Stochastic Calculus Notes

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Contents

1 Discrete Time Models

1.1 The One Period Model

1.1.1 Model Setup

- The One Period Model has assets are traded only at times $t = 0$ and $t = 1$.
- N financial assets traded in this market.
- Price of asset i at time t is S_i^t.
- Prices of assets at time t are summarized in the price vector

$$\boldsymbol{s}^t = \begin{bmatrix} S_1^t \\ \vdots \\ S_N^t \end{bmatrix}$$

- To emphasize that $\boldsymbol{s}^1$ is not known at time $t = 0$, it is random, and therefore is written as $\tilde{\boldsymbol{s}}^1$.
- The sample space (state space) for outcomes at time $t = 1$ is finite and denoted $\Omega = \{\omega_1, \ldots, \omega_M\}$
- $S_1^i(\omega_j)$ denotes the price per unit of asset i at time $t = 1$ if ω_j occurs.
- We may therefore define the matrix $\boldsymbol{D}$ as

$$\boldsymbol{D} = \begin{bmatrix} s^1(\omega_1) & s^1(\omega_2) & \ldots & s^1(\omega_M) \end{bmatrix} = \begin{bmatrix} S_1^1(\omega_1) & S_1^1(\omega_2) & \ldots & S_1^1(\omega_M) \\ S_2^1(\omega_1) & S_2^1(\omega_2) & \ldots & S_2^1(\omega_M) \\ \vdots & \vdots & \ddots & \vdots \\ S_N^1(\omega_1) & S_N^1(\omega_2) & \ldots & S_N^1(\omega_M) \end{bmatrix} \in \mathbb{R}^{N \times M}$$

  where column vector i denotes the price of each of the N assets if ω_i occurred and row vector j denotes the price of asset j in all of the M possible states.

- We also defined the augmented matrix $\bar{\boldsymbol{D}}$ as

$$\bar{\boldsymbol{D}} = \begin{bmatrix} -\boldsymbol{s}^0 & \boldsymbol{D} \end{bmatrix} \in \mathbb{R}^{N \times (M+1)}$$

  which includes the cash flows of purchasing each of the n assets at time $t = 0$. This matrix summarizes all the information about the prices of all assets, in all states of the world, at all points in time.

- Portfolio holdings are denoted by the vector $\boldsymbol{h}$ as

$$\boldsymbol{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix} \in \mathbb{R}^N$$

  where h_i denotes the number of shares held of asset i.

- The value of the portfolio at time $t = 0$ is a scalar defined as

$$V^0 = \boldsymbol{h}^T \boldsymbol{s}^0 = \begin{bmatrix} h_1 & \dots & h_N \end{bmatrix} \begin{bmatrix} S_1^1 \\ \vdots \\ S_N^1 \end{bmatrix} \in \mathbb{R}$$

- The value of the portfolio for any of the M outcomes at time $t = 1$ is defined as

$$\boldsymbol{V}^1 = \boldsymbol{h}^T \boldsymbol{D} \in \mathbb{R}^{1 \times M}$$

- It follows that

$$\begin{bmatrix} -V^0 & \boldsymbol{V}^1 \end{bmatrix} = \boldsymbol{h}^T \bar{\boldsymbol{D}} \in \mathbb{R}^{1 \times (M+1)}$$

1.1.2 Assumptions

- These assumptions are unrealistic in the real world, but allows us to make models in a simpler manner.

Friction Free Market Assumptions

- Short positions, as well as fractional holdings are allowed.
- There is no bid ask spread (i.e. the selling price is equal to the buying price of all assets).
- There are no transaction costs.
- There market is completely liquid (i.e. it is always possible to buy and/or sell unlimited quantities on the market).
- Price linearity: The cost of c units of an asset is c times the cost of one unit.
- Payoff linearity: The payoff of c units of an asset is c times the cost of one unit.
- Linearity across assets: The price and payoff of a portfolio of different assets is the sum of the prices and payoffs of the individual assets.

Behavioral Assumptions

- Agents prefer more over less.
- Agents are in agreement over the states that are possible and the states that are not possible. Nothing is assumed as to the probability of which each agent thinks each possible outcome will occur with, as this discrepancy is how agents trade against each other.

1.1.3 Noarbitrage

- Given the friction free and behavioral assumptions about the market, it is reasonable to assume that arbitrage portfolios do not exist in the market and that it is therefore *arbitrage free*.

Reachable Payoff Space

- The *reachable payoff space* denotes the possible payoffs at time $t = 1$ in the M different states.

- The reachable payoff space, denoted $\mathcal{R}$, is the row space (possible linear combinations of the rows) of $\boldsymbol{D}$.
$$\mathcal{R} = \{\boldsymbol{D}^T\boldsymbol{h} : \boldsymbol{h} \in \mathbb{R}^N\} \subset \mathbb{R}^M$$

- The augmented payoff space, denoted as $\bar{\mathcal{R}}$, is similarly defined as
$$\bar{\mathcal{R}} = \{\bar{\boldsymbol{D}}^T\boldsymbol{h} : \boldsymbol{h} \in \mathbb{R}^N\} \subset \mathbb{R}^{M+1}$$

- An *arbitrage portfolio*, $\boldsymbol{h}$, is a portfolio for which $\bar{\boldsymbol{D}}^T\boldsymbol{h} > 0$ (at least one element in the vector is strictly positive).

- Note the implications on $\bar{\boldsymbol{D}} = \begin{bmatrix}-\boldsymbol{s}^0 & \boldsymbol{D}\end{bmatrix}$. If $\bar{\boldsymbol{D}}^T\boldsymbol{h} > 0$, this means we had nonnegative cashflows even at time $t = 0$. Put simply, this means that we don't pay anything today (and maybe even earn something today) and have the possibility of positive cashflows at time $t = 1$ in at least one of the states.

- A market is said to be *arbitrage free* if there exists no arbitrage portfolios. In other words, there may be many states of the world with 0 cash flows, but if there exists at least one state with strictly positive cash flows and there are none where you pay anything, there is arbitrage.

Market Completeness

- Intuitively, a complete market is one where any desired payoff is replicable.

- The market is complete is $\mathcal{R}$ is all of $\mathbb{R}^M$. This can be shown if $\text{Rank}(\boldsymbol{D}) = M$.

- If $N \geq M$, then the market may be complete (depends on the row rank).

- If $N < M$, the market is not complete since we can't span all of $\mathbb{R}^M$ with only N rows.

State Price Vectors

- A *state price vector*, denoted ψ, prices all the payoffs in the reachable payoff space correctly and is denoted as
$$\psi = \begin{bmatrix}\psi_1 \\ \vdots \\ \psi_M\end{bmatrix} \in \mathbb{R}^{M\times 1}$$

- To get a 1 dollar payment at time $t = 1$ in state i, ψ_i is the amount needed to pay today at time $t = 0$.

- Formally, it is defined as
$$V^0 = \boldsymbol{V}^1\psi$$
$$\boldsymbol{s}^0 = \boldsymbol{D}\psi$$

- Connecting the SPV to the augmented reachable payoff space, the larger $\bar{\mathcal{R}}$ is, the more restriction are imposted on the SPV (since it needs to price more portfolios correctly) and the fewer state price vectors will therefor exist.

1.1.4 Arrow-Debreu Securities

- These securities are useful as once you have them, they allow you to build any payoff vector.
- *Arrow-Debreu securities* pay a fixed payout of 1 unit in a specified state at time $t = 1$ and no payout in all other $M - 1$ states. In other words,

$$\delta_i = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \end{bmatrix}, \; i = 1, \dots, M$$

  where δ_i pays a unit of the consumption good in state i at time $t = 1$.

- The price of δ_i at time $t = 0$ is $\delta_i \psi = \psi_i$, which makes sense as ψ_i denotes the price of δ_i at time $t = 0$.
- δ_0 denotes a time $t = 0$ Arrow-Debreu security that pays one unit at time $t = 0$.

1.1.5 Fundamental Theorems of Asset Pricing

First Fundamental Theorem of Asset Pricing

- States that the market is arbitrage free if and only if there exists a strictly positive state price vector, $\psi >> \mathbf{0}$.

Second Fundamental Theorem of Asset Pricing

- Given a market that admits no arbitrage, the state price vector is unique if and only if the market is complete.

1.1.6 Law of One Price

- This is a weaker condition than no arbitrage.
- LOOP is said to hold if the price today of two portfolios that make the same future payoff in all states must be the same. For example, if

$$\boldsymbol{h}_1^T \boldsymbol{D} = \boldsymbol{h}_2^T \boldsymbol{D} \implies \boldsymbol{h}_1^T \boldsymbol{s}^0 = \boldsymbol{h}_2^T \boldsymbol{s}^0$$

- LOOP holds if and only if there exists a state price vector ψ.

1.1.7 Types of Arbitrage

Type 1 Arbitrage

- Initial value of portfolio is nonpositive, future value is nonnegative in all states, and strictly positive in at least one state.
- This means that at time $t = 0$, we pay nothing to take it, and we have the possibility of generating strictly positive cash flows at time $t = 1$.
- Put simply, type 1 arbitrage generates strictly positive cash flows in the future in some state.

Type 2 Arbitrage

- Initial value of portfolio is negative (you get paid to take it) and future value of portfolio is nonnegative in all states.
- This means that at time $t = 0$, we have a strictly positive cash flow, and we will never have negative cash flows at time $t = 1$.
- Put simply, type 2 arbitrage generates strictly positive cash flows immediately.

Type 1 and 2 Arbitrage

- Initial value of portfolio is negative (type 2) and future value is strictly positive in at least one state.
- This means that at time $t = 0$, we are paid to take the portfolio and at time $t = 1$ we have a strictly positive probability of generating a positive cash flow.

1.1.8 Risk Neutral Probabilities

- Artifical probabilities (not true probabilities) that are useful from a pricing perspective but have no real world intepretations as probabilities.
- The reason we call these probabilities is that they behave like probabilities (they satisfy the Kolmogorov axioms of probability).
- They summarize the information in the state price vector.

Setup

- A r.v. is a function $\tilde{X} : \Omega \to \mathbb{R}$. In other words, it is a function from the sample space to a real number.
- For now, we focus on discrete sample spaces.
- The expectation of a r.v. is
$$E_{\mathbb{P}}[\tilde{X}] = \sum_i \tilde{X}(\omega_i)\mathbb{P}(\omega_i)$$
where we subscript E with the probability space that is being used.
- It is possible to have two probability measures $\mathbb{P}$ and $\mathbb{Q}$ defined over the same sample space.
  - If $\mathbb{P}$ and $\mathbb{Q}$ are in agreement over which events have a probability of 0, then they are equivalent.
$$\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$$
  - $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$, denoted $\mathbb{Q} << \mathbb{P}$, if $\mathbb{P}(A) = 0 \implies \mathbb{Q}(A) = 0$.
  - If $\mathbb{Q} << \mathbb{P}$, we can define the *Radon-Nikodym* derivative (likelihood ratio) of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ (how much $\mathbb{Q}$ probability is assigned to an event w.r.t to how much $\mathbb{P}$ probability is assigned to an event)
$$L(A) = \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\mathbb{Q}(A)}{\mathbb{P}(A)},\ \mathbb{P}(A) > 0$$
from this is follows that, for any r.v. $\tilde{X}$
$$E_{\mathbb{P}}[L\tilde{X}] = E_{\mathbb{Q}}[\tilde{X}]$$

which can be seen by

$$E_{\mathbb{Q}}[\tilde{X}] = \sum_i x_i q_i = \sum_i x_i q_i \frac{p_i}{p_i} = \sum_i x_i \frac{q_i}{p_i} p_i = \sum_i x_i L_i p_i = E_{\mathbb{P}}[L\tilde{X}]$$

which gives a methodology to switch between the $\mathbb{Q}$ probability measure and the $\mathbb{P}$ probability measure.

Risk Neutral Setup

- We can now do some redefining of the model using these risk neutral probabilities
- For portfolio $\boldsymbol{h} \in \mathbb{R}^N$, the payoff in the form of a r.v. at time $t = 1$ if ω_i occurred is defined

  $$\tilde{V}^1(\omega_i) = (\boldsymbol{h}^T \boldsymbol{D})_i$$

  Instead of having a vector $\boldsymbol{V}^1$, we now $\tilde{V}^1$ r.v.
- We assume the vector of ones, $\mathbf{1}$ is in the payoff space, that is, $\mathbf{1} \in \mathcal{R}$. This implies that there is a risk free asset, or some portfolio that generates exactly the payoffs of the risk free asset, and the price of that portfolio can determine the unique risk free rate (unique since we assume there is no arbitrage, if it there were multiple risk free rates, there would be arbitrage), defined

  $$R = 1 + r = \frac{1}{\hat{\psi}}$$

  where

  $$\hat{\psi} = \sum_{i=1}^{M} \psi_i = \mathbf{1}\psi$$

  for state price vector ψ.
- We can now define the *risk-neutral probabilities*, $q_1, \ldots, q_M$, where

  $$q_i = \frac{\psi_i}{\psi}$$

  and these q's make up the corresponding equivalent probability measure $\mathbb{Q}$.
- The higher the value q_i is, the more we are willing to pay for that state.
- In the form of a discounted expectation, it now follows that

  $$V^0 = \frac{1}{R} E_{\mathbb{Q}}[\tilde{V}^1]$$

  for all payoffs in $\mathcal{R}$. This is just a reinterpreation, but a very important one as it allows us to use fancy stochastic methods for thinking about asset pricing.
- Similarly, it follows that

  $$V^0 = \frac{1}{R} E_{\mathbb{P}}[L\tilde{V}^1]$$

Risk Neutral Probabilities

- The measure $\mathbb{Q}$ is called the *risk-neutral measure* or *martingale measure.*

- A one period random process, $X_t,\ t = 0, 1$ is called a *martingale* (or $\mathbb{Q}$-martingale) if its value at time $t = 0$ is equal to its expected value at time $t = 1$, that is

$$E_{\mathbb{Q}}[X_1] = X_0$$

  In other words, the best estimate of the value tomorrow is the value today.

- Defining the random process M, by $M_0 = 1$ and $M_1 = \frac{1}{R}L$, it then follows that

$$M_0 V^0 = E_{\mathbb{P}}[M_1 V^1]$$

  which also shows that MV is a $\mathbb{P}$-martingale.

- M is called suitably called the *stochastic discount factor*, as it is a discount factor (the price today as a function of the price tomorrow) and it is stochastic.

- As long as there is noarbitrage, there is always a stochastic discount factor.

