

# Introduction to Linear Algebra Notes

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# 1 Vectors and Matrices

## 1.1 Vectors and Linear Combinations

**Vector Length:** For a vector  $\mathbf{v} \in \mathbb{R}^n$ , its length is:

$$\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

In words, the length of a vector is the square root of the sum of the squared components.

Given two vectors in  $\mathbb{R}^2$   $\mathbf{v}, \mathbf{w}$  with their tail starting from the origin

- If they lie on the same line, the vectors are *linearly dependent*.
- If they do not lie on the same line, the vectors are *linearly independent*.

Therefore, the combinations  $c\mathbf{v} + d\mathbf{w}$  fill the  $x - y$  plane unless  $\mathbf{v}$  is in line with  $\mathbf{w}$ .

To fill  $m$ -dimensional space, we need  $m$  independent vectors, with each vector having  $m$  components.

## 1.2 Lengths and Angles from Dot Products

**Dot Product:** For two vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ , their dot product is:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$$

The dot product of two vectors tells us what amount of one vector goes in the direction of another. It tells us how much these vectors are working together.

- $\mathbf{v} \cdot \mathbf{w} > 0$ : The vectors point in somewhat similar directions. In other words, the angle between the two vectors is less than 90 degrees.
- $\mathbf{v} \cdot \mathbf{w} = 0$ : The vectors are perpendicular. In other words, the angle between the two vectors is 90 degrees.
- $\mathbf{v} \cdot \mathbf{w} < 0$ : The vectors point in somewhat opposing directions. In other words, the angle between the two vectors is greater than 90 degrees.

Dot Product Rules (for two vectors,  $\mathbf{v}, \mathbf{w}$ ):

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w})$

**Cosine Formula:** If  $\mathbf{v}$  and  $\mathbf{w}$  are nonzero vectors, then:

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

**Unit Vectors:** A vector is a unit vector if its length is 1.

For a vector  $\mathbf{u} \in \mathbb{R}^n$ :

$$\|\mathbf{u}\| = 1$$

For any vector  $\mathbf{v} \in \mathbb{R}^n$ , as long as  $\mathbf{v} \neq 0$ , dividing  $\mathbf{v}$  by its length will result in a unit vector. In other words:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

**Cauchy-Schwarz Inequality:**

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$$

In words, the absolute value of the dot product of two vectors is no greater than the product of their lengths.

**Triangle Inequality:**

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

In words, the length of any one side (in this case  $\|\mathbf{v} + \mathbf{w}\|$ ) of a triangle is at most the sum of the length of the other triangle sides.

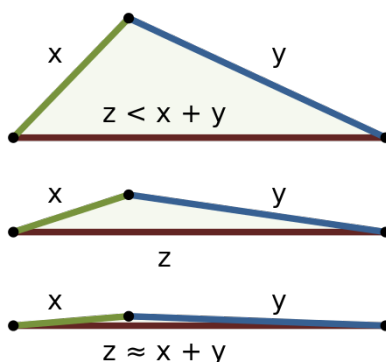


Figure 1: This Squeeze Theorem

### 1.3 Matrices and Their Column Spaces

**Independence:** Columns are independent when each new column is a vector that we don't already have as a combination of previous columns. The only combination of columns that produces  $A\mathbf{x} = (0, 0, 0)$  is  $\mathbf{x} = (0, 0, 0)$ .

**Column Space:** The column space,  $\mathbf{C}(A)$ , contains all vectors  $A\mathbf{x}$ . In other words, it contains all combinations of the columns.

The **span** of the columns of  $A$  is the column space.

**Rank:** The number of independent columns of a matrix. This is equivalent to saying the rank is the number of pivots in a matrix.

Rank Rules:

- $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$

## 1.4 Matrix Multiplication $AB$ and $CR$

To multiply two matrices  $AB$ , take the dot product of each row of  $A$  with each column of  $B$ . The number in row  $i$ , column  $j$  of  $AB$  is (row  $i$  of  $A$ )  $\cdot$  (column  $j$  of  $B$ ).

When  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times p}$ :

- $AB \in \mathbb{R}^{m \times p}$
- $mp$  dot products are needed to carry out the matrix multiplication (one for each entry in the matrix  $AB$ ).

Matrix Multiplication Rules:

- Associative:  $(AB)C$
- Distributive:  $A(B + C) = AB + AC$
- Not Commutative: In general  $AB \neq BA$

## 2 Solving Linear Equations $A\mathbf{x} = \mathbf{b}$

### 2.1 Elimination and Back Substitution

For a matrix  $A \in \mathbb{R}^{n \times n}$ , there are three outcomes for  $A\mathbf{x} = \mathbf{b}$ :

1. No solution

- $\mathbf{b}$  is not in the column space of  $A$
- This occurs when the columns of  $A$  are dependent and  $\mathbf{b}$  is not in  $C(A)$  (~~32~~ = any value but 32).

