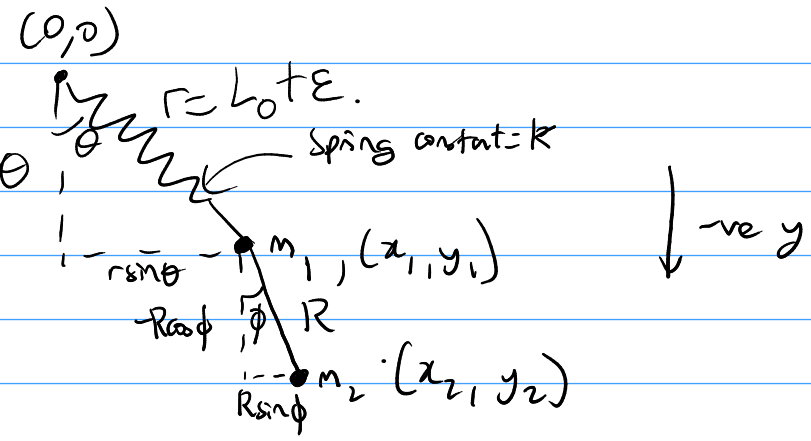
$K, R, L_0, m_1, m_2 = 0$



$$x_1 = r\sin\theta, \quad y_1 = -r\cos\theta$$
$$x_2 = r\sin\theta + R\sin\phi, \quad y_2 = -r\cos\theta - R\cos\phi$$

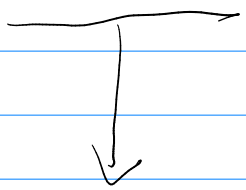
$$\dot{x}_1 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta, \quad \dot{y}_1 = -\dot{r}\cos\theta + r\dot{\theta}\sin\theta$$
$$\dot{x}_2 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta + R\dot{\phi}\cos\phi, \quad \dot{y}_2 = -\dot{r}\cos\theta + r\dot{\theta}\sin\theta + R\dot{\phi}\sin\phi$$

$$L = \underbrace{\tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2)}_{KE_1} - \underbrace{m_1 g y_1}_{GPE_1} - \underbrace{\tfrac{1}{2}k\left(\sqrt{x_1^2 + y_1^2} - L_0\right)^2}_{EPE_1} + \underbrace{\tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)}_{KE_2} - \underbrace{m_2 g y_2}_{GPE_2}$$

$$\dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + R^2\dot{\phi}^2\cos^2\phi + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + 2\dot{r}\dot{\phi}R\sin\theta\cos\phi$$
$$+ 2rR\dot{\theta}\dot{\phi}\cos\theta\cos\phi \ (= \dot{x}_2^2)$$

$$+ \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta + R^2\dot{\phi}^2\sin^2\phi - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta - 2\dot{r}R\dot{\phi}\cos\theta\sin\phi$$
$$+ 2rR\dot{\theta}\dot{\phi}\sin\theta\sin\phi \ (= \dot{y}_2^2)$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 + R^2\dot{\phi}^2 + 2R\dot{r}\dot{\phi}\sin(\theta - \phi) + 2Rr\dot{\theta}\dot{\phi}\cos(\theta - \phi)$$



$$L = \frac{1}{2}m_1\left[\dot{r}^2 + r^2\dot{\theta}^2\right] + m_1 g r\cos\theta - \frac{1}{2}kr^2 + krL_0 - \frac{1}{2}kL_0^2$$

$$+\frac{1}{2}m_2\left[\dot{r}^2 + r^2\dot{\theta}^2 + R^2\dot{\phi}^2 + 2R\dot{r}\dot{\phi}\sin(\theta-\phi) + 2Rr\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right]$$

$$- m_2 g\left(-r\cos\theta - R\cos\phi\right)$$

$$L = \frac{1}{2}m_1\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 + m_1 g r\cos\theta - \frac{1}{2}kr^2 + Krl_0 - \frac{1}{2}kL_0^2$$

$$+\frac{1}{2}m_2\dot{r}^2 + \frac{1}{2}m_2 r^2\dot{\theta}^2 + \frac{1}{2}m_2 R^2\dot{\phi}^2 + m_2 R\dot{r}\dot{\phi}\sin(\theta-\phi)$$

$$+ m_2 Rr\dot{\theta}\dot{\phi}\cos(\theta-\phi) + m_2 g r\cos\theta + m_2 g R\cos\phi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

$$\frac{\partial L}{\partial r} = m_1\dot{\theta}^2 r + m_2\dot{\theta}^2 r + m_1 g\cos\theta + m_2 g\cos\theta$$
$$-kr + kL_0 + m_2 R\dot{\theta}\dot{\phi}\cos(\theta-\phi)$$

$$\frac{\partial L}{\partial \dot{r}} = m_1\dot{r} + m_2\dot{r} + m_2 R\dot{\phi}\sin(\theta-\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m_1\ddot{r} + m_2\ddot{r} + m_2 R\dot{\phi}\sin(\theta-\phi) + m_2 R\ddot{\phi}\cos(\theta-\phi)$$
$$- m_2 R\dot{\phi}^2\cos(\theta-\phi)$$

$$m_1\dot{\theta}^2 r + m_2\dot{\theta}^2 r + m_1 g\cos\theta + m_2 g\cos\theta - k(r-L_0)$$
$$= m_1\ddot{r} + m_2\ddot{r} + m_2 R\ddot{\phi}\sin(\theta-\phi) - m_2 R\dot{\phi}^2\cos(\theta-\phi)$$

