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ORIGINAL ARTICLE

Projection neural networks for sample-regular product optimization model

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Abstract
Reducing product fit uncertainty is a critical strategy for retailers to enhance sales and profitability in e-commerce. This paper proposes an optimization-based product sampling strategy to improve sales promotion and profits using recurrent neural networks (RNN) method. Unlike conventional selling modes, the strategy allows consumers to purchase discounted product samples while receiving coupons for subsequent full-priced purchases. A profit optimization model is formulated incorporating constraints such as production costs. The methodological breakthrough lies in introducing RNN to solve this constrained optimization problem, offering a novel approach to dynamic pricing and marketing strategy optimization. In addition, a projection method has been introduced to avoid the additional operation of normalizing product prices in existing methods. The proposed RNN-based framework ensures real-time adaptability, robustness, and efficient convergence, which addresses the complexities of sample distribution and coupon redemption and provides a new perspective on pricing strategies for retailers. The feasibility and effectiveness of the model are demonstrated through numerical simulations, providing valuable insights for retailers seeking promotion-driven pricing strategies.

Keywords Recurrent neural networks · Optimization with constraints · Projection operator · Sale promotion

1 Introduction

With the development of digital technology and the widespread embrace of online shopping platforms, the retail industry has experienced a significant shift towards the digital realm. This trend has gained substantial momentum [1], as evidenced by the findings of the 53rd Statistical Report on the Development of the Internet in China by the China Internet Network Information Center (CNNIC). However, this transition from offline to online sales has introduced a new challenge: product fit uncertainty. Although brand images and promotional videos can mitigate consumers' uncertainty perceptions, the lack of physical interaction with products in online shopping leads to uncertainty in product matching [2]. Additionally, consumers may incur losses in time and money due to frequent product returns caused by mismatches between their expectations and reality [3]. On the other hand, intense competition among merchants has led to unethical practices, such as hiring individuals to post false reviews under competitors' products contents, which discourage potential buyers and lead to financial losses to affected retailers [4].

In e-commerce, how to deal with product fit uncertainty has become a vital challenge. Consumers have to rely on products descriptions and images without the ability to physically evaluate the products, which limit their confidence in decision-making [5]. To overcome this, some retailers have introduced the sample-sending model-an innovative approach allowing consumers to purchase a small sample at a low cost to assess whether the product meets their expectations [6]. This strategy reduces the shipping cost losses associated with product fit uncertainty while simultaneously enhancing consumer confidence in full-size

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products purchase behavior. Moreover, when consumers purchase a sample, they will receive a discount coupon applicable to the full-sized product, further incentivizing them to complete the purchase [7]. Prior studies on promotional incentives, such as [8, 9], emphasize the role of price perceptions and trial costs in shaping purchase behavior, demonstrating behavioral consistency with the trial-to-purchase pathway postulated in the product sample-sending model.

In modern corporate marketing strategies, Philip Kotler, widely regarded as the father of marketing management, asserts that promotion is a mature and effective marketing tool [10]. Retailers utilize various promotional strategies-such as discounts, coupons, and rebates-to expand their market share, increase sales, and boost profitability [11]. The foundational work by [12, 13] further underscores the theoretical and practical mechanisms through which promotions influence consumer decision-making. However, as e-commerce grows, traditional physical retailers struggle to attract new consumers with conventional promotional tools, necessitating new strategies like sample-sending. The promotion significantly reduces consumers' trial-and-error costs when purchasing a full-sized product, making them more willing to explore new options while decreasing the return rate and associated costs for both consumers and merchants [14]. Furthermore, this strategy mitigates the impact of false and malicious reviews by enabling consumers to assess product quality for themselves [4]. Recent advancements in digital promotions, as explored by [15], highlight the importance of platform-specific strategies, which resonate with the sample-sending model's integration of coupons and cross-channel effects. Merchants, in turn, can analyze consumer preferences based on sample purchases and refine their marketing strategies, ultimately fostering a win-win situation [16].

Optimizing the sample-sending model presents unique challenges that require innovative solutions. Although previous research has explored optimization techniques in e-commerce, the dynamic nature of sample-sending need to be developed. Unlike traditional e-commerce models, the sample-sending model involves continuous interactions among consumers, retailers, and products, requiring advanced optimization methods. [17] proposed an adaptive neural dynamics method that can effectively solve nonsmooth optimization problems and reduce computational complexity. Studies such as [18] demonstrate the value of real-time promotional adjustments in dynamic retail environments, offering insights applicable to sample-sending optimization. The recurrent neural networks (RNN) model offers a promising solution to address these complexities, providing real-time problem-solving capabilities suited to the dynamic nature of sample-sending optimization [3]. Because RNN have fast convergence and good robustness, great progress has been made in artificial intelligence applications. For example, neurodynamic approach [19] have been applied to multi-agent systems, further reducing the communication resource consumption of multi-agent systems. However, their use in e-commerce optimization remains largely unexplored [1]. The meta-analysis by [20] on discount effectiveness and the comparative study by [21] on coupon versus free shipping strategies further validate the need for context-specific optimization frameworks. By leveraging an efficient parallel solver, RNN can effectively manage complex sample-sending models and facilitate optimal decision-making in e-commerce management. Recent advances in neurodynamic optimization have demonstrated RNNs' effectiveness in handling dynamic constraints, particularly in time-sensitive applications [22–25]. Our work extends these principles to e-commerce pricing contexts.

The objective of this paper is to optimize the sample-sending model involving dynamic interactions among consumers, retailers, and products, formulated as a constrained optimization problem. The main contributions of this research are as follows:

1. This study explores a novel type of sample-sending model which is getting a trail in low price and return coupons, which can be considered sending free trails. The strategy can provide a chance for consumers to subsequent purchases, which is different from previous research [26]. This study can also help retailers lower the threshold for consumer decision-making, increase user stickiness, and reduce invalid trials.
2. To the best of our knowledge, it is the first time that RNN is introduced into commodity pricing strategy model. The unique structure of RNN enables precise analysis of dynamic factors during sample delivery, including changes in consumer demand and the resulting market fluctuations [2]. This is consistent with the findings of [27], which highlights the importance of dynamic program effectiveness in promotional development.
3. It is rigorous proved the Lyapunov stability of the proposed RNN and its convergence to the optimal solution in the sample-regular optimization model, helping merchants maximize profits through optimized sample allocation and pricing level, complementing earlier work Unlike traditional normalization [4], it is introduced a projection operator that directly constrains the neural network's state solutions to constraint sets, eliminating quantization and enhancing computational efficiency, stability, and accuracy.

In conclusion, the sample-sending model represents a major innovation in e-commerce, Through a trial-and-error framework, the mechanism enables consumers to evaluate product suitability with minimal upfront risk, thereby optimizing their subsequent full-price purchase decisions. By integrating RNN-based optimization, this model can be further refined to maximize merchant profitability while enhancing consumer confidence and satisfaction, catalyzing the structure of online retailing. [1].

The rest of this paper is as follows. Section 2 consists of three sample—sending models and optimization models. In Sect. 3, we study the Recurrent neural networks and Convergence Analysis. The numerical simulation is presented in Sect. 4. Finally, Sect. 5 summarizes the full paper.

2 The construction of models

In this section, a profit maximization model is established for retailers in through a constrained optimization framework, incorporating both price elasticity of consumer demand and profit considerations. Table 1 is the parameters used in this paper.

The model is constructed based on the following fundamental assumptions:

The market size is normalized to 1, representing the total potential consumer demand as a single unit. The reservation utility v, defined as the maximum price consumers willing to pay for the commodity, follows a uniform distribution across the interval [0, 1]. The purchasing decision rule is established such that consumers will purchase the product (price is p) if and only if their reservation utility exceeds the market price ($v > p$). Therefore, the consumer demand $D = \int_v^1 1\mathrm{d}p = 1 - v$ represents the market share of this product.

Building upon these assumptions, this section proposes an innovative sales strategy that fundamentally differs from traditional approaches. This strategy dynamically adjusts subsequent sales promotions based on the market feedback obtained from distributed product samples, thereby enabling more responsive and adaptive market engagement.