- Given the discounted process $\hat{V}_0 = V_0$ and $\hat{V}^1 = \frac{\tilde{V}}{R}$, we have

$$\hat{V}_0 = E_{\mathbb{Q}}[\hat{V}_1]$$

  therefore $\mathbb{Q}$ is called the *equivalent martingale measure.* In other words, if the expectation of a r.v. tomorrows is equal to its value today, it is a martingale (the best expectation of tomorrow's value is today's value).

1.1.9 Risk Neutral Fundamental Theorems

- We can reformulate the fundamental theorems as a statement about risk neutral measures and equivalent martingale measures.

Risk Neutral First Fundamental Theorem

- The market is arbitrage free if and only if there exists an equivalent martingale measure, $\mathbb{Q}$, such that the price of any asset in $\mathcal{R}$ is given by $V^0 = \frac{1}{R}E_{\mathbb{Q}}[\tilde{V}^1]$

Risk Neutral Second Fundamental Theorem

- Given a market that admits no arbitrage, the equivalent martingale measure is unique if and only if the market is complete.

1.1.10 Summary

- V_0 may be written in many equivalent manners
  - Using the state price vector

$$V_0 = \boldsymbol{V}^1\psi$$

  - Using equivalent martingale measure/risk neutral probabilities

$$V_0 = \frac{1}{R}E_{\mathbb{Q}}[\tilde{V}^1]$$

    where $q_i = \frac{\psi_i}{\hat{\psi}}, R = \frac{1}{\hat{\psi}}$.

We show how to get from $V_0 = \boldsymbol{V}^1\psi$ to $_0 = \frac{1}{R}E_{\mathbb{Q}}[\tilde{V}^1]$

$$\begin{aligned}
\boldsymbol{V}^1\psi &= \sum V^1(\omega_i)\psi_i \\
&= \sum V^1(\omega_i)\psi_i \frac{\sum \psi_i}{\sum \psi_i} \\
&= \sum V^1(\omega_i)\frac{\psi_i}{\sum \psi_i}\sum \psi_i \\
&= \sum V^1(\omega_i) q_i \sum \psi_i \\
&= E_{\mathbb{Q}[\tilde{V}^1]} \sum \psi_i \\
&= E_{\mathbb{Q}[\tilde{V}^1]} \frac{1}{\frac{1}{\sum \psi_i}} \\
&= \frac{1}{R} E_{\mathbb{Q}[\tilde{V}^1]}
\end{aligned}$$

– Using the likelihood ratio

$$V_0 = \frac{1}{R}E_{\mathbb{P}}[L\tilde{V}^1]$$

where $L_i = \frac{q_i}{p_i}$.

– Using the stochastic discount factor

$$V_0 = E_{\mathbb{P}}\left[\frac{M_1}{M_0}\tilde{V}\right]$$

where $(M_1)_i = \frac{L_i}{R} = \frac{\psi_i}{p_i},\ M_0 = 1$

1.2 The Binomial Model

1.2.1 Notation

- S: value of the underlying asset.
- C: value of the call option.
- P: value of the put option.
- K: exercise price of the option.
- r: one period net risk free interest rate.
- R: one period gross risk free interest rate $(1 + r)$.

1.2.2 Options

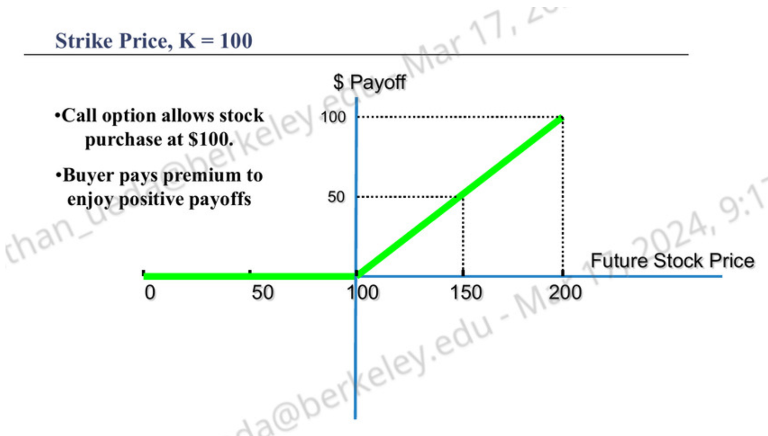

Figure 1: Call payoff at expiration.

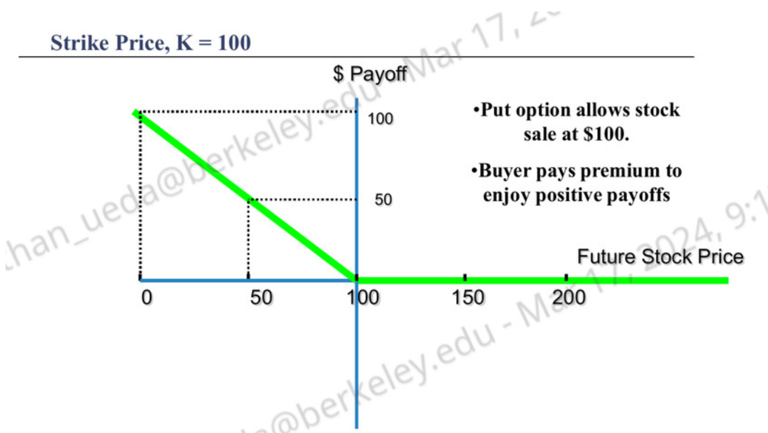

Figure 2: Put payoff at expiration.

Formulas for Options Payoffs

- Call: $C = \text{Max}[0, S - K]$
- Put: $P = \text{Max}[0, K - S]$

1.2.3 One Period, Two State Binomial Model

- Two points in time: $t = 0$ and $t = 1$
- Two assets: a bond and a stock
- Price of bond at time t is denoted B_t
- Price of stock at time t is denoted S_t

- The bond price is deterministic and given by

$$B_0 = 1$$

$$B_1 = R$$

where $R \geq 1$.

- The stock price is a stochastic process where

$$S_0 = S$$

$$S_1 = \begin{cases} uS, & \text{with probability } p_u \\ dS, & \text{with probability } p_d \end{cases}$$

where $u \geq d$.

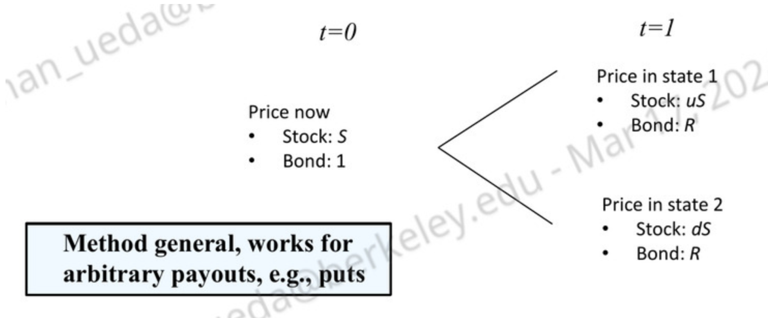

Figure 3: Price dynamics

Given the following 1 period, 2 state binomial tree market above. This can be setup as

$$\boldsymbol{s}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} u & d \\ R & R \end{bmatrix}$$

For this 2×2 market, answer the following questions.

1. What are the conditions for noarbitrage?
   - No arbitrage if $d < R < u$: If R is greater than u and d, then there would be no reason to take on the risk of the stock since more could be earned risk free with R. If R is less than u and d, then the stock is superior.
   - Another possibility for a market with no arbitrage would be if $d = R = u$. The market would not be complete, but there would be no arbitrage.

2. What are the conditions for a complete market?
   Approach: If $\text{Rank}(\boldsymbol{D}) = M$, market is complete.

   $u \neq d$. If $u = d$, $\text{Rank}(\boldsymbol{D}) = 1 < M = 2$.

3. What are the state prices?
Approach: Find the state price vector. Since this is a 2×2 market, we can just invert $\boldsymbol{s}^0 = \boldsymbol{D}\psi$ and solve for $\psi = \boldsymbol{D}^{-1}s^0$ to get the state prices.

Recall
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Therefore
$$\boldsymbol{D}^{-1} = \frac{1}{uR - Rd} \begin{bmatrix} R & -d \\ -R & u \end{bmatrix} = \boldsymbol{D}^{-1} = \frac{1}{(u-d)R} \begin{bmatrix} R & -d \\ -R & u \end{bmatrix}$$

$$\psi = \boldsymbol{D}^{-1}s^0 = \frac{1}{(u-d)R} \begin{bmatrix} R & -d \\ -R & u \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{(u-d)R} \begin{bmatrix} R - d \\ u - R \end{bmatrix} = \begin{bmatrix} \frac{R-d}{(u-d)R} \\ \frac{u-R}{(u-d)R} \end{bmatrix}$$

4. What are the risk neutral probabilities?
Approach: The risk neutral measure is $\mathbb{Q}$, where each $q_i = \frac{\psi_i}{\hat{\psi}}$, and $\hat{\psi} = \sum_{i=1}^{M} \psi_i$.

Calculate
$$\hat{\psi} = \psi_1 + \psi_2 = \frac{R-d}{(u-d)R} + \frac{u-R}{(u-d)R} = \frac{R-d+u-R}{(u-d)R} = \frac{u-d}{(u-d)R} = \frac{1}{R}$$
$$q_u = \frac{\psi_1}{\hat{\psi}} = \frac{R-d}{u-d}$$
$$q_d = \frac{\psi_2}{\hat{\psi}} = \frac{u-R}{u-d}$$
Therefore
$$\begin{bmatrix} q_u \\ q_d \end{bmatrix} = \begin{bmatrix} \frac{R-d}{u-d} \\ \frac{u-R}{u-d} \end{bmatrix}$$
The q_u is the q for an up move, is typically what we call the risk neutral probability.

5. What is the price the call option?
$$C_0 = C_u\psi_1 + C_d\psi_2 = \frac{1}{R}E_{\mathbb{Q}}[\tilde{C}_1] = \frac{1}{R}(C_u q_u + C_d q_d)$$

6. Payoff for call will be either
$$C_u = \max(uS - K, 0)$$
or
$$C_d = \max(dS - K, 0)$$

1.2.4 Basic Probability

- Assume sample space $\Omega = \{\omega_1, \ldots, \omega_M\}$
- A set of events $\{B_1, \ldots, B_m\}$ forms a *partition* of Ω if $\cup_i B_i = \Omega$ and $B_i \cap B_j = \emptyset$ when $i \neq j$
- Conditional probability is $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- For a partition of Ω $\{B_1, \ldots, B_m\}$, LOTP states that for any event A, $P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)P(B_i)$
- Law of iterated expectations states for a r.v. $\tilde{X}$, $E[E[\tilde{X}|B]] = E[\tilde{X}]$. Essentially what this is saying is that the best forcast of the forcast for $\tilde{X}$ given B is simply the unconditional expectation of $\tilde{X}$.
- Bayes rule states $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

1.2.5 General n Step Binomial Tree Method

- The general unconditional risk neutral pricing formula for the general n step binomial tree method is
$$C(0) = \frac{1}{R^n} \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} C(n,k)$$
which discounts by R n times, where the are $\binom{n}{k}$ total paths to a terminal node, each path having a $q^k(1-q)^{n-k}$ is the probability of reaching one of those terminal paths. So $\sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k}$ get the total probability of reaching the terminal nodes from any of the paths multiplied by $C(n,k)$.

1.3 The General Multi-Period Model

- This model has no restriction on the number of states, the number of assets, and is not restricted to being binomial.

Information Revelation in Multi-Period Economies

- Consider the following three period economy with 11 states

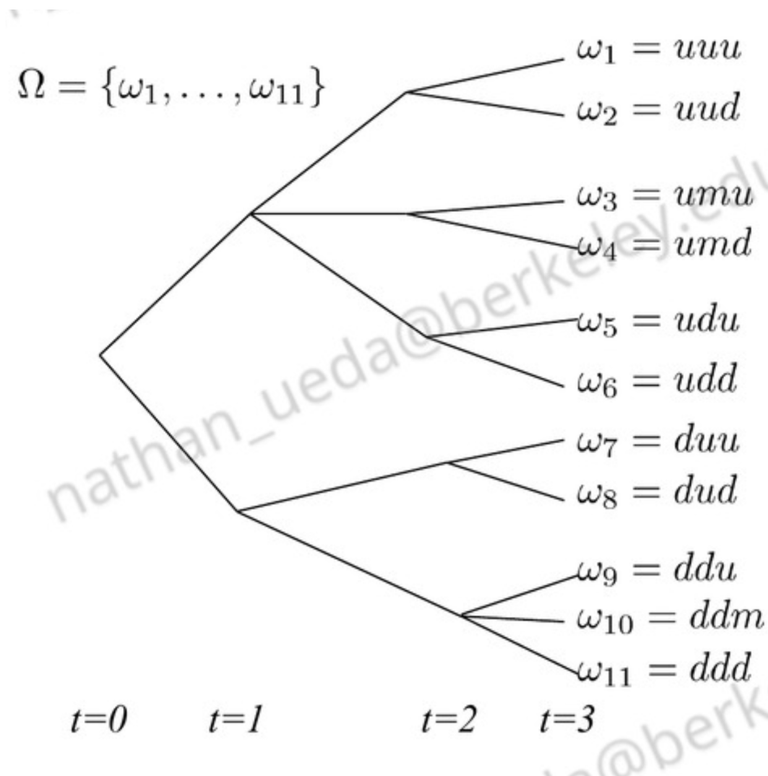

Figure 4: 11 state, 3 period economy

- We want to introduce a way to rebalance and have the formulas to keep track of the values dynamics at any point in time.
- Naively, a trading strategy is a function of time and the state space

$$h : \{0, \dots, T\} \times \Omega \to \mathbb{R}^N$$

  and the strategy must be careful not to allow for taking future (at the time unknown) into account.
- Therefore, in the dynamic setting, we need to define trading strategies so that they only use available information.

