Chapitre 10 : Problèmes aux limites unidimensionnels

Chap 10 : Pbm aux limites 1D

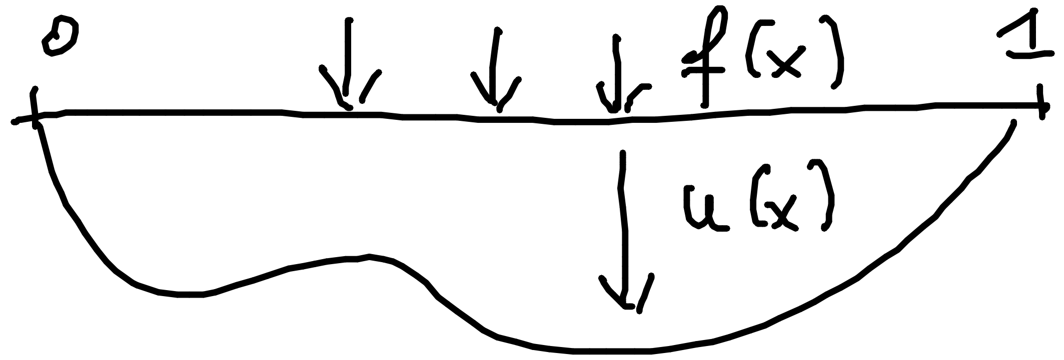
Pbm modèle :

donné $f : [0,1] \to \mathbb{R}$, $x \to f(x)$

cherché $u : [0,1] \to \mathbb{R}$, $x \to u(x)$ tq

$$\begin{cases} -u''(x) = f(x) & 0 < x < 1 \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

Corde élastique tendue, pincée $x=0$ et $x=1$



problème modèle (toy pbm)

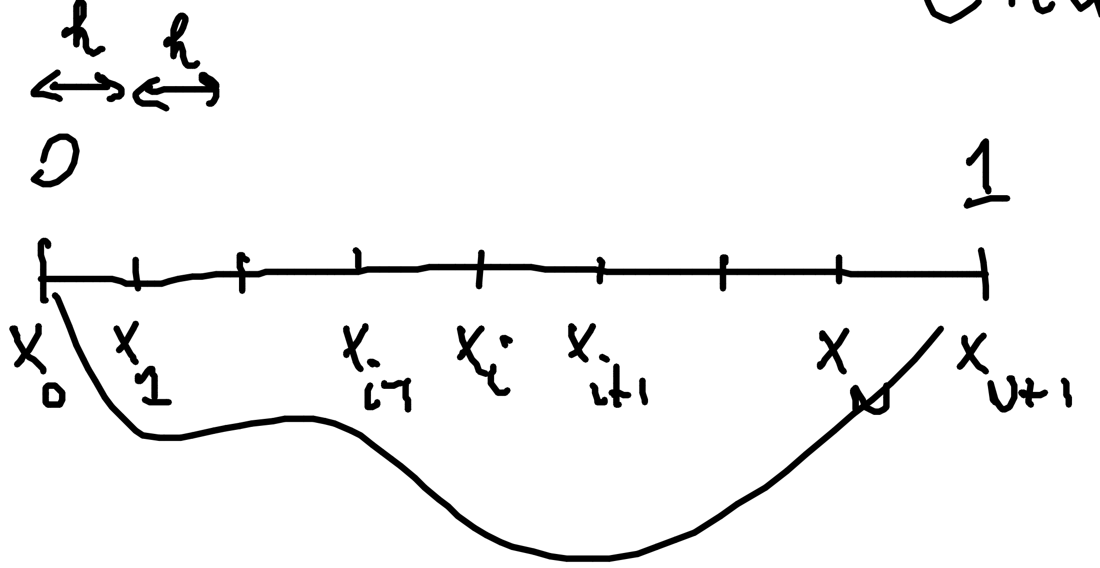
$$-u''(x) + c(x)\,u(x) = f(x) \qquad c \geq 0$$

$$-\frac{d}{dx}\left(c(x)\,\frac{du}{dx}(x)\right) = f(x) \qquad c(x) > 0$$

Attention ne pas confondre avec pbm à valeur initiale chap 9 :

$$\begin{cases} \ddot{u}(t) = f(u(t), \dot{u}(t), t) & t > 0 \\ u(0) \\ \dot{u}(0) \end{cases}$$

