Mathematics
An undergraduate notebook

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Preface

> Hold to the now, the here,
> through which all future plunges
> to the past.
>
> Ulysses, James Joyce, Episode 9

I started this notebook my Math 240 (Calculus II) class at Christopher Newport University on Monday 13$^{\text{th}}$ February, 2012. I had picked up the basics of LaTeX in my free hour before class because I wanted to learn how to type mathematical documents. Why? Because LaTeX is cool.

I started to care about math because it serves as the logical foundation for physics. I realized quickly in my studies that I did not know enough math, and I did not know it rigorously enough to truly understand physics. I cared about physics because it is a prerequisite for understanding the basics of computer engineering, my major.

This is now the longest document I've ever written. It has grown to represent a sizable portion of my college education at this time. It's also the first time I've developed a sustainable organizational system for my notes. Everything before this, and everything besides this, lies in stacks of scattered legal pads in at least four different locations.

My goal is to finally organize my thoughts and conceptualize this material in a way I have never even attempted before.

Whatever it takes.

Nathan Typanski

A note to the reader

This text is a work in progress.

Everything in this document is subject to change.

No claim is made as to the accuracy of any of the information contained herein. There may be mistakes, inaccuracies, or outright lies included among otherwise relevant and complete content. Always check with a reputable source (e.g. a math book).

A note on references

Whenever a number is appended to a sentence in brackets ([11], for example), it means the preceeding section has a citation in the bibliography and can be examined in the original source. This is very commonly used in simple tables that are reproduced here, or in sections of text where the material is not different enough from its source text.

The references for this text are not fully established in the official text. There are still a number of places where citations may be missing, or provided only by name and without a complete entry in the bibliography. However, I am regularly going back through this text and completely rewriting sections that have been copied verbatim (for example, a couple of proofs in the appendix are not my work, though I am prudent to say so outright). This is largely because this material is presented in draft form, and many of the citations are provided in the LaTeX source but not yet finalized to the reader. Over time, all instances of this will be adequately removed.

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Part I.

Discrete Mathematics

This is the part of this text which won't focus on numbers. Were it not for specific examples, no numbers would exist in this part altogether. This part sets the stage for the logical theory behind mathematics. We will start with *propositional logic*, a very simple form of logic that establishes the groundwork for logic statements. From there, we will describe *predicate logic*, essentially a more powerful variant of propositional logic. We will then analyze the idea of mathematical *proofs*, eventually working our way toward *set theory*, the groundwork of what most people know as mathematics.

This is your last chance. After this, there is no turning back. You take the blue pill—the story ends, you wake up in your bed and believe whatever you want to believe. You take the red pill—you stay in Wonderland and I show you how deep the rabbit-hole goes.

(Morpheus, The Matrix*)*

1. Propositional calculus

In this chapter we will encounter a system, called *propositional calculus*, for reasoning about things and determining whether they are true or false.

This isn't the same "calculus" as in common usage. Here, "calculus" comes from the Latin word for "pebble", which was once used for counting.[1]

1.1. Propositions and truth

Not every conceivable sentence has a definite truth value. For example, consider:

Shakespeare was a great playwright.

You might disagree with the sentiment—and certainly, for all his fame, Shakespeare's "greatness" is a matter of some subjectivity. On the other hand, we may assert the truth value of certain things:

This sentence is composed of words.

Those sentences to which we can assign a truth value will be called *propositions*. Such that our reason is not muddied by the details of language and subjectivity, we will introduce *variables* to which we may assign these propositions. This will be our first dealing with the tool of mathematical abstraction—the ability to separate the process of reason from the objects of reason, and this will greatly expand our power of thought. We will see that through reason we can find truths far greater and more universal than any individual happening, once sufficient abstraction and logic have laid the groundwork for thought.

It helps to be systematic about things. While intuition is a powerful trick, it only carries us so far: rigor is needed to go the rest of the way. As such, many definitions I will label explicitly.

Definition 1.1 (proposition)**.** A **proposition** is a sentence that makes a statement and has a truth value.

Example 1.1. The following sentence is a proposition:

$$\text{“My house is red.”} \tag{1.1}$$

The way we know that Eq. (1.1) is a proposition is because it has a theoretically knowable truth value.

If we agree that we may assign a symbol to mean the same thing as Eq. (1.1), such that everywhere we write

"My house is red."

we could have just as well written that symbol, then that symbol is called a *propositional variable*. The most formal way we may assert such an assignment is by writing a sentence like the following:

$$\text{Let } p \text{ be "My house is red."} \tag{1.2}$$

Given an assignment like Eq. (1.2), we call the symbol p a **propositional variable**, and the sentence "My house is red" its **value**. We will then write that p is TRUE if my house is, indeed, red—or that p is FALSE if that is not the case.

In most systems of mathematics, there are certain conventions for variables. In propositional calculus, the first variable we will often use is p, followed by q, then r, s, and then whatever feels convenient if we have used up all of those.

Generally speaking, smaller letters mean "lesser" abstraction, so P is more abstract than p, and fancier fonts like $\mathscr{P}$ are more abstract than even P. This is just a rule of thumb, of course, so don't take it for granted.

The simplest thing we can do next is *negation*. If we can propose the truth of some propositional variable p, we just as well may propose the alternative!

Definition 1.2 (negation)**.** Let p be a proposition. The **negation** $\neg p$ of p is the statement "it is not the case that p." The truth value of $\neg p$ is the opposite of the truth value of p.

Before we go further, it's worth mentioning something we're taking for granted: the *Law of Excluded Middle*:

> For the proposition 'it may be' implies a twofold possibility, while, if either of the two former propositions is true, the twofold possibility vanishes. For if a thing may be, it may also not be, but if it is necessary that it should be or that it should not be, one of the two alternatives will be excluded. It remains, therefore, that the proposition 'it is not necessary that it should not be' follows from the proposition 'it may be'. For this is true also of that which must necessarily be.[2]

This axiom, proposed by Aristotle, is still taken to be valid by most mathematicians today—though there is a "minority school of mathematicians," called Intuitionists, who "do precisely that."[11]

Regardless, we will take it to be self-evident that the Law of Excluded Middle holds, and move on with more interesting things.

Example 1.2. Let p be "my house is red." Then $\neg p$ is "it is not the case that my house is red," or simply "my house is not red."

The symbol $\neg$ in definition 1.2 is called a *unary operator*. Operators are things that act on variables or other symbols, called their **operands**. A **unary operator** is an operator that has only one operand.

We also have **binary operators**, like *conjunction*, which work on two operands:

Definition 1.3 (conjunction)**.** Let p and q be propositions. The **conjunction** $p \wedge q$ of p and q, is the proposition "p and q." The conjunction is TRUE when both p and q are TRUE and FALSE otherwise.

The conjunction operator $\wedge$ is an example of a binary operator. The number of operands for an operator is call its **arity**.

Example 1.3. Let p be "Kevin likes Sarah" and let q be "Sarah likes Kevin.' Then $p \wedge q$ is "Kevin likes Sarah and Sarah likes Kevin," or "Kevin and Sarah like each other." This statement would be FALSE if either of the two did not like the other.

Definition 1.4 (disjunction)**.** Let p and q be propositions. The **disjunction** of p and q, written $p \vee q$, is the proposition "p or q." It is TRUE when either p or q are TRUE and FALSE otherwise.

Example 1.4. Let p denote "Kevin hates bagels" and q be the assertion that "Kevin hates poppy seeds." The proposition $p \vee q$ is the statement "Kevin hates bagels or poppy seeds."

Remark. Note, here, that there is an implicit cue in the language that Kevin could, indeed, hate both bagels and poppy seeds. The statement would be TRUE if he were distasteful toward either one of them, or both. In Ex. 1.5 we will see language cues toward the other usage of English *or*.

Definition 1.5 (exclusive or)**.** The **exclusive or** $p \oplus q$ of propositional variables p and q is TRUE when exactly one of p and q is TRUE, and FALSE otherwise.

Example 1.5. Let p mean "I could sleep in all day today" and q mean "I could go to work." Then the statement $p \oplus q$ is TRUE when only one of p or q is TRUE. It is the sentence "I could go to work, or I could sleep in all day today." Since it does not make sense to do both at once, the *exclusive or* is implied by our choice of words.

Remark. We should keep an eye out for this distinction between disjunction and exclusive or, as it requires careful attention to the subtleties in our use of language.

Definition 1.6 (conditional statement)**.** The **conditional statement**

$$p \implies q \tag{1.3}$$

is the proposition "if p, then q." It is FALSE when p is TRUE and q is FALSE, and TRUE otherwise.

In such a conditional statement, we say that q is *necessary* for p, and that p is *sufficient* for q. The intuition here is that if p is TRUE, then q is definitely TRUE, but the truth of q does not imply the truth of p.

Definition 1.7 (hypothesis)**.** In a conditional statement, p is called the **hypothesis** (or *antecedent* or *premise*).

Definition 1.8 (conclusion)**.** In a conditional statement, the variable q is called the **conclusion** or *consequence*.

Other ways of writing the conditional statement are shown in Table 1.1.

Example 1.6. Let p be "Joe broke his arm" and let q be "Joe will go to the hospital."

If one were to state that $p \implies q$, they mean "If Joe broke his arm, then he will go to the hospital." This conditional is TRUE if Joe only goes to the hospital, if he only breaks his arm, and if both events occur. The only situation in which it is FALSE is if Joe breaks his arm, but does not go to the hospital.

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless p"	

Table 1.1.: Other ways of writing the conditional statement $p \implies q$.

Definition 1.9 (converse). The **converse** of $p \implies q$ is $q \implies p$.

Definition 1.10 (contrapositive). The **contrapositive** of (1.3) is the propositional statement

$$\neg q \implies \neg p. \tag{1.4}$$

Theorem 1. The contrapositive statement (1.4) is logically equivalent to the conditional statement (1.3).

Theorem 1 is the first example we have seen of a mathematical statement that asserts something about the mathematical constructs we have presented thus far. When we make such an assertion, it is useful to demonstrate that our assertion holds some clout. To this end, we will introduce the notion of *truth tables*.

Definition 1.11. A **truth table** shows all of the possible values of the propositional variables in at least one propositional statement, and the respective values of those propositional statements when those variables take on such values.

Truth tables can be used to demonstrate the logical relationships between statements and demonstrate some useful property about those relationships.

Proof of Theorem 1. We will prove Theorem 1 using a truth table.

p	q	$p \implies q$	$\neg q \implies \neg p$
FALSE	FALSE	TRUE	TRUE
FALSE	TRUE	TRUE	TRUE
TRUE	FALSE	FALSE	FALSE
TRUE	TRUE	TRUE	TRUE

Table 1.2.: A truth table for $p \implies q$ and $\neg q \implies \neg p$.

In Table 1.2, we have shown that for all possible values of p and q, the conditional statement $p \implies q$ holds the same truth value as the contrapositive statement $\neg q \implies \neg p$. □

This property of contrapositives is especially useful when we are trying to prove a statement, as its contrapositive is often easier to prove than the statement itself. This subject is detailed further in Section 4.2.1.

Definition 1.12 (inverse). The **inverse** of the conditional statement (1.3) is

$$\neg p \implies \neg q, \tag{1.5}$$

and it has the opposite truth value of the original statement.

Definition 1.13. The **biconditional statement**, (also called *bi-implication*) for p and q is the statement that is TRUE when both p and q have the same truth value, and FALSE otherwise. The biconditional is written

$$p \iff q \tag{1.6}$$

and means "p if and only if q."

"p is necessary and sufficient for q"
"if p, then q, conversely"
"p iff q"

In mathematical theorems, definitions, and related elements later in this text, we will often see the biconditional operator written *iff*.

Definition 1.14 (compound proposition). A **compound proposition** is a proposition that can be broken down into simpler propositions.

We will sometimes refer to compound propositions with a capital letter, like P instead of p or Q instead of q, when we wish to emphasize that P and Q

are compound propositions. However, this is just a general rule—the context will usually make it clear which form of proposition we are discussing.

Definition 1.15 (logical equivalence). The compound propositions P and Q are **logically equivalent** if $P \iff Q$ is a tautology. We will write

$$P \equiv Q \tag{1.7}$$

to mean that P and Q are logically equivalent.

Remark. The symbol $\equiv$ is not a logical connection, and $p \equiv q$ is not a compound proposition, but rather it is the statement that $p \iff q$ is a tautology. While $\iff$ is an operator applied to propositional variables, $\equiv$ is an operator on entire propositional statements.

Definition 1.16 (tautology). A **tautology** is a compound proposition that is always TRUE, regardless of the truth values of the variables that occur in it.

Definition 1.17 (contradiction). A **contradiction** is a compound proposition that is always FALSE.

Definition 1.18 (contingency). A **contingency** is a compound proposition that is neither a tautology nor a contradiction. It can be either TRUE or FALSE.

Theorem 2. The biconditional statement (1.6) is logically equivalent to

$$(p \implies q) \land (q \implies p). \tag{1.8}$$

That is,

$$p \iff q \equiv (p \implies q) \land (q \implies p). \tag{1.9}$$

Proof of Theorem 2. We will prove this by truth table.

p	q	$p \implies q$	$q \implies p$	$(p \implies q) \wedge (q \implies p)$	$p \iff q$
FALSE	FALSE	TRUE	TRUE	TRUE	TRUE
FALSE	TRUE	TRUE	FALSE	FALSE	FALSE
TRUE	FALSE	FALSE	TRUE	FALSE	FALSE
TRUE	TRUE	TRUE	TRUE	TRUE	TRUE

Table 1.3.: Truth table proof of Proof of Theorem 2.

Since the last two columns in Table 1.3 have the same truth value, we can be certain that (1.9) holds. $\square$

1.2. Precedence

For more complex propositions, we need rules to tell which logical operators come first when we read them.

The *not* operator ($\neg$) takes the highest precedence. Then conjunction ($\wedge$), followed by disjunction ($\vee$). Then conditional ($\implies$), and finally bi-conditional ($\iff$) operators.

precedence	symbol	meaning
1	$\neg$	not
2	$\wedge$	and
3	$\vee$	or
4	$\implies$	conditional
5	$\iff$	biconditional

Table 1.4.: Logical operator precedence.

Example 1.7. Is the statement $p \wedge q \vee r$ equivalent to the statement

a. $(p \wedge q) \vee r$,

b. or rather, to $p \wedge (q \vee r)$?

Solution.
Let's look at Table 1.4. It states that $\wedge$ ("and") operators come before $\vee$ ("or") operators. Therefore,

$$p \wedge q \vee r \equiv (p \wedge q) \vee r,$$

and (a) is the correct answer.

1.3. DeMorgan's Laws

DeMorgan's Laws are an important concept in propositional logic and boolean algebra.

Definition 1.19 (DeMorgan's Laws). **DeMorgan's Laws** state that:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \tag{1.10}$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \tag{1.11}$$

These can be proven using a **truth table**. In order to construct a truth table, we must display all possible values of the propositional variables, and the corresponding values of the propositional statement for each combination. Intermediary steps may aid one's understanding, though they are not necessary in the final result.

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

Table 1.5.: A proof of DeMorgan's first law.

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

Table 1.6.: A proof of DeMorgan's second law.

Extending DeMorgan's laws by the association laws for disjunction and conjunction, shown in Table 1.8, they become:

$$\neg\left(\bigwedge_{n=1}^{n} p_n\right) = \bigvee_{n=1}^{n} \neg p_n \tag{1.12}$$

$$\neg\left(\bigvee_{n=1}^{n} p_n\right) = \bigwedge_{n=1}^{n} \neg p_n \tag{1.13}$$

Using DeMorgan's Laws, we can *negate conjunctions and disjunctions*. In computer science, we use DeMorgan's Laws to simplify boolean expressions.

Example 1.8. Negate the following statement: "Miguel has a cellphone and he has a laptop."

Solution.
Let p be "Miguel has a cell phone." Let q be "Miguel has a laptop." The negation of $p \wedge q$ is

$$\neg p \vee \neg q,$$

which means "Miguel does not have a cell phone or he does not have a laptop."

1.4. Useful Logical Equivalences

In a proof or simplification of propositional logic statements, propositions like $p \implies q$ are difficult for us to work with. We have very few laws or equivalences which work directly with them, so we often must convert them into an equivalent form using other operators.

We can convert $p \implies q$ to a proposition using only the $\neg$ and $\vee$ operators using the logical equivalence

$$p \implies q \equiv \neg p \vee q, \tag{1.14}$$

which we will prove using a truth table in Table 1.7.

p	q	$p \implies q$	$\neg p \vee q$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

Table 1.7.: A proof of $p \implies q \equiv \neg p \vee q$.

Proposition	Name
$p \wedge T \equiv p$ $p \vee f \equiv p$	identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	idempotent laws
$\neg(\neg p)$	double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	associative laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	distributive laws
$\neg(p \vee q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	DeMorgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	absorbtion laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	negation laws

Table 1.8.: Useful logical equivalence laws.

1.5. Proving Logical Equivalences

We can prove logical equivalences by using the rules in Table 1.8, and showing how one step leads to another until we reach something we know to be TRUE. Alternatively, we could start with a statement that we know to be TRUE and work our way to the logical equivalence we are trying to prove. Both of these proof methods are valid, though not especially rigorous, as they rely upon rules that we may not have established the truth value of with the mathematical rigor required to consider them *formal proofs*.

Example 1.9. Show that $\neg(p \implies q)$ and $p \wedge \neg q$ are logically equivalent.

Solution.

1	$\neg(p \implies q) \equiv \neg(p \implies q)$	
2	$\neg(p \implies q) \equiv \neg(\neg p \vee q)$	$p \implies q \equiv \neg p \vee q$
3	$\neg(p \implies q) \equiv \neg(\neg p) \wedge \neg q$	DeMorgan's Law
4	$\neg(p \implies q) \equiv p \wedge \neg q$	Double negation

Alternatively, they can be proven using truth tables, as described in Section 1.10.

1.6. Propositional Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that make it TRUE. When no such assignments exists, it is **unsatisfiable**. A particular assignment of truth values that shows a compound proposition satisfiable is a **solution**.

To show a compound proposition is satisfiable, we need only demonstrate one possible solution. However, to demonstrate it unsatisfiable, we would need to show every possible assignment of truth values and show why they make it FALSE. Thus, reasoning is often more useful than truth tables in showing a compound proposition is unsatisfiable.

2. Predicates and Quantifiers

Predicates are based on the idea that we can replace parts of our propositions with variables in order to separate our discussion of logic from the irrelevant details of the problem.

We saw in the previous chapter how the specifics of propositions seemed largely irrelevant; most examples were useless gibberish like "my house is red" with all of the focus on the theoretical relationships between these statements.

Predicate logic is a more powerful system of logic which allows us to state "there exists" and "for all" using logic.

2.1. Predicate Logic

Propositional logic is too simple for us to make many types of conclusions. Instead, we use *predicate logic*, which allows us to make general statements about objects and their properties. Predicate logic is a generalization of propositional logic in which variables may be assigned to individual parts of statements, and then we can peroform analysis on the statements in general—instead of in just one specific instance.

Definition 2.1. A **term** is a variable.

Definition 2.2. A **propositional function**, $P(x)$, is a type of **predicate** in predicate logic.

The important thing about propositional functions is that their truth value depends on the value of a variable, x. A propositional function becomes a proposition when a value is assigned to x, and then it has a truth value and we can evaluate it.

Example 2.1.

$$P(n) = \text{“}n \text{ is prime”}$$

Remark. $P(n)$, a propositional function with a truth value, is different from the numerical function $p(n)$. When we talk about functions in the context of propositional logic, we must be careful not to confuse them with their possible numerical counterparts.

2.1.1. Domain of Discourse

Just as values for a variable must be stated in order for a propositional function to have a truth value, a **domain of discourse** must be specified in addition to the universal quantification. This is often referred to as just the *domain* of the function.

For example, for propositional functions talking about numbers, we often assume $D \to \mathbb{R}$.[1]

2.2. Quantification

The **universal quantification** of $P(x)$

$$\forall x P(x) \tag{2.1}$$

is the statement "$P(x)$ for all values of x in the domain."

To show that the universal quantification of $P(x)$ is false for a domain, simply find a single value of x for which $P(x)$ is false.

2.3. Existential Quantification

If we wish to state that an element exists in a domain, we use the *existential quantification* of a propositional function.

The **existential quantification** of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$." We use the notation

$$\exists x P(x)$$

for the existential quantification of $P(x)$.

Note. In order to show that the existential quantification of $P(x)$ is false, we must show that $P(x)$ is false for every possible value of x in the domain.

[1] We use D as shorthand referring to the domain of discourse of a function. $\mathbb{R}$ means "all real numbers." That is, all rational and irrational numbers. It does not include, for example, complex numbers which include i, the "imaginary unit."

2.3.1. Uniqueness Quantifier

A specific case of existential quantification is defined by the **uniqueness quantifier**, $\exists!$ or $\exists_1$. The notation

$$\exists! x P(x)$$

is the statement "There exists a unique x such that $P(x)$ is true." The downside to the uniqueness quantifier is that the rules of inference for existential quantification cannot be used on it. Since propositional logic can be used to express uniqueness already, we should try to avoid use of uniqueness quantification.

To demonstrate uniqueness using propositional logic, we make a statement such as the following:

$$\exists x\Big(P(x) \land \forall y\big(P(y) \implies (x = y)\big)\Big)$$

2.4. Logical Equivalence of Quantified Propositions

In order for two statements involving predicates and quantifiers to be logically equivalent, they must have the same truth value regardless of the values of their propositional variables and the domain of discourse used.

DeMorgan's Laws are an important logical equivalence even when quantified propositions are discussed. As stated in our definition of logical equivalence, they hold regardless of the values of their variables.

2.4.1. DeMorgan's Laws for Quantifiers

DeMorgan's Laws for quantifiers allow us to radically simplify logical expressions involving quantifiers.

$$\neg\exists x P(x) \equiv \forall x \neg P(x) \tag{2.2}$$

$$\neg\forall x P(x) \equiv \exists x \neg P(x) \tag{2.3}$$

Example 2.2. For example, let's take *Euler's conjecture*,[2] first proposed in 1769.[3]

[2] Pronounced "oiler."

[3] Eventually disproved in 1987. Solution at the end of the example.

Let us first define the propositional function $P(a, b, c, d)$.[4]

$$P(a, b, c, d) ::= a^4 + b^4 + c^4 = d^4$$

Now, Euler proposed that there are no positive integers $a, b, c,$ and d such that $P(a, b, c, d)$ is true. We state this by writing

$$E(a, b, c, d) ::= \forall a \in \mathbb{Z}^+ \forall b \in \mathbb{Z}^+ \forall c \in \mathbb{Z}^+ \forall d \in \mathbb{Z}^+ \big(\neg P(a, b, c, d)\big).$$

Let's break this apart. The "$a \in \mathbb{Z}^+$ is used to describe our *domain of discourse*. $\mathbb{Z}^+$ refers to the set of all positive integers. In general use, we can simplify this statement by writing

$$E(a, b, c, d) ::= \forall a, b, c, d, \in \mathbb{Z}^+ \big(\neg P(x)\big)$$

but for our purposes, we want to work with the original proposition, because we wish to use DeMorgan's Laws on it.

Using DeMorgan's first law for quantifiers, equation (2.2), we can change the last part of this proposition:

$$\forall d \in \mathbb{Z}^+ \big(\neg P(a, b, c, d)\big) \equiv \neg \exists d \in \mathbb{Z}^+ \big(P(a, b, c, d)\big)$$

Now, we continue up the chain, reversing each of the negated statements as if everything to the right of the negation sign were one single proposition. Here's our new statement:

$$E(a, b, c, d) :: = \forall a \in \mathbb{Z}^+ \forall b \in \mathbb{Z}^+ \forall c \in \mathbb{Z}^+ \neg \exists d \in \mathbb{Z}^+ \big(P(a, b, c, d)\big)$$

Now, continuing DeMorgan's Laws,

$$E(a, b, c, d) :: = \forall a \in \mathbb{Z}^+ \forall c \in \mathbb{Z}^+ \neg \exists c \in \mathbb{Z}^+ \exists d \in \mathbb{Z}^+ \big(P(a, b, c, d)\big)$$
$$E(a, b, c, d) :: = \forall a \in \mathbb{Z}^+ \neg \exists c \in \mathbb{Z}^+ \exists c \in \mathbb{Z}^+ \exists d \in \mathbb{Z}^+ \big(P(a, b, c, d)\big)$$

Arriving finally at

$$E(a, b, c, d) :: = \neg \exists a \in \mathbb{Z}^+ \exists c \in \mathbb{Z}^+ \exists c \in \mathbb{Z}^+ \exists d \in \mathbb{Z}^+ \big(P(a, b, c, d)\big),$$

which is logically equivalent to the original $E(a, b, c, d)$ we proposed.

This shows that if just one of the variables in $E(a, b, c, d)$ cannot be said to exist, then the entire proposition becomes false.

It turns out, in contrast to *Euler's conjecture*, a solution to $P(a, b, c, d)$ can be

[4] $::=$ is used to mean "equals by definition," and is sometimes used in order to contrast with regular "equals."

found. With the values $a = 95800$, $b = 217519$, $c = 414560$, and $d = 422481$, $P(a, b, c, d)$ is true.

2.5. Order of Quantifiers

Assuming a domain of discourse of all real numbers, the quantification

$$\exists y \forall x Q(x, y) \tag{2.4}$$

denotes the proposition "There is a real number y such that for every real number x, $Q(x, y)$."

By contrast, the quantification

$$\forall x \exists y Q(x, y) \tag{2.5}$$

states that "For every real number x there is a real number y such that $Q(x, y)$."

3. Rules of Inference

> I don't want to believe. I want to know.
>
> — Carl Sagan

Proofs are used to establish the truth of mathematical statements. In order to make a proof, we must use the **rules of inference** to establish the truth of more complicated logical arguments. An **argument** is a sequence of propositions that ends with a conclusion. A **valid** argument is one in which the last proposition follows from those propositions before it.

When we are writing mathematical proofs, it's not common to actually cite the rules of inference in our text. However, they should form the logical connectors between the claims we make in our proofs and should be present in the implicit form. Becoming familiar with these rules, and how to use them, will allow us to both write more coherent proofs and to avoid logic errors in our writing.

3.1. Rules of Inference for Propositions

We will present the rules of inference using a variant of *Fitch diagrams*. Each step in a Fitch diagram includes a number for the step, a proposition or conclusion, and a justification for the step.

3.1.1. *Modus ponens*

1	p	assumption
2	$p \to q$	assumption
3	q	$\big(p \wedge (p \to q)\big) \to q$

Another way you might see this written is

$$\begin{array}{ll} 1. & p \to q \\ 2. & p \\ \hline \therefore & q \end{array}$$

where the three dots ($\therefore$) is read as "therefore". We will usually try to stick with the fitch diagrams.

Modus ponens is Latin for "mode that affirms," and comes from the tautology $\big(p \wedge (p \rightarrow q)\big) \rightarrow q$. It is the simplest valid **argument**, a sequence of statements that ends with a conclusion.

3.1.2. *Modus tollens*

1	$\neg q$	assumption
2	$p \rightarrow q$	assumption
3	$\neg p$	$\big(\neg q \wedge (p \rightarrow q)\big) \rightarrow \neg p$

3.1.3. Hypothetical syllogism

1	$p \rightarrow q$	assumption
2	$q \rightarrow r$	assumption
3	$p \rightarrow r$	$\big((p \rightarrow q) \wedge (q \rightarrow r)\big) \rightarrow (p \rightarrow r)$

A hypothetical syllogism is sometimes thought of as *double modus ponens*.

3.1.4. Disjunctive syllogism

1	$p \vee q$	assumption
2	$\neg p$	assumption
3	q	$(p \vee q) \wedge \neg p \rightarrow q$

3.1.5. Addition

1	p	assumption
2	$p \vee q$	$p \rightarrow (p \vee q)$

3.1.6. Simplification

1	$p \wedge q$	assumption
2	p	$(p \wedge q) \rightarrow p$

3.1.7. Conjunction

1	p	assumption
2	q	assumption
3	$p \wedge q$	$\big((p) \wedge (q)\big) \to (p \wedge q)$

3.1.8. Resolution

1	$p \vee q$	assumption
2	$\neg p \vee r$	assumption
3	$q \vee r$	$\big((p \vee q) \wedge (\neg p \vee r)\big) \to (q \vee r)$

3.2. Rules of Inference for Quantified Statements

3.2.1. Universal Generalization

Universal generalization states that given $P(c)$ for all elements c in the domain, $\forall x P(x)$ is true.

1	$P(c)$ for some arbitrary c	assumption	(3.1)
2	$\forall x P(x)$	universal generalization	

3.2.2. Universal Instantiation

Universal instantiation states that given $\forall x P(x)$, $P(c)$ is true for a particular element c in the domain.

1	$\forall x \big(P(x) \to Q(x)\big)$	proposition	(3.2)
2	$P(a)$	universal instantiation	

3.2.3. Existential Generalization

Existential generalization concludes that, given a particular element c for which $P(c)$ is known to be true, $\exists x P(x)$.

3.2.4. Existential Instantiation

Existential instantiation states that if $\exists x P(x)$ is true, $P(c)$ for some element c.

3.2.5. Universal *Modus Ponens*

Universal *modus ponens* combines universal instantiation (Section 3.2.2) and *modus ponens* (Section 3.1.1) to tell us that if $\forall x(P(x) \to Q(x))$ is true, and if $P(a)$ is true for a particular element a in the domain of the universal quantifier, then $Q(a)$ must also be true.

$$\begin{array}{r|l} 1 & \forall x(P(x) \to Q(x)) \\ 2 & P(a), \text{ where } a \text{ is a particular element in the domain} \\ 3 & Q(a) \end{array} \tag{3.3}$$

3.2.6. Universal *Modus Tollens*

Universal *modus tollens* states that

$$\begin{array}{r|l} 1 & \forall x(P(x) \to Q(x)) \\ 2 & \neg Q(a), \text{ where } a \text{ is a particular element in the domain} \\ 3 & \neg P(a) \end{array} \tag{3.4}$$

4. Proofs

We use proofs to establish the truth of mathematical statements. There are a number of types of these statements, of varying importance.

Before we prove our statement, it is called a **conjecture**.

A **fact** or **result** is the simplest type of statement we might prove. Sometimes proofs are required, but more often they are simply demonstrated, implied, or obvious.

Theorems are where proofs start to become important. They are the foundations of complex mathematical logic. Some theorems are less important than others, and we call these **lemmata**. A **lemma** is a less important theorem, often used in proving our main theorem. After our theorem is proven, we can often draw **corollaries**, theorems that follow easily from our proven theorem.

Some proofs are **formal proofs**, which use the rules of inference to establish truth. These proofs are the "simplest" in that they involve the simplest forms of logic, though they can become incredibly complex as we attempt to write formal proofs for more and more complex theorems.

Thus, we most often will encounter **informal proofs**. These proofs establish truth using language and logic that makes sense to humans.

4.1. Direct Proof

A **direct proof** starts with known facts, and the statement we are trying to prove follows directly from them.

In using a direct proof for proving an implication

$$P \implies Q$$

we start by assuming P is true, then using the *rules of inference* (Chapter 3), reach the conclusion Q through direct logical steps from P.

Example 4.1. Give a direct proof of the theorem

"If $n + 1$ is an odd integer, then $n + 3$ is an odd integer."

Proof. If $n + 1$ is an odd integer, then we can write it in the form

$$n + 1 = 2k_1 + 1$$

Now, we subtract 1 from both sides.

$$n = 2k_1$$

Which shows that n is even. If we add 2 to both sides,

$$n + 2 = 2k_1 + 2$$

An even number plus 2 is still even, so we could write $2k_1 + 2$ as $2k_2$.

$$n + 2 = 2k_2$$

Now we add 1 to each side to reach our conclusion.

$$n + 3 = 2k_2 + 1$$

Any number in the form $2k + 1$ is odd. We have proven that if $n + 1$ is an odd integer, then $n + 3$ is also an odd integer. □

4.2. Indirect Proof

An **indirect proof** is, basically, any proof that is not a *direct proof*. There are a variety of ways of accomplishing a proof that are not direct proofs, and many of them are dramatically simpler or easier to understand than the direct proof for a given theorem would be.

4.2.1. Proof by Contrapositive

The first method of indirect proof we will discuss is the **proof by contrapositive**. It is often more useful to work with the contrapositive something than the original statement.

Remembering our definition of the contrapositive from Section 1.1, let's cover some examples.

Definition 4.1. The **contrapositive** of the statement $P \to Q$ is the statement $\neg Q \to \neg P$.

Example 4.2. Find the contrapositive of the following:

> "If it is snowing then it is cold."

Solution.
Assign propositional variables to both of the statements.

> Let s be "it is snowing." Let c be the statement "it is cold." The statement we are trying to prove is
>
> $$P = s \to c.$$
>
> Its contrapositive, therefore, is
>
> $$C = \neg c \to \neg s$$
>
> which is the statement "if it is not cold then it is not snowing." This is logically equivalent to our original statement.
>
> $$P \equiv C$$

Example 4.3. Prove the following, by contrapositive:

> "If a number squared is odd, then the number itself is odd."

Proof. We will use proof by contraposition.[1]

> Let $S(x)$ be "x squared is odd."
>
> Let $O(x)$ be "x is odd."

In mathematical language, this is[2]

$$S(x) = x^2 = 2k_2 + 1$$
$$O(x) = x = 2k_1 + 1$$

where k is any integer.

So we are trying to prove
$$S(x) \to O(x).$$

The contrapositive of this statement is

$$\neg O(x) \to \neg S(x).$$

Now, remembering that all even integers can be expressed in the form

[1] Same thing as proof by contrapositive (definition 4.1).
[2] $2k + 1$ is the simplest mathematical expression for a generic odd integer.

$n = 2k$, let's find the negations of each of our propositional functions.

$$\neg S(x) = x^2 = 2k_2$$
$$\neg O(x) = x = 2k_1$$

So the statement we are trying to prove is

$$\neg O(x) \implies \neg S(x)$$
$$x = 2k_1 \implies x^2 = 2k_2$$

From the first proposition,

$$x = 2k_1$$

Square both sides of the equation.

$$x^2 = 2k_1^2$$
$$x^2 = (2k_1)(2k_1)$$
$$x^2 = 4k_1$$

Because all even integers are divisible by 2, $4k_1$ could also be written as $2k_2$.

$$x^2 = 2k_2$$

which is the conclusion we were trying to reach. From this, we can conclude that

$$\neg O(x) \to \neg S(x)$$

and therefore, by contrapositive,

$$S(x) \to O(x).$$ □

4.3. Proof by Contradiction

To use **proof by contradiction** to prove a proposition P, we start by assuming P is false. From this, we derive a logical contradiction.

This might sound strange, but it makes sense: if we can prove that something not being true is a violation of logical rules, then we have just proven that the thing must exist.

In other words, if we can prove that $\neg P \implies R \wedge \neg R$, then since $R \wedge \neg R$ is obviously false, we conclude that P must be true.

Example 4.4. Prove the following:

There is no largest odd number.

Proof. Let P be the statement "There is no largest odd number." Then $\neg P$ is the statement "There is a largest odd number." Call this largest odd number x. We know that odd numbers are of the form $2k+1$. Since $2x+1 > x$, and $2x+1$ is odd, $\neg P$ must be false. Therefore, P is true: there is no largest odd number. $\square$

5. Sets

A **set** is an unordered collection of objects, called its *elements*. We generally write sets in capital letters, such as A, to contrast them from the elements contained in them.

We write $a \in A$ to state that a is an element of the set A.

There are a couple ways we could define sets. One of them is to write out all of the elements, like:

$$A = \{n, z, k, d\},$$

or to write out a few of them, creating a pattern which is obvious:

$$B = \{1, 2, 3, 4, \ldots 999\}.$$

The more common way is to use **set builder notation**. This associates all elements of a set with a certain propositional function, which is true for all elements of the set. We construct sets using this method as follows:

$$S = \left\{x \in \mathbb{R} \middle| P(x)\right\}$$

This constructs a set S which contains all real numbers x for which $P(x)$ is true. The set $\left\{x \middle| P(x)\right\}$ is called the **extension** of the predicate P. Some important sets in mathematics are listed in Table 5.1.

