WEEK 8

Class

There will be a take-home midterm across the period Saturday 26 October through Friday 1 November, due at 3pm.

We recalled the definition of a *topological space*:

Definition 0.1. A *topological space* is
- ("underlying set") a set X,
- ("topology (on X)") a collection $\mathcal{T}$ of subsets of X,

for which
- $\emptyset, X \in \mathcal{T}$,
- ("closed under (the formation of) unions") For $\{U_\alpha\}$ a collection of members of $\mathcal{T}$, the union $\bigcup\limits_\alpha U_\alpha$ is also a member of $\mathcal{T}$,
- ("closed under (the formation of) finite intersections") For $\{U_\alpha\}$ a finite collection of members of $\mathcal{T}$, the intersection $\bigcap\limits_\alpha U_\alpha$ is also a member of $\mathcal{T}$,

Notation 0.2. We often denote a topological space $(X, \mathcal{T})$ simply as its underlying set X if $\mathcal{T}$ is understood. In this case, we use the notation $\mathcal{T}_X$ for its understood topology.

Example 0.3. Let $n \geq 0$. Consider the topologiacl space $(\mathbb{R}^n, \mathcal{T}_{\mathbb{R}^n})$ whose underlying set is $\mathbb{R}^n$ and whose topology consists of those $U \subset \mathbb{R}^n$ that are *open*. Here, $U \subset \mathbb{R}^n$ being *open* means, for each $x \in U$ there is an $\varepsilon > 0$ for which the ε-ball about x

$$\mathsf{B}_\varepsilon(x) \ \subset \ U$$

is contained in U.
(In class, we verified that $\mathcal{T}_{\mathbb{R}^n}$ is indeed a topology.)

Terminology 0.4. Let X be a topological space. Lifting from the above example, we often say a subset $U \subset X$ is "open" in place of saying "U is a member of $\mathcal{T}_X$".

There are two key constructions for coming up with new topological spaces from old ones.

Construction 0.5. Let X be a topolgoical space. Let $i \colon A \longrightarrow X$ be an injective map between sets. The *subspace* topology $\mathcal{T}_A$ on A consists of those subsets $U \subset A$ for which there is a member $V \in \mathcal{T}_X$ such that $U = A \cap V$.
(In class, we verified that $\mathcal{T}_A$ is indeed a topology.)

Notation 0.6. Let $f \colon S \to T$ be a map between sets. For $P \subset T$ a subset, the *preimage (of P by f)* is the subset

$$f^{-1}(P) \ := \ \Big\{ s \in S \mid f(s) \in P \Big\} \ \subset \ S\ .$$

Date: 20 October 2019.

(Note that $f^{-1}(P)$ makes sense even if f isn't injective nor surjective.)

Construction 0.7. Let X be a topolgoical space. Let $q\colon X \longrightarrow Z$ be an surjective map between sets. The *quotient* topology $\mathcal{T}_Z$ on Z consists of those subsets $U \subset Z$ for which $q^{-1}(U) \in \mathcal{T}_X$.
(In class, we verified that $\mathcal{T}_Z$ is indeed a topology.)

Through these two constructions, we're able to construct a topology on a whole lot of sets. For example, any subset $X \subset \mathbb{R}^n$ can be endowed with a subspace topology. Thereafter, for $\sim$ an equivalence relation on X, the set of equivalence classes $X_{/\sim}$ can be endowed with a quotient topology. This is the scheme for constructing topological spaces.

Let's demonstrate this scheme through a couple examples.

- Recall the *(real) projective plane*:
$$\mathbb{RP}^2 \ := \ \Big\{ V \subset \mathbb{R}^3 \mid V \text{ is a 1-dimensional subvector space} \Big\} .$$
There is an intuitive topology on $\mathbb{RP}^2$. Indeed, you can imagine wiggling a line around, such as by rotating it a bit about the origin. But the question is how to endow $\mathbb{RP}^2$ with a topology.

  Note that every 1-dimensional subvector space of $\mathbb{R}^3$ admits an orthonormal basis (through such a basis needn't be unique). This is to say there is a surjective (but not necessarily injective) map
$$\mathsf{Span}\colon \mathbb{S}^2 \longrightarrow \mathbb{RP}^2 , \qquad x \mapsto \mathsf{Span}\{x\} . \tag{1}$$
Now, the standard inclusion $\mathbb{S}^2 \subset \mathbb{R}^3$ endows $\mathbb{S}^2$ with a subspace topology. Thereafter, the surjective map (1) endows $\mathbb{RP}^2$ with a quotient topology!

