WEEK 12

LINEAR ALGEBRA

In Week11 notes, we recalled various notions in linear algebra. Here is one more such notion.

- **A basis for an n-dimensional vector space V is the same as a linear isomorphism $V \cong \mathbb{R}^n$.** Let V be a finite-dimensional vector space. A basis $\mathcal{B} = \{v_1, \dots, v_n\} \subset V$ for V determines, and is determined by, a linear isomorphism
$$L_{\mathcal{B}} \colon \mathbb{R}^n \cong V \colon [-]_{\mathcal{B}} .$$
Indeed, given a basis $\mathcal{B}$, define
$$L_{\mathcal{B}} \colon V \longrightarrow V , \qquad (c_1, c_2, \dots, c_n) \mapsto c_1 \cdot v_1 + c_2 \cdot v_2 + \cdots + c_n \cdot v_n .$$
This map is surjective exactly because $\mathsf{Span}\{v_1, v_2, \dots, v_n\} = V$. Let's see why this map is injective. This is to show that its kernel consists solely of the zero-vector. This is so exactly because $\mathcal{B}$ is linearly independent.
$$\mathsf{Ker}(L_{\mathcal{B}}) = \Big\{(c_1, \dots, c_n) \mid c_1 \cdot v_1 + \cdots + c_n \cdot v_n = 0\Big\} = \Big\{(0, \dots, 0)\Big\} ,$$
where the last equality is exactly linear independence of $\mathcal{B}$. So $L_{\mathcal{B}}$ is a linear isomorphism. The inverse to $L_{\mathcal{B}}$ is the map that assigns to $v \in V$ the unique coefficients $[v]_{\mathcal{B}} := (c_1, \dots, c_n)$ for which $v = c_1 \cdot v_1 + \cdots + c_n \cdot v_n)$ is a linear combination.

  Conversely, given a linear isomorphism $\mathbb{R}^n \xrightarrow{L} V$, the subset $\{L(e_1), \dots, L(e_n)\} \subset V$ is a basis.

  For example, consider the vector space $\mathbb{R}^{\{\pm\}}$ of maps $\{\pm\} \to \mathbb{R}$ from this set $\{\pm\}$ with two elements. Consider the two elements
$$\alpha \colon \{\pm\} \longrightarrow \mathbb{R} , \qquad \alpha(-) = 1 \ , \ \alpha(+) = -1$$
and
$$\beta \colon \{\pm\} \longrightarrow \mathbb{R} , \qquad]beta(-) = 0 \ , \ \beta(+) = 1 .$$
The subset $\{\alpha, \beta\} \subset \mathbb{R}^{\{\pm\}}$ is a basis. This basis corresponds to the linear isomorphism:
$$\mathbb{R}^2 \xrightarrow{\cong} \mathbb{R}^{\{\pm\}} , \qquad (c, d) \mapsto c\alpha + d\beta .$$
For another demonstration, the linear isomorphism
$$\mathbb{R}^{\{\pm\}} \xrightarrow{\cong} \mathbb{R}^2 , \qquad \omega \mapsto \big(\omega(-), \omega(+)\big) ,$$
corresponds to the basis $\{e_-, e_+\} \subset \mathbb{R}^{\{\pm\}}$ where
$$e_-(-) = 1 \qquad e_-(+) = 0 \qquad \text{and} \qquad e_+(-) = 0 \qquad e_+(+) = 1 .$$

