# Recognizing Sumsets is NP-Complete

Amir Abboud, Nick Fischer, Ron Safier, *Nathan Wallheimer*

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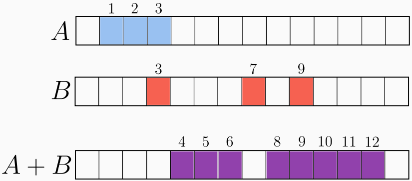
For $A, B \subseteq [0, M] := \{0, 1, \ldots, M\}$, let their sum be $A + B = \{a + b | a \in A, b \in B\}$.



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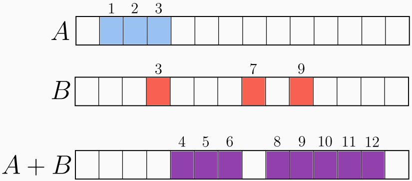
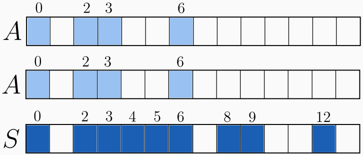



## Definition (Sumset)

A set $S \subseteq [0, M]$ is called a *sumset* if $S = A + A$ for some $A \subseteq [0, M]$.



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Given a set $S \subseteq [0, M]$ of size $n$, $M = poly(n)$, decide whether there exists a set $A \subseteq [0, M]$ such that $S = A + A$.

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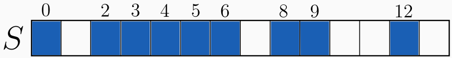
- Gain a better understanding of *factoring* problems.

# Sumset Recognition Algorithms



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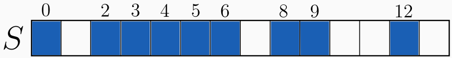


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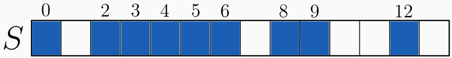


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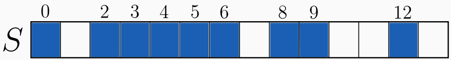


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**Can we do better?**

# Our Contribution: Recognizing Sumsets is NP-Complete

**Theorem (Reduction from 3-SAT)**

*There is a polynomial-time reduction that, given a 3-SAT formula $\phi$ with $n$ variables, outputs a set $S \subseteq [0, O(n^4)]$, such that $S$ is a sumset if and only if $\phi$ is satisfiable.*

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**Open Question:** Resolve the gap between $2^{O(n)}$ and $2^{\Omega(n^{1/4})}$.
**Remark:** The results also apply for Sumset Recognition over $\mathbb{F}_p^d$ for any prime $p$.

# Proof Outline

We are given a 3-SAT formula $\phi$ with $n$ variables and $m$ clauses.
**Goal:** Design a set $S := S(\phi)$, and sets $\{A_\alpha\}_{\alpha \in \{0,1\}^n}$, such that:

$$\phi \text{ is satisfiable} \quad \iff \quad S \text{ is a sumset.}$$

Moreover, $S = A_\alpha + A_\alpha$ for any satisfying assignment $\alpha \in \{0,1\}^n$.

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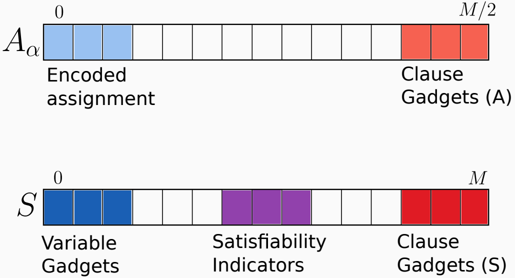

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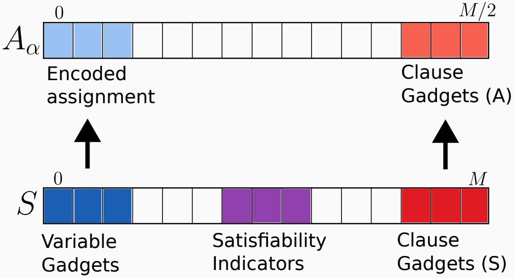