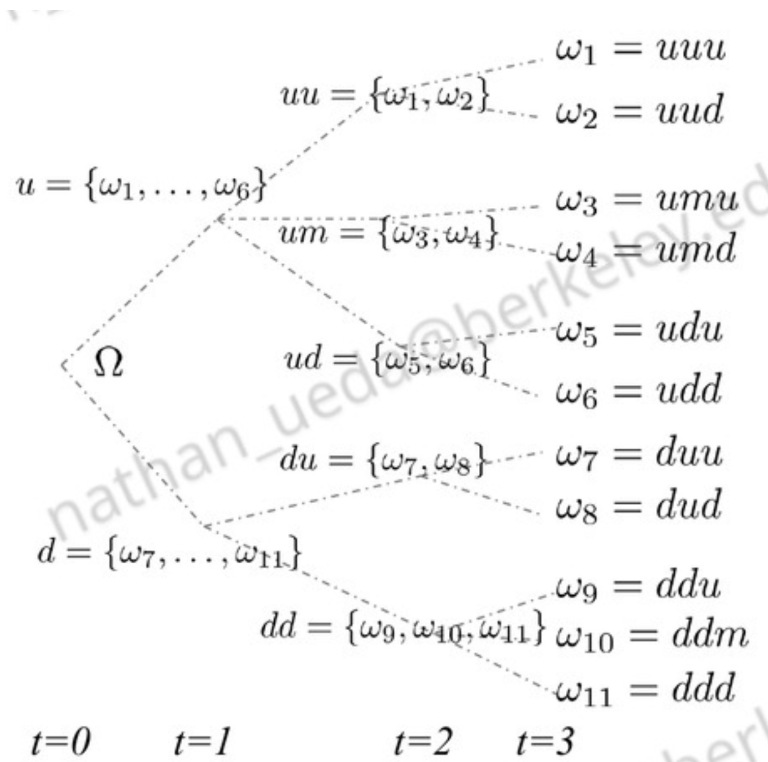

Figure 5: 11 state, 3 period economy, showing

- In this economy, we define four σ-algebras
  - $\mathcal{F}_0 = \sigma(\{\Omega\})$
  - $\mathcal{F}_1 = \sigma(\{u, d\})$
  - $\mathcal{F}_2 = \sigma(\{uu, um, ud, du, dd\})$
  - $\mathcal{F}_3 = \sigma(\Omega)$
- Note that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$
- At $\mathcal{F}_0$ we have no information and at $\mathcal{F}_3$ we have the exact state that occurred.
- A natural interpretation is that $\mathcal{F}_t$ represents how much refined the known information is at time t.
- A *filtration*, $\underline{\mathcal{F}}$ on the probability space $\Omega, \mathcal{F}, \mathbb{P}$ is an increasing sequence of σ-algebras, $\mathcal{F}_0 \subset \ldots \subset \mathcal{F}_T \subset \mathcal{F}$. As T increases, the σ- algebras becomes more refined since more information is known.
- An $\underline{\mathcal{F}}$-*adapted* (random) process, $X : T \times \Omega \to \mathbb{R}$, where we, with a slight abuse of notation, write $T = \{0, 1, \ldots, T\}$, is a process, such that X_t is $\mathcal{F}_t$-measurable for all t. Intuitively, this definition takes into account that X_t can only depend on the information that is available at time t. If the process is not adapted, then it cheats and uses future information. If it is adapted, it is only using the available information.

- An $\underline{\mathcal{F}}$-martingale is an adapted process, m, such that for all $0 \leq s \leq t, E[m_t|\mathcal{F}_s] = m_s$. In other words, the best forecast is not expected to change (the best forecast is what it is right now). It follows immediately from the law of iterated expectations that Y_t defined above is a martingale.

- A set $A \in \Omega$ is said to be $\mathcal{F}$-measurable if $A \in \mathcal{F}$.

The Multiperiod Security Market

- Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $\underline{\mathcal{F}} = (\mathcal{F}_0, \ldots, \mathcal{F}_T)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (i.e. we know nothing) and $\mathcal{F}_T = \sigma(\Omega)$ (i.e. we know everything), and a market with N traded assets.

- Each security is a claim to an adapted *dividend process*. Formally, we define the vector valued adapted process as $\delta_t \in \mathbb{R}^N$, where $(\delta_t)_i$ represents the dividends paid by asset i at time t.

- Each security is associated with an adapted *price process*, representing the price of the security, right after the dividend is paid (*ex dividend*). Formally, we defined the vector valued adapted process as $\boldsymbol{s}_t \in \mathbb{R}^N$, where $(\boldsymbol{s}_t)_i$ represents the price of asset i at time t, just after dividends have been paid.

- The before-dividend (*cum dividend*) price proess is $\boldsymbol{s}_t + \delta_t$.

- The market is summarized by the dividend price pair $(\delta, \boldsymbol{s})$. This pair gives the price and dividend amount for every possible state that may occur. This is the multiperiod version of the s^0 and $\boldsymbol{D}$ we had in the single period setting.

- In the multiperiod model, we separate the value we get from dividends and price appreciation.

- A *trading strategy* is a vector valued adapted process $\boldsymbol{h}_t \in \mathbb{R}^N$ where $\boldsymbol{h}_{-1} = 0$. $\boldsymbol{h}_{-1}$ is an artifical time $t = -1$ where we have nothing. The trading strategy can now trade over time.

- The *payoff process* generated by a trading strategy $\delta_t^{\boldsymbol{h}} \in \mathbb{R}$ is an adapted process defined by

$$\delta_t^{\boldsymbol{h}} = \boldsymbol{h}_{t-1}'(\boldsymbol{s}_t + \delta_t) - \boldsymbol{h}_t'\boldsymbol{s}_t.$$

  that represents how much money the process generates over time. $\boldsymbol{h}_{t-1}'$ is how much of each stock invested at $t-1$ multiplied by $(\boldsymbol{s}_t + \delta_t)$ which is the price of the stock at time t right before the dividend (cum dividend), δ_t, is paid.

- Put simply, $\boldsymbol{h}_{t-1}'(\boldsymbol{s}_t + \delta_t)$ is the value of our portfolio cum-dividend based on how we invested at time $t-1$ and $\boldsymbol{h}_t'\boldsymbol{s}_t$ is the value of the portfolio after we rebalanced the portfolio, at time t, and their difference is the amount we get through the rebalancing.

- In the case of assets that do not pay dividends, this reduces to

$$\delta_t^{\boldsymbol{h}} = (\boldsymbol{h}_{t-1} - \boldsymbol{h}_t)'\boldsymbol{s}_t = -(\boldsymbol{h}_t - \boldsymbol{h}_{t-1}) = -(\Delta\_\boldsymbol{h}_t)'\boldsymbol{s}_t$$

- The *value process* of a trading strategy is $V_t^{\boldsymbol{h}} = \boldsymbol{h}_t'\boldsymbol{s}_t$.

- Given, an adapted *consumption process* (which is a plan for how much we plan to comsume), c_t, a trading strategy is said to be *self-financed* if $\delta_t^{\boldsymbol{h}} = c_t$ for $t = 1, \ldots, T$. In other words, a trading strategy is self-financed if we set up the consumption process so that we get exactly the same payoff process, $\delta_t^{\boldsymbol{h}}$ as you wish to consume, c_t.

- We mainly focus on the cases where we are not consuming anything, that is $c_t = 0$.
- The below figure justifies the formula for the payoff process cum dividend

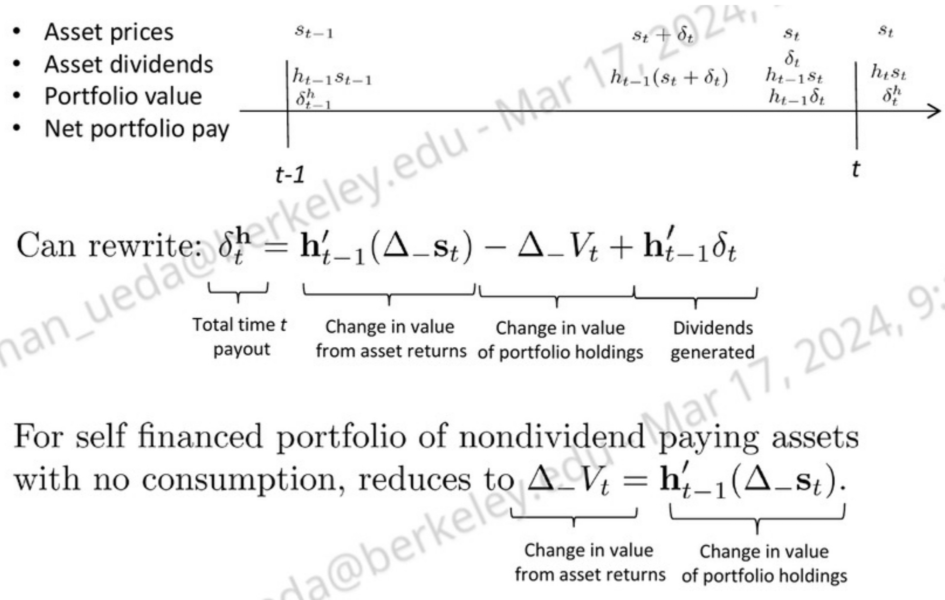

Figure 6: At time $t-1$, we just received our dividend payment so the porfolio value is currently $\boldsymbol{h}'_{t-1}\boldsymbol{s}_{t-1}$. Next, we have the cum dividend price process. Next, dividends are paid, in the amount $\boldsymbol{h}_{t-1}\delta_t$, which can then be reinvested into the portfolio $\boldsymbol{h}_t$, the rebalanced portfolio.

- Deriving the new rewritten formula

$$
\begin{aligned}
\delta_t^{\boldsymbol{h}} &= \boldsymbol{h}'_{t-1}(\Delta_- \boldsymbol{s}_t) - \Delta_- V_t + \boldsymbol{h}'_{t-1}\delta_t \\
&= \boldsymbol{h}'_{t-1}(\boldsymbol{s}_t - \boldsymbol{s}_{t-1}) - (\boldsymbol{h}'_t\boldsymbol{s}_t - \boldsymbol{h}'_{t-1}\boldsymbol{s}_{t-1}) + \boldsymbol{h}'_{t-1}\delta_t \\
&= \boldsymbol{h}'_{t-1}\boldsymbol{s}_t - \boldsymbol{h}'_{t-1}\boldsymbol{s}_{t-1} - \boldsymbol{h}'_t\boldsymbol{s}_t + \boldsymbol{h}'_{t-1}\boldsymbol{s}_{t-1} + \boldsymbol{h}'_{t-1}\delta_t \\
&= \boldsymbol{h}'_{t-1}\boldsymbol{s}_t - \boldsymbol{h}'_t\boldsymbol{s}_t + \boldsymbol{h}'_{t-1}\delta_t \\
\delta_t^{\boldsymbol{h}} &= \boldsymbol{h}'_{t-1}(\boldsymbol{s}_t + \delta_t) - \boldsymbol{h}'_t\boldsymbol{s}_t
\end{aligned}
$$

- We have $V_t = \mathbf{h}'_t \mathbf{s}_t$.
- So,

$$\begin{aligned}
\Delta_- V_t &= \mathbf{h}'_t \mathbf{s}_t - \mathbf{h}'_{t-1}\mathbf{s}_{t-1} \\
&= (\mathbf{h}_{t-1} + \Delta_- \mathbf{h}_t)'(\mathbf{s}_{t-1} + \Delta_- \mathbf{s}_t) - \mathbf{h}'_{t-1}\mathbf{s}_{t-1} \\
&= \mathbf{h}'_{t-1}\mathbf{s}_{t-1} + \mathbf{h}'_{t-1}\Delta_- \mathbf{s}_t + \Delta_- \mathbf{h}'_t(\mathbf{s}_{t-1} + \Delta_- \mathbf{s}_t) - \mathbf{h}'_{t-1}\mathbf{s}_{t-1} \\
&= \mathbf{h}'_{t-1}\Delta_- \mathbf{s}_t + \Delta_- \mathbf{h}'_t(\mathbf{s}_{t-1} + \Delta_- \mathbf{s}_t).
\end{aligned}$$

- Thus,

$$\boxed{\underbrace{\Delta_- V_t}_{\text{Total change in value of portfolio}} = \underbrace{\mathbf{h}'_{t-1}\Delta_- \mathbf{s}_t}_{\text{Change from portfolio returns}} + \underbrace{\Delta_- \mathbf{h}'_t(\mathbf{s}_{t-1} + \Delta_- \mathbf{s}_t)}_{\text{Change from rebalancing (after returns realized)}}.}$$

_ueda@berkeley.edu - Mar 17, 2024, 9:

Figure 7: Denotes the change in the value of the portfolio. Has relevance to Ito's formula in a much simpler manner.

- A trading strategy, $\boldsymbol{h}$, provides an arbitrage if $\delta_t^{\boldsymbol{h}} \geq 0$ for all t, for all $\omega \in \Omega$ (no negative cash flows anywhere), and $\delta_t^{\boldsymbol{h}} > 0$ for some t and $\omega \in \Omega$ (a strictly positive cash flow somewhere), and $\boldsymbol{h}_T = 0$ (must liquidate entire portfolio).
- Define $\bar{L}$, the (augmented) space of adapted processes, Θ, the space of trading strategies, $\Theta = \{\boldsymbol{h} : \boldsymbol{h} \text{ is a trading strategy}\}$, and $\bar{\mathcal{R}}$, the augmented marketed (reachable) space, $\bar{\mathcal{R}} = \{\delta^{\boldsymbol{h}} : \boldsymbol{h} \in \Theta\}$ (all the different possible payouts that are adaptaive at any point in time in any state). Then we have the natural embedding:

$$\bar{\mathcal{R}} \subset \bar{L} \subset \mathbb{R}^{T \times \Omega}$$

  where $\mathbb{R}^{T \times \Omega}$ is all states at any point in time, regardless of being adaptive.
- Similarly, we define the space of adapted processes from $t = 1, \ldots, T, L = \{l_t : \{1, \ldots, T\} \times \Omega \to \mathbb{R} : l_t = y_t,\ t = 1, \ldots, T, \text{ for some } y \in \bar{L}\}$ and the marketed space $\mathcal{R} = \{m_t : \{1, \ldots, T\} \times \Omega \to \mathbb{R} : m_t = y_t,\ t = 1, \ldots, T, \text{ for some } y \in \bar{\mathcal{R}}\}$ with the natural embedding

$$\mathcal{R} \subset L \subset \mathbb{R}^{(T-1) \times \Omega}$$

- The relation of $\mathcal{R}$ to $\bar{\mathcal{R}}$ is the same as in the single period binomial model, that is, $\mathcal{R}$ does not take into account time $t = 0$ cash flows while $\bar{\mathcal{R}}$ does.
- The reachable payoff space is the different types of payoffs we can generate by having an adaptive trading strategy.
- The market is said to be complete if $\mathcal{R} = L$.

- A *stochastic discount factor* (the prices today as a function of future prices), M_t, is a strictly positive adaptive process, such that for all assets, i,

$$(s_t)_i = \frac{1}{M_t} E^t_{\mathbb{P}} \left[\sum_{k=t+1}^{T} M_k (\delta_k)_i + M_T (\boldsymbol{s}_T)_i \right], \ t < T$$

- Assume a short term risk free asset: For all $t < T$, for all states B^K_t, there is a trading strategy $\boldsymbol{h}$, which invests 1 at t and receives R^k_t ad $t+1$. This defines an adapted *short term process*

$$R_t = 1 + r_t$$

- The n-period compounded discount rate is

$$R_{t,t+n} = R_t R_{t+1}, \ldots, R_{t+n-1}$$

  showing that the risk free rate may change at each period.