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- **Linear maps between finite-dimensional vector spaces are the same as matrices, at least after choosing bases for the domain and codomain.** Let $V \xrightarrow{T} W$ be a linear map between finite-dimensional vector spaces. Let $\mathcal{B} = \{v_1, \ldots, v_m\} \subset V$ be a basis, and let $\mathcal{C} = \{w_1, \ldots, w_n\} \subset W$ be a basis. Then there is a unique $n \times m$ matrix A for which, for each $v \in V$,
$$[T(v)]_{\mathcal{C}} = A[v]_{\mathcal{B}} \ .$$
Said another way,
$$[-]_{\mathcal{C}} \circ T \ = \ A[-]_{\mathcal{B}} \ .$$[1]
Explicitly, for $1 \leq i \leq m$, the ith column of A is the vector $[T(v_i)]_{\mathcal{C}} \in \mathbb{R}^n$, which is the $\mathcal{C}$-coordinate vector of the value $T(v_i)$ of T on the ith element of the basis $\mathcal{B}$. In other words, the (j,i)-entries A^j_i of the matrix A are the real numbers defined by the expression
$$T(v_i) \ = \ A^1_i w_1 + A^2_i w_2 + \cdots + A^n_i w_n \ .$$
Notation 0.1. We often give the matrix A the notation $[T]^{\mathcal{C}}_{\mathcal{B}}$. This way the dependence of A on T as well as the bases $\mathcal{B}$ and $\mathcal{C}$ are evident. If the role of the bases $\mathcal{B}$ and $\mathcal{C}$ aren't terribly important, we might simply denote
$$A \ =: \ [T]^{\mathcal{C}}_{\mathcal{B}} \ =: \ [T] \ .$$
- **Images and Kernels are Column Spaces and Null Spaces.** Let $V \xrightarrow{T} W$ be a linear map between finite-dimensional vector spaces. We seek to make as (computationally) explicit as possible, the subspaces
$$\mathsf{Ker}(T) \ \subset \ V \qquad \text{and} \qquad \mathsf{Im}(T) \ \subset \ W \ .$$
To do this, choose a basis $\mathcal{B}$ for V and the basis $\mathcal{C}$ of W. Let $[T]^{\mathcal{C}}_{\mathcal{B}}$ be the unique matrix as in the previous point. Then
$$\mathsf{Ker}(T) \ = \ \mathsf{null}([T]^{\mathcal{C}}_{\mathcal{B}}) \qquad \text{and} \qquad \mathsf{Im}(T) \ = \ \mathsf{col}([T])^{\mathcal{C}}_{\mathcal{B}}) \ .$$
- A pair of linear maps among vector spaces
$$U \xrightarrow{S} V \xrightarrow{T} W$$
is *exact* if there is an equality
$$\mathsf{Im}(S) \ = \ \mathsf{Ker}(T)$$
between subsets of V; in other words
$$\mathsf{col}([S]) \ = \ \mathsf{null}([T]) \ .$$
A sequence of linear maps among vector spaces
$$V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots$$

[1]In diagrams, this is to say that this diagram *commutes*:
$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "{[-]_{\mathcal{B}}}"'] & W \arrow[d, "{[-]_{\mathcal{C}}}"'] \\
\mathbb{R}^m \arrow[r, "x \mapsto Ax"'] & \mathbb{R}^n,
\end{tikzcd}$$
which simply means that the values of either composite map agree.

is *exact at* V_k if $V_{k-1} \xrightarrow{T_{k-1}} V_k \xrightarrow{T_k} V_{k+1}$ is exact. Such a sequence of linear maps is *exact* if, for each $k > 0$, it is exact at V_k.

CLASS

We established the following theorem, a corollary of which is a version of the inclusion-exclusion principle for $|\pi_0(-)|$.

Theorem 0.2 (Mayer-Vietoris Sequence). *Let X be a topological space. Let $A, B \in \mathcal{T}_X$ be elements in the topology of X for which $A \cup B = X$. The sequence of linear maps among vector spaces,*

$$0 \xrightarrow{\zeta} \mathsf{H}^0(X) \xrightarrow[\omega \mapsto \left(\omega_{|A}, \omega_{|B}\right)]{\Phi} \mathsf{H}^0(A) \oplus \mathsf{H}^0(B) \xrightarrow[(\alpha,\beta) \mapsto \alpha_{|A\cap B} - \beta_{|A \cap B}]{\Psi} \mathsf{H}^0(A \cap B) ,$$

is exact, *which is to say the image map equals the kernel of the next:*

(1) *The image of ζ equals the kernel of Φ:*
$$\mathsf{Im}(\zeta) \ = \ \mathsf{Ker}(\Phi) .^2$$
(2) *The image of Φ equals the kernel of Ψ:*
$$\mathsf{Im}(\Phi) \ = \ \mathsf{Ker}(\Psi) .$$

Corollary 0.3. *Consider the same set-up as the main theorem above. There is a canonical isomorphism between vector spaces:*

$$\mathsf{H}^0(X) \ \cong \ \mathsf{Ker}\Big(\mathsf{H}^0(A) \oplus \mathsf{H}^0(B) \xrightarrow{\Psi} \mathsf{H}^0(A \cap B)\Big) .$$

In particular, provided the set $\pi_0(X)$ of path-components is finite, there is an equality

$$|\pi_0(X)| = \mathsf{nullity}(\Phi) \ := \ \dim\Big(\mathsf{Ker}(\Psi)\Big) .$$

