29.3.19

Corrigé de la Série 15

1. (a) $z = 5 + 12i \;\Rightarrow\; |z| = \sqrt{25+144} = 13$, et $\cos\varphi = \dfrac{5}{13}$; $\sin\varphi = \dfrac{12}{13}$;

(b) $z = \sqrt{3} + i \;\Rightarrow\; |z| = 2$; $\varphi = \dfrac{\pi}{6}$;

(c) $z = \dfrac{1 + i\tan\alpha}{1 - i\tan\alpha} = \dfrac{\cos\alpha + i\sin\alpha}{\cos\alpha - i\sin\alpha} = \dfrac{(\cos\alpha + i\sin\alpha)^2}{(\cos\alpha - i\sin\alpha)(\cos\alpha + i\sin\alpha)} = \cos 2\alpha + i\sin 2\alpha$.

d'où $|z| = 1$ et $\varphi = 2\alpha$.

2. (a) $z = -2 = [2;\pi]$;

(b) $z = 7i - \dfrac{3}{i} = 7i + 3i = 10i = \left[10; \dfrac{\pi}{2}\right]$;

(c) $z = -1 + i = \left[\sqrt{2}; \dfrac{3\pi}{4}\right]$;

(d) $z = \sqrt{3} + i = \left[2; \dfrac{\pi}{6}\right]$;

(e) $z = \dfrac{1}{1-i} = \dfrac{1+i}{2} = \left[\dfrac{\sqrt{2}}{2}; \dfrac{\pi}{4}\right]$;

(f) $z = -3\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right) = 3\left(\cos\left(\dfrac{\pi}{4}+\pi\right) + i\sin\left(\dfrac{\pi}{4}+\pi\right)\right) = \left[3; \dfrac{5\pi}{4}\right] = \left[3; -\dfrac{3\pi}{4}\right]$.

3. (a) $z = \left[5; -\dfrac{\pi}{2}\right] = 5\left(\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right) = -5i$

(b) $z = \left[2; \dfrac{\pi}{8}\right] = 2\left(\cos\left(\dfrac{\pi}{8}\right) + i\sin\left(\dfrac{\pi}{8}\right)\right) = \sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}$

car : $\cos\left(\dfrac{\pi}{8}\right) = \sqrt{\dfrac{1}{2}\left(1 + \cos\left(\dfrac{\pi}{4}\right)\right)}$ et $\sin\left(\dfrac{\pi}{8}\right) = \sqrt{\dfrac{1}{2}\left(1 - \cos\left(\dfrac{\pi}{4}\right)\right)}$

(c) $z = [\pi; \pi - t] = \pi[\cos(\pi - t) + i\sin(\pi - t)] = -\pi\cos t + i\pi\sin t$

(d) $z = \dfrac{\left[2; -\frac{\pi}{4}\right]}{\left[\frac{1}{2}; \frac{\pi}{4}\right]} = \left[4; -\dfrac{\pi}{2}\right] = -4i$

(e) $z = \dfrac{\left[2; -\frac{\pi}{3}\right]^4}{\left[4; \frac{\pi}{4}\right]} = \dfrac{\left[16; -\frac{4\pi}{3}\right]}{\left[4; \frac{\pi}{4}\right]} = \left[4; -\dfrac{19\pi}{12}\right] = \left[4; \dfrac{5\pi}{12}\right]$

Calcul de $\cos\left(\dfrac{5\pi}{12}\right)$ et $\sin\left(\dfrac{5\pi}{12}\right)$:

$$\cos\left(\frac{5\pi}{12}\right) = \cos\left[\frac{1}{2}\left(\frac{5\pi}{6}\right)\right] = +\sqrt{\frac{1+\cos\left(\frac{5\pi}{6}\right)}{2}},$$

positivité du cos : $\quad \cos\left(\frac{5\pi}{12}\right) = \sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}}$

et de même, $\sin\left(\frac{5\pi}{12}\right) = \sqrt{\frac{2+\sqrt{3}}{4}}$

$\Rightarrow z = 4\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right] = \sqrt{8-4\sqrt{3}} + i\sqrt{8+4\sqrt{3}}$

Or : $8 \pm 4\sqrt{3} = 8 \pm 2\sqrt{12} = (\sqrt{6} \pm \sqrt{2})^2$

ce qui finalement nous donne : $\quad z = (\sqrt{6}-\sqrt{2}) + i(\sqrt{6}+\sqrt{2})$.