- An *equivalent martingale measure* $\mathbb{Q}$ satisfies

$$(\boldsymbol{s}_t)_i = E^t_{\mathbb{Q}} \left[\sum_{k=t+1}^{T} \frac{(\delta_k)_i}{R_{t,t+k}} + \frac{(\boldsymbol{s}_T)_i}{R_{t,T}} \right], \ t < T$$

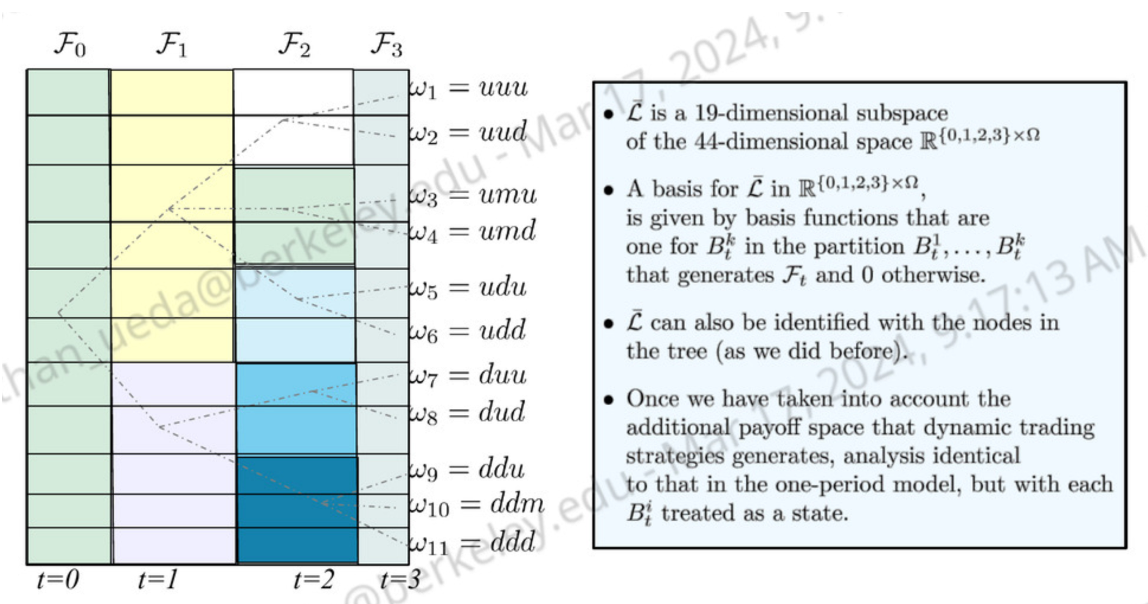

Figure 8: The augmented payoff space is 19 dimensional

1.3.1 Multiperiod Fundamental Theorems

- First fundamental theorem: The market admits no arbitrage if and only if there is a stochastic discount factor.

- First fundamental theorem: The market admits no arbitrage if and only if there is an equivalent martingale measure.
- Second fundamental theorem: Given a market that admits no arbitrage. Then the market is complete if and only if the equivalent martingale measure is unique.

2 Continuous Time Models

- The motivation for introducing the continuous time model is because it will be very useful in deriving nice solutions and being able to analyze problems in a very tractable manner.
- In the continuous time space, we unfortunately have to give up the partition representation of the sigma algebra (how information is incorporated into the market over time). The reason being is because with a partition we need something where there is a smallest piece of information, but in a continuous time, there is no smallest since they can be arbitrarily small.
- A filtration in the continuous space is a nondecreasing family of σ-algebras, $\{\mathcal{F}_t\}$, $\mathcal{F}_s \subset \mathcal{F}_t$, $s < t$, such that $\mathcal{F}_t \subset \mathcal{F}$ for all t.
- In the same way it is in the discrete model, a stochastic process is adapted if X_t is $\mathcal{F}_t$-measurable for all t.
- A stochastic process is a martingale if for all $s < t$, $E[X_t|\mathcal{F}_s] = X_s$. This means the same thing as it did in the discrete time setting, namely, that its expected value at any future time is whatever its value currently is. Note that conditioning on $\mathcal{F}_s$ just means that we are conditioning on all the information available at time s.

2.1 Brownian Motion

- Brownian motion is a physical phenomenon in which some quantity if constantly undergoing small, random fluctuations.
- It is essentially the continuous time version of a random walk (which of course is in discrete time).
- A stochastic process, W_t, $t \geq 0$ is said to be a *standard Brownian motion* (a Wiener process), if it satisfies the following four properties:
  1. It starts at the origin: $W_0 = 0$
  2. It is normally distributed $\sim N(0, t)$ and the increment for all $s < t$ is also normally distributed, that is, $W_t - W_s \sim N(0, t - s)$. Notice that the variance is equal to the length of the time period.
  3. It has independent increments: for all $r < s \leq t < u$, $W_u - W_t$ is independent of $W_s - W_r$. This means that whatever happens between W_s and W_r has no effect on what happens between W_t and W_u.
  4. It has continuous sample paths: W_t is a continuous function of t.
- Clearly, a Wiener process is a martingale process (has no drift, the expected value given all prior information is always the mean, which is 0).

- A *Brownian motion* is a stochastic process, $X_t = X_0 + \mu t + \sigma W_t$ where μ and $\sigma > 0$ are constants, and W_t is a standardized Brownian motion. This unstandardized version of Brownian motion allows us to add some drift, μt, some scaling of the volatility, and some initial starting point, X_t.
- A *geometric Brownian motion* (GBM) is a stochastic process, $Y_t = e^{X_t}$, where X_t is a Brownian motion. This is what we get when we take e to the power of a Brownian motion, X_t, giving it a log normal distribution.
- Brownian motions are continuous, but not very smooth and almost surely nowhere differentiable.

2.2 Multivariate Brownian Motion

- Given k-vector of independent standard Brownian motions, $\boldsymbol{W}_t = \begin{bmatrix} W_t^1 \\ \vdots \\ W_t^k \end{bmatrix}$ and vectors $\mu \in \mathbb{R}^k$, and $\Sigma = \sigma\sigma^T$, where $\sigma \in \mathbb{R}^{k \times k}$ is invertible
  - The process $\boldsymbol{Z}_t = \boldsymbol{Z}_0 + \mu t + \sigma \boldsymbol{W}_t$ is a k-dimensional Brownian motion.
  - $\boldsymbol{Z}_t \sim N(\mu t, \Sigma t)$, where $\Sigma = \sigma\sigma^T$.
  - The process $\boldsymbol{Y}_t = \begin{bmatrix} Y_t^1 \\ \vdots \\ Y_t^k \end{bmatrix}$, where $Y_t^i = e^{(\boldsymbol{Z}_t)_i}$ is a k-dimensional geometric Brownian motion.

2.3 Discrete Time Trading to Bridge Gap to Continuous Time Trading

- Here we look at how to define continuous time trading between $t = 0$ and $t = T$.
- We start with the discrete formula for a self financed (this is when consumption equals payoff, that is, $\delta_t^{\boldsymbol{h}} = c_t$) portfolio

$$\delta_t^{\boldsymbol{h}} = c_t = \boldsymbol{h}'_{t-1}(\Delta_- \boldsymbol{s}_t) - \Delta_- V_t + \boldsymbol{h}'_{t-1}\delta_t$$

- if we have no dividends, we can derive the following

$$\begin{aligned} c_t &= \boldsymbol{h}'_{t-1}(\Delta_- \boldsymbol{s}_t) - \Delta_- V_t + \boldsymbol{h}'_{t-1}\delta_t \\ c_t &= \boldsymbol{h}'_{t-1}(\Delta_- \boldsymbol{s}_t) - \Delta_- V_t \\ \Delta_- V_t &= \boldsymbol{h}'_{t-1}(\Delta_- \boldsymbol{s}_t) - c_t \end{aligned}$$

- Now, we divide $[0, T]$ into n sub periods of length $\Delta t = \frac{T}{n}$.
- Therefore, consumption at time $k\Delta t = c_k \Delta t$, which is talking about consumption per unit of time (i.e. driving 60 mph).
- We then rewrite

$$\begin{aligned} \Delta_- V_t &= \boldsymbol{h}'_{t-1}(\Delta_- \boldsymbol{s}_t) - c_t \\ \Delta_- V_{k\Delta t} &= \boldsymbol{h}'_{(k-1)\Delta t}\Delta_- \boldsymbol{s}_{k\Delta t} - c_{k\Delta t}\Delta t \end{aligned}$$

- Although we can theoretically sum the differences to get the change in portfolio value

$$V_T - V_0 = \sum_{k=1}^{n} \Delta_{-} V_{k\Delta t} = \sum_{k=1}^{n} \boldsymbol{h}'_{(k-1)\Delta t} \Delta_{-} s_{k\Delta t} - \sum_{k=1}^{n} c_{k\Delta t} \Delta t$$

We prefer to take limits, $\Delta t \to 0$ and rewrite as

$$V_T - V_0 = \int_{t=0}^{T} dV_t = \int_{t=0}^{T} \boldsymbol{h}'_t \, d\boldsymbol{s}_t - \int_{t=0}^{T} c_t \, dt$$

- Therefore, we have 2 terms we needs to be able to take the limit of.

- Consider a simple model with two assets: a standard Brownian motion stock price, S, and a bond with constant price, B (interest rate is zero and therefore always has the value 1), and a time step Δt.

- $S_{k\Delta t}$ are the values of the standard Brownian motion, evaluated at discrete points $S_{k\Delta t} = W_{k\Delta t}$

$$S_{k\Delta t} \sim N(0, k\Delta t)$$
$$B_{k\Delta t} = 1$$

- So,

$$S_{(k+1)\Delta t} = S_{k\Delta t} + \sqrt{\Delta t}\xi_{k\Delta t}, \ \xi_{k\Delta t} \sim N(0,1) \text{ i.i.d.}$$
$$B_{(k+1)\Delta t} = B_{k\Delta t}$$

2.4 Stochastic Integrals

- Brownian motions will be used for modeling asset dynamics, and because they are so wiggly, the formulas the *calculus* that is is going to have to be different than standard calculus.

- Instead of using the classical Riemann-Stieltjes integral, we use the Ito integral.

- The Ito integral is much more restrictive than the Riemann in that, instead of taking the minimum and maximums when summing, we're always taking the left endpoint when summing (since otherwise the process is not adaptive). Therefore, we are force to only look at the left most endpoint since that is the only value we know.

- Important Itô calculus properties:
  - Martingale property

$$E\left[\oint_0^T a_t \, dW_t\right] = 0$$

  - Ito isometry

$$E\left[\left(\oint_0^T a_t \, dW_t\right)^2\right] = \int_0^T E[a_t^2] \, dt$$

Itôs Lemma

- Assume X_t is an Ito process that satisfies $dX = u\,dt + v\,dW$, where W is a Brownian motion, and that $Y_t = g(t, X_t)$ for some smooth function g. Then dY_t can be written

$$dY_t = g_t\,dt + g_x\,dX + \frac{1}{2}g_{xx}(dX)^2$$

where $(dX_t)^2$ is computed according to the rules

$$dt \times dt = 0$$

$$dt \times dW_t = 0$$

$$dW_t \times dW_t = dt$$

which leads to

$$(dX_t)^2 = v^2\,dt$$

2.5 Transformations

1. If given a geometric Brownian motion

$$\frac{dX_t}{X_t} = \mu\,dt + \sigma\,dW_t$$

and take the log, you need to subtract $\frac{1}{2}\sigma^2$ from the drift term

$$d\ln X_t = (\mu - \frac{1}{2}\sigma^2)\,dt + \sigma\,dW_t$$

2. If given an ordinary Brownian motion

$$dX_t = \mu\,dt + \sigma\,dW_t$$

and transform to a geometric Brownian motion, you need to add $\frac{1}{2}\sigma^2$ to the drift term

$$\frac{de^{X_t}}{e^{X_t}} = (\mu + \frac{1}{2}\sigma^2)\,dt + \sigma\,dW_t$$

2.6 Multidimensional Uncorrelated Itôs Lemma

- Let $\boldsymbol{X}_t$ be a k-dimensional Ito process given by

$$d\boldsymbol{X}_t = \mu\,dt + \sigma\,d\boldsymbol{W}_t$$

where $\mu_t \in \mathbb{R}^k, \sigma_t \in \mathbb{R}^{k\times k}, \Sigma = \sigma\sigma^T, \sigma_i = [\sigma_{i,1}, \ldots, sigma_{i,k}]$ and $\boldsymbol{W}_t$ is a standardized k-dimensional Brownian motion. Let $g(t,x) \in C^2([0,\infty] \times \mathbb{R}^k)$. Then $Y_t = g(t, \boldsymbol{X}_t)$ is an Ito process, and

$$dY_t = \left(g_t + \sum_i g_{x_i}\mu_i + \frac{1}{2}\sum_{i,j} g_{x_i x_j}\Sigma_{ij}\right)\,dt + \sum_i g_{x_i}\sigma_i\,d\boldsymbol{W}_t$$

In the $k = 2$ dimensional case, this would take the form as follows

$$\begin{aligned} d\boldsymbol{X}_t =& \mu\, dt + \sigma\, d\boldsymbol{W}_t \\ \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} =& \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix} \end{aligned}$$

$$\Sigma = \sigma\sigma^T = \begin{bmatrix} \sigma_{11}^2 + \sigma_{12}^2 & \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} \\ \sigma_{21}\sigma_{11} + \sigma_{22}\sigma_{21} & \sigma_{21}^2 + \sigma_{22}^2 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Now we can apply Ito

$$\begin{aligned} dY_t &= \left(g_t + \sum_i g_{x_i}\mu_i + \frac{1}{2}\sum_{i,j} g_{x_i x_j}\Sigma_{ij} \right) dt + \sum_i g_{x_i}\sigma_i\, d\boldsymbol{W}_t \\ dY_t &= g_t\, dt + (g_{x_1}\, dx_1 + g_{x_2}\, dx_2) + (\frac{1}{2}[g_{x_1x_1}\Sigma_{11} + g_{x_2x_2}\Sigma_{22} + 2g_{x_1x_2}\Sigma_{12}]\, dt) \\ dY_t &= g_t\, dt + g_{x_1}\, dx_1 + g_{x_2}\, dx_2 + \frac{1}{2}[g_{x_1x_1}\Sigma_{11} + g_{x_2x_2}\Sigma_{22} + 2g_{x_1x_2}\Sigma_{12}]\, dt \end{aligned}$$

We can then take the stochastic integral of dY

$$\oint_0^T = g(T, \boldsymbol{X}_T) - g(0, \boldsymbol{X}_0)$$

with the hessian $H = \begin{bmatrix} g_{x_1x_1} & g_{x_1x_2} \\ g_{x_2x_1} & g_{x_2x_2} \end{bmatrix}$

2.7 Itôs Version of the Product Rule

- Define quadratic covariance of 2 Ito processes as $\langle X, Y \rangle_t = \int dXdY$, where $dXdY = \sigma_X\sigma_Y\rho_{xy}\, dt$
- The Ito version of the product rule (Leibniz' formula) now takes the following forms
  - Integral Form: $X_tY_t - X_0Y_0 = \int X\, dY + \int Y\, dX + \langle X, Y \rangle_t$
  - Differential Form: $d(XY) = X\, dY + Y\, dX + dXdY$.

