WEEK 11

CLASS

Last week we initiated a discussion about *cohomology*: it is an assignment of a vector space $\mathsf{H}^0(X)$ to a topological space X. As we'll see, the dimension $\dim\big(\mathsf{H}^0(X)\big) = |\pi_0(X)|$ agrees with the number of path-components of X. Generally, each vector space determines a number, or at least an ordinal: its dimension. But unlike numbers, it makes sense to consider a linear map between two vector spaces: for two numbers, we've only got a way to compare their size (via inequalities). So assigning to a vector space to a topological space (such as $X \mapsto \mathsf{H}^0(X)$) is more flexible than assigning a number to a topological space (such as $X \mapsto \pi_0(X)$). Also, the entire subject of linear algebra is at our disposal when we consider vector spaces. All told, this will offer computational advantage for identifying numbers of path-components.

There is a precise sense in which *cohomology* is "derived" from locally constant functions. So we're concerning ourselves at first with what a locally constant function on a topological space is, and how the collection of all such is naturally organized as a vector space. First, then, let's recall some concepts concerning vector spaces.

0.1. Vector spaces.

Informally, a vector space is a set in which addition and scaling by real numbers makes sense.

Definition 0.1. A *vector space* is

- a set V,
- a map $V \times V \xrightarrow{+} V$,
- a map $\mathbb{R} \times V \xrightarrow{\cdot} V$;

these data satisfy the following conditions:

1. Associativity (of addition):
   $v + (v' + v'') = (v + v') + v''$ for each $v, v', v'' \in V$.
2. Identity (of addition):
   there is an element $0 \in V$ for which $0 + v = v = v + 0$ for each $v \in V$.
3. Inverses:
   for each $v \in V$, there is an element $-v \in V$ for which $v + (-v) = 0 = (-v) + v$.
4. Commutative:
   $v + v' = v' + v$ for each $v, v' \in V$.
5. Identity (of scaling):
   $1 \cdot v = v$ for each $v \in V$.
6. Distribution (for addition):
   $t \cdot (v + v') = (t \cdot v) + (t \cdot v')$ for each $t \in \mathbb{R}$ and $v, v' \in V$.

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(7) Associativity (of scaling):
$(st)\cdot v = s\cdot(t\cdot v)$ for each $s,t\in\mathbb{R}$ and $v\in V$.
(8) Distribution (for scaling):
$(s+t)\cdot v = (s\cdot v)+(t\cdot v)$ for each $s,t\in\mathbb{R}$ and $v\in V$.

Remark 0.2. I encourage you to not be too distracted by the above definition, since 8 conditions that $(V,+,\cdot)$ must satisfy are typically straightforward to verify in natural examples. So I encourage you to think of a vector space as a set V equipped with an addition rule $+$ and a scaling rule $\cdot$.

Remark 0.3. Typically, for $(V,+,\cdot)$ a vector space, we'll just denote it as its underlying set V. In practice, such a set V will have a preferred vector space structure $(+,\cdot)$.

Example 0.4. Let I be a set. Consider the set

$$\mathbb{R}^I \ := \ \mathsf{Map}(I,\mathbb{R}) \ := \ \left\{I \xrightarrow{\alpha} \mathbb{R}\right\}$$

of all $\mathbb{R}$-valued maps from I. Value-wise addition and scaling make this set into a vector space:

$$(\alpha+\beta)\colon I \longrightarrow \mathbb{R}\ , \qquad i\mapsto (\alpha+\beta)(i) := \alpha(i)+\beta(i)$$

and

$$(t\cdot\alpha)\colon I \longrightarrow \mathbb{R}\ , \qquad i\mapsto t\alpha(i)\ .$$

From here on, we'll always understand $\mathbb{R}^I := \mathsf{Map}(I,\mathbb{R})$ as this vector space.

