1 Stochastic Volatility Model

Define the model as follows.

$$\begin{cases} y_{t+1} = y_t + \mu\Delta + \sqrt{v_t\Delta}\epsilon^y_{t+1} \\ v_{t+1} = v_t + \kappa(\theta - v_t)\Delta + \sigma_v\sqrt{v_t\Delta}\epsilon^v_{t+1} \end{cases} \tag{1}$$

$$\begin{aligned} &\epsilon^y_{t+1} \sim N(0,1) \\ &\epsilon^v_{t+1} \sim N(0,1) \\ &corr(\epsilon^y_{t+1}, \epsilon^v_{t+1}) = \rho \end{aligned} \tag{2}$$

We have observations, $(y_t)_{t=0}^T$, latent volatility parameters, $(v_t)_{t=0}^T$, and model parameters, $\Theta = \{\mu, \kappa, \theta, \sigma_v, \rho\}$.

We can also write the model as follows.

$$\left(\begin{array}{c} y_{t+1} - y_t \\ v_{t+1} - v_t \end{array} \right) | v_t, \Theta \sim \mathcal{N}\left(\left[\begin{array}{c} \mu\Delta \\ \kappa(\theta - v_t)\Delta \end{array} \right], v_t\Delta \left[\begin{array}{cc} 1 & \sigma_v\rho \\ \sigma_v\rho & \sigma_v^2 \end{array} \right] \right) \tag{3}$$

$$\begin{aligned} &\prod_{t=0}^{T-1} p(y_{t+1} - y_t, v_{t+1} - v_t | v_t, \Theta) \\ =& \prod_{t=0}^{T-1} \frac{1}{2\pi v_t\Delta\sigma_v\sqrt{1-\rho^2}} \times exp\left[-\frac{1}{2(1-\rho^2)} \left(\frac{(y_{t+1} - y_t - \mu\Delta)^2}{v_t\Delta} - \frac{2\rho(y_{t+1} - y_t - \mu\Delta)(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)}{v_t\Delta\sigma_v} \right.\right. \\ &\left.\left. + \frac{(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)^2}{v_t\Delta\sigma_v^2} \right) \right] \end{aligned} \tag{4}$$

2 Prior Distributions

Consider the following prior distributions.

$$\begin{aligned} &\mu \sim N(0, F^2) \\ &\kappa \sim N(0, G^2) \text{ truncated at } 0 \\ &\theta \sim N(0, H^2) \text{ truncated at } 0 \end{aligned}$$

Following Jacquier, Polson, Rossi (1994), reparameterize (ρ, σ_v) as (ϕ_v, ω_v).

$$\begin{cases} \phi_v = \sigma_v\rho \\ \omega_v = \sigma_v^2(1-\rho^2) \end{cases} \tag{5}$$

Choose the following prior distributions.

$$\begin{aligned} \phi_v | \omega_v &\sim N(0, \frac{1}{2}\omega_v) \\ \omega_v &\sim IG(a, b) \end{aligned}$$

Note that:

$$\omega_v + \phi_v^2 = \sigma_v^2(1-\rho^2) + \sigma_v^2\rho^2 = \sigma_v^2 \tag{6}$$

$$\rho = \frac{\phi_v}{\sigma_v} = \frac{\phi_v}{\sqrt{\omega_v + \phi_v^2}} \tag{7}$$

To recover the original parameterization set:

$$\begin{cases} \sigma_v^2 = \omega_v + \phi_v^2 \\ \rho = \frac{\phi_v}{\sqrt{\omega_v + \phi_v^2}} \end{cases} \tag{8}$$

3 Joint Distribution

$$p(y, v, \Theta) = p(y, v|\Theta)p(\Theta) \tag{9}$$

$$p(\Theta) = p(\mu)p(\kappa)p(\theta)p(\phi_v|\omega_v)p(\omega_v) \tag{10}$$

$$= \left(\frac{1}{\sqrt{2\pi}F}e^{\frac{-(\mu-f)^2}{2F^2}}\right)\left(\frac{\frac{1}{G}\phi(\frac{\kappa-g}{G})}{1-\Phi(-\frac{\kappa}{G})}\right)1_{\kappa>0}\left(\frac{\frac{1}{G}\phi(\frac{\theta-g}{G})}{1-\Phi(-\frac{\theta}{G})}\right)1_{\theta>0}\left(\frac{1}{\sqrt{\pi\omega_v}}e^{\frac{-\phi_v^2}{\omega_v}}\right)\left(\frac{1}{b^a\Gamma(a)}\omega^{a-1}e^{\frac{-\omega_v}{b}}\right) \tag{11}$$

Refer to Equation (3) for $p(y, v|\Theta)$.