Table 1 The parameters used in this paper

V	Consumer's valuation of the products, $V \sim U[0, 1]$
v	The valuation of products by a consumer
U	Utility function
c	Unit costs of these two products
1	Market size standardization
p_s	Price of the product sample
p_{ud}	Coupon only be used to buy one of two products
p_{us}	Coupon only be used to buy two products together
θ	Matching probability, $\theta \in (0, 1)$
λ	Combination cost ratio translation
δ	Coefficient of incomplete satisfaction

Remark 2.1 The normalization of market size is a fundamental method for studying marketing strategies, which is consistent with [13, 14]. In addition, it has been a long-standing conclusion in market research and analysis that the evaluation function V of consumers for products satisfies a uniform distribution, which aligns with common practice in pricing optimization studies[12]. While assuming uniform utility distribution simplifies analysis, our projection operator's set-constrained mechanism ensures feasibility preservation across distribution types.

2.1 The sample optimization model

In this section, it is considered the bundled sale of two products. Retailers promote sales by giving out two coupons to each sample they sell. After purchasing a sample for a certain price, consumers will receive two coupons: one that can only be used to purchase one of the two products, and the other that can only be used when both products are purchased together.

Suppose p_s is used to represent the price of the product sample, p_{ud} is used to represent a coupon that can only be used when buying one of two products, and p_{us} is used to represent a coupon that can only be used when buying two products together. The unit costs of production cost for these two products are recorded as c_1 and $c_2 (0 < c_i < p_i, i = 1, 2)$. The production cost of the sample can be obtained after the pro rata conversion of the production cost of regular-sized product. Assuming that λ is the combination cost ratio translation, c_a is the sum of shipping, packaging, advertising and other costs of product sample, Then the total cost of product sampling is

$$\lambda(c_1 + c_2) + c_a.$$

According to the above conditions, the lowest psychological price of consumers can meet $v = p_s$. Then, the demand function of consumers denoted as

$$D^s = 1 - v = 1 - p_s,$$

Consequently, the total profit of the distributed sample can be expressed as

$$\begin{aligned}\pi_1 &= D^s[p_s - \lambda(c_1 + c_2) - c_a] \\ &= (1 - p_s)[p_s - \lambda(c_1 + c_2) - c_a].\end{aligned}$$

The sample distribution optimization model can be summarized as the following convex optimization problems.

$$\begin{aligned} min \quad & -\pi_1(p_s) = -(1-p_s)[p_s - \lambda(c_1+c_2) - c_a] \\ s.t. \quad & 0 \le p_s \le 1 \end{aligned}$$

Lemma 2.1 The objective function of the sample distribution optimization model is a convex function, and there exists a maximum value of profit function π_1.

The proof is presented in the appendix.

2.2 The regular optimization model after receiving the sample

When consumers are satisfied with either one of products or both products after using the sample, they will use the complimentary coupons to buy the regular products. The formal price of the first product is p_1, the formal price of the second product is p_2, the total cost of producing the first product is recorded as c_1, and the total cost of producing the second product is recorded as c_2, and c_b is the sum of shipping, packaging, advertising and other costs of Regular-sized product.

Consider the scenario where the ordinary consumer is satisfied with only one of the products. In this situation, let δ represent the coefficient of incomplete satisfaction. δ_1 means that the consumer is only satisfied with the first product, and δ_2 means that the consumer is only satisfied with the second product. $\theta_1 \in (0,1)$ denotes the probability that all these two products meet the needs of consumers (referred to as the matching probability), $\theta_2 \in (0,1)$ denotes the probability that only the first product meets the needs of consumers and $\theta_3 \in (0,1)$ denotes the probability that only the second product meets the needs of consumers. It follows that $\theta_1+\theta_2+\theta_3=1$. Here, π and $U_i(i=1,2,3)$ are used to represent the retailer's profit function and the consumer's utility function, respectively.

From this, it can be deduced that when the individual is satisfied with both products, the utility function is

$$U_1 = v - (p_1 + p_2 - p_{us}).$$

When the individual is only satisfied with the first product, the utility function is

$$U_2 = \delta_1 v - (p_1 - p_{ud}).$$

When the individual is only satisfied with the second product, the utility function is

$$U_3 = \delta_2 v - (p_2 - p_{ud}).$$

Then the utility function of consumers can be denoted as

$$\begin{aligned} U = & \theta_1[v-(p_1+p_2-p_{us})] + \theta_2[\delta_1 v - (p_1 - p_{ud})] \\ & + \theta_3[\delta_2 v - (p_2 - p_{ud)}]. \end{aligned}$$

Therefore, according to the definition of demand function and the v solved by the above formula, the demand function of consumers when purchasing products of conventional sizes is expressed as follows.

$$D^F = 1 - \frac{(\theta_1+\theta_2)p_1 + (\theta_1+\theta_3)p_2 - \theta_1 p_{us} - (\theta_2+\theta_3)p_{ud}}{\theta_1 + \theta_2\delta_1 + \theta_3\delta_2}.$$

Consequently, the total profit of the regular-sized product can be expressed as

$$\begin{aligned} \pi_2 = & [\theta_1(p_1+p_2-p_{us}-c_1-c_2-c_b-c_c) \\ & + \theta_2(p_1-p_{ud}-c_1-c_b) + \theta_3(p_2-p_{ud}-c_2-c_c)]D^F. \end{aligned}$$

The optimization model of the regular product can be summarized as the following convex optimization problem:

$$\begin{aligned} min \quad & -\pi_2(p_1,p_2,p_{us},p_{ud}) \\ s.t. \quad & p_1+p_2-p_{us} \ge c_1+c_2+c_b+c_c \\ & p_1 - p_{ud} \ge c_1 + c_b \\ & p_2 - p_{ud} \ge c_2 + c_c \\ & 0 \le p_1 \le 1,\ 0 \le p_2 \le 1,\ 0 \le p_{us} \le 1,\ 0 \le p_{ud} \le 1 \end{aligned}$$

Lemma 2.2 The objective function of the formal attire optimization model is a convex function, and there exists a maximum value of profit function π_2.

The proof is included in the appendix.

Combining the above two models, the changes in consumers' demand functions under different circumstances are shown in the following figure (Fig. 1). Assuming π to be the total profit, then the overall profit function π can be expressed as

$$\pi = \pi_1 + \pi_2,$$

which satisfies conditions

$$\begin{cases} p_1+p_2-p_{us} & \ge c_1+c_2+c_b+c_c \\ p_1 - p_{ud} & \ge c_1+c_b \\ p_2 - p_{ud} & \ge c_2 + c_c \end{cases}$$

In summary, the sample-regular optimization model can be expressed as

$$\begin{aligned} min \quad & -\pi(p) \\ s.t. \quad & g(p) \ge 0 \\ & p \in \Omega \end{aligned} \tag{1}$$

where

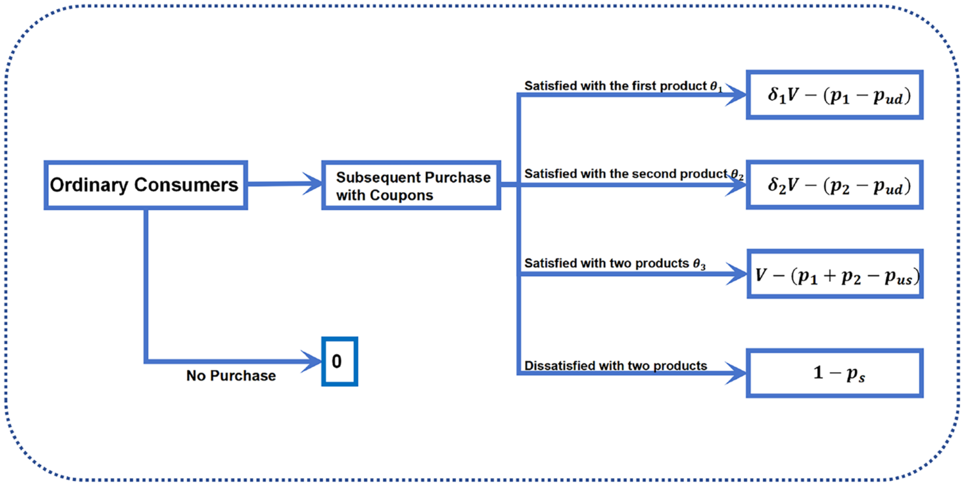

Fig. 1 Consumer demand function in sample-regular product optimization model

$$
\begin{aligned}
&p = \mathrm{col}(p_1, p_2, p_s, p_{us}, p_{ud}) \in \mathbb{R}^5,\\
&\Omega = [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1],\\
&g = \mathrm{col}(g_1, g_2, g_3) \in \mathbb{R}^3,\ \pi(p) = \pi_1(p) + \pi_2(p),\\
&g_1 = p_1 + p_2 - p_{us} - (c_1 + c_2 + c_b + c_c),\\
&g_2 = p_1 - p_{ud} - (c_1 + c_b),\\
&g_s = p_2 - p_{ud} - (c_2 + c_c).
\end{aligned}
$$