Remark 0.5. Indeed, think about it in the case that $I=\{1,2,\dots,n\}$. Then an element in $\mathsf{Map}\Big(\{1,2,\dots,n\},\mathbb{R}\Big)$ is a list $r_1,r_2,\dots,r_n\in\mathbb{R}$ of real numbers. Think of these real numbers as the coordinates of a vector in $\mathbb{R}^n$. Indeed, there is a canonical linear isomorphism

$$\mathsf{Map}\Big(\{1,2,\dots,n\},\mathbb{R}\Big) \ \cong\ \mathbb{R}^n\ , \qquad \big(\{1,\dots,n\}\xrightarrow{i\mapsto r_i}\mathbb{R}\big)\mapsto \sum_{i=1,\dots,n} r_i e_i\ .$$

In a sense, this justifies the notation $\mathbb{R}^{\{1,\dots,n\}}$.

Observation 0.6. Let I be a set. For each $i\in I$, consider the map

$$I\xrightarrow{\delta_i}\mathbb{R}\ , \qquad i'\mapsto \delta_i(i')\ ,$$

where $\delta_i(i')=0$ if $i\neq i'$ and $\delta_i(i')=1$ if $i=i'$. So,

$$\Big\{\delta_i\Big\}_{i\in I}\ \subset\ \mathbb{R}^I$$

is a collection of elements in $\mathbb{R}^I := \mathsf{Map}(I,\mathbb{R})$, one for each element in I. Provided I is finite, this collection is a *basis*:

- **Span.** Let $I\xrightarrow{\alpha}\mathbb{R}$ be a map. Then

$$\alpha\ =\ \sum_{i\in I}\alpha(i)\delta_i\ .$$

Indeed, $\alpha(i')=\alpha(i')\delta_{i'}(i')=\sum_{i\in I}\alpha(i)\delta_i(i')$ for each $i\in I$.

- **Linearly independent.** Suppose
$$\sum_{i\in I} c_i\delta_i \;=\; 0\;.$$
Let $i'\in I$. Then $c_{i'} = c_{i'}\delta_{i'}(i') = \sum_{i\in I} c_i\delta_i(i') = 0$.

Because the definition of 'dimension' is the cardinality of a basis, the above Observation immediately grants the following result.

Corollary 0.7. *Let I be a finite set. The dimension*
$$\mathsf{dim}\Big(\mathbb{R}^I\Big) \;=\; |I|$$
is the cardinality of I.

Observation 0.8.
- Let $I\xrightarrow{f} J$ be a map between sets. The map
$$f^*\colon \mathbb{R}^J \xrightarrow{-\circ f} \mathbb{R}^I\;, \qquad (J\xrightarrow{\alpha}\mathbb{R}) \mapsto (I\xrightarrow{\alpha\circ f}\mathbb{R})$$
is linear.
- Let $I\xrightarrow{f} J\xrightarrow{g} K$ be a pair of maps among sets. There is an equality between the two linear maps from $\mathbb{R}^K$ to $\mathbb{R}^I$:
$$(g\circ f)^* \;=\; f^*\circ g^*\;.$$
- Let I be a set. There is an equality between the two linear maps from $\mathbb{R}^I$ to itself:
$$(\mathsf{id}_I)^* \;=\; \mathsf{id}_{\mathbb{R}^I}\;.$$
- Let $f\colon I\to J$ be a map between sets. If f is a bijection, then the linear map $f^*\colon \mathbb{R}^J\to\mathbb{R}^I$ is a linear isomorphism.
- If I and J are bijective, then $\mathbb{R}^J$ and $\mathbb{R}^I$ are linearly isomorphic. In particular, if $\mathbb{R}^J$ is not isomorphic with $\mathbb{R}^J$ then I is not bijective with J.

Remark 0.9. Let $S\subset I$ be a subset of a set. Consider the inclusion map
$$f\colon S\longrightarrow I\;, \qquad s\mapsto s\;.$$
Then the linear map
$$f^*\colon \mathbb{R}^I\longrightarrow\mathbb{R}^S\;, \qquad \alpha\mapsto \alpha_{|S}\;,$$
evaluates simply as restriction.

Here are some basic notions concerning vector spaces.