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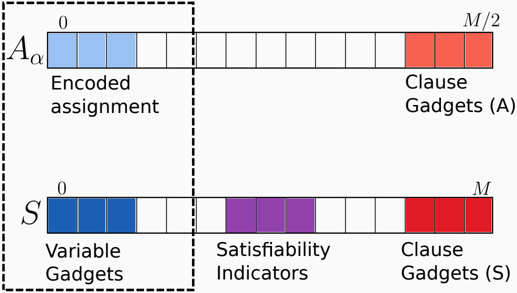
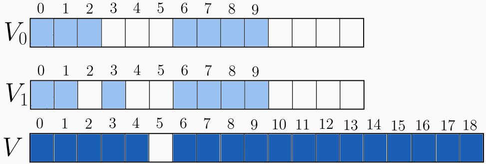


# Variable Gadgets

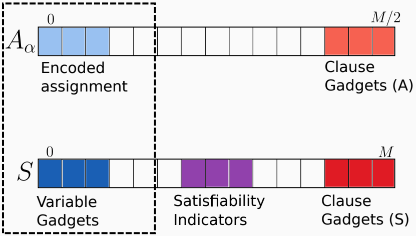
**Goal:** Design a constant sized set $V$ with only two representations: $V = V_0 + V_0$ and $V = V_1 + V_1$.

$$V_0 = \{0, 1, 2, 6, 7, 8, 9\}$$
$$V_1 = \{0, 1, 3, 6, 7, 8, 9\}$$
$$V = [0, 18] \setminus \{5\}$$




**Remark:** This is the smallest variable gadget.

# Variable Gadgets

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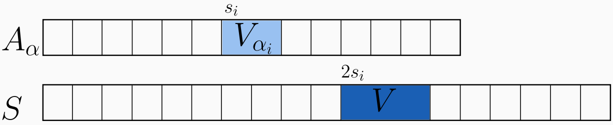

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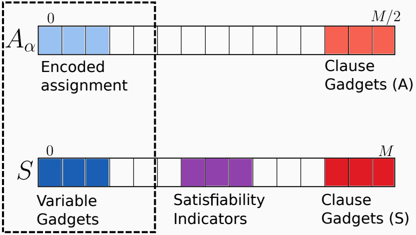


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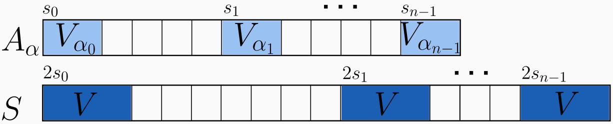

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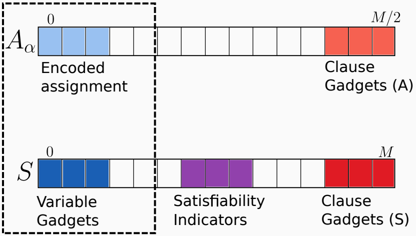


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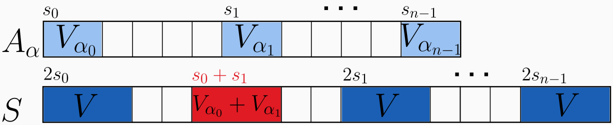
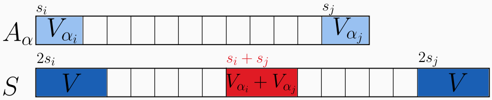



The cross-term $V_{\alpha_0} + V_{\alpha_1}$ is an unwanted by-product.

# Cross-Terms

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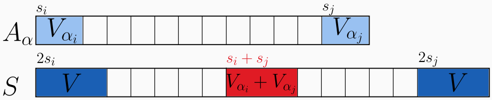


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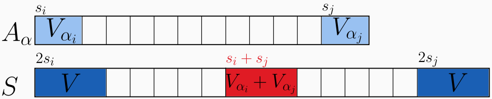


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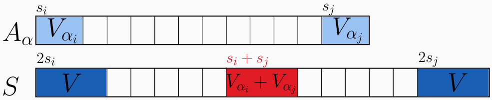


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2. Having $V_{\alpha_i} + V_{\alpha_j}$ in $S$ will make $S$ dependent on $\alpha$. The set $S$ has to be *oblivious* to $\alpha$.

