MIDTERM

Instructions. The use of any text/literature, including the internet, is allowed. The use of other humans, including through the internet, is *not* allowed. You are to adhere to the Student Conduct Code at all times.

Submit your solution to each of the following 5 problems.
Your score will be based on correctness, completeness, and effectiveness of communication. Ample credit will be given to solutions that display thought process, or examples/instances.

Notation.

- Unless explicitly stated otherwise, a subset of a Euclidean space is understood as endowed with a subspace topology.
- Let $n \geq 0$. Denote
$$\mathbb{D}^n := \Big\{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \Big\} \subset \mathbb{R}^n$$
and
$$\mathbb{B}^n := \Big\{ x \in \mathbb{R}^n \mid \|x\| < 1 \Big\} \subset \mathbb{R}^n$$
and
$$\mathbb{S}^{n-1} := \Big\{ x \in \mathbb{R}^n \mid \|x\| = 1 \Big\} \subset \mathbb{R}^n .$$
- Let $k > 0$ be a positive integer. Let $X \subset \mathbb{R}^n$ be a subset of a Euclidean space. Denote
$$\mathsf{Conf}_k(X) := \Big\{ (x_1, \ldots, x_k) \in X^{\times k} \mid i \neq j \implies x_i \neq x_j \Big\} \subset X^{\times k} \subset (\mathbb{R}^n)^{\times k} .$$

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(1) (a) Verify that, for each $(x_1, x_2) \in \mathsf{Conf}_2(\mathbb{D}^2)$, there exists a unique $(y_1, y_2) \in \mathsf{Conf}_2(\mathbb{S}^1)$ with the following properties.
- The four points $y_1, x_1, x_2, y_2 \in \mathbb{R}^2$ belong to a common line in $\mathbb{R}^2$.
- If $x_1 \in \mathbb{S}^1$ then $y_1 = x_1$. If $x_1 \in \mathbb{D}^2 \smallsetminus \mathbb{S}^1$ then, on the above line, x_1 is between y_1 and x_2.
- If $x_2 \in \mathbb{S}^1$ then $y_2 = x_2$. If $x_2 \in \mathbb{D}^2 \smallsetminus \mathbb{S}^1$ then, on the above line, x_2 is between x_1 and y_2.

(b) Consider the map

$$f\colon \mathsf{Conf}_2(\mathbb{D}^2) \longrightarrow \mathsf{Conf}_2(\mathbb{S}^1)$$

whose value $f(x_1, x_2)$ is the unique element $(y_1, y_2) \in \mathsf{Conf}_2(\mathbb{S}^1)$ as above. Prove, or disprove, that f is continuous.

(2) Let $\Gamma = (V, E, s, t)$ be an abstract (directed) graph in which V and E are finite and not empty. Suppose Γ satisfies the following condition.
- For each $v, w \in V$ there exists $u \in V$ and $a, b \in E$ for which
$$v = s(a)\ ,\ t(a) = u = s(b)\ ,\ t(b) = w\ .$$

Consider the set
$$|\Gamma| \ := \ \Big(V \sqcup \big(E \times [0,1]\big)\Big)_{/\sim}$$
of equivalence classes, where $\sim$ is generated by declaring $(e,0) \sim s(e)$ and $(e,1) \sim t(e)$ for each $e \in E$. Consider the quotient map
$$q\colon V \sqcup \big(E \times [0,1]\big) \longrightarrow |\Gamma|\ .$$
Through this quotient map, endow $|\Gamma|$ with the quotient topology. [1] [2]

Prove, or find a counter-example to, the following assertion:

The cardinality $\Big|\pi_0(|\Gamma|)\Big| = 1$.

[1] So a subset $U \subset |\Gamma|$ is in this quotient topology if and only if, for each $e \in E$, the preimage $q^{-1}(U) \cap \{e\} \times [0,1] \subset \{e\} \times [0,1] = [0,1]$ is open.

[2] Note that this quotient map restricts as a bijection $V \sqcup \big(E \times (0,1)\big) \xrightarrow{\cong} |\Gamma|$. So one can think of this quotient topology on $|\Gamma|$ as a topology on the set $V \sqcup \big(E \times (0,1)\big)$.

