Introduction to Linear Algebra Notes

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1 Vectors and Matrices

1.1 Vectors and Linear Combinations

Vector Length: For a vector $v \in \mathbb{R}^n$, its length is:

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

In words, the length of a vector is the square root of the sum of the squared components.

Given two vectors in $\mathbb{R}^2$ v, w with their tail starting from the origin

- If they lie on the same line, the vectors are *linearly dependent*.
- If they do not lie on the same line, the vectors are *linearly independent*.

Therefore, the combinations $c\mathbf{v} + d\mathbf{w}$ fill the $x - y$ plane unless v is in line with w.

To fill m-dimensional space, we need m independent vectors, with each vector having m components.

1.2 Lengths and Angles from Dot Products

Dot Product: For two vectors $v, w \in \mathbb{R}^n$, their dot product is:

$$v \cdot w = v_1 w_1 + \cdots + v_n w_n$$

The dot product of two vectors tells us what amount of one vector goes in the direction of another. It tells us how much these vectors are working together.

- $v \cdot w > 0$: The vectors point in somewhat similar directions. In other words, the angle between the two vectors is less than 90 degrees.
- $v \cdot w = 0$: The vectors are perpendicular. In other words, the angle between the two vectors is 90 degrees.
- $v \cdot w < 0$: The vectors point in somewhat opposing directions. In other words, the angle between the two vectors is greater than 90 degrees.

Dot Product Rules (for two vectors, v, w):

- $\boldsymbol{v} \cdot \boldsymbol{w} = w \cdot v$
- $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$
- $(c\boldsymbol{v}) \cdot \boldsymbol{w} = c(\boldsymbol{v} \cdot \boldsymbol{w})$

Cosine Formula: If v and w are nonzero vectors, then:

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}$$

Unit Vectors: A vector is a unit vector if its length is 1.
For a vector $u \in \mathbb{R}^n$:

$$\|u\| = 1$$

For any vector $v \in \mathbb{R}^n$, as long as $v \neq 0$, dividing v by its length will result in a unit vector. In other words:

$$u = \frac{v}{\|v\|}$$

Cauchy-Schwarz Inequality:

$$|v \cdot w| \leq \|v\|\|w\|$$

In words, the absolute value of the dot product of two vectors is no greater than the product of their lengths.

Triangle Inequality:

$$\|v + w\| \leq \|v\| + \|w\|$$

In words, the length of any one side (in this case $\|v + w\|$) of a triangle is at most the sum of the length of the other triangle sides.

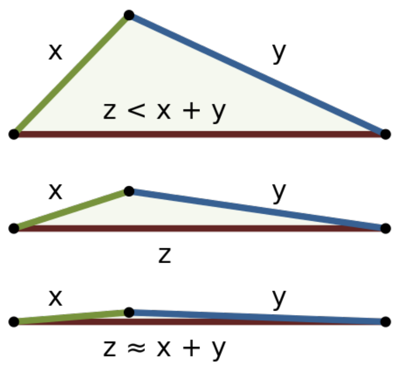

Figure 1: This Squeeze Theorem

1.3 Matrices and Their Column Spaces

Independence: Columns are independent when each new column is a vector that we don't already have as a combination of previous columns. The only combindation of columns that produces $A\boldsymbol{x} = (0, 0, 0)$ is $\boldsymbol{x} = (0, 0, 0)$.

Column Space: The column space, $\mathbf{C}(A)$, contains all vectors $A\boldsymbol{x}$. In other words, it contains all combinations of the columns.
The **span** of the columns of $\boldsymbol{A}$ is the column space.

Rank: The number of independent columns of a matrix. This is equivalent to saying the rank is the number of pivots in a matrix.

Rank Rules:

- $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$

1.4 Matrix Multiplication AB and CR

To multiply two matrices AB, take the dot product of each row of A with each column of B. The number in row i, column j of AB is (row i of A) $\cdot$ (column j or B).

