in other words

$$\iiint_\Omega \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz$$

$$= \iint_{\partial\Omega} (F_1 \nu_1 + F_2 \nu_2 + F_3 \nu_3)\, dS$$

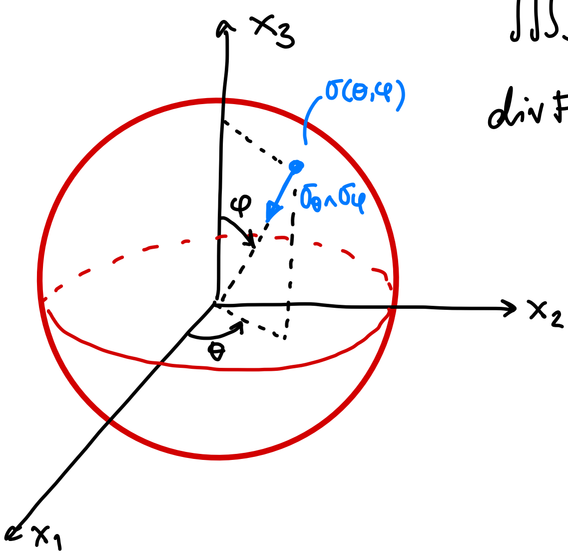
11/11/2021

3.3.3 Examples

- Example 1: verify the divergence theorem for

$\Omega = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1 \}$ and

$F(x,y,z) = (xy, y, z)$



$$\iiint_\Omega \operatorname{div} F(x,y,z)\, dx\,dy\,dz$$

$$\operatorname{div} F(x,y,z) = y + z$$

spherical coords

$$\iiint_\Omega (y+z)\,dx\,dy\,dz = \int_0^1\int_0^{2\pi}\int_0^{\pi} (r\sin\varphi\sin\theta + 2)\, r^2 \sin\varphi \, dr\, d\theta\, d\varphi$$

$$= \int_0^1 r^3 dr \underbrace{\int_0^{2\pi} \sin\theta\, d\theta}_{\overset{\shortparallel}{0}} \int_0^{\pi} \sin^2\varphi\, d\varphi + 2\int_0^1 r^2 dr \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi$$

$$= \frac{2}{3}\, 2\pi\, [-\cos\varphi]\Big|_0^{\pi} = \frac{8\pi}{3}$$

$$\iint_{\partial\Omega} F\cdot\nu\, ds =$$

Parameterization (Example 1 § 3.1.2):

$$\sigma(\theta,\varphi) = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi),\quad A = \,]0,2\pi[\times]0,\pi[$$

$$\sigma_\theta \wedge \sigma_\varphi = \begin{pmatrix} -\sin^2\varphi\cos\theta \\ -\sin^2\varphi\sin\theta \\ -\sin\varphi\cos\varphi \end{pmatrix}$$ this is inner normal of Ω

outer unit normal $\nu = -\dfrac{\sigma_\theta\wedge\sigma_\varphi}{\|\sigma_\theta\wedge\sigma_\varphi\|}$

$$\iint_{\partial\Omega} F\cdot\nu\, ds = \iint_A F(\sigma(\theta,\varphi))\cdot \underbrace{\nu\, \|\sigma_\theta\wedge\sigma_\varphi\|}_{-\sigma_\theta\wedge\sigma_\varphi}\, d\theta\, d\varphi$$

$F(x,y,z) = (xy,\ y,\ z)$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin^2\varphi\cos\theta\sin\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

$$\cdot\ (\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin^4\varphi\cos^2\theta\sin\theta + \sin^3\varphi\sin^2\theta + \sin\varphi\cos^2\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\cos^2\theta\sin\theta\, d\theta \int_0^{\pi}\sin^4\varphi\, d\varphi + \int_0^{2\pi}\sin^2\theta\, d\theta\int_0^{\pi}\sin^3\varphi\, d\varphi$$

