CHAPTER 3: APPLICATIONS OF FOURIER ANALYSIS

3.1 Introduction

3.1.1 Motivation

To apply the results and properties of Fourier series and Fourier transform to the resolution of some differential equations.

$$F :]a,b[\times \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m+1 \text{ times}} \longrightarrow \mathbb{R}^N$$

(small error during lecture)

Find $u :]a,b[\to \mathbb{R}^n$ s.t.

$$F(t, u(t), u'(t), u''(t), \ldots, u^{(m)}(t)) = 0$$

If $N=1 \longrightarrow$ ODE

If $N \geq 2 \longrightarrow$ PDE

Example: Cauchy problem.

$u : [0, +\infty] \to \mathbb{R}$

$u''(t) + u'(t) + u(t) = 1 \quad \forall\, t \in]0, +\infty[$

$$\left\{\begin{array}{l} u(0) = u_0 \\ u'(0) = v_0 \end{array}\right\} \text{ Initial conditions.}$$

Example: Sturm-Liouville problem

$u : [0, L] \to \mathbb{R} \qquad \lambda \in \mathbb{R}.$

$u''(t) + \lambda u(t) = 0$

$u(0) = u(L) = 0$ } Dirichlet conditions.

Example:

$u : \mathbb{R} \to \mathbb{R}$

$u' + u * g = f$ or $u + u * g = f$

$\int_{-\infty}^{+\infty} |u(x)|\, dx < \infty$

Recall: $u * g = \int_{-\infty}^{+\infty} u(t)\, g(x-t)\, dt.$

Example:

$u(x) + u(x-\pi) = f(x)$

3.2 Applications of Fourier series

$u(0) = u(T)$, $u(x) = u(x+T)$, or

is T-periodic

Heat diffussion

$(x,y) = R(\cos\theta, \sin\theta)$

$$\frac{\partial^2 u}{\partial \theta^2} = R^2 f(R\cos\theta, R\sin\theta)$$

f: source term.

$u(\theta=0) = u(\theta=2\pi)$

$u'(\theta=0) = u'(\theta=2\pi)$

u: temperature of annular plate.

• Example 1: let be $\alpha \neq \pm 1$, $\alpha \in \mathbb{R}$.

and $f: \mathbb{R} \to \mathbb{R}$, piecewise-defined, 2π-periodic

Find $u: [0, 2\pi] \to \mathbb{R}$ s.t.

$u(t) + \alpha u(t-\pi) = f(t)$ $\forall$ $t \in]0, 2\pi[$

$u(0) = u(2\pi)$ s.t. u is 2π-periodic.

u is the unknown, α and f are known.

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nt) + b_n \sin(nt) \right]$$

a_0, a_n, b_n are unknowns.

$$f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos(nt) + \beta_n \sin(nt) \right]$$

$\alpha_0, \alpha_n, \beta_n$ are known.

$$u(t-\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nt - n\pi) + b_n \sin(nt - n\pi) \right]$$

$$\left(\begin{array}{l} \cos(nt - n\pi) = (-1)^n \cos(nt) \\ \sin(nt - n\pi) = (-1)^n \sin(nt) \end{array} \right)$$

$$u(t-\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[(-1)^n a_n \cos(nt) + (-1)^n b_n \sin(nt) \right]$$

$$\frac{1+\alpha}{2} a_0 + \sum_{n=1}^{\infty} \Big[(1+\alpha(-1)^n) \cos(nt) + (1+\alpha(-1)^n) \sin(nt) \Big]$$

$$= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos(nt) + \beta_n \sin(nt) \right]$$

$$a_0 = \frac{1}{1+\alpha} \alpha_0, \quad a_n = \frac{1}{1+(-1)^n \alpha} \alpha_n, \quad b_n = \frac{1}{1+(-1)^n \alpha} \beta_n$$

If, eg., $f(t) = \cos t + 3 \sin(2t) + 4 \cos(5t)$

$$\alpha_1 = 1, \quad \alpha_5 = 4, \quad \beta_2 = 3$$

$$u(t) = \frac{1}{1-\alpha} \cos t + \frac{4}{1-\alpha} \cos(5t) + \frac{3}{1+\alpha} \sin(2t)$$

Example 2: let $f : \mathbb{R} \to \mathbb{R}$ defined by

$f(x) = x^2 - \frac{1}{12}$ if $x \in [-1/2, 1/2[$ extended by 1-periodicity. Find u s.t.

