WEEK 1

1. How to do well in this class

Here are some suggestions for how to do well in this class.

- Ask questions. In particular, not asking questions when you're lost or not following is a strategy for failure.
- Read the textbook, and do so regularly. In particular, avoid setting intentions to read large portions of the textbook just before/during exercise sets or take-home exams.
- Think about the material in advance of lectures. It is generally accepted that receiving information after pondering/guessing about it increases retention of the information significantly.
- Take advantage of the various resources available to you. These include:
  - The textbook.
  - The internet (such as Wikipedia).
  - Me, via lectures and office hours.
  - Your fellow students!
- Engage with the content of the course like it is your **one and only chance** to learn it. In particular, avoid operating as if you'll have a second chance to learn the material, for that is a strategy for mediocrity.
- Be skeptical of, and challenge, any authority on the content, such as me or the textbook.
- It is your responsibility to address how this content fits into the landscape of mathematics, and human knowledge more generally. For instance, the question "why is this material important?" is something you, and only you, are responsible for addressing to your satisfaction – I, and the other resources, are more than happy to help you in this quest. Indeed, that important question relies on the prior notion of what is important; and this is something that is private to each person. Discovering what criteria qualify an answer to this question is, perhaps, the most instructive part of asking it.
- Take responsibility over your education. Avoid being passive. Avoid operating as if I, or this educational system, is responsible your reception of this content.
- In your off-time, I encourage each of you to inspect the history of this academic institution, both in the general sense of 'universities and the degrees/certificates they grant' and of MSU in particular. You might notice that the existence of such institutions are in response to local- and global-policy and history; also, such institutions are new, and there is little promise they'll persist forever. I believe it's in each of your interest to channel such perspectives actionably, to empower you to use these institutions per your own agenda, and to not expect for them to have your interests in mind, per se.

Date: 28 August 2018.

The first class commenced by asking each of you how you'd communicate to a layperson "what is mathematics?"/"what characterizes it among other discliplines of human thought?" I'd respond by saying that mathematics is characterized by the following sequence of conceptual developments.

(1) Identify patterns within, and regulation of, experience;
(2) Isolate/reduce this regulation, abstract it, articulate in terms of known terms;
(3) Develop a model, predict phenomena.
(4) (Note:there's no mention of numbers, equations, formulae. Such are typical, but not essential, means for articulating/modeling phenomena.)

We experienced a few 'tricks' that I hoped instilled the sense that our experience can be surprisingly regulated. [1] The subject of Topology is a good medium to exercise this perspective on math: it's not obviously mathematical in the standard sense, yet enough so that progress/demonstration of this program can be made in short order.

Class

We had a general introduction: discussing the syllabus and expectations. I interpret the syllabus as a contract between each of you and me/MSU, so please consider it seriously. In particular, consider the grading scheme, and give me feedback by **the close of this Friday** if you care for it to be different in some way. Importantly, you're expected to do assigned reading **before each class**.

Here are some important notions we covered in class:

- Sources of examples of topological spaces.
  - Spaces of solutions to systems of equations.
    - For example:

$$\left\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2-z^2=0 \text{ and } x+y+2z=0\right\} \subset \mathbb{R}^3 .$$

    - For example:

$$T(x)=x ,$$

      such as where $x\in\mathbb{R}^N$ is the state of the ecosystem in Yellowstone (N is the number of species in Yellowstone, and for s such a species, the sth component x_s is the population of s), and where $T\colon\mathbb{R}^N\to\mathbb{R}^N$ is models the monthly change of the population. So the above equation is a state of the ecosystem that is in equilibrium.
    - For example

$$X^TX=S$$

      where S is a fixed $n\times n$ matrix, and the variable X is an $n\times n$ matrix, with X^T its transpose. This is a succinct way to write n^2 equations in n^2 real numbers; in the case that $S=\mathsf{id}_{n\times n}$, such a solution X is an orthogonal matrix.
    - Let you're imagination of 'solutions to systems of equations' grow from these examples.