Definition 0.10.
- A map $F\colon V\to W$ is *linear* if
  - $F(v+v') = F(v)+F(v')$ for each $v,v'\in V$.
  - $F(t\cdot v) = t\cdot F(v)$ for each $t\in\mathbb{R}$ and each $v\in V$.
- Let $F\colon V\to W$ be a linear map.
  - The *kernel (of F)* is the subset
  $$\mathsf{Ker}(F) \;:=\; \Big\{v\in V \mid F(v)=0\Big\} \;\subset\; V\;.$$
  - The *image (of F)* is the subset
  $$\mathsf{Im}(F) \;:=\; \Big\{w\in W \mid \text{ there exists } v\in V \text{ such that } F(v)=w\Big\} \;\subset\; W\;.$$

– The *cokernel (of F)* is the quotient space
$$\mathsf{coKer}(F) := W_{/\sim}$$
where $w \sim w'$ means $w' - w \in \mathsf{Im}(F)$. In other words/notation:
$$\mathsf{coKer}(F) := \frac{W}{\mathsf{Im}(F)} .$$
In other words/notation:
$$\mathsf{coKer}(F) := \mathsf{Im}(F)^{\perp} \subset W$$
is the orthogonal complement of $\mathsf{Im}(F)$ in W.

- Let V and W be vector spaces. Their *direct sum* is the vector space
$$V \oplus W$$
whose underlying set is $V \times W$, and whose addition rule is given by
$$(v,w) + (v',w') := (v+v', w+w') ,$$
and whose scaling rule is given by
$$t \cdot (v,w) := (t \cdot v, t \cdot w) .$$

Remark 0.11. In the case that $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$, then a linear map $F\colon V \to W$ is exactly the same thing as a linear transformation. A main theorem in Linear Algebra (M333) states that, for $F\colon \mathbb{R}^m \to \mathbb{R}^n$ a linear transformation, there a unique $n \times m$ matrix $[F]$ for which, for each $v \in \mathbb{R}^m$, the value of F on v is the product of the matrix $[F]$ and v:
$$F(v) = [F]v .$$
As so, the Kernel and the Image and the coKernel are familiar; they are respectively the null space the column space and the orthogonal complement of the column space:
$$\mathsf{Ker}(F) = \mathsf{null}([F]) \qquad \text{and} \qquad \mathsf{Im}(F) = \mathsf{col}([F]) \qquad \text{and} \qquad \mathsf{coKer}(F) = \mathsf{col}(F)^{\perp} \subset \mathbb{R}^n .$$

0.2. Locally constant functions.

Definition 0.12. Let X be a topological space. The *0th cohomology (of X)* is the vector space
$$\mathsf{H}^0(X) := \mathsf{Map}^{\text{l.c.}}(X,\mathbb{R}) := \mathsf{Map}^{\text{locally constant}}(X,\mathbb{R})$$
$$:= \left\{ X \xrightarrow{\alpha} \mathbb{R} \mid x \underset{\text{path}}{\sim} x' \implies \alpha(x) = \alpha(x') \right\} \subset \mathsf{Map}(X,\mathbb{R}) ,$$
as it is endowed with value-wise addition and scaling. [1]

Remark 0.13. Note that $\mathsf{H}^0(X)$ depends on the topology of X. Indeed, the topology of X determines which maps $[0,1] \to X$ are *continuous*, thereby defining the equivalence relation $\underset{\text{path}}{\sim}$.

The next result connects the set of path-components of a topological space with its 0th cohomology.

[1] In words, the 0th cohomology of X is the set of all real-valued functions on X that are *locally constant*, where a real-valued function f on X is *locally constant* if f evaluated identically on two points in X that are connected by a path in X.