$u'(x) = f(x) \quad \forall\ x \in]-1/2, 1/2[$

$u(-1/2) = u(1/2)$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n\pi)^2} \cos(2\pi n x)$$

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \right]$$

$$u'(x) = \sum_{n=1}^{\infty} \left[-2\pi n a_n \sin(2\pi n x) + 2\pi n b_n \cos(2\pi n x) \right]$$

$$= f(x) = \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{(n\pi)^2} \cos(2\pi n x) + 0 \sin(2\pi n x) \right]$$

$a_0 \in \mathbb{R}$ is free , $a_n = 0$.

$$b_n = \frac{(-1)^n}{2(\pi n)^3}$$

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{2(\pi n)^3} \sin(2\pi n x) \right]$$

6.3 Applications of Fourier transform

Recall:

$$\mathcal{F}(af + bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g)$$

$$\mathcal{F}(f')(\alpha) = i\alpha\,\mathcal{F}(f)(\alpha)$$

$$\mathcal{F}(f * g)(\alpha) = \sqrt{2\pi}\,\mathcal{F}(f)(\alpha)\,\mathcal{F}(g)(\alpha)$$

	$f(y)$ $w \in \mathbb{R}$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{si } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert\alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{si } y > 0 \\ 0, & \text{sinon} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$
7	$f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2 y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = y e^{-w^2 y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert \right) e^{-\lvert w\alpha\rvert}$

3.3.1 Examples:

- Example 1: Find $u(x)$ s.t.

$$u''(x) - u(x) = e^{-|x|}$$

$$\int_{-\infty}^{+\infty} \left| e^{-|x|} \right| dx < \infty$$

$$\mathcal{F}(u''(x) - u(x))(\alpha) = \mathcal{F}(e^{-|x|})(\alpha)$$

7	$f(y) = \dfrac{e^{-\lvert wy \rvert}}{\lvert w \rvert}$	$(w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\dfrac{2}{\pi}} \dfrac{1}{\alpha^2 + w^2}$

$e^{-|x|} \longrightarrow$ row 7 of the Table for $w=1$

$$\mathcal{F}(e^{-|x|}) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

$$\mathcal{F}(u''(x) - u(x))(\alpha) = \mathcal{F}(u''(x))(\alpha) - \mathcal{F}(u(x))(\alpha)$$

$$= (i\alpha)^2 \hat{u}(\alpha) - \hat{u}(\alpha) = -(\alpha^2+1)\hat{u}(\alpha)$$

$$-(\alpha^2+1)\hat{u}(\alpha) = \hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

$$\hat{u}(\alpha) = -\sqrt{\frac{2}{\pi}}\ \frac{1}{(1+\alpha^2)^2}$$

$$\mathcal{F}^{-1}(\hat{u}(\alpha)) = u(x)$$

$$\underbrace{\mathcal{F}^{-1}(\hat{u}(\alpha))(x)}_{u(x)} = \mathcal{F}^{-1}\left(-\sqrt{\frac{2}{\pi}}\ \frac{1}{(1+\alpha^2)^2}\right)(x)$$

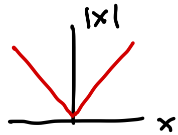
$$\int_{-\infty}^{+\infty} \left| -\sqrt{\frac{2}{\pi}}\ \frac{1}{(1+\alpha^2)^2} \right| d\alpha < \infty$$

Unfortunately, $\dfrac{1}{(1+\alpha^2)^2}$ is not in the table

WolframAlpha: $\mathcal{F}^{-1}\left(-\sqrt{\frac{2}{\pi}}\ \frac{1}{(1+\alpha^2)^2}\right) = -\frac{1}{2} e^{-|x|}(1+|x|)$

$$u(x) = -\frac{1}{2} e^{-|x|}(1+|x|)$$

$|x|$ is not differentiable at $x=0$.