$\mathbb{N}$	the set of *natural numbers*
$\mathbb{Z}$	the set of *integers*
$\mathbb{Z}^+$	the set of *positive integers*
$\mathbb{Q}$	the set of *rational numbers*
$\mathbb{R}$	the set of *real numbers*
$\mathbb{R}^+$	the set of *positive real numbers*
$\mathbb{C}$	the set of *complex numbers*

Table 5.1.: Important sets.

5.1. Properties of Sets

What are some of the properties of sets? How do we compare them? How do we write about these relationships? These are the questions that are answered in this section.

Two sets are **equal** iff[1] they have the same elements. Therefore, if A and B are sets, then they are equal iff

$$\forall\big(x \in A \iff x \in B\big).$$

In English, two sets A and B are equal if and only if every element in a implies it also exists in B, and vice versa. We then write $A = B$ to show that they are equal sets.

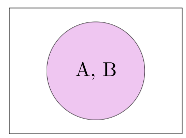

Figure 5.1.: $A = B$.

We have two ways to write an **empty set**. We could write $\emptyset$ or $\{\}$.

A set with only one element is called a **singleton set**.

The set A is a **subset** of set B iff

$$\forall x(x \in A \implies x \in B)$$

We then write $A \subseteq B$ to show that A is a subset of B. To demonstrate that A is a subset of B, show that if x belongs to A then it also belongs to B. To show that A is not a subset of B, that is, $A \nsubseteq B$, find a single element $x \in A$ such that $x \notin B$.[2]

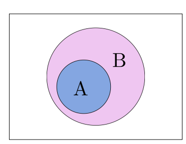

Figure 5.2.: $A \subseteq B$

If the set A is a subset of the set B but $A \neq B$, we write $A \subset B$ and say that

[1] "If and only if," see Section 1.1.
[2] The symbol $\notin$ means *not in*.

A is a **proper subset** of B. That is, A is a proper subset of B iff

$$\forall x(x \in A \implies x \in B) \wedge \exists x(x \in Bx \notin A).$$

This means, if the element x exists in set A, then it also exists in set B, but there is at least one element x in B that is not in A.

5.1.1. Set Operations

For a set A and a set B, the following operations apply:

The **cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Hence,

$$A \times B = \big\{(a, b) | a \in A \wedge b \in B\big\}.$$

Example 5.1. Find $A \times B$ when $A = \{1, 2\}$ and $B = \{a, b, c\}$.

Solution.

$$A \times B = \big\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\big\}.$$

The **intersection** of A and B, denoted $A \cap B$, is the set containing those elements in both A and B, but not just one. If the intersection of A and B is $\emptyset$, we say the two sets are **disjoint**:

$$A \cap B = \big\{x \in A | x \in B\big\}.$$

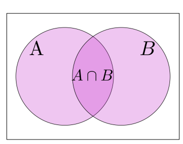

Figure 5.3.: $A \cap B$

The **union** of A and B, denoted $A \cup B$, is the set containing all elements in either or both A and B:

$$A \cup B = \big\{x | x \in A \vee x \in B\big\}.$$

Proposition	**Name**
$A \cap U = A$ $A \cup \emptyset = A$	identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	domination laws
$A \cup A = A$ $A \cap A = A$	idempotent laws
$\left(A^c\right)^c = A$	involution law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cup C$	associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	distributive laws
$\left(A \cup B\right)^c = A^c \cap B^c$ $\left(A \cap B\right)^c = A^c \cup B^c$	DeMorgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	absorbtion laws
$A \cup A^c = U$ $A \cap A^c = \emptyset$	complement laws

Table 5.2.: Useful set identities.

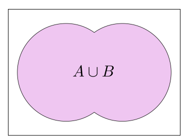

Figure 5.4.: $A \cup B$

The **complement** of A and B, denoted $A \setminus B$, is the set containing those elements in A but not in B:

$$A \setminus B = \left\{x \in A \middle| x \notin B\right\}.$$

It is sometimes called the *component of B with respect to A*. This phrasing is used because the difference of A and B is different from the difference of B and A.

The **absolute complement** of A is denoted A^c, and exists only if a universal set[3] $\mathbf{U}$ is defined, and is found by

$$A^c = \mathbf{U} \setminus A.$$

[3]A set containing all sets, including itself.

5.2. The Well-Ordering Principle

The **well-ordering principle** states that

Theorem 3. Every nonempty set of nonnegative integers has a smallest element.

We can use this to prove propositional statements associated with universal quantifiers.

To prove that

$$\forall n \in \mathbb{N}\big(P(n)\big)$$

First, we define a set C such that

$$C ::= \left\{n \in \mathbb{N}^{+} \middle| \neg P(n)\right\}$$

That is, we define the set C to be the set of integers for which $P(n)$ is false. We assume that this set is nonempty—that is, there is at least one nonnegative integer for which $P(n)$ is false. By the well-ordering principle, there is a smallest element, n, in C. From this, we reach a contradiction. For example, we could show that n can be used to find an element smaller than itself. Now, we conclude that the set C must be empty.

Example 5.2. Prove that the sum of all integers from 1 to n is

$$\frac{n(n+1)}{2}.$$

In basic algebraic notation, we could write this sum as

$$1 + 2 + 3 + \cdots + n$$

But we have a way of simplifying this using something called **sigma notation**. We would write it as follows:

$$1 + 2 + 3 + \cdots + n \sum_{1}^{n} \frac{n(n+1)}{2}$$

Proof. Now, we define a set C such that

$$C ::= \left\{n \in \mathbb{N} \middle| \sum_{i=1}^{n} i \neq \frac{n(n+1)}{2}\right\}$$

By the *well-ordering principle*, C has a smallest element, which we will call

c. Since c is the smallest counterexample,

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2}$$

holds for all $n < c$ but not for $n = c$. So, that sum should hold for $c - 1$:

$$\begin{aligned}
1 + 2 + 3 + \cdots + (c-1) &= \frac{(c-1)\big[(c-1)+1\big]}{2} \\
1 + 2 + 3 + \cdots + (c-1) &= \frac{(c-1)c}{2} \\
1 + 2 + 3 + \cdots + (c-1) &= \frac{c^2 - c}{2}
\end{aligned}$$

Now, we add c to both sides.

$$1 + 2 + 3 + \cdots + (c-1) + c = \frac{c^2 - c}{2} + c$$

Simplify.

$$\begin{aligned}
1 + 2 + 3 + \cdots + (c-1) + c &= \frac{c^2 - c}{2} + (c)\left(\frac{2}{2}\right) \\
1 + 2 + 3 + \cdots + (c-1) + c &= \frac{c^2 - c + 2c}{2} \\
1 + 2 + 3 + \cdots + (c-1) + c &= \frac{c^2 + c}{2}
\end{aligned}$$

Factor out c from the right-hand side.

$$1 + 2 + 3 + \cdots + (c-1) + c = \frac{c(c+1)}{2}$$

Which means that

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2}$$

holds for c, which contradicts our definition of C. Therefore $P(n)$ holds for all $n \in \mathbb{N}$. $\square$

[6]

A few theorems come from the *well-ordered principle*:[4]

Theorem 4. Any set of integers with a lower bound is well-ordered.

[4]Proofs will come. From [6, 29].

Theorem 5. Any nonempty set of integers with an upper bound has a maximum integer.

6. Recursion

6.1. Recursive Definitions

Definition 6.1. **Recursive form** defines a set, an equation, or a process by defining a starting set or a value and giving a rule for continuing to build the set, equation, or process based on previously defined terms.

The key for recursion is the *rule for continuing to build* the set, equation, or process. This is what allows us to do the new element, new equation, or new process based on previously defined terms.

A recursive definition has two parts.

Definition 6.2. In the **basis step**, we must define values for some finite number of elements. For sets, we state the *basic building blocks* of the set. for functions, state the values of the function on the basic building blocks.

Definition 6.3. The remaining elements in the recursive definition are defined by the **recurrence relation**. For sets, we show how to build new things from the old with some basic construction rules. For functions, we show how to compute the value of a function on the new elements of that set.

6.1.1. Recursively Defined Functions

Let us create a recursive definition of the function F, defined on nonnegative integers. To give a recursive definition of F:

1. *Basis*. Specify F(0).
2. *Recursive step*. Give a rule for defining $F(n+1)$ from F evaluated at smaller values.

Example 6.1.

$$f(0) = 1$$
$$f(n) = f(n-1) + 2$$

Example 6.2.

$$g(0) = 1$$
$$g(k+1) = g(k) + 2$$

Example 6.3.

$$a_0 = 1$$
$$a_n = a_{n-1}$$

Example 6.4. Find the recursive form of $n!$, the function given by

$$n! = \prod_{k=1}^{n} k \tag{6.1}$$

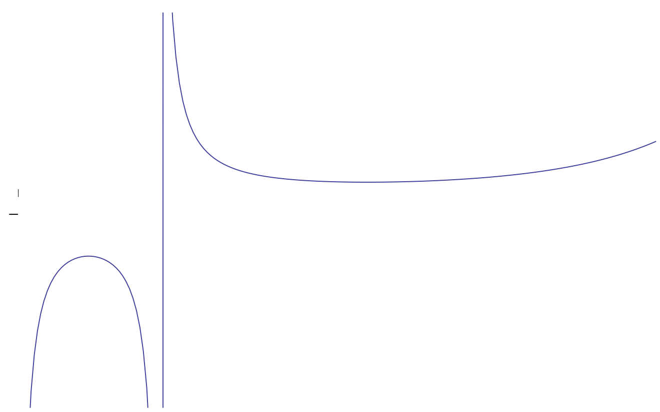

Figure 6.1.: A plot of $n!$. Its behavior is much harder to describe in the negatives, so we normally just treat it as having a domain of $n \geq 0$.

Solution.
The basis step in either the *closed form* or *recursive form* definition for $n!$ is that $0! = 1$. In equation (6.1), it is implied under the convention that the product of no numbers at all is one[1]

[1]This is called the **empty product** or **nullary product**, and is responsible for providing the

So in order to define a recursive form for $n!$, we must start with the definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \end{cases} \tag{6.2}$$

Now that we have the basis step, to get the *recursive step* we will look at a few instances of the factorial function:

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1 \\ f(2) &= 2 \\ f(3) &= 6 \\ f(4) &= 24 \\ f(5) &= 120 \\ &\vdots \end{aligned}$$

If we are careful, we'll notice that we can factor a n from our result on each instance.

$$\begin{aligned} f(1) &= 1 \cdot 1 \\ f(2) &= 2 \cdot 1 \\ f(3) &= 3 \cdot 2 \\ f(4) &= 4 \cdot 6 \\ f(5) &= 5 \cdot 24 \\ &\vdots \end{aligned}$$

We notice that $f(n)$, for any $n > 1$, is given by n times the term before it. By writing this out, we get our *recursive definition for factorials*.

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \times f(n-1) & \text{if } n > 0 \end{cases} \tag{6.3}$$

As is the case with factorials, *recursive form* often offers the advantage that it is very intuitive for humans to understand. Its downside is that it is very seldom computationally faster than its *closed-form* alternative. For this reason, we should attempt to find closed-form solutions to recursive definitions where possible or necessary.

Generally speaking, given a recursive function on a test, we should be able to find a closed-form representation and vice-versa.

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multiplicative identity 1.

Example 6.5. Find a recursive definition of the **Fibonacci sequence**:

$$1, 1, 2, 3, 5, 813, 21, 34, \ldots$$

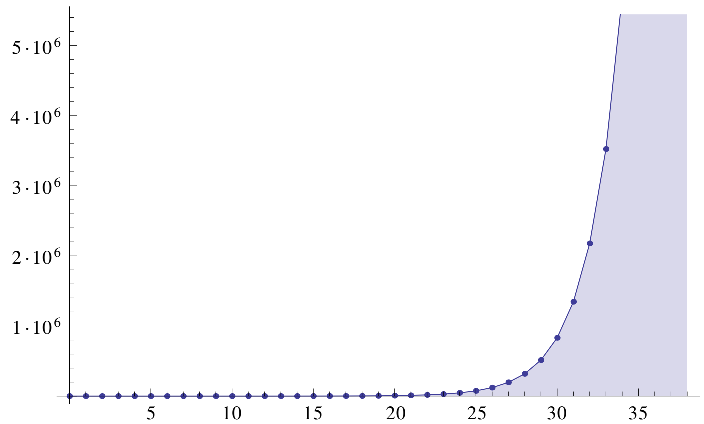


Figure 6.2.: A plot of the Fibonacci sequence.

The Fibonacci sequence is often explained using the analogy of rabbits on an island.

> "A young pair of rabbits (one for each sex) is placed on an island. After they are 2 months old, each pair of rabbits produces another pair each month. The number of pairs of rabbits after n months is $f(n)$."

Solution.
Notice that we need **two** initial conditions to define this recurrence relation.

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ f(n) + f(n-1) & \text{for } x > 1 \end{cases} \tag{6.4}$$

Note. This definition requires two initial conditions. It is very important in recursive definitions to have the right number of initial conditions.

7. Counting

The basis of much of counting is the idea of the *binomial coefficient*.

Definition 7.1. The **binomial coefficient** of n and k, read "n choose k," is written

$$\binom{n}{k}$$

and refers to the number of subsets with k elements that we could find for a set of n elements.

We will often see these with the binomial formula,

$$(a+b)^n = \sum_{n=0}^{\infty} a^{n-k} b^k. \tag{7.1}$$

Part II.

Mathematical Analysis

Here we are going to study functions, series, sequences, and applications of these concepts. The numbers we will find this section are almost always going to be *real numbers*, which basically means they behave the way we expect numbers to behave and we are able to plot them on number lines, graph them, and all that fun stuff.

Most of this part is going to focus on what schools typically call "calculus," and it covers the mathematics I have learned in my first three semesters as an undergraduate at Christopher Newport University.

8. Algebra

In order to understand functions, we need to understand basic algebra. It will give us a powerful set of tools that we can use to solve problems down the road, like partial fraction decomposition (Section 14.8).

8.1. Laws

If a, b, and c are any numbers,[1]then the following laws hold:[2]

8.1.1. Associative law for additon

The associative law extends our ability to discuss the operation $(+)$ on any two elements to three elements, without changing the order of these elements:

$$a + (b + c) = (a + b) + c. \tag{8.1}$$

It follows from this (though the proof is somewhat complicated, see [10, p. 4]), that we may write sums without regard for parentheses. This means that we may write, for instance

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n,$$

without any ambiguity as to what order the operation must be performed.

8.1.2. Existence of an additive identity

The identity element for addition is 0. This means that the sum of any element and 0 is always the original element. We write this:

$$a + 0 = 0 + a = a. \tag{8.2}$$

[1] I am sure these all hold for real numbers, and presumably for complex as well, though other number systems may have different laws. I have not explored these possibilities.

[2] Most of this is from the opening chapter of [10], but bits and pieces are collected from elsewhere and cited as such.

8.1.3. Existence of additive inverses

$$a + (-a) = (-a) + a = 0. \tag{8.3}$$

In this case, we mean that every element in the set $\mathbb{R}$ of real numbers has an inverse with respect to the operation $(+)$.[8, p. 14]

8.1.4. Commutative law for addition

This states that the value of $a + b$ or $b + a$ is independent of the order in which a and b are taken.[8, p. 14]

$$a + b = b + a. \tag{8.4}$$

8.1.5. Associative law for multiplication

The associative law for multiplication is analagous with the one for addition from Section 8.1.1.

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c. \tag{8.5}$$

8.1.6. Existence of a multiplicative identity

Multiplication of real numbers has an identity element, 1, such that multiplying any number by this element gives us the original number:

$$a \cdot 1 = 1 \cdot a = a, \qquad \text{for } 1 \neq 0. \tag{8.6}$$

The notation here is a little strange. We know that 1 is the identity element for multiplication, but it also refers to the number 1, so why do we state that $1 \neq 0$? Of course one is not equal to zero!

The reason for this is that we are talking about the *element* 1, this being the identity element for multiplication, and not simply the *number* 1. We may just as well have written:

$$a \cdot e = e \cdot a = a, \qquad \text{for } e \neq 0,$$

but writing 1 instead of e as in Eq. (8.6) here makes sense, since 1 is, in fact, both the number and the element in question.

8.1.7. Existence of multiplicative inverses

For every element a in $\mathbb{R}$, there is an element a^{-1} in $\mathbb{R}$ such that $a \cdot a^{-1}$ gives us the identity element from Section 8.1.6.

$$a \cdot a^{-1} = a^{-1} \cdot a = 1, \qquad \text{for } a \neq 0. \tag{8.7}$$

8.1.8. Commutative law for multiplication

As with the commutative law for addition (Section 8.1.4), this states that the value of $a * b$ or $b * a$ is independent of the order in which a and b are taken.[8, p. 14]

$$a \cdot b = b \cdot a. \tag{8.8}$$

8.1.9. Distributative law

The distributive law is a relationship between multiplication and addition. It allows us to manipulate the order of application when we are combining these two operations.

$$a \cdot (b + c) = a \cdot b + a \cdot c. \tag{8.9}$$

8.2. Inequality

When we say a is *less than* b, we write $a < b$, and take it to mean the same thing as saying b is *greater than* a ($b > a$).[10, p. 9] Thus the numbers a satisfying $a > 0$ are called *positive*, while those numbers a satisfying $a < 0$ are called *negative*.

8.3. More laws

Theorem 6 (Trichotomy law)**.** For every number a, one and only one of the following holds:

1. $a = 0$,
2. a is in the collection P,
3. $-a$ is in the collection P.

[10, p. 9]

Theorem 7 (Closure under addition)**.** If a and b are in P, then $a + b$ is in P. [10, p. 9]

Theorem 8 (Closure under multiplication)**.** If a and b are in P, then $a \cdot b$ is in P. [10, p. 9]

Definition 8.1. For any number a, we define the *absolute value* $|a|$ of a to be:[10, p. 11]

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0. \end{cases} \tag{8.10}$$

9. Functions

Definition 9.1 (function). A **function** f from a set D to a set R is an assignment of exactly one element $f(x) \in R$ to each element $x \in D$. If f is a function from D to R, we write

$$f : D \to R, \tag{9.1}$$

and say that f *maps* D to R. A function has four properties:

1. A *domain* D and a *codomain* R, both sets,
2. For all x in D, there exists some corresponding $f(x)$ in R which we call the *value* of the function f *at* x,
3. D, R, and each value $f(x)$ for each element x are determined completely by f,
4. The data constituting D, R, and the value $f(x) \forall x \in D$ completely determine f.

One technique that can help us understand functions is by describing them with words, including mathematical language. We can discuss the behavior of a function in general, describing its properties, or we can discuss its behavior in more specific terms, like when we evaluate a function at a certain value.

We can also talk about functions using visual language, by way of plots, arrow diagrams (Figure 9.1), and tables.

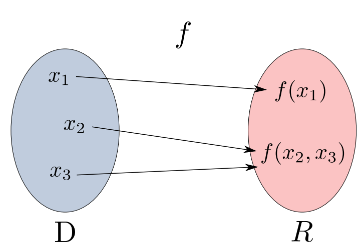


Figure 9.1.: A function f mapping from D to R. Note how multiple x values share the same $f(x)$ value.

We will combine these techniques in an effort to paint a complete picture of its behavior.

9.1. Properties of Functions

Here, we will discuss a number of the properties that will be of interest when we are describing functions. Not all of these properties will exist for every type of function. For example, not all functions will have an inverse (described in Section 9.3), but the nonexistence of an inverse then becomes an important property of the function.

9.1.1. Domain and Range

Definition 9.2 (domain). The set D of all possible input values on which a function f is defined is called the **domain** of f.

Definition 9.3 (range). Given a function f defined on a domain D, the set of all values of $f(x)$ as x varies throughout D is called the **range** or *codomain* R of f.

In certain areas of mathematics, including calculus, it is common to define functions without supplying an explicit domain. Instead, we will implicitly use the *natural domain* in these cases.

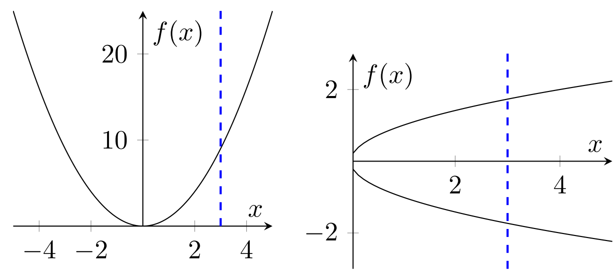
Definition 9.4 (natural domain). For a function $f : D \to R$, where $f(x)$ is the value of f at x, the **natural domain** is the set of all values

$$\{x \mid f(x) \in \mathbb{R} \land x \in D \cap \mathbb{R}\}. \tag{9.2}$$

Definition 9.5 (real-valued). A function is said to be **real-valued** when its domain is exactly equal to the set of $\mathbb{R}$ of real numbers.

That is, the domain is equal to to the set of real numbers. This means that we can put an arbitrary real number into the function and it will return a real y-value.

The **vertical line test** for a function is based on the idea that if a is in the domain of the function f then the vertical line $x = a$ will intersect the graph of f at a single point $\big(a, f(a)\big)$.



(a) This passes the vertical line test (b) This fails the vertical line test

Figure 9.2.: Examples of the vertical line test.

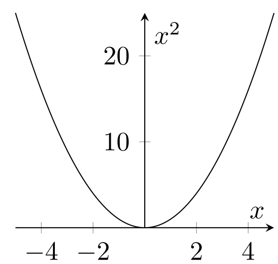


Figure 9.3.: $f(x) = x^2$ is an even function.

9.1.2. Dependent and Independent Variables

The letter x in the notation $y = f(x)$ is called the **independent variable** of the function, representing the input value of f.

The letter y in the notation $y = f(x)$ is called the **dependent variable**. It varies with respect to change in the dependent variable of the function.

9.1.3. Even and Odd Functions

For a function f in the form $y = f(x)$, we describe its type of symmetry by calling the function **even** or **odd**.

Definition 9.6 (even function)**.** An **even function** f is one where the property $f(-x) = f(x)$ holds for all values in the domain.

Example 9.1. An example of an even function is the function f, where $f(x) = x^2$. A plot of this function is shown in Figure 9.3.

Definition 9.7 (odd function)**.** An **odd function** f is one where the prop-

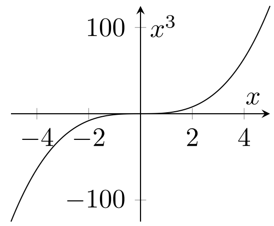
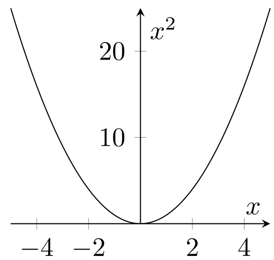
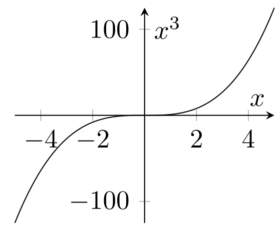


Figure 9.4.: $f(x) = x^3$ is an odd function.



(a) The function $f(x) = x^2$ is not *one-to-one* because there are two possible x-values that can produce each given y-value.



(b) The function $f(x) = x^3$ is *one-to-one* because every given y-value is mapped from a unique x-value.

erty $f(-x) = -f(x)$ holds for all values in its domain.

Example 9.2. An example of an odd function is $f(x) = x^3$. A plot of this function is shown in Figure 9.4.

9.1.4. Surjective, Injective, and Bijective Functions

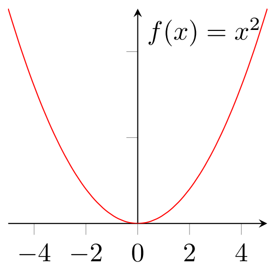
Definition 9.8 (injective)**.** If each $f(x)$ value produced by a function f can only be obtained by one unique x value, then we say f is **injective**, or *one-to-one*.

$f : D \to R$ is injective or one-to-one iff

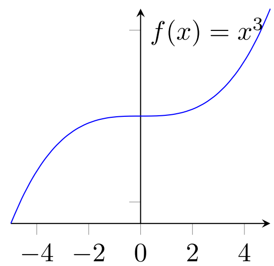
$$\forall (x_1 \wedge x_2 \in D)\big[f(x_1) = f(x_2) \to x_1 = x_2\big].$$

Remark. This also means that for injective functions, $x_1 \neq x_2 \to f(x_1) \neq f(x_2)$.

A function $y = f(x)$ is one-to-one iff its graph intersects each horizontal line



(c) The function $f(x) = x^2$ is not *surjective* because the values $(-\infty, 0)$ are never reached in its range.



(d) The function $f(x) = x^3$ is surjective because all y values from $(-\infty, \infty)$ have corresponding x-values.

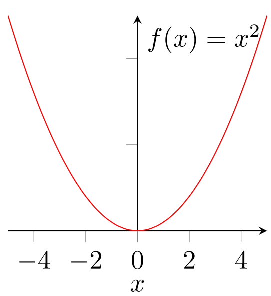
at most once.

Definition 9.9 (surjective)**.** $f : D \to R$ is **surjective** or *onto* iff

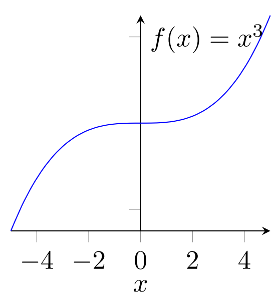
$$\forall y \in R \exists x \in D\big[f(x) = y\big]. \tag{9.3}$$

Examples of the surjective property of functions are shown in Figure **??**.

Definition 9.10. A function $f : A \to B$ is **bijective** iff it is *both injective and surjective*.



(e) The function $f(x) = x^2$ is not bijective.



(f) The function $f(x) = x^3$ is bijective.

9.1.5. Graphs

Definition 9.11. If f is a function with a domain D, then its **graph** is the set

$$\Big\{\big(x, f(x)\big)\Big|x \in D\Big\},$$

that is, it is the set of all points $(x, f(x))$ where x is in the domain of the function.[1]

If (x, y) is a point on f, then $y = f(x)$ is the height of the graph above point x. This height might be positive or negative, depending on the sign of $f(x)$. We use this height relationship to plot functions.

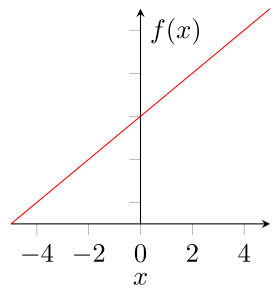


Figure 9.5.: A plot of the function $f(x) = x + 2$

9.2. Composition of Functions

Definition 9.12. If f and g are functions, then the **composite** function $f \circ g$, "f composed with g", is defined by

$$(f \circ g)(x) = f\big(g(x)\big).$$

Remark. The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f. See figure 11.2 for an example of a function which produces indefinite y-values for real x-values around $x = 0$.

Example 9.3. If $f(x) = x^2$ and $g(x) = 1 - \sqrt{x}$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution.

[1]Here, the difference between the words *graph* and *plot* is sometimes confusing. Technically speaking, a *graph* is the set defined explicitly here, while a function's *plot* refers to any pictorial representation of a data set. However, since the usage is inconsistent in this text, these formal definitions will usually not apply. It can be safely assumed that as long as we are within the realm of real numbers, all uses of either *graph* or *plot* hereafter simply refer to the pictorial representation of a function's graph in the form of a curve on the cartesian plane.

We know that $(f \circ g)(x)$ is just $f(g(x))$ so

$$f(g(x)) = (1 - \sqrt{x})^2.$$

For $(g \circ f)(x)$ it is the opposite and

$$g(f(x)) = 1 - \sqrt{x^2},$$

which is equivalent to saying

$$g(f(x)) = 1 - |x|.$$

9.3. Inverse Functions

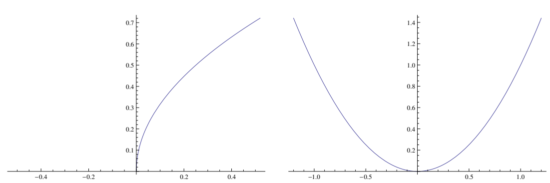
An inverse function undoes, or inverts, the effects of an original function. They are useful for producing inverse trigonometric functions–functions that are transcendental.

Definition 9.13. Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

In order for an inverse function $f^{-1}(x)$ to exist for a function $f(x)$, the original function $f(x)$ must be one-to-one. Otherwise, the resulting "inverse function" would not be a function: more than one output would be produced from only one input, and it would not pass the vertical line test.



(a) A plot of $f(x) = \sqrt{x}$. (b) A plot of $f^{-1}(\sqrt{x}) = x^2$.

Remark. By definition, either composite of a function and its inverse will return the identity function, where $y = x$. For example:

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x,$$

or

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x.$$

9.3.1. Finding Inverse Functions

To find the inverse of a function $f(x)$, replace $f(x)$ with y and solve for x in terms of y. Then, interchange x and y.

Example 9.4. Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution.
First, we solve the function for x in terms of y.

$$y = \frac{1}{2}x + 1$$

Multiply both sides by 2.

$$2y = x + 2$$

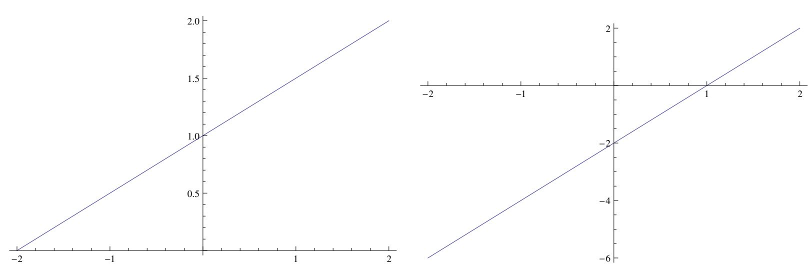
Now subtract 2 from both sides, and swap the left and right sides of the equation.

$$x = 2y - 2$$

Now we swap x and y.

$$y = 2x - 2$$

The inverse of the function $f(x) = \frac{1}{2}x + 1$ is the function $f^{-1}(x) = 2x - 2$.



(c) A plot of $f(x) = \frac{1}{2}x + 1$. (d) A plot of $f^{-1}(x) = 2x - 2$.

Figure 9.6.: Plots of functions from Example 9.4.

Now we can test this: $f(1) = (1/2)(1) + 1 = 3/2$ and $f^{-1}(3/2) = 2(3/2) -$

$2 = 1$. Same for $f(0) = (1/2)(0) + 1 = 1$; $f^{-1}(1) = 2(1) - 2 = 0$, and for any other x value we could choose.

10. Trigonometry

10.1. Sine, Cosine, and Tangent

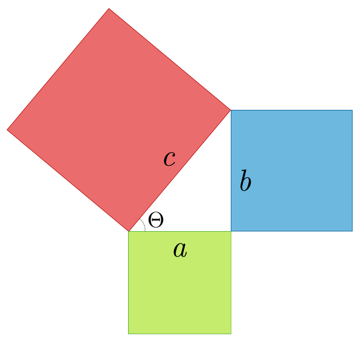


Figure 10.1.: A right triangle.

Definition 10.1. The **hypotenuse** is the side opposite the right angle.

Definition 10.2. The **sine** of $theta$, $\sin\theta$, is equal to the length of the opposite side divided by the length of the hypotenuse.

Definition 10.3. The **cosine** of $theta$, $\cos\theta$, is equal to the adjacent side divided by the hypotenuse.

Definition 10.4. The **tangent** of $theta$, $\tan\theta$, is equal to the opposite over the adjacent side.

In Figure 10.1:

$$\sin\theta = \frac{b}{c} \qquad \cos\theta = \frac{a}{c} \qquad \tan\theta = \frac{b}{a}$$

10.2. Trigonometric Identities

Our first Pythagorean identity is just derived from the Pythagorean theorem.

Theorem 9. In any right triangle, the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares whose sides are the two legs (the two sides that meet at a right angle).

Because the radius of the circle is 1, then the length of side c must also equal 1. From this, we can conclude that $\sin^2 x + \cos^2 x$ must, indeed, equal 1.

Why? From the picture, it is clear that both c and r are equal to 1. Using the Pythagorean theorem (Theorem 9):

Proof.

$$a^2 + b^2 = c^2$$
$$a \cdot a + b \cdot b = c \cdot c$$

we divide both sides by 1

$$\frac{b}{1} \cdot \frac{b}{1} + \frac{a}{1} \cdot \frac{a}{1} = \frac{c \cdot c}{1}$$

because $c = 1$

$$\frac{b}{c} \cdot \frac{b}{c} + \frac{a}{c} \cdot \frac{a}{c} = 1$$

remembering our definitions of sine and cosine

$$\sin\theta \cdot \sin\theta + \cos\theta \cdot \cos\theta = 1$$
$$\sin^2\theta + \cos^2\theta = 1$$

□

We now have our first trigonometric identity.

$$\sin^2 x + \cos^2 x = 1 \tag{10.1}$$

The other ones, we either have to memorize or learn to derive (described in Section 10.4).

$$\sin(x + y) = \sin x \cos y + \sin y \cos x \tag{10.2}$$

$$\sin(x - y) = \sin x \cos y - \sin y \cos x \tag{10.3}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{10.4}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y \tag{10.5}$$

$$\sin -x = -\sin x \tag{10.6}$$

$$\sin -x = -\sin x \tag{10.7}$$

$$\sin 2x = 2\big(\sin x \cos x\big) \tag{10.8}$$

$$\cos 2x = \cos^2 x - \sin^2 x \tag{10.9}$$

10.3. Reciprocal functions

The reciprocals of $\sin$, $\cos$, and $\tan$ all have names as well:

Definition 10.5. The **cosecant**, $\csc\theta$ is the reciprocal of $\sin\theta$:

$$\csc\theta = \frac{1}{\sin\theta} \tag{10.10}$$

Definition 10.6. The **secant**, $\sec\theta$ is the reciprocal of $\csc\theta$:

$$\sec\theta = \frac{1}{\cos\theta} \tag{10.11}$$

Definition 10.7. The **cotangent**, $\cot\theta$ is the reciprocal of $\tan\theta$:

$$\cot\theta = \frac{1}{\tan\theta} \tag{10.12}$$

10.4. Deriving Trigonometric Identities

Recall in Figure 10.1, $\sin\theta = b/c$ and $\cos\theta = a/c$. Therefore,

$$\frac{\sin\theta}{\cos\theta} = \frac{b/c}{a/c},$$

which we may simplify into

$$\frac{\sin\theta}{\cos\theta} = \frac{bc}{ac},$$

and cancel like terms to get

$$\frac{\sin\theta}{\cos\theta} = \frac{b}{a}.$$

Recall that $\tan\theta = b/a$ and we see that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}. \tag{10.13}$$

Now, taking the reciprocal of both sides of Eq. (10.13):

$$\frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

Finally, remembering Eq. (10.12),

$$\cot\theta = \frac{\cos\theta}{\sin\theta}. \tag{10.14}$$

We recall that using the Pythagorean theorem and a triangle with a hypotenuse of length 1, we can obtain Eq. (10.1):

$$\sin^2 x + \cos^2 x = 1.$$

Dividing by $\sin^2 x$ gives us:

$$\begin{aligned} \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} &= \frac{1}{\sin^2 x}, \\ 1 + \cot^2 x &= \csc^2 x, \end{aligned}$$

and dividing by $\cos^2 x$, we see that

$$\begin{aligned} \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} + &= \frac{1}{\cos^2 x}, \\ \tan^2 x + 1 &= \sec^2 x. \end{aligned} \tag{10.15}$$

We should memorize the following two identities:

$$\sin a + b = \sin a \cos a + \sin b \cos b, \tag{10.16}$$

and

$$\cos a + b = \cos a \cos b + \sin a \sin b. \tag{10.17}$$

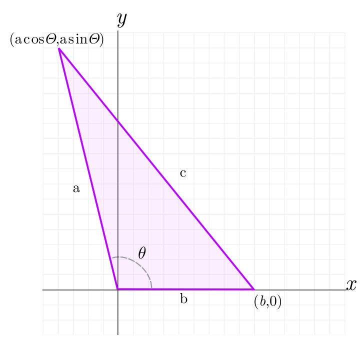
From equation (10.16) we can infer that if both a and b are the same then

$$\sin 2\theta = 2\sin\theta\cos\theta, \tag{10.18}$$

and using the same reasoning, (10.17) gives us

$$\begin{aligned}\cos 2\theta &= \cos\theta\cos\theta - \sin\theta\sin\theta,\\ \cos 2\theta &= \cos^2\theta - \sin^2\theta. \end{aligned}\tag{10.19}$$

We can generalize the Pythagorean theorem by deriving the **law of cosines**:



In the above figure, we use the distance formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{10.20}$$

$$\begin{aligned}
c &= \sqrt{(b - a\cos\theta)^2 + (0 - a\sin\theta)^2}\\
c &= \sqrt{(b - a\cos\theta)^2 + (a\sin\theta)^2}\\
c^2 &= (b - a\cos\theta)^2 + (a\sin\theta)^2\\
c^2 &= (b - a\cos\theta)(b - a\cos\theta) + (a\sin\theta)(a\sin\theta)\\
c^2 &= b^2 - ab\cos\theta - ab\cos\theta + (a\cos\theta)^2 + (a\sin\theta)^2\\
c^2 &= b^2 - 2ab\cos\theta + (a\cos\theta)^2 + (a\sin\theta)^2
\end{aligned}$$

Now we use the distance formula to find the length of a,

$$a = \sqrt{(a\cos\theta)^2 + (a\sin\theta)^2}$$

and square our result

$$a^2 = (a\cos\theta)^2 + (a\sin\theta)^2$$

which now substitutes nicely into our original equation

$$c^2 = b^2 - 2ab\cos\theta + (a\cos\theta)^2 + (a\sin\theta)^2$$
$$c^2 = b^2 - 2ab\cos\theta + a^2$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta \tag{10.21}$$

This is the *law of cosines*, a general form of the Pythagorean theorem that works with more than just right triangles.

