1.3.1 Periodicity and parity

- **Theorem:** let $f : \mathbb{R} \to \mathbb{R}$ a T-periodic function such that f and f' are piecewise defined. Then:

  a) Its Fourier series Ff is also T-periodic

  b) If f is an even function (i.e. $f(-x) = f(x)$ $\forall x \in \mathbb{R}$) we have $b_n = 0$ $\forall n \geq 1$ and $(Ff)(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T} x\right)$ is also an even function.

  c) If f is an odd function (i.e., $f(x) = -f(-x)$ $\forall x \in \mathbb{R}$) we have $a_n = 0$ $\forall n \geq 0$ and $(Ff)(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T} x\right)$ is also an odd function.

- __Proof:__ for the sake of simplicity, we assume $T = 2\pi$

  a) let $N \in \mathbb{N}^*$. The partial Fourier serie of order N is:

$$F_N f(x+2\pi) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(n(x+2\pi)) + b_n \sin(n(x+2\pi)) \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(nx) + b_n \sin(nx) \right] = F_N f(x)$$

Then $Ff(x+2\pi) = \lim_{N \to \infty} F_N f(x+2\pi) = \lim_{N \to \infty} F_N f(x) = Ff(x)$

b) We have:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= -\frac{1}{\pi}\int_{\pi}^{0} \underbrace{f(-y)}_{f(y)}\,\underbrace{\sin(-ny)}_{-\sin(ny)}\,dy + \frac{1}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$y=-x$ for 1st integ.

$$= \frac{1}{\pi}\int_{\pi}^{0} f(y)\sin(ny)\,dy + \frac{1}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = 0 \quad \forall n \geq 1$$

c) The same idea as for b), but using $f(-x) = -f(x)$ and $\cos(-nx) = \cos(nx)$.

1.3.2 Parceval identity

- Theorem: let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic function such that f and f' are piecewise-defined. Then:

$$\frac{2}{T}\int_0^T [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are the Fourier coefficients.

- Parceval identity proof: for the sake of simplicity we assume $T = 2\pi$ and we assume that $f(x)$ is continuous.

In that case $f(x) = Ff(x)\ \forall x \in \mathbb{R}$ and

$$\frac{1}{\pi}\int_0^{2\pi} [f(x)]^2 dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\, Ff(x) dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x) \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \right\} dx$$

$$= \frac{a_0}{2} \underbrace{\frac{1}{\pi}\int_0^{2\pi} f(x) dx}_{=a_0} + \sum_{n=1}^{\infty} a_n \underbrace{\left[\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx) dx \right]}_{=a_n}$$

$$+ \sum_{n=1}^{\infty} b_n \underbrace{\left[\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx) dx \right]}_{=b_n} = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$\square$

1.4.1 Fourier cosine series

- Theorem:

Let $f:[0,L]\to\mathbb{R}$ be a $\overbrace{\text{continuous}}^{\text{For the sake of simplicity}}$ function such that f' is piecewise-defined. Then, the series

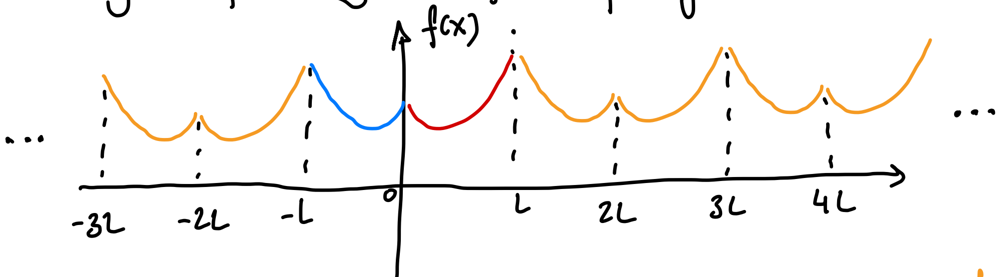
$$F_c f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L} x\right) \quad \text{with}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n}{L}x\right) \quad \text{for } n=0,1,2,\dots$$

it is called <u>Fourier cosine series of f</u> and it converges to f in the interval $[0,L]$. We have $f(x) = F_c f(x)\ \forall x\in[0,L]$

- <u>Proof</u>:

We extend the definition by parity to the interval $[-L,0]$ imposing $f(x)=f(-x)\ \forall x\in[-L,0]$. After, we extend it by $2L$-periodicity to $\mathbb{R}$ for computing Fourier series:



- extension by parity
- extension by 2L-periodicity

We denote as $g : \mathbb{R} \to \mathbb{R}$ the function obtained by this double extension: g is continuous, even, and $2L$-periodic such that g' is piecewise-defined.

We can then compute the Fourier series of g. We have $b_n = 0$ $\forall n \geq 1$ (because g is even) and

$$Fg(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L} x\right) = g(x) \quad \text{with}$$

$$a_n = \frac{2}{2L} \int_0^{2L} g(x) \cos\left(\frac{\pi n}{L} x\right) dx = \frac{1}{L} \int_0^{L} g(x) \cos\left(\frac{\pi n}{L} x\right) dx$$

$$+ \frac{1}{L} \int_L^{2L} g(x) \cos\left(\frac{\pi n}{L} x\right) dx$$

$$\int_0^{2L} g(x) \cos\left(\frac{\pi n}{L} x\right) \underset{\substack{x = 2L - y \\ dx = -dy}}{=} - \int_L^0 \underbrace{g(2L-y)}_{\substack{= g(-y) = g(y) \\ (g \text{ is } 2L\text{-periodic and even})}} \underbrace{\cos\left[\frac{\pi n}{L}(2L-y)\right]}_{= \cos\left(\frac{n\pi}{L} y\right)} dy$$

$$= \int_0^L g(y) \cos\left(\frac{n\pi}{L} y\right).$$

Then:

$$a_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{\pi n}{L}x\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n}{L}x\right)dx$$

because $g(x) = f(x)$ on $[0,L]$

Finally, we obtain that $\forall x \in [0,L]$ we have:

$$f(x) = g(x) = Fg(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L}x\right) = F_c f(x)$$

$\square$

1.4.2 Fourier sine series

- Theorem:

For the sake of simplicity

Let $f:[0,L]\rightarrow\mathbb{R}$ be a $\overbrace{\text{continuous}}$ function such that $f(0)=f(L)=0$ and f' is piecewise-defined. Then, the series

$$Fs\,f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L}x\right) \quad \text{with}$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n}{L}x\right)dx \quad \text{for } n=1,2,\dots \text{ is}$$

denoted as <u>Fourier sine series</u> of f and it converges to f in the interval $[0,L]$. We have: $f(x)=Fs\,f(x)\ \forall\, x\in[0,L]$

- <u>Hint for the proof</u>: similar procedure as for the Fourier cosine series: we extend f as an odd function to the interval $[-L,0]$, imposing $f(x)=-f(-x)$. After, we extend it by $2L$-periodicity to $\mathbb{R}$. The condition $f(0)=f(L)=0$ guarantees that the extended function g is continuous ($g\in C^0(\mathbb{R})$)