$$Ff(x) = \frac{1}{2}\left[f(x+0) + f(x-0) \right] \quad \forall x \in \mathbb{R}.$$

$$\lim_{\substack{t \to x \\ t > x}} f(t) \doteq f(x+0) \qquad \lim_{\substack{t \to x \\ t < x}} f(t) \doteq f(x-0)$$

If f is continuous in x then

$f(x+0) = f(x-0) = f(x)$ and $(Ff)(x) = f(x)$.

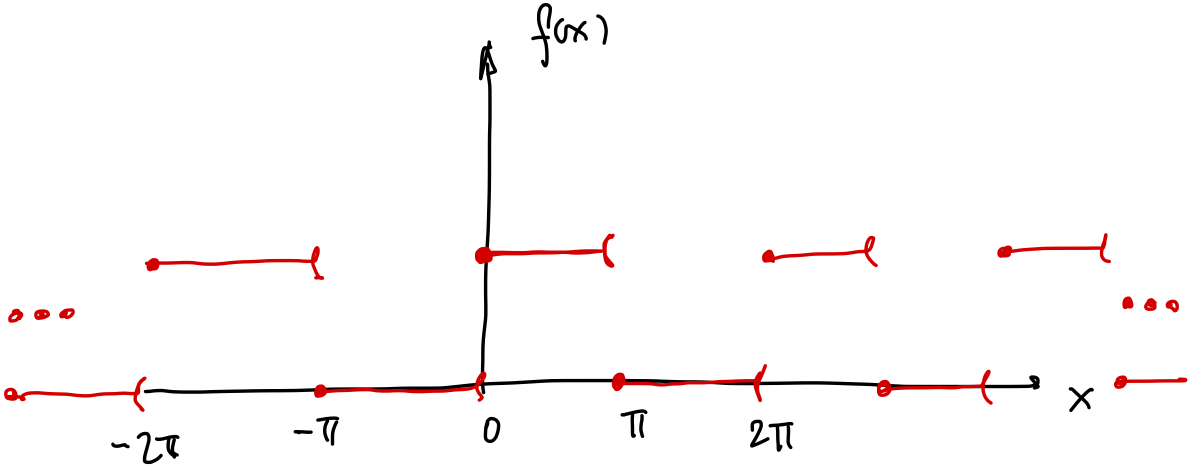
$$\frac{1}{2}\left(f(x+0) + f(x-0) \right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right]$$

1.2.4 Examples:

- Example 1: let $f: [0, 2\pi[\to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [\pi, 2\pi[\end{cases}$$

extended by 2π-periodicity to $\mathbb{R}$.



Compute the Fourier series Ff and compare Ff and f on $[0, 2\pi]$

$$a_0 = \frac{1}{\pi}\int_0^{\overset{=T}{2\pi}} f(x) = \frac{1}{\pi}\int_0^{\pi} dx + \int_{\pi}^{2\pi} 0\, dx = 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \cos(nx)\,dx$$

$\left(\frac{2}{T}\right)$

$T = 2\pi$

$$= \frac{1}{\pi}\left.\frac{\sin(nx)}{n}\right|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

$$= -\frac{1}{\pi}\,\frac{\cos(nx)}{n}\Big|_0^{\pi} = -\frac{1}{n\pi}\left[\cos(n\pi) - \cos(0)\right] =$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

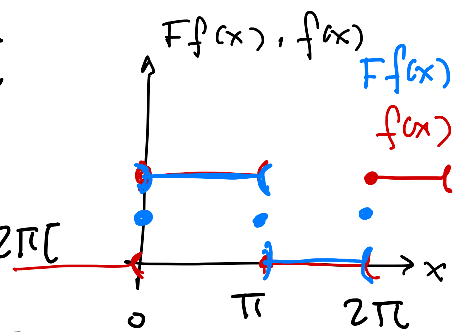
$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$= \frac{1}{2} + \sum_{n\ \text{odd}} \frac{2}{n\pi}\sin(nx) = \frac{1}{2} + \sum_{k=0}^{\infty}\frac{2}{\pi}\,\frac{\sin((2k+1)x)}{2k+1}$$