4 Posterior Distributions

4.1 Marginal Posterior Distribution of μ

$$p(\mu|\kappa, \theta, \sigma_v, \rho, y, v)$$

$$\propto p(y, v|\mu, \kappa, \theta, \sigma_v, \rho)p(\mu) \tag{12}$$

$$= \prod_{t=0}^{T-1} \frac{1}{2\pi v_t\Delta\sigma_v\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(y_{t+1}-y_t-\mu\Delta)^2}{v_t\Delta} - \frac{2\rho(y_{t+1}-y_t-\mu\Delta)(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)}{v_t\Delta\sigma_v}\right.\right. \tag{13}$$

$$\left.\left.+\frac{(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)^2}{v_t\Delta\sigma_v^2}\right)\right]\left(\frac{1}{\sqrt{2\pi}F}e^{\frac{-(\mu-f)^2}{2F^2}}\right)$$

$$\propto exp\left[-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{(y_{t+1}-y_t-\mu\Delta)^2}{v_t\Delta} - 2\rho\sum_{t=0}^{T-1}\frac{(y_{t+1}-y_t-\mu\Delta)(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)}{v_t\Delta\sigma_v}\right) - \frac{(\mu-f)^2}{2F^2}\right] \tag{14}$$

$$\text{Let } C_{t+1} = y_{t+1} - y_t, \text{ and } D_{t+1} = v_{t+1} - v_t - \kappa(\theta - v_t)\Delta \tag{15}$$

$$=exp\left[-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{(C_{t+1}-\mu\Delta)^2}{v_t\Delta}-2\rho\sum_{t=0}^{T-1}\frac{(C_{t+1}-\mu\Delta)D_{t+1}}{v_t\Delta\sigma_v}\right)-\frac{(\mu-f)^2}{2F^2}\right] \tag{16}$$

$$=exp\left[-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{C_{t+1}^2-2C_{t+1}\mu\Delta+(\mu\Delta)^2}{v_t\Delta}\right)+\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{C_{t+1}D_{t+1}-\mu\Delta D_{t+1}}{v_t\Delta\sigma_v}\right)-\frac{(\mu-f)^2}{2F^2}2\right] \tag{17}$$

$$\propto \exp\left[-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{-2C_{t+1}\mu\Delta+(\mu\Delta)^2}{v_t\Delta}\right)+\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{-\mu\Delta D_{t+1}}{v_t\Delta\sigma_v}\right)-\frac{\mu^2-2\mu f+f^2}{2F^2}\right] \tag{18}$$

$$=\exp\left[\left\{-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{\Delta^2}{v_t\Delta}\right)-\frac{1}{2F^2}\right\}\mu^2+\left\{-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{-2C_{t+1}\Delta}{v_t\Delta}\right)+\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{-\Delta D_{t+1}}{v_t\Delta\sigma_v}\right)+\frac{f}{F^2}\right\}\mu\right] \tag{19}$$

$$=\exp\left[-\frac{1}{2}\left\{\frac{\Delta}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{1}{v_t}\right)+\frac{1}{F^2}\right\}\mu^2+\left\{\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{2C_{t+1}}{v_t}\right)-\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{D_{t+1}}{v_t\sigma_v}\right)+\frac{f}{F^2}\right\}\mu\right] \tag{20}$$

$$\text{Let } W = \frac{\Delta}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{1}{v_t}\right)+\frac{1}{F^2} \tag{21}$$

$$\begin{aligned}\text{Let } S &= \frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{2C_{t+1}}{v_t}\right)-\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{D_{t+1}}{v_t\sigma_v}\right)+\frac{f}{F^2}\\ &= \frac{1}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{1}{v_t}\left[C_{t+1}-\rho\frac{D_{t+1}}{\sigma_v}\right]\right)+\frac{f}{F^2}\end{aligned} \tag{22}$$

$$=exp\left[-\frac{1}{2}W\mu^2+S\mu\right] \tag{23}$$

$$=exp\left[-\frac{1}{2}\left(W\mu^2-2S\mu\right)\right] \tag{24}$$

$$=exp\left[-\frac{1}{2}W\left(\mu^2-2\frac{S}{W}\mu\right)\right] \tag{25}$$

$$=exp\left[-\frac{1}{2}W\left(\mu^2-2\frac{S}{W}\mu+\left\{\frac{S}{W}\right\}^2-\left\{\frac{S}{W}\right\}^2\right)\right] \tag{26}$$

$$=exp\left[-\frac{1}{2}W\left(\left\{\mu-\frac{S}{W}\right\}^2-\left\{\frac{S}{W}\right\}^2\right)\right] \tag{27}$$

$$\propto exp\left[-\frac{1}{2}\frac{\left(\mu-\frac{S}{W}\right)^2}{\frac{1}{W}}\right] \tag{28}$$

$$\tag{29}$$

$\mu \sim N\left(\frac{S}{W},\frac{1}{W}\right)$, where W and S are defined in Equations (21) and (22).