$$+ 2\pi\int_0^{\pi}\sin\varphi\cos^2\varphi\, d\varphi$$

$$\int_0^{2\pi}\cos^2\theta\sin\theta\, d\theta = \left[-\frac{1}{3}\cos^3\theta\right]\Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi}\sin^2\theta\, d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)\, d\theta = \pi - \frac{1}{4}\left[\sin 2\theta\right]\Big|_0^{2\pi} = \pi - 0 = \pi$$

$$\int_0^{\pi}\sin^3\varphi\, d\varphi = \underbrace{-\sin^2\varphi\cos\varphi\Big|_0^{\pi}}_{=0} + 2\int_0^{\pi}\sin\varphi\cos^2\varphi\, d\varphi = \frac{4}{3}$$

$u = \sin^2\varphi$

$v = -\cos\varphi$

$u\,dv = \sin^3\varphi\, d\varphi$

$$\int_0^{\pi}\sin\varphi\cos^2\varphi\, d\varphi = \frac{2}{3}$$

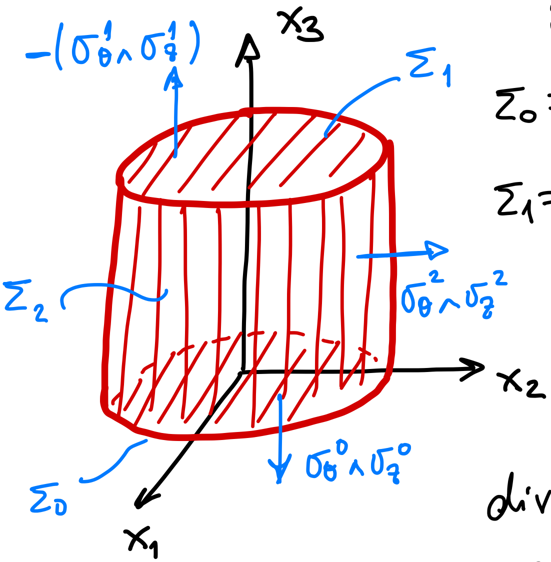
$$\iint_{\partial\Omega} F\cdot\nu\, dS = 0 + \frac{4}{3}\pi + 2\pi\frac{2}{3} = \frac{8\pi}{3} = \iiint_\Omega \operatorname{div} F\, dxdydz$$

✓

∘ Example 2: Verify the divergence theorem for

$\Omega = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2<1,\ 0\le z\le 1\}$

and $F(x,y,z) = (x^2, 0, 0)$.



$\partial\Omega = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$

$\Sigma_0 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2<1 \text{ and } z=0\}$

$\Sigma_1 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2<1 \text{ and } z=1\}$

$\Sigma_2 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=1$
and $0<z<1\}$

$\operatorname{div} F(x,y,z) = 2x$

$$\iiint_\Omega \operatorname{div} F\, dxdydz = \int_0^{2\pi}\int_0^1\int_0^1 \underbrace{2r\cos\theta}_{2x}\, r\, dr\, d\theta\, dz$$

cylindrical coords

$$= \underbrace{\int_0^{2\pi}\cos\theta\, d\theta}_{"0"} \int_0^1 r^2 dr \int_0^1 dz = 0.$$

We want to compute $\iint_{\partial\Omega} F \cdot \nu \, dS$

$$= \iint_{\Sigma_0} F \cdot \nu \, dS + \iint_{\Sigma_1} F \cdot \nu \, dS + \iint_{\Sigma_2} F \cdot \nu \, dS$$

- Parameterization for Σ_0

$$\sigma^0(\theta, r) = (r\cos\theta, r\sin\theta, 0) \quad , A_0 =]0, 2\pi[\times]0, 1[$$

$$\sigma^0_\theta \wedge \sigma^0_r = \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}$$

$$\nu = \frac{\sigma^0_\theta \wedge \sigma^0_r}{\|\sigma^0_\theta \wedge \sigma^0_r\|} \longrightarrow \nu \|\sigma^0_\theta \wedge \sigma^0_r\| = \sigma^0_\theta \wedge \sigma^0_r$$

$$\iint_{\Sigma_0} F \cdot \nu \, dS = \int_0^{2\pi} \int_0^1 F(\sigma^0(r,\theta)) \cdot \overbrace{\nu \, \|\sigma_\theta \wedge \sigma_r\|} \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r^2\cos^2\theta, 0, 0) \cdot (0, 0, -r) \, dr \, d\theta = 0.$$