Chap10 : Méthode de différences finies



N entier pos (grand) $\quad h = \frac{1}{N+1}$ pas d'espace (petit)

$$x_i = ih \qquad i = 0, 1, \dots, N, N+1.$$

But : calculer des valeurs u_i approx de $u(x_i)$ $\quad i = 1, 2, \dots, N$

$$-u''(x_i) = f(x_i) \qquad i = 1, 2, \dots, N$$

$$-\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} = f(x_i) + O(h^2)$$

Formule de diff. finie centrée (chap2)

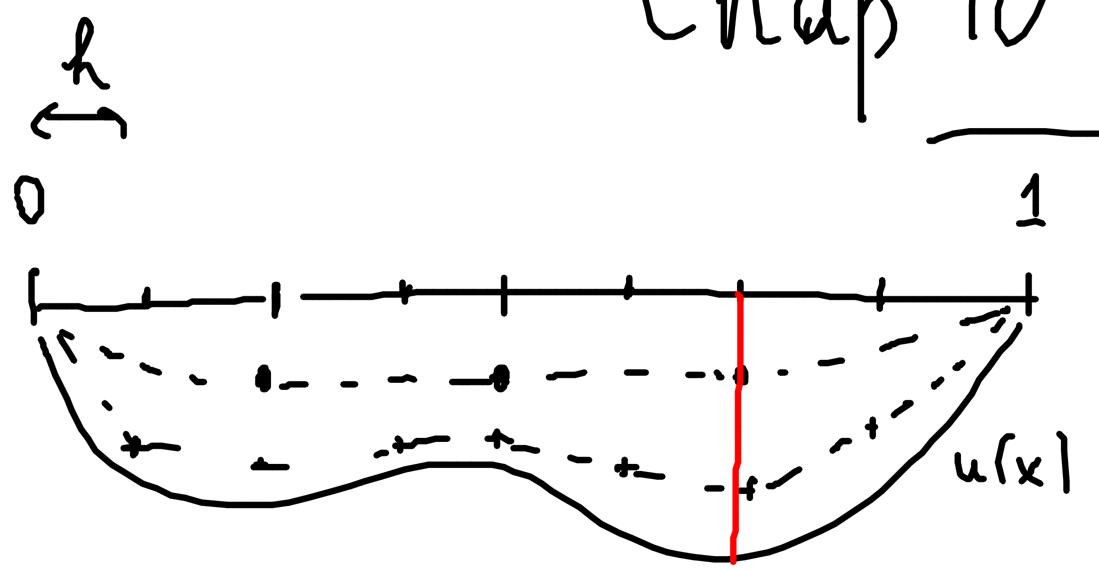
schéma :
$$\begin{cases} \dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i) & i = 1, \dots, N \\ u_0 = 0 \\ u_{N+1} = 0 \end{cases}$$

système linéaire :

$$\underbrace{\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & (0) \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ (0) & & & -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_N \end{pmatrix}}_{\vec{u}} = \underbrace{\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_N) \end{pmatrix}}_{\vec{f}}$$

A est sym. déf. pos. $\qquad A = LL^T$

Chap 10 : Méthode de différences finies (suite)



$u_i \xrightarrow{h\to 0} u(x_i)$?

l'erreur est divisée par $2^2=4$ si h est divisé par 2. $O(h^2)$

Thm : $u\in \mathcal{C}^4[0,1]$ $\exists C>0$ $\forall\, 0<h<1$ $\max\limits_{1\le i\le N}|u_i-u(x_i)| \le Ch^2$.