By rearranging (10.19) and substituting from (10.1) we can derive power reduction identities for $\cos^2\theta$

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\cos 2\theta + \sin^2\theta &= \cos^2\theta \\
\cos 2\theta + (1 - \cos^2\theta) &= \cos^2\theta \\
\cos 2\theta + 1 &= \cos^2\theta + \cos^2\theta \\
\cos 2\theta + 1 &= 2\cos^2\theta \\
\cos^2\theta &= \frac{\cos 2\theta + 1}{2} && (10.22)
\end{aligned}$$

and for $\sin^2\theta$

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\sin^2\theta &= \cos^2\theta - \cos 2\theta \\
\sin^2\theta &= (1 - \sin^2\theta) - \cos 2\theta \\
2\sin^2\theta &= 1 - \cos 2\theta \\
\sin^2\theta &= \frac{1 - \cos 2\theta}{2} && (10.23)
\end{aligned}$$

10.4.1. Examples

Example 10.1. Find the value of

$$\cos\frac{11\pi}{12}.$$

Solution.
We first rewrite the problem to fit one of our trigonometric identities, then use equation (10.4) to break apart $\cos(11\pi/12)$.

$$\cos\frac{11\pi}{12} = \cos\frac{\pi}{4} + \frac{2\pi}{3}$$

$$= \cos\frac{\pi}{4}\cos\frac{2\pi}{3} - \sin\frac{\pi}{4}\sin\frac{2\pi}{3}$$

Unlike in the original problem, we can easily simplify our new expression.

$$\begin{aligned}
&= \cos\frac{\pi}{4}\cos\frac{2\pi}{3} - \sin\frac{\pi}{4}\sin\frac{2\pi}{3} \\
&= \frac{\sqrt{2}}{2}\frac{-1}{2} - \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} \\
&= \frac{-\sqrt{2}}{4} - \frac{\sqrt{6}}{4} \\
&= \frac{-\sqrt{2}}{4}\left(1+\sqrt{3}\right)
\end{aligned}$$

Example 10.2. Prove the following trigonometric identity:

$$\frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

Proof. We first cross-multiply:

$$(1-\cos x)(1+\cos x) = sin^2 x$$

then, using equation (10.1), we realize that we can replace $\sin^2 x$ with $1-\cos^2 x$.

$$(1-\cos x)(1+\cos x) = 1-\cos^2 x$$

Now we simplify.

$$\begin{aligned}
1-\cos x+\cos x-\cos^2 &= 1-\cos^2 x \\
1-\cos^2 x &= 1-\cos^2 x
\end{aligned}$$

□

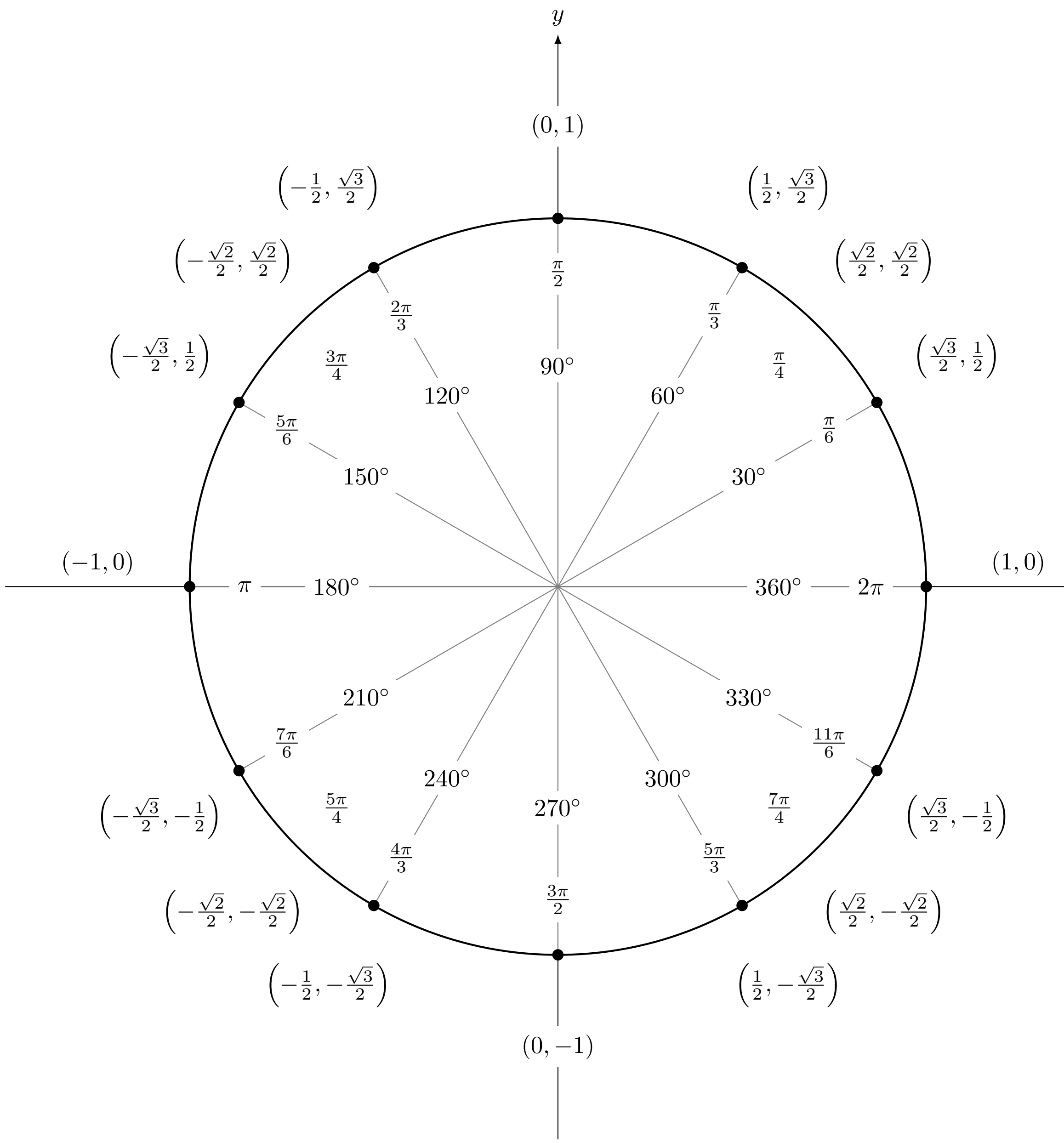
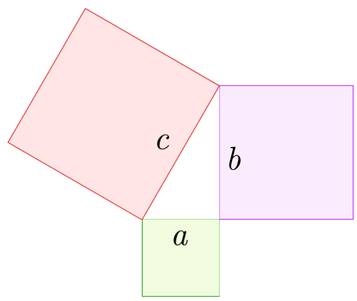
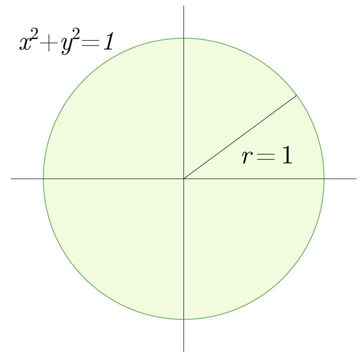


Figure 10.2.: The unit circle, thanks to [3].



(a) The Pythagorean Theorem states that the sum of the areas square a and square b is equal to the area of square c.



(b) The equation for a unit circle is $a^2 + b^2 = 1$. This creates a circle with a radius of 1.

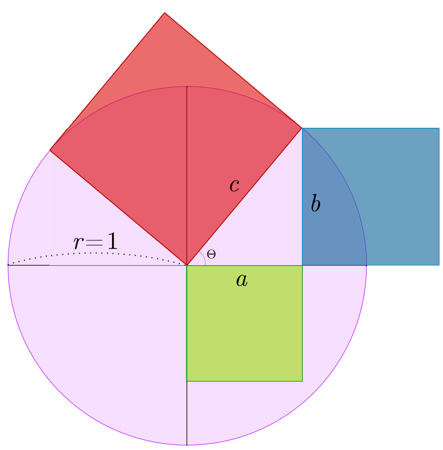


Figure 10.3.: The Pythagorean squares on a unit circle. Note that the length of side c must be 1 because the radius of the circle is 1.

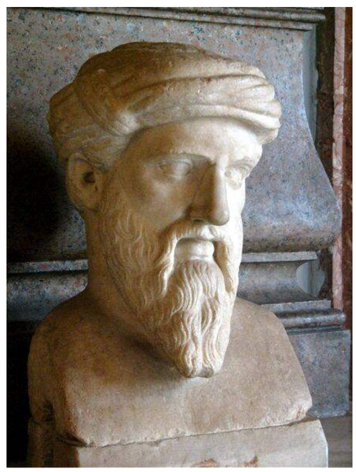


Figure 10.4.: *Busto di Pitagora. Copia romana di originale greco. Musei Capitolini, Roma*. Original uploader was *Galilea* at `http://de.wikipedia.org`. File used under the terms of the Creative Commons Attribution-Share Alike 3.0 Unported license.

11. Limits

We already know that we may evaluate functions at a certain value in order to garner some information from them. In many cases, however, this might not be all we want to know. The values of $f(x) = \sqrt{x}$ (note: not $\sqrt{|x|}$) and $g(x) = \sin x$ at $x = 0$ are both zero, but they are quite different *kinds* of zero. Limits allow us to place in words these properties of functions, empowering our language to more generally describe a functions' behavior.

11.1. Understanding Limits

If we have a function f, defined such that

$$f(x) = \frac{x^2 - 1}{x - 1} \qquad (x \neq 1), \tag{11.1}$$

this function is not defined at $x = 1$, so we cannot directly discuss its behavior at $x = 1$. However, we may still wish to know how the function

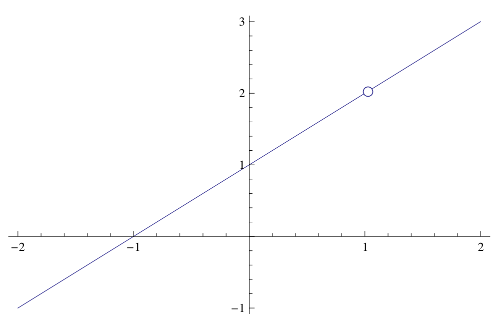


Figure 11.1.: A plot of Eq. (11.1). Note the hole at $x = 1$.

behaves *around* $x = 1$. This is where limits come in handy. They discuss the behavior of a function near a specific point, without any consideration for how the function behaves exactly *at* that point. To find the behavior of f around $x = 1$, we take the limit of f as x approaches 1.

Definition 11.1. $\lim_{x \to a} f(x) = L$ if and only if $f(x)$ gets *arbitrarily* close to L as x gets *sufficiently* close to a.

In this case, **arbitrarily close** implies that we can make the number as close as we could possibly request it. **Sufficiently close** means that we can find a number x where, for every number after (or before) it, we are past our arbitrary value of closeness. We would write this as:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

In this case, we can evaluate this using basic algebra to simplify the original function,

$$\begin{aligned} f(x) &= \frac{x^2-1}{x-1}, && x \neq 1 \\ &= \frac{(x+1)(x-1)}{x-1}, && x \neq 1 \\ &= x + 140, && x \neq 1 \end{aligned}$$

Remark. We must make the statement that $x \neq 1$ whenever $x+1$ is in the denominator of a function, because if x could ever be equal to 1, it would make the value of $f(x)$ undefined at this point.

Then we define a new function equivalent to the above, but where $1 \in D$. We then evaluate our new function at $x = 1$.

$$g(x) = x + 1$$

Which gives us

$$\begin{aligned} \lim_{x \to 1} \frac{x^2-1}{x-1} &= g(1) \\ &= 3. \end{aligned}$$

It turns out we have a number of rules that describe how we can go about evaluating a limit.

Theorem 10 (Limit Laws)**.** If L, M, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \textit{and} \lim_{x \to c} g(x) = M$$

then we have the following rules:

Sum Rule:	$\lim_{x\to c}(f(x)+g(x)) = L+M$
Difference Rule:	$\lim_{x\to c}(f(x)-g(x)) = L-M$
Constant Multiple Rule:	$\lim_{x\to c}(k\cdot f(x)) = k\cdot L$
Quotient Rule:	$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad m\neq 0$
Power Rule:	$\lim_{x\to c}[f(x)]^n = L^n, \quad n\in \mathbf{Z}_+$
Root Rule:	$\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n\in Z_+$

If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$. This is because for the rules including exponents, for $\lim x \to cf(x) = L < 0$ to be true we would require imaginary numbers.

Not all functions have limits defined everywhere on their domain. For example, $h(x) = \sin\frac{1}{x}$ oscillates indefinitely as $x\to 0$, as shown in Figure 11.2.

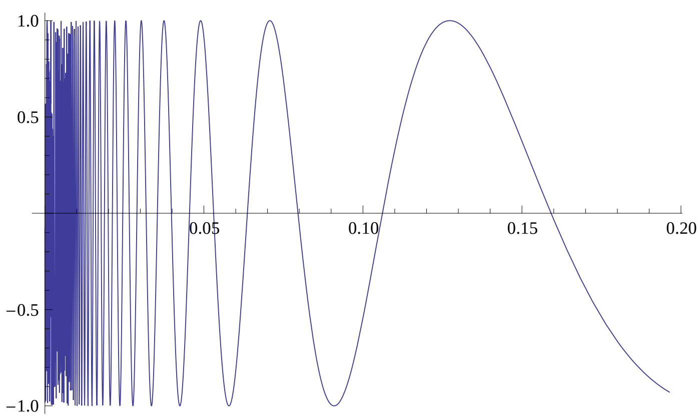


Figure 11.2.: The limit of $h(x)$ as $x\to 0$ does not exist.

As $x\to 0$, it is impossible to say whether $f(x)$ is approaching 1 or -1. This is an example of an **oscillating discontinuity**. There are more situations in which a limit does not exist.

In a **infinite discontinuity**, the graph jumps to ∞ or $-\infty$ at certain $x-values$. No limit can exist here.

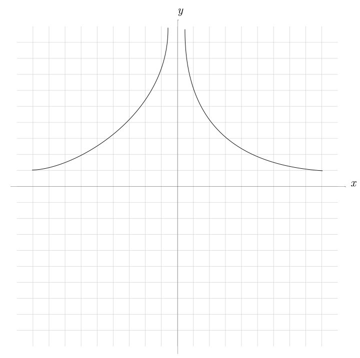


Figure 11.3.: An infinite discontinuity at $x = 0$.

The other situation which can cause a limit to not exist at a point is a **jump discontinuity**.

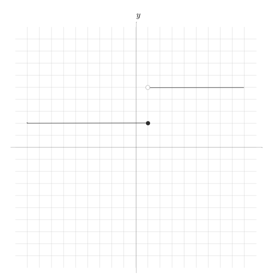


Figure 11.4.: A jump discontinuity.

11.1.1. Side Limits

In places where a limit does not exist, we can still talk about the limit from just one side or another. Note, however, that if $\lim_{x\to a} f(x)$ at a point, then the left an right limits also exist and must be equal to one another.

Theorem 11. $\lim_{x\to a^-} f(x) = L$ iff $f(x)$ gets arbitrarily close to L as x comes sufficiently close to a, but $x < a$.

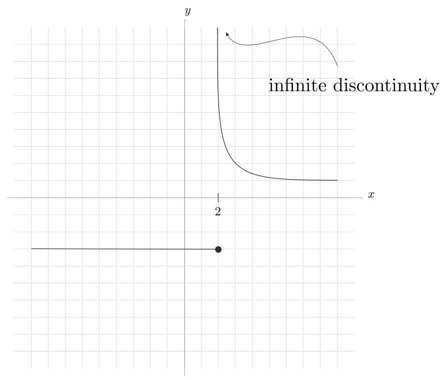


Figure 11.5.: The lefthand limit at $x = 2$ exists, though the righthand limit does not.

Theorem 12. $lim_{x \to a^+} f(x) = L$ iff $f(x)$ gets arbitrarily close to L as x gets sufficiently close to a, but $x > a$.

11.1.2. The Sandwich Theorem

Let us look at a more complicated example of a limit. Suppose we have the functions

$$h(x) = |x|,$$
$$f(x) = x \sin \frac{1}{x},$$
$$\text{and } g(x) = -|x|,$$

where at $x = 0$, $g(x) \leq f(x) \leq h(x)$ at 0.

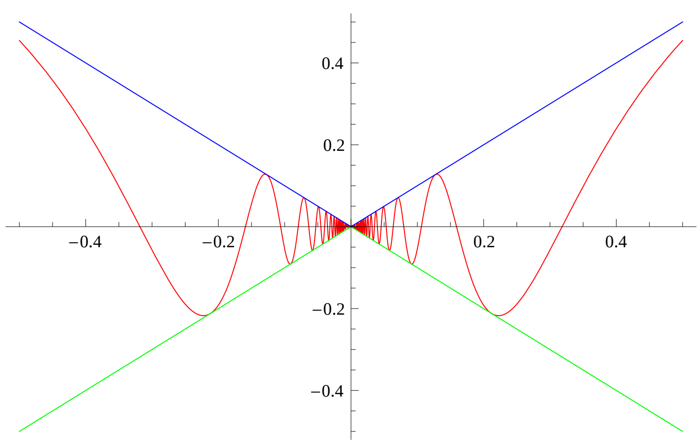


Figure 11.6.: A plot of $h(x)$, $f(x)$, and $g(x)$

We can conclude, intuitively, that the limit of $f(x)$ as $x \to 0$ must be 0 even though we cannot evaluate $f(x)$ at 0, as seen in Figure 11.2. We can generalize this kind of behavior as a theorem, the *sandwich theorem*.

Theorem 13 (The Sandwich Theorem). If

$$g(x) \leq f(x) \leq h(x)$$

holds for all $x \neq c$ in some open interval containing a number c, and

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L,$$

then $\lim_{x\to c} f(x)$ also equals L.

11.2. *Epsilon-Delta* Definition of a Limit

The statement

$$\lim_{x\to a} f(x) = L$$

is the statement

$$\forall(\varepsilon > 0)\exists(\delta > 0)\forall x\Big(0 < |x-a| < \delta \implies \big|f(x) - L\big| < \varepsilon\Big), \tag{11.2}$$

where the domain for the variables for δ and ε consists of all positive real numbers and for x consists of all real numbers.

11.3. Evaluating Limits

Let's look at a really complicated example of a limit,[1] to make sure we truly understand them. For piecewise-defined functions, limits can get especially interesting. Here's an example:

Example 11.1.

$$f(x) = \begin{cases} x & \text{if } x \in [-1, 0)\,, \\ -x & \text{if } x \in (0, 1)\,, \\ x - 1 & \text{if } x \in [1, 2]\,. \end{cases}$$

1. Does $\lim_{x\to 0} f(x)$ exist?

[1]Credit for this example goes to Dr. Dobrescu's Math 140 class at CNU, Spring 2011.

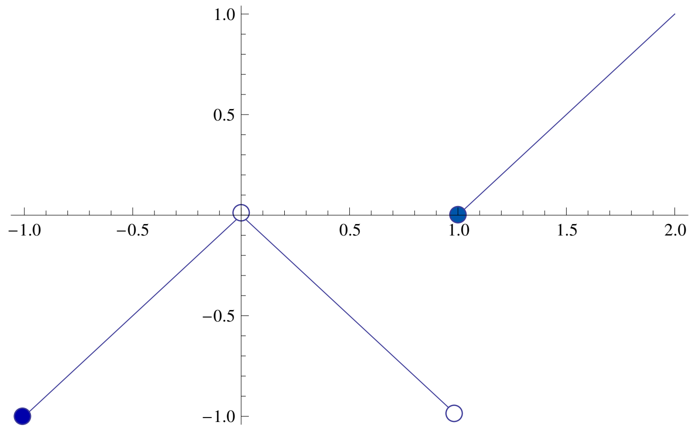


Figure 11.7.: A graph of $f(x)$

Solution.
Yes.

2. Does $\lim_{x\to 0} f(x) = 0$?

   Solution.
   Yes.

3. Does $\lim_{x\to 0} f(x) = 1$?

   Solution.
   No. $\lim_{x\to 0} f(x) = 0$.

4. Does $\lim_{x\to 1} f(x) = 1$?

   Solution.
   No. $\lim_{x\to 1} f(x)$ does not exist.

5. Does $\lim_{x\to 1} f(x) = 0$?

   Solution.
   No. The limit does not exist.

6. Can we take $\lim_{x\to x_0} f(x)$ for every $\big(x_0 \in (-1,1)\big)$?

   Solution.
   Yes.

7. Does $\lim_{x\to 1} f(x)$ exist?

Solution.
Yes.

Example 11.2. Find the value of

$$\lim_{x\to 4} \frac{4x - x^2}{2 - \sqrt{x}}.$$

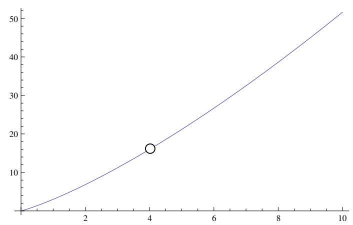


Figure 11.8.: A plot of $f(x) = \frac{4x-x^2}{2-\sqrt{x}}$.

Solution.
First we multiply by the conjugate of the denominator to remove the radical from it. For more detail on conjugates, see Section 19.3.

$$\lim_{x\to 4} \frac{4x - x^2}{2 - \sqrt{x}} = \lim_{x\to 4} \frac{4x - x^2}{2 - \sqrt{x}} \cdot \frac{2 + \sqrt{x}}{2 + \sqrt{x}}$$

Then we simplify.

$$\begin{aligned} &= \lim_{x\to 4} \frac{x\left(4 - x\right)\left(2 + \sqrt{x}\right)}{4 - x} = \lim_{x\to 4} \frac{x\left(2 + \sqrt{x}\right)}{1} \\ &= 16 \end{aligned}$$

Example 11.3. Evaluate

$$\lim_{x\to -1} \frac{\sqrt{x^2+8} - 3}{x+1}.$$

Solution.

$$\lim_{x\to -1} \frac{\sqrt{x^2+8} - 3}{x+1} = \lim_{x\to -1} \frac{\sqrt{x^2+8} - 3}{x+1} \cdot \frac{\sqrt{x^2+8} + 3}{\sqrt{x^2+8} + 3} = \lim_{x\to -1} \frac{x^2+8-9}{(x+1)(\sqrt{x^2+8}+3}$$

$$= \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3}$$
$$= \frac{-1}{3}$$

Example 11.4. Suppose we wished to evaluate the limit

$$\lim_{\theta\to 0} \frac{\sin\theta}{5\theta}.$$

Solution.
Well, first we should recognize that $\sin\theta$ is not the kind of function that is growing indefinitely: it oscillates forever between two values, 1 and -1.

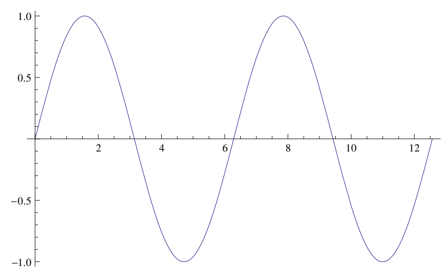


Figure 11.9.: A plot of $\sin\theta$.

But dividing by larger and larger values is going to make this function's **amplitude**, its distance from the y-value of 0, much smaller.

Using Theorem 13,[2] we can see that this limit approaches 1.

11.4. Indeterminate Forms

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3} \tag{11.3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$

[2] How?

under discussion by applying l'Hospital's Rule, which is detailed further in Section 12.5.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0, and 1^∞, which cannot be evaluated in a consistent way; these are called indeterminate forms as well.

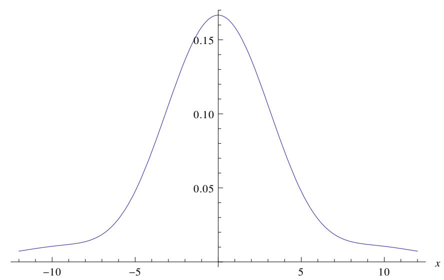


Figure 11.10.: A plot of Eq. (11.3).

11.5. Continuity of Functions

Definition 11.2. A function is **continuous** at $x = a$ iff

- $f(a)$ is defined.
- $\lim_{x \to a} f(x)$ exists.
- $\lim_{x \to a} f(x) = f(a)$.

11.6. The Mean Value Theorem

The following is a simplified case of the mean value theorem: The **mean value theorem** states that, for a continuous function f, if we have an $f(a)$ which is negative and a $f(b)$ which is positive, then f crosses the x-axis somewhere on the interval (a, b).

12. Derivatives

Say we are trying to find the rate of change of the function $f(x) = x^3$. That is, essentially, its slope.

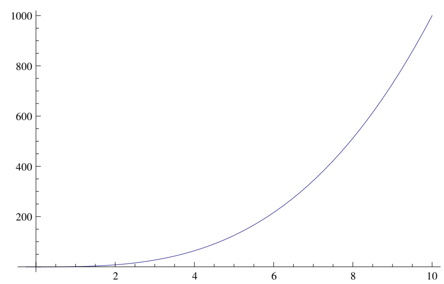
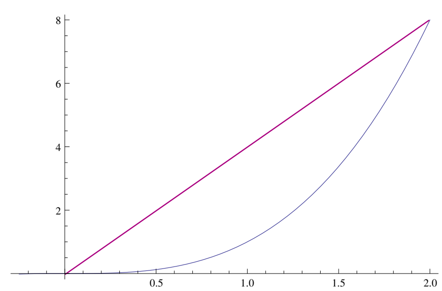
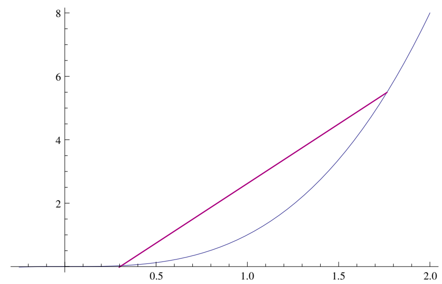


Figure 12.1.: A plot of $f(x) = x^3$.

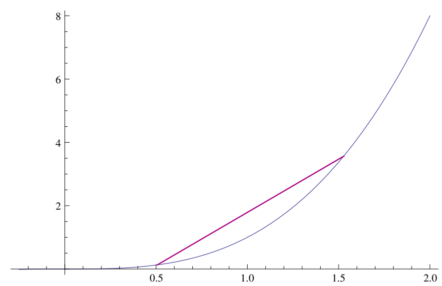
We'd like to be able to do this with just a straight line, but that's simply not possible. We could, however, draw a straight line that approximates this function around a specific value, say $x = 1$. We can just draw a line that approximates the function, to try to get it right:

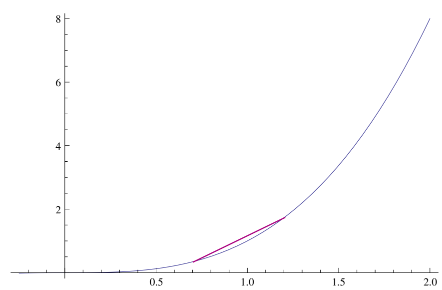


But that's not a very accurate impression of our slope. To make it better, we could draw a line between two points that are *even closer* to $x = 1$.

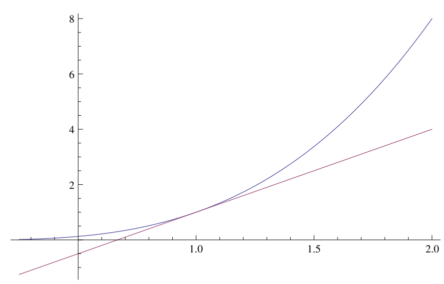


And to improve this further, we can continue to draw lines closer and closer to that point.





Until we are drawing lines between two points which are an *infinitely small* distance apart from one another.



The slope of this line is the *derivative* of our function at $x = 1$.

We see that, around $x = 1$, our slopes are very close. But as we get further from that x-value, we don't have a very good approximation of the slope anymore. This is because a derivative describes the *instantaneous rate of change* of the function *at that point*. It's not the slope of the entire function, it's just the slope at one, specific, selected point.

The **derivative** of a function $f(x)$ at $x = a$ is used to describe its *instantaneous slope* at the value $x = a$.

12.1. Slopes

The entire idea of a derivative is based on the idea behind the slope of a graph.

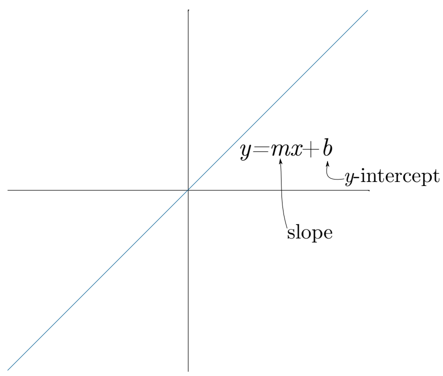


Figure 12.2.: The slope-intercept form equation for a line.

The slope, m, of a given straight line $f(x) = mx + b$ is given by its *change in y values divided by its change in x values*

$$m = \frac{\Delta y}{\Delta x}, \tag{12.1}$$

which we can find using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \tag{12.2}$$

which is equivalent to

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{12.3}$$

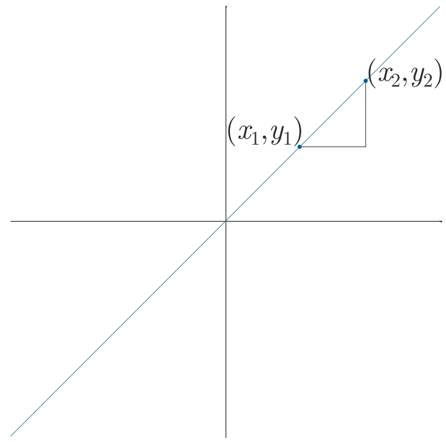


Figure 12.3.: How we determine slope in a graph.

12.2. The Difference Quotient

The *difference quotient* works the same way as the slope formula, except we treat how far apart our two x values are as a variable.

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{12.4}$$

So $\Delta x = x_2 - x_1$, and $x = x_1$.

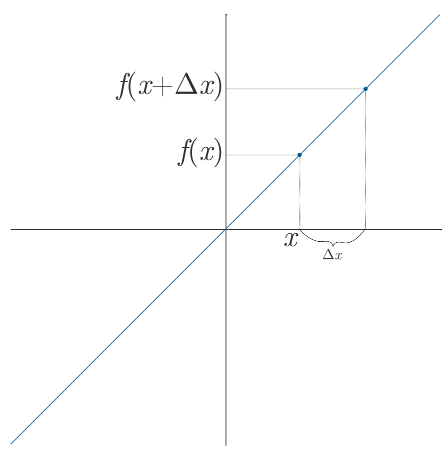


Figure 12.4.: A visual representation of the difference quotient on a line.

You'll often see the difference quotient written with $\Delta x \to h$ as

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x + h) - f(x)}{h} \tag{12.5}$$

12.2.1. Limit Definition of Derivatives

The difference quotient is used in the **limit definition of a derivative**, which is

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{12.6}$$

12.3. Notation Used with Derivatives

There are a number of ways to denote the derivative of a function $y = f(x)$. The most common notation early on in calculus classes is the *prime* notation y' and f' because it is the simplest. Later on, you often see $\mathrm{d}/\mathrm{d}x$ style notation, because it is more specific. It tells us not only which function we are differentiating, but with respect to which variable. A notable characteristic of this "d notation" is the similarity it provides to our equation for slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} \qquad\qquad y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

In the case of slope, we're talking about specific y and x values, but in the case of a derivative, we're talking about *infinitesimals*, and that particular distinction allows us to discuss the behavior of functions that aren't linear, but change their behavior over time.

12.4. Tangent Lines

Remember our line that gave us the slope of $f(x) = x^3$ at $x = 1$? That was called a *tangent line* to $f(x)$ at that point. How do we find out what the equation for that line is?

Well, we use the derivative of the function, calculated at $x = 1$, to determine the slope of our tangent line.

$$f(x) = x^3$$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Remembering back to Section 9.2,

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

Expand $(x+h)^3$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)(x+h)(x+h)-x^3}{h} \\
f'(x) &= \lim_{h\to 0} \frac{(x^2+hx+hx+h^2)(x+h)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{(x^2+2hx+h^2)(x+h)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{x^3+2hx^2+h^2x+x^2h+2h^2x+h^3-x^3}{h} \\
f'(x) &= \lim_{h\to 0} \frac{3hx^2+3h^2x+h^3}{h} \\
f'(x) &= \lim_{h\to 0} 3x^2+3hx+h^2
\end{aligned}
$$

Since we are taking the limit as $h \to 0$,

$$f'(x) = 3x^2$$

If we calculate this derivative at $x = 1$, we get

$$
\begin{aligned}
f'(x)\Big|_{x=1} &= 3(1)^2 \\
&= 3
\end{aligned}
$$

So the slope of our tangent line is 3.

Note. The symbol $\big|_{x=1}$ means to evaluate the preceding term at the value $x = a$. It is often seen referring to a function's derivative at a particular point, which is different from the function's derivative in general.

Now we use the point-slope formula (Section 19.2) with $m = 3$ to find the equation for our tangent line.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y &= mx - mx_1 + y_1 \\
y &= 3x - 3 \cdot 1 + 1 \\
y &= 3x - 2
\end{aligned}
$$

Which tells us that the equation for the tangent line to our function at $x = 1$ is $y = 3x - 2$.

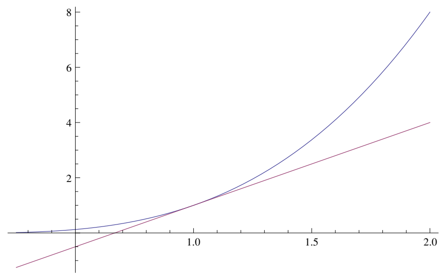


Figure 12.5.: A plot of x^3 and $3x - 2$.

12.4.1. Derivative Rules

But that was an awful lot of work just to get one little derivative! It would be handy if we could generalize the derivative of functions like this, so that we don't have to do it again in the future. Luckily, we can, and this generalization is what we call the *power rule*.

The **power rule** states that

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1} \tag{12.7}$$

Example 12.1. Find the derivative of $f(x) = x^9$ with respect to x.

Solution.
We subtract 1 from the value of our exponent and make the original exponent the coefficient of the variable. This means that

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 9x^8.$$

Example 12.2. The power rule also works for negative exponents. Let's try finding the derivative of $f(x) = \frac{1}{x}$ with respect to x.