4.2 Marginal Posterior Distribution of κ

$$
\begin{aligned}
& p(\kappa|\mu,\theta,\sigma_v,\rho,y,v) \\
\propto & p(y,v|\mu,\kappa,\theta,\sigma_v,\rho)p(\kappa) && (30)\\
= & \prod_{t=0}^{T-1} \frac{1}{2\pi v_t\Delta\sigma_v\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(y_{t+1}-y_t-\mu\Delta)^2}{v_t\Delta} - \frac{2\rho(y_{t+1}-y_t-\mu\Delta)(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)}{v_t\Delta\sigma_v}\right.\right. \\
& \left.\left. + \frac{(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)^2}{v_t\Delta\sigma_v^2}\right)\right]\left(\frac{2}{\sqrt{2\pi}}e^{-\kappa^2/2}\right)1_{\kappa>0} \\
\propto & exp\left[-\frac{1}{2(1-\rho^2)}\left(-2\rho\sum_{t=0}^{T-1}\frac{(y_{t+1}-y_t-\mu\Delta)(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)}{v_t\Delta\sigma_v} + \sum_{t=0}^{T-1}\frac{(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)^2}{v_t\Delta\sigma_v^2}\right) - \frac{\kappa^2}{2}\right]1_{\kappa>0} && (31)
\end{aligned}
$$

$$
\text{Let } C_{t+1} = y_{t+1}-y_t-\mu\Delta, \text{ and } D_{t+1} = v_{t+1}-v_t \qquad (32)
$$

$$
\begin{aligned}
= & exp\left[-\frac{1}{2(1-\rho^2)}\left(-2\rho\sum_{t=0}^{T-1}\frac{C_{t+1}(D_{t+1}-\kappa(\theta-v_t)\Delta)}{v_t\Delta\sigma_v} + \sum_{t=0}^{T-1}\frac{(D_{t+1}-\kappa(\theta-v_t)\Delta)^2}{v_t\Delta\sigma_v^2}\right) - \frac{\kappa^2}{2}\right]1_{\kappa>0} && (33)\\
= & exp\left[\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{C_{t+1}D_{t+1}-C_{t+1}\kappa(\theta-v_t)\Delta}{v_t\Delta\sigma_v}\right) - \frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{D_{t+1}^2-2D_{t+1}\kappa(\theta-v_t)\Delta+\kappa^2(\theta-v_t)^2\Delta^2}{v_t\Delta\sigma_v^2}\right) - \frac{\kappa^2}{2}\right] && (34)\\
\propto & exp\left[\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{-C_{t+1}\kappa(\theta-v_t)\Delta}{v_t\Delta\sigma_v}\right) - \frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{-2D_{t+1}\kappa(\theta-v_t)\Delta+\kappa^2(\theta-v_t)^2\Delta^2}{v_t\Delta\sigma_v^2}\right) - \frac{\kappa^2}{2}\right]1_{\kappa>0} && (35)\\
= & exp\left[\left\{-\frac{1}{2(1-\rho)^2}\left(\sum_{t=0}^{T-1}\frac{(\theta-v_t)^2\Delta^2}{v_t\Delta\sigma_v^2}\right) - \frac{1}{2}\right\}\kappa^2 + \left\{\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{-C_{t+1}(\theta-v_t)\Delta}{v_t\Delta\sigma_v}\right) - \frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{-2D_{t+1}(\theta-v_t)}{v_t\Delta\sigma_v^2}\right.\right.\right. && (36)\\
= & exp\left[-\frac{1}{2}\left\{\frac{\Delta}{\sigma_v^2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{(\theta-v_t)^2}{v_t}\right)+1\right\}\kappa^2 + \left\{\frac{1}{2\sigma_v^2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{2D_{t+1}(\theta-v_t)}{v_t}\right) - \frac{\rho}{\sigma_v(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{C_{t+1}(\theta-v_t}{v_t}\right.\right.\right. && (37)
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } W = & \frac{\Delta}{\sigma_v^2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{(\theta-v_t)^2}{v_t}\right)+1 && (38)\\
\text{Let } S = & \frac{1}{2\sigma_v^2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{2D_{t+1}(\theta-v_t)}{v_t}\right) - \frac{\rho}{\sigma_v(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{C_{t+1}(\theta-v_t)}{v_t}\right) \\
= & \frac{1}{\sigma_v(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{\theta-v_t}{v_t}\left[\frac{D_{t+1}}{\sigma_v}-\rho C_{t+1}\right]\right) && (39)
\end{aligned}
$$

