CHAPTER 2: CURVILINEAR INTEGRALS, CONSERVATIVE FIELDS, AND GREEN'S THEOREM

2.1 Curves in $\mathbb{R}^n$

2.1.1 Recalls

let $n > 1$

- Definition 1: $\Gamma \subset \mathbb{R}^n$ is regular simple curve if

there exist ($\exists$) an interval $[a,b] \subset \mathbb{R}$ and a function

$$\gamma : [a,b] \longrightarrow \mathbb{R}^n$$
$$t \longmapsto \gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$$

such that

- $\gamma([a,b]) = \Gamma = \{ x \in \mathbb{R}^n : \exists\, t \in [a,b] \text{ with } x = \gamma(t) \}$
- γ is injective on $[a,b[$: $\forall\, t_1, t_2 \in [a,b[$ with $t_1 \neq t_2 \implies \gamma(t_1) \neq \gamma(t_2)$
- $\gamma \in C^1([a,b], \mathbb{R}^n)$
- $\|\gamma'(t)\| \overset{\text{def}}{=} \left[\gamma_1'(t)^2 + \gamma_2'(t)^2 + \dots + \gamma_n'(t)^2 \right]^{1/2}$
- $\|\gamma'(t)\| \neq 0 \quad \forall\, t \in [a,b]$

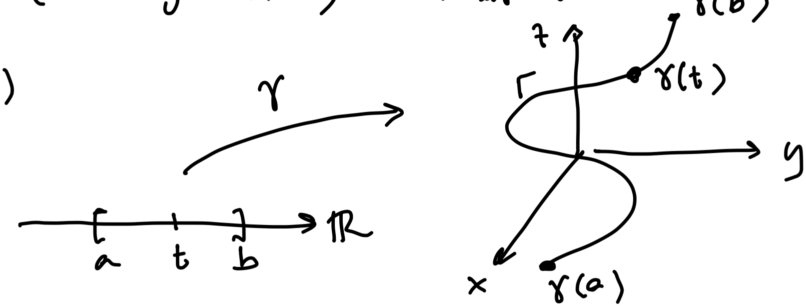
Remarks:

1) γ is a parameterization of Γ given by t.

2) $\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$ is the "position" vector on Γ at the "instant" t.

3) $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t))$ is the tangent vector ("velocity" vector) on Γ at t.

4)



- Definition 2: $\Gamma \subset \mathbb{R}^n$ is a <u>closed simple regular curve</u> if it is a s.r.c. and $\gamma(a) = \gamma(b)$

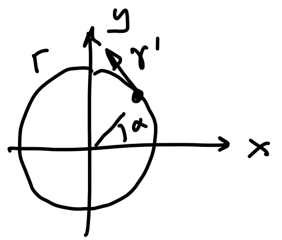
- Definition 3: $\Gamma \subset \mathbb{R}^n$ is a piecewise s.r.c. if there exist $\Gamma_1, \Gamma_2, \Gamma_3, \dots, \Gamma_k$ regular simple curves.

  s.t. $\Gamma = \bigcup_{i=1}^{k} \Gamma_i$

2.1.2 Examples

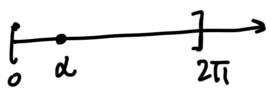
- Example

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \gamma(t) = (\cos t, \sin t)$$



Γ: circle in $\mathbb{R}^2$, radius $R = 1$, and center $(0,0)$.

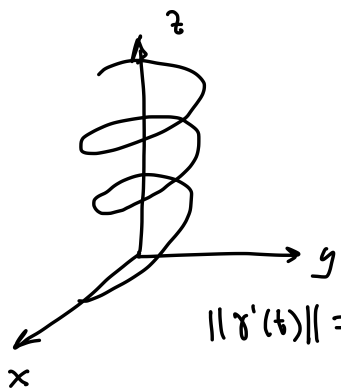
$$\gamma'(t) = (-\sin t, \cos t)$$



$$\|\gamma'(t)\| = 1 \neq 0 \quad \forall\, t \in [0, 2\pi]$$

- Example 2:

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \gamma(t) = (3\cos t, 3\sin t, 4t)$$