Solution.
To find this derivative, we first recognize that $x^{-1} \equiv 1/x$. Meaning

$$f(x) = \frac{1}{x} = x^{-1}.$$

We may now simply use the power rule to find that

$$f'(x) = -1 \cdot x^{-2},$$

Which we may now simplify:

$$f'(x) = -\frac{1}{x^2}.$$

Derivative rules are just shorthands for working things out manually using the limit definition in equation (12.6). They all come from equation (12.6) and greatly simplify the work we have to do in calculating derivatives.

Before we go further, let's cover some important properties to know when finding derivatives. First, the *derivative of a sum* is the same as a *sum of derivatives*.

Theorem 14 (Sum Rule for Derivatives)**.** For two differentiable functions $u(x)$ and $v(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x}[u(x) + v(x)] = \frac{\mathrm{d}}{\mathrm{d}x}u(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

Proof. Start with the limit definition of a derivative.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\big[u(x) + v(x)\big] &= \lim_{h\to\infty} \frac{\big[u(x+h) + v(x+h)\big] - \big[u(x) - v(x)\big]}{h} \\
&= \lim_{h\to\infty} \frac{u(x+h) - u(x) + v(x+h) - v(x)}{h} \\
&= \lim_{h\to\infty} \frac{u(x+h) - u(x)}{h} + \lim_{h\to\infty} \frac{v(x+h) - v(x)}{h} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}u(x) + \frac{\mathrm{d}}{\mathrm{d}x}v(x)
\end{aligned}$$

□

Example 12.3. Find the derivative of $f(x) = x^2 + x$.

Solution.
We now know that

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2 + x)$$

is equivalent to

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}x^2 + \frac{\mathrm{d}}{\mathrm{d}x}x.$$

Simply use the power rule to differentiate.

$$f'(x) = 2x + 1 \cdot x^0$$

Since any number raised to the power 0 is just 1,

$$f'(x) = 2x + 1.$$

When differentiating functions involving constant multiples, we may move the constants out of the derivative. This means that, if $g(x)$ is a differentiable function and k is a constant multiple,

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(kg(x)\big) = k\frac{\mathrm{d}}{\mathrm{d}x}\big(g(x)\big).$$

This is known as the **constant multiple rule**.

The derivative of a constant is always zero, because constant has no rate of change.

Theorem 15 (Constant Rule for Derivatives)**.** For any constant number k, we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}k = 0.$$

This is called the **constant rule for derivatives**.

We will use both of these new rules in the following example.

Example 12.4. Find the derivative of the function $f(x) = 3x^3 + 1$.

Solution.
We just learned to differentiate sums, so we know to differentiate this in separate pieces.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}3x^3 + \frac{\mathrm{d}}{\mathrm{d}x}1$$

We use the power rule on the first term and the constant rule on the second term to reach our derivative.

$$\begin{aligned} f'(x) &= 9x^2 + 0 \\ &= 9x^2 \end{aligned}$$

Now, say we wished to differentiate this function $f(x)$ a second time. We have a couple of ways we could write this:

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x} \qquad\qquad f''(x)$$

and we would call this the **second derivative** of $f(x)$ with respect to x.

$$f''(x) = 18x$$

Theorem 16 (Product Rule for Derivatives)**.** The **product rule** tells us that for two differentiable functions $f(x)$ and $g(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot g(x) = f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x} + g(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}. \tag{12.8}$$

It's worth noting that because of the commutative law for addition, it doesn't matter which derivative we place first.

Theorem 17 (Quotient Rule for Derivatives)**.** The **quotient rule** states that for a ratio of two differentiable functions $f(x)$ and $g(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\dfrac{\mathrm{d}f(x)}{\mathrm{d}x} - f(x)\dfrac{\mathrm{d}g(x)}{\mathrm{d}x}}{g(x)^2}. \tag{12.9}$$

We should be careful with this one. Unlike the with the product rule, there is no leniency offered with the quotient rule's order of operations:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f(x)\dfrac{\mathrm{d}g(x)}{\mathrm{d}x} - g(x)\dfrac{\mathrm{d}f(x)}{\mathrm{d}x}}{g(x)^2}. \tag{12.10}$$

The **chain rule** is used to differentiate composite functions, which look like $(f \circ g)(x)$ and mean $f\big(g(x)\big)$. This type of functions is introduced and further explained in Section 9.2.

Theorem 18 (Chain Rule for Derivatives)**.** The chain rule states that if $y = f(u)$ and $u = g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}. \tag{12.11}$$

We will want to develop an intuitive understanding for when to use the chain rule. Some derivatives appear to require the chain rule, when in fact they can be differentiated in much simpler ways.

Here is an example of such a problem:

Example 12.5. Find the derivative of $f(x)$, where

$$f(x) = \frac{1}{x^3}.$$

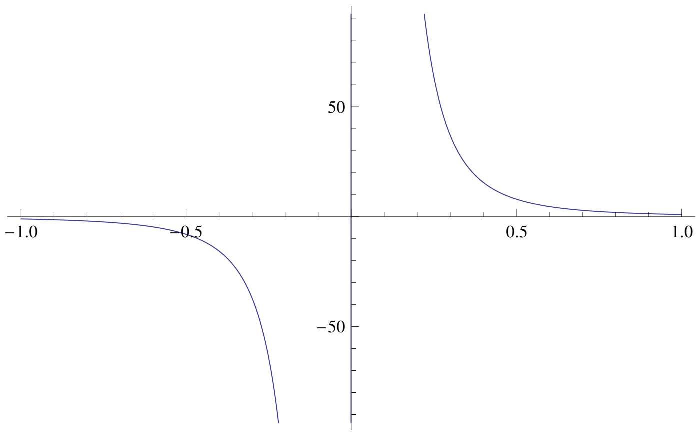


Figure 12.6.: A plot of $f(x) = \frac{1}{x^3}$.

Solution.
This is not a chain rule problem. Although you could think of it as one with $f(x) = 1/x$ and $g(x) = x^3$, it is easier to remember that factors can be moved from the numerator to the denominator simply by multiplying their exponents by -1.

$$\begin{aligned} f(x) &= \frac{1}{x^3} \\ &= x^{-3} \\ \frac{\mathrm{d}}{\mathrm{d}x} f(x) &= -3x^{-4} \end{aligned}$$

Example 12.6. Find the derivative of $f(x)$, where

$$f(x) = \left(\frac{1}{\sqrt{x}}\right)^5$$

This problem combines the chain rule and product rule. We use the chain rule with

$$g(x) = \frac{1}{\sqrt{x}} \qquad\qquad f(x) = (g(x))^5$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{x}}\right)^5 \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(g(x))^5 \\ &= 5g(x)^4 \cdot \frac{\mathrm{d}}{\mathrm{d}x}g(x)\end{aligned}$$

Now we find $\frac{\mathrm{d}}{\mathrm{d}x}g(x)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}g(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{x}} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^{1/2}} = \frac{\mathrm{d}}{\mathrm{d}x}x^{-1/2} \\ &= \frac{-1}{2}x^{-3/2} = \frac{-1}{2x^{3/2}} \\ &= \frac{-1}{2\sqrt{x^3}}\end{aligned}$$

and plug that back into our original derivative, along with $g(x) = \frac{1}{\sqrt{x}}$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f(x) &= 5g(x)^4 \cdot \frac{\mathrm{d}}{\mathrm{d}x}g(x) \\ &= 5\left(\frac{1}{\sqrt{x}}\right)^4 \cdot \frac{-1}{2\sqrt{x^3}}\end{aligned}$$

and simplify

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f(x) &= 5\left(\frac{1}{\sqrt{x}}\right)^4 \cdot \frac{-1}{2\sqrt{x^3}} = 5\left[\frac{1^4}{(\sqrt{x})^4}\right] \cdot \frac{-1}{2\sqrt{x^3}} \\ &= 5\left[\frac{1}{(x^{1/2})^4}\right] \cdot \frac{-1}{2\sqrt{x^3}} = 5\left[\frac{1}{x^{4/2}}\right] \cdot \frac{-1}{2\sqrt{x^3}} \\ &= 5\left[\frac{1}{x^2}\right] \cdot \frac{-1}{2\sqrt{x^3}} = \frac{5}{x^2} \cdot \frac{-1}{2\sqrt{x^3}} \\ &= \frac{-5}{2x^2\sqrt{x^3}} = \frac{-5}{2x^2 \cdot x^{3/2}} \\ &= \frac{-5}{2x^{4/2} \cdot x^{3/2}} \\ &= \frac{-5}{2x^{7/2}}\end{aligned}$$

Trigonometric Derivative Rules

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x \tag{12.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\cot x \csc x \tag{12.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x \tag{12.14}$$

Many of these trig derivatives, such as the derivative of $\tan x$, can be found from simpler derivative rules.

Example 12.7.

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x$$

Solution.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)\cos x - \frac{\mathrm{d}}{\mathrm{d}x}(\cos x)\sin x}{(\cos x)^2} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

12.4.2. Differentiability

A function $f(x)$ is **differentiable** at a point x_0 if a tangent line to its curve exists at that point and is not vertical. Functions are not differentiable at breaks, immediate bends, cusps, or places with vertical tangents.

12.4.3. Linearization of Functions

One application of derivatives is *linearization* of functions. This is handy for performing calculations that, otherwise, would be ridiculously difficult without a calculator. The key is to treat the derivative of the function as the slope of a straight line, and then to consider the y-values of that line *good enough* approximations of the original function within a small enough interval.

In order to linearize a function $f(x)$, we must be capable of calculating its derivative at some other, nearby value $x = a$. Then, we use the point-slope formula (see Section 19.2) to find the equation for a line that approximates our function around that value.

We then write the linearization of this function $f(x)$ at $x = a$ as $L_a(x)$, and state that

$$L_a(x) \approx f(x)$$

on some interval I.

Example 12.8. Find a linearization of the function $f(x) = \sqrt{x}$ and use it to approximate the value of $f(4.001)$.

Solution.
We must first find a value on $f(x)$ close to $x = 4.001$ that shouldn't be a problem for us. The obvious solution is to use $a = 4$, such that $f(a) = \sqrt{4} = 2$. This gives us the point $(4, 2)$ on the plot of $f(x)$.

Great, but we still need the slope of our new line. To find it, we must take the derivative of our original function. Convert the square root into an exponent.

$$f(x) = \sqrt{x} = x^{1/2} \tag{12.15}$$

Use the power rule.

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{1}{2x^{1/2}} \tag{12.16}$$

$$= \frac{1}{2\sqrt{x}} \tag{12.17}$$

Now we evaluate this at $x = a = 4$ to get the slope at our known point.

$$\left.\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right|_{x=4} = \frac{1}{2\sqrt{4}} \tag{12.18}$$

$$= \frac{1}{2 \cdot 2} \tag{12.19}$$

$$= \frac{1}{4} \tag{12.20}$$

This is the slope of $f(x)$ at $x = a$.

Now, we use this slope in the point-slope formula (Section 19.2) to write our linearized function.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ L_a(x) - f(a) &= f'(a)(x - a) \\ L_a(x) - 2 &= \frac{1}{4}(x - 4) \\ L_a(x) &= \frac{1}{4}x - 1 + 2 \end{aligned}$$

$$L_a(x) = \frac{x}{4} + 1$$

Now that we have our linearized function, which should approximate $f(x)$, we can find $L_a(4.001)$.

$$\begin{aligned} L_a(4.0001) &= \frac{4.001}{4} + 1 \\ &= 1.00025 + 1 \\ &= 2.00025 \end{aligned}$$

This means that $f(4.001)$ either equals, or is extremely close to 2.00025. In our case, it just so happens that our approximation was perfect: $f(4.001) = 2.00025$. Not bad for what could have been a nasty calculation—we found it easily, entirely by hand!

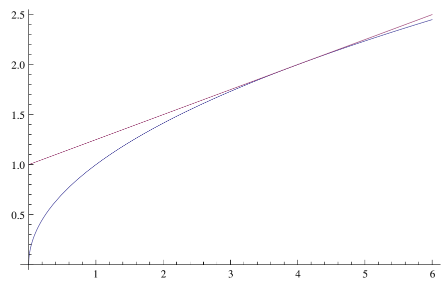


Figure 12.7.: A plot of $f(x)$ and $L_a(x)$.

12.4.4. Graphing Functions

To graph a function:[1]

1. Find all **critical values**. Critical values are locations where $\frac{dy}{dx} = 0$ or is undefined.[2]

2. Find all **points of inflection**. These are locations where the second derivative ($\frac{d^2y}{dx}$) is zero or undefined.

3. Determine where $\frac{dy}{dx}$ is *positive* to find where $f(x)$ is *increasing*. Remember that a derivative signifies the slope of a graph, so a positive derivative implies that the graph is increasing. This can be achieved either by testing values on each side of a *critical value*, or by intuitive understanding. For example, $\frac{dy}{dx} = 3x^2$ is positive everywhere

[1] This section is copied from my notes from Dr. Dobrescu's Math 140 class, taken in Fall 2011 at Christopher Newport University.

[2] At locations where $\frac{dy}{dx} = 0$, the tangent line is horizontal.

except at $x = 0$, so $f(x) = y$ is increasing everywhere except for at its horizontal tangent at $x = 0$.[3]

4. Determine where $\frac{\mathrm{d}y}{\mathrm{d}x}$ is *negative* to find where $f(x)$ is *decreasing*.
5. Determine the sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x}$ on both sides of all *points of inflection*. The graph is *concave up* where $\frac{\mathrm{d}^2y}{\mathrm{d}x}$ is positive, and *concave down* where $\frac{\mathrm{d}^2y}{\mathrm{d}x}$ is negative.

12.5. l'Hospital's Rule

l'Hospital's Rule uses derivatives to calculate limits of fractions whose numerators and denominators both approach the same indeterminate for, which can be either zero or ∞. Limits involving transcendental functions often require some use of the rule for their calculation.

Theorem 19 (L'Hospital's Rule)**.** Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} \overset{\mathrm{H}}{=} \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{12.21}$$

assuming that the limit on the right side of this equation exists.

Remark. L'Hopital's Rule does not apply when either the numerator or the denominator has a finite nonzero limit.

The proof of this theorem is found in Section 20.3.

Figure 12.8.: Augustin Louis Cauchy, 1901.

Theorem 20 (Cauchy's Mean Value Theorem)**.** Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose

[3] x^2 can never produce a negative number, because a negative times a negative or a positive times a positive is always positive

$g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{12.22}$$

12.6. Derivatives of Inverses of Differentiable Functions

Theorem 21 (The Derivative Rule for Inverses)**.** If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $\left(f^{-1}\right)'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\big(f^{-1}(b)\big)}, \tag{12.23}$$

Which may also be written

$$\left.\frac{\mathrm{d}f^{-1}}{\mathrm{d}x}\right|_{x=b} = \frac{1}{\left.\dfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=f^{-1}(b)}}. \tag{12.24}$$

Proof. Because a function applied to the inverse of itself should return its own input value, we can start with this relationship.

$$f\big(f^{-1}(x)\big) = x$$

From here, take the derivative of both sides. Since the derivative of a variable representing a constant is always 1,

$$\frac{\mathrm{d}}{\mathrm{d}x} f\big(f^{-1}(x)\big) = 1.$$

Applying the chain rule to the lefthand side of the equation gives us

$$f'\big(f^{-1}(x)\big) \cdot \frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = 1$$

Now, we divide each side of the equation by $f'\big(f^{-1}(x)\big)$ to solve for the derivative only.

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'\big(f^{-1}(x)\big)}. \qquad \square$$

13. Transcendental Functions

A **transcendental function** is a function that does not satisfy a polynomial equation whose coefficients are themselves polynomials. In other words,

> [...] a transcendental number is a (possibly complex) number that is not algebraic—that is, it is not a root of a non-zero polynomial equation with rational coefficients.[1]

A transcendental function is a function that "transcends" algebra in the sense that it cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and root extraction.

Examples of transcendental functions include the *exponential function*, the *logarithm*, and the *trigonometric functions*.

Formally,

Definition 13.1. An analytic function $f(z)$ of the real or complex variables $z_1, \ldots, z_n$ is **transcendental** if the $n+1$ functions $z_1, \ldots, z_n$ are algebraically independent. [14]

13.1. Natural Logarithms

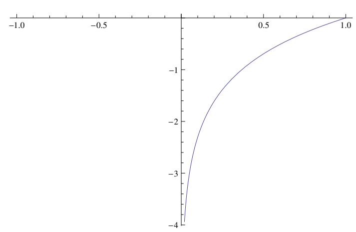


Figure 13.1.: A plot of $f(x) = \ln x$.

Definition 13.2. The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t, \qquad x \in \mathbb{N} \tag{13.1}$$

[1] http://en.wikipedia.org/w/index.php?title=Transcendental_number&oldid=609933437

Definition 13.3. The **number** e is that number in the domain of the natural logarithm satisfying

$$\ln e = 1$$

It is roughly equal to

$$2.718281828459045235360287471352662497757247093699959 5\ldots$$

13.1.1. Algebraic Properties of the Natural Logarithm

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

Product Rule	$\ln bx = \ln b + \ln x$
Quotient Rule	$\ln \frac{b}{x} = \ln b - \ln x$
Reciprocal Rule	$\ln \frac{1}{x} = -\ln x$
Power Rule	$\ln x^r = r \ln x \qquad \forall r \in \mathbb{R}$

13.2. Logarithmic Identities

$$\begin{aligned} a^x a^y &= a^{x+y} & \log_a(uv) &= \log_a u + \log_a v \\ (a^x)^y &= a^{xy} & \log_a(u^y) &= y \log_a u \\ a^{-x} &= \frac{1}{a^x} & \log_a\left(\frac{1}{u}\right) &= -\log_a u \\ \frac{a^x}{a^y} &= a^{x-y} & \log_a \frac{u}{v} &= \log_a u - \log_a v \end{aligned}$$

The number e and its relationship to logarithms becomes especially important in integration, where we manipulate its properties in calculus to solve equations and integrate functions we would not otherwise be able to handle.

The inverse equations for e^x and $\ln x$ are

$$\forall (x > 0)\left[e^{\ln x} = x\right] \tag{13.2}$$

$$\forall x \left[\ln(e^x) = x\right] \tag{13.3}$$

The derivative of e^x is very special, and it is

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x \, \mathrm{d}x. \tag{13.4}$$

13.3. Hyperbolic Functions

Both $\cos x$ and $\sin x$ come from the formula for a circle.

$$x^2 + y^2 = r^2 \tag{13.5}$$

But we can define other useful functions using the equation for a hyperbola.

$$x^2 - y^2 = 1 \tag{13.6}$$

Namely, $\cosh x$ and $\sinh x$.

In 13.6, let

$$y \to \frac{e^x - e^{-x}}{2}$$

to get $\sinh x$. Let

$$x \to \frac{e^x + e^{-x}}{2}$$

to find $\cosh x$.

We can prove that these still satisfy equation 13.6:

Proof.

$$\begin{aligned}
1 &= x^2 - y^2 \\
1 &= \left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
1 &= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}
\end{aligned}$$

□

14. Integration

If we have a function, $f(x)$, whose graph is shown in Figure 14.1. and we wish to find the area under its curve, we can do so using a technique called *integration*. As in Chapter 12, where we approximated the slope of a curve using straight lines, we are going to approximate the area of this irregular shape using simple rectangles, seen in Figure 14.2. A better estimate

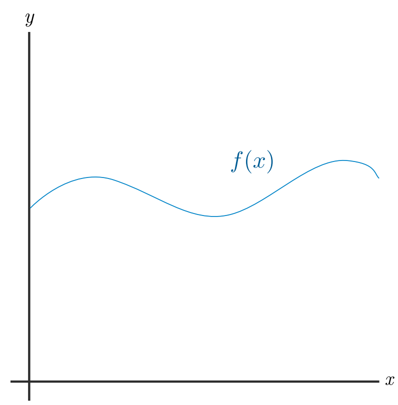


Figure 14.1.: A curved line.

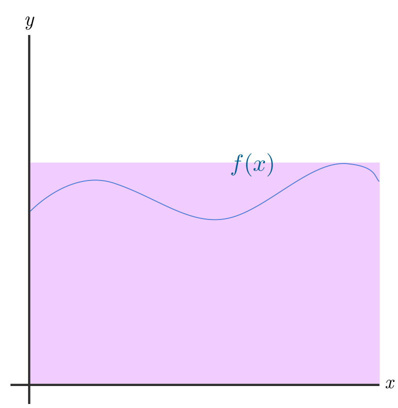


Figure 14.2.: Approximating the area under the curve with one box.

would use two boxes, as in Figure 14.3. Then we could sum the area of both boxes for a closer approximation. Use more boxes for more and more

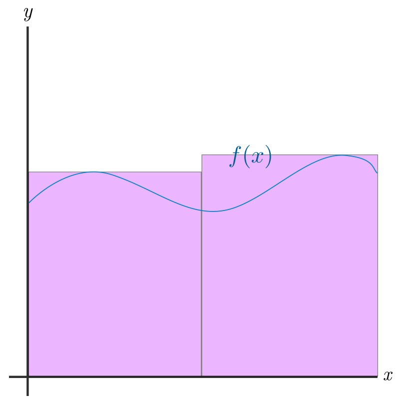


Figure 14.3.: Approximating the area under the curve with two boxes.

accurate approximations of the area under the curve, like in Figure 14.4. So,

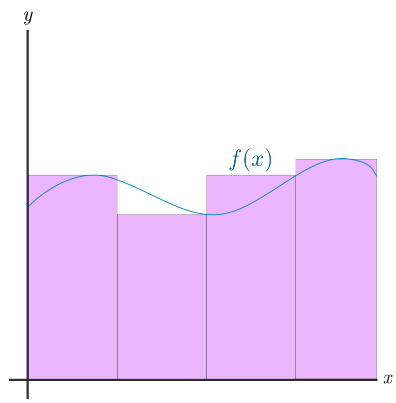


Figure 14.4.: Adding more boxes increases the accuracy of our estimate.

if we want to actually calculate something like this, it's handy to define a value Δx and let this represent the width of your boxes. This way we are just measuring different heights, and our width is always constant (Figure 14.5).

14.1. Antiderivatives

Definition 14.1. A function F is an **antiderivative** of f on an interval I if for every x in I, the derivative of F at x is equal to the value of f at x. That is,

$$\forall(x \in I)\big[F'(x) = f(x)\big]. \tag{14.1}$$

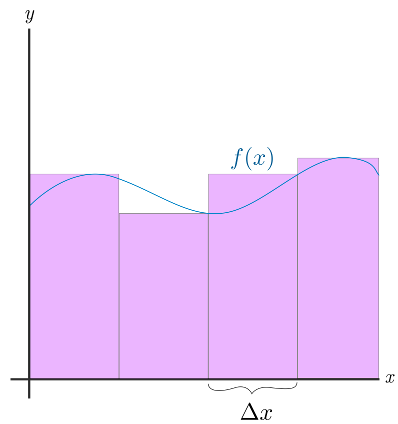


Figure 14.5.: There is a constant width Δx denoting the width of each of these boxes.

An antiderivative is related to a family of functions, all of which have the same derivative, f. This is because when taking the derivative of a function F which includes a constant C such as

$$F(x) = x + C$$

the constant will disappear upon differentiation. Thus, for a function f defined on an interval I, its most general antiderivative will be of the form

$$F(x) + C \tag{14.2}$$

to account for the possibility of such a constant.

This "function," $F(x) + C$, therefore, does not actually refer to one function in particular, or even a *handful* thereof, but rather a *family of functions* all of which have a derivative equal to $f(x)$.

This family of antiderivatives of f is called the **indefinite integral** of f with respect to x, and written as

$$\int f(x)\,\mathrm{d}x \tag{14.3}$$

where f is the **integrand** and x is the **variable of integration**.

Example 14.1. Find the antiderivative of

$$f(x) = 3x^2. \tag{14.4}$$

Solution.
We should think of a function whose derivative yields the function $3x^2$. It seems clear that this function is related to the power rule, and reversing that rule would give us x^3. Now let's take the derivative of this and see if it works:

$$F(x) = x^3,$$
$$F'(x) = 3x^2.$$

What is interesting here is that $3x^2$ is not the only function for which its derivative is $3x^2$. We know from Theorem 15, the constant rule for derivatives, that the derivative of any constant is zero. From this, we know that

$$F(x) = 3x^2 + 1,$$

or

$$F(x) = 3x^2 + 20,$$

or even

$$F(x) = 3x^2 + 39250,$$

all satisfy our definition for an antiderivative of $f(x)$.

For our example, the generalized antiderivative of Eq. (14.4) is

$$F(x) = x^3 + c.$$

We would plot this using **integral curves**, which represent a variety of possible versions of the function for different values of c.

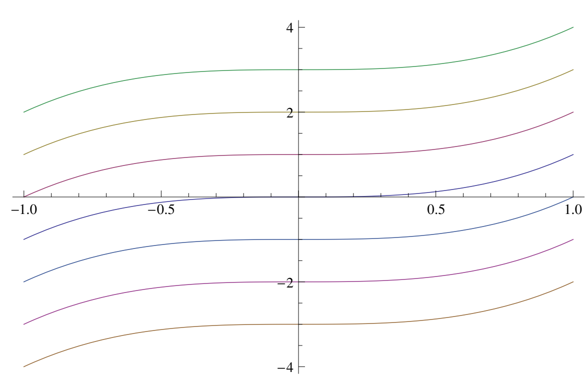


Figure 14.6.: A few possible integral curves for $F(x) = x^3 + c$.

14.2. Fundamental Theorem of Calculus

Theorem 22 (Fundamental Theorem of Calculus, pt. 1)**.** If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x). \tag{14.5}$$

[7, p. 276]

Theorem 23 (Fundamental Theorem of Calculus, pt. 2)**.** If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_b^a f(x)\,\mathrm{d}x = F(b) - F(a). \tag{14.6}$$

[7, p. 277]

14.3. Basic Integration Formulas

The following are our basic integrals. We can use this table for reference from time to time, but as we develop our integration skills, we will become increasingly proficient at using them on-the-spot.

Some are more important than others. For a more complete understanding of these rules, refer to a calculus textbook.

$$\int k\,\mathrm{d}x = kx + c \qquad \text{(any number } k\text{)}$$

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + c$$

$$\int e^x\,\mathrm{d}x = e^x + c$$

$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + c \qquad (a > 0,\ a \neq 1)$$

$$\int \sin x\,\mathrm{d}x = -\cos x + c$$

$$\int \cos x\,\mathrm{d}x = \sin x + c$$

$$\int \sec^2 x\,\mathrm{d}x = \tan x + c$$

$$\int \csc^2 x\,\mathrm{d}x = -\cot x + c$$

$$\int \sec x \tan x\,\mathrm{d}x = \sec x + c$$

$$\int \csc x \cot x\,\mathrm{d}x = -\csc x + c$$

$$\int \tan x\,\mathrm{d}x = \ln|\sec x| + c$$

$$\int \cot x\,\mathrm{d}x = \ln|\sin x| + c$$

$$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + c$$

$$\int \csc x\,\mathrm{d}x = -\ln|\csc x + \cot x| + c$$

$$\int \sinh x\,\mathrm{d}x = \cosh x + c$$

$$\int \cosh x \, \mathrm{d}x = \sinh x + c \qquad \int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + c$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 + x^2}} = \arcsin \frac{x}{a} + c \quad (a > 0) \qquad \int \frac{\mathrm{d}x}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \left|\frac{x}{a}\right| + c$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a} + c \quad (a > 0) \qquad \int \frac{\mathrm{d}x}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + c \quad (x > a > 0)$$

The above collection of integration formulae comes from [7, p. 435].

14.4. Integration by Substitution

Integration by substitution is our simplest technique for evaluating integrals that don't seem to fit into one of our basic rules. To do these, we must have a grasp on usage of the fundamental integrals from Section 14.3. Let's look at an example.

Example 14.2. Integrate

$$\int 2e^{2x} \, \mathrm{d}x.$$

Initially, we don't know how to solve this. We know that $\int e^x \, \mathrm{d}x$ is just $e^x + C$, but the constant multiplier in the exponent throws us off. Notice, however, that the derivative of $2x$, the offending element, is clearly present in the problem: it is 2. This allows us to use a handy trick to make the integrand resemble one that we know how to evaluate.

Simply let $u = 2x$. Note that if we perform this substitution, we would get half the equation in terms of u and the other half in terms of x. We obviously don't know how to handle this, so we take the derivative of u, giving us $\mathrm{d}u = 2$.

Now we may make our substitutions.

$$\begin{aligned} \int 2e^{2x} \, \mathrm{d}x &= \int e^u \, \mathrm{d}u \\ &= e^u + C \end{aligned}$$

Now we just write this in terms of our original variables.

$$= e^{2x} + C$$

And we're done.

The concept of u-substitution isn't tricky. We should know how to perform substitutions at this point. What's hard, and what we'll need to practice, is which values to select for our u and $\mathrm{d}u$. Remember: look for a piece of the integrand whose derivative is also present in the integrand, and just give it a shot.

If your first substitution fails, don't be discouraged–just try again.

Oftentimes, we won't be lucky enough to see the *exact* derivative in our integrand. If we're lucky, however, we can usually get within a constant of it, and then we can tweak the integrand to suit our needs. Let's see what this means.

Example 14.3. Integrate

$$\int x(x^2 + 3/2)^9 \,\mathrm{d}x.$$

Solution.
We can try $u = x^2$, $\mathrm{d}u = 2x$. Unfortunately, we still don't know how to evaluate $(u + 3/2)^9$ if we do that. It was a good guess, though, and got us closer to an answer.

Try letting $u = x^2 + 3/2$ and $\mathrm{d}u = 2x\,\mathrm{d}x$ instead. This gets us *really* close, except there's no 2 in our integrand.

From here, there are two ways to fix this. One is to simply make the righthand side of $\mathrm{d}u = 2x\,\mathrm{d}x$ look how we want it. To do this, we simply divide both sides by two. This gives us

$$u = x^2 + 3/2 \, \frac{\mathrm{d}u}{2} = x\,\mathrm{d}x$$

Now we're getting somewhere. We can substitute this into our original integral:

$$\begin{aligned}\int x(x^2 + 3/2)^9 \,\mathrm{d}x &= \int u^9 \frac{\mathrm{d}u}{2} \\ &= \frac{1}{2}\int u^9 \,\mathrm{d}u,\end{aligned}$$

which we know how to integrate. It becomes

$$= \frac{1}{2}\frac{u^{10}}{10} + C.$$

Finally, we just replace the u terms with their equivalents in terms of x, making our answer

$$\int x(x^2+3/2)^9\,\mathrm{d}x = \frac{x^2+3/2}{20} + C.$$

Example 14.4. Integrate

$$\int (1-x)^9\,\mathrm{d}x.$$

Solution.
Let $u = 1-x$ and $\mathrm{d}u = -\,\mathrm{d}x$.

$$\begin{aligned}\int (1-x)^9\,\mathrm{d}x &= -\int u^9 \\ &= -\frac{u^{10}}{10} + C \\ &= -\frac{(1-x)^{10}}{10} + C\end{aligned}$$

14.5. Integration By Parts Formula

Integration by parts is a method of separating out elements of integrals that we don't know how to evaluate, and taking their derivative instead.

Its formula comes from our definition for the derivative product rule, Eq. (12.8):

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x)g(x)\big] = f'(x)g(x) + f(x)g'(x) \tag{14.7}$$

If we take the indefinite integral of each side,

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\big[f(x)g(x)\big]\,\mathrm{d}x = \int \big[f'(x)g(x) + f(x)g'(x)\big]\,\mathrm{d}x$$

Now separate out the terms in the righthand side.

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\big[f(x)g(x)\big]\,\mathrm{d}x = \int f'(x)g(x)\,\mathrm{d}x + \int f(x)g'(x)\,\mathrm{d}x$$

From here, we note that the integral and derivative in the lefthand side cancel each other out, if we use the fundamental theorem of calculus.

$$f(x)g(x) = \int f'(x)g(x)\,\mathrm{d}x + \int f(x)g'(x)\,\mathrm{d}x$$

Now, subtract the first integral from both sides.

$$f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x = \int f(x)g'(x)\,\mathrm{d}x$$

Swap the left and righthand sides, and we have the formula for integration by parts!

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \tag{14.8}$$

For convenience, we often rewrite this formula in terms of $u,\ \mathrm{d}u,\ \mathrm{d}v$, and v as follows:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \tag{14.9}$$

The real goal, here, is to always make $\int v\,\mathrm{d}u$ easier to evaluate than $\int u\,\mathrm{d}v$. It doesn't really matter what u and v are. Learning when to use integration by parts can be tricky, and learning which parts are best assigned where is even more difficult. Generally, this takes a fair amount of examples, coupled with considerable practice for one to develop an efficient intuitive understanding.

Example 14.5. Integrate

$$\int x\cos x\,\mathrm{d}x.$$

Solution.
We use integration by parts (Eq. 14.9) with

$$\begin{aligned} u &= x & \qquad \mathrm{d}v &= \cos x \\ \mathrm{d}u &= \mathrm{d}x & v &= \sin x \end{aligned}$$

So our integral becomes something we can integrate.

$$\begin{aligned} \int x\cos x\,\mathrm{d}x &= x\sin x - \int \sin x\,\mathrm{d}x \\ &= x\sin x + \cos x + C \end{aligned}$$

A note on selecting u and $\mathrm{d}v$: it is usually easiest to pick $\mathrm{d}v$ first to be everything we know how to integrate, including the $\mathrm{d}x$, and then let u be the rest. This is just a guideline, however, and one we will frequently have to bend.