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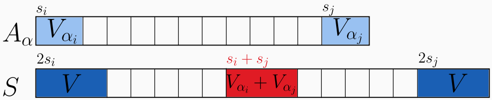


**Solution:** Mask the cross-terms with complete intervals. Let $I = [\min(V), \max(V)]$ and $R = [\frac{\min(V)}{2}, \frac{\max(V)}{2}]$, so $R + R = I$.

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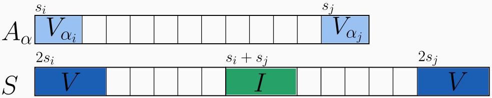


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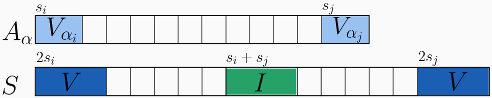


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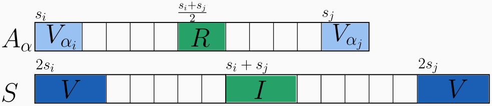


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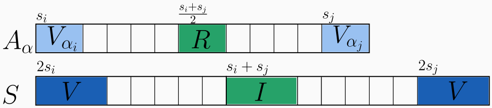


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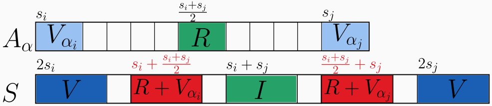


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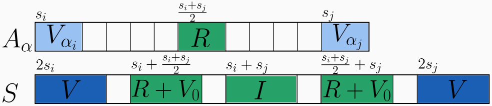


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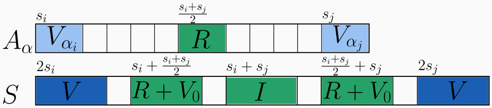


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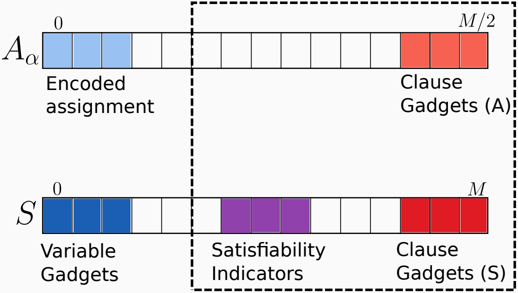
**Masking (informally)**

Whenever our design of $S$ and $A_\alpha$ fails because $A_\alpha + A_\alpha$ contains garbage terms, we can employ masking to fix it.

# Clause Gadgets

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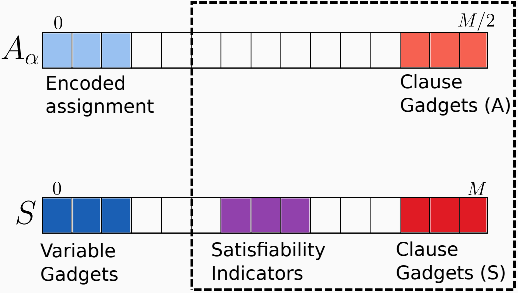


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**Remark:** To enforce $G_k$ into $A_\alpha$, let $S$ contain $G_k + G_k$.

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**Example (The gadget $G_k$)**

Let $C_k = (x_i \vee \bar{x}_j \vee x_\ell)$.

# Clause Gadgets

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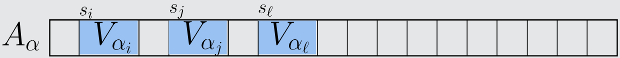
Let $C_k = (x_i \vee \bar{x}_j \vee x_\ell)$.

