Introduction to Linear Algebra Notes

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1 Vectors and Matrices

1.1 Vectors and Linear Combinations

Vector Length: For a vector $v \in \mathbb{R}^n$, its length is:

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

In words, the length of a vector is the square root of the sum of the squared components.

Given two vectors in $\mathbb{R}^2$ v, w with their tail starting from the origin

- If they lie on the same line, the vectors are *linearly dependent*.
- If they do not lie on the same line, the vectors are *linearly independent*.

Therefore, the combinations $c\mathbf{v} + d\mathbf{w}$ fill the $x - y$ plane unless v is in line with w.

To fill m-dimensional space, we need m independent vectors, with each vector having m components.

1.2 Lengths and Angles from Dot Products

Dot Product: For two vectors $v, w \in \mathbb{R}^n$, their dot product is:

$$v \cdot w = v_1 w_1 + \cdots + v_n w_n$$

The dot product of two vectors tells us what amount of one vector goes in the direction of another. It tells us how much these vectors are working together.

- $v \cdot w > 0$: The vectors point in somewhat similar directions. In other words, the angle between the two vectors is less than 90 degrees.
- $v \cdot w = 0$: The vectors are perpendicular. In other words, the angle between the two vectors is 90 degrees.
- $v \cdot w < 0$: The vectors point in somewhat opposing directions. In other words, the angle between the two vectors is greater than 90 degrees.

Dot Product Rules (for two vectors, v, w):

- $v \cdot w = w \cdot v$
- $u \cdot (v + w) = u \cdot v + u \cdot w$
- $(cv) \cdot w = c(v \cdot w)$

Cosine Formula: If v and w are nonzero vectors, then:

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}$$

Unit Vectors: A vector is a unit vector if its length is 1.
For a vector $u \in \mathbb{R}^n$:

$$\|u\| = 1$$

For any vector $v \in \mathbb{R}^n$, as long as $v \neq 0$, dividing v by its length will result in a unit vector. In other words:

$$u = \frac{v}{\|v\|}$$

Cauchy-Schwarz Inequality:

$$|v \cdot w| \leq \|v\|\|w\|$$

In words, the absolute value of the dot product of two vectors is no greater than the product of their lengths.

Triangle Inequality:

$$\|v + w\| \leq \|v\| + \|w\|$$

In words, the length of any one side (in this case $\|v + w\|$) of a triangle is at most the sum of the length of the other triangle sides.

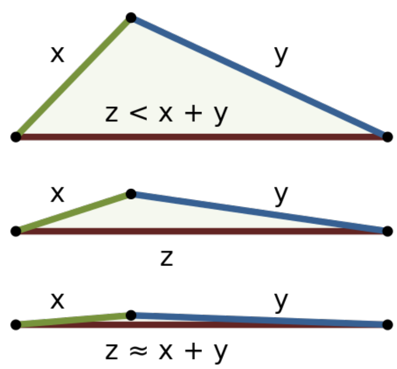

Figure 1: This Squeeze Theorem