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 16 \\ \text{32} \end{bmatrix}$$

2. Exactly 1 solution

- $A$  has independent columns and an inverse matrix  $A^{-1}$

3. Infinitely many solutions

- Columns of  $A$  are dependent.
- This occurs when the columns of  $A$  are dependent and  $\mathbf{b}$  is in  $C(A)$

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 32 \end{bmatrix}$$

**Elimination:** A system that allows us to determine if  $A\mathbf{x} = \mathbf{b}$  has no solution, 1 solution, or infinitely many solutions. The goal of elimination is to transform  $A$  to an upper triangular matrix,  $U$ .

Elimination allows us to discover the number of pivots in  $A \in \mathbb{R}^{n \times n}$  by creating  $U$ . If there are  $n$  pivots in  $U$ ,  $U$  has full rank. This implies  $A$  has exactly one solution.

**Back Substitution:** If  $U$  has full rank, back substitution allows us to find the solution.

## 2.2 Elimination Matrices and Inverse Matrices

### Elimination

- The basic elimination step subtracts a multiple of  $\ell_{ij}$  of equation  $j$  from equation  $i$

### Inverse Matrices

- If  $A$  is invertible, the one and only solution to  $A\mathbf{x} = \mathbf{b}$  is  $\mathbf{x} = A^{-1}\mathbf{b}$
- Only square matrices can have inverses
- Invertible  $\equiv$  non-singular  $\equiv$  non-zero determinant  $\equiv$  independent columns
- Not invertible  $\equiv$  singular  $\equiv$  zero determinant  $\equiv$  dependent columns
- If a matrix is invertible, the inverse is unique
- A triangular matrix has an inverse so long that it has no zero diagonal entries

## 2.3 Matrix Computations and $A = LU$

Elimination without row exchanges factors  $A$  into  $LU$ . We can't find an  $LU$  decomposition if row exchanges are needed during elimination.

### Gauss-Jordan elimination:

- An algorithm that allows us to determine if the inverse of a matrix exists, and if it does it exist, it allows us to determine what the inverse is.
- Augment  $A$  by  $I$ , that is  $[A \ I]$ , and through elementary row operations, transform this matrix to  $[I \ A^{-1}]$

## 3 The Four Fundamental Subspaces

### 3.1 Vector Spaces and Subspaces

To be a vector space means that all linear combinations  $c\mathbf{v} + d\mathbf{w}$  of the vectors or matrices stay inside that space.

### Subspaces:

- A subspace is a vector space entirely contained within another vector space
- All linear combinations of vectors in the subspace stay in the subspace
- Every subspace contains the zero vector
- Subspaces of  $\mathbb{R}^3$ :
  - The single vector  $(0, 0, 0)$
  - Any line through  $(0, 0, 0)$
  - Any plane through  $(0, 0, 0)$

- The whole space  $\mathbb{R}^3$

### Column Space ( $C(A)$ ):

- The column space consists of all linear combinations of the columns
- To solve  $A\mathbf{x} = \mathbf{b}$  is to express  $\mathbf{b}$  as a combination of the columns
- The right side,  $\mathbf{b}$ , has to be in the column space produced by  $A$ , or  $A\mathbf{x} = \mathbf{b}$  has no solution

### Row Space ( $C(A^T)$ ):

- The row space of  $A$  is the column space of  $A^T$
- The rank of  $A = \text{rank of } A^T$

### Span:

- The span of vectors  $\mathbf{v}$  and  $\mathbf{w}$  is the set of all of their linear combinations.
- In other words, this tells us, given some set of vectors, which vectors are able to be created by taking a linear combination of the vectors in the set. It is the vector space we can reach (span) by taking linear combinations of the set of vectors.
- Independence is not required by the word span.

## 3.2 Computing the Nullspace by Elimination: $A = CR$

### Nullspace ( $N(A)$ ):

- Contains all solutions  $\mathbf{x}$  to  $A\mathbf{x} = 0$  including  $\mathbf{x} = 0$
- If  $A$  is invertible, then the nullspace contains only the zero vector (no special solutions)
- If  $A$  has  $n$  columns,  $r$  of which are independent, then there are  $n - r$  vectors in the nullspace
- Elimination does not change the nullspace (though it does change the column space)
- If  $n > m$  (more columns than rows), then there is at least one free variable (can't have  $n$  independent columns in  $\mathbb{R}^m$ ). Therefore, there is at least one nonzero solution
- The dimension of the nullspace is the number of free variables in a matrix

