Série 8b – solution

Problème 8b.1 – Charge concentrée et distribuée

On considère une poutre AB, encastrée en A, avec une force ponctuelle et une force distribuée. Le moment d'inertie à l'axe neutre est $I_{z,y_0} = 3.35\ \mathrm{m}^4$. Trouver:

(a) Les forces de réactions au point A et B.
(b) La force de cisaillement V(x)
(c) Le moment de flexion M(x)
(d) La flèche w(x) de la poutre.

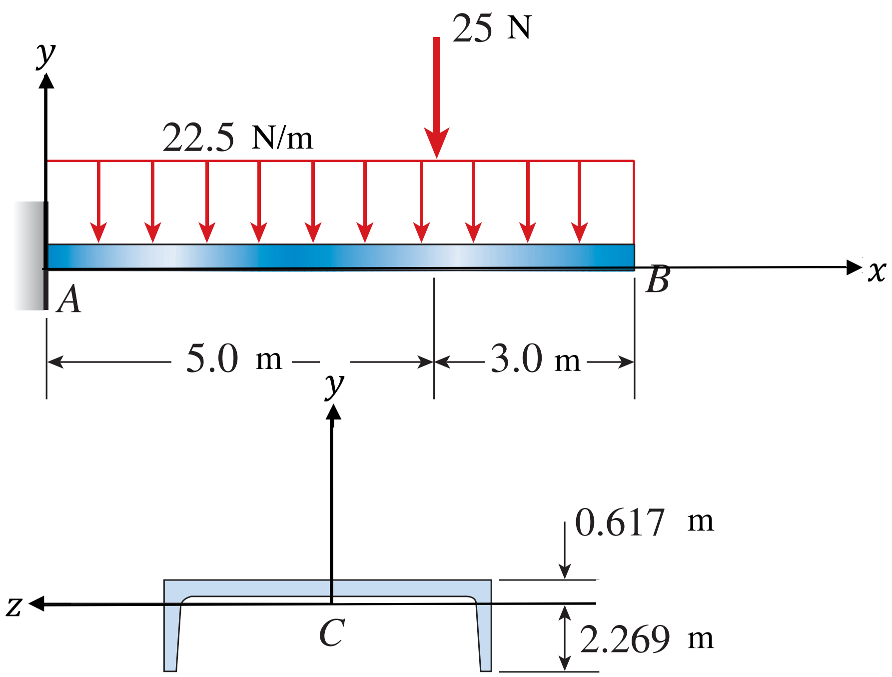

Figure Le cantilever et sa section transverse. L'origine C est située sur l'axe neutre.

Solution

What is given?

Distributed Load: $q = 22.5 \ \mathrm{N/m}$
Concentrated Load: $F = 25 \ \mathrm{N}$
Length: $L = 8 \ \mathrm{m}$

Assumptions

The material is homogeneous and isotropic.
The cross-section of the cantilever in the ZY plane remains un-deformed along the length of the cantilever.

What is asked?

(a) The reaction forces at A.
(b) The Shear Force Diagram
(c) The Bending Moment Diagram.
(d) The deflection equation of the cantilever.

Principles and formula

(a) Reaction forces at A

We calculate the reaction forces from the free body diagram of the entire cantilever:

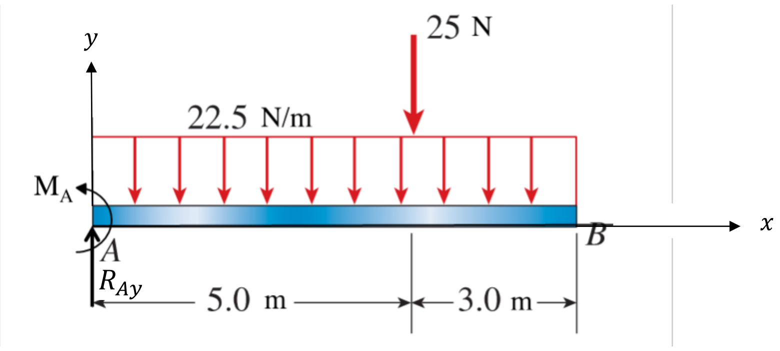

Figure 9.5.2 | Free body diagram of the cantilever.

Summing the forces we get:

$$\Sigma F_y = R_A - F - \int_0^L q dx = 0 \rightarrow R_A = 25 + (22.5)(8) = 205 \ \mathrm{N} \tag{0.0.1}$$

From the equilibrium of the moments M_y, we get the reaction moment M_A:

$$\Sigma M_y = 0 \rightarrow M_A - \left(\int_0^L q dx \cdot \frac{L}{2} \right) - F \cdot 5 = 0 \rightarrow M_A = 845 \ \mathrm{N \cdot m} \tag{0.0.2}$$

Which gives us:

$$M_A = 845 \ \mathrm{N \cdot m}, \quad R_A = 205 \ \mathrm{N} \tag{0.0.3}$$

(b) Shear Force Diagram

For this cantilever there are two external loads applied: one uniformly distributed along the whole length and the second which is a point load applied at $x = 5 \ m$.

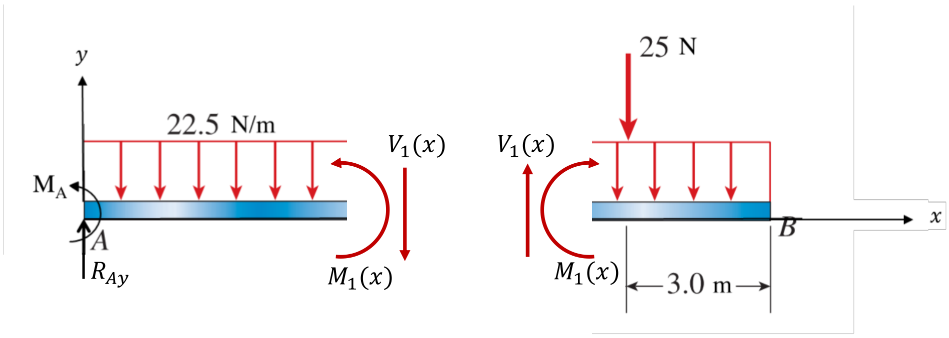

Figure 9.5.3 | Free boy diagram and internal forces at $x < 5\ m$

We write the equilibrium of the forces for the left part of the beam shown in Figure 9.5.3:

$$V_1(x) = R_A - qx = (205 - 22.5x)\ \mathrm{N} \tag{0.0.4}$$

From C to D ($x \geq 5\ m$):

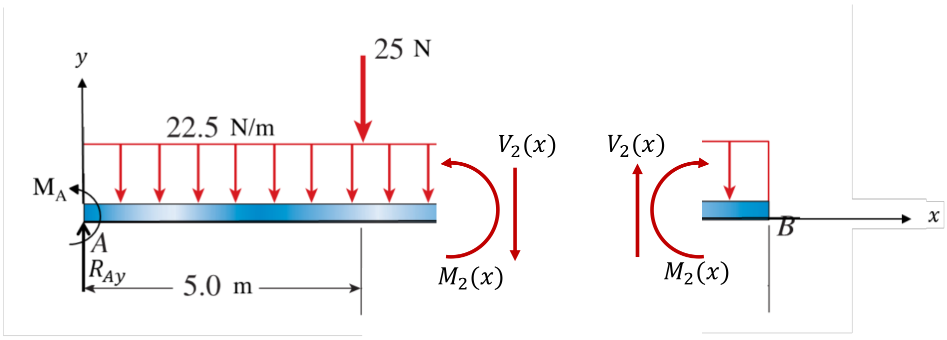

Figure 9.5.4 | Free boy diagram and internal forces at $x \geq 5\ m$

As we did before, we write the equilibrium of the forces for the left part of the beam shown in Figure 9.5.4:

$$V_2(x) = R_A - qx - F = (180 - 22.5x)\ \mathrm{N} \tag{0.0.5}$$

We are now able to draw the Shear Force Diagram:

