WEEK 7

CLASS

This past week's classes were devoted to examining P_4 and P_5 (see the Week 5 notes for the definition of P_n, for general n), as well as the projective plane $\mathbb{RP}^2$. Each class went over the specific problems of the homework, emphasizing a few take-aways.

- We drew each P_4^ℓ as a subset of the torus. We noticed that, for $\ell \in \mathbb{R}$, this subset P_4^ℓ of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is the level set for the (continuous!) map

$$L\colon \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{R}~, \qquad (z_3, z_4) \mapsto \mathsf{dist}(z_4, z_3)~, \qquad or~(\theta, \phi) \mapsto \mathsf{dist}(e^{i\phi}, 1 + e^{i\theta})~,$$

  which measures the length of the stick connecting the 4th point and the 3rd point.
- "Level sets" come up often, and deserve some notation.

  Notation 0.1. Let $f\colon X \to Y$ be a map between sets. For $B \subset Y$ a subset,

$$f^{-1}(B) ~:=~ \Big\{x \in X \mid f(x) \in B\Big\} ~\subset~ X~.$$

  Note that this notation doesn't mean f has an inverse, or anything like that. Rather, $f^{-1}(B)$ is simply the subset of X consisting of those elements in X that f carries to B.
- We quickly examined P_5.
  - We observed the projection

$$P_5 \longrightarrow P_4^{\mathsf{US}}~, \qquad (z_3, z_4, z_5) \mapsto (z_3, z_5)~.$$

    In words, this map forgets the 4th point.
  - We observed that this map is
    * 0-to-1 over P_4^ℓ for $\ell > 2$,
    * 1-to-1 over P_4^2,
    * 2-to-1 over P_4^ℓ for $0 < \ell < 2$,
    * circle-to-1 over P_4^0.

    So we can break P_5 up into chambers, according to the above. Then we can glue these chambers together along their interfaces, to identify P_5. Namely, the above observations reveal that the image of $P_5 \to P_4^{\mathsf{US}}$ is a torus without an open disk. Furthermore, removing the resulting circle boundary of this image, and removing two points (P_4^0), reveals that P_5 remove 5 circles is a 2-fold disjoint union of a torus remove 3 points. Further examination reveals that P_5 is the result of gluing these two 3-punctured tori are glued toether via three open cylinders ($\mathbb{S}^1 \times (-1,1)$). The result is a genus 4 surface!

Date: 12 October 2019.

- P_6^ℓ has been understood pretty well. But P_n^ℓ, for any ℓ or $n > 6$, has not been identified by anybody ever! Maybe one of you can tell the rest of the world what P_7 is!
  (See the paper announced through D2L.)
- We observed that $\mathbb{RP}^2$ can be described as the result of gluing a Mobius band with a 2-disk along their common circle-boundary.

Reading

By Friday 11 October. §3, just the material about *graphs* (pages 44-45). §5, through the definition of a polyhedron (pages 69-73). Be especially familiar with these aspects.

- The definition of a *graph*, and of an *isomorphism* between graphs.
- The definition of a *polyhedron*, and of a *symmetry* between polyhedrons.

Exercises

These are due by **5pm on Friday 18 October**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) Consider the map

$$\mathsf{Proj}\_ : \mathbb{RP}^2 \longrightarrow \mathsf{Mat}_{3\times 3} \cong \mathbb{R}^9 , \qquad V \mapsto \mathsf{Proj}_V ,$$

whose value on V is the linear transformation $\mathsf{Proj}_V : \mathbb{R}^3 \to \mathbb{R}^3$ which is orthogonal projection onto V.

(a) Prove that $\mathsf{Proj}\_$ is injective.

(b) Prove that the composite map

$$\mathbb{S}^2 \xrightarrow{\mathsf{Span}} \mathbb{RP}^2 \xrightarrow{\mathsf{Proj}\_} \mathbb{R}^9$$

is continuous.

(c) Consider the equivalence relation $\sim$ on the 2-sphere $\mathbb{S}^2$ that declares $x \sim y$ to mean $y = \pm x$. Construct a bijection between the set of equivalence classes and the projective plane:

$$\mathbb{S}^2_{/\sim} \cong \mathbb{RP}^2 .$$

(2)

Definition 0.2. An *(abstract) graph* Γ consists of the following data.

(a) A set V,

(b) A set E,

(c) A map between sets $s\colon E \to V$,

(d) A map between sets $t\colon E \to V$.

Let $\Gamma = (V,E,s,t)$ be a graph. Suppose Γ has the following properties.

- Both of the sets V and E are finite.
- For each $e \in E$, the two values $s(e) \neq t(e)$ are not equal.
- The map $(s,t)\colon E \xrightarrow{e \mapsto \left(s(e),t(e)\right)} V \times V$ is injective.

Consider the set of equivalence classes

$$|\Gamma| \;:=\; \Big(V \bigsqcup E \times [0,1]\Big)_{/\sim} \;,$$

where $\sim$ is the equivalence relation that declares $s(e) \sim (e,0)$ and $t(e) \sim (e,1)$ for each $e \in E$. Construct an injection from this set of equivalence classes into a Euclidean space,

$$f\colon |\Gamma| \longrightarrow \mathbb{R}^n \;,$$

with the feature that the image $X := f(|\Gamma|) \subset \mathbb{R}^n$, with the subset $f(V) \subset X$, is a graph (in the sense introduced in class).

(3) Consider the set of equivalence classes

$$\mathbb{R}^2_{/\sim}$$

where $(x,y) \sim (x',y')$ means both $x'-x$ and $y'-y$ is an integer. Construct an injection

$$f\colon \mathbb{R}^2_{/\sim} \longrightarrow \mathbb{C}^2$$

with the following two properties.

- the image is $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2$;
- the composition

$$\mathbb{R}^2 \xrightarrow{\text{quotient}} \mathbb{R}^2_{/\sim} \xrightarrow{f} \mathbb{C}^2$$

  is continuous.