Example 14.6.

$$\int x^3 \cos 3x \, \mathrm{d}x$$

Solution.
We use integration by parts.

$$\begin{aligned}
\int x^3 \cos 3x \, \mathrm{d}x =& \frac{1}{3}x^3 \sin 3x - \int x^2 \sin 3x \, \mathrm{d}x \\
=& \frac{1}{3}x^3 \sin 3x - \left[\frac{-1}{3}x^2 \cos 3x + \frac{2}{3}\int x \cos 3x \, \mathrm{d}x\right] \\
=& \frac{1}{3}x^3 \sin 3x + \frac{1}{3}x^2 \cos 3x - \frac{2}{3}\left[\frac{1}{3}x \sin 3x - \frac{1}{3}\int \sin 3x \, \mathrm{d}x\right] \\
=& \frac{1}{3}x^3 \sin 3x + \frac{1}{3}x^2 \cos 3x - \frac{2}{9}x \sin 3x - \frac{2}{27}\cos 3x + C
\end{aligned}$$

Example 14.7. A tough integration by parts problem. I originally could not integrate this myself, and presented the problem to Dr. James Martin[1], Ph.D. in Applied Mathematics from Brown University, and my professor at CNU of the time. Not surprisingly, he managed to perform the integration. I have chosen to include the problem here, for the surprising beauty of the cancellation of terms that resulted.

$$\int \frac{xe^{2x}}{(2x+1)^2} \, \mathrm{d}x$$

Solution.
We use integration by parts as follows:

$$\begin{aligned}
u &= e^{2x} & \mathrm{d}v &= \frac{x}{(2x+1)^2} \\
\mathrm{d}u &= 2e^{2x} & v &= \frac{1}{4+8x} + \frac{\ln(1+2x)}{4}
\end{aligned}$$

To get:

$$\int \frac{xe^{2x}}{(2x+1)^2} \, \mathrm{d}x = \frac{e^{2x}}{4+8x} + \frac{e^{2x}\ln|1+2x|}{4} - \int 2e^{2x}\left[\frac{1}{4+8x} + \frac{\ln(1+2x)}{4}\right] \mathrm{d}x$$

[1] `www.pcs.cnu.edu/~jamie/`

Then we simplify the integrand.

$$\begin{aligned}\int \frac{xe^{2x}}{(2x+1)^2}\,\mathrm{d}x =& \frac{e^{2x}}{4+8x} + \frac{e^{2x}\ln|1+2x|}{4} - \int \frac{2e^{2x}}{4+8x}\,\mathrm{d}x - \frac{1}{2}\int e^{2x}\ln|1+2x|\,\mathrm{d}x \\ =& \frac{e^{2x}}{4+8x} + \frac{e^{2x}\ln|1+2x|}{4} - \frac{1}{2}\int \frac{e^{2x}}{1+2x}\,\mathrm{d}x - \frac{1}{2}\int e^{2x}\ln|1+2x|\,\mathrm{d}x\end{aligned}$$

We use integration by parts again on the second integral as follows:

$$\begin{aligned} u =& \ln|1+2x| & \mathrm{d}v =& e^{2x} \\ \mathrm{d}u =& \frac{2}{1+2x}\,\mathrm{d}x & v =& \frac{1}{2}e^{2x}\end{aligned}$$

$$= \frac{e^{2x}}{4+8x} + \frac{e^{2x}\ln|1+2x|}{4} - \frac{1}{2}\left[\frac{1}{2}e^{2x}\ln|1+2x| - \frac{1}{2}\int \frac{2e^{2x}}{1+2x}\,\mathrm{d}x\right] - \frac{1}{2}\int \frac{e^{2x}}{1+2x}\,\mathrm{d}x$$

We simplify our result, and all but one of the terms cancel:

$$\begin{aligned} =& \frac{e^{2x}}{4+8x} + \frac{e^{2x}\ln|1+2x|}{4} - \frac{1}{4}e^{2x}\ln|1+2x| + \frac{1}{2}\int \frac{e^{2x}}{1+2x}\,\mathrm{d}x - \frac{1}{2}\int \frac{e^{2x}}{1+2x}\,\mathrm{d}x \\ =& \frac{e^{2x}}{4+8x} + \frac{e^{2x}\ln|1+2x|}{4} - \frac{e^{2x}\ln|1+2x|}{4} + C \\ =& \frac{e^{2x}}{4+8x} + C\end{aligned}$$

Example 14.8. Integrate

$$\int \ln x\,\mathrm{d}x.$$

Solution.
Since $\int \ln x\,\mathrm{d}x$ can be written as $\int \ln x \cdot 1\,\mathrm{d}x$, we can use integration by parts as follows:

$$\begin{aligned} u =& \ln x & \mathrm{d}v =& \mathrm{d}x \\ \mathrm{d}u =& \frac{\mathrm{d}x}{x} & v =& x\end{aligned}$$

Which gives us the integral

$$\begin{aligned}\int \ln x\,\mathrm{d}x &= x\ln x - \int \frac{x\,\mathrm{d}x}{x} \\ &= x\ln x - \int \mathrm{d}x \\ &= x\ln x - x + C\end{aligned}$$

14.5.1. Integration by Parts Cheat Sheet

This is a set of guidelines[2] advising whichever part of an integral comes first in this list should be assigned to u.

1. Logarithms
2. Inverse functions
3. Algebraic
4. Trigonometric
5. Exponential

Example 14.9. Integrate:

$$\int 3x^2 \cos x^3 \, \mathrm{d}x$$

Solution.
This is not an integration by parts problem. Because we can spot the derivative of x^3 clearly in the integrand, we can let $u = x^3$ and $\mathrm{d}u = 3x^2 \, \mathrm{d}x$.

$$\begin{aligned}\int 3x^2 \cos x^3 \, \mathrm{d}x &= \int \cos u \, \mathrm{d}u \\ &= \sin u + C \\ &= \sin x^3 + C\end{aligned}$$

Example 14.10. Integrate

$$\int x \cos x \, \mathrm{d}x$$

Solution.
Use integration by parts with

$$\begin{aligned} u &= x & \mathrm{d}v &= \cos x \, \mathrm{d}x \\ \mathrm{d}u &= \mathrm{d}x & v &= \sin x \end{aligned}$$

[2]Proposed by Herbert Kasube of Bradley University.

Which gives us

$$\begin{aligned}\int x\cos x\,\mathrm{d}x &= x\sin x - \int \sin x\,\mathrm{d}x\\ &= x\sin x + \cos x + C\end{aligned}$$

Example 14.11. Integrate

$$\int xe^x\,\mathrm{d}x$$

Solution.
Use integration by parts with

$$\begin{aligned} u &= x & \mathrm{d}v &= e^x\,\mathrm{d}x\\ \mathrm{d}u &= \mathrm{d}x & v &= e^x\end{aligned}$$

$$\begin{aligned}\int xe^x\,\mathrm{d}x &= xe^x - \int e^x\,\mathrm{d}x\\ &= xe^x - e^x + C\end{aligned}$$

Example 14.12. Integrate

$$\int x\ln x\,\mathrm{d}x$$

Solution.
Use integration by parts with

$$\begin{aligned} u &= \ln x & \mathrm{d}v &= x\,\mathrm{d}x\\ \mathrm{d}u &= \frac{\mathrm{d}x}{x} & v &= \frac{x^2}{2}\end{aligned}$$

Which gives us

$$\begin{aligned}\int x\ln x &= \frac{x^2\ln x}{2} - \int \frac{x^2\,\mathrm{d}x}{2x}\\ &= \frac{x^2\ln x}{2} - \frac{1}{2}\int x\,\mathrm{d}x\\ &= \frac{x^2\ln x}{2} - \frac{x^2}{4} + C\end{aligned}$$

Example 14.13.

$$\int xe^{x^2}\,\mathrm{d}x$$

Solution.
This looks like an integration by parts problem at first, but again it is just a regular u-substitution problem. Let $u = x^2$ and $\mathrm{d}u = 2x\,\mathrm{d}x$.

$$\begin{aligned}\int xe^{x^2}\,\mathrm{d}x &= \frac{1}{2}e^u\,\mathrm{d}u \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{x^2} + C\end{aligned}$$

14.6. Trigonometric Integrals

To solve trigonometric integrals (that is, integrals already involving trigonometric functions), we often have to do a fair amount of playing around with trigonometric identities in order to make them possible. A technique we can use to further this is separating out trigonometric operators that are in powers, such that we may perform substitution on them.

Take the following integral, for example.

Example 14.14. Integrate.

$$\int \cot^2 x \csc^4 x\,\mathrm{d}x$$

Solution.
Separate a $\csc^2 x$ from the integrand.

$$\int \cot^2 x \csc^4 x\,\mathrm{d}x = \cot^2 x \csc^2 x \csc^2 x\,\mathrm{d}x$$

Remembering our basic trig identity

$$\sin^2 x + \cos^2 x = 1$$

we can divide through by $\sin^2 x$ to get

$$1 + \cot^2 x = \csc^2 x$$

and substitute that into our new integral

$$\int \cot^2 x \csc^2 x \csc^2 x \, \mathrm{d}x = \int \cot^2 x (1 + \cot^2 x) \csc^2 x \, \mathrm{d}x$$

Now we let $u = \cot x$ and $\mathrm{d}u = \csc^2 x \, \mathrm{d}x$.

$$= \int u^2 (1 + u^2) \, \mathrm{d}u$$

which we can do.

$$= \frac{u^3}{3} + \frac{u^5}{5} + C$$

Now replace our old u-substitutions

$$= \frac{\cot^3 x}{3} + \frac{\cot^5 x}{5} + C$$

We can also use trigonometric identities to eliminate square roots in integrals:

Example 14.15.

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} \, \mathrm{d}x$$

To eliminate the square root, we use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

With $\theta = 2x$, this becomes:

$$1 + \cos 4x = 2 \cos^2 2x$$

Therefore,

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} \, \mathrm{d}x = \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 2x} \, \mathrm{d}x = \int_0^{\frac{\pi}{4}} \sqrt{2} \sqrt{\cos^2 2x} \\
&= \sqrt{2} \int_0^{\frac{\pi}{4}} |\cos 2x| \, \mathrm{d}x = \sqrt{2} \int_0^{\frac{\pi}{4}} \cos 2x \, \mathrm{d}x \\
&= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} [1 - 0] \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

Example 14.16.

$$\int \tan x \sec^4 x \, \mathrm{d}x$$

Solution.
Set aside a $\sec^2 x$

$$\int \tan x \sec^4 x \, \mathrm{d}x = \int \tan x \sec^2 x \sec^2 x \, \mathrm{d}x$$

Remembering our basic trig identity

$$\sin^2 x + \cos^2 x = 1$$

we can divide through by $\cos^2 x$ to get

$$\tan^2 x + 1 = \sec^2 x$$

and substitute that into our new integral

$$\int \tan x \sec^2 x \sec^2 x \, \mathrm{d}x = \int \tan x (\tan^2 x - 1) \sec^2 x \, \mathrm{d}x$$

Now let $u = \tan x$ and $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$

$$\begin{aligned} \int \tan x (\tan^2 x - 1) \sec^2 x \, \mathrm{d}x = & \int u(u^2 - 1) \, \mathrm{d}u \\ = & \frac{u^3}{3} - \frac{u^2}{2} + C \\ = & \frac{\tan^3 x}{3} - \frac{\tan^2 x}{2} + C \end{aligned}$$

Example 14.17.

$$\int \sin^7 x \cos^2 x \, \mathrm{d}x$$

Solution.

$$\begin{aligned} \int \sin^7 x \cos^2 x \, \mathrm{d}x &= \int \sin^6 x \cos^2 x \sin x \, \mathrm{d}x \\ &= (\sin^2 x)^3 \cos^2 x \sin x \, \mathrm{d}x \end{aligned}$$

because $\sin^2 x = 1 - \cos^2 x$

$$= (1 - \cos^2 x)^3 \cos^2 x \sin x \, \mathrm{d}x$$

let $u = \cos x$ and $\mathrm{d}u = -\sin x$.

$$\begin{aligned}
&= -\int (1-u^2)^3 u^2 \, \mathrm{d}u \\
&= -\int (1-u^2)(1-u^2)(1-u^2)u^2 \, \mathrm{d}u \\
&= -\int (1-2u^2+u^4)(1-u^2)u^2 \, \mathrm{d}u \\
&= -\int (u^2 - 2u^4 + u^6 - u^4 + 2u^6 - u^8) \, \mathrm{d}u \\
&= \frac{-u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C \\
&= \frac{-\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + \frac{\cos^5 x}{5} - \frac{2\cos^7 x}{7} + \frac{\cos^9 x}{9} + C
\end{aligned}$$

Example 14.18.

$$\int \cos^7 x \sin^2 x \, \mathrm{d}x$$

Solution.

$$\int \cos^7 x \sin^2 x \, \mathrm{d}x = \int \cos^6 x \sin^2 x \cos x \, \mathrm{d}x$$

Because $\cos^2 x = 1 - \sin^2 x$:

$$= \int (1 - \sin^2 x)^3 \sin^2 x \cos x \, \mathrm{d}x$$

Now let $u = \sin x$ and $\mathrm{d}u = \cos x \, \mathrm{d}x$.

$$\begin{aligned}
&= \int (1-u^2)^3 u^2 \, \mathrm{d}u \\
&= \int (u^2 - 2u^4 + u^6 - u^4 + 2u^6 - u^8) \, \mathrm{d}u \\
&= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} - \frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C \\
&= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} - \frac{\sin^5 x}{5} + \frac{2\sin^7 x}{7} - \frac{\sin^9 x}{9} + C \\
&= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + \frac{3\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
\end{aligned}$$

Example 14.19.

$$\int \sin^{15} x \, \mathrm{d}x$$

Solution.

$$\begin{aligned}\int \sin^{15} x \, \mathrm{d}x &= \int sin^{14} x \sin x \, \mathrm{d}x = \int (\sin^2 x)^7 \sin x \, \mathrm{d}x \\ &= \int (1 - \cos^2 x)^7 \sin x \, \mathrm{d}x\end{aligned}$$

Let $u = \cos x$ and $\mathrm{d}u = -\sin x \, \mathrm{d}x$.

$$\begin{aligned}&= -\int (1-u^2)^7 \, \mathrm{d}u = -\int \mathrm{d}u + \int u^{14} \, \mathrm{d}u = -u + \frac{u^{15}}{15} + C \\ &= -\cos x + \frac{\cos^{15} x}{15} + C\end{aligned}$$

Example 14.20.

$$\int \sin^2 x \cos^2 x \, \mathrm{d}x$$

Solution.
Remembering our power reduction identities from (10.22) and (10.23):

$$\begin{aligned}\int \sin^2 x \cos^2 x \, \mathrm{d}x &= \int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} \, \mathrm{d}x = \frac{1}{4} \int (1 - \cos^2 2x) \, \mathrm{d}x \\ &= \frac{1}{4} \int \left(\frac{2}{2} - \frac{1+\cos 4x}{2} \right) \mathrm{d}x = \frac{1}{4} \int \left(\frac{2 - (1+\cos 4x)}{2} \right) \mathrm{d}x \\ &= \frac{1}{4} \int \left(\frac{1 - \cos 4x)}{2} \right) \mathrm{d}x = \frac{1}{8} \int 1 - \cos 4x \, \mathrm{d}x \\ &= \frac{1}{8} \int \mathrm{d}x - \frac{1}{8} \int \cos 4x \, \mathrm{d}x \\ &= \frac{x}{8} - \frac{1}{32} \sin 4x + C\end{aligned}$$

Example 14.21.

$$\int \tan^6 x \sec^4 x \, \mathrm{d}x$$

Solution.

$$\int \tan^6 x \sec^4 x \, \mathrm{d}x = \int \tan^6 x \sec^2 x \sec^2 x \, \mathrm{d}x$$

Remembering that $\sec^2 x = 1 + \tan^2 x$:

$$= \int \tan^6 x (1 + \tan^2 x) \sec^2 x \, \mathrm{d}x$$

Now, letting $u = \tan x$ and $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$:

$$\begin{aligned} &= \int u^6 (1 + u^2) \, \mathrm{d}u = \int u^6 \, \mathrm{d}u + \int u^8 \, \mathrm{d}u = \frac{u^7}{7} + \frac{u9}{9} + C \\ &= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C \end{aligned}$$

Example 14.22.

$$\int \cot^8 x \csc^4 x \, \mathrm{d}x$$

Solution.
We integrate this in the same way as the previous example, by setting aside $\csc^2 x$.

$$\int \cot^8 x \csc^4 x \, \mathrm{d}x = \int \cot^8 x \csc^2 x \csc^2 x \, \mathrm{d}x$$

Now we substitute with the identity $1 + \cot^2 x = \csc^2 x$.

$$= \cot^8 x \csc^2 x (1 + \cot^2 x) \, \mathrm{d}x$$

Now we let $u = \cot x$ and $\mathrm{d}u = \csc^2 x \, \mathrm{d}x$.

$$\begin{aligned} &= \int u^7 (1 + u^2) \, \mathrm{d}u \\ &= \int u^8 \, \mathrm{d}u + \int u^{10} \, \mathrm{d}u \\ &= \frac{u^9}{9} + \frac{u^{11}}{11} + C \\ &= \frac{\cot^9 x}{9} + \frac{\cot^{11} x}{11} + C \end{aligned}$$

14.7. Trigonometric Substitution

Trigonometric substitution is the substitution of trigonometric functions for other expressions. One may use the trigonometric identities to simplify certain integrals containing radical expressions.

There are two important trig identities for us to know in this section:

$$1 - sin^2\theta = cos^2\theta, \tag{14.10}$$

and

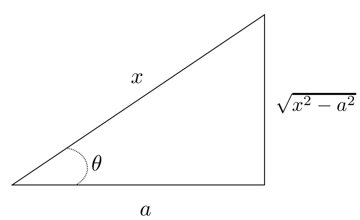
$$1 + tan^2\theta = sec^2\theta. \tag{14.11}$$

We may find Eq. (14.10) and Eq. (14.11) using the process Section 10.4 and simple algebraic rearrangement of the terms.

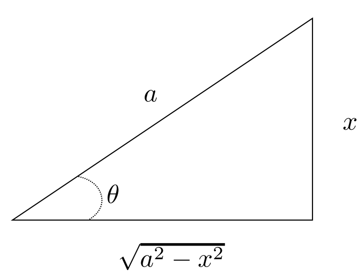
We will find trigonometric substitution useful for problems that look this:

$$\int \frac{dx}{x\sqrt{x^2-4}}$$

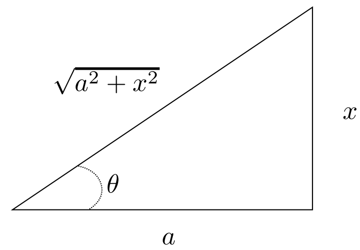
For integrands with $\sqrt{x^2-a^2}$, let $x = a\sec\theta$ and let $\mathrm{d}x = a\sec\theta\tan\theta\,\mathrm{d}\theta$.



For integrands with $\sqrt{a^2-x^2}$, let $x = a\sin\theta$ and $\mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$.



For integrands with $\sqrt{x^2+a^2}$, let $x = a\tan\theta$ and $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$.



Example 14.23. Integrate

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-4}}$$

Solution.
Let $x = 2\sec\theta$ and $\mathrm{d}x = 2\sec\theta\tan\theta\,\mathrm{d}\theta$.

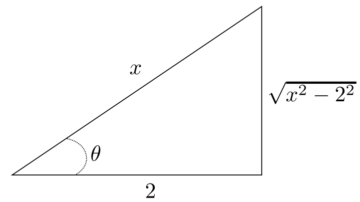
$$\begin{aligned}\int \frac{\mathrm{d}x}{x\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta\,\mathrm{d}\theta}{2\sec\theta\sqrt{4\sec^2\theta-4}} \\ &= \int \frac{tan\theta\,\mathrm{d}\theta}{\sqrt{4\sec^2\theta-4}}\end{aligned}$$

Now we use the trigonometric identity $1+\tan^2\theta = \sec^2\theta$.

$$\begin{aligned} &= \int \frac{\tan\theta\,\mathrm{d}\theta}{\sqrt{4(1+\tan^2\theta)-4}} \\ &= \int \frac{\tan\theta\,\mathrm{d}\theta}{\sqrt{4\tan^2\theta}} \\ &= \int \frac{\tan\theta\,\mathrm{d}\theta}{2\tan\theta} \\ &= \int \frac{1}{2}\,\mathrm{d}\theta \\ &= \frac{1}{2}\theta + C\end{aligned}$$

Now we draw a triangle, using the numbers from our original substitution:

$$= \frac{1}{2}\operatorname{arcsec}\frac{x}{2} + C$$



Example 14.24.

$$\int x^3 \sqrt{4 - x^2}\, \mathrm{d}x$$

Solution.
Although perhaps solvable using integration by parts, we will use trigonometric substitution.

$$\text{let } x = 2\sin\theta$$
$$\text{let } \mathrm{d}x = 2\cos\theta\, \mathrm{d}\theta$$

$$\begin{aligned}\int x^3 \sqrt{4 - x^2}\, \mathrm{d}x = \int 8\sin^3\theta\, 2\cos\theta\, 2\cos\theta\ \mathrm{d}\theta = 32 \int \sin^3\theta \cos^2\theta\, \mathrm{d}\theta \\ =& 32 \int \sin^2\theta \cos^2\theta \sin\theta\, \mathrm{d}\theta \\ =& 32 \int (1 - \cos^2\theta)\cos^2\theta \sin\theta\, \mathrm{d}\theta\end{aligned}$$

Let $u = \cos\theta$ and $\mathrm{d}u = -\sin\theta\, \mathrm{d}\theta$.

$$\begin{aligned}=& -32 \int (1 - u^2)\, u^2\, \mathrm{d}u \\ =& \frac{-32u^3}{3} + \frac{32u^5}{5} + C \\ =& \frac{-32\cos^2\theta}{3} + \frac{32\cos^5\theta}{5} + C \\ =& \frac{-32}{3}\left(\frac{\sqrt{4 - x^2}}{2}\right) + \frac{32}{5}\left(\frac{\sqrt{4 - x^2}}{2}\right)^5 + C\end{aligned}$$

Example 14.25. Integrate

$$\int \frac{1}{\sqrt{x^2 + 2x + 5}}\, \mathrm{d}x$$

Solution.
Begin by "completing the square:"

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2x+5}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{x^2+2x+1+4}} \\ &= \int \frac{1}{\sqrt{(x+1)^2+4}}\end{aligned}$$

Now let $u = x + 1$ and $\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}&= \int \frac{\mathrm{d}u}{\sqrt{u^2+4}} \\ &= \arcsin\frac{x+1}{4} + C\end{aligned}$$

14.8. Partial Fraction Decomposition

Multiply the denominator by both sides of the new equation:

$$denominator*\left[\frac{x^3+2x^2+x+8}{(x^2+4)^2}\right] = \left[\frac{Ax+B}{(x^2+4)^2} + \frac{Cx+D}{(x^2+4)^1}\right]*denominator$$

Now solve for A, B, and C.

$$\begin{aligned}x^3+2x^2+x+8 &= (Ax+B) + (Cx+D)(x^2+4) \\ x^3+2x^2+x+8 &= (Ax+B) + Cx^3 + 4Cx + Dx^2 + 4D\end{aligned}$$

Now:

$$\begin{array}{r|l} x^2 \text{ terms} & 2 = D \\ x \text{ terms} & 1 = A + 4C \\ \text{constant terms} & 8 = B + 4D \end{array}$$

14.8.1. Factoring

$$(F)^3 + (L)^3 = (F+L)(F^2 - FL + L^2) \tag{14.12}$$

$$(F)^3 - (L)^3 = (F-L)(F^2 + FL + L^2) \tag{14.13}$$

Note: you cannot factor a sum of perfect squares.

Remark. When factoring quadratics, consider the discriminant (from the quadratic formula at Eq. (19.1)):

$$10x^2 - x - 2$$
$$b^2 - 4ac$$
$$(-1)^2 - 4(10)(-2)$$

14.8.2. Examples

Example 14.26. Use partial fraction decomposition on the expression

$$\frac{x+3}{x^2-3x+2}.$$

Solution.
First, we must factor the denominator, finding that

$$x^2 - 3x + 2 = (x+3)(x-1). \tag{14.14}$$

Partial fraction decomposition tells us that we can break apart our original expression into terms of the form

$$\frac{A}{x+3} + \frac{B}{x-1}.$$

To solve for A and B, we multiply our original factored denominator by Equation (14.14), which gives us the equation

$$x + 3 = A(x-1) + B(x+3). \tag{14.15}$$

We use the distributive property on the right side of this equation, then collect common terms and factor, giving us

$$x + 3 = x(A+B) + (3B-a). \tag{14.16}$$

Algebra tells us that we can equate the coefficients of terms with like factors, giving us the system of equations

$$3 = 3B - A,$$
$$1 = A + B.$$

This system contains two equations and two variables, so we can solve for

A and B. We find that $A = -1$ and $B = 2$, making our final answer

$$\frac{x+3}{x^2-3x+2} = \frac{2}{x-1} - \frac{1}{x+3}. \tag{14.17}$$

Example 14.27.

$$\begin{aligned}\frac{x+3}{x^4+3x^3+6x^2+12x+8} &= \frac{x+3}{(x^2+4)(x+1)(x+2)} \\ &= \frac{A}{x+1} + \frac{b}{x+2} + \frac{Cx+D}{x^2+4}\end{aligned}$$

Example 14.28.

$$\begin{aligned}\frac{x+3}{x^4+5x^2+4} &= \frac{x+3}{(x^2+4)(x^2+1)} \\ &= \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+1}\end{aligned}$$

Example 14.29.

$$\begin{aligned}\frac{1}{x^2-1} &= \frac{1}{(x-1)(x-1)} \\ &= \frac{A}{x+1} + \frac{B}{x-1}\end{aligned}$$

Example 14.30.

$$\begin{aligned}\frac{1}{x^2-x} &= \frac{1}{x(x-1)} \\ &= \frac{A}{x} + \frac{B}{x-1}\end{aligned}$$

Example 14.31.

$$\begin{gathered}\frac{9x^2+7x-4}{x^3-3x^2-4x} \\ \frac{9x^2+7x-4}{x(x^2-3x-4)} \\ \frac{9x^2+7x-4}{x(x-4)(x+1)} \\ \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+1}\end{gathered}$$

Example 14.32.

$$\frac{x+3}{x^2-4} \neq \frac{A}{x+2} + \frac{B}{x-2}$$
$$\frac{x+3}{x^2-4} = \frac{x^3+0x^2+0x+4}{x^2-4}$$
$$\frac{x+3}{x^2-4} = x + \underbrace{\frac{4x+4}{x^2-4}}_{\frac{A}{x+2}+\frac{B}{x-2}}$$

Example 14.33.

$$\int \frac{8x^4+6x^2-3x+1}{2x^2-x+2}$$

Solution.
Use long division to simplify as follows:

$$\begin{array}{r|l}
 & 4x^2+2x \\ \hline
2x^2-x+2 & 8x^4 \qquad +6x^2-3x+1 \\
 & -8x^4+4x^3-8x^2 \\ \hline
 & 4x^3-2x^2-3x \\
 & -4x^3+2x^2-4x \\ \hline
 & -7x+1
\end{array}$$

$$\frac{8x^4+6x^2-3x+1}{2x^2-x+2} = 4x^2+2x+\frac{\overbrace{-7x+1}^{Ax+B}}{2x^2-x+2}$$

So

$$\int \frac{8x^4+6x^2-3x+1}{2x^2-x+2} = \int 4x^2+2x+\frac{-7x+1}{2x^2-x+2}$$

Then solve.

Example 14.34.

$$\int \frac{dx}{x^2+2x}$$

Solution.
Note that

$$\left[\frac{1}{x^2+2x}\right] \text{ is } \left[\frac{A}{x}+\frac{B}{x+2}\right]$$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2+2x} &= \int \frac{\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x+2}\,\mathrm{d}x \\
&= \int \frac{\frac{1}{2}}{x}\,\mathrm{d}x + \int \frac{\frac{1}{2}}{x+2}\,\mathrm{d}x \\
&= \frac{1}{2}\ln|x| + \frac{-1}{2}\int \frac{\mathrm{d}u}{u} \\
&= \frac{1}{2}\ln|x| + \ln|x+2|
\end{aligned}$$

Example 14.35. Write the partial fraction decomposition of

$$\frac{1}{x^4+2x^2+1}$$

Solution.
It's a trinomial, so we know we have to try binomial times binomial–and that works.

$$\frac{1}{x^4+2x^2+1} = \frac{1}{(x^2+1)(x^2+1)} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1}$$

Multiply both sides by the denominator

$$= Ax + B + (x^2+1)(Cx+D)$$

Now, solve for A, B, C, and D.

$$\begin{aligned}
1 &= Ax + B + (x^2+1)(Cx+D) \\
1 &= B + D \\
0 &= A + C \\
0 &= C \\
0 &= D
\end{aligned}$$

So A is 0, B is 1, C is 0, and D is 0.

Example 14.36. Integrate

$$\int \frac{1}{(x+1)(x-1)}.$$

Solution.

$$\int \frac{1}{(x+1)(x-1)} = \int \frac{A}{x+1} + \frac{B}{x-1}\,\mathrm{d}x$$

Solve for A and B.

$$\begin{aligned} &= \int \frac{-1/2}{x+1} + \frac{1/2}{x-1} \\ &= \frac{-1}{2}\int \frac{\mathrm{d}x}{x+1} + \frac{1}{2}\int \frac{\mathrm{d}x}{x-1} \end{aligned}$$

Example 14.37. Integrate

$$\int \frac{x^2}{x+8}\,\mathrm{d}x$$

Solution.
Use long division to simplify the integrand:

$$\begin{array}{r} x \quad - 8 \\ x+8\overline{)\; x^2 \qquad\qquad} \\ \underline{-\, x^2 - 8x} \\ -\, 8x \qquad \\ \underline{8x + 64} \\ 64 \end{array}$$

$$\begin{aligned} \int \frac{x^2}{x+8}\,\mathrm{d}x &= \int x - 8 + \frac{64}{x-8}\,\mathrm{d}x \\ &= \frac{1}{2}x^2 - 8x + 64\ln|x+8| + C \end{aligned}$$

Example 14.38. Integrate

$$\int \frac{1}{x^2-5x-6}\,\mathrm{d}x$$

Solution.
Use partial fraction decomposition:

$$\frac{1}{x^2-5x-6} = \frac{1}{(x-6)(x+1)}$$

$$
\begin{aligned}
\frac{1}{(x-6)(x+1)} &= \frac{A}{x-6} + \frac{B}{x+1} \\
1 &= A(x+1) + B(x-6) \\
1 &= Ax + A + Bx - 6B \\
1 &= Ax + Bx + A - 6B
\end{aligned}
$$

If $x = 0$, then $A - 6B = 1$. If $x = 1$, then $2A - 5B = 1$. We can substitute the first equation into the second to get

$$
\begin{aligned}
2(1 + 6B) - 5B &= 1 \\
2 + 12B - 5B &= 1 \\
B &= -1/7
\end{aligned}
$$

$$
\begin{aligned}
A - 6B &= 1 \\
A - 6(1/7) &= 1 \\
A &= 1/7
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{x^2 - 5x - 6} &= \frac{1/7}{x-6} - \frac{1/7}{x+1} \\
\int \frac{1}{x^2 - 5x - 6}\,\mathrm{d}x &= \int \frac{1/7}{x-6} - \frac{1/7}{x+1}\,\mathrm{d}x \\
&= \frac{1}{7}\int \frac{1}{x-6} - \frac{1}{x+1}\,\mathrm{d}x \\
&= \frac{1}{7}\left(\ln|x-6| - \ln|x+1|\right) + C
\end{aligned}
$$

14.9. Simpson's Rule

Example 14.39. Estimate

$$\int_1^5 \frac{1}{x}\,\mathrm{d}x$$

using the trapezoidal rule with four trapezoids.

Remark. For one trapezoid:

$$\frac{\Delta x}{2}(y_0 + y_1)$$

For the entire Trapezoid Rule:

$$\frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + y_n)$$

For full-blown Simpson's Rule:

$$\frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

Remark. The question might be seen as: "... using the trapezoidal rule with $n = 4$."

Solution.

$$\begin{aligned}\int_1^5 \frac{1}{x}\,\mathrm{d}x \approx & \frac{\overbrace{\Delta x}^{1}}{2}\left(\frac{1}{1} + 2\frac{1}{2} + 2\frac{1}{3} + 2\frac{1}{4} + 1\frac{1}{5}\right)\\ =&\frac{1}{2}(2^{1/2} + 2/3 + 1/5)\\ =&\frac{1}{2}(5/2 + 13/15)\\ =&\frac{101}{60}\end{aligned}$$

Example 14.40. Estimate

$$\int_1^5 \frac{1}{x}\,\mathrm{d}x$$

using Simpson's Rule with $n = 8$.

Solution.

$$\int_1^5 \frac{1}{x}\,\mathrm{d}x \approx \frac{\overbrace{\Delta x}^{1/2}}{3}\left(\frac{1}{1} + 4\frac{1}{3/2} + 2\frac{1}{2} + 4\frac{1}{5/2} + \dots\right)$$

Not a real problem, so not finished.

14.10. Improper Integrals

Definition 14.2. *Improper integrals* occur in cases where you are integrating across an infinite domain or infinite range.

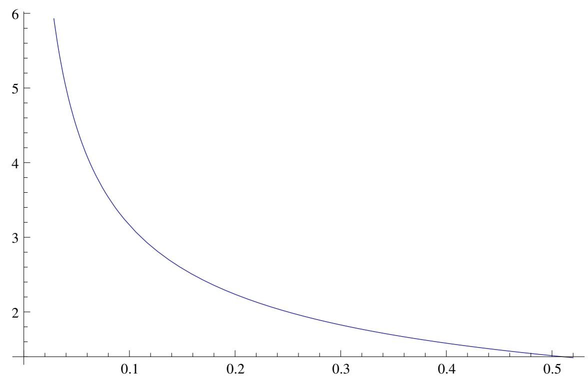


Figure 14.7.: $\frac{1}{\sqrt{x}}$ is an example of a function that can produce improper integrals.

For most integrals of this type, we can usually rewrite the integral in terms of something we know how to evaluate. For example, in

$$\int_2^6 \frac{1}{(x-6)^2}\,\mathrm{d}x, \tag{14.18}$$

we have a vertical asymptote at $x = 6$, as shown in Figure 14.8.

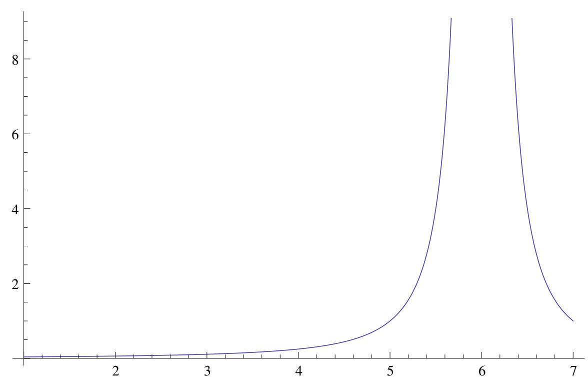


Figure 14.8.: A plot of $\frac{1}{(x-6)^2}$.

So instead of integrating from 2 to 6, we simply integrate from 2 to t, and then take the limit as $t \to 6^-$ of our answer.

$$\begin{aligned}\int_2^t \frac{1}{x+6}\,\mathrm{d}x =& \left.\frac{(x-6)^{-1}}{-1}\right|_2^t \\ =& \lim_{t\to 6^-} \frac{(t-6)^{-1}}{-1} - \frac{(2-6)^{-1}}{-1}\end{aligned}$$

this limit diverges to ∞, so our integral is undefined.

14.10.1. Infinite Range

Example 14.41. Integrate:

$$\int_1^9 \frac{1}{x-4}\,\mathrm{d}x \tag{14.19}$$

Solution.
This one is very similar to Eq. (14.18). Attempting to integrate "toward" either side of positive 4 results in an infinite range: while $x \to 4^-$, the expression tends toward $-\infty$, and while $x \to 4^+$, toward $+\infty$. Approaching this explicitly, we first attempt the indefinite integral:

$$\int \frac{1}{x-4}\,\mathrm{d}x.$$

Taking $u = x - 4,\ \mathrm{d}u = \mathrm{d}x$, we get

$$\int \frac{1}{x-4}\,\mathrm{d}x = \ln|x-4| + C,$$

which diverges toward $-\infty$ as we approach either side of 4. Regardless, to show this, we now evaluate the definite integral in terms of limits,

$$\lim_{t\to 4^-} \ln|x-4|\Big|_1^t + \lim_{s\to 4^+} \ln|x-4|\Big|_s^9$$

This expands into:

$$\lim_{t\to 4^-} \ln|x-4|\Big|_t - \ln|x-4|\Big|_1 + \lim_{s\to 4^+} \ln|x-4|\Big|_s - \ln|x-4|\Big|_9$$

which is the same as:

$$\lim_{t\to 4^-} \ln|x-4|\Big|_t - \ln 3 + \lim_{s\to 4^+} \ln|x-4|\Big|_s - \ln 5$$

because $\log_a u + \log_a v = \log_a uv$,

$$\lim_{t\to 4^-} \ln|x-4|\Big|_t + \lim_{s\to 4^+} \ln|x-4|\Big|_s - \ln 15$$

however, $\ln|x|$ diverges toward $-\infty$ as $x \to 0$, and hence Eq. (14.19) diverges.

Example 14.42. Integrate:

$$\int_4^{39} \frac{1}{\sqrt{x-4}}\,\mathrm{d}x \tag{14.20}$$

Unlike Ex. 14.41, this integral is only a problem as $x \to 4^+$, thanks to the square root in the denominator. Of course, the value of $1/(4-4)$ is still undefined.

When we integrate Eq. (14.20), however, it does not diverge. Let $u = x - 4$, $\mathrm{d}u = \mathrm{d}x$ and we find

$$\int \frac{1}{\sqrt{x-4}} = 2\sqrt{x-4} + C,$$

which we see would not be a problem to evaluate with definite integrals.

Example 14.43.

$$\int_{-4}^{0} \ln|x|\,\mathrm{d}x$$

Left approach problem spot.

Example 14.44.

$$\int_{-6\pi}^{10\pi} \sec x\,\mathrm{d}x$$

Problems with right and left approach.