### Echelon Form ( $R$ ):

- Echelon form requirements:
  1. All rows having only zeros are at the bottom
  2. The leading entry (pivot) for each nonzero row (leftmost nonzero entry), is on the right of the leading entry of every row above
- The result of Gaussian elimination on any matrix, square or otherwise
- If the matrix is square and invertible, echelon form is an upper triangular matrix,  $U$
- Can be viewed as a generalization of upper triangular form for rectangular matrices

- Example:

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

### Reduced Row Echelon Form Requirements ( $R_0$ ):

- In row echelon form
- The leading entry (pivot) of each nonzero row is a 1
- Each column containing a leading 1 has zeros in all entries above the leading 1.
- Example:

$$\begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Algorithm (must be in row echelon form):
  1. Divide pivots by themselves to make them all 1.
  2. Zero out all entries above and below the pivots.

The result will have the identity matrix in the pivot columns (and the remaining columns will be the special columns)

### Finding the Nullspace:

Goal: Given some matrix  $A$ , find all solutions  $A\mathbf{x} = 0$

1. Start with some matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

2. Do elimination to echelon form

$$R = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Do elimination from row echelon form to reduced row echelon form to find the pivot columns and free columns

$$R_0 = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Takeaways:

- $r = 2$  = number of pivots (coming from columns 1 and 3)
- 2 free variables (coming from columns 2 and 4)
  - Free variables, which in this case are  $x_2$  and  $x_4$  can take on any value
  - There is one special solution for each free variable



- The special solution for each free variable is found by setting that free variable to 1 and the rest of the free variables to 0
4. Find the special solutions for the free variables using back substitution

Recall our system of linear equations:

$$x_1 + 2x_2 - 2x_4 = 0$$

$$x_3 + 2x_4 = 0$$

- (a) Let  $x_2 = 1, x_4 = 0$ . Equation 2 gives us  $x_3 = 0$  and equation 1 gives us  $x_1 = -2$ . The result is a vector in the nullspace:

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- (b) Let  $x_2 = 0, x_4 = 1$ . Equation 2 gives us  $x_3 = -2$  and equation 1 gives us  $x_1 = 2$ . The result is a vector in the nullspace:

$$s_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Takeaways:

- These vectors form a basis for the nullspace of  $A$ .
- Any multiple of these 2 vectors (special solutions) are in the nullspace.
- The nullspace contains exactly all the linear combinations of the special solutions.
- In other words, for any 2 constants  $c, d$ , the nullspace is:

$$cs_1 + ds_2$$

### 3.3 The Complete Solution to $Ax = b$

$Ax = b$  is solvable when  $b$  is in  $C(A)$

**Finding the Complete Solution to  $Ax = b$ :**

1. Augment matrix  $A$  to account for the non-zero  $b$  on the right side

$$A \rightarrow [A \ b]$$

2. Reduce  $[A \ b] \rightarrow [U \ c]$

$$[A \ b] = \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = [U \ c]$$

- Columns 1 and 3 will be the pivot columns ( $x_1, x_3$ ) and columns 2 and 4 will be the free columns (free variables  $x_2, x_4$ )
- Condition for solvability:  $b_3 + b_2 - 5b_1 = 0$

3. Reduce  $[U \ \mathbf{c}] \rightarrow [R_0 \ \mathbf{d}]$

$$[U \ \mathbf{c}] \rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 1 & 1 & \frac{1}{2}b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 4b_1 - \frac{3}{2}b_2 \\ 0 & 0 & 1 & 1 & \frac{1}{2}b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = [R_0 \ \mathbf{d}]$$

4. Find a particular solution (there is only one),  $\mathbf{x}_p$ , for

$$A\mathbf{x} = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}$$

Note that this  $\mathbf{b}$  was given in the problem

- (a) Set all free variables to 0
- $$x_1 + 2x_2 + 2x_4 = 4b_1 - \frac{3}{2}b_2$$
- $$x_3 + x_4 = \frac{1}{2}b_2 - b_1$$

Let  $x_2 = 0, x_4 = 0$

$$x_1 = 4b_1 - \frac{3}{2}b_2$$

$$x_3 = \frac{1}{2}b_2 - b_1$$

- (b) Solve  $A\mathbf{x} = \mathbf{b}$  for the pivot variables

$$\mathbf{x}_p = \begin{bmatrix} 4b_1 - \frac{3}{2}b_2 \\ 0 \\ \frac{1}{2}b_2 - b_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4(0) - \frac{3}{2}(6) \\ 0 \\ \frac{1}{2}(6) - 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