Proof. We prove the first statement, that $\mathsf{Im}(\zeta) = \mathsf{Ker}(\Phi)$. Because the domain of ζ is $0 = \{0\}$, then $\mathsf{Im}(\zeta) = \{0\}$ consists only of the zero-element of $\mathsf{H}^0(X)$. By definition of the vector space $\mathsf{H}^0(X)$, this zero-element is the constant map $X \xrightarrow{\mathsf{const}_0} \mathbb{R}$ at $0 \in \mathbb{R}$. So statement (1) is the statement that $\mathsf{Ker}(\Phi) = \{\mathsf{const}_0\}$ consists solely of the constant map at 0. Well, we just unpack definitions:

$$\begin{aligned}
\mathsf{Ker}(\Phi) \ &:= \ \Big\{X \xrightarrow{\omega} \mathbb{R} \mid (\omega_{|A}, \omega_{|B}) = (0,0) \in \mathsf{H}^0(A) \oplus \mathsf{H}^0(B)\Big\} \\
&= \ \Big\{X \xrightarrow{\omega} \mathbb{R} \mid \omega_{|A} = 0 \in \mathsf{H}^0(A) \text{ and } \omega_{|B} = 0 \in \mathsf{H}^0(B)\Big\} \\
&= \ \Big\{X \xrightarrow{\omega} \mathbb{R} \mid \omega_{|A} = \mathsf{const}_0 \in \mathsf{H}^0(A) \text{ and } \omega_{|B} = \mathsf{const}_0 \in \mathsf{H}^0(B)\Big\} \\
&= \ \Big\{X \xrightarrow{\omega} \mathbb{R} \mid \omega(x) = 0 \forall x \in A \text{ and } \omega(x) = 0 \forall x \in B\Big\} \\
&= \ \Big\{X \xrightarrow{\omega} \mathbb{R} \mid \omega(x) = 0 \forall x \in A \text{ and } x \in B\Big\}
\end{aligned}$$

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[2](Note that, because the domain of ζ is the zero-vector space, then the image $\mathsf{Im}(\zeta) = \{0\}$ consists solely of the zero-element (which is $X \xrightarrow{\mathsf{const}_0} \mathbb{R}$, the constant map at 0) in the codomain of ζ. So this condition $\mathsf{Im}(\zeta) = \mathsf{Ker}(\Phi)$ is the condition that $\mathsf{Ker}(\Phi) = \{0\}$. This is exactly to say that Φ is injective.)

$$
\begin{aligned}
&= \left\{X \xrightarrow{\omega} \mathbb{R} \mid \omega(x) = 0 \forall x \in A \cup B\right\} \\
&= \left\{X \xrightarrow{\omega} \mathbb{R} \mid \omega(x) = 0 \forall x \in X\right\} \\
&= \left\{X \xrightarrow{\omega} \mathbb{R} \mid \omega = \mathsf{const}_0\right\} \\
&= \left\{\mathsf{const}_0\right\} \\
&= \left\{0\right\} .
\end{aligned}
$$

This completes the proof of (1).

We now establish (2). We first show an inclusion $\mathsf{Im}(\Phi) \subset \mathsf{Ker}(\Psi)$: Let $\omega \in \mathsf{H}^0(X)$. We must show that $\Phi(\omega) \in \mathsf{Ker}(\Psi)$. By definition of kernels, this is to show that $\Psi(\Phi(\omega)) = 0$. By definition of Φ and of Ψ, this is to show

$$
\left(\omega_{|A}\right)_{|A\cap B} - \left(\omega_{|B}\right)_{|A\cap B} = \mathsf{const}_0 .
$$

This is to show

$$
\left(\omega_{|A}\right)_{|A\cap B} = \left(\omega_{|B}\right)_{|A\cap B} .
$$

By definition of restrictions, this is simply to show that

$$
\omega(x) = \omega(x)
$$

for all $x \in A \cap B$, which is a tautology.

