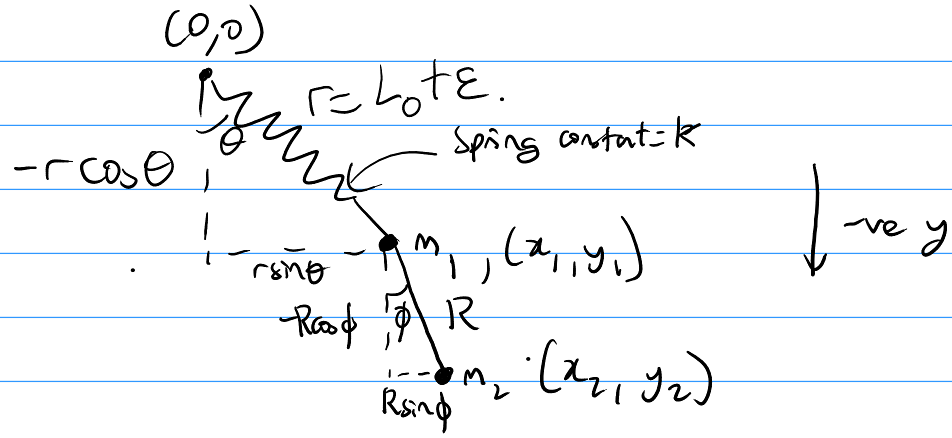
$K, R, L_0, \dot{m}_1, \dot{m}_2 = 0$



$$x_1 = r\sin\theta, \quad y_1 = -r\cos\theta$$
$$x_2 = r\sin\theta + R\sin\phi, \quad y_2 = -r\cos\theta - R\cos\phi$$

$$\dot{x}_1 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta, \quad \dot{y}_1 = -\dot{r}\cos\theta + r\dot{\theta}\sin\theta$$
$$\dot{x}_2 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta + R\dot{\phi}\cos\phi, \quad \dot{y}_2 = -\dot{r}\cos\theta + r\dot{\theta}\sin\theta + R\dot{\phi}\sin\phi$$

$$L = \overset{KE_1}{\frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right)} - \overset{GPE_1}{m_1 g y_1} - \overset{EPE_1}{\frac{1}{2}k\left(\sqrt{x_1^2+y_1^2} - L_0\right)^2} +$$
$$\overset{KE_2}{\frac{1}{2}m_2\left(\underline{\dot{x}_2^2 + \dot{y}_2^2}\right)} - \overset{GPE_2}{m_2 g y_2}$$

$$\dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + R^2\dot{\phi}^2\cos^2\phi + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + 2\dot{r}\dot{\phi}R\sin\theta\cos\phi$$
$$+ 2rR\dot{\theta}\dot{\phi}\cos\theta\cos\phi \quad (= \dot{x}_2^2)$$

$$+ \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta + R^2\dot{\phi}^2\sin^2\phi - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta - 2\dot{r}R\dot{\phi}\cos\theta\sin\phi$$
$$+ 2rR\dot{\theta}\dot{\phi}\sin\theta\sin\phi \quad (= \dot{y}_2^2)$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 + R^2\dot{\phi}^2 + 2R\dot{r}\dot{\phi}\sin(\theta-\phi) + 2Rr\dot{\theta}\dot{\phi}\cos(\theta-\phi)$$

$$L = \frac{1}{2} m_1 \left[\dot{r}^2 + r^2 \dot{\theta}^2 \right] + m_1 g r \cos\theta - \frac{1}{2} k r^2 + k r L_0 - \frac{1}{2} k L_0^2$$

$$+ \frac{1}{2} m_2 \left[\dot{r}^2 + r^2 \dot{\theta}^2 + R^2 \dot{\phi}^2 + 2R \dot{r} \dot{\phi} \sin(\theta - \phi) + 2 R r \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right]$$

$$- m_2 g \left(- r\cos\theta - R\cos\phi \right)$$

$$\boxed{\begin{aligned} L = &\frac{1}{2} m_1 \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\theta}^2 + m_1 g r \cos\theta - \frac{1}{2} k r^2 + k r L_0 - \frac{1}{2} k L_0^2 \\ &+ \frac{1}{2} m_2 \dot{r}^2 + \frac{1}{2} m_2 r^2 \dot{\theta}^2 + \frac{1}{2} m_2 R^2 \dot{\phi}^2 + m_2 R \dot{r} \dot{\phi} \sin(\theta - \phi) \\ &+ m_2 R r \dot{\theta} \dot{\phi} \cos(\theta - \phi) + m_2 g r \cos\theta + m_2 g R \cos\phi \end{aligned}}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

$$\frac{\partial L}{\partial r} = m \dot{\theta}^2 r + m_2 \dot{\theta}^2 r + m_1 g \cos\theta + m_2 g \cos\theta - kr + kL_0 + m_2 R \dot{\theta} \dot{\phi} \cos(\theta - \phi)$$

$$\frac{\partial L}{\partial \dot{r}} = m_1 \dot{r} + m_2 \dot{r} + m_2 R \dot{\phi} \sin(\theta - \phi)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m_1 \ddot{r} + m_2 \ddot{r} + m_2 R \ddot{\phi} \sin(\theta - \phi) + m_2 R \dot{\phi} \dot{\theta} \cos(\theta - \phi) - m_2 R \dot{\phi}^2 \cos(\theta - \phi)$$

