Math

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1 Stochastic Matrices

1.1 Stochastic Matrices

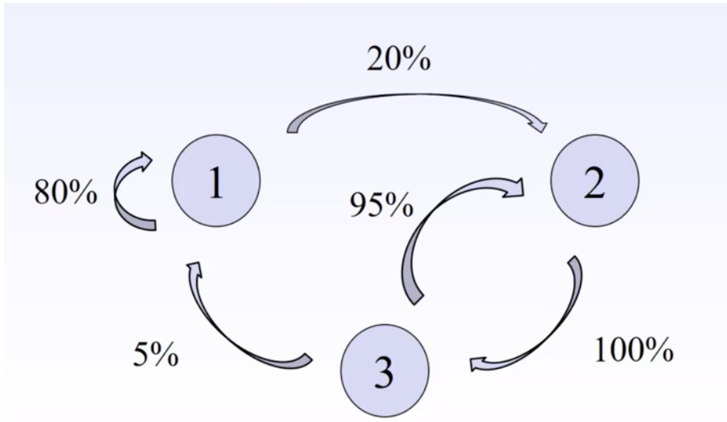
- Informal definition: A square matrix that represent transitions between different random states. Elements within a stochastic matrix represent probabilities. This implies that each element must be nonnegative and each row must sum to 1.
- Formal definition: A matrix $\Phi \in R^{N \times N}$ is a stochastic matrix if:
  - all elements are positive: $\Phi_{ij} \geq 0$ for all i, j
  - each row sums to 1: $\Sigma_j \Phi_{ij} = 1$ for all i

1.1.1 Intepretations

- $P(s_{t+1} = n | s_t = m) = \Phi_{mn}$: This says that the element Φ_{mn} represents the probability of the next state, s_{t+1} being equal to n given that the current state, s_t is the mth state for all possible states $s \in \{1, \ldots, N\}$.

1.1.2 Examples

1. 3-State Markov Process: Stochastic Matrix $\Phi = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0 & 1 \\ 0.05 & 0.95 & 0 \end{bmatrix}$



Each node is a state. Each edge represents the probability of transitioning from some state to another.

1.2 Discrete Kolmogorov Backward Equations

- Informal definition: A formula to represent the expected value of some payoff function (i.e. for an option) depending upon what state we are in the future.
- Formal definition: For function $F : \{1, \ldots, N\} \to R$, define $F(j) = f_j, f \in R^N$.
  - F is the 'payoff function' and f_j is the payoff, given we are in state j at some future specified time.

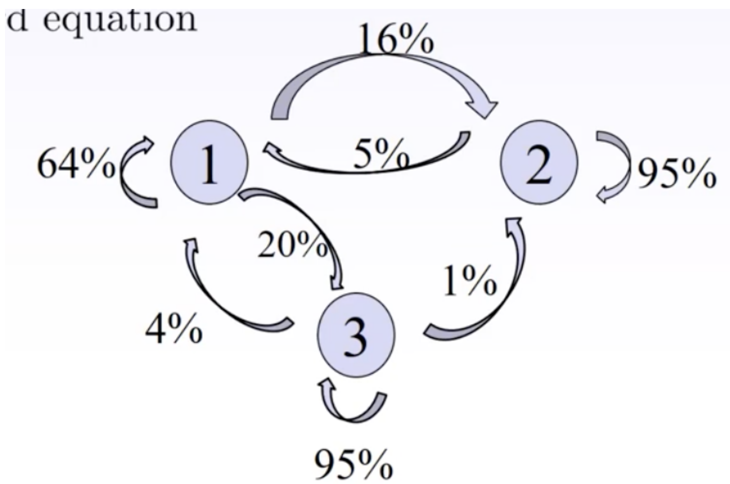
1.2.1 Intepretations

1. $\Phi^2 = \Phi\Phi$: The transition probabilities between two periods, t and $t+2$.
2. Φ^n: The transition probabilities between n periods, t and $t+n$.
3. $[\Phi^{T-t}]_{ij} = P(s_T = j|s_t = i)$: The probability of being in state j after t periods, given that we are in state i currently (at time t) is the ij-th element of the stochastic matrix Φ^{T-t}.
4. $E[F(s_T)|s_t = j] = [\Phi^{T-t}f]_j$: The expected payoff at period T, given that we are currently in state $s_t = j$ at period t, is the j-th element of the vector resulting from taking the product of Φ^{T-t} and the payoff function f.
   - $[\Phi^{T-t}f]_j = \Phi^{T-t}_{j1}f_1 + \cdots + \Phi^{T-t}_{jN}f_N$

1.2.2 Examples

1. 3-State Markov Process: Stochastic Matrix $\Phi = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0 & 1 \\ 0.05 & 0.95 & 0 \end{bmatrix}$,

   $\Phi^2 = \begin{bmatrix} 0.64 & 0.16 & 0.2 \\ 0.05 & 0.95 & 0 \\ 0.04 & 0.01 & 0.95 \end{bmatrix}$



Each node is a state. Each edge represents the probability of ending up in the state at the end of the arrow, given we are in the state at the tail of the arrow 2 periods prior.

1.3 Discrete Kolmogorov Forward Equations

- Informal definition: An equation useful in determining the distribution of us being in a specific state t periods into the future.
- Formal definition:
  - Probability vector: a vector $p \in R^N$ such that:
    * $p_i \geq 0$
    * $\Sigma_i p_i = 1$
  - Stationary distribution: a probability vector for a stochastic matrix, Φ, such that $p = \Phi^* p$ (i.e. p is a left eigenvector of Φ^* with an eigenvalue of 1) (Φ^* is used to denote the transpose).
  - Discrete Kolmogorov Forward Equation: if the probability vector at t is p^t, then at time T, the probability vector for being in any given state is $(\Phi^*)^{T-t} p^t$.

- To ensure a stochastic matrix has a unique stationary distribution, two criteria must be fulfilled:

  1. The matrix must be **irreducible**. As stochastic matrix is said to be irreducible if, for each i, j, there exists $k > 0$ such that $(\Phi^k)_{ij} > 0$. In other words, there exists some path that has a non-zero probability between any two states.
  2. The Markov chain is **aperiodic**. The **period** of state i, is k_i, where k_i is the greatest common divisor of the set $\{n|(\Phi^n)_{ii} > 0\}$. If $k_i = 1$ for $i = 1, \ldots, n$, then Φ is aperiodic. In other words, to be aperiodic means there is no cyclicality in the number of steps it takes to get back to some starting state.
  3. **Perron-Frobenius Theorem**: Consider an irreducible, aperiodic stochastic matrix Φ. Then Φ has a unique stationary distribution with strictly positive elements.
     - All other left eigvenvalues of $\Phi, \lambda_2, \ldots, \lambda_N$, have $|\lambda_i| < 1$.
     - All other left eigenvectors of Φ have at least one nonpositive element.

1.3.1 Intepretations

1. $\Phi^{1000} = \begin{bmatrix} 0.111 & 0.444 & 0.444 \\ 0.111 & 0.444 & 0.444 \\ 0.111 & 0.444 & 0.444 \end{bmatrix}$: This is the transition matrix 1000 periods into the future. Each column has the same values. This means regardless of which state we start in, we have the same probability of ending up in any state 1000 periods later. This implies there is a limiting distribution $\mathbf{p} = (0.111, 0.444, 0.444)'$ regardless of the initial state. Note: there is not always a limiting distribution.