- Parameterization for Σ_1

$$\sigma^1(\theta, r) = (r\cos\theta, r\sin\theta, 1) \quad , A_1 =]0, 2\pi[\times]0, 1[$$

$$\sigma^1_\theta \wedge \sigma^1_r = \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} \quad \text{is an inner vector}$$

$$\nu \|\sigma^1_\theta \wedge \sigma^1_r\| = -\sigma^1_\theta \wedge \sigma^1_r$$

$$\iint_{\Sigma_1} F\cdot\nu\, dS = \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta, 0, 0)\cdot(0,0,+r)\, dr\, d\theta = 0.$$

- Parameterization of Σ_2

$\sigma^2(\theta, z) = (\cos\theta, \sin\theta, z)$, $A_2 =]0, 2\pi[\times]0,1[$

$\sigma^2_\theta \wedge \sigma^2_z = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$ an outer normal

$\nu \|\sigma^2_\theta \wedge \sigma^2_z\| = \sigma^2_\theta \wedge \sigma^2_z = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$

$$\iint_{\Sigma_2} F\cdot\nu\, dS = \int_0^1 dz \int_0^{2\pi} (\cos^2\theta, 0, 0)\cdot(\cos\theta, \sin\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} \cos^3\theta\, d\theta = \int_0^{2\pi} \cos\theta \cos^2\theta\, d\theta = \int_0^{2\pi} \cos\theta(1-\sin^2\theta)\, d\theta$$

$$= \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{=0} - \int_0^{2\pi} \sin^2\theta \cos\theta\, d\theta = 0.$$

$$\iiint_\Omega \operatorname{div} F\, dx\, dy\, dz = \iint_{\partial\Omega} F\cdot\nu\, dS.$$

3.3.4 Corollary

- Corollary: If the domain $\Omega \subset \mathbb{R}^3$ and the normal fields $\nu : \partial\Omega \to \mathbb{R}^3$ are like those required by the divergence theorem, then:

$$vol(\Omega) = \frac{1}{3} \iint_{\partial\Omega} (F \cdot \nu)\, dS = \iint_{\partial\Omega} (G_i \cdot \nu)\, dS \qquad i = 1, 2, 3$$

$F(x,y,z) = (x,y,z)$

$G_1(x,y,z) = (x,0,0)$, $G_2(x,y,z) = (0,y,0)$

$G_3(x,y,z) = (0,0,z)$

Hint of the proof:

$$\operatorname{div} F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

$$\iiint_\Omega \operatorname{div} F\, dx\, dy\, dz = 3 \iiint_\Omega 1\, dx\, dy\, dz = 3\, vol(\Omega)$$

$$vol(\Omega) = \frac{1}{3} \iiint_\Omega \operatorname{div} F\, dx\, dy\, dz \underset{\text{Diverg. thm.}}{=} \frac{1}{3} \iint_{\partial\Omega} F \cdot \nu\, dS$$

3.4 Stokes' theorem

3.4.1 Motivation

- Goal: generalize Green's theorem for fields in $\mathbb{R}^3$

2D	3D
Green's theorem	Stokes' theorem

- Green's theorem (§ 2.4.2)

$$\iint_B \operatorname{curl} G(x,y)\,dx\,dy = \iint_{\partial B} G \cdot dl \quad \text{for } B \subset \mathbb{R}^2$$