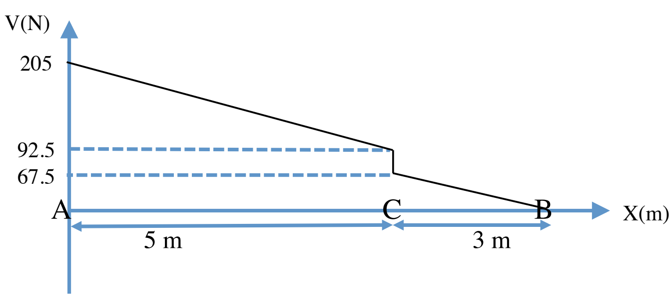

Figure 9.5.5 | Shear force diagram

(c) Bending Moment Diagram

As we do not have any concentrated moment loads, the bending moment at any point along the beam is given by:

$$M - M(x=0) = M + M_A = \int_0^x V(x')dx' \tag{0.0.6}$$

From A to C ($x < 5$ m):

$$M_1(x) = -M_A + \int_0^x V_1(x')\,dx' = -M_A + \int_0^x (R_A - qx')\,dx' = (-845 + 205x - 11.25x^2)\ \mathrm{N\cdot m} \tag{0.0.7}$$

From C to B ($x \geq 5$ m):

$$\begin{aligned} M_2(x) - M_2(x=5) &= \int_5^x V_2(x')\,dx' = -101.25 + \int_5^x (180 - 22.5x')dx' \\ &= -101.25 + 180(x-5) - 11.25(x^2 - 25) \end{aligned} \tag{0.0.8}$$

$$M_2(x) = -720 + 180x - 11.25\,x^2\ \mathrm{N\cdot m} \tag{0.0.9}$$

Using these values, we draw the Bending Moment Diagram:

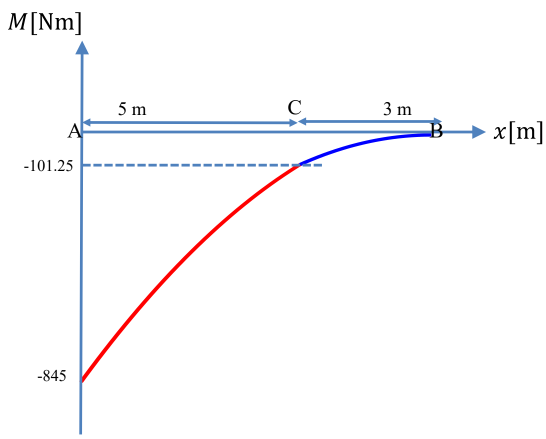

Figure 9.5.6 | Bending moment diagram

(d) Deflection of the cantilever

The curvature of the beam due to bending moments at any point is given by:

$$\frac{d^2w}{dx^2} = \frac{M(x)}{EI} \tag{0.0.10}$$

Hence we can calculate the deflection equation of the beam by double integration of Eq. (0.0.10):

$$w = \frac{1}{EI}\int_0^x \left(\int_0^{x'} M(x'')dx'' \right) dx' \tag{0.0.11}$$

To do this, we need to be careful and first calculate the derivative of the deflection, which must be continuous at $x = 5$ m. We need to remember that the moment is defined in two parts.

Therefore we obtain:

For $x \leq 5$:

$$w'(x) - w'(0) = \frac{1}{EI}\int_0^x M_1(x^{'})dx^{'} = \frac{1}{EI}\int_0^x (-845 + 205x' - 11.25x'^2)dx'$$
$$= \frac{1}{EI}\left(-845x + 205\frac{x^2}{2} - 11.25\frac{x^3}{3}\right) \tag{0.0.12}$$
beam clamped at x=0 $\rightarrow w'(0) = 0$

For $x > 5$:

$$w'(x) - w'(x = 5) = \frac{1}{EI}\int_5^x M_2(x^{'})dx^{'} = \frac{1}{EI}\int_5^x (-720 + 180x' - 11.25x'^2)dx' \tag{0.0.13}$$

$$w'(x) = w'(x = 5) + \frac{1}{EI}(1818.75 - 720x + 90x^2 - 3.75x^3) \tag{0.0.14}$$

$$w'(x) = \frac{1}{EI}(-312.5 - 720x + 90x^2 - 3.75x^3) \tag{0.0.15}$$

And now we integrate again to compute the deflection.
For $x \leq 5$:

$$w(x) - w(x = 0) = \int_0^x w^{'(x^{'})}dx^{'} = \frac{1}{EI}\left(-845\frac{x^2}{2} + 205\frac{x^3}{6} - 11.25\frac{x^4}{12}\right) \tag{0.0.16}$$

$$w(x = 0) = 0 \text{ and } w(x) = \frac{-5070x^2+410x^3-11.25x^4}{12EI} \tag{0.0.17}$$

For $x > 5$:

$$w(x) - w(x = 5) = \int_5^x w'(x')dx' \tag{0.0.18}$$

$$w(x) - w(x = 5) = \frac{1}{EI}(7398.4375 - 312.5x - 360x^2 + 30x^3 - 0.9375x^4) \tag{0.0.19}$$

$$w(x) = \frac{6250 - 3750x - 4320x^2 + 360x^3 - 11.25x^4}{12EI} \tag{0.0.20}$$

Problème 8b.2 – Calcul de la déflection à partir des moments (1)

On considère la poutre AB de longueur $L = 12$ m utilisé précédemment dans un problème.

Les diagrammes de force de cisaillement relative et de moment en flexion sont montrés sur la figure 9.2.1.

Sachant que :

$$M_1(x) = 25\,x \text{ kNm.} \quad 0<x<L/3 \tag{0.0.1}$$

$$M_2(x) = -35\,x + 240 \text{ kNm.} \quad L/3 <x <2L/3 \tag{0.0.2}$$

$$M_3(x) = -35\,x + 420 \text{ kNm.} \quad 2L/3 <x <L \tag{0.0.3}$$

Calculer la déflection w(x) le long de la poutre.

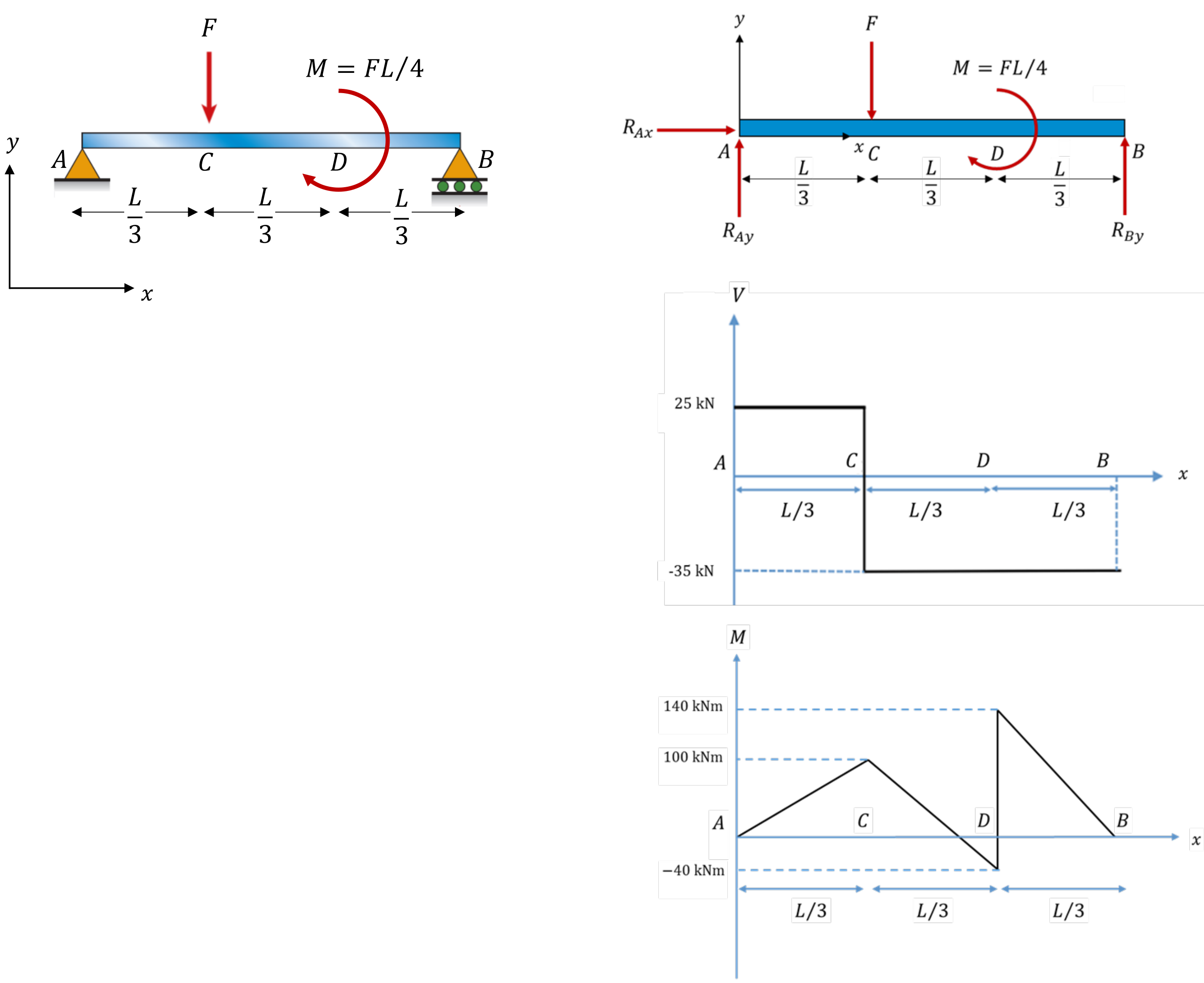

Figure 9.2.1 | Diagrammes des forces, de cisaillement et de moment en flexion relatifs à la poutre AB.

