Stochastic Calculus Notes

Nathan Ueda

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Contents

1 Discrete Time Models

1.1 The One Period Model

1.1.1 Model Setup

- N financial assets traded in this market.
- The assets are traded only at times $t = 0$ and $t = 1$.
- Price of asset i at time t is S_i^t.
- Prices of assets at time t are summarized in the price vector

$$\boldsymbol{s}^t = \begin{bmatrix} S_1^t \\ \vdots \\ S_N^t \end{bmatrix}$$

- To emphasize that $\boldsymbol{s}^1$ is not known at time $t = 0$, it is random, and therefore is written as $\tilde{\boldsymbol{s}}^1$.
- The sample space (state space) for outcomes at time $t = 1$ is finite and denoted $\Omega = \{\omega_1, \ldots, \omega_M\}$
- $S_1^i(\omega_j)$ denotes the price per unit of asset i at time $t = 1$ if ω_j occurs.
- We may therefore define the matrix $\boldsymbol{D}$ as

$$\boldsymbol{D} = \begin{bmatrix} s^1(\omega_1) & s^1(\omega_2) & \ldots & s^1(\omega_M) \end{bmatrix} = \begin{bmatrix} S_1^1(\omega_1) & S_1^1(\omega_2) & \ldots & S_1^1(\omega_M) \\ S_2^1(\omega_1) & S_2^1(\omega_2) & \ldots & S_2^1(\omega_M) \\ \vdots & \vdots & \ddots & \vdots \\ S_N^1(\omega_1) & S_N^1(\omega_2) & \ldots & S_N^1(\omega_M) \end{bmatrix} \in \mathbb{R}^{N \times M}$$

  where column vector i denotes the price of each of the N assets if ω_i occurred and row vector j denotes the price of asset j in all of the M possible states.

- We also defined the augmented matrix $\bar{\boldsymbol{D}}$ as

$$\bar{\boldsymbol{D}} = \begin{bmatrix} -\boldsymbol{s}^0 & \boldsymbol{D} \end{bmatrix} \in \mathbb{R}^{N \times (M+1)}$$

- Portfolio holdings are denoted by the vector $\boldsymbol{h}$ as

$$\boldsymbol{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix} \in \mathbb{R}^N$$

  where h_i denotes the number of shares held of asset i.

- The value of the portfolio at time $t = 0$ is a scalar defined as

$$V^0 = \boldsymbol{h}^T \boldsymbol{s}^0 = \begin{bmatrix} h_1 & \ldots & h_N \end{bmatrix} \begin{bmatrix} S_1^1 \\ \vdots \\ S_N^1 \end{bmatrix} \in \mathbb{R}$$

- The value of the portfolio for any of the M outcomes at time $t = 1$ is defined as

$$\boldsymbol{V}^1 = \boldsymbol{h}^T \boldsymbol{D} \in \mathbb{R}^{1\times M}$$

- It follows that

$$\begin{bmatrix} -V^0 & \boldsymbol{V}^1 \end{bmatrix} = \boldsymbol{h}^T \bar{\boldsymbol{D}} \in \mathbb{R}^{1\times(M+1)}$$

1.1.2 Assumptions

- These assumptions in the real world, but allows us to make models in a simpler manner.

Friction Free Market Assumptions
Behavioral Assumptions

- Agents prefer more over less.
- Agents are in agreement over the states that are possible and the states that are not possible. Nothing is assumed as to the probability of which each agent thinks each possible outcome will occur with, as this discrepancy is how agents trade against each other.

1.1.3 Noarbitrage

- Given the friction free and behavioral assumptions about the market, it is reasonable to assume that arbitrage portfolios do not exist in the market and that it is therefore *arbitrage free*.