[1] the David Copperfield thing, the explanation having to do with pairity; the carrabiners and rope trick; the tie trick.

  - Spaces of arrangements (also known as *moduli spaces*).
    * The set of arrangements of a pen in the room is naturally organized as a topological space. It is 5-dimensional (3 for where its tip is; 2 for which direction it's pointing (think of spherical coordinates)).
    * The set of arrangements of 3 pens in the room whose 'direction'-lines all intersect, is naturally organized as a topological space. It is $(3+2)+(3+1)+(3+0) = 12$-dimensional.
    * The set of arrangements of a unit cube in the room is naturally organized as a topological space. It is $3+3 = 6$-dimensional.
    * The set of arrangements of a k-dimensional sub-vector-space of $\mathbb{R}^n$ is naturally organized as a topological space. It is $k(n-k)$-dimensional. Even the case that $k = 1$ and $n = 3$ is very interesting.
    * Let you're imagination of 'arrangement spaces' grow from these examples.
  - Polytope complexes. These are topological spaces obtained by gluing together polytopes along faces.
    * A data set is a finite subset $S \subset \mathbb{R}^N$. Such could arise, for instance, as 10,000 images of my dog sleeping on my couch, or 10,000 images of a pencil on a desk, each taken from various positions in the room. Indeed, a single black&white photo is a gray-scale (i.e., a real number) assigned to each pixel of a digital camera (say it's resolution is $10^9 \times 10^9$ pixels). So a black&white photo is an element in $(\mathbb{R}^{10^9})^2 = \mathbb{R}^{10^18}$. So 10,000 black&white photos is a subset $S \subset \mathbb{R}^{10^18}$ with 10,000 elements in it. The problem at hand is to distinguish the two data sets – the dog photos and the pencil photos – in such a way that when I upload a new photo, a machine can decide if it's a dog- or a pencil-photo. One idea is to construct, for each such finite subset $S \subset \mathbb{R}^N$, a topological space X_S; then use topological invariants of X_S to distinguish among such S's.
      The construction of X_S will depend on a choice, $r > 0$, of a radius. Given such a choice, declare X_S to have one vertex for each element in S, an edge connecting $s_0 \in S$ and $s_1 \in S$ if the distance $\mathsf{dist}(s_0, s_1) \leq r$; a triangle if $\mathsf{diam}\{s_0, s_1, s_2\} \leq r$; a 3-simplex if $\mathsf{diam}\{s_0, s_1, s_2, s_3\} \leq r$; etc. So we've build X_S as a union of basic shapes (polytopes), each possibly glued to another along a face.
    * Rubik's Cube. Construct a topological space R in which there is one vertex for each possible state of a Rubik's Cube. Two states are connected by an edge if there is a sequence of moves relating them. There is an (oriented) triangle with boundary edges $(\sigma_{01}, \sigma_{12}, \sigma_{02})$ for if the sequence σ_{01} followed by the sequence σ_{12} is the sequence σ_{02}. There is an (oriented) tetrahedron ... etcetera.
    * Let you're imagination of 'polytope complexes' grow from these examples.
- Now, given a topological space X, we seek to address each of the following questions. Consider how to interpret the question through each of the examples above.

  - Is X empty? (Ie, what is $\pi_{-1}(X)$?)
  - Is X connected? If not, how many connected components are there? (Ie, what is $\pi_0(X)$?)
  - Is X simply-connected? If not, how many paths, up to homotopy (relative to end points), are there? (Ie, what is $\pi_1(X)$?)
  - Is X compact? Relatedly, does every continuous real-valued function on X attain its maximum? (This is an articulation of optimization...)

In this class, we'll define *topological space* and *continuous maps* among them, supply loads of (sources of) examples of them, and consider *topological properties* of such. Along the way, we'll repeatedly undergo the process of making informal intuitions mathematically rigorous.

Next we covered §1, and the first part of Appendix A, of the textbook.