2.8 Continuous Time Porfolio Model

Total Change in Value of Portfolio

- Assume N-asset trading strategy according to diffusion process, $\boldsymbol{h}_t$, and asset value diffusion process, $\boldsymbol{s}_t$.
- The value of the portfolio at time t is $V_t = \boldsymbol{h}_t'\boldsymbol{s}_t$.

- So, by Ito, we have

$$dV_t = \boldsymbol{h}_t'\, d\boldsymbol{s} + d\boldsymbol{h}_t' \boldsymbol{s}_t + d\boldsymbol{h}_t' d\boldsymbol{s}_t$$
$$dV_t = \boldsymbol{h}_t'\, d\boldsymbol{s}_t + d\boldsymbol{h}_t'(\boldsymbol{s}_t + d\boldsymbol{s_t})$$

- Compared to the discrete relationship for the total change in value of portfolio

$$\Delta_{-} V_t = \boldsymbol{h}_{t-1}' \Delta_{-} \boldsymbol{s}_t + \Delta_{-} \boldsymbol{h}_t' (\boldsymbol{s}_{t-1} + \Delta_{-} \boldsymbol{s}_t)$$

Portfolio Payoff

- In discrete time, we used the following formula to denote the portfolio payoff

$$\delta_t^{\boldsymbol{h}} = \boldsymbol{h}_{t-1}'(\Delta_{-} \boldsymbol{s}_t) - \Delta_{-} V_t + \boldsymbol{h}_{t-1}' \delta_t$$

  or equivalently

$$\delta_t^{\boldsymbol{h}} = \boldsymbol{h}_{t-1}'(\boldsymbol{s}_t + \delta_t) - \boldsymbol{h}_t' \boldsymbol{s}_t$$

- For the continuous time version, we start by defining the cumulative dividend process of stocks from 0 to t, $\Theta_t \in \mathbb{R}^N$ and the gain process, $G_t = \boldsymbol{s}_t + \Theta_t$.
- We then define the cumulative payoff process of a porfolio from 0 to t, $F_t^{\boldsymbol{h}} \in \mathbb{R}$
- This allows us to create the following relationship

$$dV + dF^{\boldsymbol{h}} = \boldsymbol{h}'\, d\boldsymbol{s} + \boldsymbol{h}'\, d\Theta = \boldsymbol{h}'\, dG$$

- We can then define the (instantaneous) porfolio payoff in continuous time as follows

$$dF_t^{\boldsymbol{h}} = -d\boldsymbol{h}_t'(\boldsymbol{s}_t + d\boldsymbol{s}_t) + \boldsymbol{h}'\, d\Theta_t$$

  which is the porfolio payout equal to the payments from rebalancing the portfolio with the payments from stock dividends added.
- If the stocks pay no dividends ($\Theta = d\Theta = 0$), we have

$$dF_t^{\boldsymbol{h}} = -d\boldsymbol{h}_t'(\boldsymbol{s}_t + d\boldsymbol{s}_t)$$

- If the stocks pay no dividends ($\Theta = d\Theta = 0$) and we have self financing trading strategy ($dF^{\boldsymbol{h}} = 0$), we have

$$(d\boldsymbol{h})'\boldsymbol{s} = -(d\boldsymbol{h})'(d\boldsymbol{s})$$
$$dV = \boldsymbol{h}'\, d\boldsymbol{s}$$

2.9 Stochastic Differential Equations

- A SDE is of the form

$$dX = \mu(t, X)\, dt + \sigma(t, X_t)\, dW_t$$

3 Discrete Time Models Examples

3.1 Example

Given the following one period, two state model, answer the questions below.

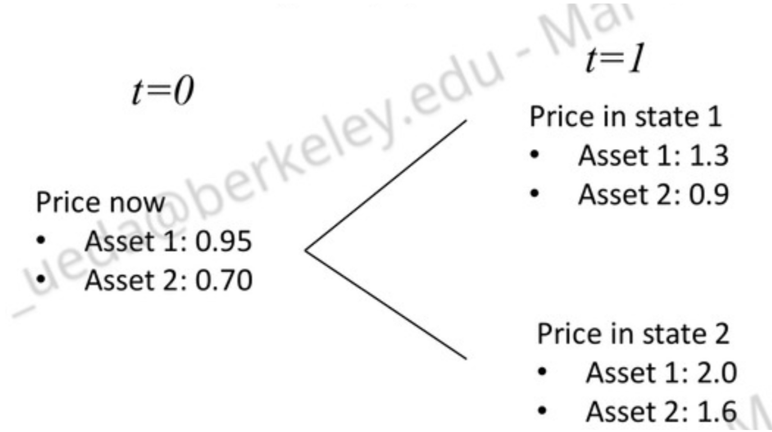

Figure 9: One Period, Two State Model

1. Is the market complete?
   Approach: True if Rank($\boldsymbol{D}$) $= M$ (since this is a square matrix, this is also true if $\boldsymbol{D}$ is invertible).

   $$\boldsymbol{D} = \begin{bmatrix} 1.3 & 2.0 \\ 0.9 & 1.6 \end{bmatrix}$$

   Rank($\boldsymbol{D}$) $= 2 = M \rightarrow$ the market is complete.

   An equivalent check is that $\det(\boldsymbol{D}) = ad - bc = (1.3 \times 1.6) - (2.0 \times 0.9) = 0.28 \neq 0 \rightarrow \boldsymbol{D}$ is invertible and therefore the market is complete.

2. What are the state prices?
   Approach: Find the state price vector.

   $$\boldsymbol{s}^0 = \begin{bmatrix} 0.95 \\ 0.70 \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} 1.3 & 2.0 \\ 0.9 & 1.6 \end{bmatrix}$$

   $$\boldsymbol{D}^{-1} = \begin{bmatrix} 1.3 & 2.0 \\ 0.9 & 1.6 \end{bmatrix}^{-1} = \frac{1}{(1.3 \times 1.6) - (2.0 \times 0.9)} \begin{bmatrix} 1.6 & -2.0 \\ -0.9 & 1.3 \end{bmatrix} = \frac{1}{0.28} \begin{bmatrix} 1.6 & -2.0 \\ -0.9 & 1.3 \end{bmatrix} = \begin{bmatrix} 5.714 & -7.143 \\ -3.214 & 4.643 \end{bmatrix}$$

   $$\boldsymbol{s}^0 = \boldsymbol{D}\psi$$

   Since $\boldsymbol{D}$ is square and invertible, we can find the SPV by solving for

   $$\psi = \boldsymbol{D}^{-1}\boldsymbol{s}^0 = \begin{bmatrix} 5.714 & -7.143 \\ -3.214 & 4.643 \end{bmatrix} \begin{bmatrix} 0.95 \\ 0.70 \end{bmatrix} = \begin{bmatrix} 0.4282 \\ 0.1968 \end{bmatrix}$$

   where $\psi_1 = 0.4282$ is the price for state 1 and $\psi_2 = 0.1968$ is the price for state 2.

3. What is the price of an asset that pays \$1 in each state of the world?
   Approach: Since we have ψ and the market is complete, we can price any payoff structure. We want to price the payoff where we are paid \$1 in each state.

$$\boldsymbol{V}^1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$
$$\psi_1 \times 1 + \psi_2 \times 1 = 0.4282 + 0.1968 = 0.625$$

   If we wanted, we could easily calculate R^f as $\frac{1}{0.625} = 1.6$ since we are paying 0.625 at $t = 0$ to get \$1 at $t = 1$.

3.2 Example

Given the following one period, two state model, answer the questions below.

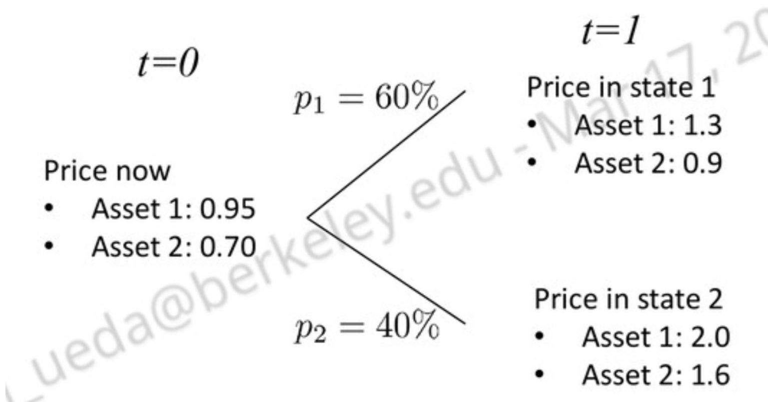

Figure 10: One Period, Two State Model

From example 1, we are also given

$$\psi = \begin{bmatrix} 0.4282 \\ 0.1968 \end{bmatrix}$$
$$R^f = \frac{1}{\hat{\psi}} = \frac{1}{0.625} = 1.6$$

1. Find the equivalent martingale measure.
   Approach: The martingale measure is $\mathbb{Q}$, where each $q_i = \frac{\psi_i}{\hat{\psi}}$, and $\hat{\psi} = \sum_{i=1}^{M} \psi_i$.

$$\hat{\psi} = \psi_1 + \psi_2 = 0.4282 + 0.1968 = 0.625$$
$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{\psi_1}{\hat{\psi}} \\ \frac{\psi_2}{\hat{\psi}} \end{bmatrix} = \begin{bmatrix} 0.4282/0.625 \\ 0.1968/0.625 \end{bmatrix} = \begin{bmatrix} 0.685 \\ 0.315 \end{bmatrix}$$

2. Find the likelihood ratio.
   Approach: The likelihood ratio for each state A is $L(A) = \frac{\mathbb{Q}(A)}{\mathbb{P}(A)}$.

$$L(\omega_1) = \frac{q_1}{p_1} = \frac{0.685}{0.60} = 1.142$$

$$L(\omega_2) = \frac{q_2}{p_2} = \frac{0.315}{0.40} = 0.7875$$

3. Find the stochastic discount factor.
   Approach: The stochastic discount factor may be defined as $M_1 = \frac{1}{R}L$.

$$M_1(\omega_1) = \frac{1}{R}L(\omega_1) = \frac{1}{1.6}1.142 = 0.714$$
$$M_1(\omega_2) = \frac{1}{R}L(\omega_2) = \frac{1}{1.6}0.7875 = 0.492$$

3.3 Example

Answer the questions for the following economy

$$\boldsymbol{s}^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; \boldsymbol{D} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 3 & 3 & 3 & 3 & 7 \end{bmatrix}$$

1. Is the market complete?
   Approach: If Rank($\boldsymbol{D}$) $= M$, market is complete. Also, if there are fewer assets than there are states, that is $N < M$, the market cannot possibly be complete since the rank can be at most $\min\{N, M\}$.

   Since $N < M$, we automatically know the market is not complete.

2. Is there an arbitrage opportunity?
   Approach: We could either think of this as trying to find strictly positive SPV and then use the FTAP or we could try to create an arbitrage and see if we could do it (this is what we will do in this example since there aren't many states and assets to check and it is an illustrative example this way).

   Recall: An arbitrage opportunity exists if there is some portfolio $\boldsymbol{h}$ such that $\boldsymbol{h}^T\bar{\boldsymbol{D}} > 0$.

   The higher M is, the more states we have to check, notice that really, in this example, there are only 3 states we need to check (since states 1-4 have exactly the same payoffs), which make up the new reduced matrix

$$\bar{\boldsymbol{D}} = \begin{bmatrix} -1 & 5 & 5 \\ -1 & 3 & 7 \end{bmatrix}$$

   We now want to try to find some $\boldsymbol{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ such that $\boldsymbol{h}^T\bar{\boldsymbol{D}} > 0$.

   This provides us with 3 constraints:

$$\boldsymbol{h}^T\bar{\boldsymbol{D}} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} -1 & 5 & 5 \\ -1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} -h_1 - h_2 & 5h_1 + 3h_2 & 5h_1 + 7h_2 \end{bmatrix} > 0$$

   Splitting these constraints apart, these say

$$-h_1 - h_2 > 0 \rightarrow h_1 + h_2 \leq 0$$

$$5h_1 + 3h_2 \geq 0$$

$$5h_1 + 7h_2 \geq 0$$

For $\boldsymbol{h}^T \bar{\boldsymbol{D}} > 0$ hold, we also need one of the above to be a strict inequality.

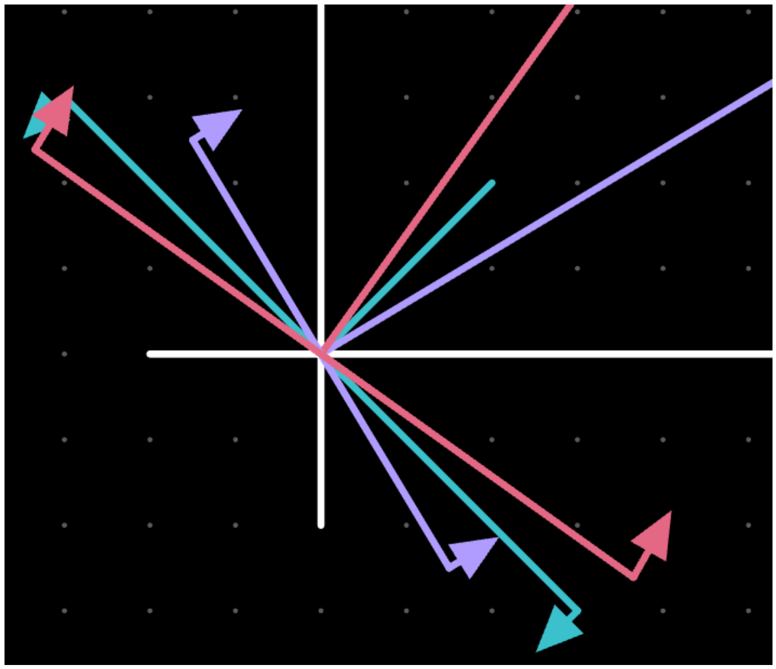

Figure 11: Turquoise corresponds to vector [1, 1], purple corresponds with vector [5,3], and red corresponds with vector [5,7]. The arrows denote the direction where the constraint is met for each vector.

