WEEK 2

CLASS

We considered several examples of subsets of Euclidean spaces. We pondered which, among these, were homeomorphic with one another. In doing so, we reflected on *topological properties*: cardinality, connectedness/connected components, compactness (i.e., closedness and boundedness), etc. Properties that are not topological include 'angles' and 'boundedness'. We then defined the condition of *continuity* on a map between subsets of Euclidean spaces. We defined what a *homeomorphism* is. Through some in-class exercises, we considered examples of maps, and subsets of Euclidean spaces, and confronted the difficulty of how to show each is *continuous*, and *homeomorphic* with another. This consideration prompts the upcoming developments: how to find a host of examples of continuous maps, and of homeomorphic subsets of Euclidean spaces.

Here are some important notions we covered in class:

- The definition of *continuity* (from Week 1 notes). Importantly, a *continuous* map $f\colon X \to Y$ between subsets of Euclidean spaces is a mapping from the underlying set of X to the underlying set of Y that satisfies a *condition*. So to name a continuous map is to name a map between sets, then check this condition.
- I could have remarked that it is perhaps more fruitful, for both understanding the definition of *continuiuty* as well as for beliving a given map $f\colon X \to Y$ is or is not *continuous*, to isolate the *contrapositive* of the condition of continuity.

  Remark 0.1. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be subsets of Euclidean spaces. Let $f\colon X \to Y$ be a map between sets. The map f is *not* continuous if there is an element $x \in X$ and a positive number $\varepsilon > 0$ for which, for any positive number $\delta > 0$, there is an element $x' \in X$ with $\mathsf{dist}_{\mathbb{R}^n}(x', x) < \delta$ yet $\mathsf{dist}_{\mathbb{R}^k}(f(x'), f(x)) \geq \varepsilon$.

  Indeed, think about why the map
  $$\mathbb{R} \longrightarrow \mathbb{R} ,$$
  given by $f(x) := 0$ if $x \leq 0$ and $f(x) = 1$ if $x > 0$, is not continuous.
- We showed that the following classes of maps are continuous. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be subsets. Let $f\colon X \to Y$ be a map between sets.
  - **Constant maps.** Let $y_0 \in Y$ be an element. The *constant map (at $y_0 \in Y$)* is the map
    $$\mathsf{const}_{y_0}\colon X \longrightarrow Y , \qquad x \mapsto y_0 .$$
    This map is continuous: ... take δ to be any positive number ... .

Date: 5 September 2019.

– **Projections.** For each $1 \leq i \leq n$, the projection map

$$\mathsf{pr}_i \colon \mathbb{R}^n \longrightarrow \mathbb{R} , \qquad (x_1, \ldots, x_n) \mapsto x_i ,$$

is continuous: ... take $\delta = \varepsilon$ ... .

– **Products.** Let $Z \subset \mathbb{R}^\ell$ be another subset. Let

$$F \colon X \longrightarrow Y \times Z , \qquad x \mapsto F(x) ,$$

be a map between sets. This map F is continuous if and only if each of the projections

$$\mathrm{pr}_Y \circ F \colon X \xrightarrow{F} Y \times Z \xrightarrow{\mathrm{pr}_Y} Y \qquad \text{and} \qquad \mathrm{pr}_Z \circ F \colon X \xrightarrow{F} Y \times Z \xrightarrow{pr_Z} Z$$

is continuous.

– **Inclusions.** Suppose $n = k$ and $X \subset Y$. The inclusion

$$\mathsf{inc} \colon X \hookrightarrow Y , \qquad x \mapsto x ,$$

is continuous.

– The addition map

$$\mathsf{add} \colon \mathbb{R}^2 \longrightarrow \mathbb{R} , \qquad (x,y) \mapsto x + y ,$$

is continuous. Indeed, let $(x,y) \in \mathbb{R}^2$. Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{2}$. We now check: Let $(x',y') \in \mathbb{R}^2$ with $\mathsf{dist}_{\mathbb{R}^2}\big((x',y'),(x,y)\big) < \delta$. Then

$$\mathsf{dist}_{\mathbb{R}}\big(\mathsf{add}(x',y'), \mathsf{add}(x,y)\big) \;=\; |(x'+y') - (x+y)|$$

$$=\; |(x'-x) + (y'-y)| \underset{\text{triag ineq}}{\leq} |x'-x| + |y'-y|$$

$$\leq\; \sqrt{(x'-x)^2 + (y'-y)^2} + \sqrt{(x'-x)^2 + (y'-y)^2} \;<\; \delta \;+\; \delta = \varepsilon .$$