- The subject of topology, in a sense, is that of crafting definitions to make intuitions rigorous concerning a priori known "topological properties". [2] So the pedagological challenge is to simultaneously present
  - the objects of study: topological spaces;
  - the maps among these objects: continuous maps;
  - properties of the objects that are respected by these maps: topological properties.

  Most pedagological approaches present these three pillars in this sequence. This textbook attempts to present these features simultaneously, which is more true to the development of the subject, though at the expense of thorough and clear development. So paying attention in class will be singularly important.
- For the first part of the course, the objects of study will be only a certain class of topological spaces: subsets
$$X \subset \mathbb{R}^n$$
  of some Euclidean space.
- The maps among these objects are *continuous maps*:

  Definition 1.1. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be subsets of Euclidean spaces. A *continuous map (from X to Y)* is a map between sets,
$$f\colon X \longrightarrow Y~, \qquad x \mapsto f(x)~,$$
  [3] satisfying the following property.
  - For each $x \in X$, and for each $\varepsilon > 0$, there is a $\delta > 0$ for which $\mathsf{dist}_X(x,x') < \varepsilon$ implies $\mathsf{dist}_Y(f(x), f(x')) < \varepsilon$.

  Making this definition intuitive is a worthy exercise! Here are a few ways you might do so.

[2] In fact, it is this spirit, more than the specific content, of brings topology into the larger landscape of mathematics and physics: abstract objects, abstract morphisms between them; abstract properties/invariants of these objects respected by these morphisms.

[3] So, f is a rule for how to assign to each element in X an element in Y.

- One might say a map $f\colon X \to Y$ between sets is continuous if a small perturbation of an element $x \in X$ results in a small perturbation in the element $f(x)$. The issue with this definition is that 'perturbation' isn't a precise notion.
- Alternatively, one might say that if $x \in X$ and $x' \in X$ are close, then so are $f(x) \in Y$ and $f(x') \in Y$. The issue with this is that 'close' isn't a precise notion.
- But we can exploit that the sets X and Y are presented as subsets of Euclidean spaces, and Euclidean space has a notion of *distance* between any two of its elements. So one might say if $\mathsf{dist}_X(x,x')$ is small then so is $\mathsf{dist}_Y(f(x),f(x'))$. But, here, the issue is that 'small' isn't precise.
- So one might say that, for $\mathsf{dist}_Y(f(x),f(x')) < \varepsilon$, for some specified number $\varepsilon > 0$, then $\mathsf{dist}_X(x,x') < \delta$ for some other number $\delta > 0$ which very well might depend on x and ε. This is one way to articulate these attempts precisely.

Later, we'll see different, though equivalent, definitions of *continuity* which don't explicitly make reference to "distance".

Reading

By Wednesday 28 August. §1, up to Theorem 1.1 (on page 6) . Be especially familiar with these aspects.

- The convention (for now) that when the book says "set" it means "subset of a Euclidean space.

By Wednesday 4 September. All of §1, the first part of Appendix A concerning continuous maps and homeomorphisms.

Exercises

These are due at **5pm on Wednesday 4 September**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) Give an example of a bijection that is not a homeomorphism.

(2) From §1 of textbook:
- 1.2,
- 1.4,
- 1.5,
- 1.7 xor 1.9 (for these last two, it's your choice, but don't submit both).

(So you're to submit 4 problems from what's above.)

(3) Do one, and only one, of these two problems.

(a) Prove that

$$X \ := \ \left\{(x,y,z) \mid x^2+y^2-z^2=0 \text{ and } z=1\right\} \ \subset \ \mathbb{R}^3$$

and

$$Y \ := \ \left\{(x,y,z) \mid x^2+y^2+z^2=1 \text{ and } x+y+z=\sqrt{3}\right\}$$

are not homeomorphic.

(b) Prove that

$$X \;:=\; \Big\{(x,y) \mid y - \sin(2\pi x) = 0 \text{ and } y = 1\Big\} \;\subset\; \mathbb{R}^3$$

and

$$Y \;:=\; \Big\{(x,y,z) \mid y - 2\pi x = 0\Big\}$$

are not homeomorphic.