The image shows there are indeed no portfolios that satisfy all 3 constraints (other than the origin, which is irrevelant since that is the case where we don't have any portfolio) and therefore there is no way to satisfy $\boldsymbol{h}^T \bar{\boldsymbol{D}} > 0$. This allows us to conclude there is no arbitrage.

3. Is there a strictly positive state price vector?
   Approach: By FTAP, we know the market if arbitrage free iff there exists a strictly positive SPV. Since we have shown the market is arbitrage free, we know there is a strictly positive SPV.

   Yes

3.4 Example

Answer the questions for the following economy

$$\boldsymbol{s}^0 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} 5 & 5 \\ 6 & 9 \\ 8 & 11 \\ 9 & 10 \\ 11 & 6 \end{bmatrix}$$

1. Is the market complete?
   Approach: If Rank($\boldsymbol{D}$) $= M$, market is complete.
   Rank($\boldsymbol{D}$) $= 2 = M$ therefore, the market is complete.

2. Is there an arbitrage opportunity?
   Approach: To show there exists an arbitrage, we just need to show one example, that is, one portfolio such that $\boldsymbol{h}^T\bar{\boldsymbol{D}} > 0$

   Let

$$\boldsymbol{h} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{h}^T\bar{\boldsymbol{D}} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 5 \\ -2 & 6 & 9 \\ -3 & 8 & 11 \\ -4 & 9 & 10 \\ -5 & 11 & 6 \end{bmatrix} = \begin{bmatrix} -2+2 & 10-6 & 10-9 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix} > 0$$

   Therefore, we have shown that there is arbitrage in this economy since we have found a portfolio such that $\boldsymbol{h}^T\bar{\boldsymbol{D}} > 0$.

3. Is there a strictly positive state price vector?
   Approach: By FTAP, we know the market if arbitrage free iff there exists a strictly positive SPV. Since we have shown the market is not arbitrage free, we know there is not a strictly positive SPV.

3.5 Example

Given the following 1 period, 2 state binomial tree market (denoted by the image below).

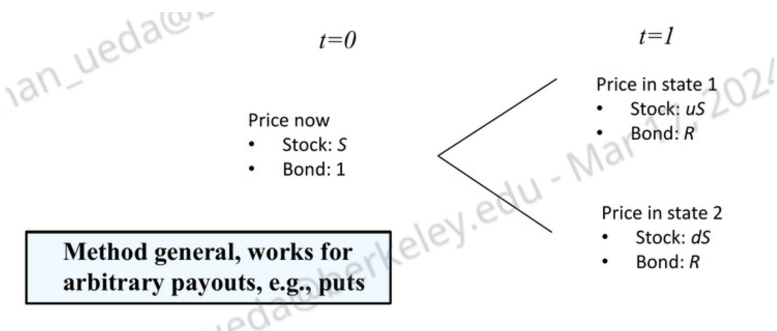

Figure 12: Price dynamics

Note that an equivalent interprestion, since payoff is in $\boldsymbol{\mathcal{R}}$, should be possible to form a *replciating portfolio*, $\boldsymbol{h} = \begin{bmatrix} \Delta \\ B \end{bmatrix}$, where Δ is the number of shares bought at B is how much is put into the bond. Conditions we want for the payoff are

$$\Delta uS + BR = C_u$$

$$\Delta dS + BR = C_d$$

The price today of the portfolio must be equal to the price of the call for there to be no arbitrage, that is

$$C = \Delta S + B$$

Solve for Δ

$$\begin{aligned}
(\Delta uS + BR) - (\Delta dS + BR) &= C_u - C_d \\
\Delta S(u-d) &= C_u - C_d \\
\Delta &= \frac{C_u - C_d}{S(u-d)}
\end{aligned}$$

Plug Δ into the first equation

$$\begin{aligned}
\Delta uS + BR &= C_u \\
\frac{C_u - C_d}{S(u-d)} uS + BR &= C_u \\
\frac{C_u - C_d}{(u-d)} u + BR &= C_u \\
C_u u - C_d u + BR(u-d) &= C_u(u-d) \\
C_u u - C_d u + BR(u-d) &= C_u u - C_u d \\
BR(u-d) &= C_u u - C_u d - C_u u + C_d u \\
BR(u-d) &= C_d u - C_u d \\
B &= \frac{C_d u - C_u d}{R(u-d)}
\end{aligned}$$

Plug in Δ and B into the equation for C

$$\begin{aligned}
C &= \Delta S + B \\
C &= \frac{C_u - C_d}{S(u-d)} S + \frac{C_u u - C_u d}{R(u-d)} \\
C &= \frac{C_u - C_d}{S(u-d)} S \frac{R}{R} + \frac{C_u u - C_u d}{R(u-d)} \\
C &= \frac{RC_u - RC_d}{R(u-d)} + \frac{C_u u - C_u d}{R(u-d)} \\
C &= \frac{RC_u - RC_d + C_u u - C_u d}{R(u-d)} \\
C &= C_u \frac{R-d}{R(u-d)} + C_d \frac{u-R}{R(u-d)}
\end{aligned}$$

3.6 Example

Given the following 2 period binomial model

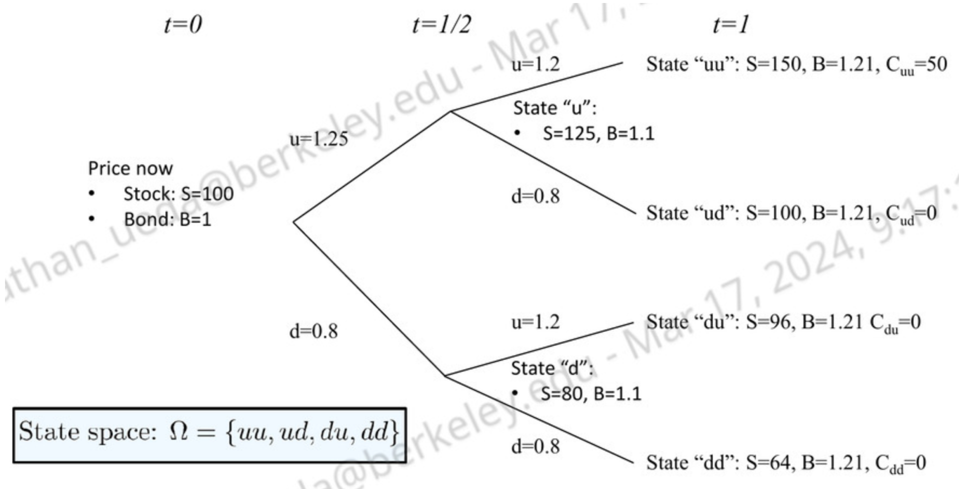

Figure 13: Two period binomial model

Approach: To analyze this, the easiest method would be to split it into a series of single period models. For the structure of this tree, that would be 3 single period models. Solving for each of these will give us the unique value of the call option at time $t = 0$.

We start by analyzing the dynamics from $t = 1/2$ to $t = 1$ after an up move.

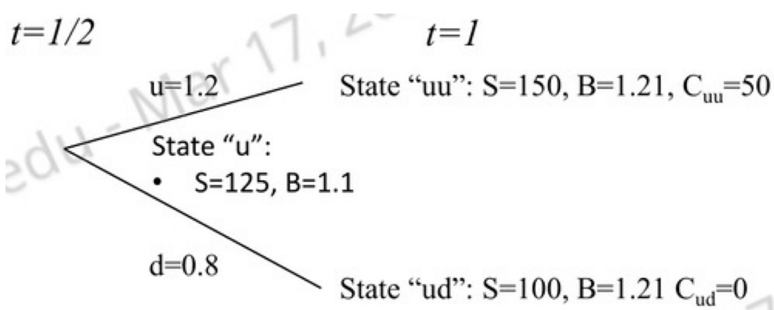

Figure 14: The one period model at time $t = 1/2$ to $t = 1$ after an up move

$$\boldsymbol{s}^0 = \begin{bmatrix} 125 \\ 1 \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} 150 & 100 \\ 1.1 & 1.1 \end{bmatrix}$$

1. Calculate the risk neutral probability:

$$q_u = \frac{R-d}{u-d} = \frac{1.1-0.8}{1.2-0.8} = \frac{3}{4}$$

2. Calculate the price of the call option after the up move, C_u at time $t = 1/2$

$$C_u = \frac{1}{R}(C_{uu}q_u + C_{ud}(1-q_u)) = \frac{1}{1.1}(50 \times 0.75 + 0 \times 0.25) = 34.1$$

3. Calculating the replicating portfolio
   Approach: Since we just had an up move, we are now calculating this at time $t = 1/2$. The replicating portfolio here states how to replicate a portfolio that has payments at $t = 1$ of $C_{uu} = 50$ after two up total up-moves and $C_{ud} = 0$ after an up-move followed by a down-move.

   Recall the formulas to calculate the amount of shares and bonds.

$$\Delta = \frac{C_u - C_d}{S(u - d)}$$

$$B = \frac{C_d u - C_u d}{R(u - d)}$$

   Apply these to our scenario and solve

$$\Delta_u = \frac{C_{uu} - C_{ud}}{S(u - d)} = \frac{50 - 0}{125 \times (1.2 - 0.8)} = \frac{50}{50} = 1$$

$$B_u = \frac{C_{ud} u - C_{uu} d}{R(u - d)} = \frac{(0 \times 1.2) - (50 \times 0.8)}{1.1(1.2 - 0.8)} = -\frac{40}{.44} = -90.91$$

Second, we will analyzing the dynamics from $t = 1/2$ to $t = 1$ after a down move.

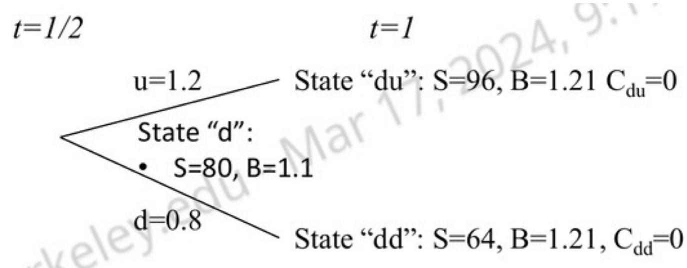

Figure 15: The one period model at time $t = 1/2$ to $t = 1$ after a down move

$$\boldsymbol{s}^0 = \begin{bmatrix} 80 \\ 1 \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} 96 & 64 \\ 1.1 & 1.1 \end{bmatrix}$$

1. Calculate the risk neutral probability:

$$q_u = \frac{R - d}{u - d} = \frac{1.1 - 0.8}{1.2 - 0.8} = \frac{3}{4}$$

2. Calculate the price of the call option after the down move, C_d at time $t = 1/2$

$$C_d = \frac{1}{R}(C_{du} q_u + C_{dd}(1 - q_u)) = \frac{1}{1.1}(0 \times 0.75 + 0 \times 0.25) = 0$$

3. Calculating the replicating portfolio
   Approach: Since we just had a down move, we are now calculating this at time $t = 1/2$. The replicating portfolio here states how to replicate a portfolio that has payments at $t = 1$ of $C_{du} = 0$ after a down move followed up an up move and $C_{dd} = 0$ after two down moves.

Recall the formulas to calculate the amount of shares and bonds.

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$B = \frac{C_d u - C_u d}{R(u-d)}$$

Apply these to our scenario and solve

$$\Delta_d = \frac{C_{du} - C_{dd}}{S(u-d)} = \frac{0-0}{80 \times (1.2-0.8)} = 0$$

$$B_d = \frac{C_{dd}u - C_{du}d}{R(u-d)} = \frac{(0 \times 1.2) - (0 \times 0.8)}{1.1(1.2-0.8)} = 0$$

Lastly, we will analyzing the dynamics at time $t = 0$.

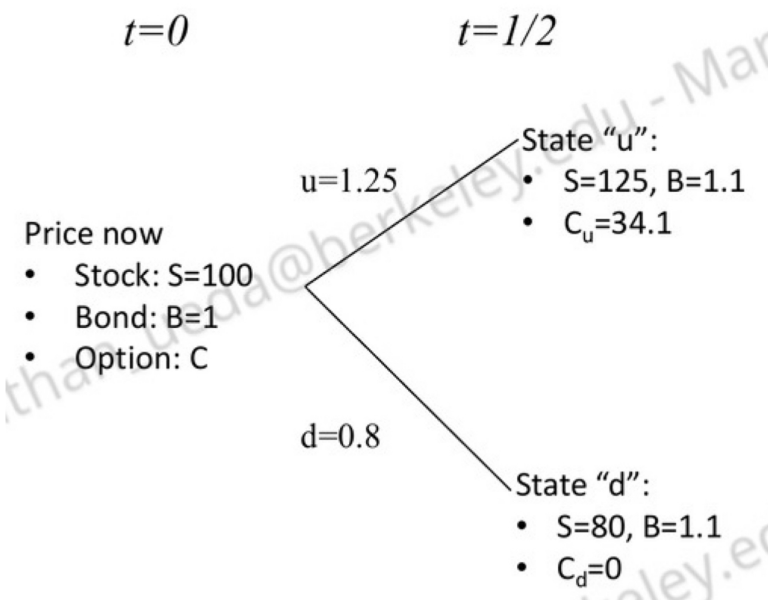

Figure 16: The one period model at time $t = 0$ to $t = 1/2$

$$\boldsymbol{s}^0 = \begin{bmatrix} 100 \\ 1 \end{bmatrix}, \; \boldsymbol{D} = \begin{bmatrix} 125 & 80 \\ 1.1 & 1.1 \end{bmatrix}$$

1. Calculate the risk neutral probability:

$$q_u = \frac{R-d}{u-d} = \frac{1.1-0.8}{1.25-0.8} = \frac{2}{3}$$

2. Calculate the price of the call option, C at time $t = 0$

$$C = \frac{1}{R}(C_u q_u + C_d(1-q_u)) = \frac{1}{1.1}(34.1 \times \frac{2}{3} + 0 \times \frac{1}{3}) = 20.67$$

3. Calculating the replicating portfolio
   Approach: Since we haven't moved yet, we are calculating this at time $t = 0$. The replicating portfolio here states how to replicate a portfolio that has payments at $t = 1/2$ of $C_u = 34.1$ after an up-move and $C_d = 0$ after a down-move.

   Recall the formulas to calculate the amount of shares and bonds.