Comme d'hab, 2 options pour arriver à la flèche

a) Trouver F et M. puis méthode sections, V(x), M(x) et enfin w(x) par double intégration
b) Superpositions et formulaire

Solution – Première Option

1. Conditions au bord
   a. $w_1(x=0) = 0$
   b. $w_3(x=L)=0$
2. Continuity:
   a. $w_1(x=L/3) = w_2(x=L/3)$
   b. $w_2(x=2L/3) = w_3(x=2L/3)$
   c. $w'_1(x=L/3) = w'_2(x=L/3)$
   d. $w'_2(x=2L/3) = w'_3(x=2L/3)$

The curvature of the beam due to the bending moment at any point is given by:

$$\frac{d^2w}{dx^2} = \frac{M(x)}{EI} \tag{0.0.4}$$

Hence we can calculate the deflection equation of the beam by double integration of (0.0.4):

$$w = \frac{1}{EI}\int_0^x \left(\int_0^{x\prime} M(x'')dx''\right)dx' \tag{0.0.5}$$

We first calculate $w'(x)$, the derivative of the deflection, which must be **continuous** at $x = \frac{L}{3} = 4$ m (point C), and also at $x = \frac{2L}{3} = 8$ m (point D). We obtain:

For $x \leq \frac{L}{3}$:

$$w'(x) - w'(0) = \frac{1}{EI}\int_0^x (25\,x')dx' \tag{0.0.6}$$

$$w'(x) = \frac{1}{EI}\left(25\frac{x^2}{2}\right) + w'(0) \tag{0.0.7}$$

For $\frac{L}{3} < x < 2\frac{L}{3}$:

$$w'(x) - w'\left(x = \frac{L}{3}\right) = \frac{1}{EI}\int_{L/3}^x (-35\,x' + 240)dx' \tag{0.0.8}$$

$$w'(x) = w'\left(x = \frac{L}{3}\right) + \frac{1}{EI}\left(-\frac{35\,x^2}{2} + 240x - 680\right) \tag{0.0.9}$$

$$w'(x) = \frac{1}{EI}\left(-\frac{35\,x^2}{2} + 240x - 480\right) + w'(0) \tag{0.0.10}$$

$$w(x) - w(x=0) = \int_0^x w'(x')dx' = \int_0^x \left(\frac{1}{EI}\left(25\frac{x'^2}{2}\right) + w'(0)\right)dx' \tag{0.0.14}$$

$$w(x=0) = 0 \; and \; w(x) = \frac{25x^3}{6EI} + w'(0)x \tag{0.0.15}$$

For $\frac{L}{3} < x < 2\frac{L}{3}$:

$$w(x) - w\left(x = \frac{L}{3}\right) = \int_{L/3}^x w'(x')dx' = \int_{L/3}^x \left(\frac{1}{EI}\left(-\frac{35\,x'^2}{2} + 240x' - 480\right) + w'(0)\right)dx' \tag{0.0.16}$$

$$w(x) - w\left(x = \frac{L}{3}\right) = \frac{1}{EI}\left(\frac{-35x^3}{6} + 120x^2 - 480x + \frac{1120}{3}\right) + w'(0)(x - L/3) \tag{0.0.17}$$

$$w(x) = \frac{1}{EI}\left(\frac{-35x^3}{6} + 120x^2 - 480x + 640\right) + w'(0)x \tag{0.0.18}$$

For $\frac{2L}{3} < x < L$:

$$w(x) - w\left(x = \frac{2L}{3}\right) = \int_{2L/3}^x w'(x')dx' = \int_{2L/3}^x \left(\frac{1}{EI}\left(-\frac{35\,x'^2}{2} + 420x' - 1920\right) + w'(0)\right)dx' \tag{0.0.19}$$

$$w(x) - w\left(x = \frac{2L}{3}\right) = \frac{1}{EI}\left(\frac{-35x^3}{6} + \frac{420x^2}{2} - 1920x + \frac{14080}{3}\right) + w'(0)(x - \frac{2L}{3}) \tag{0.0.20}$$

$$w(x) = \frac{1}{EI}\left(\frac{-35x^3}{6} + 210x^2 - 1920x + 6400\right) + w'(0)x \tag{0.0.21}$$

Applying the condition that the deflection at point B should be zero $w(x = L) = 0$, we can obtain $w'(x = 0)$:

$$w(x = L) = 0 \rightarrow w'(x = 0) = -\frac{880}{3EI} \tag{0.0.22}$$

Which yields a final expression for the deflection.

$$w(x) = \begin{cases} \frac{25x^3}{6EI} - \frac{880}{3EI}x & ;\; x \leq \frac{L}{3} \\ \frac{1}{EI}\left(\frac{-35x^3}{6} + 120x^2 - \frac{2320}{3}x + 640\right) & ;\frac{L}{3} \leq x \leq \frac{2L}{3} \\ \frac{1}{EI}\left(\frac{-35x^3}{6} + 210x^2 - \frac{6640}{3}x + 6400\right) & ;\frac{2L}{3} \leq x \leq L \end{cases} \tag{0.0.23}$$

Solution – Deuxième option

Another way to solve this problem is to split the problem into two and apply superposition. The two separate problems to be solved are shown below.

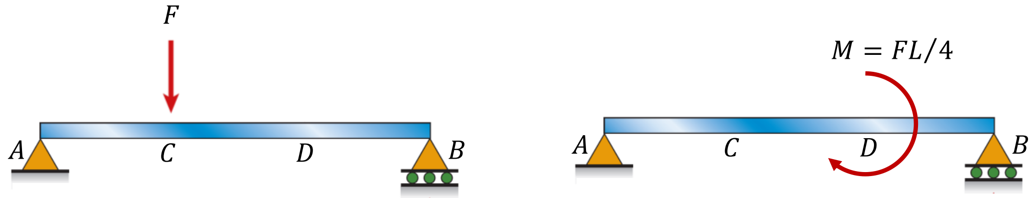

Figure 9.2.2 | The two independent problems to be solved and later the solutions added.

De l'énoncé (V(x) et M(x), on voit que F =60 kN et M= -180 kN.m. on donne L=12 m

If we check in the help formulas, eg Appendix G in Geer and Goodno book, Table G2.5 that is equivalent to the case in the left, giving a deflection of:

$$w_F(x) = \begin{cases} -\dfrac{Fx}{9EI}\left(\dfrac{5L^2}{9} - x^2\right) = -\dfrac{20x}{3EI}(80 - x^2); x \le \dfrac{L}{3} \\ -\dfrac{F(L-x)}{18EI}\left(-\dfrac{L^2}{9} + 2xL - x^2\right) = -\dfrac{10}{3EI}(-192 + 304x - 36x^2 - x^3); \dfrac{L}{3} \le x \end{cases} \tag{0.0.24}$$

For the problem on the right, we check the same table G2.9: (and use change of variable for the side that is not given)

$$w_M(x) = \begin{cases} \dfrac{Mx}{6LEI}\left(\dfrac{2L^2}{3} - x^2\right) = \dfrac{5x}{2EI}(96 - x^2); x \le \dfrac{2L}{3} \\ \dfrac{M(L-x)}{6LEI}\left(-\dfrac{4L^2}{3} + 2xL - x^2\right) = -\dfrac{5}{2EI}(-2304 + 480x - 36x^2 + x^3); x \ge \dfrac{2L}{3} \end{cases} \tag{0.0.25}$$

