Introduction to Linear Algebra Notes

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Contents

1 Vectors and Matrices

1.1 Vectors and Linear Combinations

Vector Length: For a vector $v \in \mathbb{R}^n$, its length is:

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

In words, the length of a vector is the square root of the sum of the squared components.

Given two vectors in $\mathbb{R}^2$ v, w with their tail starting from the origin

- If they lie on the same line, the vectors are *linearly dependent*.
- If they do not lie on the same line, the vectors are *linearly independent*.

Therefore, the combinations $c\mathbf{v} + d\mathbf{w}$ fill the $x - y$ plane unless v is in line with w.

To fill m-dimensional space, we need m independent vectors, with each vector having m components.

1.2 Lengths and Angles from Dot Products

Dot Product: For two vectors $v, w \in \mathbb{R}^n$, their dot product is:

$$v \cdot w = v_1 w_1 + \cdots + v_n w_n$$

The dot product of two vectors tells us what amount of one vector goes in the direction of another. It tells us how much these vectors are working together.

- $v \cdot w > 0$: The vectors point in somewhat similar directions. In other words, the angle between the two vectors is less than 90 degrees.
- $v \cdot w = 0$: The vectors are perpendicular. In other words, the angle between the two vectors is 90 degrees.
- $v \cdot w < 0$: The vectors point in somewhat opposing directions. In other words, the angle between the two vectors is greater than 90 degrees.

Dot Product Rules (for two vectors, v, w):

- $\boldsymbol{v} \cdot \boldsymbol{w} = w \cdot v$
- $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$
- $(c\boldsymbol{v}) \cdot \boldsymbol{w} = c(\boldsymbol{v} \cdot \boldsymbol{w})$

Cosine Formula: If v and w are nonzero vectors, then:

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}$$

Unit Vectors: A vector is a unit vector if its length is 1.
For a vector $u \in \mathbb{R}^n$:

$$\|u\| = 1$$

For any vector $v \in \mathbb{R}^n$, as long as $v \neq 0$, dividing v by its length will result in a unit vector. In other words:

$$u = \frac{v}{\|v\|}$$

Cauchy-Schwarz Inequality:

$$|v \cdot w| \leq \|v\|\|w\|$$

In words, the absolute value of the dot product of two vectors is no greater than the product of their lengths.

Triangle Inequality:

$$\|v + w\| \leq \|v\| + \|w\|$$

In words, the length of any one side (in this case $\|v + w\|$) of a triangle is at most the sum of the length of the other triangle sides.

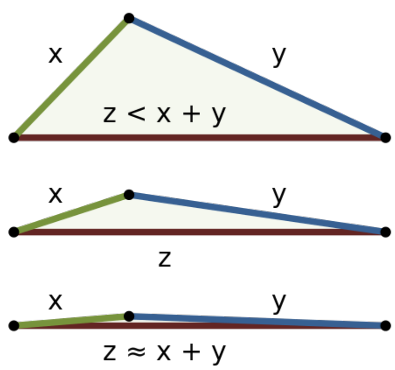

Figure 1: This Squeeze Theorem

1.3 Matrices and Their Column Spaces

Independence: Columns are independent when each new column is a vector that we don't already have as a combination of previous columns. The only combindation of columns that produces $A\boldsymbol{x} = (0, 0, 0)$ is $\boldsymbol{x} = (0, 0, 0)$.

Column Space: The column space, $\mathbf{C}(A)$, contains all vectors $A\boldsymbol{x}$. In other words, it contains all combinations of the columns.
The **span** of the columns of $\boldsymbol{A}$ is the column space.

Rank: The number of independent columns of a matrix. This is equivalent to saying the rank is the number of pivots in a matrix.

Rank Rules:

- $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$

1.4 Matrix Multiplication AB and CR

To multiply two matrices AB, take the dot product of each row of A with each column of B. The number in row i, column j of AB is (row i of A) $\cdot$ (column j or B).

When $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$:

- $AB \in \mathbb{R}^{m \times p}$
- mp dot products are needed to carry out the matrix multiplication (one for each entry in the matrix AB).

Matrix Multiplication Rules:

- Associative: $(\boldsymbol{AB})\boldsymbol{C}$
- Distributive: $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{BC}$
- Not Commutative: In general $\boldsymbol{AB} \neq \boldsymbol{BA}$

2 Solving Linear Equations Ax = b

2.1 Elimination and Back Substitution

For a matrix $A \in \mathbb{R}^{n \times n}$, there are three outcomes for $A\boldsymbol{x} = \boldsymbol{b}$:

1. No solution
   - $\boldsymbol{b}$ is not in the column space of A
   - This occurs when the columns of A are dependent and $\boldsymbol{b}$ is not in $C(A)$.

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 16 \\ \cancel{32} \end{bmatrix}$$

2. Exactly 1 solution
   - A has independent columns and an inverse matrix A^{-1}
3. Infinitely many solutions
   - Columns of A are dependent.
   - This occurs when the columns of A are dependent and $\boldsymbol{b}$ is in $C(A)$

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 32 \end{bmatrix}$$

Elimination: A system that allows us to determine if $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, 1 solution, or infinitely many solutions. The goal of elimination is to transform A to an upper triangular matrix, U.

Elimination allows us to discover the number of pivots in $A \in \mathbb{R}^{n \times n}$ by creating U. If there are n pivots in U, U has full rank. This implies A has exactly one solution.

Back Substitution: If U has full rank, back substituion allows us to find the solution.

