xrightarrow{\omega} \mathbb{R}) \mid \forall a \in E,\ \omega(t(a)) - \omega(s(a)) = 0\Big\} \\ &= \Big\{(V \xrightarrow{\omega} \mathbb{R}) \mid \forall a \in E,\ \omega(t(a)) = \omega(s(a))\Big\}\ . \end{aligned}$$

Proof. We proceed by induction on the cardinality of E. For the base case, suppose $E = \emptyset$ has cardinality zero. Then the topological space $|\Gamma|$ is V, with its discrete topology. So there is a canonical bijection between sets $V \cong \pi_0\big(|\Gamma|\big)$. So there is an isomorphism between vector spaces

$$\mathsf{H}^0(|\Gamma|) \;\cong\; \mathbb{R}^{\pi_0(|\Gamma|)} \;\cong\; \mathbb{R}^V \;\cong\; \mathsf{Ker}\Big(\mathbb{R}^V \xrightarrow{!} 0\Big)\ ,$$

where 0 is the zero vector space, and ! is the unique linear map to it (ie, it's the constant map at 0). Since $\mathbb{R}^E = \mathbb{R}^{\emptyset} = 0$, this establishes the statement of the Lemma in this case that $E = \emptyset$.

Now assume the cardinality $|E| > 0$ is positive. Let $a \in E$ be an element (such an element exists, by assumption). Denote $E' := E \smallsetminus \{a\}$. Denote the restrictions of the maps s and t:

$$s' \colon E' \xrightarrow{s_{|E'}} V \qquad \text{and} \qquad t' \colon E' \xrightarrow{t_{|E'}} V\ .$$

Consider the graph $\Gamma' := (V,E',s',t')$; it is an abstract directed graph with the same set V of vertices as Γ, but with one fewer edge. By definition of the geometric realization, there is a canonical surjective continuous map

$$q\colon |\Gamma'| \sqcup \{a\} \times [0,1] \longrightarrow |\Gamma|\ .$$

Take

$$A := q\Big(|\Gamma'| \sqcup \{a\} \times \big([0,\tfrac{1}{2}) \sqcup (\tfrac{1}{2},1]\big)\Big) \subset |\Gamma| \qquad \text{and} \qquad B := q\big(\{a\} \times (0,1)\big) \subset |\Gamma|\ .$$

In other words, the subset $A \subset |\Gamma|$ is the result of removing the center of the edge of $|\Gamma|$ corresponding to a; the subset $B \subset |\Gamma|$ is the interior of that edge. So $A \cup B = |\Gamma|$, and $A, B \in \mathcal{T}_{|\Gamma|}$ are in the topology of the geometric realization (ie, both A and B are "open"). So the Mayer-Vietoris theorem applies, and gives an identification

$$\mathsf{H}^0\big(|\Gamma|\big) \;\cong\; \mathsf{Ker}\Big(\mathsf{H}^0(A) \oplus \mathsf{H}^0(B) \xrightarrow{\Psi} \mathsf{H}^0(A \cap B)\Big)\ . \tag{1}$$

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We thusly seek to identify each term in this expression.

Well, by definition of A, there is a continuous map
$$|\Gamma'| \longrightarrow A \ .$$
This continuous map induced a map between sets
$$\pi_0\Big(|\Gamma'|\Big) \longrightarrow \pi_0(A) \ .$$
By inspection (ie, just thinking about it), this map is surjective, and this map is injective. Consequently, the induced linear map
$$\mathsf{H}^0\big(|\Gamma'|\big) \cong \mathbb{R}^{\pi_0\big(|\Gamma'|\big)} \longrightarrow \mathbb{R}^{\pi_0(A)} \cong \mathsf{H}^0(A)$$
is an isomorphism between vector spaces.

Next, by definition of the topology on $|\Gamma|$, the map q restricts as a homeomorphism
$$q_|\colon (0,1) \xrightarrow{\cong} B \ .$$
Since $\pi_0\big((0,1)\big) \cong \{a\}$ is a singleton, then
$$\mathsf{H}^0(B) \ \cong \ \mathbb{R}^{\{a\}} = \mathsf{Span}\{e_a\}$$
is 1-dimensional. We're denoting a generator for this vector space as e_a, which is the locally constant real-valued map $(0,1) \to \mathbb{R}$ that is constant at 1.