The final deflection will be the sum of Eq. (0.0.24) and (0.0.25)

$$w_{total}(x) = \begin{cases} -\dfrac{Fx}{9EI}\left(\dfrac{5L^2}{9} - x^2\right) + \dfrac{Mx}{6LEI}\left(\dfrac{2L^2}{3} - x^2\right); x \le \dfrac{L}{3} \\ -\dfrac{F(L-x)}{18EI}\left(-\dfrac{L^2}{9} + 2xL - x^2\right) + \dfrac{Mx}{6LEI}\left(\dfrac{2L^2}{3} - x^2\right); \dfrac{L}{3} \le x \le \dfrac{2L}{3} \\ -\dfrac{F(L-x)}{18EI}\left(-\dfrac{L^2}{9} + 2xL - x^2\right) + \dfrac{M(L-x)}{6LEI}\left(-\dfrac{4L^2}{3} + 2xL - x^2\right); x \ge \dfrac{2L}{3} \end{cases} \tag{0.0.26}$$

then simplifying:

$$w_{total}(x) = \begin{cases} \dfrac{1}{EI}\left(\dfrac{25}{6}x^3 - \dfrac{880}{3}x\right); x \le \dfrac{L}{3} \\ \dfrac{1}{EI}\left(640 - \dfrac{2320}{3}x + 120x^2 - \dfrac{35}{6}x^3\right); \dfrac{L}{3} \le x \le \dfrac{2L}{3} \\ \dfrac{1}{EI}\left(6400 - \dfrac{6640}{3}x + 210x^2 - \dfrac{35}{6}x^3\right); x \ge \dfrac{2L}{3} \end{cases} \tag{0.0.27}$$

Which equals our previous result above in Eq. (0.0.23).

- Problème 8b.3 – Extraire le diagramme des forces à partir de la flèche

On considère une poutre de longueur $2L = 2$ m avec une rigidité en flexion EI constante le long de la poutre. L'équation de déflection de la poutre est donnée par :

$$pour\ 0 < x < L \qquad w(x) = \frac{1}{EI}\left[-\frac{1}{24}q_0x^4 + \frac{3}{32}q_0Lx^3 - \frac{5}{96}q_0L^2x^2\right] \tag{0.0.1}$$

$$et$$

$$pour\ L < x < 2L \quad w(x) = \frac{1}{EI}\left[-\frac{1}{24}q_0x^4 + \frac{3}{32}q_0Lx^3 - \frac{5}{96}q_0L^2x^2 + \frac{1}{24}q_0(x-L)^4 + \frac{1}{12}q_0L(x-L)^3\right] \tag{0.0.2}$$

Déterminer :

(a) Le moment de flexion $M_z(x)$ le long de la poutre
(b) La force de cisaillement V(x) le long de la poutre
(c) Les forces de réactions aux supports (et leur positions)
(d) Dessiner le diagramme des forces

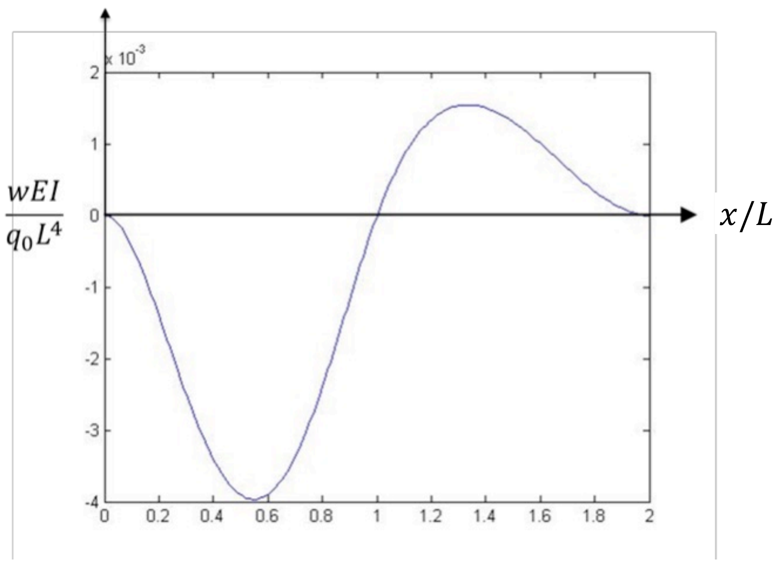

Figure 9.1.1 | Déflection de la poutre

Solution

What is given?

Deflection equation of the beam
Beam length $2L = 2$ m

Assumptions

The material is homogeneous and isotropic

What is asked?

(e) Bending moment diagram
(f) Shear force diagram
(g) Reaction forces
(h) Free body diagram

Principles and formula

(a) Bending Moment diagram

The curvature of the beam due to bending moments at any point is given by:

$$\frac{d^2w}{dx^2} = \frac{M(x)}{EI} \tag{0.0.3}$$

Hence by differentiation of the deflection equation, we get the bending moment:

For x<L:

$$\frac{d^2\left(\frac{1}{EI}\left[-\frac{1}{24}q_0x^4 + \frac{3}{32}q_0Lx^3 - \frac{5}{96}q_0L^2x^2\right]\right)}{dx^2} = \frac{M(x)}{EI} \tag{0.0.4}$$

$$\frac{1}{EI}\left[-\frac{1}{2}q_0x^2 + \frac{9}{16}q_0Lx - \frac{5}{48}q_0L^2\right] = \frac{M(x)}{EI} \tag{0.0.5}$$

$$M(x) = -\frac{1}{2}q_0x^2 + \frac{9}{16}q_0Lx - \frac{5}{48}q_0L^2 \tag{0.0.6}$$

For $x \geq L$

$$\frac{d^2\left(\frac{1}{EI}\left[-\frac{1}{24}q_0x^4 + \frac{3}{32}q_0Lx^3 - \frac{5}{96}q_0L^2x^2 + \frac{1}{24}q_0(x-L)^4 + \frac{1}{12}q_0L(x-L)^3\right]\right)}{dx^2} = \frac{M(x)}{EI} \tag{0.0.7}$$

$$\frac{1}{EI}\left[-\frac{1}{2}q_0x^2 + \frac{9}{16}q_0Lx - \frac{5}{48}q_0L^2 + \frac{1}{2}q_0(x-L)^2 + \frac{1}{2}q_0L(x-L)\right] = \frac{M(x)}{EI} \tag{0.0.8}$$

$$M(x) = -\frac{1}{2}q_0x^2 + \frac{9}{16}q_0Lx - \frac{5}{48}q_0L^2 + \frac{1}{2}q_0(x-L)^2 + \frac{1}{2}q_0L(x-L) = \frac{1}{48}L(-5L+3x)q_0 \tag{0.0.9}$$

In $x = L$

$$M(x = L_-) = M(x = L_+) = -\frac{q_oL^2}{24} \tag{0.0.10}$$

The shear force at any point along the beam is given by:

$$\frac{dM(x)}{dx} = V(x) \tag{0.0.11}$$

For $(x < L)$:

$$V(x) = \frac{d\left[-\frac{1}{2}q_0x^2 + \frac{9}{16}q_0Lx - \frac{5}{48}q_0L^2\right]}{dx} \tag{0.0.12}$$

$$V(x) = -q_0x + \frac{9}{16}q_0L \tag{0.0.13}$$

For $(x \geq L)$:

$$V(x) = \frac{d\left[-\frac{1}{2}q_0x^2 + \frac{9}{16}q_0Lx - \frac{5}{48}q_0L^2 + \frac{1}{2}q_0(x-L)^2 + \frac{1}{2}q_0L(x-L)\right]}{dx} \tag{0.0.14}$$

$$V(x) = -q_0x + \frac{9}{16}q_0L + q_0(x-L) + \frac{1}{2}q_0L = \frac{1}{16}q_0L \tag{0.0.15}$$

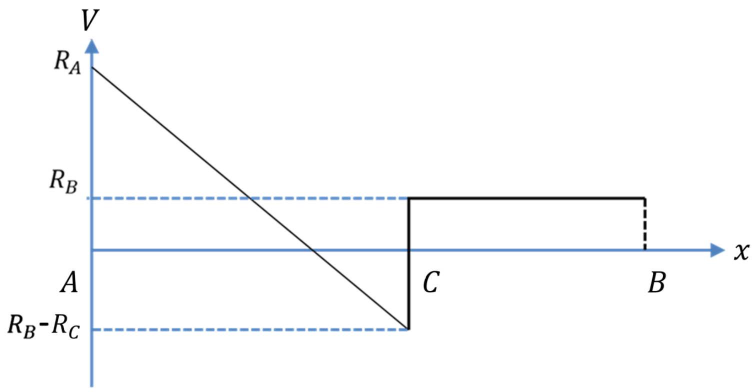

Figure 9.1.3 | Shear force diagram

(c) Reaction forces

From the shear force diagram of Figure 9.1.3 we notice that a uniform distributed load is applied between A and C, while no loads are applied between $C$and B.