We now show an inclusion $\mathsf{Im}(\Phi) \supset \mathsf{Ker}(\Psi)$. So let $(\alpha, \beta) \in \mathsf{Ker}(\Psi)$. By definition of images, we must construct $\omega \in \mathsf{H}^0(X)$ for which $\Phi(\omega) = (\alpha, \beta)$. By definition of kernel, $\Psi(\alpha, \beta) := \alpha_{|A\cap B} - \beta_{|A\cap B} = \mathsf{const}_0$; this is to say that $\alpha_{|A\cap B} = \beta_{|A\cap B}$. Well, define

$$
\omega \colon X \longrightarrow \mathbb{R} , \qquad x \mapsto \alpha(x) \text{ if } x \in A \qquad x \mapsto \beta(x) \text{ if } x \in B .
$$

This map is well-defined precisely because $A \cup B = X$ and, by assumption, $\alpha(x) = \beta(x)$ for all $x \in A \cap B$. By construction, $\omega_{|A} = \alpha$ and $\omega_{|B} = \beta$. By definition of Φ, this is to say $\Phi(\omega) = (\alpha, \beta)$, as desired.

□

The above Theorem implies a version of the inclusion-exclusion principle for $|\pi_0(-)|$.

Corollary 0.4 (Inclusion-Exclusion bounds)**.** *Let X be a topological space. Let $A, B \in \mathcal{T}_X$ be elements in the topology of X for which $A \cup B = X$. Suppose $\pi_0(X)$ and $\pi_0(A)$ and $\pi_0(B)$ and $\pi_0(A \cap B)$ are finite. There are inequalities:*

$$
|\pi_0(A)| + |\pi_0(B)| - |\pi_0(A \cap B)| \leq |\pi_0(X)| \leq |\pi_0(A)| + |\pi_0(B)| .
$$

Proof. Statement (1) of the Theorem gives that $\Phi \colon \mathsf{H}^0(X) \xrightarrow{\cong} \mathsf{Im}(\Phi)$ is an isomorphism. Statement (2) of the Theorem states that $\mathsf{Im}(\Phi) = \mathsf{Ker}(\Psi)$. Therefore,

$$
\begin{aligned}
&\dim(\mathsf{H}^0(X)) \\
&= \dim(\mathsf{Im}(\Phi)) \\
&= \dim(\mathsf{Ker}(\Psi)) \\
&= \dim(\mathsf{H}^0(A) \oplus \mathsf{H}^0(B)) - \dim(\mathsf{Im}(\Psi))
\end{aligned}
$$

$$= \dim(\mathsf{H}^0(A)) + \dim(\mathsf{H}^0(B)) - \dim(\mathsf{Im}(\Psi)) ,$$

where the 3rd equality is the Rank-Nullity Theorem. Now, since $\{0\} \subset \mathsf{Im}(\Psi) \subset \mathsf{H}^0(A \cap B)$, then $0 \leq \dim(\mathsf{Im}(\Psi)) \leq \dim(\mathsf{H}^0(A \cap B))$. Substituting this inequality into the above string of equalities gives

$$\dim(\mathsf{H}^0(A)) + \dim(\mathsf{H}^0(B)) \geq \dim(\mathsf{H}^0(X)) \geq \dim(\mathsf{H}^0(A)) + \dim(\mathsf{H}^0(B)) - \dim(\mathsf{H}^0(A \cap B)) .$$

Finally, using that $\dim(\mathsf{H}^0(Z)) = |\pi_0(Z)|$ for any topological space Z, the desired inequalities among cardinalities follows.

□

Remark 0.5. We'll see two examples below of these results. Really, they're the same example but the first concerns only the Corollary, whereas the second concerns the Theorem. These examples demonstrate that the Theorem is *stronger* than the Corollary.

Example 0.6. Consider the topological space $\mathbb{S}^1$. Consider the two subsets of $\mathbb{S}^1$:

$$A := \left\{ z \in \mathbb{S}^1 \mid z \neq 1 \right\} \qquad \text{and} \qquad B := \left\{ z \in \mathbb{S}^1 \mid z \neq -1 \right\} .$$

Note the homeomorphisms

$$f \colon (0, 2\pi) \xrightarrow[\cong]{t \mapsto e^{it}} A \qquad \text{and} \qquad g \colon (-\pi, +\pi) \xrightarrow[\cong]{t \mapsto e^{it}} B ,$$

as well as

$$h \colon (-\pi, 0) \sqcup (0, +\pi) \xrightarrow[\cong]{t \mapsto e^{it}} A \cap B .$$