a regular domain, ∂B positively oriented and

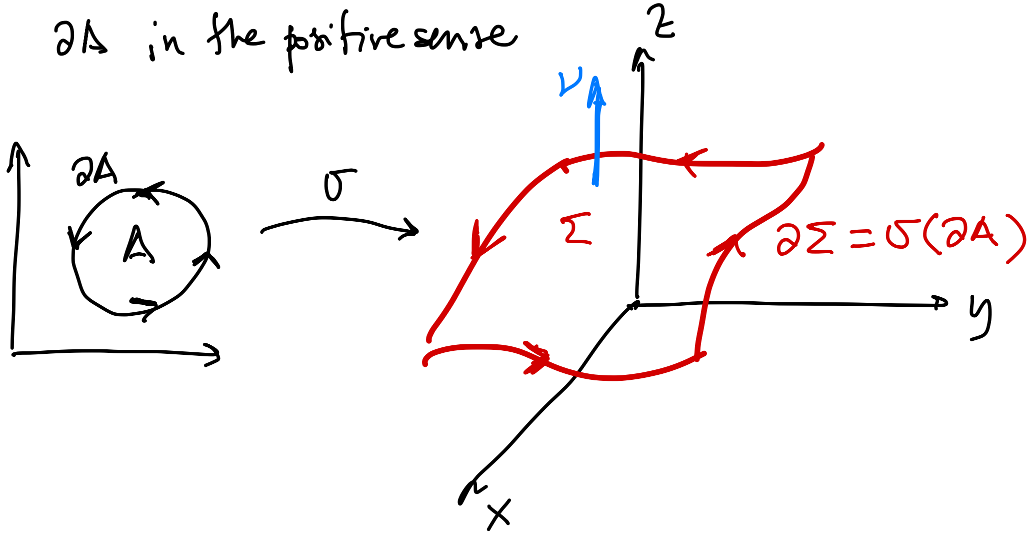
$G : \overline{B} \to \mathbb{R}^2$ a vector field $C^1(B, \mathbb{R}^2)$

3.4.2 Determination of the boundary of a surface and its sense of circulation

1) If $\Sigma \subset \mathbb{R}^3$ is a regular surface and

$\sigma : \overline{A} \to \Sigma$ is a parameterization of Σ, then $\partial\Sigma = \sigma(\partial A)$ is independent of the parameterization chosen

2) The sense of circulation of $\partial\Sigma$ is induced by the parameterization. It is obtained by circulating ∂A in the positive sense



3) If the surface Σ is (piecewise) regular and parameterized by σ, then we have $\sigma(\partial A) = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_m$ then, we proceed in the following way:

1. We eliminate from $\sigma(\partial A)$ the curves Γ that are reduced to a single point ($\text{length}(\Gamma)=0$)
2. We eliminate the curves Γ that are circulate twice (and with different sense).

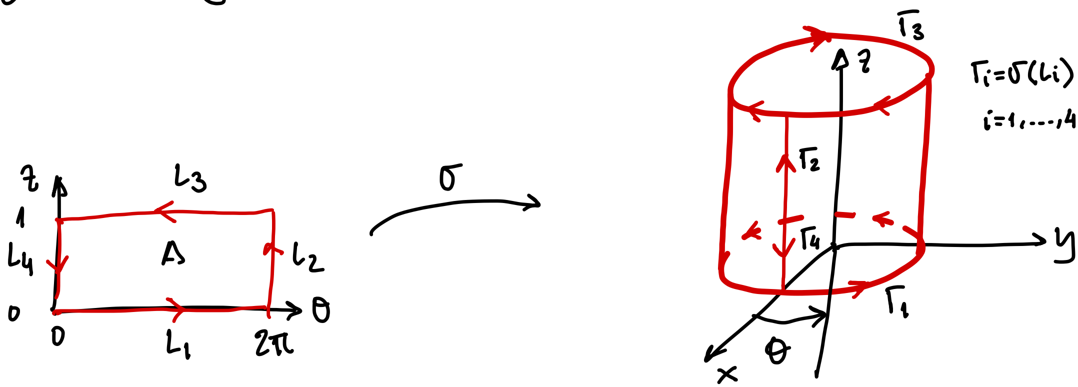
What remains is the boundary of Σ.