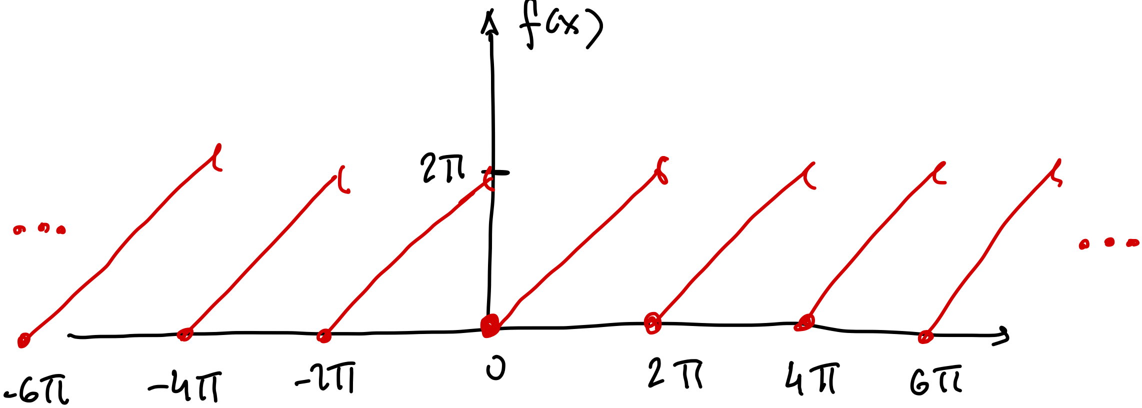
$n = 2k+1$, n is an odd number for $k = 0, 1, 2, \ldots$

Apply Dirichlet: 25/11/2021

$$Ff(x) = \begin{cases} \frac{1}{2}\left(f(0^-) + f(0^+)\right) = \frac{1}{2}(0+1) = \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{2}(1+1) = 1 & \text{if } x \in \left]0, \pi\right[\\ \frac{1}{2}(1+0) = \frac{1}{2} & \text{if } x = \pi \\ \frac{1}{2}(0+0) = 0 & \text{if } x \in \left]\pi, 2\pi\right[\\ \frac{1}{2}(0+1) = \frac{1}{2} & \text{if } x = 2\pi \end{cases}$$



- Example 2: let $f : [0, 2\pi[\to \mathbb{R}$ defined by $f(x) = x$ be a function extended by 2π-periodicity to $\mathbb{R}$. Compute the Fourier coefficients, and compare $Ff(x)$ to $f(x)$ on $[0, 2\pi]$



Fourier coefficients: $T = 2\pi$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{2\pi} x\,dx = 2\pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\,dx = \frac{1}{\pi} \int_0^{2\pi} x \cos(nx)\,dx$$

$u = x$

$v = \frac{\sin(nx)}{n}$

$$= \frac{1}{\pi} \underbrace{x \frac{\sin(nx)}{n} \Big|_0^{2\pi}}_{=0} - \frac{1}{n\pi} \underbrace{\int_0^{2\pi} \sin(nx)\,dx}_{=0}$$

$a_n = 0.$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin(nx)\,dx$$

$u = x$

$v = -\frac{\cos nx}{n}$

$$= -\frac{1}{\pi}\, x\, \frac{\cos(nx)}{n}\Big|_0^{2\pi} + \frac{1}{n\pi}\underbrace{\int_0^{2\pi}\cos(nx)\,dx}_{=0}$$

$$= -\frac{1}{\pi}\,\frac{2\pi}{n} = -\frac{2}{n}$$

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

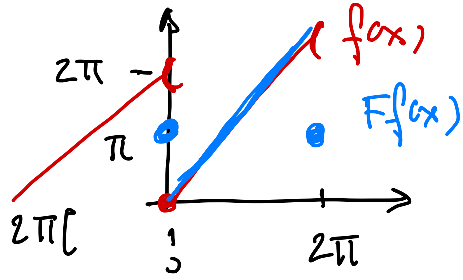
$$= \frac{2\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\sin(nx) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

Comparison: $Ff(x)$ -vs- $f(x)$ on $[0, 2\pi]$

$\rightarrow$ Dirichlet theorem.

$$Ff(x) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} = \frac{1}{2}\left[f(x+0) + f(x-0)\right]$$

$$= \begin{cases} \frac{1}{2}[0+2\pi] = \pi & \text{if } x=0 \\ \frac{1}{2}(x+x) = x = f(x) & \text{if } x \in]0, 2\pi[\\ \frac{1}{2}(2\pi + 0) = \pi & \text{if } x = 2\pi \end{cases}$$