Reachable Payoff Space

- The *reachable payoff space* denotes the possible payoffs at time $t = 1$ in the M different states.
- The reachable payoff space, denoted $\mathcal{R}$, is the row space (possible linear combinations of the rows) of $\boldsymbol{D}$.

$$\mathcal{R} = \{\boldsymbol{D}^T \boldsymbol{h} : \boldsymbol{h} \in \mathbb{R}^N\} \subset \mathbb{R}^M$$

- The augmented payoff space, denoted as $\bar{\mathcal{R}}$, is similarly defined as

$$\bar{\mathcal{R}} = \{\bar{\boldsymbol{D}}^T \boldsymbol{h} : \boldsymbol{h} \in \mathbb{R}^N\} \subset \mathbb{R}^{M+1}$$

- An *arbitrage portfolio*, $\boldsymbol{h}$, is a portfolio for which $\bar{\boldsymbol{D}}^T \boldsymbol{h} > 0$ (at least one element in the vector is strictly positive).
- A market is said to be *arbitrage free* if there exists no arbitrage portfolios. In other words, there may be many states of the world with 0 cash flows, but if there exists at least state with strictly positive cash flows there is arbitrage and there are none where you pay anything.
- Note the implications on $\bar{\boldsymbol{D}} = \begin{bmatrix} -\boldsymbol{s}^0 & \boldsymbol{D} \end{bmatrix}$. If $\bar{\boldsymbol{D}}^T \boldsymbol{h} > 0$, this means we had nonnegative cashflows even at time $t = 0$. Put simply, this means that we don't pay anything today (and maybe even earn something today) and have the possibility of positive cashflows at time $t = 1$ in at lesat one of the states.

Market Completeness

- The market is complete is $\mathcal{R}$ is all of $\mathbb{R}^M$. This can be shown if $\text{Rank}(\boldsymbol{D}) = M$.
- If $N \geq M$, then the market may be complete (depends on the row rank).
- If $N < M$ the market is not complete since we can't span all of $\mathbb{R}^M$ with only N rows.

State Price Vectors

- A *state price vector*, denoted ψ, prices all the payoffs in the reachable payoff space correctly and is denoted as

$$\psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_M \end{bmatrix} \in \mathbb{R}^{M \times 1}$$

- To get a 1 dollar payment at time $t = 1$ in state i, ψ_i is the amount needed to pay today at time $t = 0$.
- Formally, it is defined as

$$V^0 = \boldsymbol{V}^1 \psi$$
$$\boldsymbol{s}^0 = \boldsymbol{D} \psi$$

1.1.4 Arrow-Debreu Securities

- *Arrow-Debreu securities* pay a fixed payout of 1 unit in a specified state at time $t = 1$ and no payout in all other $M - 1$ states. In other words,

$$\delta_i = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \end{bmatrix},\ i = 1, \dots, M$$

  where δ_i pays a unit of the consumption good in state i at time $t = 1$.
- The price of δ_i at time $t = 0$ is $\delta_i \psi = \psi_i$, which makes sense as ψ_i denotes the price of δ_i at time $t = 0$.
- δ_0 denotes a time $t = 0$ Arrow-Debreu security that pays one unit at time $t = 0$.

1.1.5 Fundamental Theorems of Asset Pricing

First Fundamental Theorem of Asset Pricing

- States that the market is arbitrage free if and only if there exists a strictly positive state price vector, $\psi >> \boldsymbol{0}$.

Second Fundamental Theorem of Asset Pricing

- Given a market that admits no arbitrage, the state price vector is unique if and only if the market is complete.

1.1.6 Law of One Price

- This is a weaker condition than no arbitrage.
- LOOP is said to hold if the price today of two portfolios that make the same future payoff in all states must be the same. For example, if

$$\boldsymbol{h}_1^T \boldsymbol{D} = \boldsymbol{h}_2^T \boldsymbol{D} \implies \boldsymbol{h}_1^T \boldsymbol{s}^0 = \boldsymbol{h}_2^T \boldsymbol{s}^0$$

- LOOP holds if and only if there exists a state price vector ψ.