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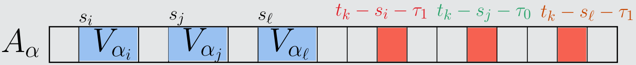


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Let $t_k \gg s_n$, and set:

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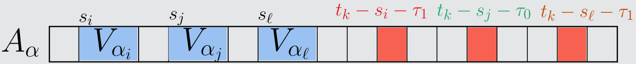


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## Observation

$A_\alpha + A_\alpha$ contains the set $(s_i + V_{\alpha_i}) + (t_k - s_i - \tau_1)$.

- $s_i$ and $-s_i$ cancel out.
- $-\tau_1$ cancels if and only if $V_{\alpha_i} = V_1$, i.e., if and only if $\alpha_i$ satisfies the clause.

Hence, $t_k$ is in the set if and only if $\alpha$ satisfies the clause.

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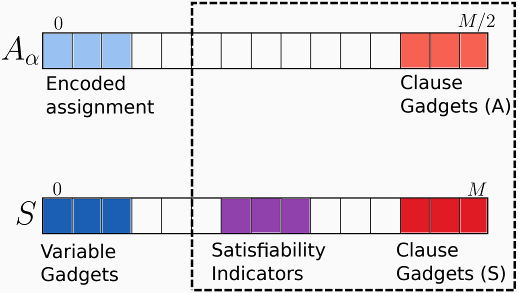


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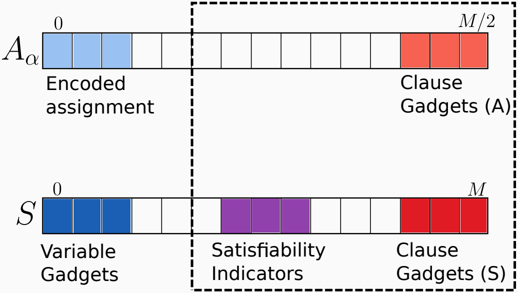


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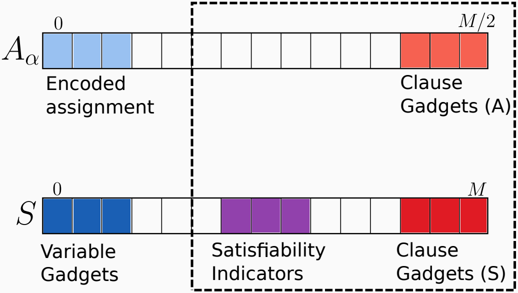


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**Problem:** How could $S$ enforce $G_k$ into $A_\alpha$?

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For any set $G \subseteq [0, M]$, the set $G^\star := G \cup \{4M\}$ is such that $G^\star + G^\star = R + R \iff R = G^\star$.

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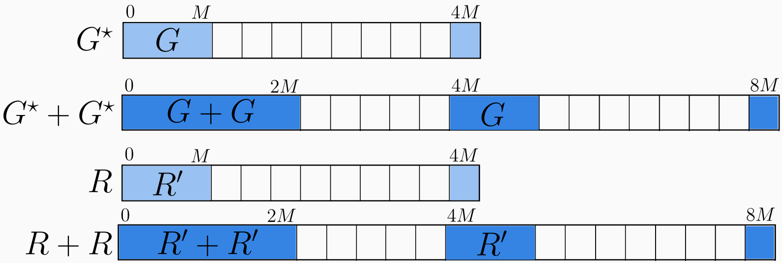
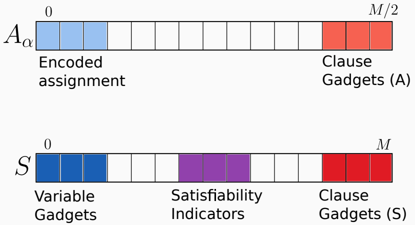


# Masking, Positioning, and Wrapping it all Together




Our actual construction is more modular and uses generalized masking and positioning:

- Masking: Can ignore garbage terms in $A_\alpha + A_\alpha$.
- Positioning: Can enforce that $S = A + A$ if and only if $A = A_\alpha$ for some assignment $\alpha$.

# Open Questions

- Average-case complexity?
- Close the gap between $2^{\Omega(n^{1/4})}$ and $2^{O(n)}$?
- Approximation versions? E.g. $\min_A |S \Delta (A + A)|$.