Γ: circular helix in $\mathbb{R}^3$

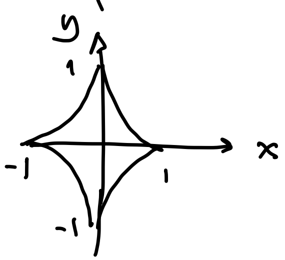
Radius $= 3$, pitch : 8π

$$\gamma'(t) = (-3\sin t, 3\cos t, 4)$$

$$\|\gamma'(t)\| = [\underbrace{(-3\sin t)^2 + (3\cos t)^2}_{9} + 4^2]^{1/2} = \sqrt{9 + 16} = 5 \;\forall t$$

- Example 3:

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \gamma(t) = (\cos^3 t, \sin^3 t)$$



Γ: astroide in $\mathbb{R}^2$

$$\gamma'(t) = (-3\cos^2 t \sin t,\ 3\sin^2 t \cos t)$$

$$\|\gamma'(t)\| = \left[9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t\right]^{1/2}$$

$$= \left[9\cos^2 t \sin^2 t\, \underbrace{(\cos^2 t + \sin^2 t)}_{1}\right]^{1/2} = 3\,|\cos t \sin t|$$

for $t = 0, \pi/2, 3\pi/2, \pi \longrightarrow \|\gamma'(t)\| = 0$

2.2 Curvilinear integrals

2.2.1 Definition

Let $\Gamma \subset \mathbb{R}^n$, $n > 1$, be r.s.c with a parameterization

$$\begin{aligned} \gamma : [a, b] &\to \mathbb{R}^n \\ t &\mapsto \gamma(t) \end{aligned}$$

- Definition 1: let $\begin{aligned} f : \Gamma \subset \mathbb{R}^n &\to \mathbb{R} \\ x &\longmapsto f(x) \end{aligned}$ be a continuous scalar field. The integral of f along Γ is defined as

$$\int_\Gamma f \, dl = \int_a^b f(\gamma(t))\, \|\gamma'(t)\| \, dt$$

Remark: The curve's length can be computed by setting $f = 1$. We have $\text{length}(\Gamma) = \int_\Gamma 1\, dl = \int_a^b \|\gamma'(t)\|\, dt$

• Definition 2: let $F: \Gamma \to \mathbb{R}^n$
$$x \mapsto F(x) = (F_1(x), \dots, F_n(x))$$

a continuous vector field. The integral of F along Γ

$$\int_\Gamma F \cdot dl = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_a^b \left(\sum_{i=1}^n F_i(\gamma(t))\, \gamma_i'(t) \right) dt$$

Remark: this integral computes the work required for moving a particle, under the action of a force F along Γ

• Definition 3: If Γ is a piecewise r.s. curve then

$\Gamma = \bigcup_{i=1}^K \Gamma_i$, $\int_\Gamma f\, dl = \sum_{i=1}^K \int_{\Gamma_i} f\, dl$ and $\int_\Gamma F \cdot dl = \sum_{i=1}^K \int_{\Gamma_i} F \cdot dl$

2.2.2. Examples

• Example 1: Compute $\int_\Gamma f\, dl$ for

a) $f: \mathbb{R}^2 \to \mathbb{R}$
$$(x,y) \mapsto f(x,y) = \sqrt{x^2 + 4y^2}$$ and the arc Γ

parameterized as: $\gamma: [0,1] \to \mathbb{R}^2$
$$t \mapsto \gamma(t) = \left(t, \frac{t^2}{2}\right)$$

$\gamma'(t) = (1, t)$, and $\|\gamma'(t)\| = \sqrt{1+t^2}$. Then

$$\int_\Gamma f\, dl = \int_0^1 f(\gamma(t))\, \|\gamma'(t)\|\, dt = \int_0^1 \sqrt{t^2 + 4\left(\frac{t^2}{2}\right)^2}\, \sqrt{1+t^2}\, dt$$

$$= \int_0^1 t\, \sqrt{1+t^2}\, \sqrt{1+t^2}\, dt = \int_0^1 t\,(1+t^2)\, dt = \int_0^1 (t+t^3)\, dt$$

$$= \frac{1}{2} t^2 \Big|_0^1 + \frac{t^4}{4}\Big|_0^1 = \frac{1}{2}(1^2 - 0^2) + \frac{1}{4}(1^4 - 0^4) = \frac{3}{4}$$

b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the curve Γ

$(x,y) \longmapsto f(x,y) = x$

$$\Gamma = \{ (x,y) \in \mathbb{R}^2 : y = \cosh x \quad \forall\, x \in [0,1] \}$$