Theorem 0.14. *Let X be a topological space. Consider the quotient map $X \xrightarrow{q} \pi_0(X)$. There is a canonical linear isomorphism between vector spaces:*

$$q^*\colon \mathbb{R}^{\pi_0(X)} \xrightarrow{\cong} \mathsf{H}^0(X)~, \qquad \big(\pi_0(X) \xrightarrow{\alpha} \mathbb{R}\big) \mapsto \big(X \xrightarrow{\alpha\circ q} \mathbb{R}\big)~.$$

Proof. A priori, the named map takes values in $\mathbb{R}^X$:

$$q^*\colon \mathbb{R}^{\pi_0(X)} \longrightarrow \mathbb{R}^X~, \qquad \big(\pi_0(X) \xrightarrow{\alpha} \mathbb{R}\big) \mapsto \big(X \xrightarrow{\alpha\circ q} \mathbb{R}\big)~.$$

This map is evidently linear, since addition and scaling in both the domain vector space and the codomain vector space is value-wise. We'll now show that this map takes values in the subspace $\mathsf{H}^0(X) \subset \mathbb{R}^X$. Let $\big(\pi_0(X) \xrightarrow{\alpha} \mathbb{R}\big) \in \mathbb{R}^{\pi_0(X)}$. We must show $q^*\alpha$ has the property that $x \underset{\text{path}}{\sim} x'$ implies $q^*\alpha(x) = q^*\alpha(x')$. By definition of q, the assumption that $x \underset{\text{path}}{\sim} x'$ implies $q(x) = q(x')$. Therefore $q^*\alpha(x) = \alpha(q(x)) = \alpha(q(x')) = q^*\alpha(x')$.

So the named map indeed takes values in $\mathsf{H}^0(X)$:

$$q^*\colon \mathbb{R}^{\pi_0(X)} \longrightarrow \mathsf{H}^0(X)~.$$

We now show this map is both injective and surjective. Let's show q^* is injective. Let $\alpha,\beta \in \mathbb{R}^{\pi_0(X)}$ be elements for which $q^*(\alpha) = q^*(\beta)$. We must show that $\alpha = \beta$. Well, being maps, this is to show, for each $[x] \in \pi_0(X)$, that $\alpha([x]) = \beta([x])$. By definition of q, note that $q(x) = [x]$. So $\alpha([x]) = \alpha(q(x)) = q^*\alpha(x) \underset{\text{assumption}}{=} q^*\beta(x) = \beta(q(x)) = \beta([x])$.
We now show that q^* is surjective. So let $\omega \in \mathsf{H}^0(X)$. We must construct $\big(\pi_0(X) \xrightarrow{\alpha} \mathbb{R}\big) \in \mathbb{R}^{\pi_0(X)}$ for which $q^*\alpha = \omega$. Take

$$\alpha\colon \pi_0(X) \longrightarrow \mathbb{R}~, \qquad [x] \mapsto \omega(x)~.$$

Let's verify that this is well-defined (in other words, this map doesn't depend on the choice of element $x \in [x]$). So let $x' \in [x]$ be another element. We must show $\omega(x) = \omega(x')$. Well, $x' \in [x]$ implies $x \underset{\text{path}}{\sim} x'$. By assumption that ω is locally constant, $\omega(x) = \omega(x')$, as desired. Finally, by construction $q^*\alpha = \omega$.

□

Corollary 0.15. *Let X be a topological space. Suppose $\pi_0(X)$ is finite. Then the two numbers*

$$\mathsf{dim}\big(\mathsf{H}^0(X)\big) ~=~ |\pi_0(X)|$$

agree.

We finish this subsection by recording some basic facts about how 0th cohomology interacts with continuous maps.

Observation 0.16.

- Let $X \xrightarrow{f} Y$ be a continuous map between topological spaces. Then the map

  $$\mathsf{H}^0(f)\colon \mathsf{H}^0(Y) \longrightarrow \mathsf{H}^0(X)~, \qquad \alpha \mapsto f^*\alpha~,$$

  is a linear map.

- Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be continuous maps among topological spaces. Then there is an equality between linear maps from $\mathsf{H}^0(Z)$ to $\mathsf{H}^0(X)$:
$$\mathsf{H}^0(g \circ f) \;=\; \mathsf{H}^0(f) \circ \mathsf{H}^0(g)\ .$$
- Let X be a topological space. There is an equality between linear maps from $\mathsf{H}^0(X)$ to $\mathsf{H}^0(X)$:
$$\mathsf{H}^0(\mathsf{id}_X) \;=\; \mathsf{id}_{\mathsf{H}^0(X)}\ .$$

The above Observation has the following consequence.

Corollary 0.17. *Let $f\colon X \to Y$ be a continuous map between topological spaces. If f is a homeomorphism, then $\mathsf{H}^0(f)$ is a linear isomorphism. In particular, if X and Y are homeomorphic then $\mathsf{H}^0(Y)$ and $\mathsf{H}^0(X)$ are linearly isomorphic; in other words, if $\mathsf{H}^0(Y)$ and $\mathsf{H}^0(X)$ are* not *linearly isomorphic then X is* not *homeomorphic with Y.*

0.3. **Inclusion-Exclusion principle.** The inclusion-exclusion principle supplies a method for identifying the cardinality of a finite set in terms of cardinalities of subsets of it whose union is entire.

Theorem 0.18 (Inclusion-Exclusion Principle [2])**.** *Let S be a finite set. Let $A, B \subset S$ be subsets for which $A \cup B = S$. Then there is an equality involving cardinalities:*
$$|S| \;=\; |A| + |B| - |A \cap B|\ .$$[3]
In particular, there is a bound:
$$|S| \;\leq\; |A| + |B|\ .$$

As we're most interested in the cardinality $|\pi_0(X)|$ for X a topological space, we seek to have an inclusion-exclusion principle concerning subsets of X. This is articulated as the Corollary below. Next week we'll discuss, and prove, the Theorem below, and draw that Corollary as a consequence of it.

Theorem 0.19 (Mayer-Vietoris Sequence)**.** *Let X be a topological space. Let $A, B \in \mathcal{T}_X$ be elements in the topology of X for which $A \cup B = X$. The sequence of linear maps*
$$\mathsf{H}^0(X) \xrightarrow[\Phi]{\omega \mapsto \left(\omega_{|A}, \omega_{|B}\right)} \mathsf{H}^0(A) \oplus \mathsf{H}^0(B) \xrightarrow[\Psi]{(\alpha,\beta) \mapsto \alpha_{|A\cap B} - \beta_{|A\cap B}} \mathsf{H}^0(A \cap B)$$
has the following properties.

1. *The first linear map Φ is injective.*
2. *The image of the first linear map Φ equals the kernel of the second linear map Ψ:*
$$\mathsf{Im}(\Phi) \;=\; \mathsf{Ker}(\Psi)\ .$$

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[2]Check out the "Inclusion-Exclusion Principle" on the internet for some applications of it to some neat counting problems.

[3]More generally, let $A_1, A_2, \dots, A_n \subset S$ be subsets for which $\bigcup_{i=1,\dots,n} A_i = S$. Then
$$|S| \;=\; \sum_{0 \leq k \leq n} (-1)^{k+1} \sum_{I \in \binom{n}{k}} |\bigcap_{i \in I} A_i|\ .$$

Corollary 0.20 (Inclusion-Exclusion bounds)**.** *Let X be a topological space. Let $A, B \in \mathcal{T}_X$ be elements in the topology of X for which $A \cup B = X$. Suppose $\pi_0(X)$ and $\pi_0(A)$ and $\pi_0(B)$ and $\pi_0(A \cap B)$ are finite. There are inequalities:*

$$|\pi_0(A)| + |\pi_0(B)| - |\pi_0(A \cap B)| \;\leq\; |\pi_0(X)| \;\leq\; |\pi_0(A)| + |\pi_0(B)| \;.$$

Example 0.21. Consider the topological space $\mathbb{S}^1$. Consider the two subsets of $\mathbb{S}^1$:

$$A := \left\{z \in \mathbb{S}^1 \mid z \neq 1\right\} \qquad \text{and} \qquad B := \left\{z \in \mathbb{S}^1 \mid z \neq -1\right\} .$$