5. Find the nullspace

Recall our system of linear equations (and set to 0 since we're solving nullspace):

$$x_1 + 2x_2 + 2x_4 = 0$$

$$x_3 + x_4 = 0$$

- (a) Let  $x_2 = 1, x_4 = 0$ . Equation 2 gives us  $x_3 = 0$  and equation 1 gives us  $x_1 = -2$ . The result is a vector in the nullspace:

$$\mathbf{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- (b) Let  $x_2 = 0, x_4 = 1$ . Equation 2 gives us  $x_3 = -1$  and equation 1 gives us  $x_1 = -2$ . The result is a vector in the nullspace:

$$\mathbf{s}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

For any 2 constants  $c_1, c_2$ , the nullspace is:

$$\mathbf{x}_n = c_1 \mathbf{s}_1 + c_2 \mathbf{s}_2$$

6. Sum the particular and nullspace solutions to get the complete solution

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$$

$$\mathbf{x} = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Properties for every matrix  $A$  with full column rank ( $r = n$ ) (a pivot in every column):

- All columns of  $A$  are pivot columns (independent). No free variables
- The nullspace,  $N(A)$ , contains only the zero vector  $\mathbf{x} = 0$
- There are 0 or 1 solutions to  $A\mathbf{x} = \mathbf{b}$

Properties for every matrix  $A$  with full row rank ( $r = m$ ) (a pivot in every row):

- All rows of  $A$  are pivot rows and  $R_0$  has no zero rows ( $R_0 = R$ ).  $n - r$  free variables
- $A\mathbf{x} = \mathbf{b}$  has a solution for every right side  $\mathbf{b}$
- The column space of  $A$  is the whole space  $\mathbb{R}^m$
- If  $m < n$  then  $A\mathbf{x} = \mathbf{b}$  is undetermined (many solutions)

Possibilities for Linear Equations:

1.  $r = m, r = n$  : square and invertible  $\rightarrow A\mathbf{x} = \mathbf{b}$  has 1 solution
2.  $r = m, r < n$  : short and wide  $\rightarrow A\mathbf{x} = \mathbf{b}$  has  $\infty$  solutions
3.  $r < m, r = n$  : tall and thin  $\rightarrow A\mathbf{x} = \mathbf{b}$  has 0 or 1 solution
4.  $r < m, r < n$  : not full rank  $\rightarrow A\mathbf{x} = \mathbf{b}$  has 0 or  $\infty$  solutions

### 3.4 Independence, Basis, and Dimension

**Linear Independence** (below are different ways of saying a matrix has independent columns)

- Having independent vectors essentially means there are no extra, unnecessary vectors. In other words, each vector is necessary, as no vector is a linear combination of the other vectors
- Different ways of saying a matrix has independent columns
  - The columns of  $A$  are linearly independent when the only solution to  $A\mathbf{x} = \mathbf{0}$  is  $\mathbf{x} = \mathbf{0}$
  - The columns of  $A$  are linearly independent when there is no combination of the columns of  $A$ , except for the zero vector, solve  $A\mathbf{x} = \mathbf{0}$
  - The columns of  $A$  are linearly independent when its nullspace,  $N(A)$ , contains only the zero vector

#### Span

- The span of a set of vectors is defined as the set of all linear combinations of The set of vectors
- Saying that vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  span a space means the space consists of all linear combinations of those vectors
- In  $\mathbb{R}^n$ 
  - $n - 1$  independent columns, cannot span all of  $\mathbb{R}^n$  (that being said, they do still span a space; however, the space they span is just not all of  $\mathbb{R}^n$ . In fact, they would even form a basis for this space (since the columns are independent))
  - $n$  independent columns are needed to span all of  $\mathbb{R}^n$
  - $n + 1$  vectors cannot be independent (only  $n$  pivots, so even if we have  $n$  pivots, one vector will still be dependent)
- The vectors,  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , may or may not be independent
  - For example, the columns of a matrix span its column space. These column vectors may be dependent

#### Basis

- In the special case that vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are independent (meaning we have just the right amount of vectors to describe the space, any less vectors and we wouldn't fully define the space, any more would be redundant), we say these vector form a basis for the space. In other words, a basis is a special case of a span
- More formally, a basis for a space is a sequence of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  that have 2 properties
  1. the vectors are independent
  2. the vectors span the space
- The vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are a basis for  $\mathbb{R}^n$  exactly when they are the columns of an  $n$  by  $n$  invertible matrix
- To see if some vectors form a basis we would

1. Put the vectors in the columns of a matrix
  2. Do elimination
  3. See if all the columns are pivot columns, if so, they form a basis, else, they don't
- All bases (plural for basis) for a vector space have the same number of vectors
    - For example, there are many sets of vectors that form a basis for  $\mathbb{R}^n$  and they all have  $n$  vectors

### **Dimension**

- The number of vectors in any and every basis is the dimension of the space
- The dimension of the column space equals the rank of the matrix which equals the number of pivot columns

### **3.5 Dimensions of the Four Subspaces**