Example 14.45. This is considered a "Classic."

$$\int_a^b \frac{1}{(b-x)^p}\,\mathrm{d}x \qquad \text{and} \qquad \int_a^b \frac{1}{(x-a)^p}\,\mathrm{d}x$$

Example 14.46. Integrate:

$$\int_{-6}^{-2} \frac{1}{(x+6)^1}\,\mathrm{d}x$$

Solution.

$$\int_t^{-2} \frac{1}{x+6}\,\mathrm{d}x = \ln|x+6|\Big|_t^{-2}$$

$$= \lim_{t \to -6^+} \ln|-2+6| - \underbrace{\ln|-6^+ + 6|}_{\text{Diverges.}}$$

Example 14.47. Integrate:

$$\int_1^3 \frac{1}{\sqrt{x-1}} = \left.\frac{(x-1)^{1/2}}{1/2}\right|_t^3$$
$$= \lim_{t \to 1+} \frac{(3-1)^{1/2}}{1/2} - \frac{(t-1)^{1/2}}{1/2}$$

Example 14.48. Integrate.

$$\int_4^{39} \frac{1}{\sqrt{x-4}}\,\mathrm{d}x$$

Substitute with $4 \to t$:

$$\int_t^{39} \frac{1}{\sqrt{x-4}}\,\mathrm{d}x$$

And evaluate:

$$\int_t^{39} \frac{1}{\sqrt{x-4}}\,\mathrm{d}x = \lim_{t \to 4^+} \left.\frac{(x-4)^{1/2}}{1/2}\right|_t^{39}$$
$$= \lim_{t \to 4^+} \frac{(39-4)^{1/2}}{1/2} - \frac{(t-4)^{1/2}}{1/2}$$
$$= \frac{(39-4)^{1/2}}{1/2}$$

Example 14.49. Integrate.

$$\int_1^{29} \frac{1}{x-4}\,\mathrm{d}x$$

Solution.
Substitute $29 \to t$:

$$\int_1^{29} \frac{1}{x-4}\,\mathrm{d}x = \int_1^t \frac{1}{x-4}\,\mathrm{d}x + \int_t^{29} \frac{1}{x-4}\,\mathrm{d}x$$
$$= \ln|x-4|\Big|_1^t + \ln|x-4|\Big|_t^{29}$$
$$= \ln|t-4| - \ln|1-4| + \ln|29-4| - \ln|t-4|$$

$$= \lim_{t \to 4^-} \left[\underbrace{\ln|t-4|}_{\frac{1}{-\infty}} - \ln|1-4| \right] + \lim_{t \to 4^+} \left[\ln|29-4| - \underbrace{\ln|t-4|}_{\frac{1}{\infty}} \right]$$

Diverges.

Example 14.50. Integrate:

$$\int_0^5 \ln x \, \mathrm{d}x$$

Solution.

$$\begin{aligned} \int_t^5 \ln x \, \mathrm{d}x &= x \ln x - x \Big|_t^5 \\ &= 5 \ln 5 - 5 - t \ln t + t \end{aligned}$$

Now evaluate the limit as:

$$\begin{aligned} \int_0^5 \ln x \, \mathrm{d}x &= \lim_{t \to 0^+} 5 \ln 5 - 5 - t \ln t + t \\ &= 5 \ln 5 - 5 \end{aligned}$$

14.10.2. Infinite Domain

For these, the integrand is well behaved throughout the integration interval. The problem is that we are integrating across an interval $[a, b]$ where either a or b is ∞ or $-\infty$.

$$\int_a^\infty f(x) \, \mathrm{d}x \tag{14.21}$$

We would treat this an indefinite integral:

$$\int f(x) \, \mathrm{d}x = F(x)$$

Then evaluate it as follows:

$$\lim_{t \to \infty} F(x) \Big|_a^t$$

$$\int_{-\infty}^{b} f(x)\,\mathrm{d}x \tag{14.22}$$

Like the first one, we would treat this as an indefinite integral:

$$\int f(x)\,\mathrm{d}x = F(x)$$

Then evaluate it as follows:

$$\lim_{t\to\infty} F(x)\Big|_{-\infty}^{b}$$

Others, still, look like this:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \tag{14.23}$$

Treat this as a combination of two of the above integrals. Rewrite it as:

$$\int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_{a}^{\infty} f(x)\,\mathrm{d}x$$

choosing some arbitrary a, which we usually choose to be 0.

Example 14.51. For $p > 1$, does the following integral converge or diverge?

$$\int_{1}^{\infty} \frac{1}{x^p}\,\mathrm{d}x$$

Solution.
The integral converges.

Example 14.52.

$$\int_{1}^{\infty} x^{-3}\,\mathrm{d}x$$

Solution.

$$\begin{aligned}\int_1^\infty x^{-3}\,\mathrm{d}x &= \lim_{t\to\infty} \frac{x^{-2}}{2}\bigg|_1^t \\ &= \lim_{t\to\infty} \frac{t^{-2}}{-2} - \frac{1^{-2}}{-2} \\ &= \lim_{t\to\infty} \frac{1}{-2t^2} + \frac{1}{2}\end{aligned}$$

The integral converges to $1/2$.

Example 14.53.

$$\int_1^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x$$

Solution.

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x &= \lim_{t\to\infty} \frac{x^{1/2}}{\frac{1}{2}}\bigg|_1^t \\ &= \lim_{t\to\infty} \frac{t^{1/2}}{\frac{1}{2}} - \frac{1^{1/2}}{\frac{1}{2}} \\ &= \lim_{t\to\infty} 2t^{1/2} - 2\end{aligned}$$

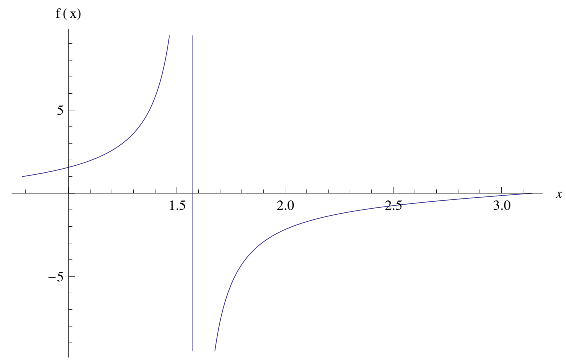
The integral diverges.

Example 14.54.

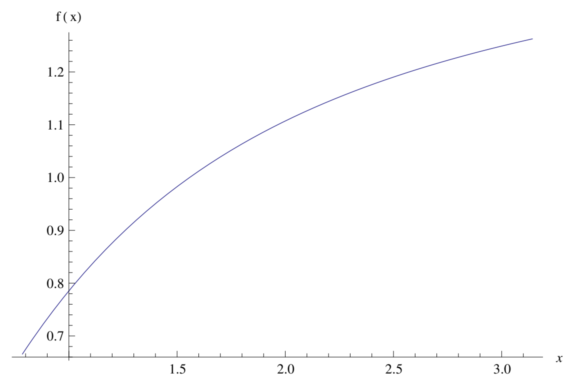
$$\int_0^\infty \frac{1}{x^2+1}$$

Solution.

$$\begin{aligned}\int_0^\infty \frac{1}{x^2+1} &= \lim_{t\to\infty} \arctan x\bigg|_0^t = \lim_{t\to\infty} \arctan t - \arctan 0 \\ &= \lim_{t\to\infty} \arctan t \\ &= \lim_{t\to\infty} \arctan t \\ &= \frac{\pi}{2}\end{aligned}$$



(a) $f(x) = \tan x$.



(b) $f(x) = \arctan x$.

Figure 14.9.: $\tan x$ compared with $\arctan x$.

Example 14.55.

$$\int_{-\infty}^{0} \frac{x^2}{x^3-1}$$

Remark. If $0 \to 1$, this would be a Type I integral.

Solution.

$$\begin{aligned}\int_{-\infty}^{0} \frac{x^2}{x^3-1} &= \int_{-\infty}^{0} \frac{1}{3}\frac{\mathrm{d}u}{u} \\ &= \lim_{t\to\infty} \frac{1}{3}\ln|x^3-1|\Big|_t^0 = \lim_{t\to\infty} \frac{1}{3}\ln|-1| - \frac{1}{3}\ln|t^3-1| \\ &= \lim_{t\to\infty} \frac{1}{3} ln 1 - \frac{1}{3}\ln|t^3-1|\end{aligned}$$

$$= \lim_{t\to\infty} -\frac{1}{3}\ln|t^3-1|$$

The integral diverges.

Example 14.56.

$$\int_{-\infty}^{0} e^x \,\mathrm{d}u$$

Remark. This is likely to converge, because of the $e^{-\infty}$.

Solution.

$$\begin{aligned}\int_{-\infty}^{0} e^x \,\mathrm{d}u &= \lim_{t\to-\infty} e^x\Big|_t^0 \\ &= \lim_{t\to-\infty} e^0 - e^t\end{aligned}$$

The integral converges to 1.

Example 14.57.

$$\int_{-\infty}^{\infty} e^{-x}\cos x \,\mathrm{d}x$$

Solution.
First, break this into two parts.

$$\int_{-\infty}^{\infty} e^{-x}\cos x \,\mathrm{d}x = \int_{-\infty}^{0} e^{-x}\cos x \,\mathrm{d}x + \int_{0}^{\infty} e^{-x}\cos x \,\mathrm{d}x$$

Then, treat it as an indefinite integral and evaluate using integration by parts:

$$\int e^{-x}\cos x \,\mathrm{d}x = \frac{e^{-x}\sin x - e^{-x}\cos x}{2}$$

Now, return to our original function.

$$\begin{aligned}\lim_{t\to\infty} e^{-x}\cos x \,\mathrm{d}x &= \lim_{t\to\infty} \frac{e^{-x}\sin x - e^{-x}\cos x}{2}\Bigg|_t^0 - \frac{e^{-x}\sin x - e^{-x}\cos x}{2}\Bigg|_0^t \\ &= \lim_{t\to\infty}\left[\frac{e^{-0}\sin -e^{-0}\cos 0}{2} - \frac{e^{-t}\sin t - e^{-t}\cos t}{2}\right] - \left[0+\frac{1}{2}\right]\end{aligned}$$

Because the left half of this diverges, the entire integral diverges.

Example 14.58. According to James E. Martin, "this is one of the classics:"

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x \tag{14.24}$$

Solution.
Consider:

$$\int_1^\infty \frac{1}{x^{1.000000001}}\,\mathrm{d}x = x^{-0.000000001}\Big|_1^\infty = 0 - \frac{1}{1^{0.000000001}}$$

Therefore:

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,\mathrm{d}x &= \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| - \ln|1| \\ &= \lim_{t\to\infty} \ln|t|\end{aligned}$$

The integral diverges.

Example 14.59.

$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = \lim_{t\to\infty} \frac{x^{-1}}{-1}\Big|_1^t$$

This integral converges.

Example 14.60.

$$\int x^2 e^x\,\mathrm{d}x$$

Solution.

$$\begin{aligned}\int x^2 e^x\,\mathrm{d}x =& x^2e^x - 2\int xe^x\,\mathrm{d}x \\ =& x^2e^x - 2\left(xe^x - \int e^x\,\mathrm{d}x\right) \\ =& x^2e^x - 2xe^x + 2x + C\end{aligned}$$

Example 14.61.

$$\int_2^4 \frac{1}{x-4}\,\mathrm{d}x$$

Solution.

$$\begin{aligned}\int_2^4 \frac{1}{x-4}\,\mathrm{d}x =& x\ln|x-4|\Big|_2^t \\ =& \ln|t-4| - \ln|-2|\end{aligned}$$

Now we go back to the original integral

$$\int_2^4 \frac{1}{x-4}\,\mathrm{d}x = \lim_{t\to 4^-} \ln|t-4| - \ln 2$$

Remembering the graph of $\ln|x|$

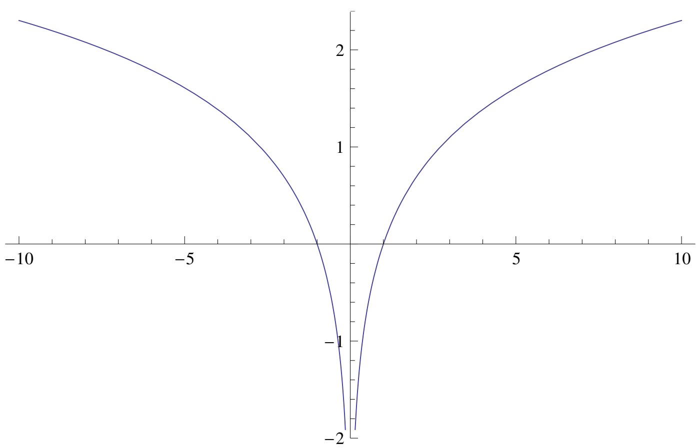


Figure 14.10.: A plot of $f(x) = \ln|x|$

we can conclude that the integral diverges toward $-\infty$.

Example 14.62.

$$\int_5^6 \frac{1}{(x-5)^{1/2}}\,\mathrm{d}x$$

Solution.

$$\begin{aligned}\int_t^6 \frac{1}{(x-5)^{1/2}}\,\mathrm{d}x =& 2\sqrt{x-5}\Big|_t^6 \\ =& 2\sqrt{6-5} - 2\sqrt{t-5}\end{aligned}$$

$$\begin{aligned}\int_5^6 \frac{1}{(x-5)^{1/2}}\,\mathrm{d}x =& \lim_{t\to 5^+} 2\sqrt{6-5} - \underbrace{2\sqrt{t-5}}_{0} \\ =& \lim_{t\to 5^+} 2\sqrt{6-5} \\ =& 2\end{aligned}$$

Example 14.63.

$$\int_{\frac{8\sqrt{3}}{3}}^{\infty} \frac{1}{x^2+64}\,\mathrm{d}x$$

Solution.

$$\begin{aligned}\int \frac{1}{x^2-64}\,\mathrm{d}x =& \int \frac{1}{64(\frac{x^2}{64}+1)} \\ =& \frac{1}{8}\arctan\frac{x}{8} + C\end{aligned}$$

$$\int_{\frac{8\sqrt{3}}{3}}^{t} \frac{1}{x^2+64}\,\mathrm{d}x = \frac{1}{8}\arctan\frac{x}{8}\Big|_{\frac{8\sqrt{3}}{3}}^{t}$$

$$\begin{aligned}\int_{\frac{8\sqrt{3}}{3}}^{\infty} \frac{1}{x^2+64}\,\mathrm{d}x =& \lim_{t\to\infty} \frac{1}{8}\arctan\frac{t}{8} - \frac{1}{8}\arctan\frac{\sqrt{3}}{3} \\ =& \frac{1}{8}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \\ =& \frac{\pi}{24}\end{aligned}$$

15. Sequences

A **sequence** is an ordered list of numbers. They are mostly important because of their application to **series**, detailed in Chapter 16.

15.1. Representing a Sequence

Definition 15.1. A **sequence** is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order. This order is important: The sequence $2, 4, 6, 8, \ldots$ is not the same as the sequence $4, 2, 8, 6, \ldots$.

When we write sequences like $\{a_n\}$, we are talking about the entire sequence. A sequence like this is described as "starting at" a certain *index*, usually $n = 1$. Most of the sequences we will be talking about will be infinite in length.

We can also write sequences using rules that describe their terms, as follows:

Example 15.1.

$$a_n = \frac{n!}{2n! + 1}$$

We can write out a few terms of the sequence

$$\left\{ \frac{1}{3}, \frac{2}{5}, \frac{6}{13}, \frac{24}{49}, \frac{120}{241}, \ldots \right\}$$

and then plot them to get a feel that the sequence is getting closer and closer to a certain number:

$$\frac{1}{2}.$$

We would thus describe this sequence as *converging to* the number $\frac{1}{2}$

Definition 15.2. The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that for all n

$$n > N \to |a_n - L| < \varepsilon$$

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, and call L the *limit* of the sequence.

To really understand this, try thinking of it this way: pick a number ε. It can be very large

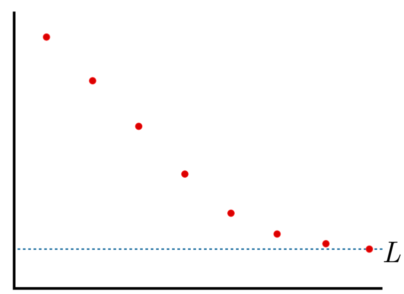
$$\varepsilon = 1000$$

or really small

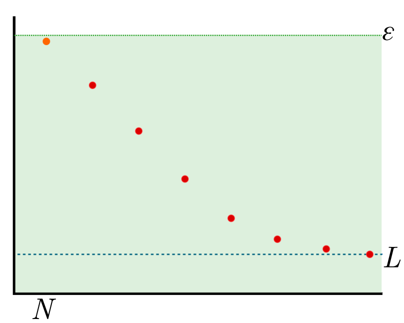
$$\varepsilon = 0.0001$$

but no matter which one we pick, we can find an index N on the sequence such that for every index past it, we are always within ε of the limit L for the sequence. Note that we don't have to know what this number L is, nor what ε is for that matter, just that these numbers exist to state that a sequence is *converging*.

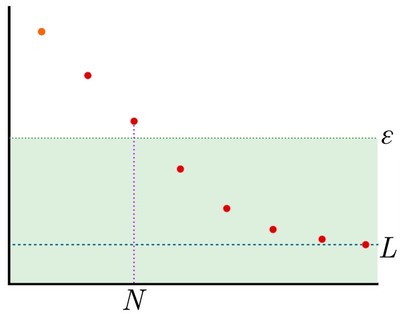
Example 15.2. Let us examine the following imaginary sequence:



Note that for a large ε, our corresponding N occurs very early in the sequence:



However, if we shrink our ε, the number N needed such that every element in the sequence after it is within ε of L gets larger. The real key, however, is that it *always exists*.

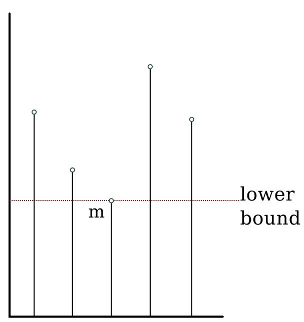


Definition 15.3. The sequence $\{a_n\}$ **diverges** if the sequence does not converge to any number L.

That is, this number L does not exist.

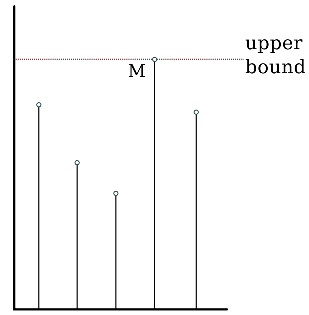
15.1.1. Bounded Sequences

Definition 15.4. If there exists a number m such that $m \leq a_n$ for every n we say the sequence is **bounded below**. The number m is called a **lower bound** for the sequence.



Definition 15.5. If there exists a number M such that $a_n \leq M$ for every n we say that the sequence is **bounded above**. The number M is called an **upper bound** for the sequence.

Definition 15.6. If the sequence is both bounded below and bounded above we call the sequence **bounded**.



15.1.2. Nondecreasing and Nonincreasing Sequences

Definition 15.7. Given a sequence $\{a_n\}$, we call the sequence **nondecreasing** if

$$\forall n(a_n \geq a_{n-1}).$$

This means that across the entire sequence, a given element in the sequence is either greater than or equal to the preceding element. A nondecreasing sequence is different from a strictly *increasing* sequence in that it allows for two elements to be equal.

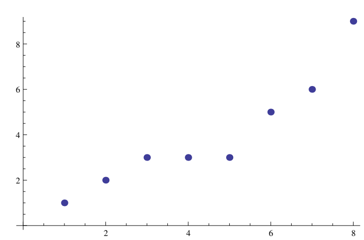


Figure 15.1.: This sequence is not *increasing* everywhere, although it is *nondecreasing*.

Definition 15.8. Given a sequence $\{a_n\}$, we call the sequence **nonincreasing** if

$$\forall n(a_n \leq a_{n-1}).$$

Definition 15.9. If $\{a_n\}$ is either nondecreasing or nonincreasing we call it **monotonic**.

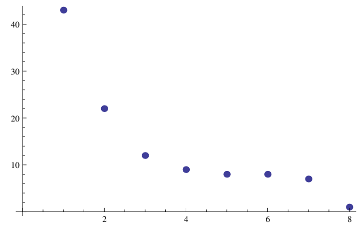


Figure 15.2.: An example of a nonincreasing sequence.

Example 15.3.

$$a_n = \frac{n+1}{2n-1}$$

Let's take a more detailed look at an example of a sequence.

To get a feel for the behavior of the sequence, let's find a few of its values:

$$\frac{n+2}{2n-1} = \left\{3, \frac{4}{3}, 1, \frac{6}{7}, \frac{7}{9}, \frac{8}{11}, \cdots\right\}$$

We could take the derivative of the similar function

$$f(x) = \frac{x+2}{2x-1}$$

to try to figure out what is happening. Since we can assume $n \geq 1$ for our sequence, for $x \geq 1$ the derivative of $f(x)$ should also be somewhat representative of a_n.

$$\begin{aligned} f'(x) &= \frac{(2x-1)\left(\frac{\mathrm{d}}{\mathrm{d}x}(x+2)\right) - (x+2)\left(\frac{\mathrm{d}}{\mathrm{d}x}(2x-1)\right)}{(2x-1)^2} \\ &= \frac{2x-2(x+2)-1}{(2x-1)^2} \\ &= -\frac{5}{1-4x+4x^2} \end{aligned}$$

Based on Table 15.1, we can conclude that the sequence is nonincreasing.

x	0	0.5	1
$f'(x)$	$-$	undefined	$-$

Table 15.1.: A sign diagram for $f'(x)$.

We can therefore describe this sequence as *monotonic*.

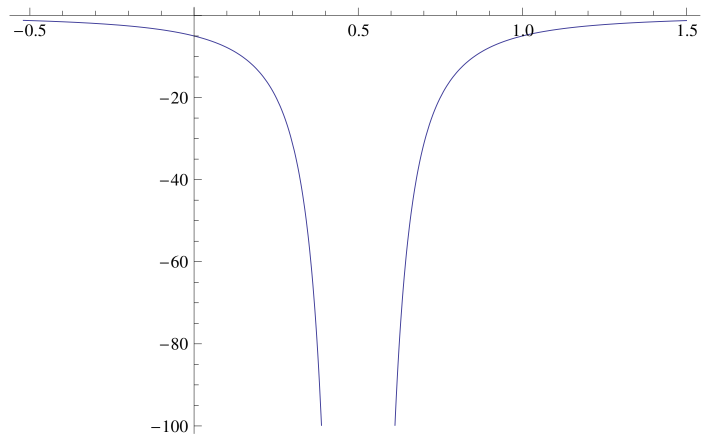


Figure 15.3.: A plot of $f'(x)$.

Furthermore, note the end-behavior of this derivative:

$$\lim_{x\to\infty} f'(x) = 0$$

This implies that, eventually, the rate of change evens off to an indefinitely small number. Presumably, this would mean that $f(x)$ has a horizontal asymptote. Based on this information, it seems reasonable to conclude that the sequence a_n is converging upon a number. We do not, however, know to what number it converges.

To find that, we will need to take the following limit:

$$\lim_{n\to\infty} a_n$$

How do we evaluate limits for sequences? As it turns out, we usually treat them exactly the same as limits of functions, which are described in Chapter 11.

Theorem 24. Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \to \lim_{n\to\infty} a_n = L.$$

Proof. Suppose that $\lim_{x\to\infty} f(x) = L$. Then for each positive number ε there is a number M such that for all x,

$$x > M \implies \left|f(x) - L\right| < \varepsilon.$$

Let N be an integer greater than M and greater than or equal to n_0. Then

$$n > N \implies a_n = f(n),$$

and

$$\left|a_n - L\right| = \left|f(n) - L\right| < \varepsilon. \qquad \square$$

[7, p. 537]

Theorem 24 allows us to use *l'Hospital's Rule* to find the limits of some sequences. We can state that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(x)$$

assuming

$$f(x) = \frac{n+1}{2n-1}.$$

And thus evaluate the following limit:

$$\begin{aligned} \lim_{n\to\infty} \frac{n+1}{2n-1} &\overset{\text{H}}{=} \lim_{n\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(n+1)}{\frac{\mathrm{d}}{\mathrm{d}x}2n-1} \\ &\overset{\text{H}}{=} \lim_{n\to\infty} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

This shows us that the sequence a_n also converges to $\frac{1}{2}$.

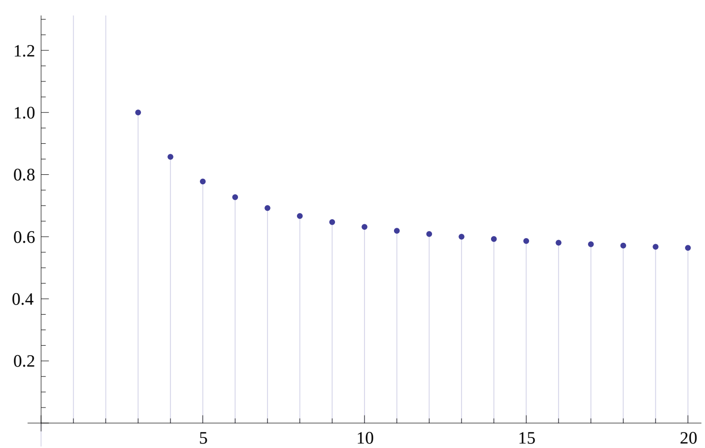


Figure 15.4.: A graph of the sequence $a_n = \frac{n+1}{2n-1}$.

Example 15.4. Demonstrate that $\{n!\}$ is nondecreasing.

Solution.
First let us look at the definition for a factorial.

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n(n-1)! & \text{if } n > 0 \end{cases} \tag{15.1}$$

Proof. Remember our definition for a nondecreasing sequence from 15.1.2:

$$\forall n(a_n \geq a_{n-1}).$$

Let us assume that this holds true for our sequence $\{n!\}$ when $n > 1$.

$$n! \geq (n-1)! \qquad n > 1$$

Divide each side by $(n-1)!$

$$\frac{n!}{(n-1)!} \geq \frac{(n-1)!}{(n-1)!} \qquad n > 1$$
$$\frac{n!}{(n-1)!} \geq 1 \qquad n > 1$$

Now, remembering our definition for the factorial,

$$\frac{n(n-1)!}{(n-1)!} \geq 1 \qquad n > 1$$
$$n \geq 1 \qquad n > 1$$

□

Example 15.5. Show that the sequence

$$\{a_n\} = \left\{ \frac{n+2}{2n-1} \right\}_{n=1}^{\infty}$$

is nonincreasing.

Solution.
This example is very similar to Example 15.3, with only a change of constant in the numerator. We will demonstrate this sequence's behavior using a different method than before, however.

Another way we can demonstrate that a sequence is decreasing is by using the definition of a *decreasing sequence*: $\forall n(a_n < a_{n-1})$. We simply substitute in n and $n-1$ on the respective sides of the inequality.

Proof.

$$\begin{aligned}
\forall n(a_n &< a_{n-1}) && n > 1 \\
\frac{n+2}{2n-1} &< \frac{(n-1)+2}{2(n-1)-1} && n > 1 \\
\frac{n+2}{2n-1} &< \frac{n+1}{2n-3} && n > 1 \\
\left(\frac{2n-3}{2n-3}\right)\left(\frac{n+2}{2n-1}\right) &< \left(\frac{n+1}{2n-3}\right)\left(\frac{2n-1}{2n-1}\right) && n > 1 \\
\frac{2n^2-3n+4n-6}{4n^2-6n-2n+3} &< \frac{2n^2+2n-n-1}{4n^2-6n-2n-3} && n > 1 \\
\frac{2n^2+n-6}{4n^2-8n+3} &< \frac{2n^2+n-1}{4n^2-8n-4} && n > 1 \\
2n^2+n-6 &< 2n^2+n-1 && n > 1 \\
-6 &< -1 && n > 1
\end{aligned}$$

□

Example 15.6. Show that

$$\left\{\frac{n}{n+1}\right\}$$

is monotonic increasing.

Solution. *Proof.*

$$\forall n > 1\,(a_n > a_{n-1})$$

$$\begin{aligned}
a_n &> a_{n-1} \\
\frac{n}{n+1} &> \frac{n-1}{n-1+1} \\
\frac{n}{n}\frac{n}{n+1} &> \frac{n-1}{n}\frac{n+1}{n+1} \\
n^2 &> (n-1)(n+1) \\
n^2 &> n^2-1 \\
0 &> -1
\end{aligned}$$

□

Example 15.7. Does

$$a_n = \frac{n+1}{2^n}$$

converge or diverge?

Solution.

Remembering that we can rewrite 2^n as $e^{n \cdot \ln 2}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{n+1}{2^n} &= \lim_{n\to\infty} \frac{n+1}{e^{n\cdot\ln 2}} \\
&\overset{\text{H}}{=} \lim_{n\to\infty} \frac{1}{\ln 2 \cdot e^{n \ln 2}} \\
&= \lim_{n\to\infty} \frac{1}{\ln 2 \cdot 2^n}
\end{aligned}$$

Using the constant multiple rule for limits of sequences, we can pull out the $\ln 2$.

$$= \frac{1}{\ln 2} \cdot \left(\lim_{n\to\infty} \frac{1}{2^n} \right)$$

$\{\frac{1}{2^n}\}$ is converging very quickly toward 0, so a_n as well must converge to 0.

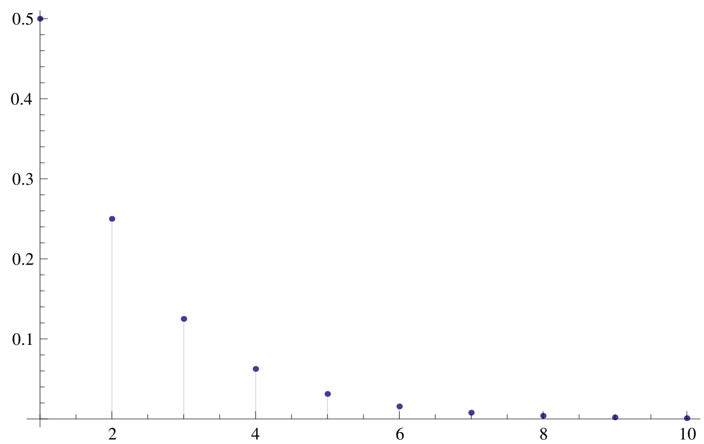


Figure 15.5.: A plot of $\{\frac{1}{2^n}\}$.

Example 15.8. Does

$$\left\{ \frac{1}{n!} \right\}$$

converge or diverge?

Solution.

$$\lim_{n\to\infty} \frac{1}{n!} = 0$$

Alternatively, we could demonstrate that the sequence is bounded and monotonic.

Example 15.9. Does

$$a_n = \frac{2^n}{n!}$$

converge or diverge?

Solution.
We can show this, intuitively, as follows:

$$a_n = \left\{ \frac{2 \times 2 \times 2 \times 2 \times 2 \times \dots}{1 + 2 + 3 + 4 + 5 + 6 + \dots} \right\}$$

But the actual proof requires the *sandwich theorem for sequences*[7, p. 536]:

Theorem 25 (The Sandwich Theorem for Sequences)**.** Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

A proof of this theorem is found in 20.2. By Theorem 25,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

Example 15.10.

$$a_n = \frac{(-1)^n}{n!}$$

Solution.
Consider

$$b_n = \left| \frac{(-1)^n}{n!} \right| = \frac{1}{n!} = 0$$

Intuitively, we can also state that a_n should converge. Using Theorem 25, we can determine that this sequence converges.

Example 15.11.

$$a_n = \frac{1}{n - 0.\bar{9}}$$

Solution.
A sequence can be unbounded, and still converge.

$0.\bar{9}$ is very close to 1, close enough that we can treat it as 1 as $n \to \infty$.

Doing so, it becomes clear that this sequence converges, as $\frac{1}{n-1}$ converges. Consider, however, the very first term, where $n = 1$. This term is *gigantic*, essentially ∞. The sequence is not bounded, yet it still manages to converge. This sequence converges to 0.

Example 15.12. Does the following sequence converge?

$$a_n = \left(\frac{n+8}{9n}\right)\left(1 - \frac{8}{n}\right)$$

Find the limit if the sequence is convergent.

Solution.
Remembering that a limit of products is a product of limits,

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{n+8}{9n}\right)\left(1-\frac{8}{n}\right) &= \lim_{n\to\infty}\left(\frac{n+8}{9n}\right)\cdot\lim_{n\to\infty}\left(1-\frac{8}{n}\right)\\ &= \frac{1}{9}\end{aligned}$$

The sequence converges to $\frac{1}{9}$.

Example 15.13. Show that the sequence

$$a_n = \frac{2}{n^2}$$

is monotonic and bounded.

Solution.
Treat the sequence as a function $f(x)$ and take the derivative.

$$f(x) = \frac{2}{x^2}$$

$$f'(x) = -4x^{-3}$$

$f'(x)$ is negative on the interval $[1, \infty)$, so $f(x)$ is decreasing on the interval $[1, \infty)$. Since the sequence goes from 1 to ∞ and is always decreasing, its upper bound must be at $n = 1$. $a_1 = 2$, the upper bound for the sequence. To find the lower bound, we take the limit as $n \to \infty$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2}{n^2} = 0$$

Which shows us that 0 is the lower bound of the sequence.

Thus, we can conclude that the sequence is monotonic and bounded, and therefore converges.

Example 15.14. Find a formula for the nth term of the sequence where a_n is calculated directly from n.

$$\frac{1}{1}, \frac{4}{2}, \frac{7}{6}, \frac{10}{24}, \frac{13}{120} \dots$$

Solution.
We basically have to just take close guesses and see if we can make a sequence that follows this.

Doing so, we will find that the answer is

$$a_n = \frac{3n-2}{n!}.$$

16. Infinite Series

We produce an infinite series by taking the sum of an infinite sequence.

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Given a sequence $\{a_n\}$, the number a_n is the n**th term** of the series. The sequence $\{s_n\}$ is the **sequence of partial sums** of the series, defined by [7, p. 544]

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
&\vdots \\
s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{\infty} a_k \\
&\vdots
\end{aligned}$$

The number s_n represents the n**th partial sum** of the infinite series.

The partial sum s_n of a series can also be given by

$$s_n = s_{n-1} + a_n.$$

If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, then we say that the series **diverges**. [7, p. 544]

Given a series, we will usually want to know whether it converges or diverges, and if it converges, then to what number?

For converging series, there are some rules that can aid us:

Theorem 26 (Rules for combining series)**.** If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then the **sum rule** states that

$$\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B.$$

The **difference rule** for series states that

$$\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B.$$

And the **constant multiple rule** tells us

$$\sum ka_n = k \sum a_n = kA,$$

where k is a constant number.

[7, p. 549]

16.1. Limit Test

When presented with a series, such as

$$\sum_{n=1}^{\infty} \frac{n!}{2n! + 1},$$

if we wish to know whether it converges or diverges, we should generally first confirm that the series does not diverge before trying to determine its convergence.

For a series like this, it is better to think of it as a sum of numbers in a sequence, $\{a_n\}$, than as a sum operator operating on a function. In this case, $\{a_n\}$ would equal

$$\frac{n!}{2n! + 1}$$

Now think about it–if this $\{a_n\}$ is converging to any number other than 0, then as our n goes toward ∞, we will always have numbers to add. The sequence will continue growing. That is, the sequence will diverge.

So we should take the following limit:

$$\lim_{n\to\infty} \frac{n!}{2n! + 1}$$

Our limit rules tell us nothing about evaluating limits involving factorials. We can, however, use a creative little trick common in evaluating limits of polynomials: divide each term in both the numerator and denominator by the term of the highest power.

$$\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{\frac{n!}{n!}}{\frac{2\cdot n!}{n!} + \frac{1}{n!}}$$
$$= \frac{1}{2 + \frac{1}{n!}}$$

Now, think about where the term $\frac{1}{n!}$ is heading as $n \to \infty$. The denominator is getting huge, and very quickly, as if there were a high degree power of x in the denominator of a limit. In that case, $1/x^k$ would become meaningless–it would essentially disappear, so $1/n!$ must also go to 0.

$$= \frac{1}{2+0}$$
$$\lim_{n\to\infty} a_n = \frac{1}{2}$$

So the limit of $\{a_n\}$ as n approached ∞ is not zero. Thus, the *nth term* of the series, no matter how large n becomes, will never be zero. The series is always adding more numbers, and that means it is **diverging**.