1.1.7 Types of Arbitrage

Type 1 Arbitrage

- Initial value of portfolio is nonpositive, future value is nonnegative in all states, and strictly positive in at least one state.
- This means that at time $t = 0$, we pay nothing to take it, and we have the possibility of generating strictly positive cash flows at time $t = 1$.
- Put simply, type 1 arbitrage generates strictly positive cash flows in the future in some state.

Type 2 Arbitrage

- Initial value of portfolio is negative (you get paid to take it) and future value of portfolio is nonnegative in all states.
- This means that at time $t = 0$, we have a strictly positive cash flow, and we will never have negative cash flows at time $t = 1$.
- Put simply, type 2 arbitrage generates strictly positive cash flows immediately.

Type 1 and 2 Arbitrage

- Initial value of portfolio is negative (type 2) and future value is strictly positive in at least one state.
- This means that at time $t = 0$, we are paid to take the portfolio and at time $t = 1$ we have a strictly positive probability of generating a positive cash flow.

1.1.8 Risk Neutral Probabilities

- Artifical probabilities (not true probabilities) that are useful from a pricing perspective but have no real world intepretations as probabilities.
- The reason we call these probabilities is that they behave like probabilities (they satisfy the Kolmogorov axioms of probability).
- They summarize the information in the state price vector.

Setup

- A r.v. is a function $\tilde{X} : \Omega \to \mathbb{R}$. In other words, it is a function from the sample space to a real number.
- For now, we focus on discrete sample spaces.
- The expectation of a r.v. is
$$E_{\mathbb{P}}[\tilde{X}] = \sum_i \tilde{X}(\omega_i)\mathbb{P}(\omega_i)$$
- It is possible to have two probability measures $\mathbb{P}$ and $\mathbb{Q}$ defined over the same sample space.
  - If $\mathbb{P}$ and $\mathbb{Q}$ are in agreement over which events have a probability of 0, then they are equivalent.
$$\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$$

- $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$, denoted $\mathbb{Q} << \mathbb{P}$, if $\mathbb{P}(A) = 0 \implies \mathbb{Q}(A) = 0$.
- If $\mathbb{Q} << \mathbb{P}$, we can define the *Radon-Nikodym* derivative (likelihood ratio) of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ (how much $\mathbb{Q}$ probability is assigned to an event w.r.t to how much $\mathbb{P}$ probability is assigned to an event)
$$L(A) = \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\mathbb{Q}(A)}{\mathbb{P}(A)},\ \mathbb{P}(A) > 0$$
from this is follows that, for any r.v. $\tilde{X}$
$$E_{\mathbb{P}}[L\tilde{X}] = E_{\mathbb{Q}}[\tilde{X}]$$

Risk Neutral Setup

- We can now do some redefining of the model using these risk neutral probabilities
- For portfolio $\boldsymbol{h} \in \mathbb{R}^N$, the payoff in the form of a r.v. at time $t = 1$ if ω_i occurred is defined
$$\tilde{V}^1(\omega_i) = (\boldsymbol{h}^T\boldsymbol{D})_i$$
Instead of having a vector $\boldsymbol{V}^1$, we now $\tilde{V}^1$ r.v.
- We assume the vector of ones, $\mathbf{1}$ is in the payoff space, that is, $\mathbf{1} \in \mathcal{R}$. This implies that there is a risk free asset, or some portfolio that generates exactly the payoffs of the risk free asset, and the price of that portfolio can determine the unique risk free rate (unique since we assume there is no arbitrage, if it there were multiple risk free rates, there would be arbitrage), defined
$$R = 1 + r = \frac{1}{\hat{\psi}}$$
where
$$\hat{\psi} = \sum_{i=1}^{M} \psi_i = \mathbf{1}\psi$$
for state price vector ψ.
- We can now define the *risk-neutral probabilities*, $q_1, \ldots, q_M$, where
$$q_i = \frac{\psi_i}{\psi}$$
and these q's make up the corresponding equivalent probability measure $\mathbb{Q}$.
- The higher the value q_i is, the more we are willing to pay for that state.
- In the form of a discounted expectation, it now follows that
$$V^0 = \frac{1}{R}E_{\mathbb{Q}}[\tilde{V}^1]$$
for all payoffs in $\mathcal{R}$. This is just a reinterpreation, but a very important one as it allows us to use fancy stochastic methods for thinking about asset pricing.
- Similarly, it follows that
$$V^0 = \frac{1}{R}E_{\mathbb{P}}[L\tilde{V}^1]$$