Note the homeomorphisms

$$f\colon (0, 2\pi) \xrightarrow[\cong]{t \mapsto e^{it}} A \qquad \text{and} \qquad g\colon (-\pi, +\pi) \xrightarrow[\cong]{t \mapsto e^{it}} B \;,$$

as well as

$$h\colon (-\pi, 0) \sqcup (0, +\pi) \xrightarrow[\cong]{t \mapsto e^{it}} A \cap B \;.$$

These homeomorphisms implement bijections among sets of path-components:

$$\pi_0((0, 2\pi)) \cong \pi_0(A) \qquad \text{and} \qquad \pi_0((-\pi, +\pi)) \cong \pi_0(B) \;,$$

as well as

$$\pi_0((-\pi, 0) \sqcup (0, +\pi)) \cong \pi_0(A \cap B) \;.$$

We conclude that the cardinalities

$$|\pi_0(A)| = 1 \qquad \text{and} \qquad |\pi_0(B)| = 1 \qquad \text{and} \qquad |\pi_0(A \cap B)| = 2 \;.$$

So the above Corollary gives the bounds:

$$0 = 1 + 1 - 2 = |\pi_0(A)| + |\pi_0(B)| - |\pi_0(A \cap B)| \;\leq\; |\pi_0(\mathbb{S}^1)| \;\leq\; |\pi_0(A)| + |\pi_0(B)| = 1 + 1 = 2 \;.$$

These bounds aren't great, but they're something.

Example 0.22. Let's pick up on the previous example, but using the above Theorem concerning 0th cohomology instead of the above Corollary concerning cardinalities of path-components. The Theorem gives that, whatever on earth is the vector space $\mathsf{H}^0(\mathbb{S}^1)$, it injects into $\mathsf{H}^0(A) \oplus \mathsf{H}^0(B)$, furthermore, the image of this injection is precisely the kernel of the linear map Ψ. Because a linear injection is an isomorphism onto its image, we can figure out what the vector space $\mathsf{H}^0(\mathbb{S}^1)$ is, at least up to isomorphism (which is good enough for identifying the dimension of $\mathsf{H}^0(\mathbb{S}^1)$. Well, through the identifications above, we have that

$$\mathsf{H}^0(A) \cong \mathsf{H}^0((0, 2\pi)) = \mathbb{R} =: \mathsf{Span}\{e_A\} \qquad \text{and} \qquad \mathsf{H}^0(B) \cong \mathsf{H}^0((-\pi, +\pi)) = \mathbb{R} =: \mathsf{Span}\{e_B\}$$

and

$$\mathsf{H}^0(A \cap B) \cong \mathsf{H}^0((-\pi, 0) \sqcup (0, +\pi)) \cong \mathsf{H}^0((-\pi, 0)) \oplus \mathsf{H}^0((0, +\pi)) \cong \mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2 = \mathsf{Span}\{e_L, e_R\} \;.$$

So the linear map

$$\Psi\colon \mathsf{Span}\{e_A, e_B\} = \mathbb{R}^2 \longrightarrow \mathbb{R}^2 = \mathsf{Span}\{e_L, e_R\}$$

in the statement of the Theorem is precisely a 2×2 matrix, $[\Psi]$. The Ath column of this matrix $[\Psi]$ is the value

$$\Psi(e_A) = 1e_L + 1e_R = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;,$$

while the Bth column of this matrix $[\Psi]$ is the value

$$\Psi(e_B) = -1e_L + -1e_R \begin{bmatrix} -1 \\ -1 \end{bmatrix} .$$

So

$$[\Psi] \ = \ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} .$$

Compute that

$$\mathsf{H}^0(\mathbb{S}^1) \underset{\text{Thm (1)}}{\cong} \mathsf{Im}(\Phi) \underset{\text{Thm (2)}}{=} \mathsf{Ker}(\Psi) \ = \ \mathsf{null}([\Psi]) = \mathsf{Span}\left\{e_A + e_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \ \cong \ \mathbb{R} .$$