2.2 Elimination Matrices and Inverse Matrices

Elimination

- The basic elimination step subtracts a multiple of ℓ_{ij} of equation j from equation i

Inverse Matrices

- If A is invertible, the one and only solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$
- Only square matrices can have inverses
- Invertible $\equiv$ non-singular $\equiv$ non-zero determinant $\equiv$ independent columns
- Not invertible $\equiv$ singular $\equiv$ zero determinant $\equiv$ dependent columns
- If a matrix is invertible, the inverse is unique
- A triangular matrix has an inverse so long that it has no zero diagonal entries

2.3 Matrix Computations and $\mathbf{A = LU}$

Elimination without row exchanges factors A into LU. We can't find an LU decomposition if row exchanges are needed during elimination.

Gauss-Jordan elimination:

- An algorithm that allows us to determine if the inverse of a matrix exists, and if it does it exist, it allows us to determine what the inverse is.
- Augment A by I, that is $[A\ I]$, and through elementary row operations, transform this matrix to $\left[I\ A^{-1}\right]$

3 The Four Fundamental Subspaces

3.1 Vector Spaces and Subspaces

To be a vector space means that all linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$ of the vectors or matrices stay inside that space.

Subspaces:

- A subspace is a vector space entirely contained within another vector space
- All linear combinations of vectors in the subspace stay in the subspace
- Every subspace contains the zero vector
- Subspaces of $\mathbb{R}^3$:
  - The single vector $(0, 0, 0)$
  - Any line through $(0, 0, 0)$
  - Any plane through $(0, 0, 0)$

- The whole space $\mathbb{R}^3$

Column Space ($C(A)$):

- The column space consists of all linear combinations of the columns
- To solve $A\boldsymbol{x} = \boldsymbol{b}$ is to express $\boldsymbol{b}$ as a combination of the columns
- The right side, $\boldsymbol{b}$, has to be in the column space produced by A, or $A\boldsymbol{x} = \boldsymbol{b}$ has no solution

Row Space ($C(A^T)$):

- The row space of A is the column space of A^T
- The rank of $A =$ rank of A^T

Span:

- The span of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is the set of all of their linear combinations.
- In other words, this tells us, given some set of vectors, which vectors are able to be created by taking a linear combination of the vectors in the set. It is the vector space we can reach (span) by taking linear combindations of the set of vectors.
- Independence is not required by the word span.

3.2 Computing the Nullspace by Elimination: A = CR

Nullspace ($N(A)$):

- Contains all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = 0$ including $\boldsymbol{x} = 0$
- If A is invertible, then the nullspace contains only the zero vector (no special solutions)
- If A has n columns, r of which are independent, then there are $n - r$ vectors in the nullspace
- Elimination does not change the nullspace (though it does change the column space)
- If $n > m$ (more columns than rows), then there is at least one free variable (can't have n independent columns in $\mathbb{R}^m$). Therefore, there is at least one nonzero solution
- The dimension of the nullspace is the number of free variables in a matrix

Echelon Form (R):

- Echelon form requirements:
  1. All rows having only zeros are at the bottom
  2. The leading entry (pivot) for each nonzero row (leftmost nonzero entry), is on the right of the leading entry of every row above
- The result of Gaussian elimination on any matrix, square or otherwise
- If the matrix is square and invertible, echelon form is an upper triangular matrix, U
- Can be viewed as a generalization of upper triangular form for rectangular matrices

- Example:

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form Requirements (R_0):

- In row echlon form
- The leading entry (pivot) of each nonzero row is a 1
- Each column containing a leading 1 has zeros in all entries above the leading 1.
- Example:

$$\begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Algorithm (must be in row echelon form):
  1. Divide pivots by themselves to make them all 1.
  2. Zero out all entires above and below the pivots.

  The result will have the identity matrix in the pivot columns (and the remaining columns will be the special columns)

Finding the Nullspace:
Goal: Given some matrix A, find all solutions $A\boldsymbol{x} = 0$

1. Start with some matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

2. Do elimination to echelon form

$$R = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Do elimination from row echelon form to reduced row echelon form to find the pivot columns and free columns

$$R_0 = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   Takeaways:

   - $r = 2 =$ number of pivots (coming from columns 1 and 3)
   - 2 free variables (coming from columns 2 and 4)
     - Free variables, which in this case are x_2 and x_4 can take on any value
     - There is one special solution for each free variable

- The special solution for each free variable is found by setting that free variable to 1 and the rest of the free variables to 0

4. Find the special solutions for the free variables using back substituion

   Recall our system of linear equations:

$$x_1 + 2x_2 - 2x_4 = 0$$

$$x_3 + 2x_4 = 0$$

   (a) Let $x_2 = 1, x_4 = 0$. Equation 2 gives us $x_3 = 0$ and equation 1 gives us $x_1 = -2$. The result is a vector in the nullspace:

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

   (b) Let $x_2 = 0, x_4 = 1$. Equation 2 gives us $x_3 = -2$ and equation 1 gives us $x_1 = 2$. The result is a vector in the nullspace:

$$s_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

   Takeaways:

   - These vectors form a basis for the nullspace of A.
   - Any multiple of these 2 vectors (special solutions) are in the nullspace.
   - The nullspace contains exactly all the linear combinations of the special solutions.
   - In other words, for any 2 constants c, d, the nullspace is:

$$cs_1 + ds_2$$

3.3 The Complete Solution to $\mathbf{Ax} = \mathbf{b}$