– The multiplication map

$$\mathsf{mult} \colon \mathbb{R}^2 \longrightarrow \mathbb{R} , \qquad (x,y) \mapsto xy ,$$

is continuous. Indeed, let $(x,y) \in \mathbb{R}^2$. Let $\varepsilon > 0$. Let $R \geq \mathsf{Max}\big\{|x|,|y|,1\big\}$. Take $\delta = \mathsf{Min}\big\{\frac{\varepsilon}{2R}, R\big\}$. Indeed, let $(x',y') \in \mathbb{R}^2$ with $\mathsf{dist}_{\mathbb{R}^2}\big((x',y'),(x,y)\big) < \delta$. Then

$$\mathsf{dist}_{\mathbb{R}}\big(\mathsf{mult}(x',y'), \mathsf{mult}(x,y)\big) \;=\; |(x'y') - (xy)|$$

$$=\; |x'y' - xy' + xy' - xy| \;=\; |(x'-x)y' - x(y'-y)|$$

$$\underset{\text{triag ineq}}{\leq} |x'-x||y'| + |x'||y'-y|$$

$$\leq\; |y'|\sqrt{(x'-x)^2 + (y'-y)^2} + |x|\sqrt{(x'-x)^2 + (y'-y)^2} \;<\; |y'|\delta \;+\; |x|\delta$$

$$\leq\; (|y| + \delta)\delta + |x|\delta \;\leq\; (R+R)\delta + R\delta \;\leq\; 3R\delta \;=\; \varepsilon .$$

READING

By Wednesday 11 September. §1-2, and Appendix A (concerning continuous maps and homeomorphisms). Be especially familiar with these aspects.

- The definition of a *homeomorphism*, and the definition of *homeomorphic*.
- The notion of a topological property.
- The definition of path-connected, togetherness, and the set of path components.
- The definition of cut-points.

EXERCISES

These are due at **5:30pm on Wednesday 11 September**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) (Optional) Let Z be a countably infinite set. Prove that Z and $Z \times Z$ have the same cardinality.

(2) (Optional) Let $f\colon S \longrightarrow T$ be a bijection between sets. Prove that there is a unique map $g\colon T \longrightarrow S$ for which

$$f \circ g = \mathrm{id}_T \qquad \text{and} \qquad g \circ f = \mathrm{id}_S \ .$$

(3) Prove that the map

$$\mathbb{R} \longrightarrow \mathbb{R} \ , \qquad x \mapsto \mathsf{Min}\Big\{ y \mid y^3 - y = x \Big\} \ ,$$

is not continuous.

(4) Prove that

$$f\colon [-1,0] \cup (1,2] \longrightarrow [0,4] \ , \qquad x \mapsto x^2 \ ,$$

is continuous, yet not a homeomorphism.

(5) Let A be a $k \times n$ matrix. Prove that the map

$$f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^k \ , \qquad x \mapsto Ax \ ,$$

is continuous.

(6) Prove that the map

$$\mathbb{R} \times (\mathbb{R} \smallsetminus \{0\}) \longrightarrow \mathbb{R} \ , \qquad (x,y) \mapsto \frac{x}{y} \ ,$$

is continuous.

(7) (Optional) Prove that the above map does not extend as a continuous map

$$\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \ .$$

(In other words, no matter how one might declare the extended values on elements in $\mathbb{R} \times \mathbb{R}$ of the form $(x,0)$, the resulting map will not be continuous.)

(8) Prove that the determinant map

$$(\mathbb{R}^n)^{\times n} \longrightarrow \mathbb{R} \ , \qquad (a_1, \ldots, a_n) \mapsto \mathsf{det}\Big(\Big[a_1 | a_2 | \ldots | a_n \Big] \Big) \ ,$$

is continuous.

(You're welcome to only do this problem in the case that $n = 3$ if you'd rather.)

(9) Let $X := \left\{(x,y,z) \mid x^2+y^2+z^2=1\right\} \subset \mathbb{R}^3$ be the *2-sphere*. Prove that the map

$$X \longrightarrow \mathbb{R}^2 \ , \qquad (x,y,z) \mapsto \frac{1}{x^2+y^2+z^2}(xyz, x+y+z) \ ,$$

is continuous.

(10) (Optional) Outline a proof for why the map

$$\mathbb{R} \longrightarrow \mathbb{R} \ , \qquad x \mapsto \cos(x)$$

is continuous.