Similarly, by definition of the topology on $|\Gamma|$, the map q restricts as a homeomorphism
$$q_|\colon (0,\frac{1}{2}) \sqcup (\frac{1}{2},1) \xrightarrow{\cong} A \cap B \ .$$
Since $\pi_0\big((0,\frac{1}{2}) \sqcup (\frac{1}{2},1)\big) \cong \{a_-,a_+\}$ has exactly two elements in it, then
$$\mathsf{H}^0(A\cap B) \ \cong \ \mathbb{R}^{\{a_-,a_+\}} = \mathsf{Span}\{e_{a_-},e_{a_+}\}$$
is 2-dimensional. Here, we're denoting the generators for this vector space as e_{a_-} and e_{a_+}: the first which is the locally constant real-valued map $(0,\frac{1}{2}) \sqcup (\frac{1}{2},1) \to \mathbb{R}$ that is constant at 1 on $(0,\frac{1}{2})$ and constant at 0 on $(\frac{1}{2},1)$; the second which is the locally constant real-valued map $(0,\frac{1}{2}) \sqcup (\frac{1}{2},1) \to \mathbb{R}$ that is constant at 0 on $(0,\frac{1}{2})$ and constant at 1 on $(\frac{1}{2},1)$.

By induction, there is an isomorphism between vector spaces
$$\mathsf{H}^0\big(|\Gamma'|\big) \ \cong \ \mathsf{Ker}\Big(\mathbb{R}^V \xrightarrow{(t')^*-(s')^*} \mathbb{R}^{E'}\Big) \ .$$
So the expression (1) is identified as
$$\mathsf{Ker}\Big(\mathsf{Ker}\big(\mathbb{R}^V \xrightarrow{(t')^*-(s')^*} \mathbb{R}^{E'}\big) \oplus \mathbb{R}^{\{a\}} \xrightarrow{\Psi} \mathbb{R}^{\{a_-,a_+\}}\Big) \ .$$
We thusly seek to identify the linear map Ψ in this expression. Well, let $(V \xrightarrow{\omega} \mathbb{R}) \in \mathsf{Ker}\big(\mathbb{R}^V \xrightarrow{(t')^*-(s')^*} \mathbb{R}^{E'}\big)$ and let $(\{a\} \xrightarrow{\beta} \mathbb{R}) \in \mathbb{R}^{\{a\}}$. The definition of Ψ, and of the

isomorphisms above, is such that

$$\Psi(\omega,\beta) \;:=\; \big(\omega(s(a))e_{a_-}+\omega(t(a))e_{a_+}\big) \;-\; \big(\beta(a)e_{a_-}+\beta(a)\big)e_{a_+}$$

$$=\; \big(\omega(s(a))-\beta(a)\big)e_{a_-} \;+\; \big(\omega(t(a))-\beta(a)\big)e_{a_+}\;.$$

This element in $\mathbb{R}^{\{a_-,a_+\}} = \mathsf{Span}\{e_{a_-},e_{a_+}\}$ is zero if and only if each of the coefficients in this linear combination of e_{a_-} and e_{a_+} is zero. So

$$\mathsf{Ker}(\Psi) \;=\; \Big\{(\omega,\beta)\in\mathsf{Ker}\big((t')^*-(s')^*\big)\oplus\mathbb{R}^{\{a\}} \mid \omega(s(a))=\beta(a) \text{ and } \omega(t(a))=\beta(a)\Big\}$$

$$=\; \Big\{(\omega,\beta)\in\mathsf{Ker}\big((t')^*-(s')^*\big)\oplus\mathbb{R}^{\{a\}} \mid \omega(s(a))=\beta(a)=\omega(t(a))\Big\}$$

(... so β is determined by ω, and thus ...)

$$\xrightarrow[\cong]{(\omega,\beta)\mapsto\omega}\; \Big\{\omega\in\mathsf{Ker}\big((t')^*-(s')^*\big) \mid \omega(s(a))=\omega(t(a))\Big\}$$

$$=\; \Big\{\omega\in\mathsf{Ker}((t')^*-(s')^*) \mid \omega(t(a))-\omega(s(a))=0\Big\}$$

$$=\; \mathsf{Ker}(t^*-s^*)\;.$$

This is what we wanted to establish.