$$=exp\left[-\frac{1}{2}W\kappa^2+S\kappa\right]1_{\kappa>0} \tag{40}$$

$$=exp\left[-\frac{1}{2}\left(W\kappa^2-2S\kappa\right)\right]1_{\kappa>0} \tag{41}$$

$$=exp\left[-\frac{1}{2}W\left(\kappa^2-2\frac{S}{W}\kappa\right)\right]1_{\kappa>0} \tag{42}$$

$$=exp\left[-\frac{1}{2}W\left(\kappa^2-2\frac{S}{W}\kappa+\left\{\frac{S}{W}\right\}^2-\left\{\frac{S}{W}\right\}^2\right)\right]1_{\kappa>0} \tag{43}$$

$$=exp\left[-\frac{1}{2}W\left(\left\{\kappa-\frac{S}{W}\right\}^2-\left\{\frac{S}{W}\right\}^2\right)\right]1_{\kappa>0} \tag{44}$$

$$\propto exp\left[-\frac{1}{2}\frac{\left(\kappa-\frac{S}{W}\right)^2}{\frac{1}{W}}\right]1_{\kappa>0} \tag{45}$$

$$\tag{46}$$

$$\kappa \sim N\left(\frac{S}{W},\frac{1}{W}\right)1_{\kappa>0}\text{ , where } W \text{ and } S \text{ are defined in Equations (38) and (39).}$$

4.3 Marginal Posterior Distribution of θ

$$\begin{aligned}
&p(\theta|\mu,\kappa,\sigma_v,\rho,y,v)\\
&\propto p(y,v|\mu,\kappa,\theta,\sigma_v,\rho)p(\theta) && (47)\\
&=\left\{\prod_{t=0}^{T-1}\frac{1}{2\pi v_t\Delta\sigma_v\sqrt{1-\rho^2}}\times exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(y_{t+1}-y_t-\mu\Delta)^2}{v_t\Delta}-\frac{2\rho(y_{t+1}-y_t-\mu\Delta)(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)}{v_t\Delta\sigma_v}+\frac{(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)^2}{v_t\Delta\sigma_v^2}\right)\right]\right\} && (48)\\
&\quad\times\left(\frac{2}{\sqrt{2\pi}}e^{-\theta^2/2}\right)1_{\theta>0}\\
&\propto exp\left[-\frac{1}{2(1-\rho^2)}\left(-2\rho\sum_{t=0}^{T-1}\frac{(y_{t+1}-y_t-\mu\Delta)(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)}{v_t\Delta\sigma_v}+\sum_{t=0}^{T-1}\frac{(v_{t+1}-v_t-\kappa(\theta-v_t)\Delta)^2}{v_t\Delta\sigma_v^2}\right)-\frac{\theta^2}{2}\right]1_{\theta>0} && (49)
\end{aligned}$$

$$\text{Let } C_{t+1}=y_{t+1}-y_t-\mu\Delta\text{, and } D_{t+1}=v_{t+1}-v_t \tag{50}$$