Remark 2.2 The proposed sample-regular optimization model (1) is primarily divided into two parts. The first focuses on determining the optimal pricing strategy within the sample optimization model, while the second aims to identify the optimal pricing for two products in the regular optimization model. The overarching objective is to maximize the merchants' profits. Notably, the regular optimization model comprehensively accounts for various consumer behaviors after utilizing samples, thereby ensuring the rationality of the proposed framework.

2.3 The discount optimization model

There exists a special case in the pricing of coupons, that is, that the discount price of the coupon is priced as a percentage of the price of the product. There is a proportional relationship between p_{us}, p_{ud} and price.

$$
\begin{cases}
p_{ud} = a_1 p_1 \text{ or } a_2 p_2,\\
p_{us} = b(p_1 + p_2),
\end{cases}
$$

By incorporating the above proportional relationship between the coupon value and the positive price into the demand function of the sample model, it is easy to get the retention effect in this case as follows.

$$
v = \frac{\theta_1(1-b)(p_1+p_2) + \theta_2(1-a_1)p_1 + \theta_3(1-a_2)p_2}{r}.
$$

Following a similar methodological approach to the derivation of the sample regularization optimization model's objective function, the total profit of the discount optimization model can be expressed as

$$
\begin{aligned}
\pi_2 = {} & [\theta_1((1-b)(p_1+p_2) - c_1 - c_2 - c_b - c_c) + \theta_2((1-a_1)p_1 - c_1 - c_b)\\
& + \theta_3((1-a_2)p_2 - c_2 - c_c)](1-v),
\end{aligned}
$$

The changes in consumers' demand functions under different circumstances are shown in the figure (Fig. 2) below. The discount optimization model can be summarized as the following optimization problems.

$$
\begin{aligned}
\mathit{min} \quad & -\pi_2(p_1, p_2)\\
\mathit{s.t.} \quad & (p_1 + p_2)(1-b) \ge c_1 + c_2 + c_b + c_c\\
& c_1 + c_b \le (1-a_1)p_1, \quad p_1 \le 2c_1\\
& c_2 + c_c \le (1-a_2)p_2, \quad p_2 \le 2c_2\\
& 0 \le p_1 \le 1,\ 0 \le p_2 \le 1
\end{aligned}
$$

Lemma 2.3 The objective function of the discount optimization model is a convex function, and there exists a maximum value of profit function π_2.

The proof is placed in the appendix.

Remark 2.3 Unlike the sample-regular optimization model, the discount optimization model directly links coupon value to product price, making it a more flexible and adaptive pricing mechanism that aligns with consumer price sensitivity.

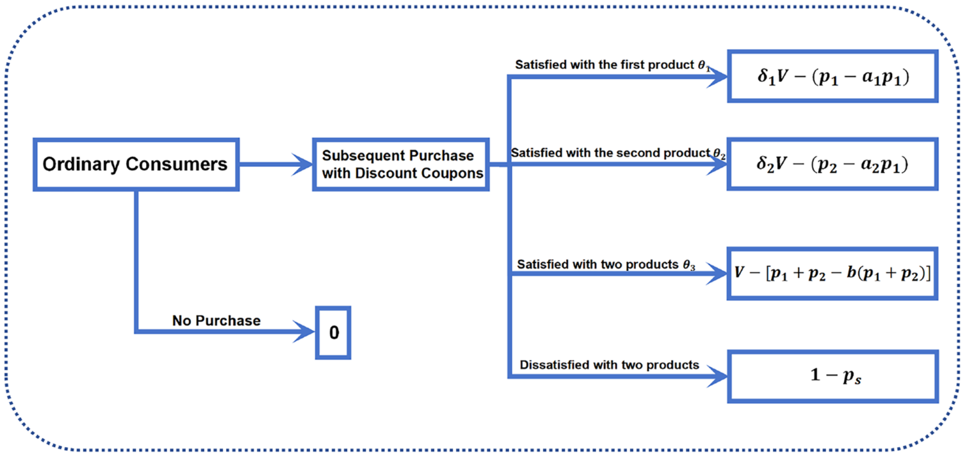

Fig. 2 Consumer demand function in discount optimization model

3 Recurrent neural networks and convergence analysis

This section proposes an effective RNN to solve the sample-regular optimization model. Notably, the discount optimization model can be considered as a special case of the sample-regular optimization model. As a result, the RNN introduced in this study is also applicable to solving the discount optimization model.

Define the following Lagrangian function of optimization model (1) as follows, for all $(p, \lambda) \in \Omega \times \mathbb{R}^3_+$, Ω has been defined in (1)

$$L(p, \lambda) = -\pi(p) - \lambda^T g(p). \tag{2}$$

(p^*, λ^*) is called a saddle point of the above mentioned Lagrangian function. If for

$$L(p^*, \lambda) \le L(p^*, \lambda^*) \le L(p, \lambda^*).$$

According to the Strong Duality Theorem, for any saddle point (p^*, λ^*), p^* is the optimal solution of the optimization model (1).

Based on the sample-regular model and the above analysis, the following RNN is proposed.

$$\begin{cases} \frac{dp}{dt} = -2(p - \widetilde{p}) \\ \frac{d\lambda}{dt} = -(\lambda - \widetilde{\lambda}) \end{cases} \tag{3}$$

Where $\widetilde{p} = P_\Omega[p + \nabla\pi(p) + \nabla g^T \widetilde{\lambda}]$ is a projection operator., $\lambda \in \mathbb{R}^3$ represents the Lagrangian operator, and $\nabla\pi(p)$ is the gradient of the objective function of the optimization model (3).

$\nabla g = col(\nabla g_1, \nabla g_2, \nabla g_3) \in \mathbb{R}^{3\times5}$, $\widetilde{\lambda}_i = max\{0, \lambda_i - g_i(p)\}$, $\widetilde{\lambda} = (\lambda_1, \lambda_2 \lambda_3) \in \mathbb{R}^3_+$.

From [22], $P_\Omega : \mathbb{R}^5 \to \Omega$ is a projection factor, and it is defined as

$$P_\Omega = arg \min_{v\in\Omega} ||p - v||.$$

If Ω is a closed convex set, then for all $p \in \mathbb{R}^5, v \in \Omega$, has

$$(p - P_\Omega)^T(P_\Omega - v) \ge 0. \tag{4}$$

Remark 3.1 The proposed RNN inherits the advantages of neural networks in constrained optimization by seamlessly integrating projection-based constraint handling and non-smooth dynamics adaptation, enabling direct enforcement of price-coupon bounds without parameter tuning. Its Lagrangian saddle-point structure ensures exact convergence to optimal solutions while maintaining robustness against market fluctuations, addressing time-sensitive pricing dynamics. Furthermore, the model's Lyapunov stability guarantees global convergence to equilibrium points, providing theoretical rigor for profit maximization in dynamic e-commerce environments with evolving consumer behaviors.

Theorem 3.1 (p^*, λ^*) is an equilibrium point of the RNN (3) if and only if it is a saddle point of the Lagrangian function (2).