□

Remark 0.2. So, given an abstract directed graph $\Gamma=(V,E,s,t)$, in which V and E are finite, its 0th cohomology can be computed as the null space of a $n\times m$ matrix, where $m=|V|$ and $n=|E|$ are the cardinalities! Namely, choosing an enumeration $V=\{v_1,\dots,v_m\}$ and $E=\{a_1,\dots,a_n\}$ determines a basis $\mathcal{B}:=\{e_{v_1},\dots,e_{v_m}\}$ for $\mathbb{R}^V$ and a basis $\mathcal{C}:=\{e_{a_1},\dots,e_{a_n}\}$ for $\mathbb{R}^E$, and thusly an $n\times m$ matrix $A:=[t^*-s^*]^{\mathcal{C}}_{\mathcal{B}}$; the above result thusly gives an isomorphism

$$\mathsf{H}^0\big(|\Gamma|\big) \;\cong\; \mathsf{null}(A)\;.$$

In particular,

$$\big|\pi_0\big(|\Gamma|\big)\big| \;=\; \mathsf{nullity}(A)\;!$$

Nullity of matrices is something a computer can easily compute. So, in this way, inputting Γ into a computer quickly outputs how many path components its geometric realization has!

Example 0.3. Take Γ to be the following abstract directed graph. The set of vertices is the set of websites. An edge from a website to another is a link from it to the other. The question of whether or not $|\Gamma|$ is path-connected is the question of weather or not it's possible to connect any two websites by clicking on links (or the reverse).

Example 0.4. Take Γ to be the following abstract directed graph. The set of vertices is the set of arrangements of a Rubiks cube. There is an edge from one arrangement to another if a simple motion of the Rubik's cube relates them. Suppose you drop a Rubik's cube, and put it back together. Now you wonder if the result can be solved. This is a question if the arrangement after dropping is in the same path-component of $|\Gamma|$ as the solved cube.

Example 0.5. We seek to distinguish between male and female marmots by listening to their chirp. This is not something we can easily assess with our human ear. So we record 500 marmots' chirp, and hope to use a computer to distinguish the chips for us. But, think about it: computers can only do what we tell them to do. So how do we instruct a computer to distinguish these audio-recordings? Well, we might hope to see a simple clustering of all the male marmot recordings from all the female marmot recordings. But what, exactly, is being 'clustered'? This is where topology comes in.

Each audio recording is, say, 10 seconds long. We compress each audio-file as follows. We record the frequency and amplitude of the audio for each $\frac{1}{100}$ of a second throughout the 10 second recording. So each file is recorded as a $10 \times 100 = 1000$ pairs of real numbers (frequency and amplitude, at each time sample). In other words, each audio file is a point in

$$\begin{aligned} &\mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \qquad 1000 \text{ times} \\ &= (\mathbb{R}^2)^{1000} = \mathbb{R}^{2000} . \end{aligned}$$

Since we took 500 audio recordings, we result in a subset (with cardinality 500)

$$S \subset \mathbb{R}^{2000}$$

of 2000-dimensional Euclidean space. Now, we hope that S is roughly clustered in two parts. This feels like saying that S has two path-components, doesn't it? Literally, this is not the case: S is a finite subset of $\mathbb{R}^{2000}$, so it has one path-component for each element in S, of which there are 500. But surely we didn't mean this literally; really, we intend to 'thicken' each point in S to, say, a ball. Then consider the union of these balls. Then identify the path-components of this union. This is a nice idea, but confronts two issues. First, by how much do we thicken? Second, what computational tools allow us to identify the path-components of the resulting union of thickened points? The answer to the latter is 'cohomology'. The answer to the former is more subtle; consider all thickening-amounts, and pay attention to those intervals of thickening-amounts that result in no change to the cardinality of path-components.