L[r]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\frac{\partial L}{\partial \theta} = -m_1 g r \sin\theta - m_2 g r \sin\theta + m_2 R r \dot{\phi}\cos(\theta-\phi) - m_2 R r \dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \dot{\theta} + m_2 r^2 \dot{\theta} + m_2 R r \dot{\phi}\cos(\theta-\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \ddot{\theta} + 2m_1 r \dot{r}\dot{\theta} + m_2 r^2 \ddot{\theta} + 2m_2 r \dot{r}\dot{\theta} - m_2 R r \dot{\phi}\dot{\theta}\sin(\theta-\phi) + m_2 R r \dot{\phi}^2 \sin(\theta-\phi) + m_2 R \dot{r}\dot{\phi}\cos(\theta-\phi) + m_2 R r \ddot{\phi}\cos(\theta-\phi)$$

$L[\theta]$

$$-m_1 g r \sin\theta - m_2 g r \sin\theta = m_1 r^2 \ddot{\theta} + 2m_1 r \dot{r}\dot{\theta} + m_2 r^2 \ddot{\theta} + 2m_2 r \dot{r}\dot{\theta} + m_2 R r \dot{\phi}^2 \sin(\theta-\phi) + m_2 R r \ddot{\phi}\cos(\theta-\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}$$

$$\frac{\partial L}{\partial \phi} = -m_2 R \dot{r}\dot{\phi}\cos(\theta-\phi) + m_2 R r \dot{\theta}\dot{\phi}\sin(\theta-\phi) - m_2 g R \sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = m_2 R^2 \dot{\phi} + m_2 R \dot{r}\sin(\theta-\phi) + m_2 R r \dot{\theta}\cos(\theta-\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = m_2 R^2 \ddot{\phi} + m_2 R \dot{r}\sin(\theta-\phi) + 2m_2 R \dot{r}\dot{\theta}\cos(\theta-\phi) - m_2 R \dot{r}\dot{\phi}\cos(\theta-\phi) + m_2 R r \ddot{\theta}\cos(\theta-\phi) - m_2 R r \dot{\theta}^2 \sin(\theta-\phi) + m_2 R r \dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

Then:

$$-m_2R^2\ddot{\phi} + m_2R\ddot{r}\sin(\theta-\phi) + 2m_2R\dot{r}\dot{\theta}\cos(\theta-\phi) + m_2R r\ddot{\theta}\cos(\theta-\phi) - m_2Rr\dot{\theta}^2\sin(\theta-\phi) = -m_2gR\sin\phi$$

The idea from here is to get $(\ddot{r}, \ddot{\theta}, \ddot{\phi})$. We use this to get $(\dot{r}, \dot{\theta}, \dot{\phi})$, then (r, θ, ϕ) and hence (x_1, y_1, x_2, y_2) iteratively.

I think the easiest way to do this is by solving a matrix; first rearrange all $L[z]$ to make $\ddot{z}$ the subject. The RHS for all $L[z]$:

A_r: $\left(\dot{\theta}^2 r + g\cos\theta\right)(m_1+m_2) - k r + k L_0 + m_2 R\dot{\phi}^2\cos(\theta-\phi)$

A_θ: $-gr\sin\theta(m_1+m_2) - m_2Rr\dot{\phi}^2\sin(\theta-\phi) - 2r\dot{r}\dot{\theta}(m_1+m_2)$

A_ϕ: $-2m_2R\dot{r}\dot{\theta}\cos(\theta-\phi) + m_2Rr\dot{\theta}^2\sin(\theta-\phi) - m_2gR\sin\phi$

	Coeff $(\ddot{r})$	Coeff $(\ddot{\theta})$	Coeff $(\ddot{\phi})$		
from $L[r]$:	m_1+m_2	0	$m_2R\sin(\theta-\phi)$	$\ddot{r}$	A_r
from $L[\theta]$:	0	$m_1r^2 + m_2r^2$	$m_2Rr\cos(\theta-\phi)$	$\ddot{\theta}$	A_θ
from $L[\phi]$:	$m_2R\sin(\theta-\phi)$	$m_2Rr\cos(\theta-\phi)$	m_2R^2	$\ddot{\phi}$	A_ϕ

M | from rearranged $L[z]$

Then:

$$\begin{bmatrix}\ddot{r}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = M^{-1}A$$