Dem : schéma : $\dfrac{-u_{i-1}+2u_i-u_{i+1}}{h^2}=f(x_i)$ $\qquad A\vec{u}=\vec{f}$

$$\frac{-u(x_{i-1})+2u(x_i)-u(x_{i+1})}{h^2}=f(x_i)+O(h^2) \qquad A\vec{w}=\vec{f}+\vec{r} \qquad \vec{w}=\begin{pmatrix}u(x_1)\\u(x_2)\\\vdots\\u(x_N)\end{pmatrix} \quad \vec{r}=\begin{pmatrix}r_1\\r_2\\\vdots\\r_N\end{pmatrix}$$

Chap 2

$|r_i|\le Ch^2$

$C=\dfrac{1}{12}\max\limits_{0\le x\le 1}|u^{(4)}(x)|$

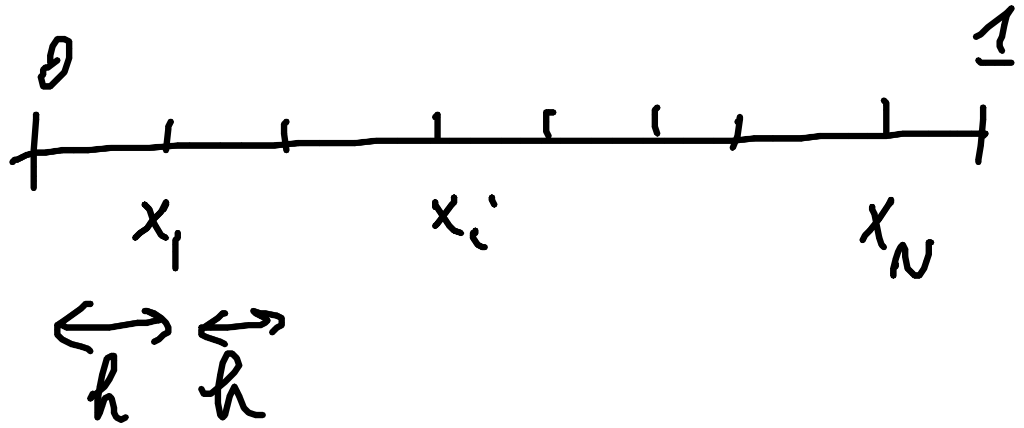
$A(\vec{w}-\vec{u})=\vec{r}$

Lemme : Soit $\vec{g}\in\mathbb{R}^N$ soit $\vec{v}$ tq $A\vec{v}=\vec{g}$, on a : $\max\limits_{1\le i\le N}|v_i| \le \dfrac{1}{8}\max\limits_{1\le i\le N}|g_i|$

$$\max_{1\le i\le N}|u(x_i)-u_i| \le \frac{1}{8}\max_{1\le i\le N}|r_i| \le \frac{1}{8.12}\max_{0\le x\le 1}|u^{(4)}(x)|\,h^2$$

Chap 10 : un problème non linéaire

$$\begin{cases} -u''(x) + x\,(u(x))^3 = f(x) & 0 < x < 1 \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$



$h = \frac{1}{N+1}$ $\quad x_i = ih \quad i = 0, 1, \dots, N+1$

u_i approx de $u(x_i)$

$$-u''(x_i) + x_i\,(u(x_i))^3 = f(x_i) \qquad i = 1, \dots, N$$

$$\frac{-u(x_{i-1}) + 2u(x_i) + u(x_{i+1})}{h^2} + x_i\,(u(x_i))^3 = f(x_i) + O(h^2) \qquad \text{FDF centrée}$$

$$\text{schéma :} \begin{cases} \dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + x_i\,(u_i)^3 = f(x_i) & i = 1, \dots, N \\ u_0 = 0 \\ u_{N+1} = 0 \end{cases}$$

Système non linéaire $\vec{F}(\vec{u}) = \vec{0}$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \qquad \vec{F}(\vec{u}) = \begin{pmatrix} F_1(u_1, u_2, \dots, u_N) \\ F_2(u_1\ u_2 \cdots u_N) \\ \vdots \\ F_N(u_1, u_2 \cdots u_N) \end{pmatrix} = \begin{pmatrix} \dfrac{2u_1 - u_2}{h^2} + x_1(u_1)^3 - f(x_1) \\ \dfrac{-u_1 + 2u_2 - u_3}{h^2} + x_2(u_2)^3 - f(x_2) \\ \vdots \\ \dfrac{-u_{N-1} + 2u_N}{h^2} + x_N(u_N)^3 - f(x_N) \end{pmatrix}$$

Méthode de Newton $\quad$ Chap 8.