Risk Neutral Probabilities

- The measure $\mathbb{Q}$ is called the *risk-neutral measure*.
- A one period random process, $X_t,\ t = 0, 1$ is called a *martingale* (or $\mathbb{Q}$-martingale) if its value at time $t = 0$ is equal to its expected value at time $t = 1$, that is

$$E_{\mathbb{Q}}[X_1] = X_0$$

  In other words, the best estimate of the value tomorrow is the value today.
- Defining the random process M, by $M_0 = 1$ and $M_1 = \frac{1}{R}L$, it then follows that

$$M_0 V^0 = E_{\mathbb{P}}[M_1 V^1]$$

  which also shows that MV is a $\mathbb{P}$-martingale.
- M is called suitably called the *stochastic discount factor*, as it is a discount factor (the price today as a function of the price tomorrow) and it is stochastic.
- As long as there is noarbitrage, there is always a stochastic discount factor.
- Given the discounted process $\hat{V}_0 = V_0$ and $\hat{V}^1 = \frac{\tilde{V}}{R}$, we have

$$\hat{V}_0 = E_{\mathbb{Q}}[\hat{V}_1]$$

  therefore $\mathbb{Q}$ is called the *equivalent martingale measure*.

1.1.9 Risk Neutral Fundamental Theorems

- We can reformulate the fundamental theorems as a statement about risk neutral measures and equivalent martingale measures.

Risk Neutral First Fundamental Theorem

- The market is arbitrage free if and only if there exists an equivalent martingale maeasure, $\mathbb{Q}$, such that the price of any asset in $\mathcal{R}$ is given by $V^0 = \frac{1}{R}E_{\mathbb{Q}}[\tilde{V}^1]$

Risk Neutral Second Fundamental Theorem

- Given a market that admits no arbitrage, the equivalent martingale measure is unique if and only if the market is complete.

1.1.10 Summary

- V_0 may be written in many equivalent manners
  - Using the state price vector

$$V_0 = \boldsymbol{V}^1 \psi$$

  - Using equivalent martingale measure/risk neutral probabilities

$$V_0 = \frac{1}{R}E_{\mathbb{Q}}[\tilde{V}^1]$$

    where $q_i = \frac{\psi_i}{\hat{\psi}}, R = \frac{1}{\hat{\psi}}$.
  - Using the likelihood ratio

$$V_0 = \frac{1}{R}E_{\mathbb{P}}[L\tilde{V}^1]$$

    where $L_i = \frac{q_i}{p_i}$.

1.2 The Binomial Model

1.2.1 Notation

- S: value of the underlying asset.
- C: value of the call option.
- P: value of the put option.
- K: exercise price of the option.
- r: one period net risk free interest rate.
- R: one period gross risk free interest rate $(1 + r)$.

1.2.2 Options

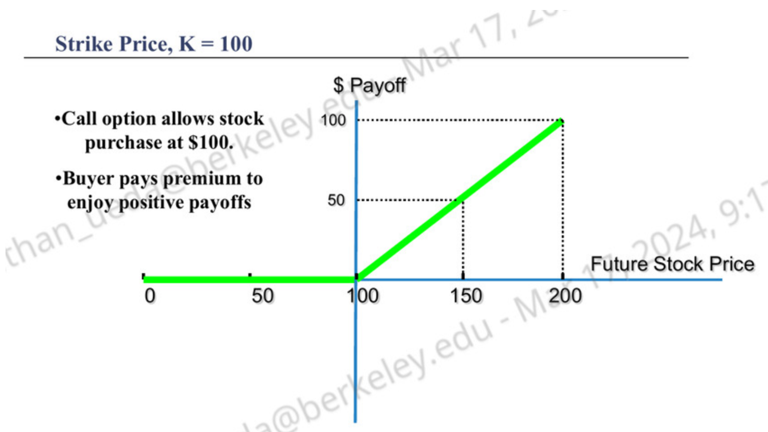

Figure 1: Call payoff at expiration.