4. $\mathrm{Re}\left(\left[\sqrt{3}; \frac{2}{3}\right]^3 \cdot [4;\varphi]\right) = \mathrm{Im}\left(\frac{[6; 1+\varphi]^2}{[3;\varphi]}\right) \;\Leftrightarrow\; \mathrm{Re}\left(\left[3\sqrt{3}; 2\right] \cdot [4;\varphi]\right) = \mathrm{Im}\left(\frac{[36; 2+2\varphi]}{[3;\varphi]}\right)$

$\Leftrightarrow\; \mathrm{Re}\left(\left[12\sqrt{3}; 2+\varphi\right]\right) = \mathrm{Im}\left([12; 2+\varphi]\right) \;\Leftrightarrow\; 12\sqrt{3}\cos(2+\varphi) = 12\sin(2+\varphi)$

$\Leftrightarrow\; \frac{\sqrt{3}}{2}\cos(2+\varphi) - \frac{1}{2}\sin(2+\varphi) = 0 \;\Leftrightarrow\; \sin\frac{\pi}{3}\cos(2+\varphi) - \cos\frac{\pi}{3}\sin(2+\varphi) = 0$

$\Leftrightarrow\; \sin\left(\frac{\pi}{3} - (2+\varphi)\right) = 0 \;\Leftrightarrow\; \frac{\pi}{3} - (2+\varphi) = 0 + k\pi \;\Leftrightarrow\; \varphi = \frac{\pi}{3} - 2 + k\pi$

avec la condition $\varphi \in [0,\pi]$, on obtient : $\qquad \varphi = \frac{4\pi}{3} - 2.$

5. (a) On a $z - i\overline{z} = 0$ et $|z| = 2\sqrt{2}$ d'où $z\overline{z} - i\overline{z}^2 = 0 \Leftrightarrow (2\sqrt{2})^2 - i\overline{z}^2 = 0$

On pose $\overline{z} = x - iy \Rightarrow$

$$8 - i(x-iy)^2 = 0 \Leftrightarrow 8 - 2xy + i(y^2 - x^2) = 0 \Leftrightarrow \begin{cases} 8 - 2xy = 0 \\ y^2 - x^2 = 0 \end{cases}$$

$$\Rightarrow x = \frac{4}{y} \;\text{ et }\; y^2 - \frac{16}{y^2} = 0 \Rightarrow y = \pm 2 \Rightarrow \begin{cases} z_1 = 2 + 2i \\ z_2 = -2 - 2i \end{cases}$$

(b) On a $2iz + \overline{z} = 0$ et $|z| = 2$ d'où $2iz\overline{z} + \overline{z}^2 = 0 \Leftrightarrow 2i2^2 + \overline{z}^2 = 0$

On pose $\overline{z} = x - iy \Rightarrow$

$$8i + (x-iy)^2 = 0 \quad\Leftrightarrow\quad 8i - 2ixy + (x^2 - y^2) = 0 \quad\Leftrightarrow \begin{cases} x^2 - y^2 = 0 \\ 8 - 2xy = 0 \end{cases}$$

$\Rightarrow \quad y = \frac{4}{x}$ et $x^2 - \frac{16}{x^2} = 0 \quad \Rightarrow x = y = \pm 2 < \quad\Rightarrow \quad$ contradiction avec $|z| = 2$:

pas de solution .

(c) On a $z^{11}=\overline{z}$ et $0<\text{Im}z<\dfrac{\sqrt{2}}{2}$; on pose $z=[r;\varphi]$ et on obtient :

$$[r^{11};11\varphi]=[r;-\varphi]\Leftrightarrow\left\{\begin{array}{l} r^{10}=1\\ 11\varphi=-\varphi+2k\pi\end{array}\right. \Leftrightarrow \quad \left\{\begin{array}{l} r=1\\ \varphi=\frac{k\pi}{6}\quad k=0,\dots,11\end{array}\right.$$

On doit avoir : $0<\sin\left(\dfrac{k\pi}{6}\right)<\dfrac{\sqrt{2}}{2} \quad \Rightarrow \quad k=1;5$ d'où les solutions :

$$\left\{\begin{array}{l} z_1=\left[1;\dfrac{\pi}{6}\right]=\dfrac{1}{2}(\sqrt{3}+i)\\[2ex] z_5=\left[1;\dfrac{5\pi}{6}\right]=\dfrac{1}{2}(-\sqrt{3}+i)\end{array}\right.$$

