WEEK 4

CLASS

We continued our development of $\pi_0(X)$, the set of path-components of a subset $X \subset \mathbb{R}^n$.

Here are some important notions we covered in class:

- There is a canonical map
$$X \longrightarrow \pi_0(X) , \qquad x \mapsto [x] .$$
It assigns to $x \in X$ the path-component of X in which x belongs. Note that this map is surjective.
- Let $f\colon X \longrightarrow Y$ be a continuous map. We showed that it induces a map between sets
$$\pi_0(f)\colon \pi_0(X) \longrightarrow \pi_0(Y) , \qquad [x] \mapsto [f(x)] .$$
We checked that this expression indeed makes sense (i.e., that it's well-defined: for $[x] = [x']$ then $[f(x)] = [f(x')]$).
- The resulting diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"] & \pi_0(Y) \end{tikzcd}$$
commutes.
- For $X \xrightarrow{f} Y \xrightarrow{g} Z$ continuous maps, then $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.
- $\pi_0(\mathsf{id}_X) = \mathsf{id}_{\pi_0(X)}$.
- We made several easy, though useful, observations from these facts above.
  - For $f\colon X \xrightarrow{\cong} Y$ a homeomorphism, the map $\pi_0(f)\colon \pi_0(X) \xrightarrow{\cong} \pi_0(Y)$ is a bijection.
    In particular, if the cardinalities $\left|\pi_0(X)\right| \neq \left|\pi_0(Y)\right|$ don't agree, then X and Y are not homeomorphic.
  - For $f\colon X \to Y$ a continuous surjection, the induced map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a surjection.
  - In particular, if there is such a continuous surjection, then there is an inequality between the cardinalities:
$$\left|\pi_0(X)\right| \geq \left|\pi_0(Y)\right| .$$
  - In particular, such a continuous surjection implies Y is path-connected whenever X is path-connected.
- We used these observations to examine the following classes of examples.

Date: 18 September 2019.

– Let $n \geq 0$.

$$\mathsf{GL}_n(\mathbb{R}) := \Big\{ A \ n\times n \text{ matrix} \,|\, A \text{ is invertible} \Big\} \subset \mathsf{Mat}_{n\times n}(\mathbb{R}) \underset{\text{as column vectors}}{=} (\mathbb{R}^n)^{\times n} = \mathbb{R}^{n^2} .$$

– Let $n \geq 0$.

$$\mathsf{O}(n) := \Big\{ A \ n\times n \text{ matrix} \mid A^T A = \mathsf{id}_{n\times n} \Big\} \subset \mathsf{Mat}_{n\times n}(\mathbb{R}) \underset{\text{as column vectors}}{=} \mathbb{R}^{n^2} .$$

– Let $n \geq 0$.

$$\mathsf{SO}(n) := \Big\{ A \in \mathsf{O}(n) \mid \det(A) > 0 \Big\} \subset \mathsf{Mat}_{n\times n}(\mathbb{R}) \underset{\text{as column vectors}}{=} \mathbb{R}^{n^2} .$$

Namely, we considered the map

$$\det\colon \mathsf{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R} \setminus \{0\} .$$

We noted that this is *continuous*, and that it is *surjecrive*. Therefore, since the codomain is not path-connected, neither is the domain.
Similarly,

$$\det\colon \mathsf{O}(n) \longrightarrow \{\pm 1\}$$

shows that $\mathsf{O}(n)$ is not path-connected.
We discussed/ponderred if $\mathsf{SO}(n)$ is path-connected.

Reading

By Wednesday 25 September. §3. Be especially familiar with these aspects.

- The definition of r-cut-points.

Exercises

These are due by **5pm on Wednesday 25 September**. You can turn your homework in directly to me, or slip it in the slot on the North wall of the Math Department's Main Office. Contact me immediately if you have any questions.

(1) (Optional) Let $G \subset \mathsf{GL}_n(\mathbb{R})$ be a subgroup (with respect to matrix multiplication). Consider the path-component $G_0 \subset G$ containing the identity matrix. Prove that $G_0 \subset G$ is a normal subgroup. Construct a bijection between the set of cosets and the set of path-components:

$$G/G_0 \cong \pi_0(G) .$$

Deduce that $\pi_0(G)$ canonically inherits a group structure.

(2) For which $n \geq 1$ is $\mathsf{SO}(n)$ path-connected? Justify your answer.

(3) Let $r > 0$. Consider the set

$$X_r := \Big\{ (a_1, a_2, \dots, a_r) \in (\mathbb{S}^1)^{\times r} \mid a_i = a_j \text{ only if } i = j \Big\} \subset (\mathbb{S}^1)^{\times r} \subset (\mathbb{R}^2)^{\times r} = \mathbb{R}^{2r} .$$

What is the cardinality of $\pi_0(X_r)$? Justify your answer.

(4) Is there a continuous surjection

$$f\colon \mathbb{S}^2 \times \mathbb{S}^1 \longrightarrow \mathsf{O}(3) \ ?$$

Justify your answer.