So, for each $\varepsilon > 0$, consider a graph $\Gamma_\varepsilon := (S, E_\varepsilon, s_\varepsilon, t_\varepsilon)$ in which the set of vertices is $V = S$ and there is an edge connecting v to v' precisely if v and v' are within ε of one another. Then

$$|\pi_0(|\Gamma_\varepsilon|)| \ = \ \mathsf{dim}(\mathsf{H}^0(|\Gamma_\varepsilon|)) \ = \ \mathsf{nullity}\Big(\mathbb{R}^S \xrightarrow{t_\varepsilon^* - s_\varepsilon^*} \mathbb{R}^{E_\varepsilon}\Big) .$$

Note, the matrix associated to the linear map $t_\varepsilon^* - s_\varepsilon^*$ is very simple: each entry is either ± 1 or 0; and, in practice, the vast majority of the entries are 0.

So we have a function

$$\mathbb{R}_{>0} \longrightarrow \mathbb{N} , \qquad \varepsilon \mapsto \mathsf{nullity}\Big(\mathbb{R}^S \xrightarrow{t_\varepsilon^* - s_\varepsilon^*} \mathbb{R}^{E_\varepsilon}\Big) .$$

Clearly, for ε sufficiently small, the value of this function is $|S|$, the cardinality of S. Clearly, for ε sufficiently large, the value of this function is 1. Also, clearly, this function is non-increasing. So we might hope that there is a large interval $[s,t] \subset \mathbb{R}_{>0}$ on which this function is constant, but not 1. The value of this function on the longest such interval is what we'll interpret to be the "number of clusters of S".

Back to our original problem, hopefully there are 2 clusters. If there are more, then perhaps we've discovered some dialects, or other features, among our marmots. In any case, once the number of clusters are identified, we might inspect the specific features (such as sex) of representatives of each cluster, in order to understand marmots' chips better.

COHOMOLOGY

(Note, this section is essentially repeated from the last set of notes.)

We've seen that 0th cohomology, $\mathsf{H}^0(-)$, is an assignment of a vector space to each topological space. The Mayer-Vietoris Theorem above reveals a sense in which $\mathsf{H}^0(-)$ satisfies a sort of inclusion-exclusion principle. Inspecting the statement of that Theorem, one might ask if there is a vector space V together with a linear map $\partial\colon \mathsf{H}^0(A\cap B)\to V$ for which the resulting sequence

$$0 \xrightarrow{\zeta} \mathsf{H}^0(X) \xrightarrow{\Phi} \mathsf{H}^0(A)\oplus\mathsf{H}^0(B) \xrightarrow{\Psi} \mathsf{H}^0(A\cap B) \xrightarrow{\partial} V$$

is exact. In fact, why stop there; let's ask if this sequence can continue as an exact sequence of linear maps among vector spaces:

$$0 \xrightarrow{\zeta} \mathsf{H}^0(X) \xrightarrow{\Phi} \mathsf{H}^0(A)\oplus\mathsf{H}^0(B) \xrightarrow{\Psi} \mathsf{H}^0(A\cap B) \xrightarrow{\partial} V \to U \to U' \to U'' \to \dots .$$

In fact, it does: for each $k\geq 0$, *kth cohomology* is an assignment $Z\mapsto \mathsf{H}^k(Z)$ of a vector space to a topological space. This assignment respects continuous maps in the sense that, a continuous map $Z\to Z'$ determines a linear map $\mathsf{H}^k(Z')\to\mathsf{H}^k(Z)$. [1] Furthermore, as in the statement of the Mayer-Vietoris Theorem, there is an **exact sequence**