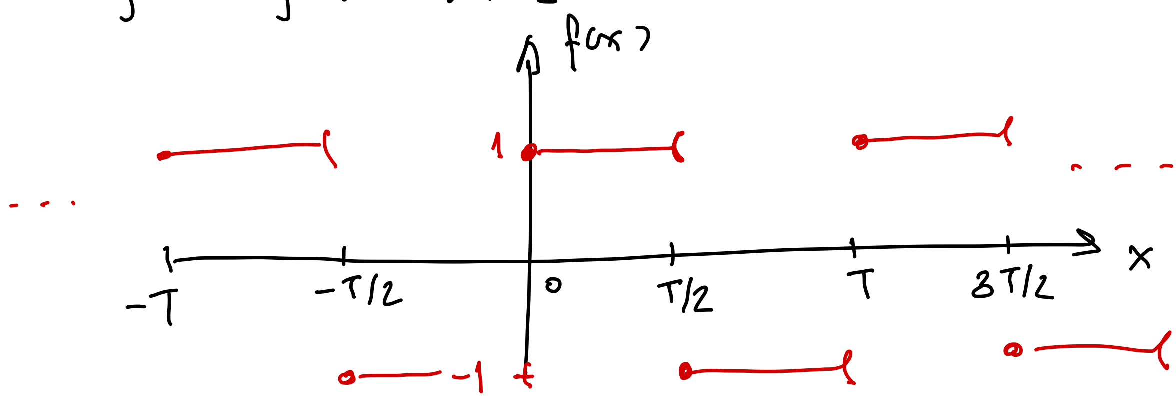


Example 3: let $f : [0, T[\to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{T}{2}[\\ -1 & \text{if } x \in [\frac{T}{2}, T[\end{cases}$$ be a function

extended by T-periodicity to $\mathbb{R}$.

Compute the Fourier series of f and compare Ff and f and $[0, T]$.



$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx = \frac{2}{T}\int_0^{T/2} 1\,dx + \int_{T/2}^{T} (-1)\,dx = 0$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi n}{T}x\right)dx = \frac{2}{T}\int_0^{T/2}\cos\left(\frac{2\pi n}{T}x\right)dx$$

$$-\int_{T/2}^{T}\cos\left(\frac{2\pi n}{T}x\right)dx =$$

$$= \frac{2}{T}\,\frac{T}{2\pi n}\left[\sin\left(\frac{2\pi n}{T}x\right)\Big|_0^{T/2} - \sin\left(\frac{2\pi n}{T}x\right)\Big|_{T/2}^{T}\right]$$

$$= \frac{1}{\pi n}\left[\sin(\pi n) - 0 - \sin(2\pi n) + \sin(\pi n)\right] = 0$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi n}{T}x\right)dx = \frac{2}{T}\int_0^{T/2}\sin\left(\frac{2\pi n}{T}x\right)dx$$

$$-\int_{T/2}^{T}\sin\left(\frac{2\pi n}{T}x\right)dx =$$

$$= \frac{2}{T}\,\frac{T}{2\pi n}\left[-\cos\left(\frac{2\pi n}{T}x\right)\Big|_0^{T/2} + \cos\left(\frac{2\pi n}{T}x\right)\Big|_{T/2}^{T}\right]$$

$$= \frac{1}{\pi n}\left[-2\cos(\pi n) + \cos(0) + \cos(2\pi n)\right]$$

$$= \frac{1}{\pi n}\left(2 - 2\cos(\pi n)\right) = \frac{2}{\pi n}\left(1 - \cos(\pi n)\right)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)\right]$$

$$= \sum_{n \text{ odd}} \frac{4}{n\pi} \sin\left(\frac{2\pi n}{T} x\right)$$

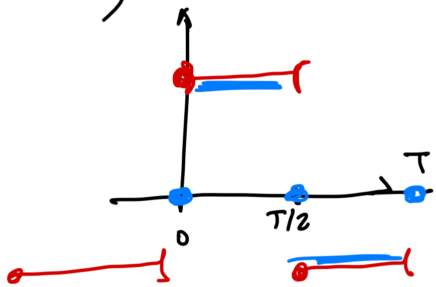
$$= \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{2\pi}{T}(2k+1)x\right)$$

$n = 2k+1$

Comparison $Ff(x)$ -vs- $f(x)$ on $[0,2\pi]$

$\rightarrow$ Dirichlet theorem.

$$Ff(x) = \frac{1}{2}\left(f(x+0) + f(x-0) \right)$$



$$\begin{cases} \frac{1}{2}(-1+1) = 0 & \text{if } x = 0 \\ \frac{1}{2}(1+1) = 1 = f(x) & \text{if } x \in]0, \frac{T}{2}[\\ \frac{1}{2}(1-1) = 0 & \text{if } x = \frac{T}{2} \\ \frac{1}{2}(-1-1) = f(x) = -1 & \text{if } x \in]\frac{T}{2}, T[\\ \frac{1}{2}(-1+1) = 0 & \text{if } x = T \end{cases}$$

• Result 2: Let $f : \mathbb{R} \to \mathbb{R}$ a T-periodic piecewise defined function:

Then: $$\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx \quad \forall\, a \in \mathbb{R}$$

1.2.2 Heuristic justification of the definition

For sake of simplicity $T = 2\pi$.