These homeomorphisms implement bijections among sets of path-components:

$$\pi_0((0, 2\pi)) \cong \pi_0(A) \qquad \text{and} \qquad \pi_0((-\pi, +\pi)) \cong \pi_0(B) ,$$

as well as

$$\pi_0((-\pi, 0) \sqcup (0, +\pi)) \cong \pi_0(A \cap B) .$$

We conclude that the cardinalities

$$|\pi_0(A)| = 1 \qquad \text{and} \qquad |\pi_0(B)| = 1 \qquad \text{and} \qquad |\pi_0(A \cap B)| = 2 .$$

So the above Corollary gives the bounds:

$$0 = 1 + 1 - 2 = |\pi_0(A)| + |\pi_0(B)| - |\pi_0(A \cap B)| \leq |\pi_0(\mathbb{S}^1)| \leq |\pi_0(A)| + |\pi_0(B)| = 1 + 1 = 2 .$$

These bounds aren't great, but they're something.

Example 0.7. Let's pick up on the previous example, but using the above Theorem concerning 0th cohomology instead of the above Corollary concerning cardinalities of path-components. The Theorem gives that, whatever on earth is the vector space $\mathsf{H}^0(\mathbb{S}^1)$, it injects into $\mathsf{H}^0(A) \oplus \mathsf{H}^0(B)$, furthermore, the image of this injection is precisely the kernel of the linear map Ψ. Because a linear injection is an isomorphism onto its image, we can figure out what the vector space $\mathsf{H}^0(\mathbb{S}^1)$ is, at least up to isomorphism (which is good enough for identifying the dimension of $\mathsf{H}^0(\mathbb{S}^1)$. Well, through the identifications above, we have that

$$\mathsf{H}^0(A) \cong \mathsf{H}^0((0, 2\pi)) = \mathbb{R} =: \mathsf{Span}\{e_A\} \qquad \text{and} \qquad \mathsf{H}^0(B) \cong \mathsf{H}^0((-\pi, +\pi)) = \mathbb{R} =: \mathsf{Span}\{e_B\}$$

and

$$\mathsf{H}^0(A \cap B) \cong \mathsf{H}^0((-\pi, 0) \sqcup (0, +\pi)) \cong \mathsf{H}^0((-\pi, 0)) \oplus \mathsf{H}^0((0, +\pi)) \cong \mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2 = \mathsf{Span}\{e_L, e_R\} .$$

So the linear map

$$\Psi\colon \mathsf{Span}\{e_A, e_B\} = \mathbb{R}^2 \longrightarrow \mathbb{R}^2 = \mathsf{Span}\{e_L, e_R\}$$

in the statement of the Theorem is precisely a 2×2 matrix, $[\Psi]$. The Ath column of this matrix $[\Psi]$ is the value

$$\Psi(e_A) = 1e_L + 1e_R = \begin{bmatrix} 1 \\ 1 \end{bmatrix} ,$$

while the Bth column of this matrix $[\Psi]$ is the value

$$\Psi(e_B) = -1e_L + -1e_R = \begin{bmatrix} -1 \\ -1 \end{bmatrix} .$$

So

$$[\Psi] \ = \ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} .$$

Compute that

$$\mathsf{H}^0(\mathbb{S}^1) \underset{\text{Thm (1)}}{\cong} \mathsf{Im}(\Phi) \underset{\text{Thm (2)}}{=} \mathsf{Ker}(\Psi) \ = \ \mathsf{null}([\Psi]) = \mathsf{Span}\left\{ e_A + e_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \ \cong \ \mathbb{R} .$$

In other words, we've established an isomorphism

$$\mathsf{H}^0(\mathbb{S}^1) \ \cong \ \mathbb{R} .$$

So

$$|\pi_0(\mathbb{S}^1)| \ = \ \mathsf{dim}\big(\mathsf{H}^0(\mathbb{S}^1)\big) \ = \ \mathsf{dim}(\mathbb{R}) \ = \ 1 .$$

COHOMOLOGY

We've seen that 0th cohomology, $\mathsf{H}^0(-)$, is an assignment of a vector space to each topological space. The Theorem above reveals a sense in which $\mathsf{H}^0(-)$ satisfies a sort of inclusion-exclusion principle. Inspecting the statement of that Theorem, one might ask if there is a vector space V together with a linear map $\mathsf{H}^0(A\cap B) \to V$ for which the resulting sequence

$$0 \xrightarrow{\zeta} \mathsf{H}^0(X) \xrightarrow{\Phi} \mathsf{H}^0(A) \oplus \mathsf{H}^0(B) \xrightarrow{\Psi} \mathsf{H}^0(A\cap B) \xrightarrow{\partial} V$$

is exact. In fact, why stop there; let's ask if this sequence can continue as an exact sequence of linear maps among vector spaces:

$$0 \xrightarrow{\zeta} \mathsf{H}^0(X) \xrightarrow{\Phi} \mathsf{H}^0(A) \oplus \mathsf{H}^0(B) \xrightarrow{\Psi} \mathsf{H}^0(A\cap B) \xrightarrow{\partial} V \to U \to U' \to U'' \to \dots .$$