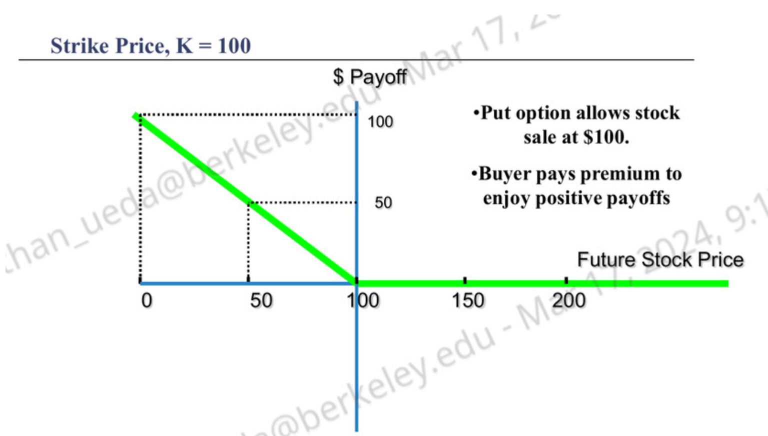

Figure 2: Put payoff at expiration.

Formulas for Options Payoffs

- Call: $C = \text{Max}[0, S - K]$
- Put: $P = \text{Max}[0, K - S]$

1.2.3 One Period, Two State Binomial Model

- Two points in time: $t = 0$ and $t = 1$
- Two assets: a bond and a stock
- Price of bond at time t is denoted B_t
- Price of stock at time t is denoted S_t
- The bond price is deterministic and given by

$$B_0 = 1$$

$$B_1 = R$$

  where $R \geq 1$.

- The stock price is a stochastic process where

$$S_0 = S$$

$$S_1 = \begin{cases} uS, & \text{with probability } p_u \\ dS, & \text{with probability } p_d \end{cases}$$

  where $u \geq d$.

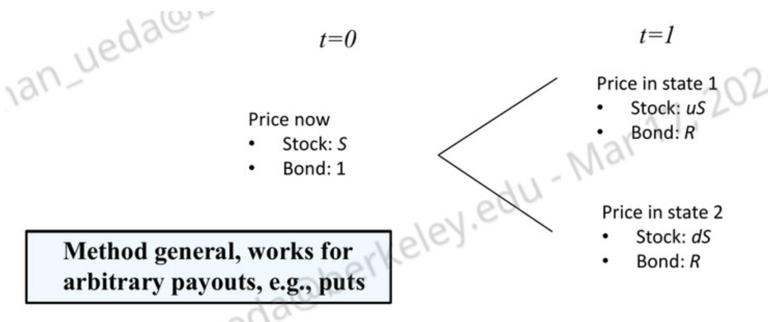

Figure 3: Price dynamics

- Within our framework, this can be setup as

$$\boldsymbol{s}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} u & d \\ R & R \end{bmatrix}$$

- Payoff for call will be either

$$C_u = \max(uS - K, 0)$$

  or

$$C_d = \max(dS - K, 0)$$

2 Appendix

2.1 Operator Overloading

For some vector $\boldsymbol{a}$, we write

- $\boldsymbol{a} \geq \mathbf{0}$ if all elements are nonnegative
- $\boldsymbol{a} > \mathbf{0}$ if at least one element is strictly positive
- $\boldsymbol{a} >> \mathbf{0}$ if all elements are strictly positive

Similarly,

- $\boldsymbol{a} \leq \mathbf{0}$ if all elements are nonpositive
- $\boldsymbol{a} < \mathbf{0}$ if at least one element is strictly negative
- $\boldsymbol{a} << \mathbf{0}$ if all elements are strictly negative