Time Series Notes

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March 28, 2024

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1 Time Series Basics

What is a time series?

- A *time series* is a sequence of observations of the same variable indexed in time order (i.e. monthly stock returns).
- Defining x_t as a r.v., a time series may be written

$$\{x_1, x_2, \ldots, x_T\} \text{ or } \{x_t\},\ t = 1, 2, \ldots, T$$

Common time series properties

- Time series are typically not i.i.d. (i.e. if GNP is unusually high today, GNP will likely be unusually high tomorrow).

Modeling time series data

- The models impose structure, so when dealing with model selection, it is important to evaluate to see if the model captures the features you believe to be present in the data.

2 Stationarity

2.1 Strongly Stationarity and Weakly Stationarity

- A process $\{x_t\}$ is *strongly stationary* or *strictly stationary* if all aspects of its behavior are unchanged by shifts in time. More formally, it is defined as the requirement that for every m and n, the distributions of $\{x_1, \ldots, x_n\}$ and $\{x_{1+m}, \ldots, x_{n+m}\}$ are the same, that is, the probability distribution of a sequence of n observations does not depend on their time origin.
- A process $\{x_t\}$ is *weakly stationary* or *covariance stationary* if its mean, variance, and covariance are unchanged by time shifts. More formally, it is defined as weakly stationary if
  - First moment is a finite constant: $E(x_t) = \mu$
  - Second moment is a finite constant: $Var(x_t) = \sigma^2$
  - $\text{Cov}(x_t, x_s) - \gamma(|t - s|)$

  In other words, the mean and variance do not change with time and the covariance between two observations depends on the time distance between them (aka having same number of observations), not the specific points.
- Strong stationarity does not imply weak stationarity and weak stationarity does not imply strong stationarity.
- Stationarity is important as a stationary process can be modeled with relatively few paramters.

2.2 Testing for Stationarity

- When a times series is observed, a natural question is whether it appears to be stationary.
- **Time series plot**: Looking at a *time series plot* (plot of the series in chronological order) may be useful. If the time series is a stationary series, it should show some random oscillation around some fixed level, a phenomenon called *mean reversion*. If the series wanders without returning repeatedly to some fixed level, then the series should not be modeled as a stationary process.

Figure 1: Stationary vs. non-stationary time series

3 White Noise

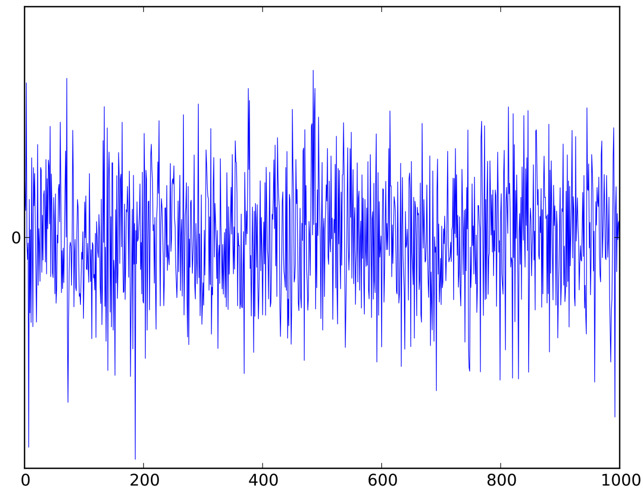

Figure 2: White noise

- The building block for time series models is the *white noise process*, denoted ϵ_t.
- In the least general case,

$$\epsilon_t \sim \text{i.i.d. } N(0, \sigma_\epsilon^2)$$

The assumption of i.i.d. have the following implications

1. No predictability: Past values of a white noise process contain no information to predict future values. Therefore, the best predictor is its mean, which is the same prediction without observing past values. More formally,

$$E(\epsilon) = E(\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2} \ldots) = 0$$

2. No autocorrelation: Each observation is independent each other. More formally,

$$E(\epsilon_t \epsilon_{t-j}) = \text{Cov}(\epsilon_t \epsilon_{t-j}) = 0$$

3. Conditional homoskedasticity: The conditional variance is a constant, and is the same variance without observing past values. More formally,

$$\text{Var}(\epsilon_t) = \text{ Var}(\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2} \ldots) = \sigma_\epsilon^2$$

4 Autocovariance and Autocorrelation

4.1 Autocovariance

- Autocovariance: Specifies the covariance between the value of a process at two times.