$$\begin{aligned}
&=exp\left[-\frac{1}{2(1-\rho^2)}\left(-2\rho\sum_{t=0}^{T-1}\frac{C_{t+1}(D_{t+1}-\kappa(\theta-v_t)\Delta)}{v_t\Delta\sigma_v}+\sum_{t=0}^{T-1}\frac{(D_{t+1}-\kappa(\theta-v_t)\Delta)^2}{v_t\Delta\sigma_v^2}\right)-\frac{\theta^2}{2}\right]1_{\theta>0} && (51)\\
&=exp\left[\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{C_{t+1}D_{t+1}-C_{t+1}\kappa(\theta-v_t)\Delta}{v_t\Delta\sigma_v}\right)-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{D_{t+1}^2-2D_{t+1}\kappa(\theta-v_t)\Delta+\kappa^2(\theta-v_t)^2\Delta^2}{v_t\Delta\sigma_v^2}\right)-\frac{\theta^2}{2}\right]1_{\theta>0} && (52)\\
&\propto exp\left[\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{-C_{t+1}\kappa(\theta-v_t)\Delta}{v_t\Delta\sigma_v}\right)-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{-2D_{t+1}\kappa(\theta-v_t)\Delta+\kappa^2(\theta-v_t)^2\Delta^2}{v_t\Delta\sigma_v^2}\right)-\frac{\theta^2}{2}\right]1_{\theta>0} && (53)\\
&=exp\left[\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{-C_{t+1}\kappa\Delta\theta+C_{t+1}\kappa\Delta}{v_t\Delta\sigma_v}\right)-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{-2D_{t+1}\kappa\Delta\theta+2D_{t+1}v_t\Delta+\kappa^2\Delta^2\theta^2-2\kappa^2\Delta^2\theta v_t+\kappa^2\Delta^2v_t^2}{v_t\Delta\sigma_v^2}\right)-\frac{\theta^2}{2}\right]1_{\theta>0} && (54)\\
&\propto exp\left[\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{-C_{t+1}\kappa\Delta\theta}{v_t\Delta\sigma_v}\right)-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{-2D_{t+1}\kappa\Delta\theta+\kappa^2\Delta^2\theta^2-2\kappa^2\Delta^2\theta v_t}{v_t\Delta\sigma_v^2}\right)-\frac{\theta^2}{2}\right]1_{\theta>0} && (55)\\
&=exp\left[\left\{-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{\kappa^2\Delta^2}{v_t\Delta\sigma_v^2}\right)-\frac{1}{2}\right\}\theta^2+\left\{\frac{\rho}{1-\rho^2}\left(\sum_{t=0}^{T-1}\frac{-C_{t+1}+\kappa\Delta}{v_t\Delta\sigma_v}\right)-\frac{1}{2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{-2D_{t+1}\kappa\Delta-2\kappa^2\Delta^2v_t}{v_t\Delta\sigma_v^2}\right)\right\}\theta\right]1_{\theta>0} && (56)\\
&=exp\left[-\frac{1}{2}\left\{\frac{\kappa^2\Delta}{\sigma_v^2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{1}{v_t}\right)+1\right\}\theta^2+\left\{\frac{1}{2\sigma_v^2(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{2D_{t+1}\kappa+2\kappa^2\Delta v_t}{v_t}\right)-\frac{\rho}{\sigma_v(1-\rho^2)}\left(\sum_{t=0}^{T-1}\frac{C_{t+1}\kappa}{v_t}\right)\right\}\theta\right]1_{\theta>0} && (57)
\end{aligned}$$

$$\text{Let } W = \frac{\kappa^2 \Delta}{\sigma_v^2(1-\rho^2)} \left(\sum_{t=0}^{T-1} \frac{1}{v_t}\right) + 1 \tag{58}$$

$$\begin{aligned}\text{Let } S &= \frac{1}{2\sigma_v^2(1-\rho^2)} \left(\sum_{t=0}^{T-1} \frac{2D_{t+1}\kappa + 2\kappa^2 \Delta v_t}{v_t}\right) - \frac{\rho}{\sigma_v(1-\rho^2)} \left(\sum_{t=0}^{T-1} \frac{C_{t+1}\kappa}{v_t}\right) \\ &= \frac{\kappa}{\sigma_v(1-\rho^2)} \left(\sum_{t=0}^{T-1} \frac{1}{v_t}\left[\frac{D_{t+1} + \kappa \Delta v_t}{\sigma_v} - \rho C_{t+1}\right]\right)\end{aligned} \tag{59}$$

$$= exp\left[-\frac{1}{2} W\theta^2 + S\theta\right] 1_{\theta>0} \tag{60}$$

$$= exp\left[-\frac{1}{2} \left(W\theta^2 - 2S\theta\right)\right] 1_{\theta>0} \tag{61}$$

$$= exp\left[-\frac{1}{2} W \left(\theta^2 - 2\frac{S}{W}\theta\right)\right] 1_{\theta>0} \tag{62}$$

$$= exp\left[-\frac{1}{2} W \left(\theta^2 - 2\frac{S}{W}\theta + \left\{\frac{S}{W}\right\}^2 - \left\{\frac{S}{W}\right\}^2\right)\right] 1_{\theta>0} \tag{63}$$

$$= exp\left[-\frac{1}{2} W \left(\left\{\theta - \frac{S}{W}\right\}^2 - \left\{\frac{S}{W}\right\}^2\right)\right] 1_{\theta>0} \tag{64}$$

$$\propto exp\left[-\frac{1}{2} \frac{\left(\theta - \frac{S}{W}\right)^2}{\frac{1}{W}}\right] 1_{\theta>0} \tag{65}$$

$$\tag{66}$$

$$\theta \sim N\left(\frac{S}{W}, \frac{1}{W}\right) 1_{\theta>0} \text{ , where } W \text{ and } S \text{ are defined in Equations (58) and (59).}$$

4.4 Joint Posterior Distribution of (ϕ_v, ω_v)

$$\begin{aligned}&p(\phi_v, \omega_v | \mu, \kappa, \theta, y, v) \\ &\propto p(y, v | \mu, \kappa, \theta, \phi_v, \omega_v) p(\phi_v | \omega_v) p(\omega_v)\end{aligned} \tag{67}$$