It is worth noting that if, indeed, the $\lim_{n\to\infty}$ of $\{a_n\}$ did equal 0, this would tell us nothing about the convergence or divergence of the series.

Theorem 27. The series $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$.

Proof. We prove this by contrapositive, described in Section 4.2.1. The statement we are trying to prove is:

> If $\lim_{n\to\infty} a_n \neq 0$ then the series diverges.

We could write this as $P \to Q$ where P represents "The limit of a_n as $n \to \infty$ is nonzero" and Q represents "the series diverges." Thus, the contrapositive of this proposition would be $\neg Q \to \neg P$. If we can prove the contrapositive, then the original statement must be true. We would $\neg Q \to \neg P$ as:

> "If a series is convergent, then $\lim_{n\to\infty} a_n = 0$."

Now, let L represent the series' sum.

$$s_n = s_{n-1} + a_n$$
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} + a_n$$
$$L = L + \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} a_n = 0$$

Because we have proven this statement, we know that our original remark, If $\lim_{n\to\infty} a_n \neq 0$ then the series diverges,

must be true. $\square$

Let's try to put this theorem to use.

Example 16.1. Does the series

$$\sum_{n=1}^{\infty} \frac{6n}{6n+1}$$

diverge or converge?

Solution.
Take $\lim_{n\to\infty} \frac{6n}{6n+1}$. This gives us

$$\lim_{n\to\infty} \frac{6n}{6n+1} \overset{\text{H}}{=} \lim_{n\to\infty} \frac{6}{6} \neq 0$$

By Theorem 27, which we just proved, the series diverges.

16.2. Harmonic Series

Example 16.2. The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called the **harmonic series**. How does this series behave?[1]

We can try finding some of its partial sums:

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&\vdots \\
s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + a_n
\end{aligned}$$

[1]Note that Theorem 27 is useless to us here. The fact that $\lim_{n\to\infty} \frac{1}{n}$ is zero does not imply anything about convergence or divergence.

Initially, it might appear as if the harmonic sequence converges. However, we would be wrong to assume as much.

Notice that as we continue to add terms, we could group them in the following way:

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots \\ &= 1 + \frac{1}{2} + \left(\frac{7}{12}\right) + \left(\frac{533}{840}\right) + \left(\frac{95549}{144144}\right) + \cdots \end{aligned}$$

Calculating the exact values of these terms,

$$\approx 1 + 0.5 + (0.583333) + (0.635424) + (0.662872) + \cdots$$

we can see that the series is increasing indefinitely, just doing it very slowly.

16.3. Geometric Series

Geometric Series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$, where the only difference is the starting value of n. The ratio r can be positive or negative.

If $r = 1$, the nth partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$.

If $r = -1$, the series diverges because the nth partial sums alternate between a and 0.

If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Multiply s_n by r.

$$s_n - rs_n = \left[a + ar + ar^2 + \cdots + ar^{n-1}\right] - \left[ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n\right]$$

Subtract rs_n from s_n.

$$s_n - rs_n = (a) + (ar - ar) + \left(ar^2 - ar^2\right) + \cdots + \left(ar^{n-1} - ar^{n-1}\right) - ar^n$$

Rearrange and the inner terms cancel.

$$s_n - rs_n = a - ar^n$$
$$s_n(1 - r) = a(1 - r^n)$$

Factor.

$$s_n = \frac{a(1 - r^n)}{1 - r}, \quad (r \neq 1).$$

If $|r| < 1$, then $r^n \to 0$ as $n \to 0$ and $s_n \to a/(1 - r)$. If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.[7, p. 546]

Example 16.3.

$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$$

Solution.
This is a sum of two series. Theorem 26 tells us that a series of a sum of converging series is a sum of series.

Thus,

$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{3^n}{6^n} + \sum_{n=1}^{\infty} \frac{2^n}{6^n},$$

assuming both of these series converge.

Let us look at the first series:

$$\sum_{n=1}^{\infty} \frac{3^n}{6^n}$$

If we write out some of the terms:

$$s_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$
$$s_n = 0 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{n-1}$$

we can see that we are handling a geometric series with $a = 0$ and $r = 1/2$.
Thus,

$$\begin{aligned}\frac{1}{2} \cdot s_n &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots + \left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^n \\ s_n - \frac{1}{2} \cdot s_n &= \left(\frac{1}{2}\right) + \left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right) + \left(\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^3\right) + \cdots + \left(\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{n-1}\right) - \left(\frac{1}{2}\right)^n\end{aligned}$$

The inner terms cancel:

$$\begin{aligned}s_n - \frac{1}{2} \cdot s_n &= \frac{1}{2} - \left(\frac{1}{2}\right)^n \\ s_n\left(1 - \frac{1}{2}\right) &= \frac{1}{2} - \left(\frac{1}{2}\right)^n \\ s_n &= \frac{1/2 - (1/2)^n}{1 - 1/2}\end{aligned}$$

Now we take the limit of both sides of the equation.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1/2 - 0}{1 - 1/2}$$

As $n \to \infty$, $(1/2)^n \to 0$.

$$\begin{aligned}&= \lim_{n\to\infty} \frac{1}{2(1 - 1/2)} = \lim_{n\to\infty} \frac{1}{2 - 1} = \lim_{n\to\infty} \frac{1}{1} \\ &= 1\end{aligned}$$

From this, we realize that that $\sum_{n=1}^{\infty} \frac{3^n}{6^n}$ converges to 1.

The second series is quite similar:

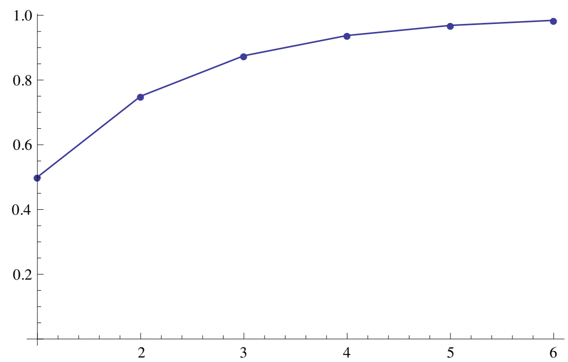
$$\sum_{n=1}^{\infty} \frac{2^n}{6^n}$$

Also a geometric series, we will find that

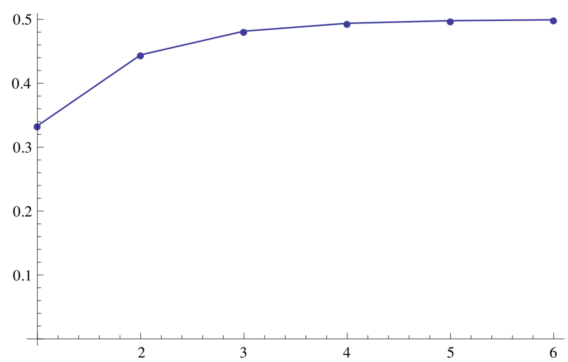
$$\begin{aligned}s_n &= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots + \left(\frac{1}{3}\right)^{n-1} \\ s_n\,(1 - 1/3) &= (1/3 - (1/3)^n) \\ s_n &= \frac{1/3 - (1 - 3)^n}{1 - 1/3}\end{aligned}$$

Taking the limit as $n \to \infty$ of both sides:

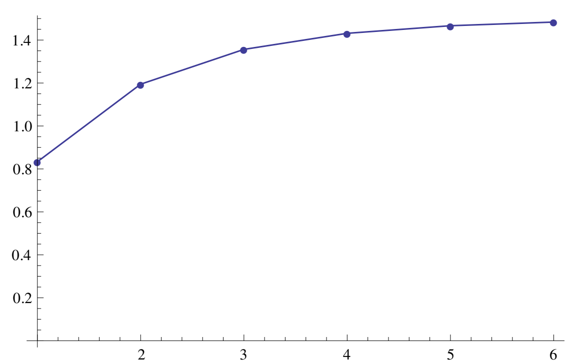
$$\begin{aligned}
\lim_{s_n \to \infty} s_n &= \lim_{s_n \to \infty} \frac{1/3 - (1-3)^n}{1 - 1/3} \\
&= \lim_{s_n \to \infty} \frac{1}{3(1 - 1/3)} \\
&= \frac{1}{2}
\end{aligned}$$



(a) A plot of $\sum_{n=1}^{\infty} \frac{3^n}{6^n}$.



(b) A plot of $\sum_{n=1}^{\infty} \frac{2^n}{6^n}$.



(c) A plot of $\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n}$.

Finally,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n} &= \sum_{n=1}^{\infty} \frac{3^n}{6^n} + \sum_{n=1}^{\infty} \frac{2^n}{6^n} \\ &= 1 + \frac{1}{2}\end{aligned}$$

$$= \frac{3}{2}$$

The sequence converges to $\frac{3}{2}$.

16.4. The Comparison Test

Theorem 28 (The Comparison Test). Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$\forall n > N \left(d_n \leq a_n \leq c_n \right)$$

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

[7, p. 559]

16.5. The Integral Test

Another helpful tool for discovering the behavior of a series is the *integral test*. In simple terms, given the following two statements:

1. $f' < 0$ over some interval $[a, \infty)$
2. $f(n)$ is always positive.[2]

The integral test states that:

(a) $\sum_{n=a}^{\infty} f(n)$ converges if $\int_a^{\infty} f(x)\,\mathrm{d}x$ converges.

(b) $\sum_{n=a}^{\infty} f(n)$ diverges if $\int_a^{\infty} f(x)\,\mathrm{d}x$ diverges.

[2] We can often see this by writing out some terms.

Proof.

$$\begin{aligned}\int_1^\infty f(x)\,\mathrm{d}x &\approx f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + \cdots + \text{with } \Delta x = 1\\ &\approx f(1) + f(2) + f(3) + \cdots\\ &\approx \sum_{n=1}^{\infty} f(n)\end{aligned}$$ □

Here is the official wording for the integral test, copied from *Thomas' Calculus* [7, p. 554]:

Theorem 29 (The Integral Test)**.** Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ ($N \in \mathbb{Z}_+$). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^\infty f(x)\,\mathrm{d}x$ both converge or both diverge.

Example 16.4. Use the integral test to show that the harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Solution.
First, we test it with Theorem 27.

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Because this limit is zero, the test proves inconclusive.

Let's check the two prerequisites for the integral test. If we let $\sum_{n=1}^{\infty} \frac{1}{n}$ be $\sum_{n=1}^{\infty}$, and $a_n = \frac{1}{n}$, then we can treat a_n as a function and make sure it is positive and always decreasing.

The function

$$f(x) = \frac{1}{x}$$

is always positive on $[1, \infty)$. Its derivative,

$$f'(x) = -\frac{1}{x^2}$$

is always negative. Thus, we can use the integral test to describe the behavior of $\sum_{n=1}^{\infty} a_n$.

$$\int_1^\infty \frac{1}{n} = \lim_{t\to\infty} \ln x \Big|_1^\infty$$

This integral diverges, so by Theorem 29 the series also diverges.

Example 16.5. Determine whether the following series is convergent or divergent.

$$\sum_{k=4}^{\infty} ke^{-k^2}$$

Solution.
We can use the integral test.

$$\begin{aligned}\int_{k=4}^{\infty} ke^{-k^2}\,\mathrm{d}k &= \frac{-1}{2}\int_{k=4}^{\infty} -2ke^{-k^2}\,\mathrm{d}k \\ &= \frac{-1}{2}e^{-x^2}\Big|_4^\infty\end{aligned}$$

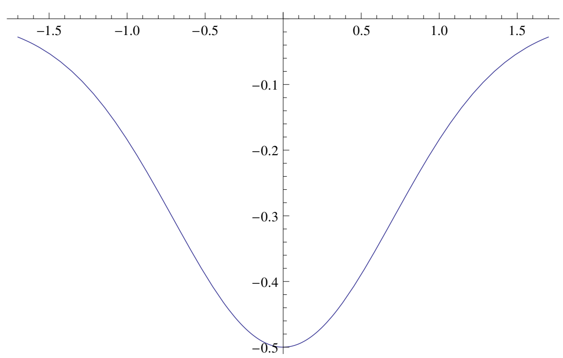


Figure 16.1.: A plot of $-\frac{1}{2}e^{-k^2}$

We can tell that this integral converges, so the series converges.

16.6. p Series

A p series is of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

At $p = 1$, it is a harmonic series and we could say it diverges because of

Theorem 29. Meanwhile, at $p = 2$, the series converges. If we take $p = -1$, the series diverges. At $p = -2$, the series diverges by Theorem 27. At $p = 1.0000000000001$, the series converges by the integral test.

We can get a feeling that $p > 1$ leads to convergence, and $p \leq 1$ leads to a diverging series.

16.7. Series Comparison Test

16.7.1. Converging Series

Example 16.6.

$$\sum_{n=1}^{\infty} \frac{5 + 2\sqrt{n}}{n^3}$$

Solution.
Let $f(x) = \frac{5+2\sqrt{x}}{x^3}$, which is always positive on $x \geq 1$. Then $f'(x) = -\frac{5(\sqrt{x}+3)}{x^4}$, which is always negative on $x \geq 1$.

$$\begin{aligned}
\int_1^{\infty} \frac{5 + 2\sqrt{x}}{x^3}\,\mathrm{d}x &= \int_1^{\infty} \frac{5}{x^3}\,\mathrm{d}x + \int_1^{\infty} \frac{2\sqrt{x}}{x^3}\,\mathrm{d}x \\
&= \int_1^{\infty} 5x^{-3}\,\mathrm{d}x + \int_1^{\infty} 2x^{1/2} \cdot x^{-3}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \int_1^{t} 5x^{-3}\,\mathrm{d}x + \lim_{t\to\infty} \int_1^{t} 2x^{-5/2}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \frac{5x^{-2}}{-2}\bigg|_1^t + \lim_{t\to\infty} \frac{2x^{-3/2}}{-3/2}\bigg|_1^t \\
&= \lim_{t\to\infty} \left[-\frac{5t^2}{2} + \frac{4(1)^{-3/2}}{3}\right] + \lim_{t\to\infty} \left[-\frac{4t^{-5/2}}{3} + \frac{4(1)^{-3/2}}{3}\right] \\
&= \lim_{t\to\infty} \left[-\frac{5t^{-2}}{2} + \frac{5}{2}\right] + \lim_{t\to\infty} \left[-\frac{4t^{-3/2}}{3} + \frac{4}{3}\right] \\
&= \lim_{t\to\infty} -\frac{5t^{-2}}{2} + \lim_{t\to\infty} \frac{5}{2} + \lim_{t\to\infty} -\frac{4t^{-3/2}}{3} + \lim_{t\to\infty} \frac{4}{3} \\
&= \frac{23}{6} - \lim_{t\to\infty} \left[\frac{5t^{-2}}{2} + \frac{4t^{-5/2}}{5}\right] \\
&= \frac{23}{6} - \lim_{t\to\infty} \left[\frac{5}{2t^2} + \frac{4}{3t^{3/2}}\right] \\
&= \frac{23}{6} + 0 \\
&= \frac{23}{6}
\end{aligned}$$

By Theorem 29, the series $\sum_{n=1}^{\infty} \frac{5+2\sqrt{n}}{n^3}$ converges.

Example 16.7.

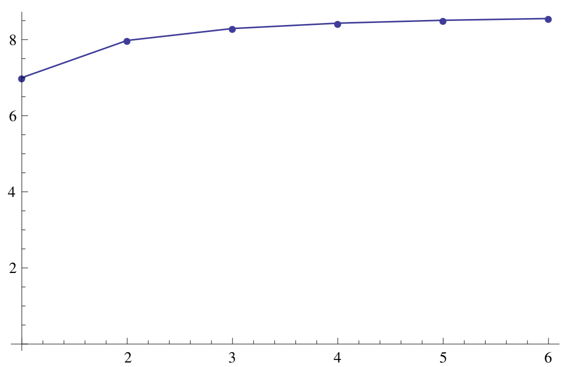
$$\sum_{n=1}^{\infty} \frac{5+2\sqrt{n}}{n^3+1}$$

We can tell that this is quite similar to Example 16.6, except we are making the denominator larger. We can tell by comparison with the original problem that this must also converge. Likewise, if we made the numerator smaller, we could treat this as converging. For a proper solution, we would write

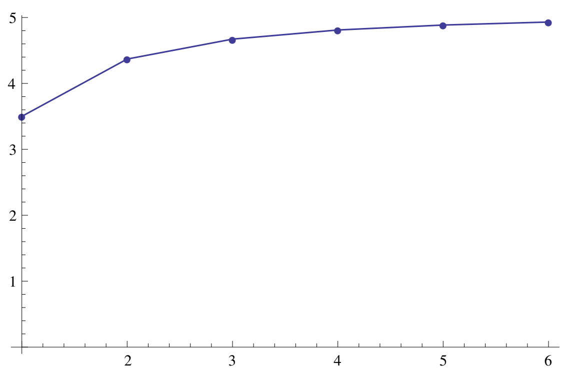
Solution.

$$\sum_{n=1}^{\infty} \frac{5+2\sqrt{n}}{n^3+1}$$

Converges by comparison with $\sum_{n=1}^{\infty} \frac{5+2\sqrt{n}}{n^3}$.



(a) A plot of $s_n = \frac{5+2\sqrt{n}}{n^3}$



(b) A plot of $s_n = \frac{5+2\sqrt{n}}{n^3+1}$

Make note, however, that making the denominator smaller does not help us in making these comparisons for convergence.

Example 16.8.

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

This is a p-series with $p = 4$, and therefore converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 2}$$

This series converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Example 16.9.

$$\sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3}$$

Note. We can use partial fraction decomposition to handle this, and it becomes a telescoping sum.

$$\sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 6}$$

Remark. Imagine we were dealing with one series that converges to $\frac{1}{2}$. In turn, we were comparing this with another series that “shoots off into the negatives.”

We would have issues comparing these in any way.

Thus, we must check that a_n and b_n are exclusively positive.

16.7.2. Diverging Series

Example 16.10.

$$\sum_{n=1}^{\infty} \frac{n}{n + 5}$$

Using theorem 27, we know that this series diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n + 5 - 1}$$

Here, we are decreasing the size of the denominator for a divergent series. We can state that $\sum_{n=1}^{\infty} \frac{n}{n+5-1}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{n}{n+5-1}$.

Example 16.11.

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$$

Diverges by theorem 27.

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$$

Again, making the denominator smaller. This diverges by comparison with the above series.

Example 16.12.

$$\sum_{n=1}^{\infty} \frac{1}{3n+1}$$

Diverges.

$$\sum_{n=1}^{\infty} \frac{15}{3n+1}$$

Diverges by comparison with the above.

Example 16.13.

$$\sum_{n=1}^{\infty} \frac{n+2}{n+1}$$

Diverges.

$$\sum_{n=1}^{\infty} \frac{n(n+2)}{n+1}$$

It doesn't have to be a constant multiple for the comparison test to work. We can still state that this diverges by comparison with the above.

16.8. Root Test

Theorem 30 (Root Test)**.** Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

then (a) the series *converges* if $\rho < 1$, **(b)** the series diverges if $\rho > 1$ or if ρ is infinite, or **(c)** the test is *inconclusive* if $\rho = 1$. [7, p. 565]

Example 16.14. Use the root test to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{(5n+7)^n}$$

Solution.
The root test tells us that

$$\rho = \lim_{n\to\infty} \frac{1}{5n+7}$$

Example 16.15.

$$\sum_{n=1}^{\infty} \frac{12}{\left(3+\frac{1}{n}^{4n}\right)}$$

Solution.
Use the root test. Take the nth root of s_n.

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left[\frac{12}{\left(3+\frac{1}{n}^{4n}\right)}\right]^{1/n} \\ &= \lim_{n\to\infty} \frac{12^{1/n}}{\left(3+\frac{1}{n}\right)^4} \\ &= \lim_{n\to\infty} \frac{12^{1/n}}{3^4} \\ &= \frac{1}{81}\end{aligned}$$

16.9. Ratio Test

Theorem 31 (The Ratio Test)**.** Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then **(a)** the series *converges* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

Example 16.16. Use the ratio test to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^n + n}{n!}$$

Solution.
First, let's look at the series:

$$\frac{4}{1} + \frac{11}{3} + \frac{30}{6} + \cdots$$

Just use the ratio test, which we can do because this series has exclusively positive terms.

$$\lim_{n\to\infty} \frac{\dfrac{3^{n+1} + (n+1)}{(n+1)!}}{\dfrac{3^n + n}{n!}} = \lim_{n\to\infty} \frac{(n!)\left(3^{n+1} + (n+1)\right)}{(3^n + n)(n+1)!}$$

It is important to note here that by factoring a $n+1$ term out of $(n+1)!$, we find that $(n+1)! = (n+1)n!$. This allows us to manipulate our limit so that some terms will cancel.

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{(n!)\left(3^{n+1} + n + 1\right)}{(3^n + n)(n+1)(n!)} \\
&= \lim_{n\to\infty} \frac{3^{n+1} + n + 1}{(3^n + n)(n+1)} \\
&= \lim_{n\to\infty} \frac{3^{n+1} + n + 1}{n3^n + n^2 + 3^n + n} \\
&= \lim_{n\to\infty} \frac{\dfrac{3^{n+1}}{3^{n+1}} + \dfrac{n}{3^{n+1}} + \dfrac{1}{3^{n+1}}}{\dfrac{n(3^n)}{3^{n+1}} + \dfrac{n^2}{3^{n+1}} + \dfrac{3^n}{3^{n+1}} + \dfrac{n}{3^{n+1}}}
\end{aligned}$$

Now, as $n \to \infty$ many of the terms disappear. $n3^n/3^{n+1}$ becomes $n/3$.

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{\frac{n}{3} + \dfrac{1}{3}} \\
&= \lim_{n\to\infty} \frac{3}{n+1} \\
&= 0
\end{aligned}$$

Thus, by 31, this series converges.

Remark.

$$\lim n \to \infty \frac{n}{3^n} \overset{\text{H}}{=} \frac{1}{\ln 3 e^{n \ln 3}}$$

Example 16.17. Use the ratio test to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^{19}}{19^n}$$

Solution.
Use the ratio test.

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{19}}{19^{n+1}}}{\frac{n^{19}}{19^n}} = \lim_{n\to\infty} \frac{19^n \cdot (n+1)^{19}}{19^{n+1} \cdot n^{19}}$$

$19^n/19^{n+1}$ becomes just 19.

$$= \lim_{n\to\infty} \frac{(n+1)^{19}}{19 \cdot n^{19}}$$

Note that there's no way the $(n+1)^{19}$ and $n^1 9$ can in any way cancel. However, were we to foil the $(n+1)^{19}$ nineteen times, the leading term would be n^{19}. The limit is, therefore, just the limit

$$= \lim_{n\to\infty} \frac{n^{19}}{19n^{19}}$$
$$= \frac{1}{19}$$

16.9.1. Examples

Example 16.18. Does

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converge or diverge?

Solution.

$$\begin{aligned} s_1 &= \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{1}{4} \\ s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\ s_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \\ s_n &= \frac{2^n - 1}{2^n} \end{aligned}$$

Example 16.19. Does

$$\sum_{n=2}^{\infty} \frac{n+10}{10n+1}$$

converge or diverge?

$$\lim_{n\to\infty} \frac{n+10}{10n+1} =$$

Divide all terms by n.

$$\frac{\frac{n}{n}}{\frac{10n}{n}} \qquad = \frac{1}{10} \neq 0$$

Therefore, the series diverges.

Example 16.20.

$$\sum_{n=0}^{\infty} \frac{4}{2^n}$$

Solution.
Theorem 27 gets us nowhere with this one. In other words,

$$\lim_{n\to\infty} \frac{4}{2^n} = 0$$

is inconclusive.

We have a geometric series.

$$4\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) = \frac{1}{1 - \frac{1}{2}}$$

So

$$\sum n = 0^{\infty} \frac{4}{2^n}$$

converges to 8.

Example 16.21. Does

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n-2}$$

converge or diverge?

Solution.
Check the limit.

$$\lim_{n\to\infty} \frac{1}{n} - \frac{1}{n-2} = \lim_{n\to\infty} \frac{n-2}{n(n-2)} - \frac{n}{n(n-2)}$$

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots$$

This is called a telescoping sum that converges to $\frac{3}{2}$.

Example 16.22.

$$\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$$

Solution.
Check the limit.

$$\lim_{n\to\infty} \frac{n+1}{2n-1} \neq 0$$

The series diverges.

Example 16.23.

$$\sum_{n=1}^{\infty} \frac{3n-1}{2n+1}$$

Check the limit.

$$\lim_{n\to\infty} \frac{3n-1}{2n+1} \neq 0$$

The series diverges.

Example 16.24.

$$\sum_{n=0}^{\infty} \frac{1}{4^n}$$

Solution.

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \dots$$

Converges to $\frac{1}{1-\frac{1}{4}}$.

Example 16.25.

$$\sum_{n=0}^{\infty} (1.075)^n$$

Solution.
$\lim_{n\to\infty} (1.075)^n \neq 0$ and the series diverges because of Theorem 27.

Example 16.26.

$$\sum_{n=1}^{\infty} \frac{2^n}{100}$$

Solution.
$\lim_{n\to\infty} \frac{2^n}{100} \neq 0$ so by Theorem 27 the series diverges.

Example 16.27.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

Solution.
2^n is geometric but the n^2 denominator means we can't treat it like a geometric series.

Using Theorem 27, we find that

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n}{n^2} &= \lim_{n\to\infty} \frac{e^{n \ln 2}}{n^2} \\
&\stackrel{\text{H}}{=} \lim_{n\to\infty} \frac{\ln 2 e^{n \ln 2}}{2n} \\
&\stackrel{\text{H}}{=} \lim_{n\to\infty} \frac{2 \ln 2 e^{n \ln 2}}{2}
\end{aligned}$$

$$\neq 0$$

so the series diverges.

Example 16.28.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$$

Solution.
Checking with 27 proves inconclusive.

We can use partial fraction decomposition on this fraction.

$$\frac{1}{n(n+3)} = \frac{A}{x} + \frac{B}{x+3}$$

We find that $A = \frac{1}{3}$ and $B = -\frac{1}{3}$

$$= \frac{1/3}{x} - \frac{1/3}{x+3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right)$$

Example 16.29.

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$$

Solution.
Check it with Theorem 27:

$$\lim_{n \to \infty} \frac{1}{(n-3)^2} = 0$$

Which is inconclusive, so we will attempt to use the integral test.

$$f(x) = \frac{1}{(x-3)^2}$$

Check the derivative:

$$f'(x) = \frac{(x-3)^2 \frac{\mathrm{d}}{\mathrm{d}x}(1) - 1 \frac{\mathrm{d}}{\mathrm{d}x}(x-3)^2}{(x-3)^4}$$

$$= \frac{-2(x-3)}{(x-3)^4}$$
$$= \underbrace{\frac{-2}{(x-3)^3}}_{\text{DNE when } x=3}$$

$f(x)$ is always positive, and $f'(x)$ is always negative, so the first two

x	$-\infty$	3	∞
$f(x)$		+	
$f'(x)$	$-$	undefined	$-$

Table 16.1.: A sign diagram of $f(x)$ and $f'(x)$

prerequisites for theorem 29 are passed.

Now consider
$$\int_4^\infty \frac{1}{(x-3)^2}\,\mathrm{d}x$$
First evaluate this as an indefinite integral:
$$\int \frac{1}{(x-3)^2}\,\mathrm{d}x = \int \frac{1}{u^2}\,\mathrm{d}u = \int u^{-2}\,\mathrm{d}u$$
$$= \frac{u^{-1}}{-1}$$
$$= \frac{-1}{(x-1)}$$

Now evaluate it in terms of the original definite integral:
$$\int_4^\infty \frac{1}{(x-3)^2}\,\mathrm{d}x = \lim_{t\to\infty} \frac{-1}{(x-1)}\bigg|_4^t$$
$$= -\lim_{t\to\infty} \frac{1}{(t-1)} + \frac{1}{3}$$
$$= -\frac{1}{3}$$

This integral converges, so the $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ must converge (though we don't know to what).

Example 16.30. Does the following series converge or diverge?
$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

Solution.
Check Theorem 27 first and foremost.

$$\begin{aligned}\lim_{n\to\infty}\frac{\ln n}{n} &\overset{\text{H}}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{1}\\ &= 0\end{aligned}$$

Next, check with the integral test, theorem 29.

We know that the $\ln x$ is positive past $x = 1$, so the second condition for use of the theorem is satisfied.

The first condition is slightly more difficult:

$$\begin{aligned}f' &= \frac{x\cdot\frac{1}{x} - \ln x}{x^2}\\ &= \frac{1-\ln x}{x^2}\end{aligned}$$

Which is > 0, so we can use the integral test.

$$\int_1^\infty \frac{\ln n}{n}\,\mathrm{d}x$$
$$\left.\frac{(\ln x)^2}{2}\right|_1^\infty$$

We can conclude that the series diverges.

Example 16.31. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty}\frac{3}{\sqrt{n}}$$

Solution.
Recognize this as a p-series with $p = 1/2$.

Note that, without the 3, we would be looking at a definitely diverging p-series.

However, multiply that diverging series by 3 and we are dealing with an even *bigger* diverging p-series.

The series, therefore, diverges by comparison with

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Example 16.32. Determine if this geometric series converges or diverges. If it converges, find its sum.

$$(1/5) + (1/5)^2 + (1/5)^3 + (1/5)^4 + \cdots$$

Solution.
Treat it as a geometric series and multiply it as such.

$$\left(1 - \frac{1}{5}\right)\left(\frac{1}{5} + (1/5)^2 + (1/5)^3 + (1/5)^4 + \cdots\right) = \frac{1}{5}$$

So now we look for our final answer:

$$\frac{\frac{1}{5}}{1 - \frac{1}{5}} = \frac{1}{4}$$

Example 16.33. If the series

$$\sum_{n=0}^{\infty} \cos 9n\pi$$

converges, then what is its sum.

Solution.
This turns out to just be a fancy way of writing

$$\sum 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Alternatively, we are talking about a cosine function. The cosine function never settles down, it constantly jumps back and forth between values. Thus, we could take the limit as $n \to \infty$ of s_n to find that it diverges.

Example 16.34. Find a formula for the nth partial sum of the series and use it to determine if the series converges.

$$\sum_{n=1}^{\infty} (\ln \sqrt{n+1} - \ln \sqrt{n})$$

Solution.

$$\begin{aligned}
s_1 &= \ln\sqrt{2} - \ln\sqrt{1} \\
s_2 &= \ln\sqrt{3} - \ln\sqrt{2} + \ln\sqrt{2} - \ln\sqrt{1} \\
s_3 &= \ln\sqrt{4} - \ln\sqrt{3} + \ln\sqrt{3} + \ln\sqrt{2} - \ln\sqrt{1} \\
s_4 &= \ln\sqrt{5} - \ln\sqrt{4} + \ln\sqrt{4} - \ln\sqrt{3} + \ln\sqrt{3} - \ln\sqrt{2} + \ln\sqrt{2} - \ln\sqrt{1} \\
s_5 &= \ln\sqrt{6} - \ln\sqrt{5} + s_4 \\
s_n &= \ln\sqrt{n+1} - \ln\sqrt{1}
\end{aligned}$$

$$\sum_{n=1}^{\infty} \text{ is } s_\infty$$

We can conclude that the series diverges.

Example 16.35. Use the limit comparison test to determine convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+7n+5}$$

Solution.

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+7n+5}}{\frac{1}{n}} = 1$$

This is about the same order of magnitude as our original problem. The numerator and the denominator balance, and the limit takes us to 1, so the series diverges.

Example 16.36. Tell whether the series converges or diverges. If it converges, give its sum.

$$1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^3 + \cdots + \left(\frac{8}{9}\right)^n + \cdots$$

Solution.
Treat this as a geometric series.

We will find the answer converges to 9.

Example 16.37. Does the series

$$\sum_{n=1}^{\infty} \frac{5}{n^2+36}$$

converge or diverge? If so, to what?

Example 16.38. If the series

$$\sum_{n=0}^{\infty} \left(\frac{3}{\sqrt{13}}\right)^n$$

converges, what is its sum?

Solution.

$$\left(1-\frac{3}{\sqrt{13}}\right)\left[1+\frac{3}{\sqrt{13}}+\left(\frac{3}{\sqrt{13}}\right)^2+\dots\right]$$

Example 16.39. Find a formula for the nth partial sum of the series.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Solution.

$$s_n = 1-\frac{1}{n+1}$$

Example 16.40. Does the series

$$\sum_{n=1}^{\infty} \frac{4e^n}{1+e^{2n}}$$

converge or diverge?

Solution.

This series is a variation of

$$\sum_{n=1}^{\infty} \frac{4e^n}{e^{2n}} = 4\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}}$$
$$= 4\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$$

Which is a geometric series.

The original series converges by comparison with the above.

Example 16.41. Use the limit comparison test to determine if the following series converges or diverges by comparison with $\sum_{n=2}^{\infty} \frac{1}{n}$.

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

Solution.
Remember that the limit comparison test is far different from the comparison test.

$$\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n}$$
$$\overset{\text{H}}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}}$$
$$= \lim_{n\to\infty} n$$

n diverges as it goes toward ∞.

16.10. Alternating Series Test

Theorem 32 (Leibniz's Test)**.** [7, p. 568] The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \leq u_{n+1} foralln \geq N$, for some integer N.

3. $u_n \to 0$

Note. Also known as the alternating series test.

Proof. Let's write out some of the terms of an alternating series, see how it's playing out:

$$u_1 - u_2 + u_3 - u_4 + \cdots$$

Remark. These could start negative, too, and we would just factor that out of the sum to make it positive and fit the theorem.

$$\begin{aligned} s_2 &= u_1 - u_2 \geq 0 \\ s_4 &= s_2 + (u_3 - u_4) \geq s_2 \\ s_6 &= s_4 + (u_5 - u_6) \geq s_4 \end{aligned}$$

So we see, at each step, that the sequence of partial sums is monotonic increasing. To know that it converges, we mus first show that it is bounded.

The sequence of partial sums gets larger and larger so the lower bound is

$$s_n = u_1 - \overbrace{(u_2 - u_3)}^{+} - \overbrace{(u_4 - u_5)}^{+}$$

The *sequence of partial sums* is monotonic and bounded by $M = u$ and $m = s_2$.

Therefore, the series converges. □

Example 16.42.

$$\sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1}$$

Although at first this appears to be an alternating series, it actually diverges by Theorem 27, the limit test.

Example 16.43.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{3n+2}{4n^2+3} \right)$$

Solution.
Check with Theorem 27, which proves inconclusive.

Take the derivative of $f(x) = (-1)^n \frac{3n}{4n-1}$. We find $f'(x)$ is negative everywhere.

$$f'(x) = 12x^2 - 9 - 24x^2 - 16x$$
$$f'(x) = -12x^2 - 16x - 9$$

By Theorem 32, this series converges.

So far, we've been able to tell to what number very few infinite sums converge. Geometric and telescoping sums, for example. For convergent alternating series, we can always estimate the limit, L, and measure the error of our estimation, the discrepancy between this estimate and the actual theoretical value of L.[3]

This error is given by

$$\pm|L - S_n| \tag{16.1}$$

16.11. Power Series

A **power series** about $x = 0$ is the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x + \cdots . + c_n x^n + \cdots$$

And a power series about $x = a$ is of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{16.2}$$

in which the center a and the coefficients $c_0, c1, c2, \cdots$ are constants.

Taking all the coefficients to be 1 in equation (16.2) gives us the **geometric series**

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^n + \cdots . \tag{16.3}$$

[7, p. 575]

[3] There's an awesome diagram about this in the textbook. Check out [7, p. 569 Fig. 10.13].