We also notice three discontinuities in the shear force diagram, respectively at points A, B and C, which we know to correspond to three punctual loads applied. We are going to call these loads $\vec{R}_A$, $\vec{R}_B$and $\vec{R}_C$. In order to calculate their values, we will choose a given direction for each of them:

$$\vec{R}_A = R_A \cdot \hat{y};\ \vec{R}_B = R_B \cdot \hat{y}; \vec{R}_C = R_C \cdot (-\hat{y}) \tag{0.0.16}$$

Substituting the value of $x = 0$ in the shear force equation we get:

$$V(0) = -q_0x + \frac{9}{16}q_0L = \frac{9}{16}q_0L \tag{0.0.17}$$

Therefore, by cutting the beam very close to the origin $x = 0$ and taking the section on the left, we can write equilibrium of forces that states:

$$\vec{R}_B + \vec{V}(2L) = R_B \cdot \hat{y} + V(x = 2L) \cdot \hat{y} = 0 \rightarrow R_B = -\frac{1}{16} q_0 L \tag{0.0.20}$$

Finally, we can check what happens at $x = L$ in the shear force, where we get:

$$V(x = L^+) = \frac{1}{16} q_0 L \tag{0.0.21}$$

$$V(x = L^-) = -q_0 L + \frac{9}{16} q_0 L = -\frac{7}{16} q_0 L \tag{0.0.22}$$

If we take a differential section of the beam just around point C, we can apply equilibrium of forces:

$$\vec{V}(L^+) + \vec{V}(L^-) + \vec{R}_C = V(L^+) \cdot (-\hat{y}) + V(L^-)\hat{y} + R_C \cdot (-\hat{y}) = 0 \tag{0.0.23}$$

$$R_C = V(L^-) - V(L^+) = -\frac{1}{2} q_0 L \tag{0.0.24}$$

Knowing these reactions, we can now verify the force equilibrium:

$$\begin{aligned} \vec{R}_A + \vec{R}_B + \vec{R}_C + q_0 L \cdot (-\hat{y}) &= R_A \cdot \hat{y} + R_B \cdot \hat{y} + R_C \cdot (-\hat{y}) + q_0 L \cdot (-\hat{y}) = \\ &= \left(\frac{9}{16} q_0 L - \frac{1}{16} q_0 L + \frac{1}{2} q_0 L - q_0 L\right) \hat{y} = 0 \end{aligned} \tag{0.0.25}$$

Remember that the values (signs) included in Eq. (0.0.30) are defined by the selection of the direction of the reactions that we did in Eq. (0.0.16).

Analysing the bending moment diagram in Figure 9.1.2 we notice discontinuities in A and B, meaning that a punctual moment is applied at those points. In both cases we choose them to point in the positive direction of the z axis:

$$\vec{M}_A = M_A \cdot \hat{z};\ \vec{M}_B = M_B \cdot \hat{z} \tag{0.0.26}$$

Cutting very close to the left clamp of the beam and taking the section on the left, we can write equilibrium of moments as:

$$\vec{M}_A + \vec{M}_{int}(x = 0) = M_A \cdot \hat{z} + M_{int}(x = 0) \cdot (\hat{z}) = 0 \rightarrow M_A = -M_{int}(x = 0) = \frac{5}{48} q_0 L^2 \tag{0.0.27}$$

And, equivalently, we cut very close to the right end of the beam and we take the section on the right:

$$\vec{M}_B + \vec{M}_{int}(x = 2L) = M_B \cdot \hat{z} + M_{int}(x = 2L) \cdot (-\hat{z}) = 0 \rightarrow M_B = M_{int}(x = 2L) = \frac{1}{48} q_0 L^2 \tag{0.0.28}$$

Knowing these moments, we can now verify the equilibrium of moments in the overall beam:

$$M_A + M_B - R_C L + R_B 2L - \frac{q_0 L^2}{2} = q_0 L^2 \left(\frac{5}{48} + \frac{1}{48} + \frac{1}{2} - \frac{1}{8} - \frac{1}{2}\right) = 0 \tag{0.0.29}$$

(d) Free Body Diagram

Finally, we can just summarize by giving all the values for the reaction moments and forces together with the Free Body Diagram in Figure 9.1.4, where we are using the choices for vector direction that we made before.

Hence the reactions are:

$$R_A = \frac{9}{16}q_0L;\ R_B = -\frac{1}{16}q_0L;\ R_C = -\frac{1}{2}q_0L; M_A = \frac{5}{48}q_0L^2; M_B = \frac{q_0L^2}{48} \tag{0.0.30}$$

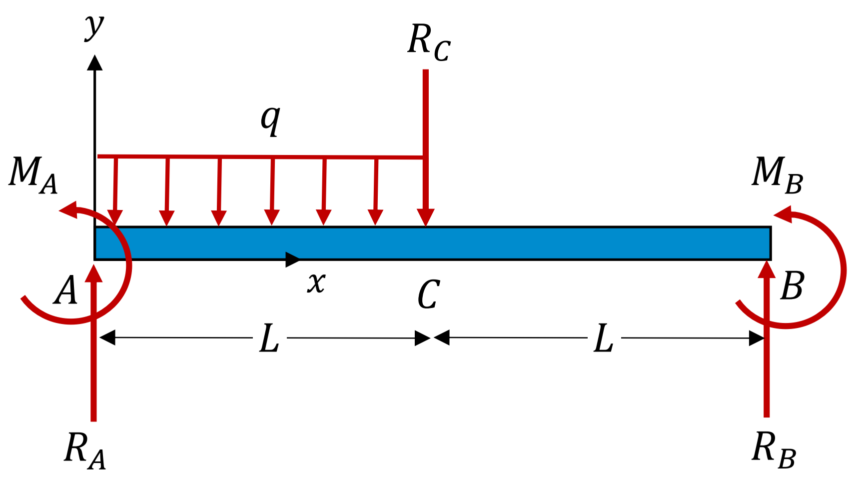

Figure 9.1.4 | Free body Diagram

Problème 8b.4 – Fléchissement avec une force axiale

Une poutre AB de longueur $L = 4$ m est supportée en ses extrémités. On impose une force F_0 et un moment M_0 au centre C. $F_0 = 30\sqrt{2}$ N avec un angle de 45°. $M_0 = 20$ N · m.

La section de la poutre est rectangulaire, de dimensions $b = 10$ cm (largeur en z), $d = 20$ cm (épaisseur en y), surface $A = 200$ cm^2. Le module de Young du matériau est $E = 200$ GPa.

Déterminer:

(e) Les forces de réaction aux points A et B.
(f) La force de cisaillement V(x)
(g) Le moment de flexion M(x)
(h) Les contraintes maxima en compression et en traction.
(i) La déflection w(x) de la poutre.

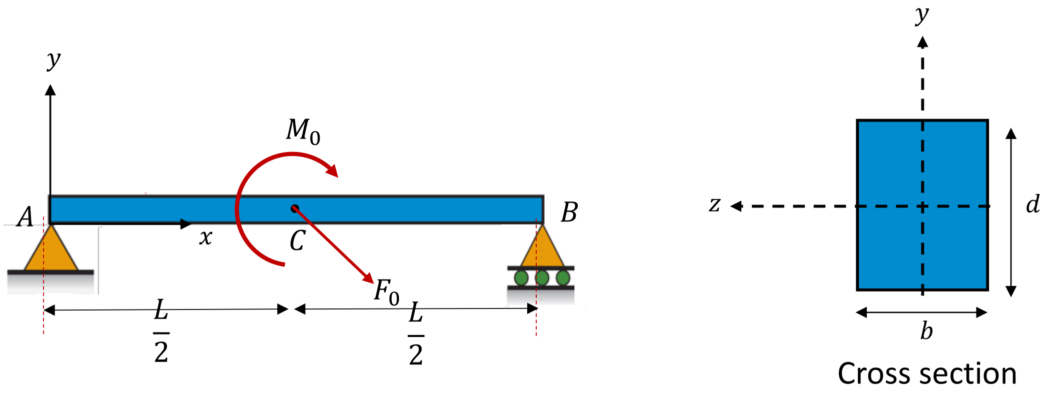

Figure 9.4.1 | Force de cisaillement et moment en flexion relatifs à la poutre AB.