Chap 10 : un problème non linéaire

Cherche $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$ tq $\vec{F}(\vec{u}) = \vec{0} = \begin{pmatrix} \frac{2u_1 - u_2}{h^2} + x_1 (u_1)^3 - f(x_1) \\ \vdots \end{pmatrix}$

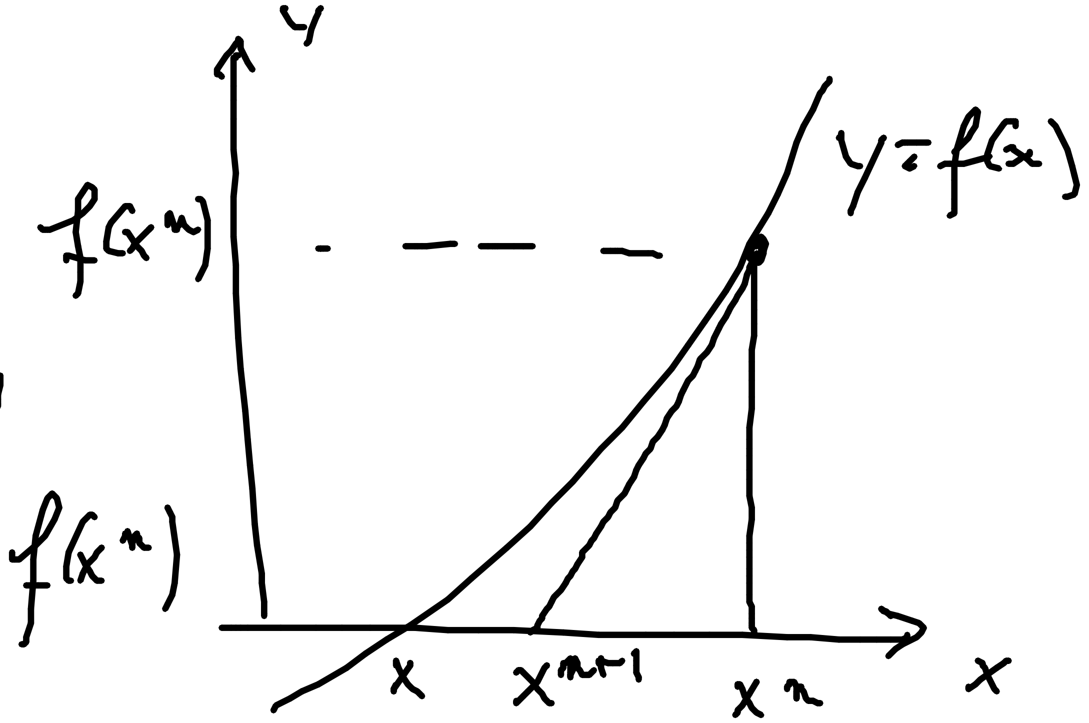
$\vec{u}^n$ connu $\vec{u}^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_N^n \end{pmatrix}$ $\quad \vec{u}^{n+1}$ tq $DF(\vec{u}^n)(\vec{u}^n - \vec{u}^{n+1}) = \vec{F}(\vec{u}^n)$