We get a parameterization $\gamma(t)$ of Γ :

$$\gamma : [0,1] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (t, \cosh t)$$

$$\left(\frac{d \cosh t}{dt} = \sinh t \qquad \frac{d \sinh t}{dt} = \cosh t \right)$$

$$\gamma'(t) = (1, \sinh t)$$

$$\|\gamma'(t)\| = \sqrt{1 + \sinh^2 t} = \cosh t$$

$$\int_\Gamma f\, dl = \int_0^1 f(\gamma(t))\, \|\gamma'(t)\|\, dt = \int_0^1 t\, \sqrt{1 + \sinh^2 t}\, dt$$

$$= t \sinh t \Big|_0^1 - \int_0^1 \sinh t\, dt = \sinh(1) - \cosh t \Big|_0^1$$

$u = t$
$dv = \cosh t\, dt$ $\left(\int u\, dv = uv - \int v\, du\right)$

$$= \sinh(1) - \cosh(1) + \overset{\cosh(0)}{1} = \frac{e-e^{-1}}{2} - \frac{e+e^{-1}}{2} + 1$$

$$= 1 - e^{-1} = \frac{e-1}{e}$$

- Example 2: To compute $\int_\Gamma F \cdot dl$ for

a) $F : \mathbb{R}^3 \to \mathbb{R}^3$
$(x,y,z) \mapsto F(x,y,z) = (x, z, y)$ and the curve

Γ parameterized like: $\gamma : [0, 2\pi] \to \mathbb{R}^3$
$t \mapsto (\cos t, \sin t, 0)$

$\gamma'(t) = (-\sin t, \cos t, 0)$

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_0^{2\pi} (\cos t, 0, \sin t) \cdot (-\sin t, \cos t, 0)\, dt = \int_0^{2\pi} -\cos t \sin t$$

$$= -\frac{1}{2} \sin^2 t \Big|_0^{2\pi} = 0$$

b) $F : \mathbb{R}^3 \to \mathbb{R}^3$
$(x,y,z) \mapsto F(x,y,z) = (x^2, y^3, z^2)$

and the curve $\Gamma = \{ (x,y,z) \in \mathbb{R}^3 : y = e^x \text{ and } z = x \text{ for } x \in [0,1] \}$

We construct $\gamma : [0,1] \to \mathbb{R}^3$
$t \mapsto (t, e^t, t)$

$$\gamma'(t) = (1, e^t, 1)$$

$$\int_\Gamma F \cdot dl = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_0^1 (t^2, e^{3t}, t^2) \cdot (1, e^t, 1)\, dt = \int_0^1 (2t^2 + e^{4t})\, dt$$

$$= \frac{2}{3} t^3 \Big|_0^1 + \frac{1}{4} e^{4t} \Big|_0^1 = \frac{2}{3} + \frac{1}{4}(e^4 - 1) = \frac{5 + 3e^4}{12}$$

- Example 3: compute the length of the circle Γ with radius R and centered at the origin:

$$\Gamma = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = R^2 \}$$

We construct
$$\gamma : [0, 2\pi] \longrightarrow \Gamma \subset \mathbb{R}^2$$
$$t \longmapsto (R \cos t, R \sin t)$$

$$\text{length}(\Gamma) = \int_\Gamma 1\, dl = \int_0^{2\pi} \underbrace{1(\gamma(t))}_{1} \|\gamma'(t)\|\, dt$$

$$\|\gamma'(t)\| = \|(-R \sin t, R \cos t)\| = R$$

$$\text{length}(\Gamma) = \int_0^{2\pi} 1 \cdot R\, dt = R\, t \Big|_0^{2\pi} = 2\pi R$$

2.3 Fields that derive from potentials

2.3.1 Description of conservative fields

- Definition: let $\Omega \subset \mathbb{R}^n$ be an open domain and

$$F : \Omega \to \mathbb{R}^n \qquad \text{a vector field}$$
$$x \mapsto F(x) = (F_1(x), \dots, F_n(x))$$

We say that F derives from a potential on Ω if

$\exists$ a scalar field $f : \Omega \to \mathbb{R}$ s.t. $f \in C^1(\Omega)$

s.t. $F = \text{grad} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$

In this case F is called a <u>conservative field</u> and

f is called the <u>potential</u>.

Remarks:

1) If the potential $\exists$, then it is defined up to constant $\alpha \in \mathbb{R}$. Because

$$\text{grad}(f + \alpha) = \text{grad} f = F \quad \forall \alpha \in \mathbb{R}.$$