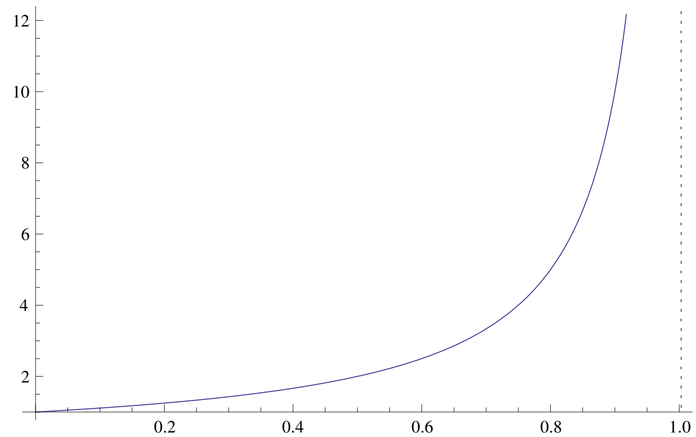


Figure 16.2.: A plot of Eq. (16.3).

This is the geometric series with the first term 1 and the ratio x. It converges to $1/(1-x)$ for $|x|<1$. As shown in Figure 16.2, the series has a vertical asymptote at $x=1$ and we cannot approximate it at $x \geq 1$.

Theorem 33 (Reciprocal Power Series)**.**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \tag{16.4}$$

Example 16.44.

$$\sum_{n=0}^{\infty} n^3 x^n$$

An example of a **power series**, where the coefficient depends on where you are adding the polynomial. Otherwise described as a *power series about* $x=0$.

Example 16.45.

$$\sum_{n=0}^{\infty} n^3 (x-5)^n$$

This is nothing more than example 16.44 shifted 5 to the right. Otherwise described as a *power series about* $x=5$.

Solution.

Ratio test.

$$\lim n \to \infty \left| \frac{(n+1)^3(x-5^{n+1}}{n^3(x-5)^n} \right| = \lim n \to \infty \left| \frac{(n+1)^3(x-5^1}{n^3} \right|$$

Factor out the polynomial and take the limit by dividing each term by the highest power.

$$\begin{aligned} &= |x-5| \cdot \lim_{n\to\infty} \frac{n^3+3n^2+3n+1}{n^3} \\ &= |x-5| \cdot 1 \\ &= |x-5| \end{aligned}$$

$$|x-5| < 1$$

This has an interval of convergence from 4 to 6.

Note. For this to be correct, we **must** consider:

$$\sum_{n=0}^{\infty} n^3(6-5)^n$$

This one diverges by the *nth term test*.

$$\sum_{n=0}^{\infty} n^3(4-5)^n$$

because the ratio test is inconclusive at $x = 1$. Although this is an alternating series, this one diverges by the *nth term test*.

The interval $(4, 6)$ is the interval of convergence. This interval is *about* $x \approx 5$.

This is sometimes called a "radius of convergence," drawn as a circle on a number line.

We want to talk about what kind of x values will result in a finite sum, or infinite sum.

Note. An interval of convergence can be $-\infty < x < \infty$. The radius of convergence would thus be ∞.

Example 16.46. An interval of convergence from $3 < x < 7$. The radius of convergence is 2.

16.12. Taylor and Maclauren Series

A **Taylor Series** is an approximation of a function using a power series.

We do so by treating this function as the sum of a power series

$$f(x) = \sum_{n=1}^{\infty} a_n (x-a)^n$$
$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots a_n(x-a)^n + \cdots$$

If we differentiate this power series, we get

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots a_n(x-a)^n + \cdots$$
$$f'(x) = 0 + a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots na_n(x-a)^{n-1} + \cdots$$

Differentiate it again, and we get

$$f''(x) = 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \cdots + (n-1)na_n(x-a)^{n-2} + \cdots$$

Further differentiation leaves us

$$f'''(x) = 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^2 + \cdots$$

So the n^{th} derivative, for all n, is

$$f^n(x) = n!a_n + \text{a sum of terms with } (x-a) \text{ as a factor.}$$

All these equations are perfect approximations of $f(x)$ at $x = a$, though repeated differentiation makes them more accurate as we get further from that value. This property allows us to state that $f'(a) = a_1$, $f''(a) = 1 \cdot 2a_2$, $f'''(a) = 1 \cdot 2 \cdot 3a_3$, and

$$f^{(n)}(a) = n!a_n.$$

This shows us that, if $f(x)$ *has* a power series representation, then its n^{th} term coefficient is

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

[7, p. 584]

Theorem 34 (Taylor's Theorem)**.** Let f be a function with derivatives of order k for $k = 1, 2, \cdots, N$ in some interval containing a as in interior point. Then $\forall(n \in \mathbb{Z})$ from 0 through N, the Taylor polynomial of order n

generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n \tag{16.5}$$

[7, p. 586]

We don't normally treat these as infinite series, as that defeats the purpose of using a series produce a simplified version of a function.

Theorem 35. The Taylor series of a function around a point $x = a$ is

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots .$$

If this Taylor Series is calculated about $x = 0$, we call it a **Maclauren Series**.

Example 16.47. Say we are trying to approximate the function

$$f(x) = e^x$$

around $x = 1$.

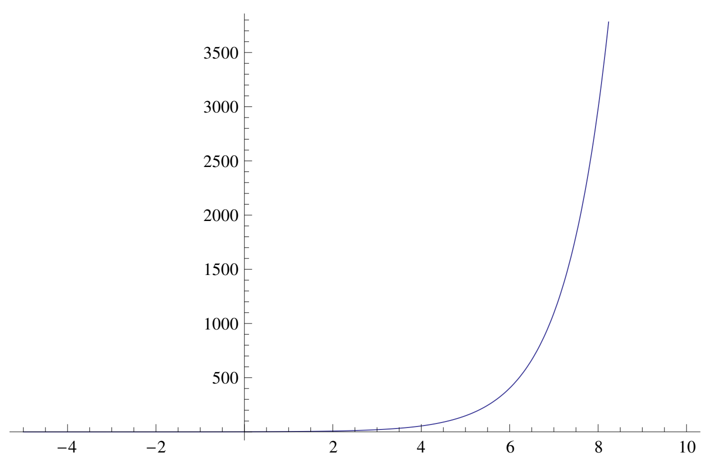


Figure 16.3.: A plot of $f(x) = e^x$.

For the simplest possible approximation, we would just calculate the function at $x = 1$ and that would be our solution. This is a constant function and is called a **0th order approximation**.

$$f(x) \approx e$$

To improve our approximation, we increase the power of our Taylor series.

$$f(x) \approx a_0 + a_1(x-a)$$
$$f(x) \approx e + e(x-1)$$

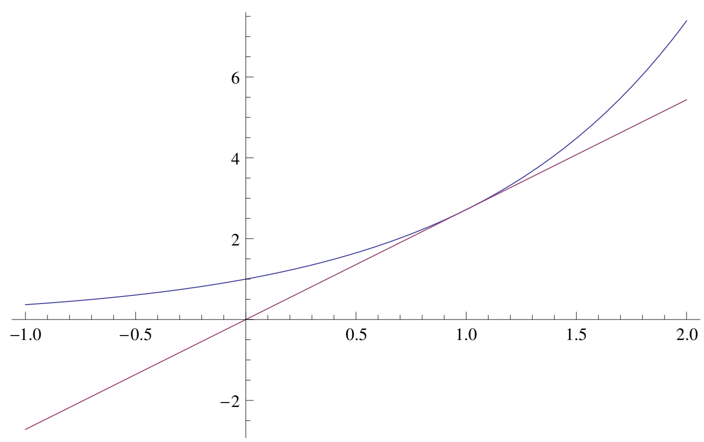


Figure 16.4.: A plot of e^x and $e + e(x-1)$.

We can improve it even further by increasing our number of terms:

$$f(x) \approx a_0 + a_1(x-a) + a_2(x-a)^2$$

We already know the first two coefficients, so

$$f(x) \approx e + e(x-a) + a_2(x-a)^2$$

To find a_2, we take the second derivative of $f(x)$ at $x = 1$ and divide that by 2!.

$$f(x) \approx e + e(x-a) + \frac{e}{2!}(x-a)^2$$

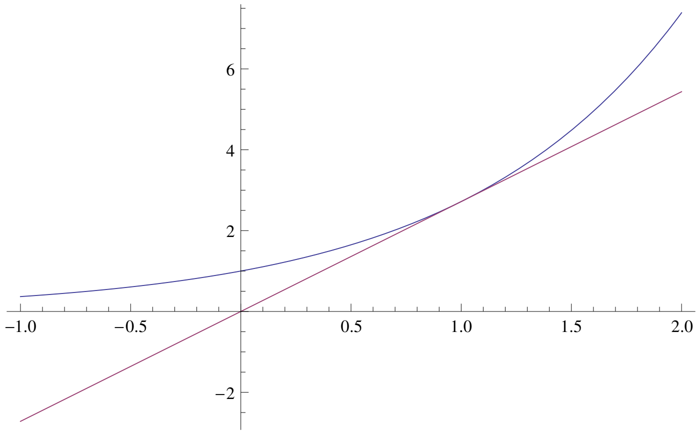


Figure 16.5.: A plot of e^x and $e + e(x-1) + \frac{e}{2!}(x-a)^2$.

Example 16.48. Find a 4$^{\text{th}}$ order Taylor series approximation of

$$f(x) = \ln x$$

at $x = 2$.

Solution.
The first two terms are easy

$$f(x) \approx \ln 2 + \frac{1}{2}(x-2) + \cdots$$

but then we must take the second derivative of $f(x)$ at $x = 2$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \ln x &= \frac{1}{x} = x^{-1} \\ \frac{\mathrm{d}}{\mathrm{d}x} x^{-1} &= -2x^{-2} \\ \left. \frac{\mathrm{d}}{\mathrm{d}x} x^{-1} \right|_{x=2} &= \frac{-2}{4} \\ &= \frac{-1}{2} \end{aligned}$$

and now we can get the third term

$$\begin{aligned} f(x) &\approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{2 \cdot 2!}(x-2)^2 + \cdots \\ f(x) &\approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \cdots \end{aligned}$$

Now we find the third derivative of $f(x)$ at $x = 2$:

$$f^{(3)}(x)\bigg|_{x=2} = \frac{1}{4}$$

$$f(x) \approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 + \cdots$$

And the fourth

$$f^{(4)}(x)\bigg|_{x=2} = \frac{8}{3}$$

Making our final answer

$$f(x) \approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4$$

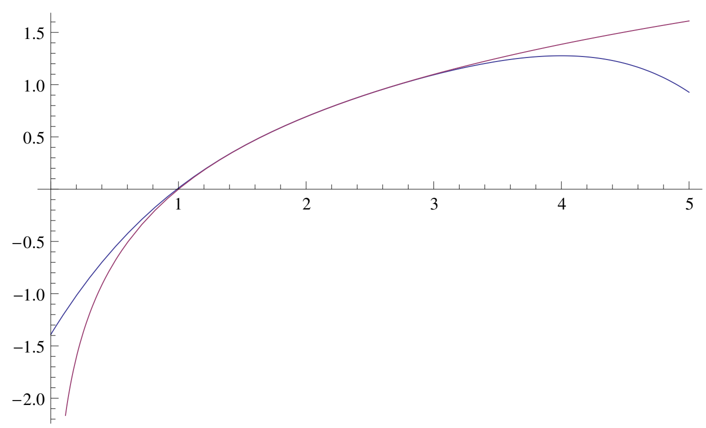


Figure 16.6.: A plot of $\ln x$ and our Taylor approximation.

17. Complex Numbers

Earl A. Coddington, professor of mathematics at UCLA, offers an extremely helpful crash-course in complex numbers in his book *An Introduction To Ordinary Differential Equations*, Chapter 0 [5]. Most of the initial knowledge in this chapter comes from my notes on that chapter, but I will attempt to provide pictures and examples where I found the source text lacking.

Definition 17.1. A **complex number** is an ordered pair of real numbers (x, y). If z is a complex number, we write

$$z = (x, y). \tag{17.1}$$

Definition 17.2. The **sum** $z_1 + z_2$ is the complex number given by

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2). \tag{17.2}$$

Definition 17.3. If $z = (x, y)$, the **negative** of z, denoted $-z$, is defined to be the number

$$-z = (-x, -y). \tag{17.3}$$

Definition 17.4. The **zero** complex number, written simply 0, is defined as

$$0 = (0, 0). \tag{17.4}$$

Since Eq. (17.2) defines complex sums in terms of just real number addition operations, and we know that these real number operations are commutative, it follows that

$$z_1 + z_2 = z_2 + z_1. \tag{17.5}$$

Likewise does the associative property of addition for real numbers hold for complex numbers:

$$(z_1 + z_2) + z_2 = z_1 + (z_2 + z_3). \tag{17.6}$$

And the number 0 provides our additive identity:

$$z + 0 = z. \tag{17.7}$$

Finally, we have an additive inverse for complex numbers

$$z + (-z) = 0. \tag{17.8}$$

For additional information on these properties as they apply to the set of real numbers, I will direct the reader to Michael Spivak's *Calculus, Third Edition*, perhaps the single greatest introduction to "real mathematics" ever written. These properties, and their importance with regard to real numbers, is detailed extensively in the first chapter.

Definition 17.5. The **difference**, $z_1 - z_2$, is defined by

$$z_1 - z_2 = z_1 + (-z_2). \tag{17.9}$$

Definition 17.6. The **product** $z_1 z_2$ is defined by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1). \tag{17.10}$$

Remark. Eq. (17.10) can be found by performing basic multiplication on the following form of the numbers:

$$\begin{aligned} z_1 &= x_1 + \mathrm{i} y_1 \\ z_2 &= x_2 + \mathrm{i} y_2 \\ z_1 z_2 &= (x_1 + \mathrm{i} y_1)(x_2 + \mathrm{i} y_2) \end{aligned}$$

In order to use this, however, we must define the following units:

Definition 17.7. The **unit** complex number is the number $(1, 0)$. This may be multiplied by any complex number $z = (x, y)$ and the product will always be z.

Definition 17.8. The **imaginary unit** is defined to be the number

$$\mathrm{i} = (0, 1).$$

From those definitions, we see that if $z = (x, y)$ we can write it in terms of its real and imaginary parts as follows:

$$z = x(1, 0) + y(0, 1), \tag{17.11}$$

which is equivalent to stating

$$z = x + \mathrm{i}y. \tag{17.12}$$

18. Ordinary Differential Equations

Definition 18.1. A **differential equation** is an equation involving derivatives.

Definition 18.2. A **direction field** tells us the slope of a function at any given place.

Example 18.1. In physics, we often define acceleration to be a vector relative to another vector, velocity. Here, we will just consider them as scalars for the sake of argument. Acceleration is a change in velocity, so

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}, \tag{18.1}$$

where v represents velocity and t represents time. Now we integrate both sides of Eq. (18.1) with respect to t,

$$\int a \,\mathrm{d}t = \int \frac{\mathrm{d}v}{\mathrm{d}t} \,\mathrm{d}t. \tag{18.2}$$

Now, assuming[1]

$$\int \frac{\mathrm{d}v}{\mathrm{d}t} \,\mathrm{d}t = \int \,\mathrm{d}v, \tag{18.3}$$

then using Eq. (18.3), we see that

$$\int a \,\mathrm{d}t = \int \,\mathrm{d}v. \tag{18.4}$$

From here we simplify, finding that

$$at + c_1 = v + c_2. \tag{18.5}$$

The constants in Eq. (18.5) are simply constants and may be combined into another constant, C. Also, the equation may be rearranged to put it in more

[1] This must be explained later, but as a warning: no, the $\mathrm{d}t$ in the derivative operation and the $\mathrm{d}t$ in the integration operation do not simply cancel.

familiar form, yielding

$$v = at + C, \tag{18.6}$$

which we recognize as the classical mechanics equation for velocity. Integrating once more, and replacing v with the definition of velocity as change in position, we find

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t &= \int (at + C)\,\mathrm{d}t, \\
\int \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t &= \int at\,\mathrm{d}t + \int C\,\mathrm{d}t, \\
x + c_3 &= a\frac{t^2}{2} + Ct + c_4.
\end{aligned}$$

Now we may simply combine the constants once more, defining C_1 to constitute the difference of c_4, and c_3,

$$x = a\frac{t^2}{2} + Ct + C_1. \tag{18.7}$$

Eq. (18.7) may be rewritten in its more common form:

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0. \tag{18.8}$$

18.1. Second-order differential equations with linear combination solutions

18.2. Linear, homogeneous

$$ay'' + by'' + cy = 0$$

Each coefficient is a constant. We come up with a characteristic equation

$$ar^2 + br + c = 0$$

Which is quadratic.

1. We get two distinct solutions, meaning $r1 \neq r2$. this implies $r1, r2 \in \mathbb{R}$ and our characteristic equation is of the form $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

2. $r1 = r2$, and $r1, r2 \in \mathbb{C}$, meaning we still have a general solution of the form $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where $r_1, \quad r_2$ are of the form $\alpha \pm \beta i$. Note that our general solution here is in what is called "linear combination form," and we will change this later.

3. $r_1 = r_2 = r$, where $r \in \mathbb{R}$. This implies a general solution of the form $y = c_1 e^{rt} + c_2 t e^{rt}$.

Example 18.2.

$$y'' + 4y' + 4y = 0 \tag{18.9}$$

(a) Find one solution, $y_1(t)$.

(b) Show that $y_2(t) = t y_1(t)$ is also a solution.

(c) Give the general solution.

Solution. (a) Characteristic equation:

$$r^2 + 4r + 4 = 0 \tag{18.10}$$
$$(r+2)^2 = 0 \tag{18.11}$$

This implies that $r = -2$, and therefore $y_1(t) = e^{-2t}$ is a solution to Eq. (18.9).

(b)

$$y_2(t) = te^{-2t} \tag{18.12}$$
$$y_2'(t) = e^{-2t} - 2te^{-2t} \tag{18.13}$$
$$y_2''(t) = -2e^{-2t} - 2e^{-2t} + 4te^{-2t} \tag{18.14}$$
$$\tag{18.15}$$

To test this, we show that

$$-2e^{-2t} - 2e^{-2t} + 4te^{-2t} + 4e^{-2t} - 8te^{-2t} + 4te^{-2t} = 0,$$

which it does, so our solution is correct.

(c) The general solution will be a linear combination of y_1 and y_2:

$$y = c_1 e^{-2t} + c_2 t e^{-2t}.$$

18.3. Principle of Superposition

If y_1, y_2 are solutions to $L(y) = y'' + g(t)y' + r(t)y = 0$, then $y = c_1 y_1 + c_2 y_2$ is a solution to $L(y) = 0$.

The proof for this is found by plugging y'' and y' into $L(y)$, and showing that the result equals zero.

Theorem 36. If y_1, y_2 are solutions to $L(y) = 0$, then $y = c_1 y_1 + c_2 y_2$ is the general solution, iff:

$$\mathrm{W}\left(y_1, y_2\right) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0 \tag{18.16}$$

Example 18.3. If $y_1 = t$ and $\mathrm{W}\left(y_1, y_2\right) = t^2 e^t$, find $y_2(t)$.

Solution.

$$\begin{aligned} y_1(t) y_2'(t) - y_2(t) y_1'(t) &= t^2 e^t \\ y_1(t) \frac{\mathrm{d} y_2(t)}{\mathrm{d} t} - y_2(t) \frac{\mathrm{d} y_1(t)}{\mathrm{d} t} &= t^2 e^t \\ t \frac{\mathrm{d} y_2(t)}{\mathrm{d} t} - y_2(t) &= t^2 e^t \\ \frac{\mathrm{d} y_2}{\mathrm{d} t} - \frac{y_2(t)}{t} &= t e^t \end{aligned}$$

Let $\mu(t) = e^{\ln t} = -1/t$ and we find that

$$\frac{y_2(t)}{t} = e^t + C.$$

18.4. Second-order linear homogeneous differential equations with constant coefficients

A **second-order linear homogeneous** differential equation with **constant coefficients** is of the form

$$a y'' + b y' + c y = 0. \tag{18.17}$$

Example 18.4. Solve the initial value problem

$$y'' + 4y' + 5y = 0, \tag{18.18}$$

where $y(0) = 1$ and $y'(0) = 0$.

Solution.

From Eq. (18.18) we have the characteristic equation

$$r^2 + 4r + 5 = 0, \tag{18.19}$$

which implies that

$$\begin{aligned}
r_{1,2} &= \frac{-4 \pm \sqrt{4^2 - 4 \times 5}}{2} \\
r_{1,2} &= \frac{-4 \pm \sqrt{4 - 5}}{2} \\
r_{1,2} &= -2 \pm \sqrt{-1} \\
r_{1,2} &= -2 \pm \mathrm{i}
\end{aligned}$$

Thus

$$y_{1,2} = \mathrm{e}^{(-2 \pm \mathrm{i})t}$$

are solutions to Eq. (18.18).

18.5. Euler's identity

$$\mathrm{e}^{it} = \cos t + \mathrm{i} \sin t \tag{18.20}$$

$$\mathrm{e}^{\pi \mathrm{i}} = -1 \tag{18.21}$$

$$\mathrm{e}^{\pi \mathrm{i}} + 1 = 0 \tag{18.22}$$

Remark. The number e is defined in the equation

$$\int_1^{\mathrm{e}} \frac{1}{t} \, \mathrm{d}t \tag{18.23}$$

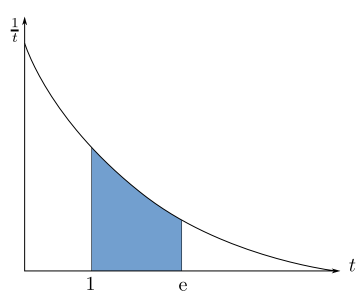


Figure 18.1.: e is the number that makes the shaded area equal to 1.

Part III.

Appendix

19. Important Concepts

19.1. Quadratic Formula

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{19.1}$$

19.2. Point-slope Formula

The **point-slope formula** allows us, given a point (x_1, y_1) and a slope m, to solve for y as a function of x.

$$y - y_1 = m(x - x_1) \tag{19.2}$$

19.3. Conjugate

In algebra, the **conjugate** of a *binomial* is another binomial formed by taking the opposite of the second term of the first binomial. For the initial binomial

$$a + b$$

its conjugate would be

$$a - b.$$

Meanwhile, for the expression

$$a^2 + b^2$$

we can factor this to produce

$$(a - b)(a + b)$$

where one expression is the conjugate of the other.

20. Proofs

20.1. Power Rule for Derivatives

The *power rule for derivatives* states that

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1} \tag{20.1}$$

Proof. To prove this, we use the limit definition of a derivative:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{20.2}$$

And assume that $f(x)$ is of the form $f(x) = x^n$.

Look at the term $(x+h)^n$. Take $n = 1$. In this case:

$$(x+h)^1 = x + h$$

Try it for $n = 2$.

$$\begin{aligned}(x+h)^2 &= (x+h)(x+h)\\ &= x^2 + 2hx + h^2\end{aligned}$$

Now for $n = 3$.

$$\begin{aligned}(x+h)^3 &= (x+h)(x+h)^2\\ &= (x+h)(x^2 + 2hx + h^2)\\ &= x^3 + 2hx^2 + h^2x + hx^2 + 2h^2x + h^3\\ &= x^3 + 3hx^2 + 3h^2x + h^3\end{aligned}$$

We are beginning to see a pattern in each of these sums: the first term is x^n, and each term after that has a common factor of h. Furthermore, it looks like there is always a term in the sum that has only one h within. Let's find this for $n = 4$ to be sure:

$$\begin{aligned}(x+h)^4 &= (x+h)(x+h)^3\\ &= (x+h)(x^3 + 3hx^2 + 3h^2x + h^3)\end{aligned}$$

$$= x^4 + 3hx^3 + 3h^2x^2 + h^3x + x^3h + 3hx^2x^2 + 3h^3x + h^4$$

Simplify.

$$(x+h)^4 = x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^3x + x^3h + h^4$$

It looks like the pattern follows for $n = 4$. We can claim that this holds for any value n, supposing that $f(x)$ is of the form x^n.

What does this mean for our limit equation (20.2)?

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Well, it means that the first term of our expanded polynomial should always cancel with the $\cdots - f(x)$ term if $n \neq 1$. It also means that we should always be able to divide our numerator by h to calculate the limit, assuming, again, that $n \neq 1$. Our limit would then look like this:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h \to 0} \frac{(x^n + n \cdot hx^{n-1} + \cdots + h^n) - x^n}{n} \qquad n \neq 1$$

x^n terms cancel.

$$= \lim_{h \to 0} \frac{n \cdot hx^{n-1} \cdots + h^n}{h} \qquad n \neq 1$$

Divide by h

$$= \lim_{h \to 0} nx^{n-1} \cdots + h^{n-1} \qquad n \neq 1, \quad h \neq 0$$

All terms following the first go to 0 as $h \to 0$, and we are left with

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = nx^{n-1}$$

Which is the same as (20.1). □

20.2. Sandwich Theorem for Sequences

I didn't write any of this. Check the citations for my sources.

Theorem 37 (The Sandwich Theorem for Sequences)**.** Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Theorem 38. If $|x| < 1, \lim_{n\to\infty} x^n = 0$.

Proof. We need to show that to each $\varepsilon > 0$ there corresponds an integer N so large that $\forall n > N\big(|x^n| < \varepsilon\big)$ Since $\varepsilon^{1/n} \to 1$, while $|x| < 1$, $\exists N\big(\varepsilon^{1/n} > |x|$. In other words,

$$\left|x^N\right| = \left|x^N\right| < \varepsilon. \tag{20.3}$$

This is the integer we seek because, if $|x| < 1$, then

$$\forall n > N\left(\left|x^n\right| < \left|x^N\right|\right) \tag{20.4}$$

Combining (20.3) and (20.4) produces $|x^n| < \varepsilon$ for all $n > N$, concluding the proof.[7, p. AP-21] ☐

Theorem 39. For any number x, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. [7, p. AP-22]

Proof. Since

$$-\frac{|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!},$$

all we need to show is that $|x|^n/n! \to 0$. We can then apply theorem 37 to conclude that $x^n/n! \to 0$.

The first step in showing that $|x|^n/n! \to 0$ is to choose an integer $M > |x|$, so that $(|x|/M) < 1$. By Theorem 38, we then have that $(|x|/M)^n \to 0$. We then restrict our attention to values of $n > M$. For these values of n, we can write

$$\begin{aligned}
\frac{|x|^n}{n!} &= \frac{|x|^n}{1 \cdot 2 \cdot \cdots \cdot M \cdot \underbrace{(M+1) \cdot (M+2) \cdot \cdots \cdot n}_{(n-M) \text{ factors}}} \\
&\le \frac{|x|^n}{M!M^{n-M}} = \frac{|x|^n \cdot M^M}{M^n \cdot M!} = \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n.
\end{aligned}$$

Thus,

$$0 \le \frac{|x|^n}{n!} \le \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n.$$

Now, the constant $M^M/M!$ does not change as n increases. Thus Theorem 37 tells us that $|x|^n/n! \to 0$ because $(|x|/M)^n \to 0$. [7, p. AP-22] ☐

20.3. L'Hospital's Rule

Theorem 40 (L'Hospital's Rule). Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \stackrel{\text{H}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

Proof. We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

But $f(a) = g(a)$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$$

As x approaches a, c approaches a because it always lies between a and x. Therefore,

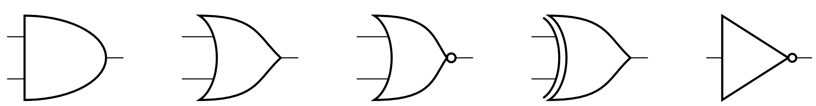
$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = lim_{c\to a} \frac{f'(c)}{g'(c)} = lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

which establishes l'Hospital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem (found in Section 12.5) to the closed interval $[x, a], x < a$.[7] □

21. Logic Circuits

> Look, everything we're putting into that box becomes ungrounded, and I don't mean grounded like to the earth, I mean, not tethered. I mean, we're blocking whatever keeps it moving forward and so they flip-flop. Inside the box it's like a street, both ends are cul-de-sacs. I mean, this isn't frame dragging or wormhole magic, this is basic mechanics and heat 101.
>
> *Primer*, 2004

A logic circuit receives input signals $p_1, p_2, \dots, p_n$, each a bit, and produces output signals $s_1, s_2, \dots, s_n$, each a bit.



(a) and (b) or (c) nor (d) xor (e) not

Figure 21.1.: Basic logic gates.

22. The Fibonacci Sequence and the Golden Ratio

22.1. The Fibonacci Sequence

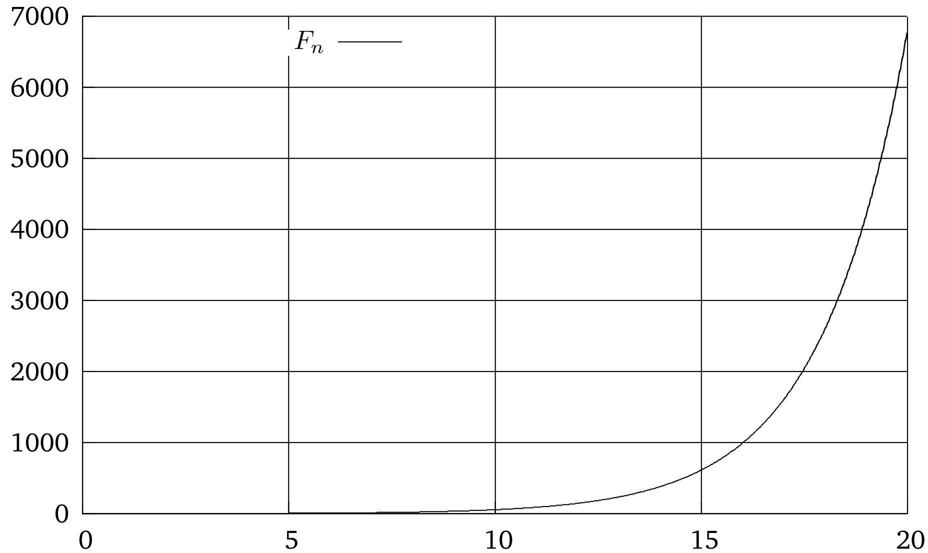


Figure 22.1.: A plot of equation 22.2.

The *Fibonacci Sequence* is the first recursive number sequence known in Europe. Its first 10 numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55 \ldots \quad .$$

A Fibonacci sequence, in general, is any sequence of numbers in which each number is the sum of the two preceeding numbers.[9]

22.1.1. History

French mathematician Edouard Lucas coined the term "Fibonacci sequence" in the 19th century. The sequence is found throughout nature, as in the spirals of sunflower heads, pine cones, snail shells, and animal horns.[9] Because of this natural prevalence, patterns based on the Fibonacci sequence are considered aesthetically pleasing. The sequence can be found in Mozart and Beethoven's works as well as in classical art and architecture. [**?** , p. 94]

22.1.2. Mathematics

Definition 22.1. The *Fibonacci numbers* are the sequence of numbers $\{F_n\}_{n=1}^{\infty}$ defined by the linear recurrence equation

$$F_n = F_{n-1} + F_{n-2}. \tag{22.1}$$

Often, we will see them defined with $F_0 = 0$.

This can be represented in the *closed form*

$$F_n = \left[\frac{\Phi^n}{\sqrt{5}}\right] \tag{22.2}$$

where $[x]$ is the *nearest integer function*. [4]

22.2. The Golden Ratio

The Fibonacci sequence and the golden ratio are closely related.

Definition 22.2. The *golden ratio*, denoted Φ, is given by the positive solution to the equation

$$\Phi^2 - \Phi - 1 = 0 \tag{22.3}$$

Using the quadratic equation (19.1) we can find that

$$\begin{aligned}\Phi =& \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} \\ =& \frac{1 \pm \sqrt{1+4}}{2} \\ =& \frac{1 \pm \sqrt{5}}{2}\end{aligned}$$

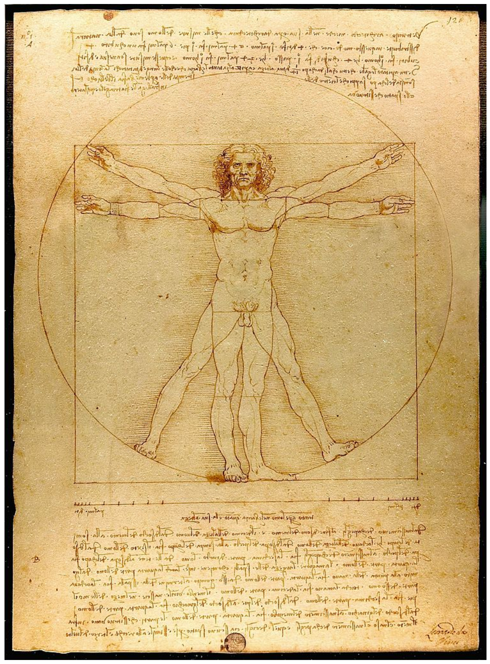


Figure 22.2.: The Vitruvian Man, said to depict ideal human proportions, bases its proportions on the golden ratio.

and taking the positive root

$$\Phi = \frac{1+\sqrt{5}}{2}$$
$$=1.6180339887498948\ldots$$

[13]

We will notice that many closed-form representations of the Fibonacci sequence use the golden ratio. For example, *Binet's Formula*

$$F_n = \frac{\Phi^n - (-\Phi)^{-n}}{\sqrt{5}} \tag{22.4}$$

derived[1] by Binet in 1843 and equation 22.2 both write F_n in terms of Φ.[12]

The ratio of consecutive terms in the Fibonacci sequence approximate the golden ratio:

$$\frac{1}{1} = 1$$

[1]Though not for the first time.

$$\frac{2}{1} = 2$$
$$\frac{3}{2} = 1.5$$
$$\frac{5}{3} = 1.\overline{6}\dots$$
$$\frac{8}{5} = 1.6$$
$$\frac{13}{8} = 1.625$$
$$\frac{21}{13} \approx 1.6153846$$

Through this, we can conclude that

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \Phi \qquad (22.5)$$

[4]

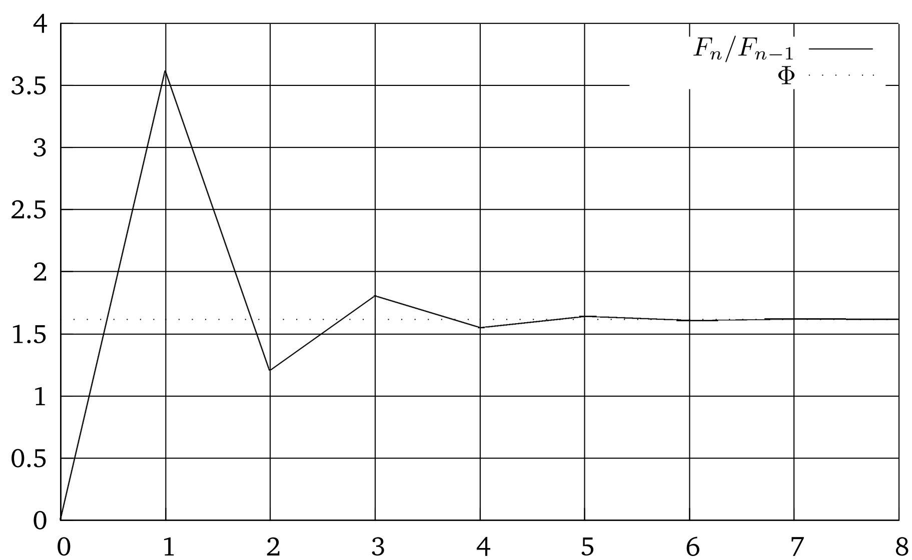


Figure 22.3.: Equation 22.5 converges to Φ.

22.3. Culture

22.3.1. Lateralus

The song "*Lateralus*" by the American rock band Tool counts out the Fibonacci sequence in its syllables:[2]

1	Black,
1	then,
2	white are,
3	all I see,
5	in my infancy,
8	red and yellow then came to be,
5	reaching out to me,
3	lets me see.
2	There is,
1	so,
1	much,
2	more and
3	beckons me,
5	to look through to these,
8	infinite possibilities.
13	As below so above and beyond I imagine,
8	drawn beyond the lines of reason.
5	Push the envelope.
3	Watch it bend.

Figure 22.4.: Maynard James Keenan's vocals.

[2] `http://en.wikipedia.org/w/index.php?title=Lateralus%20(song)&oldid=479876017`

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