$\forall \vec{u} \in \mathbb{R}^N \qquad DF(\vec{u}) = \begin{pmatrix} \frac{2}{h^2} + 3x_1(u_1)^2 & -\frac{1}{h^2} & & (0) \\ -\frac{1}{h^2} & \frac{2}{h^2} + 3x_2(u_2)^2 & -\frac{1}{h^2} & \\ & \ddots & \ddots & -1/h^2 \\ (0) & & -1/h^2 & \frac{2}{h^2} + 3x_N(u_N)^2 \end{pmatrix}$

x tq $f(x)$

$$f'(x^n) = \frac{f(x^n)}{x^n - x^{n+1}}$$

$$f'(x^n)(x^n - x^{n+1}) = f(x^n)$$



Algorithme : $\vec{u}^0$ donné

$n = 0, 1, 2, \ldots$

$A = DF(\vec{u}^n)$

$\vec{b} = \vec{F}(\vec{u}^n)$

résoud $A\vec{y} = \vec{b}$ $\qquad A = LL^T$

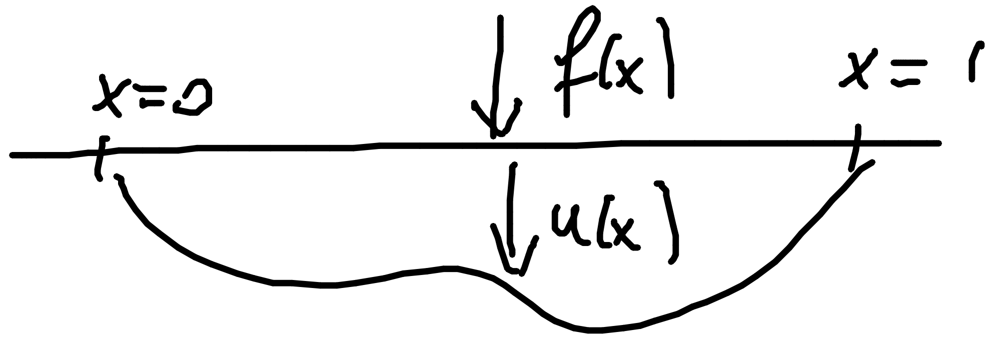
$L\vec{z} = \vec{b}$

$L^T\vec{y} = \vec{z}$

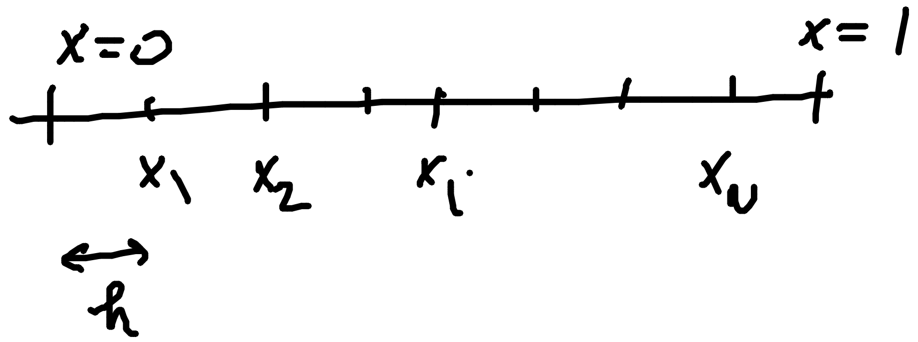
pose $\vec{u}^{n+1} = \vec{u}^n - \vec{y}$

Chap 10 : résumé

$$\begin{cases} -u''(x) = f(x) & 0 < x < 1 \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$



$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i) \qquad i = 1, \dots, N$$



$$A\vec{u} = \vec{f}$$

$$\max_{1 \leq i \leq N} |u(x_i) - u_i| = O(h^2) \qquad u \in \mathcal{C}^4[0,1]$$

$$\begin{cases} -u''(x) + x(u(x))^3 = f(x) & 0 < x < 1 \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

$\vec{u}$ tq $\vec{F}(\vec{u}) = \vec{0}$ $\qquad$ $\vec{u}^n, \vec{u}^{n+1}$ tq $DF(\vec{u}^n)(\vec{u}^n - \vec{u}^{n+1}) = \vec{F}(\vec{u}^n)$