- Covariance: A nonstandardized measure to quantify the relationship between two variables. Can take on any value. A positive value means the variables tend to move in the same direction, a negative values means the variables tend to move in the opposite direction, and a zero value means they are independent and don't move in relation to each other.

- The *autocovariance* of a series x_t is defined as

$$\gamma_j = \text{Cov}(x_t, x_{t-j})$$

- $\gamma(h) = \gamma(-h)$ since what is important is the the space between the two observations, rather than the exact observations themselves.

- $\gamma_0 = \sigma^2$

4.2 Autocorrelation

- Autocorrelation (serial correlation): The degree of correlation of the same variable between different time intervals.

- Correlation: A standardized measure to quantify the relationship between two variables. Can take on a value inclusively between -1 and 1.

- A time series with autocorrleation implies that, predictive power (i.e. knowing the price of a stock today helps forecast its price tomorrow).

- The *autocorrelation* of a series x_t is defined as

$$\rho_j = \frac{\gamma_j}{\gamma_0}$$

4.3 Testing for autocorrelation

ACF Plots

- Show a correlation between a time series and lagged versions of itself.

- The plot also includes test bounds used to test the null hypothesis that an autocorrelation coefficient is 0. The null is rejected if the sample autocorrelatioon is outside the bounds, The usual level of the test is 0.05, so one can expect to see about 1 out of 20 samples outside the bounds simply by chance.

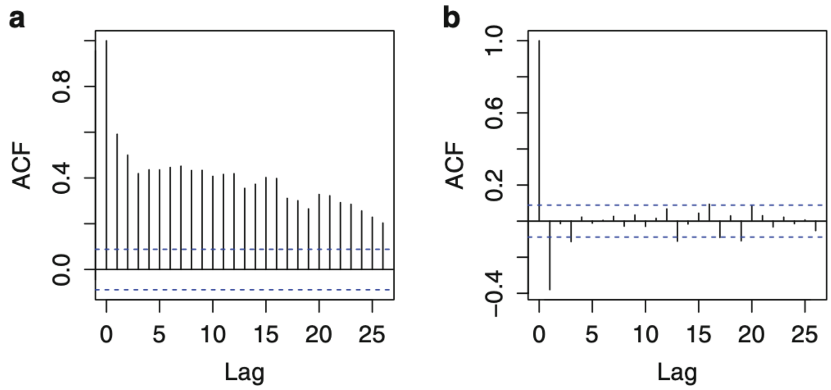

Figure 3: Sample ACF plots of (a) one-month inflation rate (decays slowly, indicating either nonstationarity or long-memory dependence) and (b) changes in the inflation rate (decays to 0 quickly, indicating that it is stationary).

Ljung-Box Test

- A *simultaneous test* is one that tests whether a group of null hypotheses area all true versus the alternative that at least one of them is false.
- The *Ljung-Box test* is a simultaneous test where the null is $\rho_1 = \rho_2 = \ldots = \rho_K$ for some K.
- If the Ljung-Box test rejects, then we conclude that one or more of $\rho_1, \rho_2, \ldots, \rho_K$ is nonzero.

5 Autoregressive Processes

- Time series models with correlation can be constructed from white noise.
- The simplest correlated stationary processess are *autoregressive processeses*, where $\{x_t\}$ is modeled as a weighted average of past observations plus a white noise "error."
- The term *autoregression* refers to the regression of the process on its own past values.

5.1 AR(1) Models

- Let $\epsilon_1, \epsilon_2, \ldots$ be WN$(0, \sigma_\epsilon^2)$. Then $\{x_t\}$ is an AR(1) process if, for some constant parameter ϕ

$$x_t = \phi x_{t-1} + \epsilon_t$$

  for all t.
- ϕx_{t-1} may be thought of as representing the memory of the past observation into the present value of the process. This is what we believe we can model and predict.
- ϕ determines the amount of feedback, where a larger absolute value results in more feedback.
- ϵ_t represents the effect of new information that cannot be modeled, hence, it is represented as white noise. This is what we cannot model nor predict.
- Properties of a Stationary AR(1) Process

- $E(x_t) = \mu$
- $\text{Var}(x_t) = \gamma_0 = \sigma_x^2 = \frac{\sigma_{\epsilon_t}^2}{1- \ phi^2}$

- The ACF of an AR(1) process depends only on one parameter, ϕ. This parsimony comes at the cost that the ACF has only a very limited range of shapes. If the ACF does not behave in one of these shapes, the AR(1) model is not suitable.

