M 476 - Homework 1

Nathan Stouffer

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Problem 1

Problem Statement: Give an example of a bijection that is not a homeomorphism.

Solution: Let $A = y \in \mathbb{R} \mid 0 < y \leq 1$. Now consider the map $f : \mathbb{R} \to \mathbb{R} \setminus A$ defined by $f(x) = \begin{cases} x & x \leq 0 \\ x+1 & x > 0 \end{cases}$
f is a bijection because it is one-to-one and onto, but f is discontinuous at $x = 0$ and, therefore, is not a homeomorphism.

Problem 2

Exercise 1.2

Problem statement: Find an explicit formula for a homeomorphism from $(0,1)$ to the real line.

Solution: Let $X = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and $Y = \{y \in \mathbb{R}\}$. A map $f : X \to Y$ exists. More specifically, $f(x) = \log(\frac{1}{x} - 1)$. f is continuous on X so we must now show that the inverse of f is continuous. The inverse of f is $g : Y \to X$ where $g(y) = \frac{1}{10^y + 1}$. The function g is continuous on Y. Therefore, f describes an explicit formula for a homeomorphism from X to Y.

Exercise 1.4

Problem statement: Show that the plane sets $\{(x,y) \mid 0 < x^2 + y^2 < 1\}$ and $\{(x,y) \mid 1 < x^2 + y^2 < 4\}$ are homeomorphic. Let D be the closed disc $\{(x,y) \mid x^2 + y^2 \leq 1\}$. Show that the plane with the origin removed is homeomorphic to the plane with the disc D removed.

Solution: First, we will show that the plane sets $\{(x,y) \mid 0 < x^2 + y^2 < 1\}$ and $\{(u,v) \mid 1 < u^2 + v^2 < 4\}$ are homeomorphic. Let X be the first set and U be the second. The map $f : X \to U$ where $f(x,y) = (4 - 3x, 4 - 3y)$. The inverse of f is $g : U \to X$ where $g(u,v) = (\frac{4-u}{3}, \frac{4-v}{3})$. Both f and g are continuous and map to the correct sets, therefore, X and U are homeomorphic.

We now show that the plane with the origin removed is homeomorphic to the plane with the disc D removed. Let X be the plan with the origin removed and U be the plane with the disc D removed. We define $f : X \to U$ where $f(x,y) = (\frac{1}{x} + 1, \frac{1}{y} + 1)$ and its inverse as $g : U \to X$ where $g(u,v) = (\frac{1}{u-1}, \frac{1}{v-1})$. Both f and g are continuous and map to the correct sets, therefore, X and U are homeomorphic.

Exercise 1.5

Problem statement: Show that the cone $\{(x,y,z) \mid x^2 + y^2 = z^2, z \geq 0\}$ is homeomorphic to the plane.

Solution: Let $X = \{(x,y,z) \mid x^2 + y^2 = z^2, z \geq 0\}$ and $U = \{(u,v) \mid (u,v) \in \mathbb{R}\}$. We map X to U by projecting the cone X onto the xy-plane. We define $f : X \to U$ by $f(x,y,z) = (x,y)$. The inverse of f is defined as $g : U \to X$ by $g(x,y) = (x, y, \sqrt{x^2 + y^2})$. Both f and g are continuous maps to their respective sets and thus it is proved that X is homeomorphic to the plane.

Exercise 1.7

Problem statement: Let C be the cube $\{(x,y,z) \mid 0 < x,y,z < 1\}$ and let B be the ball $\{(x,y,z) \mid x^2 + y^2 + z^2 < 1\}$. Show that B and C are homeomorphic.

Solution: We begin by showing that C is homeomorphic to $\mathbb{R}^3$. We define $f : C \to \mathbb{R}^3$ by $f(x,y,z) = (\log(\frac{1}{x} - 1), \log(\frac{1}{y} - 1), \log(\frac{1}{z} - 1))$ with inverse defined as $g : \mathbb{R}^3 \to C$ by $g(u,v,w) = (\frac{1}{10^u + 1}, \frac{1}{10^v + 1}, \frac{1}{10^w + 1})$. Both f and its inverse g are continuous so $C \cong \mathbb{R}^3$.

We now move on to show that $B \cong \mathbb{R}^3$. Begin by converting $\{(x,y,z) \mid x^2 + y^2 + z^2 < 1\}$ to spherical coordinates. This gives B as $\{(r,\theta,\phi) \mid r < 1, 0 < \theta < 2\pi, 0 < \phi < \pi\}$. We now define $f : B \to \mathbb{R}^3$ by $f(r,\theta,\phi) = (\frac{r}{1-r}, \theta, \phi)$ with inverse defined as $g : \mathbb{R}^3 \to B$ by $g(r,\theta,\phi) = (\frac{r}{1+r}, \theta, \phi)$. Both f and its inverse g are continuous so $B \cong \mathbb{R}^3$.

So we know that $C \cong \mathbb{R}^3$ and $B \cong \mathbb{R}^3$, therefore, by the transitive property of homeomorphism, $B \cong C$.

Problem 3

Problem statement: Prove that $X = \{(x, y) \mid y - \sin 2\pi x = 0 \text{ and } y = 1\} \subset \mathbb{R}^3$ and $Y = \{(x, y, z) \mid y - 2\pi x = 0\}$ are not homeomorphic.

Solution: First we observe that Y is continuous for all inputs (x, y, z) and has uncountably infinite elements. We then turn to X, where we notice that $y = 1$. Knowing this, $y - \sin 2\pi x = 1 - \sin 2\pi x = 0$. We then solve for x:

$$\begin{aligned}
1 &= \sin 2\pi x && \text{subtract } \sin 2\pi x \text{ from both sides} \\
\arcsin(1) &= 2\pi x && \text{take the inverse of sine of both sides} \\
\frac{\pi}{2} + n\pi &= 2\pi x && \text{where } n \in \mathbb{Z} \\
x &= \frac{1}{4} + \frac{n}{2} && \text{divide both sides by } 2\pi
\end{aligned}$$

Now note that any value of $x \in X$ is at a discrete interval with another $x_1 \in X$. This means that the number of elements in X is countably infinite. Knowing this, we can say that any map from X to Y cannot be one-to-one since the cardinality of the sets X and Y are not equal. And since no one-to-one mapping exists between the two sets, X and Y cannot be homeomorphic.