Solution

<u>What is given?</u>

Force $F_0 = 30\sqrt{2}$ N at 45°
Moment $M_0 = 20 \text{ N} \cdot \text{m}$
Length of beam $L = 4$ m

<u>Assumptions</u>

The material is homogeneous and isotropic

<u>What is asked?</u>

(a) Reaction forces at points A and B
(b) Shear force diagram
(c) Bending moment diagram
(d) Maximum tensile and compressive stresses
(e) Deflection equation of beam.

<u>Principles and formula</u>

(a) Reaction force at points A and B

We calculate the reaction forces from the free body diagram of the entire beam Figure 9.4.2.

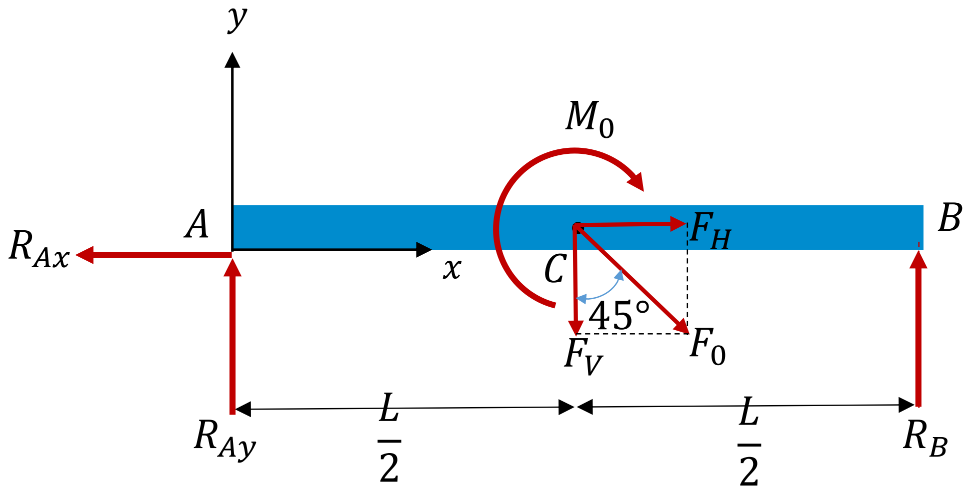

Figure 9.4.2 | Free body diagram of the beam AB.

We write the force equilibrium of equations in x and y:

$$\Sigma F_x = 0 \rightarrow -R_{Ax} + F_H = 0 \rightarrow R_{Ax} = F_H = F_0 \frac{\sqrt{2}}{2} = 30 \text{ N} \tag{0.0.1}$$

$$\Sigma F_y = 0 \rightarrow R_{Ay} + R_B - F_V = R_{Ay} + R_B - 30\sqrt{2}\frac{\sqrt{2}}{2} = R_{Ay} + R_B - 30 = 0 \tag{0.0.2}$$

We notice that we don't have reaction moments at points A and B, therefore from the equilibrium equation of the moments we obtain:

$$\Sigma M_A = 0 \implies -F_V\left(\frac{L}{2}\right) - M_0 + R_B(L) = 0 \tag{0.0.3}$$

$$R_B = \frac{F_V L + 2M_0}{2L} = \frac{4 \cdot 30 + 40}{8} = 20\ \text{N} \tag{0.0.4}$$

Which gives us:

$$R_B = 20\ \text{N} \quad \& \quad R_{Ay} = 10\ \text{N} \tag{0.0.5}$$

(b) Shear force diagram

We cut the beam at $x < L/2$ and write the force equilibrium equation for the left part of the cut:

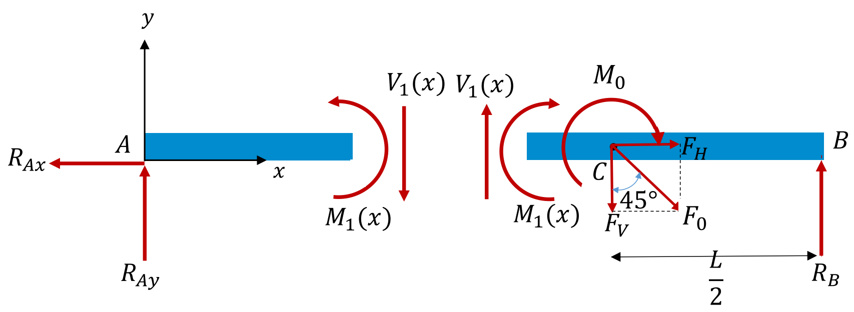

Figure 9.4.3 |Shear Force: Free body diagram for $x < L/2$.

$$-V_1(x) + R_{Ay} = 0 \tag{0.0.6}$$

$$V_1(x) = R_{Ay} = 10\ \text{N} \tag{0.0.7}$$

We do the same for $L/2 < x < L$, as shown in Figure 9.4.4, but this time we consider the right part of the cut:

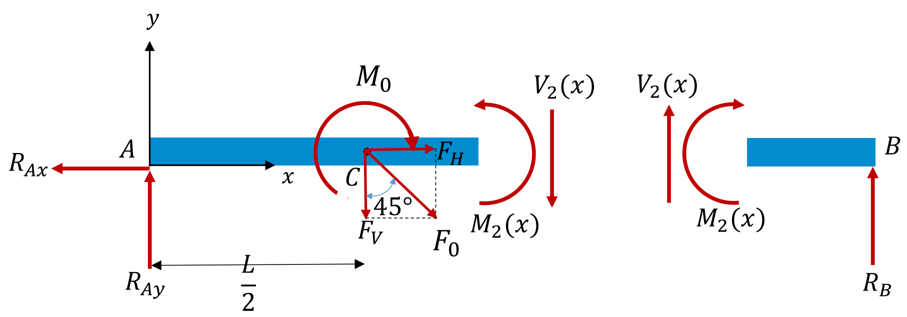

Figure 9.4.4 |Shear Force: Free body diagram for $L/2 < x < L$.

$$V_2(x) + R_B = 0 \rightarrow V_2(x) = -20\ \text{N} \tag{0.0.8}$$

To calculate the value of the shear force at point C we consider Figure 9.4.5:

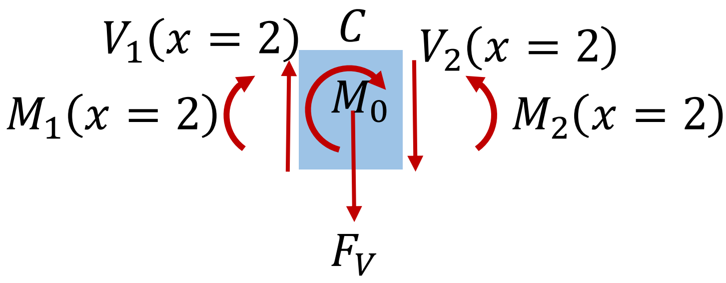

Figure 9.4.5 | Shear Force: Free body diagram for $x = C$.

$$\Delta V(x = C) = -F_V \rightarrow V_2(x = 2) = V_1(x = 2) - 30 = 10 - 30 = -20 \text{ N} \quad (0.0.9)$$

Using these values, we draw the Shear Force Diagram

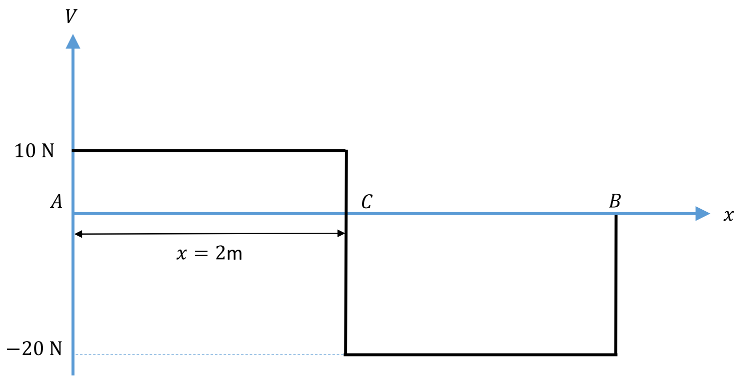

Figure 9.4.6 | Shear Force Diagram

(c) Bending Moment Diagram
On pourrait utiliser soit les coupes, soit intégrer V(x).