Using the result in Equation (7)

$$= \left\{\prod_{t=0}^{T-1} \frac{1}{2\pi v_t \Delta \sqrt{\omega_v + \phi_v^2} \sqrt{1 - \frac{\phi_v^2}{\omega_v + \phi_v^2}}}\right. \tag{68}$$

$$\left.\times exp\left[-\frac{1}{2\left(1 - \frac{\phi_v^2}{\omega_v + \phi_v^2}\right)} \left(\frac{(y_{t+1} - y_t - \mu\Delta)^2}{v_t \Delta} - \frac{2\phi_v (y_{t+1} - y_t - \mu\Delta)(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)}{(\omega_v + \phi_v^2) v_t \Delta} + \frac{(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)^2}{(\omega_v + \phi_v^2) v_t \Delta}\right)\right]\right\} \tag{69}$$

$$\times \left(\frac{1}{\sqrt{\pi \omega_v}} e^{-\phi_v^2/\omega_v}\right) \left(\frac{200^2 e^{-200/\omega_v}}{\Gamma(2) \omega_v^3}\right) \tag{70}$$

$$\propto \left\{\prod_{t=0}^{T-1} \frac{1}{\sqrt{\omega_v}} exp\left[-\frac{\omega_v + \phi_v^2}{2\omega_v} \left(\frac{(y_{t+1} - y_t - \mu\Delta)^2}{v_t \Delta} - \frac{2\phi_v (y_{t+1} - y_t - \mu\Delta)(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)}{(\omega_v + \phi_v^2) v_t \Delta} + \frac{(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)^2}{(\omega_v + \phi_v^2) v_t \Delta}\right)\right]\right\} \tag{71}$$

$$\times \left(\frac{1}{\sqrt{\omega_v}} e^{-\phi_v^2/\omega_v}\right) \left(\frac{e^{-200/\omega_v}}{\omega_v^3}\right) \tag{72}$$

$$\text{Let } C_{t+1} = \frac{y_{t+1} - y_t - \mu\Delta}{\sqrt{v_t \Delta}}, \text{ and } D_{t+1} = \frac{v_{t+1} - v_t - \kappa(\theta - v_t)\Delta}{\sqrt{v_t \Delta}} \tag{73}$$

$$=\left\{\prod_{t=0}^{T-1}\frac{1}{\sqrt{\omega_v}}exp\left[-\frac{\omega_v+\phi_v^2}{2\omega_v}\left(C_{t+1}^2-\frac{2\phi_v C_{t+1}D_{t+1}}{\omega_v+\phi_v^2}+\frac{D_{t+1}^2}{\omega_v+\phi_v^2}\right)\right]\right\}\left(\frac{1}{\sqrt{\omega_v}}e^{-\phi_v^2/\omega_v}\right)\left(\frac{e^{-200/\omega_v}}{\omega_v^3}\right) \tag{74}$$

$$=\left(\frac{1}{\omega_v}\right)^{\frac{T+7}{2}}exp\left[\left(\sum_{t=0}^{T-1}-\frac{C_{t+1}^2}{2}-\frac{\phi_v^2C_{t+1}^2}{2\omega_v}+\frac{\phi_vC_{t+1}D_{t+1}}{\omega_v}-\frac{D_{t+1}^2}{2\omega_v}\right)-\frac{\phi_v^2}{\omega_v}-\frac{200}{\omega_v}\right] \tag{75}$$

$$\propto\left(\frac{1}{\omega_v}\right)^{\frac{T+7}{2}}exp\left[\left(\sum_{t=0}^{T-1}-\frac{\phi_v^2C_{t+1}^2}{2\omega_v}+\frac{\phi_vC_{t+1}D_{t+1}}{\omega_v}-\frac{D_{t+1}^2}{2\omega_v}\right)-\frac{\phi_v^2}{\omega_v}-\frac{200}{\omega_v}\right] \tag{76}$$

$$=\left(\frac{1}{\omega_v}\right)^{\frac{T+7}{2}}exp\left[\left\{\left(\sum_{t=0}^{T-1}-\frac{C_{t+1}^2}{2\omega_v}\right)-\frac{1}{\omega_v}\right\}\phi_v^2+\left\{\sum_{t=0}^{T-1}\frac{C_{t+1}D_{t+1}}{\omega_v}\right\}\phi_v+\left(\sum_{t=0}^{T-1}-\frac{D_{t+1}^2}{2\omega_v}\right)-\frac{200}{\omega_v}\right] \tag{77}$$

