WEEK 10

Class

We explored abstract topological spaces. We did so by considering how a topology on a set X determines which maps $[0,1] \to X$ are continuous. I encourage you to found your intuition of what a topology on a set is in these terms. Specifically, a topology on a set is, essentially, a declaration of which maps $[0,1] \to X$ are continuous. After all, a map $[0,1] \to X$ is an element $x_t \in X$ for each time $0 \leq t \leq 1$. So the above intuition, then, is that a topology on X determines which such maps $[0,1] \ni t \mapsto x_t \in X$ are 'wiggling's from x_0. (Here, 'wiggling' is an informal stand-in for 'continuous'.)

This exploration was founded in a few instances of such sets X. Namely, $X = \mathbb{S}^1$ and $X = \mathbb{S}^0$. We say that some topologies on $\mathbb{S}^1$ result in $\mathbb{S}^1$ *not* being path-connected; while some topologies on $\mathbb{S}^0$ result in $\mathbb{S}^0$ being path-connected!

Understand, however, that the subject of *topology* is not to endow familiar sets with exotic topologies, rather it is to find natural topologies on sets that are not, per se, presented as subsets of Euclidean spaces.

Here are some **definitions**.

- Let X be a set. The *discrete* topology on X is
$$\mathcal{T}_{\mathsf{discrete}} := \Big\{ U \subset X \Big\} ,$$
the set of all subsets of X.
- Let X be a set. The *codiscrtete* topology on X is
$$\mathcal{T}_{\mathsf{codiscrete}} := \Big\{ \emptyset, X \Big\} ,$$
the set consisting solely of the empty set and the entire set.
- Let $(X, \mathcal{T}_X)$ be a topological space. Say $(X, \mathcal{T}_X)$ is *Hausdorff* if, for each $x \neq x' \in X$, there are elements $U, U' \in \mathcal{T}_X$ for which $x \in U$ and $x' \in U'$ yet $U \cap U' = \emptyset$.

Note that the topological space $(\mathbb{S}^0, \mathcal{T}_{\mathsf{codiscrete}})$ is not Hausdorff. Indeed, $\mathbb{S}^0 \in \mathcal{T}_{\mathsf{codiscrete}}$ is the only member that contains $-1 \in \mathbb{S}^0$; likewise, $\mathbb{S}^0 \in \mathcal{T}_{\mathsf{codiscrete}}$ is the only member that contains $+1 \in \mathbb{S}^0$. So, for $U, U' \in \mathcal{T}_{\mathsf{codiscrete}}$ for which $-1 \in U$ and $+1 \in U'$, then $U \cap U' \neq \emptyset$.
Topological spaces that are *not* Hausdorff are generally regarded as pathological, and are rarely considered in the context of topology.

Our next topic will be *cohomology*.

Date: 3 November 2019.

EXERCISES

These are due by **5pm on Friday 8 November**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) Let $(X,\mathcal{T}_X)$ be a topological space. Suppose, for each $x\in X$, that the singleton $\{x\}\in\mathcal{T}_X$ is a member of the topology. Prove that $\mathcal{T}_X$ is the discrete topology.

(2) Let $(X,\mathcal{T}_X)$ be a Hausdorff topological space.
   (a) Let $x\in X$ be an element. Prove that the complement $X\setminus\{x\}$ is a member of $\mathcal{T}_X$.
   (b) Use induction to prove that, if X is finite, then $\mathcal{T}_X$ is the discrete topology.

(3) Consider the equivalence relation on the subset

$$\{-1,+1\}\times\mathbb{R}\ \subset\ \mathbb{R}\times\mathbb{R}=\mathbb{R}^2\ .$$

given by declaring $(-1,t)\sim(+1,t)$ for each t that is a rational number. Consider the set of equivalence classes,

$$X\ :=\ \big(\{-1,+1\}\times\mathbb{R}\big)_{/\sim}\ ,$$

which we regard as a topological space via the quotient topology from the surjection $\{-1,+1\}\times\mathbb{R}\xrightarrow{\text{quotient}}X$ from the subspace topology on $\{-1,+1\}\times\mathbb{R}\subset\mathbb{R}^2$. Verify that X, with this topology, is *not* Hausdorff.

(4) Consider the maps

$$f\colon[0,2\pi)\xrightarrow{t\mapsto e^{it}}\mathbb{S}^1\qquad\text{and}\qquad g\colon[0,2\pi]\xrightarrow{t\mapsto e^{it}}\mathbb{S}^1\qquad\text{and}\qquad h\colon[0,\pi)\sqcup[3\pi,4\pi)\xrightarrow{t\mapsto e^{it}}\mathbb{S}^1\ .$$

Consider the resulting quotient topologies $\mathcal{T}_f$ and $\mathcal{T}_g$ and $\mathcal{T}_h$ on $\mathbb{S}^1$.
   (a) Prove that all of $(\mathbb{S}^1,\mathcal{T}_f)$ and $(\mathbb{S}^1,\mathcal{T}_g)$ and $(\mathbb{S}^1,\mathcal{T}_h)$ are Hausdorff.
   (b) Prove that both of the topological spaces $(\mathbb{S}^1,\mathcal{T}_f)$ and $(\mathbb{S}^1,\mathcal{T}_g)$ are path-connected. Prove that $(\mathbb{S}^1,\mathcal{T}_h)$ is *not* path-connected. Deduce that $(\mathbb{S}^1,\mathcal{T}_h)$ is not homeomorphic with either $(\mathbb{S}^1,\mathcal{T}_f)$ or $(\mathbb{S}^1,\mathcal{T}_g)$.
   (c) Convince yourself that $(\mathbb{S}^1,\mathcal{T}_f)$ is not homeomorphic with $(\mathbb{S}^1,\mathcal{T}_g)$. Supply an outline of your reasoning. As you write this up, point out gaps in your logic. Try to make those logical gaps as small, and succinct, and few, as possible.