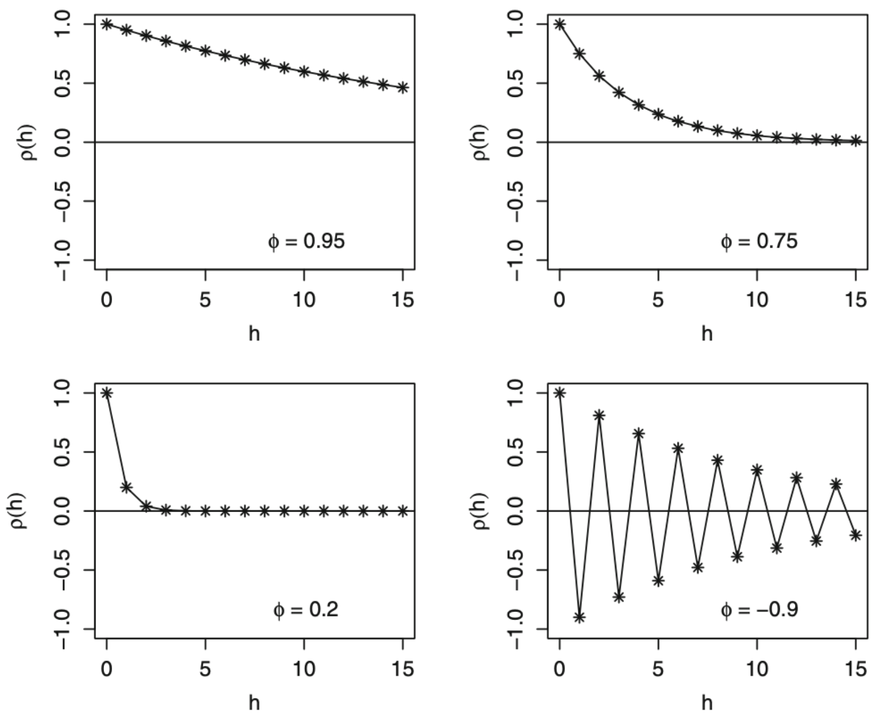

Figure 4: ACF of AR(1) processes with ϕ equal to 0.95, 0.75, 0.2, and -0.9.

- When $|\phi| < 1$ the process is stationary.
- When $\phi = 1$ the process is not stationary and takes the form of a random walk. Recall a random walk, in each period, takes a step random step that is i.i.d. from its previous steps.
- When $|\phi| > 1$ the process is non-stationary and has explosive behavior.
- A non-zero value of ϕ mean that there is some information in the previous observation, but a small value of ϕ means the prediction will not be very accurate.

5.2 AR(p) Models

- A more flexible adaptation of an AR model that is still parsimonious and regresses on the p past values.
- Let $\epsilon_1, \epsilon_2, \ldots$ be WN$(0, \sigma_\epsilon^2)$. Then $\{x_t\}$ is an AR(p) process if, for constant parameters $\phi_1, \phi_2, \ldots, \phi_{t-p}$

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p} + \epsilon_t$$

- If the AR(p) model fits the time series well, then the residuals should look like white noise. The residual autocorrelation can be detected examining the sample ACF of the residuals and using the Ljung-Box test. Any significant residual autocorrelation is a sign the AR(p) model does not fit well.
- A problem with AR models is that they often need a rather large value of p to fit a dataset.

6 Moving Average Processes

7 Misc.

- Returns are closer to i.i.d. than prices and overall exhibit more attractive statistical qualities than prices, therefore making it more sensible to study returns rather than prices.

8 Definitions

- Homoskedasticity: A condition in which the variance of the residual is constant, that is, the error term does not vary much. In other words, the variance of the data points is roughly the same for all data points.