Probability and Statistics Notes

Nathan Ueda

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1 Introduction to Probability

1.1 The History of Probability

1.2 Interpretations of Probability

Probability Interpretations

- Frequency: If an experiment is carried out many times, the frequency with which a particular outcome occurred would define its probability.
- Classical: If an outcome of some experiment must be one of n different, equally likely outcomes, the probability of each outcome is $\frac{1}{n}$.
- Subjective: An entity assigns probabilities to each possible outcome.

Probability theory does not depend on intepretation.

1.3 Experiments and Events

Probability allows us to quantify how likely an outcome is to occur.

Experiments: Any process in which the possible outcomes can be identified ahead of time.
Events: A well defined set of possible outcomes of the experiment.

Although there is controvery in regard to the proper meaning and interpration of some of the probabilities that are assigned to the outcomes of many experiements, once these probabilities are assigned, there is complete agreemenet upon the mathematical theory of probability.

Almost all work in the mathematical theory of probability is related to:

- Methods for determining probabilities of certain events from given probabilities for each possible outcome in an experiment.
- Methods for revising probabilities of events when additional relevant information is obtained.

1.4 Set Theory

Sample Space: The collection of all possible outcomes of an experiment.
Empty Set: Subset of S containing no elements, denoted $\emptyset$, representing any events that cannot occur.
Complement: For some set A, its complement, denoted A^c, is the set containing all elements of S not in A.
Union: For n sets $A_1, \ldots, A_n$, their union, denoted $A_1 \cup \ldots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$, is defined as the set containing all outcomes that belong to at least one of these n sets.
Intersection: For n sets $A_1, \ldots, A_n$, their intersection, denoted $A_1 \cap \ldots \cap A_n$ or $\bigcap_{i=1}^{n} A_i$, is defined as the set containing the elements common to all these n sets.
Disjoint/Mutually Exclusive: Two sets A and B are disjoint/mutually exclusive if they have no outcomes in common, that is, if $A \cap B = \emptyset$, representing that both A and B cannot occur.

1.5 The Definition of Probability

Axioms of Probability:

1. For every event A, $P(A) \geq 0$
2. $P(S) = 1$
3. $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Basic Theorems:

1. $P(\emptyset) = 0$
2. For every finite sequence of n disjoint events, $A_1, \ldots, A_n$, $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$
3. For every event A, $P(A^c) = 1 - P(A)$
4. If $A \subset B$, then $P(A) \leq P(B)$
5. For every event A, $0 \leq P(A) \leq 1$
6. For every two events A and B, $P(A \cap B^c) = P(A) - P(A \cap B)$
7. For every two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
8. Bonferroni Inequality: For all events $A_1, \ldots, A_n$, $P(\bigcap_{i=1}^{n} A_i) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$

1.6 Finite Sample Spaces

Simple Sample Space

- Has a finite number (n) of possible outcomes
- Each outcome has an equal probability $(\frac{1}{n})$
- If an event A has m outcomes, then $P(A) = \frac{m}{n}$

1.7 Counting Methods

Multiplication Rule: An experiment with k parts where the ith part has n_i possible outcomes (regardless of which specific outcomes have occurred in the other parts) has a sample space $S = n_1 n_2 \ldots n_k$

Permutations ($P_{n,k}$):

- Number of ways to arrange a set (order matters)
- Sampling considering n different items and making k choices from them
  - Sampling with replacement: n^k
  - Sampling without replacement: $n(n-1)\ldots(n-k+1)$
    - n options for first choice, $n-1$ options for second choice, $n-k+1$ options for kth choice

- The number of permutations of n different items is $P_{n,n} = n!$
- The number of permutations of n different items making k choices ($0 \le k \le n$) is

$$P_{n,k} = n(n-1)\dots(n-k+1)$$

$$P_{n,k} = n(n-1)\dots(n-k+1)\left(\frac{1}{1}\right)$$

$$P_{n,k} = n(n-1)\dots(n-k+1)\left(\frac{(n-k)(n-k-1)\dots 1}{(n-k)(n-k-1)\dots 1}\right)$$

$$P_{n,k} = \frac{n(n-1)\dots(n-k+1)(n-k)(n-k-1)\dots 1}{(n-k)(n-k-1)\dots 1}$$

$$P_{n,k} = \frac{n!}{(n-k)!}$$

1.8 Combinatorial Methods

Combinations ($C_{n,k}$):

- Number of subsets (order does not matter)
- Permutations may be thought of as combinations of size k chosen out of n, multiplied by the number of ways to arrange the size k subsets, $k!$. More formally, this says

$$P_{n,k} = C_{n,k}k!$$

- Combinations (binomial coefficient) are the number of distinct subsets of size k that can be chosen from a set of size n:

$$C_{n,k} = \binom{n}{k} = \frac{P_{n,k}}{k!} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!k!}$$

- Combinations without replacement: $\binom{n+k-1}{k}$

1.9 Multinomial Coefficients

The total number of different ways of dividing n elements into k groups is

$$\binom{n}{n_1, n_2, \dots, n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{k-2}}{n_{k-1}} = \frac{n!}{n_1!n_2!\cdots n_k!}$$

1.10 The Probability of a Union of Events

- Union of two events A_1 and A_2:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

- Union of three events A_1, A_2, and A_3:

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - [P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3)]$$
$$+P(A_1 \cap A_2 \cap A_3)$$

- Union of n events $A_1, \ldots, A_n$:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) -$$

$$\sum_{i<j<k<l} P(A_i \cap A_j \cap A_k \cap A_l) + \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$