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$B = \frac{C_d u - C_u d}{R(u-d)}$$

   Apply these to our scenario and solve

$$\Delta = \frac{C_u - C_d}{S(u-d)} = \frac{34.1 - 0}{100 \times (1.25 - 0.8)} = \frac{34.1}{45} = 0.758$$

$$B = \frac{C_d u - C_u d}{R(u-d)} = \frac{(0 \times 1.25) - (34.1 \times 0.8)}{1.1(1.25 - 0.8)} = -\frac{27.28}{.495} = -55.11$$

In contrast to an incomplete market where, for the initial call we are only able to get bounds for the option price, with a (dynamically) complete market we can nail down an exact figure.

3.7 Example

Given the following two period European put

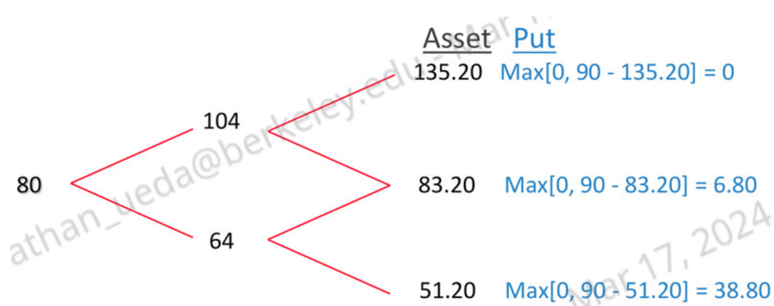

Figure 17: Two period European put labeled with the value of the stock, S

$$S = 80,\ K = 90,\ R = 1.1,\ u = 1.3,\ d = 0.8$$

Recall we are only able to exercise at the end.

Approach: Start and the end and work backwards through the tree.

$$Su = 80 \times 1.3 = 104$$

$$Sd = 80 \times 0.8 = 64$$

Put Payoff: $\text{Max}\{0, K - S\}$

Therefore at the terminals, we have

$$P_{uu} = \text{Max}\{0, 90 - 135.20\} = 0$$
$$P_{ud} = P_{du} = \text{Max}\{0, 90 - 83.20\} = 6.80$$
$$P_{dd} = \text{Max}\{0, 90 - 51.20\} = 38.80$$

With this information, we may now price the puts at the period prior, P_u, P_d, P, using the risk neutral formula $q_u = \frac{R-d}{u-d}$

$$q_u = \frac{R-d}{u-d} = \frac{1.1-0.8}{1.3-0.8} = 0.6$$

We can calculate the prices of P_u, P_d

$$P_u = \frac{1}{R}(q_u P_{uu} + (1-q_u)P_{ud}) = \frac{1}{1.1}(0.6 \times 0 + 0.4 \times 6.8) = 2.473$$

$$P_d = \frac{1}{R}(q_u P_{du} + (1-q_u)P_{dd}) = \frac{1}{1.1}(0.6 \times 6.80 + 0.4 \times 38.80) = 17.818$$

Finally, we can use P_u, P_d to price P.

$$P = \frac{1}{R}(q_u P_u + (1-q_u)P_d) = \frac{1}{1.1}(0.6 \times 2.473 + 0.4 \times 17.818) = 7.828$$

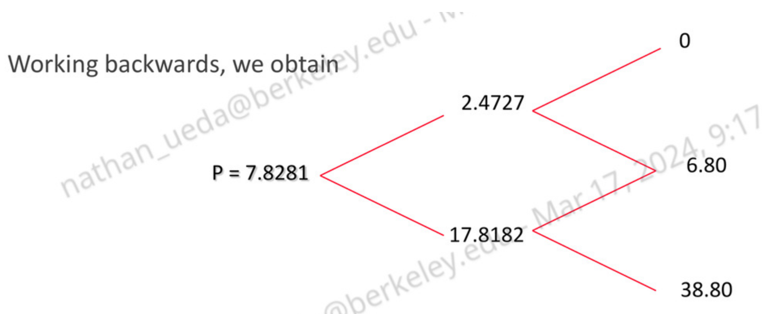

Figure 18: Two period European put labeled with the value of the puts

With an American put, we have the opportunity to exercise early. So when moving backwards, we need to check if it is more valuable to exericse the option early.

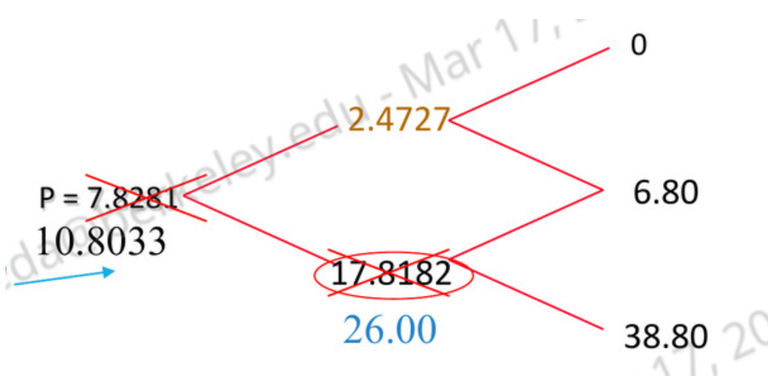

Figure 19: Two period American put labeled with the value of the puts

At time $t = 1$

$$P_u = \max\{2.473, \max\{K - S\}\} = \max\{2.473, \max\{90 - 104\}\} = 2.473$$

$$P_d = \max\{17.818, \max\{K - S\}\} = \max\{17.818, \max\{90 - 64\}\} = 26$$

Therefore, the new value of P, taking into consideration the updated value of P_u is

$$P = \frac{1}{R}(q_u P_u + (1 - q_u)P_d) = \frac{1}{1.1}(0.6 \times 2.473 + 0.4 \times 26) = 10.803$$

3.8 Example

The tree below is relevant to a prior example and is used for illustrative purposes to verify payoffs. In the prior example, we had

$$\boldsymbol{s}_0 = \begin{bmatrix} 100 \\ 1 \end{bmatrix},\ C = 20.67,\ \boldsymbol{h}^0 = \begin{bmatrix} \Delta^0 \\ B^0 \end{bmatrix} = \begin{bmatrix} 0.7578 \\ -55.1 \end{bmatrix}$$

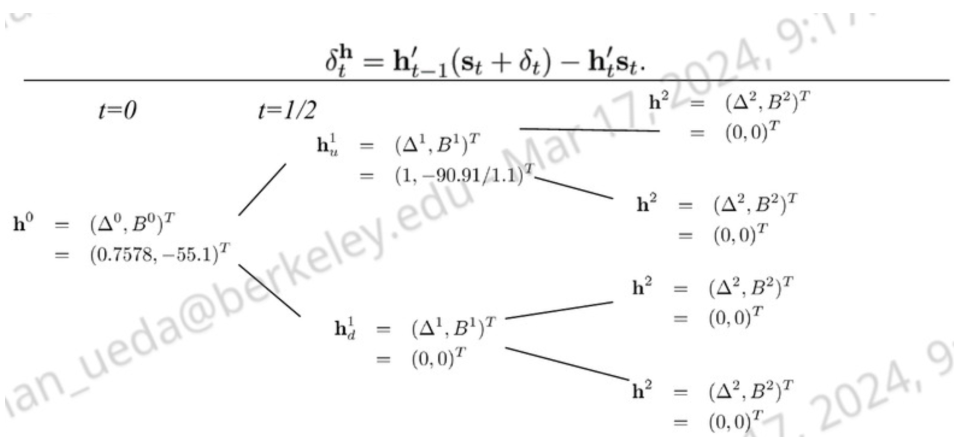

Figure 20: Multiperiod security market

By convention

$$\boldsymbol{h}_{-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can use

$$\delta_t^{\boldsymbol{h}} = \boldsymbol{h}'_{t-1}(\boldsymbol{s}_t + \delta_t) - \boldsymbol{h}'_t \boldsymbol{s}_t$$

Since there are no dividends, we can reduce the equation to

$$\delta_0^{\boldsymbol{h}} = \boldsymbol{h}'_{-1}\boldsymbol{s}_0 - \boldsymbol{h}'_0\boldsymbol{s}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.7578 & -55.1 \end{bmatrix} \begin{bmatrix} 100 \\ 1 \end{bmatrix} = -20.67$$

This states that, to hold $\boldsymbol{h}'_0\boldsymbol{s}_0$ at time $t = 0$, we need to pay 20.67 at time $t = -1$.

3.9 Example

Determine which of the following four processes are adapted w.r.t. to the filtration below.

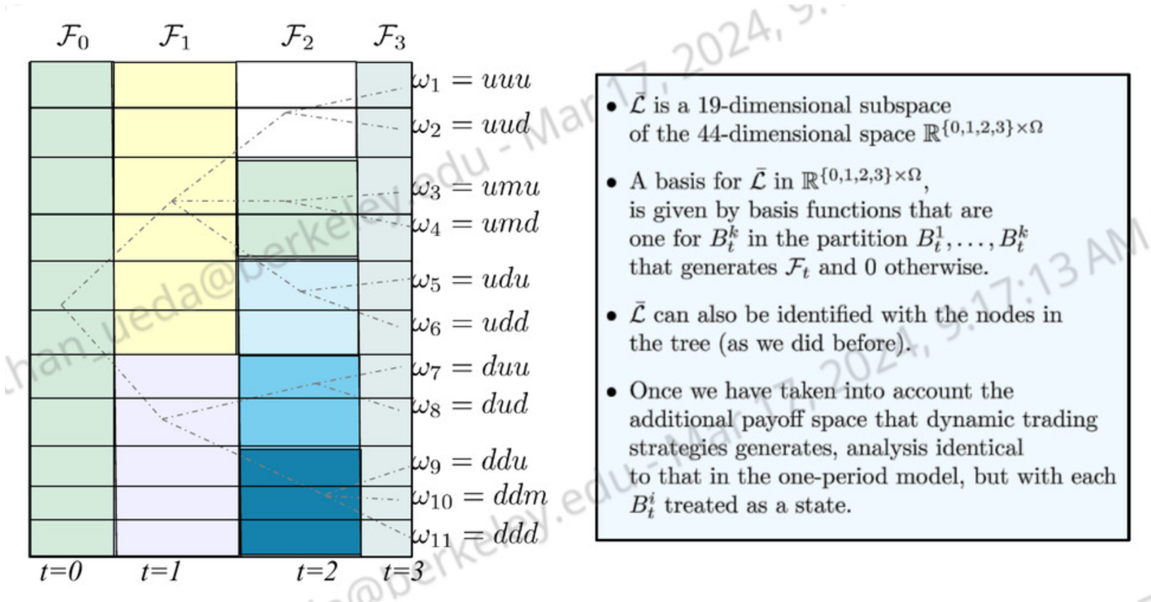

Figure 21: The augmented payoff space is 19 dimensional

$$f_t : \Omega \to \mathbb{R}$$

1. $f(t, \omega_i) = i + t$
   Here, we are at time t in state ω_i. The answer is no and an example to show this is at $t = 1$, with states ω_1 and ω_2. Regarding these 2 states, all we know at $t = 1$ is they are in the same filtration (i.e. both should evaluate to the same value since they are part of the same filtration). However, upon evaluation we get different values
   $$f(1, \omega_1) = 1 + 1 = 2$$
   $$f(1, \omega_2) = 2 + 1 = 3$$
   $$f(1, \omega_2) \neq f(1, \omega_1)$$

2. $f(t, \omega_i) = t$
   The answer here is yes since it is adaptive.

   For the next 2 questions, let χ be an indicator r.v. (has value of 1 or 0 when its corresponding statement is true)

3. $f(t, \omega_i) = (t^2 - t)\chi_{i \geq 5}$
   The answer here is yes. When $= 0$, every state is part of the same filtration which checks out and the same goes for when $t = 1$. At $t = 2$, the line we can drawn between the indicator r.v. giving us a 1 or a 0 is the line between ω_4 and ω_5 and we notice no blocks are being split up.

4. $f(t, \omega_i) = (t^2 - t)\chi_{i \geq 6}$ The answer here is no. At $t = 2$ the line between the indicator r.v. is drawn between ω_5 and ω_6, splitting up two states that should be together at $t = 2$.

4 Continuous Time Models Examples

5 Itôs Lemma Examples

5.1 Example

Calculate the Itô integral $\oint W_t \, dW_t$
The problem defines $dX_t = dW_t$ and $Y_t = g(t, X_t) = X_t^2$, which means $Y_t = W_t^2$

1. Calculate the partial derivatives

$$g_t(t, X_t) = 0$$
$$g_x(t, X_t) = 2X_t$$
$$g_{xx}(t, X_t) = 2$$

2. Plug these partials into Itôs Lemma

$$dY_t = g_t \, dt + g_x \, dX_t + \frac{1}{2} g_{xx} (dX_t)^2$$
$$dY_t = 0 \, dt + 2X_t \, dX_t + \frac{1}{2} 2 (dX_t)^2$$
$$dY_t = 2X_t \, dX_t + (dX_t)^2$$

3. Make the substitions for $X_t = W_t$ and $(dX_t)^2 = v^2 dt$, where in this case $v = 1$

$$dY_t = 2W_t \, dW_t + dt$$

5.2 Example

Let $X_t = 2 + t + e^{W_t}$. Calculate dX_t.

1. Define the function, here we have x as a function of t and W_t, $x(t, W_t)$.
2. Calculate the partial derivatives

$$x_t(t, W_t) = 1$$
$$x_w(t, W_t) = e^{W_t}$$
$$x_{ww}(t, W_t) = e^{W_t}$$

3. Plug these partials into Itôs Lemma

$$dX_t = x_t \, dt + x_w \, dW_t + \frac{1}{2} x_{ww} (dW_t)^2$$
$$dX_t = 1 \, dt + e^{W_t} \, dW_t + \frac{1}{2} e^{W_t} (dW_t)^2$$
$$dX_t = 1 \, dt + e^{W_t} \, dW_t + \frac{1}{2} e^{W_t} \, dt$$
$$dX_t = (1 + \frac{1}{2} e^{W_t}) \, dt + e^{W_t} \, dW_t$$

4. Note
$$X_0 = 2 + 0 + e^0 = 2 + 1 = 3$$
5. Then X_T is calculated as followed
$$X_T = X_0 + \oint dX_t = 3 + \oint (1 + \frac{1}{2}e^{W_t})\,dt + e^{W_t}\,dW_t$$

5.3 Example

Calculate the following stochastic integral

$$\oint_0^T t\,dW_t$$

Our goal here is to write it as an integral of something dt, $\oint$ something dt.
For this specific problem, we end up getting the same answer as we would using classical calculus (Since the dependence on W is linear, therefore its second derivative is 0 and the Ito term does not play a role).