In other words, we've established an isomorphism

$$\mathsf{H}^0(\mathbb{S}^1) \ \cong \ \mathbb{R} .$$

So

$$|\pi_0(\mathbb{S}^1)| \ = \ \dim\big(\mathsf{H}^0(\mathbb{S}^1)\big) \ = \ \dim(\mathbb{R}) \ = \ 1 .$$

Exercises

These are due by **5pm on Monday 18 November**. [4] You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) Consider the set

$$X \ := \ \Big\{(x,y) \in \mathbb{R}^2 \mid xy = 1\Big\} \ \subset \ \mathbb{R}^2 ,$$

as it's endowed with the subspace topology. Construct a basis for the vector space $\mathsf{H}^0(X)$.

(2) Let $Z \xrightarrow{f} X$ be a continuous map between topological spaces. Prove, or find a counter-example to, the following assertion.

If f is surjective, verify that the linear map

$$\mathsf{H}^0(f) \colon \mathsf{H}^0(X) \longrightarrow \mathsf{H}^0(Z)$$

is injective.

(3) Let X and Y be topological spaces. Prove, or find a counter-example to, the following assertion.

There is a linear isomorphism

$$\mathsf{H}^0(X \sqcup Y) \ \cong \ \mathsf{H}^0(X) \oplus \mathsf{H}^0(Y) .$$ [5]

[4] Remember to consult the "How to prove stuff" flowchart. Namely, given an assertion, first consider if you know all definitions in its articulation. If not, review definitions. Next, consider if you believe, or disbelieve the assertion. If you're not sure if you believe/disbelieve the assertion, work through a host of examples. This is the most important step. As you work through examples, pay attention to reasoning you're using to validate your belief/disbelief. Finally, if you do believe/disbelieve the assertion, try to construct a rigorous argument. As you write such up, consider who your audience is: are you convinced? would your friend be convinced? would your colleague be? would I be? would your (logical) enemy be?

[5] You might firs consider the case in which both X and Y are discrete topological spaces, for in that case each of the canonical quotient maps $X \to \pi_0(X)$ and $Y \to \pi_0(Y)$ is a bijection.

(4) Let X be a topological space. Let $A, B \in \mathcal{T}_X$ be element for which $A \cup B = X$. Suppose each of the sets $\pi_0(X)$ and $\pi_0(A)$ and $\pi_0(B)$ and $\pi_0(A \cap B)$ is finite.

(a) (i) Find an example of A, B, X, as above, for which

$$|\pi_0(A)| + |\pi_0(B)| - |\pi_0(A \cap B)| \;=\; |\pi_0(X)| \;.$$

(ii) Find an example of A, B, X, as above, for which

$$|\pi_0(X)| \;=\; |\pi_0(A)| + |\pi_0(B)| \;.$$

(iii) Find an example of A, B, X, as above, for which

$$|\pi_0(A)| + |\pi_0(B)| - |\pi_0(A \cap B)| \;<\; |\pi_0(X)| \;<\; |\pi_0(A)| + |\pi_0(B)| \;.$$

(b) Consider the map

$$f \colon A \sqcup B \longrightarrow X \;, \qquad a \mapsto a \qquad \text{and} \qquad b \mapsto b \;.$$

Prove that the map

$$\Phi \colon \mathsf{H}^0(X) \xrightarrow{\mathsf{H}^0(f)} \mathsf{H}^0(A) \oplus \mathsf{H}^0(B)$$

is injective.

(c) Consider the linear map

$$\Psi \colon \mathsf{H}^0(A) \oplus \mathsf{H}^0(B) \longrightarrow \mathsf{H}^0(A \cap B) \;, \qquad (\alpha, \beta) \mapsto \alpha_{|A \cap B} - \beta_{|A \cap B} \;.$$

(i) Find an example of A, B, X, as above, for which this linear map is surjective.

(ii) Find an example of A, B, X, as above, for which this linear map is *not* surjective.