Let (p^*, λ^*) be an equilibrium point of the RNN (3), then

$$\begin{cases} p^* = P_\Omega[p^* + \nabla\pi(p^*) + \nabla g(p*)^T \lambda^*] \\ \lambda^* = max\{0, \lambda^* - g(p*)\} \end{cases} \tag{5}$$

for all $(p, \lambda) \in \Omega \times \mathbb{R}^3_+$, then it is attainable from (4) that

$$\begin{aligned} &(P_\Omega[p^* + \nabla\pi(p^*) + \nabla g(p^*)^T\lambda^*] - p)^T(p^* + \nabla\pi(p^*) + \nabla g(p^*)^T\lambda^* \\ &- P_\Omega[p^* + \nabla\pi(p^*) + \nabla g(p^*)^T\lambda^*]) \ge 0. \end{aligned} \tag{6}$$

So, it can be derived from (9) and (10) that $(p - p^*)^T(-\nabla\pi(p^*) - \nabla g(p^*)^T\lambda^*) \ge 0, \forall p \in \Omega$. Since both $-\pi(p)$ and $g(p)$ are convex functions, the Lagrangian function (2) is convex with respect to p. Then

$$L(p,\lambda^*) - L(p^*,\lambda^*) \geq (-\nabla\pi(p^*) - \nabla g(p^*)^T\lambda^*)(p-p^*) \geq 0. \tag{7}$$

From (5), it can be obtained $\lambda^* \geq 0, g(p^*) \geq 0, (\lambda^*)^T g(p^*) = 0$,then

$$L(p^*,\lambda^*) - L(p^*,\lambda) = \lambda^T g(p^*) \geq 0. \tag{8}$$

Then, it can be acquired from (7) and (8) that (p^*,λ^*) is the saddle point of the Lagrangian function (4).

On the other hand, it is assumed that (p^o,λ^o) is the saddle point of the Lagrangian function, then for all $p \in \Omega, \lambda \in \mathbb{R}^3_+$,

$$L(p^o,\lambda) \leq L(p^o,\lambda^o) \leq L(p,\lambda^o). \tag{9}$$

From the right hand inequality of (9), it can be concluded that p^o is the minimum point of $L(p,\lambda^o)$, then for all $p \in \Omega$

$$(p-p^o)^T(-\nabla\pi(p^o) - \nabla g(p^o)^T\lambda^o) \geq 0. \tag{10}$$

After that it can be deduced from (4) that

$$\begin{aligned}&(p^o + \nabla\pi(p^o) + \nabla g(p^o)^T\lambda^o - P_\Omega[p^o + \nabla\pi(p^o)\\ &\quad + \nabla g(p^o)^T\lambda^o])^T(-p^o\\ &\quad + P_\Omega[p^o + \nabla\pi(p^o) + \nabla g(p^o)^T\lambda^o]) \geq 0.\end{aligned}$$

Next

$$\begin{aligned}&||p^o - P_\Omega[p^o + \nabla\pi(p^o) + \nabla g(p^o)^T\lambda^o]||^2\\ &\quad \leq (\nabla\pi(p^o) + \nabla g(p^o)^T\lambda^o)^T(-p^o + P_\Omega[p^o + \nabla\pi(p^o) + \nabla g(p^o)^T\lambda^o]).\end{aligned}$$

It can be generated from (10) that $||p^o - P_\Omega[p^o + \nabla\pi(p^o) + \nabla g(p^o)^T\lambda^o]||^2 \leq 0$, and then

$$p^o = P_\Omega[p^o + \nabla\pi(p^o) + \nabla g(p^o)^T\lambda^o]. \tag{11}$$

Then, it can be extracted from the left hand inequality of (9) that for all $\lambda \in \mathbb{R}^3_+$, one has

$$(\lambda^o - \lambda)^T g(p^o) \leq 0.$$

According to the Projection Theorem in [28], there is

$$\lambda^o = max\{0, \lambda^o - g(p^o)\}. \tag{12}$$

So, it can be derived from (11) and (12) that the saddle point (p^o,λ^o) is the equilibrium point of the RNN (3).

Theorem 3.2 If the initial state is $p(0) \notin \Omega$, the solution $p(t)$ of the RNN (3) will converge exponentially to the set constraint Ω. If the initial state is $p(0) \in \Omega$, then for all $t > 0, p(t) \in \Omega$.

Take the Lyapunov function $V(p) = \frac{1}{2}||p - P_\Omega||^2$, then the derivative of V along (2) is

$$\begin{aligned}\dot{V} &= 2(P_\Omega - p)^T(p - P_\Omega[p + \nabla\pi(p) + \nabla g^T\widetilde{\lambda}])\\ &= 2(P_\Omega - p)^T(p - P_\Omega + P_\Omega - P_\Omega[p + \nabla\pi(p) + \nabla g^T\widetilde{\lambda}])\\ &= -2||p - P_\Omega||^2 + 2(P_\Omega - p)^T(P_\Omega - P_\Omega[p + \nabla\pi(p) + \nabla g^T\widetilde{\lambda}]).\end{aligned}$$

It can be procured from (3) that $(P_\Omega - p)^T(P_\Omega - P_\Omega[p + \nabla\pi(p) + \nabla g^T\widetilde{\lambda}]) \leq 0$, then $\dot{V} \leq -2||p - P_\Omega||^2 = -4V$.It can be deduced that

$$||p(t) - P_\Omega(p(t))|| \leq ||p(0) - P_\Omega(p(0))||e^{-4t}.$$

So, if $p(0) \notin \Omega$, then the solution of the RNN (3) will converge exponentially to Ω. If $p(0) \in \Omega$, then $||p(0) - P_\Omega(p(0))|| = 0$ and $p = P_\Omega$ for all $t > 0, p(t) \in \Omega$.

Remark 3.2 Theorem 2 establishes the exponential convergence property of $x(t)$ to the feasible set Ω, ensuring that once the trajectory enters Ω, it remains within the constrained domain. This property is crucial for practical implementation, as it guarantees the stability and boundedness of the optimization process.

Theorem 3.3 For any $(p(0),\lambda(0)) \in \Omega \times \mathbb{R}^3$, the state solution of the RNN (3) converges to the optimal solution of the sample-regular optimization model (1), and the RNN (3) is Lyapunov stable.

Define $x = \mathrm{col}(p,\lambda), x^* = (p^*,\lambda^*)$. Consider the following energy function.

$$V(x) = \phi(x) - \phi(x^*) - (x - x^*)\nabla\phi(x^*) + \frac{1}{2}||x - x^*||^2. \tag{13}$$

Where $\phi(x) = -\pi(p) + \frac{1}{2}||\widetilde{\lambda}||^2$. Obviously, $V(x)$ is convex and continuously differentiable. From the convexity condition, we can obtain that $\phi(x) \geq \phi(x^*) + (x - x^*)\nabla\phi(x^*), \forall x \in K$, then there is

$$V(x) \geq \frac{1}{2}||x - x^*||^2 \text{ for all } x \in K \text{ with } K = \Omega \times \mathbb{R}^3_+.$$

since

$$max\{0, \lambda_i - g_i(p)\}^2 = [\lambda_i - g_i(p)]^+ = \begin{cases}\lambda_i - g_i(p) & \text{if } \lambda_i \geq g_i(p)\\ 0 & \text{otherwise}\end{cases}$$