In fact, it does: for each $k \geq 0$, *kth cohomology* is an assignment $Z \mapsto \mathsf{H}^k(Z)$ of a vector space to a topological space. This assignment respects continuous maps in the sense that, a continuous map $Z \to Z'$ determines a linear map $\mathsf{H}^k(Z') \to \mathsf{H}^k(Z)$. Furthermore, as in the statement of the Theorem, there is a **long exact sequence**

$$\begin{gathered} 0 \xrightarrow{\zeta} \mathsf{H}^0(X) \xrightarrow{\Phi} \mathsf{H}^0(A) \oplus \mathsf{H}^0(B) \xrightarrow{\Psi} \mathsf{H}^0(A\cap B) \\ \xrightarrow{\partial} \mathsf{H}^1(X) \xrightarrow{\Phi} \mathsf{H}^1(A) \oplus \mathsf{H}^1(B) \xrightarrow{\Psi} \mathsf{H}^1(A\cap B) \\ \xrightarrow{\partial} \mathsf{H}^2(X) \xrightarrow{\Phi} \mathsf{H}^2(A) \oplus \mathsf{H}^2(B) \xrightarrow{\Psi} \mathsf{H}^2(A\cap B) \\ \xrightarrow{\partial} \cdots \end{gathered}$$

in which each instance of the maps Φ and Ψ is given in the same way as it is concerning 0th cohomology, and the linear maps ∂ are more mysterious.

Definition 0.8. For X a topological space, its *Euler characteristic* is the alternating sum of dimensions:

$$\begin{aligned}\chi(X) \; &:= \; \dim(\mathsf{H}^0(X)) - \dim(\mathsf{H}^1(X)) + \dim(\mathsf{H}^2(X)) - \dots \\ &= \; \sum_{k\geq 0} (-1)^k \dim(\mathsf{H}^k(X)) \; .\end{aligned}$$

A consequence of the long exact sequence on cohomology that characterizes $\mathsf{H}^*(-)$ is that Euler characteristic satisfies an inclusion-exclusion principle:

$$\chi(X) \; = \; \chi(A) + \chi(B) - \chi(A \cap B) \; .$$

So, you can imagine that to compute $\mathsf{H}^k(X)$ is an inductive process involving linear algebra. Here are some cool calculations.

- $\mathsf{H}^k(\mathbb{S}^n)$ is $\mathbb{R}$ if $k = 0, n$ and otherwise it is the zero-vector space.
- $\mathsf{H}^k(X \times Y) \cong \mathsf{H}^k(X) \otimes \mathsf{H}^k(Y)$.
  So $\mathsf{H}^1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{R}^2$ and $\mathsf{H}^2(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{R}$ and $\mathsf{H}^k(\mathbb{S}^1 \times \mathbb{S}^1) = \{0\}$ for $k > 2$.
- $\mathsf{H}^2(\mathbb{RP}^2) = \{0\}$.
- $\mathsf{H}^1(P_5) \cong \mathbb{R}^8$; $\mathsf{H}^2(P_5) \cong \mathbb{R}$; $\mathsf{H}^k(P_5) = \{0\}$ for $k > 2$.
- $\mathsf{H}^k(P_n) = \{0\}$ for $k > n-3$. $\mathsf{H}^k(P_n)$ is not known $0 < k \leq n-3$ for $n > 6$.

Exercises

These are due by **5pm on Monday 25 November**. You can submit your homework to me via email, or by handing it directly to me, or you can slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) Let $\Gamma = (V, E, s, t)$ be a graph. Suppose the sets V and E are finite, and that $V \neq \emptyset$ is not empty. Enumerate $V = \{v_1, \dots, v_m\}$ and $E = \{e_1, \dots, e_n\}$. Consider the topological space $|\Gamma|$, which is the geometric realization of Γ.