  Unwind definitions to observe the following properties for this topological space $\mathbb{RP}^2$:
  - Singletons are *not* open in $\mathbb{RP}^2$.
  - There are infinitely many elements in $\mathbb{RP}^2$.
  - Because $\mathbb{S}^2$ is path-connected, and because the above continuous map (2) is surjective, then $\mathbb{RP}^2$ is path-connected!
(What an easy way to deduce as much, eh?)
- A *unit square in the plane* is a subset $S \subset \mathbb{R}^2$ for which there is a point $x \in \mathbb{R}^2$ and an orthonormal basis (u, v) such that there is an equality between subsets of $\mathbb{R}^2$:
$$S \ = \ \Big\{ x + au + bv \mid a, b \in [-1, +1] \Big\} .$$
Consider the set
$$\mathsf{Squares}$$
of (all) unit squares in the plane. There is an intuitive topology on $\mathsf{Squares}$. Indeed, you can imagine wiggling a unit square around in the plane, such as by translating it a bit, or rotating it a bit, or doing both of these types of moves simultaneously. But the question is how to endow $\mathsf{Squares}$ with a topology.

Note that, by definition of a *unit square in the plane*, every such thing is determined (though not uniquely so) by the choice of a point $x \in \mathbb{R}^2$ together with an orthonormal basis (u,v). This is to say there is a surjective (but not necessarily injective) map

$$\mathbb{R}^2 \times \mathsf{O}(2) \longrightarrow \mathsf{Squares}~, \qquad \big(x,(u,v)\big) \mapsto \big\{x+au+bv \mid a,b \in [-1,+1]\big\}~. \qquad (2)$$

Now, the standard inclusion $\mathbb{R}^2 \times \mathsf{O}(2) \subset \mathbb{R}^2 \times \mathsf{Mat}_{2\times 2} = \mathbb{R}^2 \times (\mathbb{R}^2)^{\times 2} = \mathbb{R}^6$ endows $\mathbb{R}^2 \times \mathsf{O}(2)$ with a subspace topology. Thereafter, the surjective map (2) endows $\mathsf{Squares}$ with a quotient topology!

Unwind definitions to observe the following properties for this topological space $\mathsf{Squares}$:

- Singletons are *not* open.
- There are infinitely many elements in $\mathsf{Squares}$.
- Because $\pi_0(\mathbb{R}^2 \times \mathsf{O}(2)) \cong \pi_0(\mathbb{R}^2) \times \pi_0(\mathsf{O}(2)) \cong \pi_0(\mathsf{O}(2))$ has cardinality 2, and because the above continuous map (2) is surjective, then the cardinality of $\pi_0(\mathsf{Squares})$ is at most 2. In fact, a bit more thought reveals that the cardinality of $\pi_0(\mathsf{Squares})$ is 1. (Indeed, note the role of $\mathsf{O}(2)$ could have been replaced by $\mathsf{SO}(2)$.)

You might notice that we haven't defined $\pi_0(X)$ for an arbitrary topological space. Reflecting on the definition of $\pi_0(X)$ for $X \subset \mathbb{R}^n$ a subset of a Euclidean space, recall that it's defined purely in terms of *paths* in X, which are continuous maps $[0,1] \to X$. So once we have a notion of continuous maps from $[0,1]$ to an arbitrary topological space X, then we'll have a rigorous notion of $\pi_0(X)$.

Definition 0.8. Let X and Y be topological spaces. Let $f\colon X \to Y$ be a map between their underlying sets. This map f is *continuous* if, for each $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V) \in \mathcal{T}_X$ is a member of the topology on X.

Terminology 0.9. Let X be a topological space. A *path* in X is a continuous map

$$[0,1] \longrightarrow X$$

from an interval, endowed with its subspace topology from $\mathbb{R}$.

EXERCISES

These are due by **5pm on Friday 25 October**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) Supply a reasonable definition of a *unit regular pentagon in the plane*. Endow the set Pent of unit regular pentagons in the plane with a topology with the following properties.
  - Singletons are not open.
  - There are infinitely many open subsets.
  - It is path-connected.

(2) Let $0 \leq k \leq n$. Consider the set

$$\mathsf{Gr}_k(n) ~:=~ \Big\{V \subset \mathbb{R}^n \mid V \text{ is a } k\text{-dimensional subvector space of}\Big\}~.$$

Endow $\mathsf{Gr}_k(n)$ with a topology with these features.

- Singletons are not open.
- There are infinitely many open subsets.
- It is path-connected.

(3) Let $n \geq 0$. Endow the set
$$\mathsf{Gr}(n) \; := \; \Big\{ V \subset \mathbb{R}^n \mid V \text{ is a subvector space} \Big\}$$
with the quotient topology via the surjecrtive map
$$\mathsf{Mat}_{n\times n} \longrightarrow \mathsf{Gr}(n) \;, \qquad A \mapsto \mathsf{col}(A) \;.$$
(a) Find an element $V \in Gr(n)$ for which the singleton $\{V\} \subset \mathsf{Gr}(n)$ is open.
(b) Prove that $\mathsf{Gr}(n)$, with this topology, is path-connected.

(4) Consider the set X consisting of (all) those 4-tuples (V_1, V_2, V_3, V_4) of 1-dimensional subvector spaces of $\mathbb{R}^4$ for which $i \neq j$ implies $V_i \perp V_j$ are orthogonal. Endow X with a topology with these features.
- Singletons are not open.
- There are infinitely many open subsets.

How many path-components does X, with this topology, have?