So $||\phi(\lambda)||$ is convex and continuously differentiable. It can be supported from (3), (5) and (13) that

$$
\begin{aligned}
\dot{V} =& -2(p-\widetilde{p})^T[p-p^*+\nabla\pi(p^*)+(\nabla g(p^*))^T\lambda^* \\
& -\nabla\pi(p)-(\nabla g(p))^T\widetilde{\lambda}] \\
& -(\lambda-\widetilde{\lambda})^T(\lambda-2\lambda^*+\widetilde{\lambda})) \\
=& -2(\widetilde{p}-p^*)^T[p-\widetilde{p}-\nabla\pi(p^*)-(\nabla g(p^*))^T\lambda^* \\
& +\nabla\pi(p)+(\nabla g(p))^T\widetilde{\lambda}] \\
& -2||p-\widetilde{p}||^2 \\
& -2(p-p^*)^T[-\nabla\pi(p)-(\nabla g(p))^T\widetilde{\lambda}+\nabla\pi(p^*)+(\nabla g(p^*))^T\lambda^*] \\
& -||\lambda-\widetilde{\lambda}||^2+2(\widetilde{\lambda}-\lambda^*)^T(g(p)-[g(p)-\lambda]^+) \\
=& -2(\widetilde{p}-p^*)^T[p-\widetilde{p}+\nabla\pi(p)+(\nabla g(p))^T\widetilde{\lambda}] \\
& -2(\widetilde{p}-p^*)^T[-\nabla\pi(p^*)-(\nabla g(p))^T\lambda^*] \\
& -2||p-\widetilde{p}||^2-||\lambda-\widetilde{\lambda}||^2 \\
& -2(p-p^*)^T[-\nabla\pi(p)+\nabla\pi(p^*)]-2\widetilde{\lambda}^T[g(p)-[g(p)-\lambda]^+-(\nabla g(p^*))^T(p-p^*)] \\
& +2(\lambda^*)^T[g(p)-[g(p)-\lambda]^+-(\nabla g(p^*))^T(p-p^*)] \\
\leq& -2||p-\widetilde{p}||^2-2(p-p^*)^T[-\nabla\pi(p)+\nabla\pi(p^*)] \\
& -2\widetilde{\lambda}^T g(p^*)-||\lambda-\widetilde{\lambda}||^2 \\
& -2\widetilde{\lambda}^T[g(p)-g(p^*)-(\nabla g(p))^T(p-p^*)] \\
& +2(\lambda^*)^T[g(p)-g(p^*)-(\nabla g(p^*))^T(p-p^*)],
\end{aligned}
$$

among them, $\lambda-\widetilde{\lambda}=g(p)-[g(p)-\lambda]^+$ and the optimality condition in [29] are utilized.

In addition, since $\widetilde{x}\in\Omega$, $\widetilde{\lambda}^T[g(p)-\lambda]^+=0$, $(\lambda^*)^T[g(p)-\lambda]^+\geq 0$, combined with the convex property, it can be deduced that

$$
\dot{V}\leq -2||p-\widetilde{p}||^2-||\lambda-\widetilde{\lambda}||^2\leq 0, \tag{14}
$$

which means that the RNN (3) is Lyapunov stable.

Next, from the proof of Theorem 3.2, it can be seen that $x(t)$ is bounded. This implies that there exists a convergent subsequence $x(t_k)$, $\lim_{k\to+\infty}t_k=+\infty$ such that $\lim_{k\to\infty}x(t_k)=\hat{x}$, where $\hat{x}$ is a finite point.

From LaSalle Invariant Theorem , define $Q=\{x|\dot{V}(x)=0\}$. Let S be the largest invariant set of Q such that when t approach infinity, $x(t)$ tend to M.

From (3) and (4), it can be attained that $\frac{dx}{dt}=0$ is equivalent to $\dot{V}=0$. Therefore, it can be accessed according to the Corollary in [30]. Let $K^*=\{x\in\Omega\times\mathbb{R}^3|x \text{ satisfy } (5)\}$. Then $Q\subset K^*$, so $\hat{x}\in K^*$, substitute $x^*=\hat{x}$ into (13). Similar to the derivation, it can be obtained that $V(x,\hat{x})$ is monotonically non-increasing, and $V(x,\hat{x})\geq\frac{1}{2}||x-\hat{x}||^2$ for all $x\in K$. So, there exists $t_p>0$ such that

$$
V(x,\hat{x})\leq\frac{1}{2}\epsilon^2,\ \text{for all } t>t_p,
$$

so

$$
||x(t)-\hat{x}||\leq\sqrt{2V(x,\hat{x})}\leq\sqrt{2V(x(t_p),\hat{x})}\leq\epsilon.
$$

Then

$$
\lim_{t\to\infty}x(t)=\hat{x}.
$$

In addition, combining with the expression of $V(x,x^*)$, $V(x,x^*)\geq\frac{1}{2}||x-x^*||$ and $V(x,x^*)$ converges to 0. Therefore, $\hat{x}=x^*$, that is, the state solution (p,λ) of the RNN (3) converges to (p^*,λ^*).

Remark 3.3 The use of RNN in optimization problems ensures real-time adaptability and robustness, making it particularly suitable for dynamic pricing strategies where consumer preferences and market conditions are constantly evolving. Moreover, The convergence of the proposed RNN-based framework highlights its effectiveness in solving constrained optimization problems, demonstrating its potential applicability in broader e-commerce scenarios beyond sample pricing.

4 Numerical simulations

In this section, the proposed RNN (3) is applied to sample-regular product optimization model and the discount optimization model to verify their feasibility. Moreover,

Table 2 Parameter value

parameter	(c_1, c_2)	(c_a, c_b, c_c)	$(\theta_1, \theta_2, \theta_3)$	(δ_1, δ_2)	λ	(a_1, a_2)	b
value	(0.3,0.2)	(0.1,0.1,0.1)	(0.4,0.3,0.3)	(0.88,0.88)	0.5	(0.4,0.3)	0.4

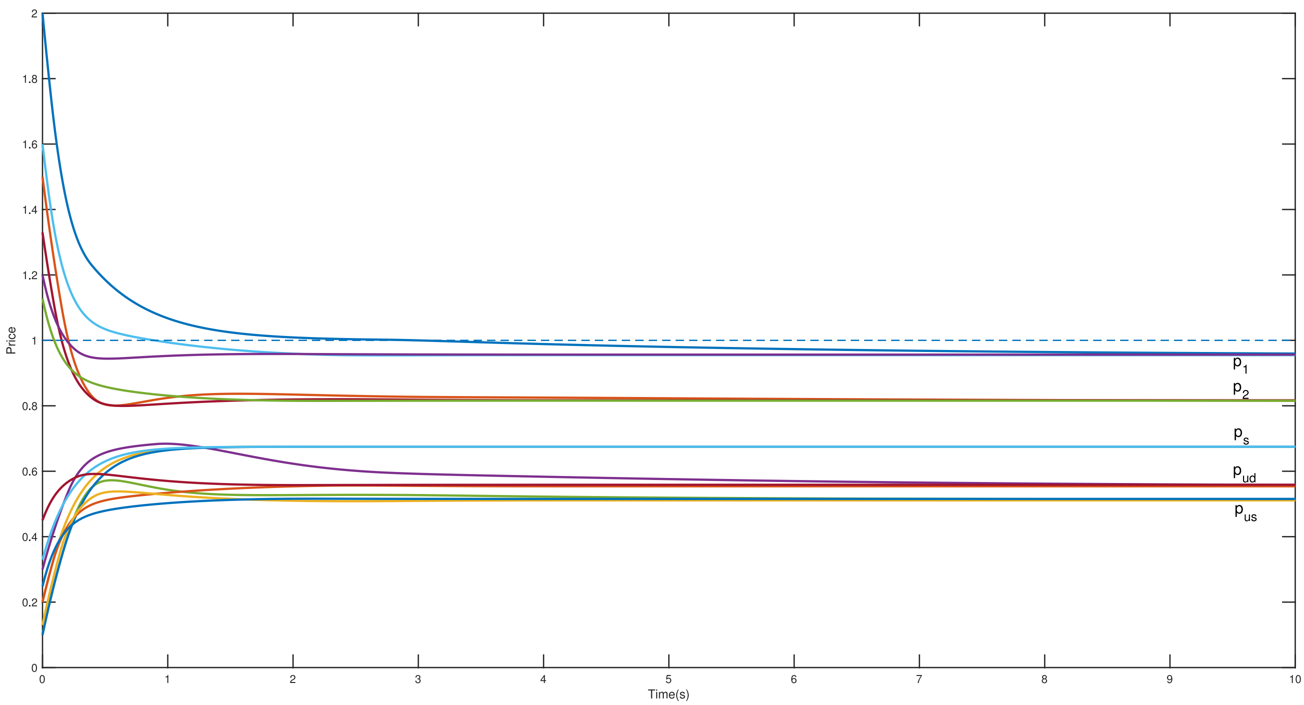

Fig. 3 Evolution of price state trajectory in RNN (3)

exploring how to set the price of samples and suits and the amount of coupons to maximize revenue. The parameter values in the optimization model are set as follows.