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\,dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\,dx$$

If $f(x) \in C^0(\mathbb{R})$, then $f(x) = Ff(x)$

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} Ff(x)\,dx$$

$$= \int_0^{2\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} \int_0^{2\pi} a_n \cos(nx)\,dx + \sum_{n=1}^{\infty} \int_0^{2\pi} b_n \sin(nx)\,dx$$

$$= \pi a_0 + \sum_{n=1}^{\infty} \left[a_n \underbrace{\frac{\sin(nx)}{n} \Big|_0^{2\pi}}_{=0} - b_n \underbrace{\frac{\cos(nx)}{n} \Big|_0^{2\pi}}_{=0} \right]$$

$$= \pi a_0 \implies a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$$

$k \in \mathbb{N}^*$

$$\int_0^{2\pi} f(x) \cos(kx)\,dx = \int_0^{2\pi} Ff(x) \cos(kx)\,dx$$

$$= \underbrace{\frac{a_0}{2} \int_0^{2\pi} \cos(kx)\,dx}_{=0} + \sum_{n=1}^{\infty} \int_0^{2\pi} a_n \cos(nx)\cos(kx)\,dx$$

$$+ \sum_{n=1}^{\infty} \underbrace{\int_0^{2\pi} b_n \sin(nx)\cos(kx)\,dx}_{=0 \quad \forall n,k}$$

$$\int_0^{2\pi} \cos(nx)\cos(kx)\,dx = \begin{cases} 0 & \text{if } n \neq k \\ \pi & \text{if } n = k \end{cases}$$

$$\int_0^{2\pi} f(x)\cos(kx)\,dx = \pi a_k$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx$$

You can do the same for obtaining b_n

(compute $\int_0^{2\pi} f(x)\sin(nx)\,dx \cdots$)

1.2.5 Complex notation for Fourier series

Euler identity $\longrightarrow$ $\begin{cases} e^{ix} = \cos x + i\sin x \\ e^{-ix} = \cos x - i\sin x \end{cases}$

The Fourier series of T-periodic piecewise-defined function $f: \mathbb{R} \to \mathbb{R}$ in complex-form is:

$$Ff(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i \frac{2\pi n}{T} x} \quad \text{where } c_n \in \mathbb{C}$$

$$\text{and } c_n = \frac{1}{T} \int_0^T f(x) e^{-i \frac{2\pi n}{T} x} dx$$

$$\left(e^a = \exp(a) \right)$$

For the sake of simplicity $T = 2\pi$, then:

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$\left[\cos(nx) = \frac{1}{2}\left(e^{inx} + e^{-inx} \right), \sin(nx) = \frac{-i}{2}\left(e^{inx} - e^{-inx} \right) \right]$$

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(-i \frac{e^{inx} - e^{-inx}}{2} \right) \right]$$

$$\left(\begin{array}{l|l} & (1) - (2) \\ (1)\ e^{ix} = \cos x + i \sin x & e^{ix} - e^{-ix} = 0 + 2i \sin x \\ (2)\ e^{-ix} = \cos x - i \sin x & \frac{1}{2}\left(e^{ix} - e^{-ix} \right) = i \sin x \end{array} \right.$$

$$-i\,\frac{1}{2}\left(e^{+ix}-e^{-ix}\right)=\underbrace{-i\,i}_{+1}\sin x\ ,\quad i^2=-1$$

$$Ff(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[\frac{a_n-ib_n}{2}e^{inx}+\frac{a_n+ib_n}{2}e^{-inx}\right]$$

$$\frac{a_0}{2}=\frac{1}{2\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{2\pi}\int_0^{2\pi}f(x)\underbrace{e^{i0x}}_{=e^0=1}dx$$

For $n\geq 1$

$$\frac{a_n-ib_n}{2}=\underbrace{\frac{1}{2\pi}\int_0^{2\pi}f(x)\cos(nx)\,dx}_{\frac{a_n}{2}}-i\underbrace{\frac{1}{2\pi}\int_0^{2\pi}f(x)\sin(nx)\,dx}_{\frac{b_n}{2}}$$

$$=\frac{1}{2\pi}\int_0^{2\pi}f(x)\left[\cos(nx)-i\sin(nx)\right]dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi}f(x)\,e^{-inx}\,dx=c_n$$

$$\frac{a_n + i b_n}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx + i\,\frac{1}{2\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\left[\cos(nx) + i\sin(nx)\right]dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{inx}\,dx = C_{-n}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-inx}\,dx = C_n$$

$$Ff(x) = C_0 + \sum_{n=1}^{\infty}\left[C_n e^{inx} + C_{-n} e^{-inx}\right]$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{inx} + \sum_{n=1}^{\infty} C_{-n} e^{-inx}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{inx} + \sum_{n=-\infty}^{-1} C_n e^{inx}$$

$$= \sum_{n=-\infty}^{+\infty} C_n e^{inx} \quad \text{with} \quad C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-inx}\,dx.$$