6. L'équation : $(z+\overline{z})z^3+4(\overline{z}^2-z^2)=0 \;\Leftrightarrow\; (z+\overline{z})z^3+4(\overline{z}-z)(\overline{z}+z)=0$

$\Leftrightarrow\; (\overline{z}+z)(z^3+4(\overline{z}-z))=0$

a) $\overline{z}+z=0 \;\Rightarrow\; x-iy+x+iy=0 \;\Rightarrow\; 2x=2\text{Re}(z)=0$: solution rejetée par la condition $2\text{Re}z>|z|$ imposée.

b) $z^3+4(\overline{z}-z)=0 \;\Leftrightarrow\; (x+iy)^3+4(-2iy)=0 \;\Leftrightarrow\; \left\{\begin{array}{l} x^3-3xy^2=0\\ 3x^2y-y^3=8y\end{array}\right.$

La solution $(x;y)=(0;0)$ ne satisfait pas non plus la condition.

$$\left\{\begin{array}{l} x^2-3y^2=0\\ 3x^2-y^2=8\end{array}\right. \Rightarrow \left\{\begin{array}{l} x^2=3y^2\\ 3(3y^2)-y^2=8\end{array}\right. \Rightarrow y^2=1 \;\Rightarrow\; y=\pm 1$$

Les deux solutions sont alors : $\quad z_1=\sqrt{3}+i$ et $z_2=\sqrt{3}-i \quad$ car $x>0$.

7. (a) $z=9i=\left[9;\dfrac{\pi}{2}\right] \quad\Rightarrow\quad \sqrt{z}=\left[3;\dfrac{\pi}{4}+k\pi\right]\quad k=0;1,$ d'où :

$$z_1=\frac{3\sqrt{2}}{2}(1+i)\quad\text{et}\quad z_2=-\frac{3\sqrt{2}}{2}(1+i).$$

(b) $z=5-12i=[13;\varphi]$ avec $\cos\varphi=\dfrac{5}{13},\quad \sin\varphi=-\dfrac{12}{13}$;

$\sqrt{z}=\left[\sqrt{13};\dfrac{\varphi}{2}+k\pi\right]\quad k=0;1$;

(c) $z = \dfrac{1}{1-i} + \dfrac{1}{i} = \dfrac{1+i}{2} - i = \dfrac{1}{2} - \dfrac{i}{2} = \left[\dfrac{\sqrt{2}}{2}; -\dfrac{\pi}{4}\right]$

$\sqrt{z} = \left[\dfrac{1}{\sqrt[4]{2}}; -\dfrac{\pi}{8} + k\pi\right] \quad k = 0; 1$

8. (a) $z = 1 - i\sqrt{3} = \left[2; \dfrac{5\pi}{3}\right] = \left[2; -\dfrac{\pi}{3}\right] \quad \Rightarrow \quad \sqrt[3]{z} = \left[\sqrt[3]{2}; -\dfrac{\pi}{9} + \dfrac{2k\pi}{3}\right], \quad k = 0; 1; 2$

$z = \dfrac{1}{(1+i)^2} = \dfrac{1}{2i} = -\dfrac{i}{2} = \left[\dfrac{1}{2}; \dfrac{3\pi}{2}\right] \quad \Rightarrow \quad \sqrt[3]{z} = \left[\dfrac{1}{\sqrt[3]{2}}; \dfrac{\pi}{2} + \dfrac{2k\pi}{3}\right] \quad k = 0; 1; 2$

(b) $z = \dfrac{\sqrt{(-1+i)^3}}{\sqrt[7]{i}} = \dfrac{\sqrt{\left[\sqrt{2}; \frac{3\pi}{4}\right]^3}}{\sqrt[7]{\left[1; \frac{\pi}{2}\right]}} = \dfrac{\sqrt{\left[\sqrt{2^3}; \frac{9\pi}{4}\right]}}{\left[1; \frac{\pi}{14} + \frac{2k\pi}{7}\right]} = \dfrac{\left[2^{3/4}; \frac{9\pi}{8} + \ell\pi\right]}{\left[1; \frac{\pi}{14} + \frac{2k\pi}{7}\right]}, \quad \begin{array}{l} \ell = 0; 1 \\ k = 0; \ldots; 6 \end{array}$

d'où :

$$z = \left[2^{3/4}; \frac{59\pi}{56} + \left(\ell - \frac{2k}{7}\right)\pi\right] \quad \ell = 0; 1, \quad k = 0; \ldots; 6$$