$$\begin{aligned} 0 \xrightarrow{\partial^0_{-1}} \mathsf{H}^0(X) \xrightarrow{\Phi^0} \mathsf{H}^0(A)\oplus\mathsf{H}^0(B) \xrightarrow{\Psi^0} \mathsf{H}^0(A\cap B) \qquad {}^{2} \\ \xrightarrow{\partial^1_0} \mathsf{H}^1(X) \xrightarrow{\Phi^1} \mathsf{H}^1(A)\oplus\mathsf{H}^1(B) \xrightarrow{\Psi^1} \mathsf{H}^1(A\cap B) \\ \xrightarrow{\partial^2_1} \mathsf{H}^2(X) \xrightarrow{\Phi^2} \mathsf{H}^2(A)\oplus\mathsf{H}^2(B) \xrightarrow{\Psi^2} \mathsf{H}^2(A\cap B) \\ \xrightarrow{\partial^3_2} \cdots \end{aligned}$$

in which each instance of the maps Φ and Ψ is given in the same way as it is concerning 0th cohomology [3], and the linear maps ∂ are more mysterious.

Definition 0.6. For X a topological space, its *Euler characteristic* is the alternating sum of dimensions of its cohomology:

$$\begin{aligned} \chi(X) \ &:= \ \dim(\mathsf{H}^0(X)) - \dim(\mathsf{H}^1(X)) + \dim(\mathsf{H}^2(X)) - \dots \\ &= \ \sum_{k\geq 0}(-1)^k\dim(\mathsf{H}^k(X)) \ . \end{aligned}$$

A consequence of the long exact sequence on cohomology that characterizes $\mathsf{H}^*(-)$ is that **Euler characteristic** (as opposed to $|\pi_0(-)|$), satisfies an inclusion-exclusion principle:

$$\chi(X) \ = \ \chi(A)+\chi(B)-\chi(A\cap B) \ !$$

[1] In the case that $Z\to Z'$ is the inclusion of a subspace, we'll denote the induced linear map as

$$\mathsf{H}^k(Z')\longrightarrow\mathsf{H}^k(Z) , \qquad \omega\mapsto\omega_{|Z} ,$$

suggesting "restriction of ω to $Z\subset Z'$".

[2] Here, $\Phi^0 := \Phi$ and $\Psi^0 := \Psi$.

[3] Namely, $\Phi^k(\omega) := (\omega_{|A},\omega_{|B})$ and $\Psi^k(\alpha,\beta) := \alpha_{|A\cap B} - \beta_{|A\cap B}$.

So, you can imagine that to compute $\mathsf{H}^k(X)$ is an inductive process involving linear algebra. Here are some cool calculations.

- $\mathsf{H}^k(\mathbb{S}^n)$ is $\mathbb{R}$ if $k = 0, n$ and otherwise it is the zero-vector space.
- $\mathsf{H}^k(X \times Y) \cong \mathsf{H}^k(X) \otimes \mathsf{H}^k(Y)$.
  So $\mathsf{H}^1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{R}^2$ and $\mathsf{H}^2(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{R}$ and $\mathsf{H}^k(\mathbb{S}^1 \times \mathbb{S}^1) = \{0\}$ for $k > 2$.
- $\mathsf{H}^2(\mathbb{RP}^2) = \{0\}$.
- $\mathsf{H}^1(P_5) \cong \mathbb{R}^8$; $\mathsf{H}^2(P_5) \cong \mathbb{R}$; $\mathsf{H}^k(P_5) = \{0\}$ for $k > 2$.
- $\mathsf{H}^k(P_n) = \{0\}$ for $k > n-3$. $\mathsf{H}^k(P_n)$ is not known $0 < k \leq n-3$ for $n > 6$.
- For $X \subset \mathbb{R}^n$ a subspace of n-dimensional Euclidean space
  $$\mathsf{H}^k(X) \ = \ \{0\}$$
  for $k \geq n$.
- Let X be a topological space. Suppose there are finitely many elements $A, B \subset X$ for which $A, B \in \mathcal{T}_X$ and $A \cup B = X$. Suppose there is a number $n \geq 0$ for which $\mathsf{H}^k(A) = \mathsf{H}^k(B) = \mathsf{H}^k(A \cap B) = \{0\}$ for all $k \geq n$. Then $\mathsf{H}^k(X) = \{0\}$ for all $k > n$, and $\mathsf{H}^n(X) = \mathsf{coKer}\big(\partial^n_{n-1}\big)$.