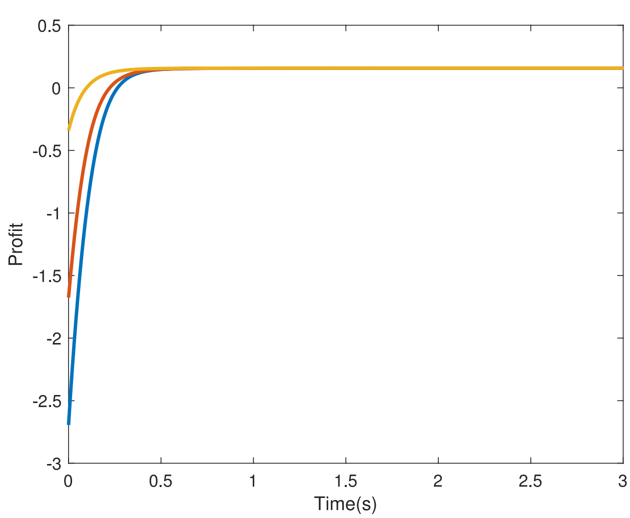

Fig. 4 Evolution of profit state trajectory in RNN (3)

4.1 The sample-regular product optimization model

In this analytical framework, merchants must strategically determine the optimal pricing levels for sample products and regular while simultaneously optimizing coupon incentives to maximize profitability. Price variations directly influence consumer purchasing power, creating dynamic market interactions that require in-depth sales model.

The proposed RNN (3) is employed to solve the sample-regular optimization model, with the corresponding numerical results being systematically presented in Figs. 3, 4.

As illustrated in Fig. 3, the pricing trajectories of regular products demonstrate adaptive convergence within the (0, 1) interval, confirming the efficacy of the projection operator in handling set constraints. The numerical results yield an optimal price vector $p = (p_1, p_2, p_s, p_{us}, p_{ud}) = (0.96, 0.81, 0.67, 0.56, 0.51)$, revealing a hierarchical pricing pattern where sample products maintain lower price points than formal merchandise,

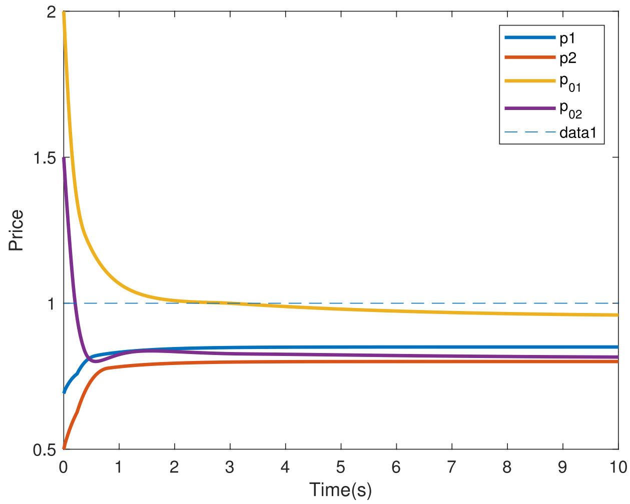

Fig. 5 Comparison of price evolution

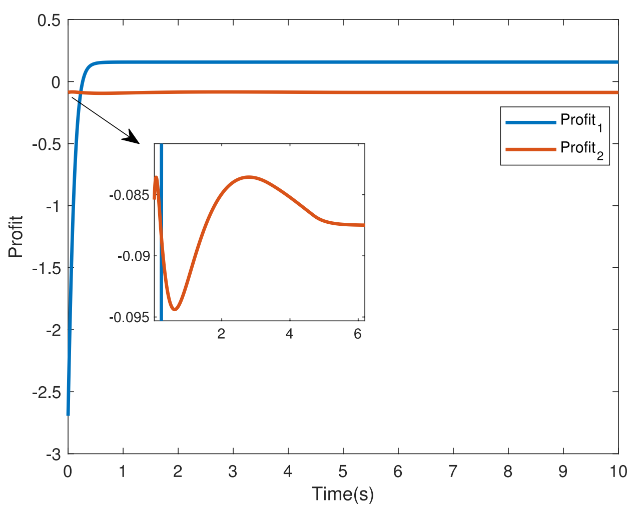

Fig. 6 Comparison of profit evolution

and the amount of the coupon is lower than the prices of the sample and the regular product. This systematic price differentiation substantiates the rationality of our pricing strategy. Furthermore, the convergent state solutions of the RNN (3) align with our theoretical framework, validating the stability of the proposed model. Figure 4 demonstrates the evolutionary process of profit maximization through iterative exploration of pricing strategies. The progressive optimization trajectory ultimately converges to the global maximum profit point, evidencing the algorithm's effectiveness in strategic pricing discovery.

In addition, the sample formal model proposed in this paper is compared with the traditional sales model, and the output results are shown in Figs. 5, 6.

According to [12–15], the demand function and utility function of consumers are both related to the pricing of goods by merchants. On the contrary, changes in consumer demand will affect the profits of businesses. Overall, changes in price and merchant profits reflect changes in consumer demand for the product and changes in market share for the product. Therefore, compared with traditional methods, this paper compares the price and merchant profit functions to demonstrate the superiority of this strategy. Figures 5, 6 show that compared with traditional sales methods, the sample-regular optimization model (1) can enable businesses to obtain higher profit margins. This phenomenon can be attributed to the dual advantages of the model in cost efficiency: while optimizing the production expenditure, it effectively reduces the return freight, thus providing a more effective sales strategy for modern commerce.

4.2 The discount optimization model

This kind of optimization model is a special case of the sample-regular product optimization model. Because there is a linear relationship between the price of coupons and the price of conventional products, it can still be solved by using the proposed RNN (3). The parameter selection of the discount optimization model is shown in Table 2, and the output of the RNN (3) is shown in Figs. 7, 8.

From the figures, it can be seen that even with different initial values, the final RNN state solution will always converge to the optimal solution of the problem, and If the initial state is $p(0) \notin \Omega$, the solution $p(t)$ of the RNN will converge exponentially to the set constraint Ω. If the initial state is $p(0) \in \Omega$, then for all $t > 0, p(t) \in \Omega$. This is consistent with theoretical analysis.

The simulation results presented in Figs. 7 and 8 demonstrate that the RNN (3) architecture effectively resolves the discount optimization model while exhibiting favorable convergence properties. Furthermore, comparative analyses reveal that the proposed discount model achieves comparable economic benefits to conventional sample-regular models when optimal discount parameters are implemented.