The bending moment at any point along the beam without concentrated moments is given by:

$$M(x) = \int_0^x V(x')dx' \quad (0.0.10)$$

Considering Figure 9.4.3, we write the bending moment equilibrium equation from A to C, relative to the left part of the cut:

$$M_1(x) = M_1(0) + \int_0^x V_1(x')dx' = 0 + \int_0^x R_{Ay}dx' = R_{Ay} \cdot x = 10x \quad (0.0.11)$$

From C to B, we write the moment equilibrium equation of the left part of the cut we have in Figure 9.4.4:

$$M_2(x) - M_2(x = 2) = \int_2^x V_2(x')dx' = \int_2^x (-20)dx' \quad (0.0.12)$$

And then we use Figure 9.4.5 to calculate that:

$$M_2(x) = 40 - 20(x-2) = 80 - 20x \quad (0.0.14)$$

Using these values, we draw the Bending Moment Diagram:

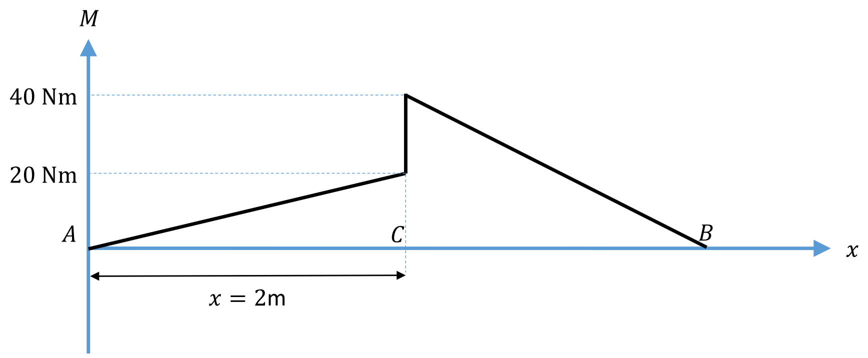

Figure 9.4.7 | Bending Moment Diagram

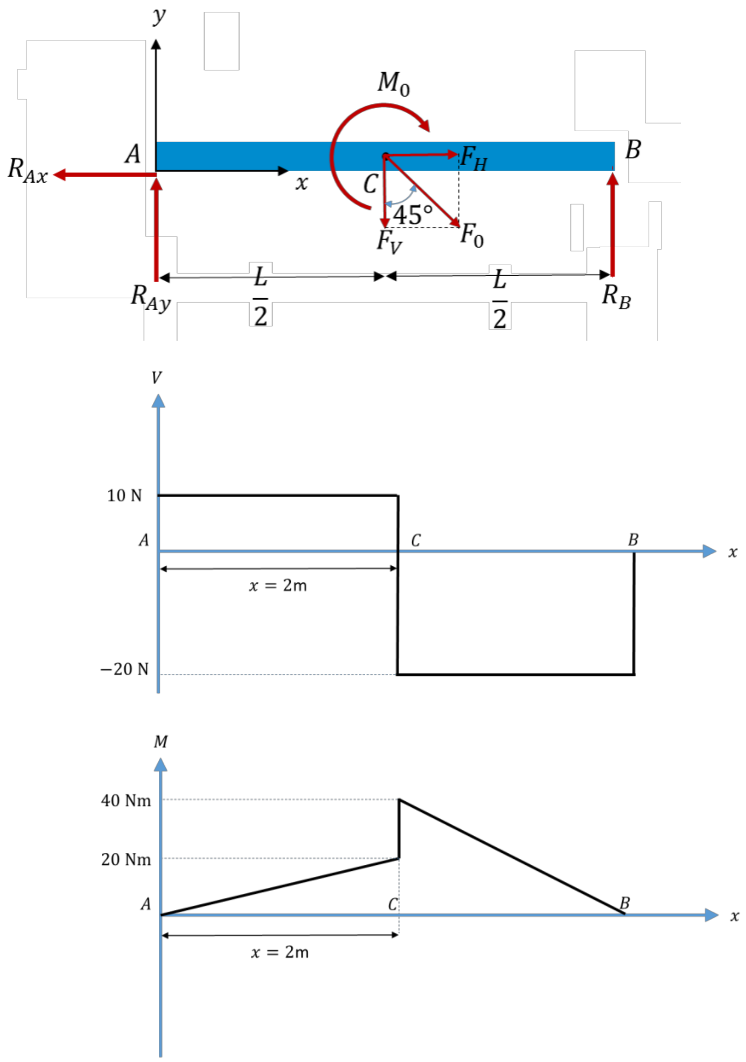

Figure 9.4.8 | Shear Force and Bending Moment diagram of the beam AB.

(d) Maximum tensile and compressive stress.

On the left half of the beam we have two contributions to the stress, a) part due to bending and b) part due to elongation generated by the axial force. On the right part of the beam, however, we only have the part due to bending. For the part due to bending, as the cross section is symmetrical, one can write:

$$\sigma_{max,t,bending} = \sigma_{max,c,bending} = \frac{M_{max}}{I_{z,y_0}}\frac{d}{2} \tag{0.0.15}$$

First of all we calculate the second moment of area about the Y axis in the yz plane, using the following formula:

$$I_{z,y_0} = \iint_A y^2 dA = \frac{bd^3}{12} \tag{0.0.16}$$

$$I_{z,y_0} = \frac{(10)(20^3)}{12}10^{-8} = 6.7 \cdot 10^{-5}\ \mathrm{m}^4 \tag{0.0.17}$$

The bending contribution is going to be maximum close to the center of the beam (at L/2) for each of the two halves.

In addition to the bending contribution, in the case of the left half, we also have an axial load to consider on top of the bending of the beam. The axial load is causing the bar to elongate, generating a tensile normal stress that is uniformly distributed in the cross section. It is equal to:

$$\sigma_{elong} = \frac{F_H}{A} = \frac{30}{0.02}\mathrm{Pa} = 1.5\ \mathrm{kPa} \tag{0.0.18}$$

We can now calculate the four different options where the maximum stress can be located:

$$|\sigma|_{bottom,left} = \frac{F_H}{A} - \frac{M\left(x = \frac{L^-}{2}\right)}{I_{z,y_0}}\frac{d}{2} = \frac{30}{0.02} + \frac{20 \cdot 0.1}{6.7 \cdot 10^{-5}} = 28500\ \mathrm{Pa} \tag{0.0.19}$$

$$|\sigma|_{top,left} = \frac{F_H}{A} + \frac{M\left(x = \frac{L^-}{2}\right)}{I_{z,y_0}}\frac{d}{2} = \left|\frac{30}{0.02} - \frac{20 \cdot 0.1}{6.7 \cdot 10^{-5}}\right| = 31500\ \mathrm{Pa} \tag{0.0.20}$$

$$|\sigma|_{top,right} = \frac{M\left(x = \frac{L^+}{2}\right)}{I_{z,y_0}}\frac{d}{2} = \frac{40 \cdot 0.1}{6.7 \cdot 10^{-5}} = 60000\ \mathrm{Pa} \tag{0.0.21}$$

$$|\sigma|_{bottom,right} = \frac{M\left(x = \frac{L^+}{2}\right)}{I_{z,y_0}}\frac{d}{2} = \frac{40 \cdot 0.1}{6.7 \cdot 10^{-5}} = 60000\ \mathrm{Pa} \tag{0.0.22}$$

So the maximum stress will be located at the top and bottom of the beam, just to the right of the central point.

(e) Deflection equation of the beam

The curvature of the beam due to bending moments at any point is given by:

$$\frac{d^2w}{dx^2} = \frac{M(x)}{EI} \tag{0.0.23}$$

Hence we can calculate the deflection equation of the beam by double integration of Eq. (0.0.23):

$$w = \frac{1}{EI}\int_0^x \left(\int_0^{x\prime} M(x'')dx''\right)dx' \tag{0.0.24}$$

When doing this double integration, we need to be careful because the first integral will generate constants that need to be carried away. Therefore, it is best to proceed in steps and calculate first the derivative of the deflection (slope) and then the deflection.