- Example 1: cylinder $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1, \; 0 \leq z \leq 1\}$

Parameterization $A = [0, 2\pi] \times [0,1]$

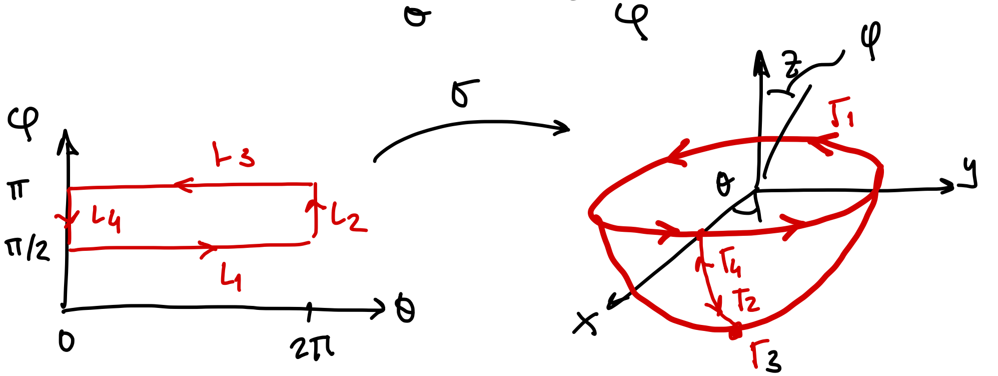
$\partial A = L_1 \cup L_2 \cup L_3 \cup L_4$, $\quad \Gamma_i = \sigma(L_i) \quad i=1,2,\dots,4$

$\partial \Sigma = \Gamma_1 \cup \Gamma_3$



• Example 2: semi-sphere (lower) $\Sigma = \{(x,y,z) \in \mathbb{R}^3 ;\ x^2+y^2+z^2=1$ and $z \le 0\}$

Parameterization $A = \underbrace{[0, 2\pi]}_{\theta} \times \underbrace{[\frac{\pi}{2}, \pi]}_{\varphi}$



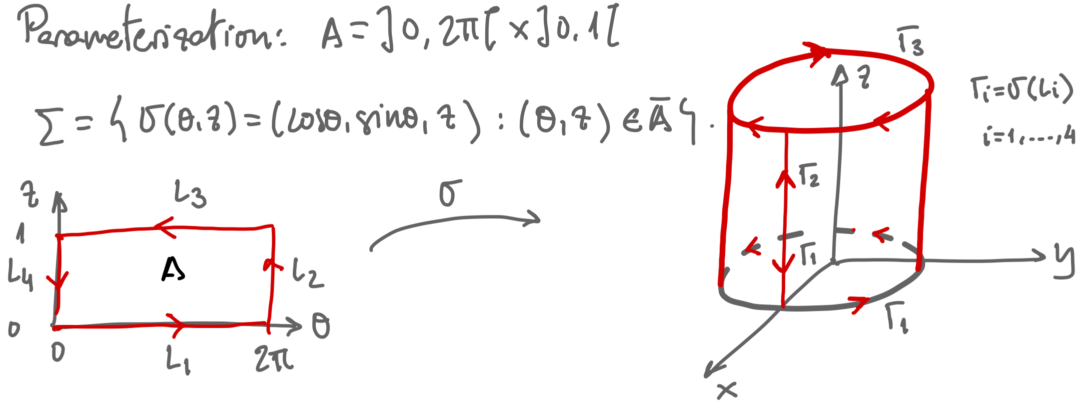
$\sigma(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) : (\theta,\varphi) \in A$.

NOTE: Hereinafter, this material was not covered during the theory lecture. I just provide specific details about examples 1 and 2 (see the video for the conceptual explanation) and an extra example using a cone

• Example 1: cylinder $\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2=1,\ 0 \le z \le 1 \}$.