When $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$:

- $AB \in \mathbb{R}^{m \times p}$
- mp dot products are needed to carry out the matrix multiplication (one for each entry in the matrix AB).

Matrix Multiplication Rules:

- Associative: $(\boldsymbol{AB})\boldsymbol{C}$
- Distributive: $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{BC}$
- Not Commutative: In general $\boldsymbol{AB} \neq \boldsymbol{BA}$

2 Solving Linear Equations Ax = b

2.1 Elimination and Back Substitution

For a matrix $A \in \mathbb{R}^{n \times n}$, there are three outcomes for $A\boldsymbol{x} = \boldsymbol{b}$:

1. No solution
   - $\boldsymbol{b}$ is not in the column space of A
   - This occurs when the columns of A are dependent and $\boldsymbol{b}$ is not in $C(A)$.

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 16 \\ \cancel{32} \end{bmatrix}$$

2. Exactly 1 solution
   - A has independent columns and an inverse matrix A^{-1}
3. Infinitely many solutions
   - Columns of A are dependent.
   - This occurs when the columns of A are dependent and $\boldsymbol{b}$ is in $C(A)$

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 32 \end{bmatrix}$$

Elimination: A system that allows us to determine if $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, 1 solution, or infinitely many solutions. The goal of elimination is to transform A to an upper triangular matrix, U.

Elimination allows us to discover the number of pivots in $A \in \mathbb{R}^{n \times n}$ by creating U. If there are n pivots in U, U has full rank. This implies A has exactly one solution.

Back Substitution: If U has full rank, back substituion allows us to find the solution.

2.2 Elimination Matrices and Inverse Matrices

Elimination

- The basic elimination step subtracts a multiple of ℓ_{ij} of equation j from equation i

Inverse Matrices

- If A is invertible, the one and only solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$
- Only square matrices can have inverses
- Invertible $\equiv$ non-singular $\equiv$ non-zero determinant $\equiv$ independent columns
- Not invertible $\equiv$ singular $\equiv$ zero determinant $\equiv$ dependent columns
- If a matrix is invertible, the inverse is unique
- A triangular matrix has an inverse so long that it has no zero diagonal entries

2.3 Matrix Computations and $\mathbf{A = LU}$

Elimination without row exchanges factors A into LU. We can't find an LU decomposition if row exchanges are needed during elimination.

Gauss-Jordan elimination:

- An algorithm that allows us to determine if the inverse of a matrix exists, and if it does it exist, it allows us to determine what the inverse is.
- Augment A by I, that is $[A\ I]$, and through elementary row operations, transform this matrix to $\left[I\ A^{-1}\right]$

3 The Four Fundamental Subspaces

3.1 Vector Spaces and Subspaces

To be a vector space means that all linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$ of the vectors or matrices stay inside that space.

Subspaces:

- A subspace is a vector space entirely contained within another vector space
- All linear combinations of vectors in the subspace stay in the subspace
- Every subspace contains the zero vector
- Subspaces of $\mathbb{R}^3$:
  - The single vector $(0, 0, 0)$
  - Any line through $(0, 0, 0)$
  - Any plane through $(0, 0, 0)$

- The whole space $\mathbb{R}^3$

Column Space:

- The column space ($C(A)$) consists of all linear combinations of the columns
- To solve $A\boldsymbol{x} = \boldsymbol{b}$ is to express $\boldsymbol{b}$ as a combination of the columns
- The right side, $\boldsymbol{b}$, has to be in the column space produced by A, or $A\boldsymbol{x} = \boldsymbol{b}$ has no solution

Row Space:

- The row space of A is the column space $C(A^T)$
- The rank of $A =$ rank of A^T

Span:

- The span of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is the set of all of their linear combinations.
- In other words, this tells us, given some set of vectors, which vectors are able to be created by taking a linear combination of the vectors in the set. It is the vector space we can reach (span) by taking linear combindations of the set of vectors.
- Independence is not required by the word span.