(3) A *unit cube in 3-space* is a subset $C \subset \mathbb{R}^3$ with the property that there exists a point $x \in \mathbb{R}^3$ and an orthonormal basis (u,v,w) for $\mathbb{R}^3$ for which the subsets of $\mathbb{R}^3$ agree:

$$C ~=~ \Big\{x + au + bv + cw \in \mathbb{R}^3 \mid a,b,c \in [-1,+1]\Big\}~.$$

Consider the subspace

$$\mathsf{Faces} ~\subset~ (\mathbb{R}^3)^{\times 6}$$

consisting of (all) those $(x_1,\dots,x_6)$ such that there is a unit cube in 3-space, C, for which $\{x_1,\dots,x_6\}$ comprise the centers of the faces of C. Find a bound $N < 6!$, if not identify, the cardinalty:

$$\Big|\pi_0(\mathsf{Faces})\Big| ~\leq~ N~;$$

completely justify your answer. [3]

[3] As a warm-up, you might think through a version of this problem in which the role of $\mathbb{R}^3$ is replaced by that of $\mathbb{R}^2$, and the role of a *unit cube in 3-space* is replaced by a *unit square in 2-space*. In this case, there are $\frac{4!}{\frac{2!\cdot 2^2}{2}} = 6$ path-components.

(4) Let X be a path-connected topological space. Let Y be a topological space whose topology $\mathcal{T}_Y$ consists of all subsets of Y. Prove, or find a counter-example to, the following assertion.

Let $f\colon X \to Y$ be a map between underlying sets. This map f is continuous if and only if f is constant. [4] [5] [6]

[4]As usual, I suggest the following strategy for this problem. First make sure you understand each term in this problem. Next, work through some instances of this problem. (For instance, you might take Y to be a circle (with this discrete topology) and X to be a closed interval (with its subspace topology from $\mathbb{R}$.)

[5]Hint: Consider the case in which $X = [0,1]$ (with its subspace topology from $\mathbb{R}$). Try to reduce your examination to this case.

[6]Hint: for each pair $y_- \neq y_+ \in Y$ of distinct elements in Y, construct a continuous map $Y \xrightarrow{g} \mathbb{S}^0$ for which $g(y_\pm) = \pm 1$.

(5) Submit one of the following two problems.

(a) A *unit cube in 3-space* is a subset $C \subset \mathbb{R}^3$ with the property that there exists a point $x \in \mathbb{R}^3$ and an orthonormal basis (u, v, w) for $\mathbb{R}^3$ for which there is an equality between subsets of $\mathbb{R}^3$:

$$C = \Big\{ x + au + bv + cw \in \mathbb{R}^3 \mid a, b, c \in [-1, +1] \Big\} .$$

Consider the set

$$\mathsf{Cubes}$$

of (all) cubes in 3-space.

(i) Endow Cubes with a topology $\mathcal{T}_{\mathsf{Cubes}}$ with the property that it has infinitely many members yet no singleton is a member. (You're not expected to display your verification that $\mathcal{T}_{\mathsf{Cubes}}$ satisfies these properties; just convince yourself it does.)

(ii) With respect to this topology, identify the cardinality

$$\Big| \pi_0(\mathsf{Cubes}) \Big| .$$

(b) A *line in 6-space* is a subset $L \subset \mathbb{R}^6$ with the property that there exists two distinct points $x \neq y \in \mathbb{R}^4$ for which there is an equality between subsets of $\mathbb{R}^6$:

$$L = \Big\{ tx + (1-t)y \in \mathbb{R}^6 \mid t \in \mathbb{R} \Big\} .$$

Consider the set

$$\mathsf{Lines}$$

of (all) lines in 6-space.

(i) Endow Lines with a topology $\mathcal{T}_{\mathsf{Lines}}$ with the property that it has infinitely many members yet no singleton is a member. (You're not expected to display your verification that $\mathcal{T}_{\mathsf{Lines}}$ satisfies these properties; just convince yourself it does.)

(ii) With respect to this topology, identify the cardinality

$$\Big| \pi_0(\mathsf{Lines}) \Big| .$$