As we see from Figure 9.4.7 the moment is defined in two parts.

For $x < 2$:

$$w'\,(x) - w'(0) = \frac{1}{EI}\int_0^x M_1(x')dx' = \frac{1}{EI}\int_0^x 10x'dx' \tag{0.0.25}$$

$$w'(x) = \frac{1}{EI}\frac{10x^2}{2} + w^{'}(0) = \frac{5}{EI}x^2 + w'(0) \tag{0.0.26}$$

For $x \geq 2$:

$$w'(x) - w'(x=2) = \frac{1}{EI}\int_2^x M_2(x')dx' = \frac{1}{EI}\int_2^x (80 - 20x')dx' \tag{0.0.27}$$

$$w'(x) = \frac{1}{EI}[80(x-2) - 10(x^2-4)] + \left(\frac{20}{EI} + w'(0)\right) = w'(0) - \frac{10}{EI}(10 - 8x + x^2) \tag{0.0.28}$$

Now we can perform the second integral:

For $x < 2$:

$$w(x) - w(x=0) = \int_0^x w'(x')dx' \tag{0.0.29}$$

$$w(x=0) = 0 \; and \; w(x) = \frac{5x^3}{3EI} + w'(0)x \tag{0.0.30}$$

For $x \geq 2$:

$$w(x) - w(x=2) = \int_2^x w'(x')dx' \tag{0.0.31}$$

$$w(x) = \frac{40}{3EI} + w'(0)\cdot 2 + w'(0)(x-2) - \frac{10}{EI}\left(10(x-2) - 4(x^2-4) + \frac{x^3-8}{3}\right) \tag{0.0.32}$$

$$w(x) = \frac{-10(-24 + 30x - 12x^2 + x^3)}{3EI} + w^{'}(0)x \tag{0.0.33}$$

Problème 8b.5 - Calcul de la déflection à partir des moments (2)

On considère une poutre AB de longueur L=4 m . Les diagrammes de force de cisaillement relative et de moment en flexion sont données sur la figure 9.3.1.

Les moments en flexion sont :

$$M_1(x) = 32x - 10x^2 \text{ N}\cdot\text{m}. \qquad 0 < \text{x} < \frac{\text{L}}{2}$$

$$M_2(x) = 24 - 8(x-2) \text{ N}\cdot\text{m} \qquad \frac{\text{L}}{2} < \text{x} < \frac{4\text{L}}{5}$$

$$M_3(x) = 14.4 - 18(x-3.2) \text{ N}\cdot\text{m} \qquad \frac{4\text{L}}{5} < \text{x} < \text{L}$$

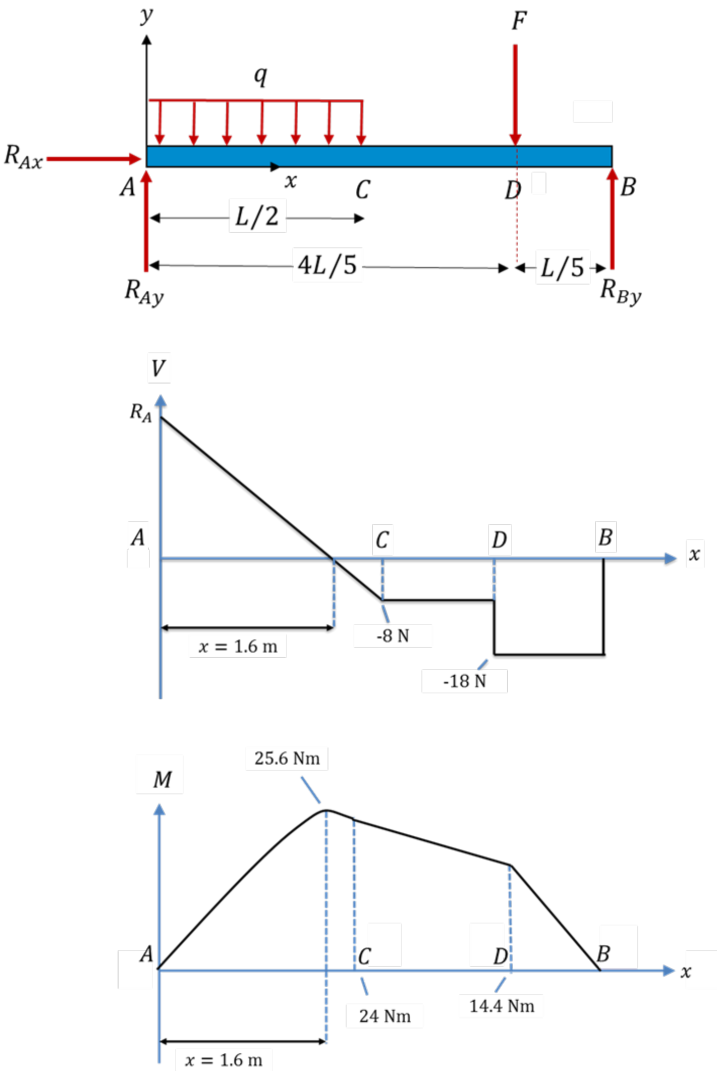

Figure 9.3.1 Diagrammes de force de cisaillement et de moment en flexion relatifs à la poutre AB.

Solution

I Conditions au bord

a. $w_1(x=0) = 0$
b. $w'_1(x=0)=0$ (encastré)

II continuity:

c. $w_1(x=L/2) = w_2(x=L/2)$
d. $w_2(x=4L/5) = w_3(x=4L/5)$
e. $w'_1(x=L/2) = w'_2(x=L/2)$
f. $w'_2(x=4L/5) = w'_3(x=4L/5)$

As for the previous problem, we calculate the deflection equations for every portion of the beam:

$$\frac{d^2w}{dx^2} = \frac{M(x)}{EI} \tag{0.0.1}$$

We first calculate the first derivative of the deflection, because we need to impose its continuity at points C and D ($x = \frac{L}{2}$ and $x = \frac{4L}{5}$ respectively). Remember that we do not know the values for the derivative at the extremities, only at x=0.

We obtain:

For $x \leq \frac{L}{2}$:

$$w'(x) - w'(0) = \frac{1}{EI}\int_0^x \left(32x' - 10x'^2\right) dx' = \frac{1}{EI}\left(16x^2 - \frac{10x^3}{3}\right) \tag{0.0.2}$$

For $\frac{L}{2} < x < \frac{4L}{5}$:

$$w'(x) - w'(x=2) = \frac{1}{EI}\int_2^x \big(24 - 8(x' - 2)\big)dx' \tag{0.0.3}$$

$$w'(x) = w'(x=2) + \frac{1}{EI}(40x - 4x^2 - 64) \tag{0.0.4}$$

$$w'(x) = \frac{1}{EI}\left(40x - 4x^2 - \frac{80}{3}\right) + w'(0) \tag{0.0.5}$$

For $\frac{4L}{5} < x < L$:

$$w'(x) - w'(x=3.2) = \frac{1}{EI}\int_{3.2}^x (14.4 - 18(x' - 3.2))dx' \tag{0.0.6}$$

$$w'(x) = w'(x=3.2) + \frac{1}{EI}(72x - 9x^2 - 138.24) \tag{0.0.7}$$

Applying the condition that the deflection at point B

$$w(x) = \frac{1}{EI}(36x^2 - 3x^3 - 77.8667x + 67.9467) + w'(0)x$$

should be zero, we can obtain how much the derivative of the deflection at x=0 is:

$$w(x=4) = 0 \rightarrow w'(x=0) = -\frac{35.12}{EI} \tag{0.0.17}$$

Which yields a final expression for the deflection:

$$w(x) = \begin{cases} \frac{1}{EI}\left(\frac{16}{3}x^3 - \frac{5}{6}x^4\right) - \frac{35.12}{EI}\,x & ;\ x \leq \frac{L}{2} \\ -\frac{4}{3EI}(-10. + 46.34x - 15.\,x^2 + x^3) & ;\frac{L}{2} \leq x \leq \frac{4L}{5} \\ \frac{1}{EI}(67.9467 - 112.987x + 36.\,x^2 - 3.\,x^3) & ;\frac{4L}{5} \leq x \leq L \end{cases} \tag{0.0.18}$$