The numerical results presented in Figs. 3, 8 demonstrate the dynamic convergence of pricing trajectories and profit optimization under the proposed RNN framework. While specific numerical values of the optimal price vector (e.g., $p_1 = 0.96$, $p_2 = 0.81$) are derived from the model's parameterization (Table 2), their hierarchical structure reflects strategic trade-offs inherent in real-world pricing. For instance, the lower sample price ($p_s = 0.67$) aligns with the objective of reducing consumer risk perception and incentivizing trial adoption, while the higher regular product prices (p_1, p_2) ensure profitability after coupon redemption. The coupon

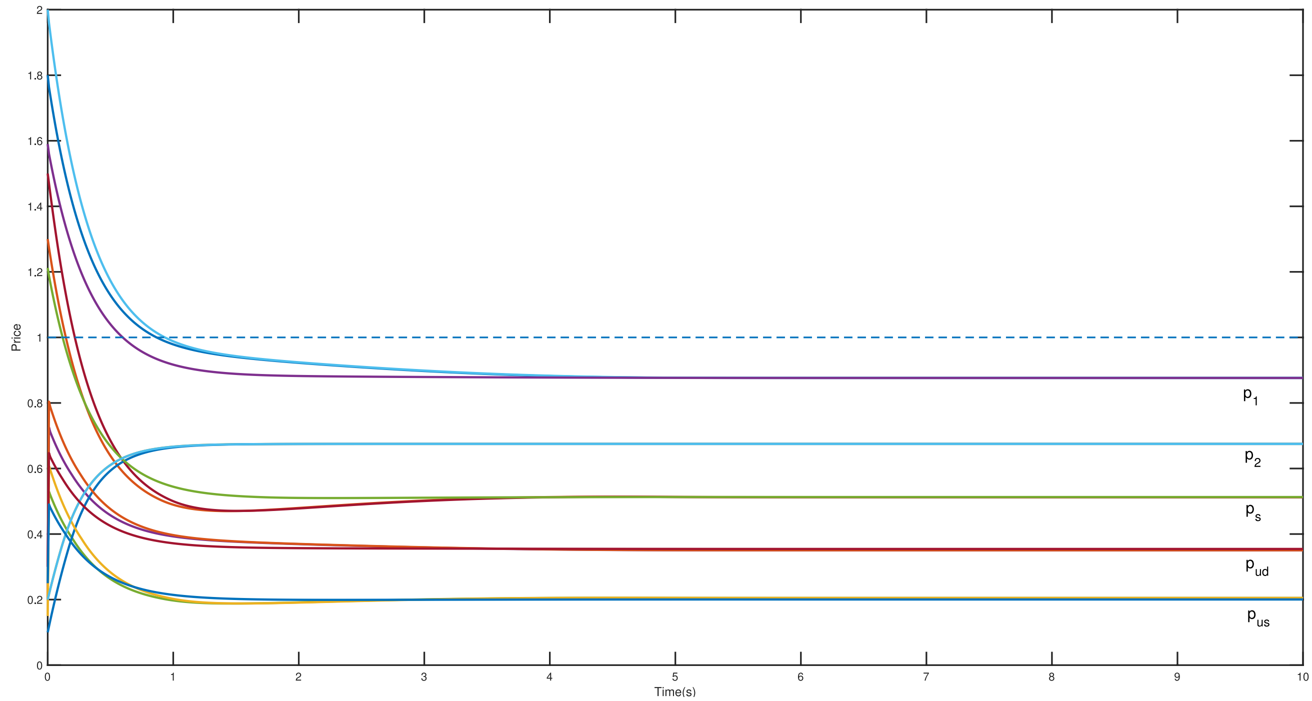

Fig. 7 Evolution of profit state trajectory in RNN (3)

values ($p_{us} = 0.56$, $p_{ud} = 0.51$) are strategically positioned between sample and regular prices to balance consumer appeal and margin preservation-a common practice in promotional pricing.

The profit evolution curves highlight the RNN's ability to navigate complex constraints (e.g., production costs, coupon dependencies) and rapidly converge to equilibrium. The steep initial ascent in profit trajectories (Figs. 4 and 8) signifies the model's efficiency in escaping suboptimal pricing regimes, while the eventual stabilization reflects the attainment of a globally optimal balance between market penetration and revenue maximization. Comparative results (Figs. 5, 6) further underscore the model's advantage over traditional strategies: the sample-regular framework achieves higher profit margins by reducing return costs and enhancing post-trial conversions, consistent with empirical observations in e-commerce. These dynamics validate the RNN's capacity to emulate real-world retailer decision-making, where adaptive pricing and promotion adjustments are critical for sustaining competitiveness. The convergence properties also align with practical requirements for real-time responsiveness in dynamic markets.

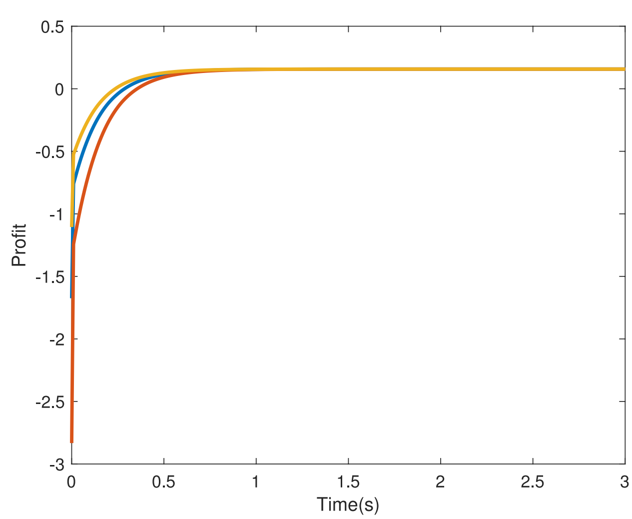

Fig. 8 Evolution of profit state trajectory in RNN (3)

5 Conclusion

This study proposes a retailer sampling model based on RNN, demonstrating the model's structure, which consists of two components: the profit function after sample distribution by retailers and the profit function from consumers' subsequent purchase. It effectively captures the interconnected consumer behavior of purchasing samples and continuing to buy regular-size products. In this approach, the RNN addresses complex computations that are challenging to perform with discrete-time methods. The model not only directly supports retailers in formulating sampling marketing strategies but also optimizes sampling ratios and pricing through profit function analysis, offering a novel framework for strategy development. Future research could explore additional scenarios, such as free trial sampling with shipping fees covered by customers. Further extensions could generalize the framework to n-dimensional product categories, investigating multi-product sampling interactions and cross-category demand dependencies under high-dimensional parameter spaces.

Appendixes

Proof of lemma 2.1

We can obtain the partial derivative of the function π_1 with respect to p_s as follows.

$$\begin{aligned}\frac{\partial \pi_1}{\partial p_s} &= -(p_s - \lambda(c_1 + c_2) - c_a) + (1 - p_s)\\ &= -2p_s + \lambda(c_1 + c_2) + c_a + 1.\end{aligned}$$

Then, the second-order partial derivative of the function π_1 with respect to p_s can be obtained as follows.

$$\frac{\partial^2 \pi_1}{\partial^2 p_s} = -2.$$

So the objective function of the formal attire optimization model is a concave function.

Proof of lemma 2.2

We can obtain the partial derivative of the function π_2 with respect to p_1 as follows.

$$\begin{aligned}\frac{\partial \pi_2}{\partial p_1} &= -\left(\frac{\theta_1 + \theta_2}{a}p_1 + \frac{\theta_1 + \theta_3}{a}p_2 - \frac{\theta_1}{a}p_{us} - \frac{\theta_2 + \theta_3}{a}p_{ud}\right)(\theta_1 + \theta_2)\\ &+ \left(\theta_1(p_1 + p_2 - p_{us}) + \theta_2(p_1 - p_{ud}) + \theta_3(p_2 - p_{ud})\right)\left(-\frac{\theta_1 + \theta_2}{a}\right).\end{aligned}$$

Then, the second-order partial derivative of the function π_2 with respect to p_1, p_2, p_{us} and p_{ud} can be obtained as follows.

$$\frac{\partial^2 \pi_2}{\partial^2 p_1} = -\frac{2(\theta_1 + \theta_2)^2}{a};\ \frac{\partial^2 \pi_2}{\partial p_1 \partial p_2} = -\frac{2(\theta_1 + \theta_2)(\theta_1 + \theta_3)}{a};$$
$$\frac{\partial^2 \pi_2}{\partial p_1 \partial p_{us}} = \frac{2\theta_1(\theta_1 + \theta_2)}{a};\ \frac{\partial^2 \pi_2}{\partial p_1 \partial p_{ud}} = \frac{2(\theta_1 + \theta_2)(\theta_2 + \theta_3)}{a}.$$

The partial derivative of the function π_2 with respect to p_2 can be obtained as follows.

$$\begin{aligned}\frac{\partial \pi_2}{\partial p_2} &= (\theta_1 + \theta_3)\left[1 - \left(\frac{\theta_1 + \theta_2}{a}p_1 + \frac{\theta_1 + \theta_3}{a}p_2 - \frac{\theta_1}{a}p_{us} - \frac{\theta_2 + \theta_3}{a}p_{ud}\right)\right]\\ &+ (\theta_1(p_1 + p_2 - p_{us}) + \theta_2(p_1 - p_{ud}) + \theta_3(p_2 - p_{ud}))\left(-\frac{\theta_1 + \theta_3}{a}\right).\end{aligned}$$