To do at home: check that $u(x)$ is a solution of

$$u''(x) - u(x) = e^{-|x|} \quad \left(\text{Hint } \frac{d|x|}{dx} = \operatorname{sign}(x) = \begin{cases} +1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}\right.$$

Example 2: Find $u(x)$ s.t.

$$9\,u(x) + \underbrace{\int_{-\infty}^{\infty} 8\,u(t)\, e^{-|x-t|}\, dt}_{\text{convolution.}} = \underbrace{e^{-|x|}}_{f(x)}$$

$$9\,u(x) + 8\, u * f\,(x) = f(x)$$

$$\mathcal{F}\left(9\,u(x) + 8\, u * f\,(x)\right)(\alpha) = \mathcal{F}\left(f(x)\right)(\alpha)$$

$$9\,\hat{u}(\alpha) + 8\,\mathcal{F}(u * f)(\alpha) = \hat{f}(\alpha)$$

$$9\,\hat{u}(\alpha) + 8\sqrt{2\pi}\,\hat{u}(\alpha)\,\hat{f}(\alpha) = \hat{f}(\alpha)$$

$$\hat{u}(\alpha)\left(9 + 8\sqrt{2\pi}\,\hat{f}(\alpha)\right) = \hat{f}(\alpha)$$

$$\hat{u}(\alpha) = \hat{f}(\alpha)\left(9 + 8\sqrt{2\pi}\,\hat{f}(\alpha)\right)^{-1}$$

7	$f(y) = \dfrac{e^{-\lvert wy\rvert}}{\lvert w\rvert}$	$(w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{\alpha^2 + w^2}$

$f(x) = e^{-|x|} \longrightarrow$ Table, row 7, $w = 1$

$$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+\alpha^2}$$

substituting $\hat{f}(\alpha)$ in the expression above

$$\hat{u}(\alpha) = \frac{1}{9}\sqrt{\frac{2}{\pi}}\,\frac{1}{\frac{25}{9} + \alpha^2}$$

$$\mathcal{F}^{-1}(\hat{u}(\alpha))(x) = u(x)$$

$$\mathcal{F}^{-1}\left(\frac{1}{9} \sqrt{\frac{2}{\pi}} \frac{1}{\frac{25}{9} + \alpha^2} \right)(x)$$

7	$f(y) = \dfrac{e^{-\lvert wy \rvert}}{\lvert w \rvert}$	$(w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\dfrac{2}{\pi}} \dfrac{1}{\alpha^2 + w^2}$

Table row 7 for $w = \sqrt{\frac{25}{9}} = \frac{5}{3}$

$$\frac{1}{9} \mathcal{F}^{-1}\left(\sqrt{\frac{2}{\pi}} \frac{1}{(5/3)^2 + \alpha^2} \right) = \frac{1}{9} \frac{e^{-\lvert 5/3 \, x \rvert}}{\lvert 5/3 \rvert}$$

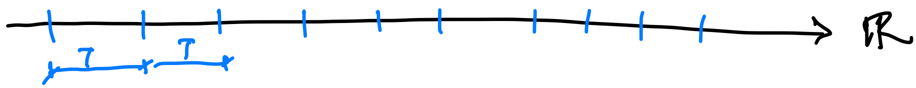
$$u(x) = \frac{1}{9} \frac{e^{-\lvert 5/3 \, x \rvert}}{\lvert 5/3 \rvert} = \frac{1}{15} e^{-5/3 \lvert x \rvert}$$

3.4 Incompatibility of methods

If we have a problem where we potentially can apply both methods, then the following hypothesis are true.

- u is piecewise defined
- u is T-periodic
- $\int_{-\infty}^{+\infty} \lvert u(x) \rvert \, dx < \infty$

$$\int_{-\infty}^{+\infty} |u(x)|\,dx = \sum_{n\in\mathbb{Z}} \int_{nT}^{(n+1)T} |u(x)|\,dx = \sum_{n\in\mathbb{Z}} \int_0^{T} |u(y+nT)|\,dy$$



$$= \sum_{n\in\mathbb{Z}} \underbrace{\int_0^{T} |u(y)|\,dy}_{I} < \infty \to I = 0$$

$u(y+nT) = u(y)$

$$I = 0 \to |u(y)| = 0 \ \forall y \in \mathbb{R} \to u(y) = 0 \ \forall y \in \mathbb{R}.$$

u can only be a trivial solution.