  (a) Prove that there is an isomorphism

  $$\mathsf{H}^0\big(|\Gamma|\big) \; = \; \mathsf{null}(A)$$

  where A is the $n \times m$ matrix in which the (j, i)-entry is

  $$A_i^j = \delta_{t(e_j)}^{v_i} - \delta_{s(e_j)}^{v_i} \; ,$$

  which takes the value 1 if and only if $s(e_j) \neq v_i = t(e_j)$, the value -1 if and only if $s(e_j) = v_i \neq t(e_j)$, the value 0 otherwise.

  (b) Conclude that the cardinality of the set of path-components

  $$\Big|\pi_0\big(|\Gamma|\big)\Big| \; = \; \mathsf{nullity}(A)$$

  is the nullity of this matrix.

(2) Consider the space

$$P_4 \;:=\; \Big\{(z_3,z_4)\in\mathbb{S}^1\times\mathbb{S}^1 \mid |z_4-(z_3+1)|=1\Big\} \;\subset\; (\mathbb{S}^1)^{\times 2}$$

of equilateral 4-gons in the plane, up to translation and rotation of the plane. Consider the two subsets of P_4:

$$A := \Big\{(z_3,z_4) \mid \text{ either } z_3 \neq \pm 1 \text{ or } z_4 \neq \pm 1\Big\}$$

and

$$B := \Big\{(z_3,z_4) \mid \text{ neither } z_3 = \pm i \text{ nor } z_4 = \pm i\Big\} .$$

(a) Explain why each of these subsets $A,B\subset \mathcal{T}_{P_4}$ is a member of the topology on P_4.

(b) Verify that the union $A\cup B = P_4$ is entire.

(c) Name bases for the vector spaces $\mathsf{H}^0(A)\oplus\mathsf{H}^0(B)$ and $\mathsf{H}^0(A\cap B)$. With respect to these bases, describe the linear map

$$\Psi\colon\, \mathsf{H}^0(A)\oplus\mathsf{H}^0(B)\longrightarrow \mathsf{H}^0(A\cap B)$$

as a matrix A.

(d) Identify the kernel $\mathsf{Ker}(\Psi)=\mathsf{null}(A)$, which is the same as the null space of the matrix of the previous point. Thereafter, identify the dimension $\dim(\mathsf{Ker}(\Psi))$.

(e) Identify the cokernel $\mathsf{coKer}(\Psi)=\mathsf{col}(A)^{\perp}$, which is the orthogonal complement of the column space of A. [3] Thereafter, identify the dimension $\dim(\mathsf{coKer}(\Psi))$.

Remark. The significance of this calculation is that the 1st cohomology of P_4 is this cokernel:

$$\mathsf{H}^1(P_4) \;=\; \mathsf{coKer}(\Psi) \;.$$

So

$$\begin{aligned}
\dim\Big(\mathsf{H}^1(P_4)\Big) &= \dim\Big(\mathsf{coKer}(\Psi)\Big)\\
&= \dim\Big(\mathsf{H}^0(A\cap B)\Big) - \dim\Big(\mathsf{Im}(\Psi)\Big)\\
&= \dim\Big(\mathsf{H}^0(A\cap B)\Big) - \dim\Big(\mathsf{H}^0(A)\oplus\mathsf{H}^0(B)\Big) + \dim\Big(\mathsf{Ker}(\Psi)\Big)\\
&= \dim\Big(\mathsf{H}^0(A\cap B)\Big) - \dim\Big(\mathsf{H}^0(A)\oplus\mathsf{H}^0(B)\Big) + \dim\Big(\mathsf{Im}(\Phi)\Big)\\
&= \dim\Big(\mathsf{H}^0(A\cap B)\Big) - \dim\Big(\mathsf{H}^0(A)\oplus\mathsf{H}^0(B)\Big) + \dim\Big(\mathsf{H}^0(P_4))\\
&= \dim\Big(\mathsf{H}^0(A\cap B)\Big) + \dim\Big(\mathsf{H}^0(P_4)\Big) - \dim\Big(\mathsf{H}^0(A)\Big) - \dim\Big(\mathsf{H}^0(B)\Big) \;.
\end{aligned}$$

[3] Note that, through the Rank-Nullity Theorem, this is identical with the null space of the transpose:

$$\mathsf{coKer}(\Psi) \;=\; \mathsf{col}(A)^{\perp} \;=\; \mathsf{null}(A^T) \;.$$