Then, the second-order partial derivative of the function π_2 with respect to p_1, p_2, p_{us} and p_{ud} can be obtained as follows.

$$\frac{\partial^2 \pi_2}{\partial p_2 \partial p_1} = -\frac{2(\theta_1 + \theta_3)(\theta_1 + \theta_2)}{a};\ \frac{\partial^2 \pi_2}{\partial^2 p_2} = -\frac{2(\theta_1 + \theta_3)^2}{a};$$
$$\frac{\partial^2 \pi_2}{\partial p_2 \partial p_{us}} = \frac{2\theta_1(\theta_1 + \theta_3)}{a};\ \frac{\partial^2 \pi_2}{\partial p_2 \partial p_{ud}} = \frac{2(\theta_1 + \theta_3)(\theta_2 + \theta_3)}{a}.$$

The partial derivative of the function π_2 with respect to p_{us} can be obtained as follows.

$$\begin{aligned}\frac{\partial \pi_2}{\partial p_{us}} &= -\theta_1\left[1 - \left(\frac{\theta_1 + \theta_2}{a}p_1 + \frac{\theta_1 + \theta_3}{a}p_2 - \frac{\theta_1}{a}p_{us} - \frac{\theta_2 + \theta_3}{a}p_{ud}\right)\right]\\ &+ (\theta_1(p_1 + p_2 - p_{us}) + \theta_2(p_1 - p_{ud}) + \theta_3(p_2 - p_{ud}))\left(\frac{\theta_1}{a}\right).\end{aligned}$$

Then, the second-order partial derivative of the function π_2 with respect to p_1, p_2, p_{us} and p_{ud} can be obtained as follows.

$$\frac{\partial^2 \pi_2}{\partial p_{us} \partial p_1} = \frac{2\theta_1(\theta_1 + \theta_2)}{a};\ \frac{\partial^2 \pi_2}{\partial p_{us} \partial p_2} = \frac{2\theta_1(\theta_1 + \theta_3)}{a};$$
$$\frac{\partial^2 \pi_2}{\partial^2 p_{us}} = -\frac{2\theta_1^2}{a};\ \frac{\partial^2 \pi_2}{\partial p_{us} \partial p_{ud}} = -\frac{2\theta_1(\theta_2 + \theta_3)}{a}.$$

The partial derivative of the function π_2 with respect to p_{ud} can be obtained as follows.

$$\frac{\partial \pi_2}{\partial p_d} = -(\theta_2+\theta_3)\left[1-\left(\frac{\theta_1+\theta_2}{a}p_1+\frac{\theta_1+\theta_3}{a}p_2-\frac{\theta_1}{a}p_{us}-\frac{\theta_2+\theta_3}{a}p_{ud}\right)\right] + (\theta_1(p_1+p_2-p_{us})+\theta_2(p_1-p_{ud})+\theta_3(p_2-p_{ud}))\left(\frac{\theta_2+\theta_3}{a}\right).$$

Then, the second-order partial derivative of the function π_2 with respect to p_1, p_2, p_{us} and p_{ud} can be obtained as follows.

$$\frac{\partial^2 \pi_2}{\partial p_{ud}\partial p_1} = \frac{2(\theta_2+\theta_3)(\theta_1+\theta_2)}{a}; \frac{\partial^2 \pi}{\partial p_{ud}\partial p_2} = \frac{2(\theta_2+\theta_3)(\theta_1+\theta_3)}{a};$$
$$\frac{\partial^2 \pi}{\partial p_{ud}\partial p_{us}} = -\frac{2\theta_1(\theta_2+\theta_3)}{a}; \frac{\partial^2 \pi}{\partial^2 p_{ud}} = 1\frac{2(\theta_2+\theta_3)^2}{a}.$$

So

$$\nabla \pi_2 = \begin{pmatrix} -\frac{2(\theta_1+\theta_2)^2}{a} & -\frac{2(\theta_1+\theta_2)(\theta_1+\theta_3)}{a} & \frac{2\theta_1(\theta_1+\theta_2)}{a} & \frac{2\theta_1(\theta_1+\theta_2)(\theta_2+\theta_3)}{a} \\ -\frac{2(\theta_1+\theta_3)(\theta_1+\theta_2)}{a} & -\frac{2(\theta_1+\theta_3)^2}{a} & \frac{2\theta_1(\theta_1+\theta_3)}{a} & \frac{2(\theta_1+\theta_3)(\theta_2+\theta_3)}{a} \\ \frac{2\theta_1(\theta_1+\theta_2)}{a} & \frac{2\theta_1(\theta_1+\theta_3)}{a} & -\frac{2\theta_1^2}{a} & -\frac{2\theta_1(\theta_2+\theta_3)}{a} \\ \frac{2(\theta_2+\theta_3)(\theta_1+\theta_2)}{a} & \frac{2(\theta_2+\theta_3)(\theta_1+\theta_3)}{a} & -\frac{2\theta_1(\theta_2+\theta_3)}{a} & -\frac{2(\theta_2+\theta_3)^2}{a} \end{pmatrix}$$

First-order principal minor is

$$-(\theta_1+\theta_2)^2.$$

Second-order principal minor is 0. Third-order principal minor is 0. Fourth-order principal minor is 0. So, π_2 is a concave function.

Proof of lemma 2.3

The partial derivative of the function π_2 with respect to p_1 can be obtained as follows.

$$\frac{\partial \pi_2}{\partial p_1} = (\theta_1(1-b)+\theta_2(1-a_1)) \left[1-\left(\frac{\theta_1(1-b)}{r}(p_1+p_2)+\frac{\theta_2(1-a_1)}{r}p_1+\frac{\theta_3(1-a_2)}{r}p_2\right)\right] + [\theta_1(1-b)(p_1+p_2)+\theta(1-a_1)p_1 + \theta_3(1-a_2)p_2]\left[-\frac{\theta_1(1-b)+\theta_2(1-a_1)}{r}\right].$$

Then, the second-order partial derivative of the function π_2 with respect to $p_1$and p_2 can be obtained as follows.

$$\frac{\partial^2 \pi_2}{\partial^2 p_1} = -\frac{2[\theta_1(1-b)+\theta_2(1-a_1)]^2}{r}; \frac{\partial^2 \pi_2}{\partial p_1 \partial p_2} = -\frac{(\theta_1(1-b)+\theta_2(1-a_1))(\theta_1(1-b)+\theta_3(1-a_2))}{r}.$$

The partial derivative of the function π_2 with respect to p_2 can be obtained as follows.

$$\frac{\partial \pi_2}{\partial p_2} = (\theta_1(1-b)+\theta_3(1-a_2)) \left[1-\left(\frac{\theta_1(1-b)}{r}(p_1+p_2)+\frac{\theta_2(1-a_1)}{r}p_1+\frac{\theta_3(1-a_2)}{r}p_2\right)\right] + [\theta_1(1-b)(p_1+p_2)+\theta_2(1-a_1)p_1 + \theta_3(1-a_2)p_2]\left(-\frac{\theta_1(1-b)+\theta_3(1-a_2)}{r}\right).$$

Then, the second-order partial derivative of the function π_2 with respect to $p_1$and p_2 can be obtained as follows.

$$\frac{\partial^2 \pi_2}{\partial p_2 \partial p_1} = -\frac{2(\theta_1(1-b)+\theta_3(1-a_2))(\theta_1(1-b)+\theta_2(1-a_1))}{r};$$
$$\frac{\partial^2 \pi_2}{\partial^2 p_2} = -\frac{2[\theta_1(1-b)+\theta_3(1-a_2)]^2}{r}.$$

First-order principal minor is

$$-\frac{2[\theta_1(1-b)+\theta_2(1-a_1)]^2}{r}.$$

Second-order principal minor is 0. So, π_2 is a concave function.

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Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that there is no Conflict of interest regarding the publication of this paper.

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