1. $$g(t, W_t) = W_t t$$
2. Find the partials
$$g_t = W_t$$
$$g_w = t$$
$$g_{ww} = 0$$
3. Plug these partials into Itôs Lemma
$$dY_t = g_t\,dt + g_w\,dW_t + \frac{1}{2}g_{ww}(dW_t)^2$$
$$dY_t = W_t\,dt + t\,dW_t + \frac{1}{2}0(dW_t)^2$$
$$dY_t = W_t\,dt + t\,dW_t$$
4. $$\int_0^T dY = \oint W\,dt + \oint t\,dW = [tW]_0^T = TW_T$$

5.4 Example

Define $Y_t = e^{X_t}$, where $X_t = \mu t + \sigma W_t$ is a Brownian motion, i.e., Y_t is a geometric Brownian motion. Derive $E[Y_t]$ using Ito's lemma. You may wish to use the ODE formula: $y' = cy \to y_t = y_0 e^{ct}$. This problem transforms from ordinary Brownian Motion to geometric Brownian motion.

1. $Y_t = g(X_t) = e^{X_t}$

2. Find the partials

$$g_t = 0$$
$$g_x = e^x$$
$$g_{xx} = e^x$$

3. Plug these partials into Itôs Lemma

$$dY_t = g_t\,dt + g_x\,dX_t + \frac{1}{2}g_{xx}(dX_t)^2$$
$$dY_t = 0\,dt + e^x\,dX_t + \frac{1}{2}e^x(dX_t)^2$$
$$dY_t = e^x\,dX_t + \frac{1}{2}e^x(dX_t)^2$$
$$dY_t = e^x(\mu\,dt + \sigma\,dW) + \frac{1}{2}e^x\sigma^2\,dt$$
$$dY_t = e^x\mu\,dt + e^x\sigma\,dW + \frac{1}{2}e^x\sigma^2\,dt$$
$$dY_t = e^x((\mu + \frac{1}{2}\sigma^2)\,dt + \sigma\,dW)$$

4. Let $e^x = y_t$, let $\hat{\mu} = \mu + \frac{1}{2}\sigma^2$

$$dY_t = y_t(\hat{\mu}\,dt + \sigma\,dW)$$
$$\frac{dY_t}{y_t} = \hat{\mu}\,dt + \sigma\,dW$$

5. Now we find the expectation

$$Y_0 = e^0 = 1$$

$$E[Y_T] - Y_0 = E\left[\oint_0^T dY_t\right]$$
$$E[Y_T] - 1 = E\left[\oint_0^T y_t(\hat{\mu}\,dt + \sigma\,dW)\right]$$

We can now use the martingale property

$$E\left[\oint_0^T at\,dW_t\right] = 0$$

$$E[Y_T] - 1 = E\left[\oint_0^T y_t(\hat{\mu}\,dt + 0)\right]$$
$$E[Y_T] - 1 = E\left[\oint_0^T y_t\hat{\mu}\,dt\right]$$

We can now use linearity to move the expectation operator inside

$$E[Y_T] - 1 = \oint_0^T E[y_t]\hat{\mu}\,dt$$

Define $m_t = E[Y_t]$ to simplify the notation

$$m_T = m_0 + \hat{\mu} \oint_0^T m_t \, dt$$
$$m_T = 1 + \hat{\mu} \oint_0^T m_t \, dt$$

Now differentiate

$$m' = 0 + \hat{\mu} m$$
$$m' = \hat{\mu} m$$

Now use the ODE formula $y' = cy \rightarrow y_t = y_0 e^{ct}$ where, in our case, we have

$$m' = \hat{\mu} m \rightarrow m_t = 1 e^{\hat{\mu} t}$$

where we previously defined $\hat{\mu} = \mu + \frac{1}{2}\sigma^2$

$$1e^{\hat{\mu}t}$$
$$E[Y_t] = e^{(\mu + \frac{1}{2}\sigma^2)t}$$

5.5 Example

Calculate $d(e^X e^Y)$. Verify that the result is the same as what you get when you define $Z = X + Y$ and calculate $d(e^Z)$.

Recall that an Ito process X_t in correlated form (using V instead of W) is given by

$$dX_t = u \, dt + v \, dV_t$$

We also define:

$$(dV_x)(dV_y) = \rho_{xy} \, dt$$
$$\sigma_{xy} = \sigma_x \sigma_y \rho_{xy}$$

Therefore for our problem, we have

$$dX = \mu_x \, dt + \sigma_x \, dV_x$$

$$dY = \mu_y \, dt + \sigma_y \, dV_y$$

$$\begin{aligned} Z &= X + Y \\ dZ &= dX + dY \\ dZ &= \mu_x \, dt + \sigma_x \, dV_x + \mu_y \, dt + \sigma_y \, dV_y \\ dZ &= (\mu_x + \mu_y) \, dt + \sigma_x \, dV_x + \sigma_y \, dV_y \end{aligned}$$

Squaring dZ, we get

$$
\begin{aligned}
(dZ)^2 &= ((\mu_x + \mu_y)\,dt + \sigma_x\,dV_x + \sigma_y\,dV_y)^2 \\
(dZ)^2 &= (\sigma_x\,dV_x + \sigma_y\,dV_y)^2 \\
(dZ)^2 &= (\sigma_x\,dV_x + \sigma_y\,dV_y)(\sigma_x\,dV_x + \sigma_y\,dV_y) \\
(dZ)^2 &= \sigma_x^2(\,dV_x)^2 + 2\sigma_{xy}(\,dV_y)^2 + \sigma_y^2(\,dV_y)^2 \\
(dZ)^2 &= \sigma_x^2\,dt + 2\sigma_{xy}\,dt + \sigma_y^2\,dt \\
(dZ)^2 &= (\sigma_x^2 + 2\sigma_{xy} + \sigma_y^2)\,dt
\end{aligned}
$$

Now calculate $d(e^Z)$

1. $g(Z) = e^Z$
2. Find the partials

$$
\begin{aligned}
g_t &= 0 \\
g_z &= e^z \\
g_{zz} &= e^z
\end{aligned}
$$

3. Plug these partials into Itôs Lemma

$$
\begin{aligned}
dZ &= g_t\,dt + g_z\,dZ_t + \frac{1}{2}g_{zz}(dZ)^2 \\
dZ &= 0\,dt + e^z\,dZ_t + \frac{1}{2}e^z(dZ)^2 \\
dZ &= e^z\,dZ + \frac{1}{2}e^z(dZ)^2
\end{aligned}
$$

4. Plug in dZ and $(dZ)^2$

$$
\begin{aligned}
dZ &= e^z((\mu_x + \mu_y)\,dt + \sigma_x\,dV_x + \sigma_y\,dV_y) + \frac{1}{2}e^z(\sigma_x^2 + 2\sigma_{xy} + \sigma_y^2)\,dt \\
dZ &= e^z((\mu_x + \mu_y)\,dt + \sigma_x\,dV_x + \sigma_y\,dV_y) + \frac{1}{2}e^z(\sigma_x^2 + 2\sigma_x\sigma_y\rho_{xy} + \sigma_y^2)\,dt \\
dZ &= e^z((\mu_x + \mu_y)\,dt + \sigma_x\,dV_x + \sigma_y\,dV_y + \frac{1}{2}e^z(\sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y\rho_{xy})\,dt)
\end{aligned}
$$

Now caluclate $d(e^X e^Y)$ using the Ito version of the product rule

1. $d(e^X e^Y) = (de^x)e^y + e^x(de^y) + (de^x)(de^y)$

2. Solve the derivatives

$$\begin{aligned}
d(e^X e^Y) &= (de^x)e^y + e^x(de^y) + (de^x)(de^y) \\
d(e^X e^Y) &= e^x(\mu_x\,dt + \sigma_x\,dV_x + \frac{1}{2}\sigma_x^2\,dt)e^y \\
&\quad + e^x(\mu_y\,dt + \sigma_y\,dV_y + \frac{1}{2}\sigma_y^2\,dt)e^y \\
&\quad + e^x(\mu_x\,dt + \sigma_x\,dV_x + \frac{1}{2}\sigma_x^2\,dt)e^y(\mu_y\,dt + \sigma_y\,dV_y + \frac{1}{2}\sigma_x^2\,dt) \\
d(e^X e^Y) &= e^{x+y}((\mu_x + \mu_y)\,dt + \sigma_x\,dV_x + \sigma_y\,dV_y + \frac{1}{2}(\sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y\rho_{xy})\,dt) \\
d(e^X e^Y) &= e^z((\mu_x + \mu_y)\,dt + \sigma_x\,dV_x + \sigma_y\,dV_y + \frac{1}{2}(\sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y\rho_{xy})\,dt)
\end{aligned}$$

6 Continuous Time Trading Examples

6.1 Example

One stock with GBM (returns are i.i.d.) dynamics for prices and constant dividend yield α. A risk free asset with constant returns

$$\begin{aligned}
\frac{dS}{S} &= \hat{\mu}\,dt + \sigma\,dW,\ S_0 = 1 \\
\frac{dB}{B} &= r\,dt,\ B_0 = 1 \\
d\Theta &= (\alpha S\,dt, 0)^T,
\end{aligned}$$

where $\alpha > 0$ and $\hat{\mu}$ and σ are constant, and r is the instantaneous risk free rate. αS may be interpreted as the fraction of the stock price that we get in dividends over a year.

Also, recall

$$S_t = e^{\mu t + \sigma W_t}$$
$$B_t = B_0 e^{rt} = e^{rt}$$

Calculate the value process, V_t, and the payoff process, $F_t^{\boldsymbol{h}}$, for the following three trading strategies

1. $\boldsymbol{h}_t = (0, e^{-rt})^T$ (investing nothing in the stock and a decreasing amount in the bond)
2. $\boldsymbol{h}_t = (e^{\alpha t}, 0)^T$
3. $\boldsymbol{h}_t = (\frac{1}{\sigma St}, -\frac{1}{\sigma Bt})^T$, assuming $\alpha = 0$

- Part 1: Solving for $\boldsymbol{h}_t = (0, e^{-rt})^T$

- Solve for V_t

$$\begin{aligned} V_t &= \boldsymbol{h}'\boldsymbol{s}_t \\ &= \begin{bmatrix} 0 & e^{-rt} \end{bmatrix} \begin{bmatrix} S_t \\ e^{rt} \end{bmatrix} \\ &= 0 + e^0 V_t & = 1 \end{aligned}$$

- Solve for $F_t^{\boldsymbol{h}}$
  From above, we know $dV_t = 0$ and since $B = e^{rt}$, we know $dB = re^{rt}\,dt$.

$$\begin{aligned} dV + dF^{\boldsymbol{h}} &= \boldsymbol{h}'\,d\boldsymbol{s} + \boldsymbol{h}'\,d\Theta \\ 0 + dF^{\boldsymbol{h}} &= \begin{bmatrix} 0 & e^{-rt} \end{bmatrix}' \begin{bmatrix} dS \\ dB \end{bmatrix} + \begin{bmatrix} 0 & e^{-rt} \end{bmatrix} \begin{bmatrix} \alpha S\,dt, \\ 0 \end{bmatrix} \\ dF^{\boldsymbol{h}} &= \begin{bmatrix} 0 & e^{-rt} \end{bmatrix}' \begin{bmatrix} dS \\ re^{rt}\,dt \end{bmatrix} + \begin{bmatrix} 0 & e^{-rt} \end{bmatrix} \begin{bmatrix} \alpha S\,dt, \\ 0 \end{bmatrix} \\ dF^{\boldsymbol{h}} &= \begin{bmatrix} 0 & e^{-rt} \end{bmatrix}' \begin{bmatrix} dS \\ re^{rt}\,dt \end{bmatrix} + \begin{bmatrix} 0 & e^{-rt} \end{bmatrix} \begin{bmatrix} \alpha S\,dt, \\ 0 \end{bmatrix} \\ dF^{\boldsymbol{h}} &= re^0\,dt + 0 \\ dF^{\boldsymbol{h}} &= r\,dt \end{aligned}$$

- Part 2: $\boldsymbol{h}_t = (e^{\alpha t}, 0)^T$
- Solve for V_t

$$\begin{aligned} V_t &= \boldsymbol{h}'\boldsymbol{s}_t \\ &= \begin{bmatrix} e^{\alpha t} & 0 \end{bmatrix} \begin{bmatrix} e^{\mu t + \sigma W_t} \\ B_t \end{bmatrix} \\ &= e^{(\alpha+\mu)t + \sigma W_t} + 0 \\ V_t &= e^{(\alpha+\mu)t + \sigma W_t} \end{aligned}$$

- Solve for $F_t^{\boldsymbol{h}}$
  From above, we know $dV_t = 0$ and since $B = e^{rt}$, we know $dB = re^{rt}\,dt$.

$$\begin{aligned} dF^{\boldsymbol{h}} &= -d\boldsymbol{h}_t'(\boldsymbol{s}_t + d\boldsymbol{s}_t) + \boldsymbol{h}'\,d\Theta_t \\ dF^{\boldsymbol{h}} &= \begin{bmatrix} -\alpha e^{\alpha t}\,dt & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{s} + d\boldsymbol{s} \\ B + dB \end{bmatrix} + \begin{bmatrix} e^{\alpha t}\,dt & 0 \end{bmatrix} \begin{bmatrix} \alpha S\,dt \\ 0 \end{bmatrix} \\ dF^{\boldsymbol{h}} &= -\alpha e^{\alpha t}\,dt(\boldsymbol{s} + d\boldsymbol{s}) + 0 + e^{\alpha t}(\alpha S\,dt) + 0 \\ dF^{\boldsymbol{h}} &= -\alpha e^{\alpha t}\boldsymbol{s}\,dt - \alpha e^{\alpha t} d\boldsymbol{s} + e^{\alpha t}\alpha S\,dt \\ dF^{\boldsymbol{h}} &= -\alpha e^{\alpha t}\boldsymbol{s}\,dt + \alpha S e^{\alpha t}\,dt \end{aligned}$$

7 Appendix

7.1 Operator Overloading

For some vector $\boldsymbol{a}$, we write

- $\boldsymbol{a} \geq \mathbf{0}$ if all elements are nonnegative
- $\boldsymbol{a} > \mathbf{0}$ if at least one element is strictly positive
- $\boldsymbol{a} >> \mathbf{0}$ if all elements are strictly positive

Similarly,

- $\boldsymbol{a} \leq \mathbf{0}$ if all elements are nonpositive
- $\boldsymbol{a} < \mathbf{0}$ if at least one element is strictly negative
- $\boldsymbol{a} << \mathbf{0}$ if all elements are strictly negative