$$=exp\left[\left\{\left(\sum_{t=0}^{T-1}-\frac{C_{t+1}^2}{2\omega_v}\right)-\frac{1}{\omega_v}\right\}\phi_v^2+\left\{\sum_{t=0}^{T-1}\frac{C_{t+1}D_{t+1}}{\omega_v}\right\}\phi_v\right]\left(\frac{1}{\omega_v}\right)^{\frac{T+7}{2}}exp\left[\left(\sum_{t=0}^{T-1}-\frac{D_{t+1}^2}{2\omega_v}\right)-\frac{200}{\omega_v}\right] \tag{78}$$

$$=exp\left[-\frac{1}{2\omega_v}\left\{\left(\sum_{t=0}^{T-1}C_{t+1}^2\right)+2\right\}\phi_v^2+\left\{\frac{\sum_{t=0}^{T-1}C_{t+1}D_{t+1}}{\omega_v}\right\}\phi_v\right]\left(\frac{1}{\omega_v}\right)^{\frac{T+5}{2}+1}exp\left[-\frac{1}{\omega_v}\left\{\left(\sum_{t=0}^{T-1}\frac{D_{t+1}^2}{2}\right)+200\right\}\right] \tag{79}$$

$$\text{Let } W=\left(\sum_{t=0}^{T-1}C_{t+1}^2\right)+2 \tag{80}$$

$$\text{Let } S=\frac{\sum_{t=0}^{T-1}C_{t+1}D_{t+1}}{\omega_v} \tag{81}$$

$$\text{Let } U=\left(\sum_{t=0}^{T-1}\frac{D_{t+1}^2}{2}\right)+200 \tag{82}$$

$$=exp\left[-\frac{1}{2\omega_v}W\phi_v^2+S\phi_v\right]\left(\frac{1}{\omega_v}\right)^{\frac{T+5}{2}+1}exp\left[-\frac{U}{\omega_v}\right] \tag{83}$$

$$=exp\left[-\frac{1}{2\omega_v}\left(W\phi_v^2-2S\phi_v\right)\right]\left(\frac{1}{\omega_v}\right)^{\frac{T+5}{2}+1}exp\left[-\frac{U}{\omega_v}\right] \tag{84}$$

$$=exp\left[-\frac{1}{2\omega_v}W\left(\phi_v^2-2\frac{S}{W}\phi_v\right)\right]\left(\frac{1}{\omega_v}\right)^{\frac{T+5}{2}+1}exp\left[-\frac{U}{\omega_v}\right] \tag{85}$$

$$=exp\left[-\frac{1}{2\omega_v}W\left(\phi_v^2-2\frac{S}{W}\phi_v+\left\{\frac{S}{W}\right\}^2-\left\{\frac{S}{W}\right\}^2\right)\right]\left(\frac{1}{\omega_v}\right)^{\frac{T+5}{2}+1}exp\left[-\frac{U}{\omega_v}\right] \tag{86}$$

$$=exp\left[-\frac{1}{2\omega_v}W\left(\left\{\phi_v-\frac{S}{W}\right\}^2-\left\{\frac{S}{W}\right\}^2\right)\right]\left(\frac{1}{\omega_v}\right)^{\frac{T+5}{2}+1}exp\left[-\frac{U}{\omega_v}\right] \tag{87}$$

$$\propto exp\left[-\frac{1}{2}\frac{\left(\phi_v-\frac{S}{W}\right)^2}{\frac{\omega_v}{W}}\right]\left(\frac{1}{\omega_v}\right)^{\frac{T+5}{2}+1}exp\left[-\frac{U}{\omega_v}\right] \tag{88}$$

$$\tag{89}$$

$$\begin{cases}\phi_v|\omega_v\sim N\left(\frac{S}{W},\frac{\omega_v}{W}\right) & \text{, where } W \text{ and } S \text{ are defined in Equations (80) and (81).}\\ \omega_v\sim IG\left(\frac{T+5}{2},U\right) & \text{, where } U \text{ is defined in Equation (82)}\end{cases}$$

4.5 Posterior Distribution of v_0

$$p(v_0|\mu,\kappa,\theta,\sigma_v,\rho,y,v)$$

$$=p(y_1-y_0,v_1-v_0|v_0) \tag{90}$$

$$=\frac{1}{2\pi v_0\Delta\sigma_v\sqrt{1-\rho^2}}\times exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(y_1-y_0-\mu\Delta)^2}{v_0\Delta}-\frac{2\rho(y_1-y_0-\mu\Delta)(v_1-v_0-\kappa(\theta-v_0)\Delta)}{v_0\Delta\sigma_v}+\frac{(v_1-v_0-\kappa(\theta-v_0)\Delta)^2}{v_0\Delta\sigma_v^2}\right)\right] \tag{91}$$

$$\propto\frac{1}{v_0}exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(y_1-y_0-\mu\Delta)^2}{v_0\Delta}-\frac{2\rho(y_1-y_0-\mu\Delta)(v_1-v_0-\kappa(\theta-v_0)\Delta)}{v_0\Delta\sigma_v}+\frac{(v_1-v_0-\kappa(\theta-v_0)\Delta)^2}{v_0\Delta\sigma_v^2}\right)\right] \tag{92}$$