L[r]

$$\boxed{\begin{aligned} &m_1 \dot{\theta}^2 r + m_2 \dot{\theta}^2 r + m_1 g \cos\theta + m_2 g \cos\theta - k(r - L_0) \\ &= m_1 \ddot{r} + m_2 \ddot{r} + m_2 R \ddot{\phi} \sin(\theta - \phi) - m_2 R \dot{\phi}^2 \cos(\theta - \phi) \end{aligned}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\frac{\partial L}{\partial \theta} = -m_1 g r \sin\theta - m_2 g r \sin\theta + m_2 R \dot{r} \dot{\phi} \cos(\theta-\phi) - m_2 R r \dot{\theta} \dot{\phi} \sin(\theta-\phi)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \dot{\theta} + m_2 r^2 \dot{\theta} + m_2 R r \dot{\phi} \cos(\theta-\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \ddot{\theta} + 2m_1 r \dot{r} \dot{\theta} + m_2 r^2 \ddot{\theta} + 2m_2 r \dot{r} \dot{\theta} - m_2 R r \dot{\phi} \dot{\theta} \sin(\theta-\phi) + m_2 R r \dot{\phi}^2 \sin(\theta-\phi) + m_2 R \dot{r} \dot{\phi} \cos(\theta-\phi) + m_2 R r \ddot{\phi} \cos(\theta-\phi)$$

$L[\theta]$

$$-m_1 g r \sin\theta - m_2 g r \sin\theta = m_1 r^2 \ddot{\theta} + 2m_1 r \dot{r} \dot{\theta} + m_2 r^2 \ddot{\theta} + 2m_2 r \dot{r} \dot{\theta} + m_2 R r \dot{\phi}^2 \sin(\theta-\phi) + m_2 R r \ddot{\phi} \cos(\theta-\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}$$

$$\frac{\partial L}{\partial \phi} = -m_2 R \dot{r} \dot{\phi} \cos(\theta-\phi) + m_2 R r \dot{\theta} \dot{\phi} \sin(\theta-\phi) - m_2 g R \sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = m_2 R^2 \dot{\phi} + m_2 R \dot{r} \sin(\theta-\phi) + m_2 R r \dot{\theta} \cos(\theta-\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = m_2 R^2 \ddot{\phi} + m_2 R \ddot{r} \sin(\theta-\phi) + 2m_2 R \dot{r} \dot{\theta} \cos(\theta-\phi) - m_2 R \dot{r} \dot{\phi} \cos(\theta-\phi) + m_2 R r \ddot{\theta} \cos(\theta-\phi) - m_2 R r \dot{\theta}^2 \sin(\theta-\phi) + m_2 R r \dot{\theta} \dot{\phi} \sin(\theta-\phi)$$

Thus:

$$L[\phi]:\quad \boxed{\begin{aligned}&-m_2R^2\ddot{\phi} + m_2R\ddot{r}\sin(\theta-\phi) + 2m_2R\dot{r}\dot{\theta}\cos(\theta-\phi)\\ &+ m_2Rr\ddot{\theta}\cos(\theta-\phi) - m_2Rr\dot{\theta}^2\sin(\theta-\phi) = -m_2gR\sin\phi\end{aligned}}$$

The idea from here is to get $(\ddot{r}, \ddot{\theta}, \ddot{\phi})$. We use this to get $(\dot{r}, \dot{\theta}, \dot{\phi})$, then (r, θ, ϕ) and hence (x_1, y_1, x_2, y_2) iteratively.

I think the easiest way to do this is by solving a matrix; first rearrange all $L[z]$ to make $\ddot{z}$ the subject. The RHS for all $L[z]$:

A_r: $(\dot{\theta}^2 r + g\cos\theta)(m_1+m_2) - kr + kL_0 + m_2R\dot{\phi}^2\cos(\theta-\phi)$

A_θ: $-gr\sin\theta(m_1+m_2) - m_2Rr\dot{\phi}^2\sin(\theta-\phi) - 2r\dot{r}\dot{\theta}(m_1+m_2)$

A_ϕ: $-2m_2R\dot{r}\dot{\theta}\cos(\theta-\phi) + m_2Rr\dot{\theta}^2\sin(\theta-\phi) - m_2gR\sin\phi$.

$$\boxed{\begin{array}{l l} & \begin{array}{ccc}\text{Coeff}(\ddot{r}) & \text{Coeff}(\ddot{\theta}) & \text{Coeff}(\ddot{\phi})\end{array}\\ \begin{array}{l}\text{from } L[r]:\\ \text{from } L[\theta]:\\ \text{from } L[\phi]:\end{array} & \underbrace{\begin{bmatrix} m_1+m_2 & 0 & m_2R\sin(\theta-\phi)\\ 0 & m_1r^2+m_2r^2 & m_2Rr\cos(\theta-\phi)\\ m_2R\sin(\theta-\phi) & m_2Rr\cos(\theta-\phi) & m_2R^2\end{bmatrix}}_{M}\begin{bmatrix}\ddot{r}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = \underbrace{\begin{bmatrix}A_r\\ A_\theta\\ A_\phi\end{bmatrix}}_{\text{from rearranged } L[z]}\end{array}}$$

Then:

$$\begin{bmatrix}\ddot{r}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = M^{-1}A.$$