A Primer for the Mathematics of Financial Engineering Notes

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Contents

1 Calculus Review, Options, Put-Call Parity

1.1 Brief Review of Differentiation

Differentiation Basics

- Differentiation is a method used to compute the rate of change of a function $f(x)$ with respect to its input x. This rate of change is known as the derivative of f with respect to x.
- The first derivative of a function $y = f(x)$ is denoted $\frac{dy}{dx}$, where dy denotes an infinitesimally small change in y and dx denotes an infinitesimally small change in x. It is defined by

  $$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

  where as h gets closer to 0, it approaches the slope at the tangent line
- The process of finding the derivative by taking this limit is known as differentiation from first principles. In practice, it is not convenient to use this method.
- The derivatives of many functions can instead be found using standard derivatives in conjunction with rules such as the chain rule, product rule, and quotient rule.
- Leibniz's Notation
  - Given a function $y = f(x)$, the first derivative of y with respect to x is denoted by

    $$\frac{dy}{dx}$$

  - The second derivative of y with respect to x is denoted by

    $$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}$$

  - The nth derivative of y with respect to x is denoted by

    $$\frac{d^ny}{dx^n}$$

  - The derivative of y with respect to x at the point $x = a$ is denoted by

    $$\frac{dy}{dx}|_{x=a}$$

- The function $f(x)$ is differentiable if it is differentiable at all points x.
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Product Rule

- If the two functions $f(x)$ and $g(x)$ are differentiable (i.e. their derivatives exist) then the product is differentiable and

  $$(fg)' = f'g + fg'$$

Quotient Rule

- If the two functions $f(x)$ and $g(x)$ are differentiable (i.e. their derivatives exist) then the quotient is differentiable and

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

- This has the mnemonic

$$\frac{\text{lo de hi} - \text{hi de lo}}{\text{lo lo}}$$

Chain Rule

- If the two functions $f(x)$ and $g(x)$ are differentiable then the composite function $f \circ g = f(g(x)) = F'(x)$ is differentiable and

$$F'(x) = f'(g(x))g'(x)$$

- It is often used for power functions, exponential functions, and logarithmic functions.

1.2 Brief Review of Integration

Integration Basics

- Integration is a way to figure out what function we differentiated to get the function $f(x)$.

Indefinite Integrals

- Given a function $f(x)$, an antiderivative of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$.
- If $F(x)$ is any antiderivative of $f(x)$ then the most general antiderivative of $f(x)$ is called an indefinite integral and is denoted

$$\int f(x)dx = F(x) + c, \quad c \text{ is an arbitrary constant}$$

  where $\int$ is the integral symbol, $f(x)$ is the integrand, x is the integration variable, and c is the constant of integration.

Indefinite Integrals

- Suppose $f(x)$ is a continuous function on $[a, b]$ and also suppose that $F(x)$ is any antiderivative for $f(x)$, then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Integration By Parts

- Easier formula

$$\int u\,dv = uv - \int v\,du$$

- To use this formula, we will need to identify u and dv and then use the formula.
- Choose u such that
  - The integral of v is possible.

- The derivative of u is better (reduced power)

- Choosing u order of operations
  - L: Log functions
  - I: Inverse trig functions
  - P: Polynomials
  - E: Exponentials
  - T: Trig functions

- It will not always be obvious what the correct values to choose are and, on occassion, the wrong choice will be made. If this happens, we can always just go back and try a different set of choices.

- Integration by parts is the counterpart for integration of the product rule.

Integration By Substitution

- Integration by substituion formula

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad \text{where } u = g(x)$$

- Integration by substitution is the couterpart for integration of the chain rule.

1.3 Differentiating Definite Integrals

1.4 Limits

1.5 L'Hopitals Rule and Connections to Taylor Expansions

1.6 Multivariable Functions

Partial Derivatives

- In practice, the partial derivative $\frac{\partial f}{\partial x_i}(x)$ is computed by considering the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ to be fixed, and differentiating $f(x)$ as a function of one variable x_i.

- Partial derivatives of higher order are defined similarly. The second order partial derivative of $f(x)$ with respect to x_i and then with respect to x_j, with $j \neq i$ is defined as

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(x)\right)$$

Gradient

- The gradient is a vector that points in the direction of maximum increase.

- An $n \times 1$ vectors that stores all the first order partial derivatives.

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables and assume that $f(x)$ is differentiable with respect to all variables $x_i,\ i = 1, \ldots, n$. The gradient ∇f of the function $f(x)$ is the following vector of n components

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix}$$

Hessian

- An $n \times n$ matrix that stores all the second order partial derivatives.
- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables. The Hessian of $f(x)$ is the following matrix of $n \times n$ components

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{bmatrix}$$