4.6 Posterior Distribution of v_T

$$
\begin{aligned}
& p(v_T|\mu, \kappa, \theta, \sigma_v, \rho, y, v) \\
=& p(y_T - y_{T-1}, v_T - v_{T-1}|v_T) && (93) \\
=& \frac{1}{2\pi v_{T-1}\Delta\sigma_v\sqrt{1-\rho^2}} && (94) \\
& \times exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(y_T - y_{T-1} - \mu\Delta)^2}{v_{T-1}\Delta} - \frac{2\rho(y_T - y_{T-1} - \mu\Delta)(v_T - v_{T-1} - \kappa(\theta - v_{T-1})\Delta)}{v_{T-1}\Delta\sigma_v} + \frac{(v_T - v_{T-1} - \kappa(\theta - v_{T-1})\Delta)^2}{v_{T-1}\Delta\sigma_v^2}\right)\right] && (95) \\
\propto& exp\left[-\frac{1}{2(1-\rho^2)}\left(-\frac{2\rho(y_T - y_{T-1} - \mu\Delta)(v_T - v_{T-1} - \kappa(\theta - v_{T-1})\Delta)}{v_{T-1}\Delta\sigma_v} + \frac{(v_T - v_{T-1} - \kappa(\theta - v_{T-1})\Delta)^2}{v_{T-1}\Delta\sigma_v^2}\right)\right] && (96)
\end{aligned}
$$

4.7 Posterior Distribution of v_t

$$
\begin{aligned}
& p(v_t|\mu, \kappa, \theta, \sigma_v, \rho, y, v) \\
=& p(y_{t+1} - y_t, v_{t+1} - v_t|v_t)p(y_t - y_{t-1}, v_t - v_{t-1}|v_{t-1}) && (97) \\
=& \prod_{t=0}^{T-1} \frac{1}{2\pi v_t\Delta\sigma_v\sqrt{1-\rho^2}} \times exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(y_{t+1} - y_t - \mu\Delta)^2}{v_t\Delta} - \frac{2\rho(y_{t+1} - y_t - \mu\Delta)(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)}{v_t\Delta\sigma_v} + \frac{(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)^2}{v_t\Delta\sigma_v^2}\right)\right] && (98) \\
& \times \prod_{t=0}^{T-1} \frac{1}{2\pi v_{t-1}\Delta\sigma_v\sqrt{1-\rho^2}} && (99) \\
& \times exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(y_t - y_{t-1} - \mu\Delta)^2}{v_{t-1}\Delta} - \frac{2\rho(y_t - y_{t-1} - \mu\Delta)(v_t - v_{t-1} - \kappa(\theta - v_{t-1})\Delta)}{v_{t-1}\Delta\sigma_v} + \frac{(v_t - v_{t-1} - \kappa(\theta - v_{t-1})\Delta)^2}{v_{t-1}\Delta\sigma_v^2}\right)\right] && (100) \\
\propto& \frac{1}{v_t} exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(y_{t+1} - y_t - \mu\Delta)^2}{v_t\Delta} - \frac{2\rho(y_{t+1} - y_t - \mu\Delta)(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)}{v_t\Delta\sigma_v} + \frac{(v_{t+1} - v_t - \kappa(\theta - v_t)\Delta)^2}{v_t\Delta\sigma_v^2}\right)\right] && (101) \\
& \times exp\left[-\frac{1}{2(1-\rho^2)}\left(-\frac{2\rho(y_t - y_{t-1} - \mu\Delta)(v_t - v_{t-1} - \kappa(\theta - v_{t-1})\Delta)}{v_{t-1}\Delta\sigma_v} + \frac{(v_t - v_{t-1} - \kappa(\theta - v_{t-1})\Delta)^2}{v_{t-1}\Delta\sigma_v^2}\right)\right] && (102)
\end{aligned}
$$

5 Simulation

To simulate a data set from the model, set $\Delta = 1$ (so that the data is evenly spaced in time) and set reasonable model parameters according to their prior distributions. For the first simulation set:

$$
\begin{aligned}
\mu &= -1.04 \\
\kappa &= 0.05 \\
\theta &= 0.87 \\
\omega_v &= 131.60 \\
\phi_v &= 5.24
\end{aligned}
\tag{103}
$$

Note that this makes:

$$
\begin{aligned}
\sigma_v^2 &= 159.02 \\
\rho &= 0.42
\end{aligned}
\tag{104}
$$