Parameterization: $A =]0,2\pi[\times]0,1[$

$\Sigma = \{ \sigma(\theta,z) = (\cos\theta, \sin\theta, z) : (\theta,z) \in \bar{A} \}$.



We have $\sigma(\partial A) = \sigma(L_1 \cup L_2 \cup L_3 \cup L_4) = \sigma(L_1) \cup \sigma(L_2) \cup \sigma(L_3) \cup \sigma(L_4)$

$= \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ with

$\Gamma_1 = \{ \gamma_1(\theta) = \sigma(\theta, 0) = (\cos\theta, \sin\theta, 0)$ with $\theta : 0 \to 2\pi \}$

circulated in counter-clockwise sense (seen from above)

$\Gamma_2 = \{ \gamma_2(z) = \sigma(2\pi, z) = (1, 0, z)$ with $z : 0 \to 1 \}$

circulated upwards

$\Gamma_3 = \{ \gamma_3(\theta) = \sigma(\theta, 1) = (\cos\theta, \sin\theta, 1)$ with $\theta : 2\pi \to 0 \}$

circulated in clockwise sense (seen from above)

$\Gamma_4 = \{ \gamma_4(z) = \sigma(0, z) = (1, 0, z)$ with $z : 1 \to 0 \}$

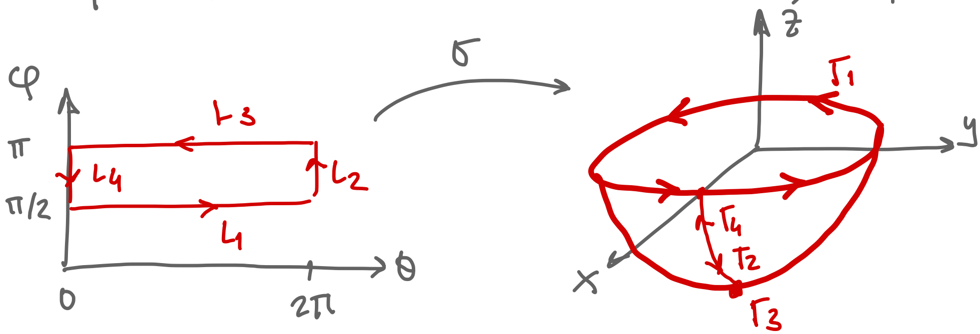
$= \Gamma_2$ circulated downwards

Applying the procedure detailed above the curves Γ_2 and Γ_4 are eliminated from $\sigma(\partial A)$ and we obtain $\partial\Sigma = \Gamma_1 \cup \Gamma_3$ with Γ_1 counter-clockwise and Γ_3 clockwise circulated (both seen from above).

• Example 2: semi-sphere $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1 \text{ and } z \le 0\}$.
(inferior)

Parameterization: $A =]0, 2\pi[\times]\frac{\pi}{2}, \pi[$ and

$\Sigma = \{\sigma(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) : (\theta,\varphi) \in \bar{A}\}$.



$\sigma(\partial A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ with $\Gamma_i = \sigma(L_i)$ for $i = 1, \dots, 4$.

$\Gamma_1 = \{ r_1(\theta) = \sigma(\theta, \pi/2) = (\cos\theta, \sin\theta, 0)$ with $\theta : 0 \to 2\pi \}$

circulated in clockwise sense (seen from above).

$\Gamma_2 = \{ r_2(\varphi) = \sigma(2\pi, \varphi) = (\sin\varphi, 0, \cos\varphi)$ with $\varphi : \frac{\pi}{2} \to \pi \}$

semi-arc crossing the "south-pole" circulating downwards.

$\Gamma_3 = \{ r_3(\theta) = \sigma(\theta, \pi) = (0, 0, -1)$ with $\theta : 2\pi \to 0 \}$

a single point $\to$ the south pole of the sphere.

$\Gamma_4 = \{ r_4(\varphi) = \sigma(0,\varphi) = (\sin\varphi, 0, \cos\varphi)$ with $\varphi : \pi \to \frac{\pi}{2} \}$

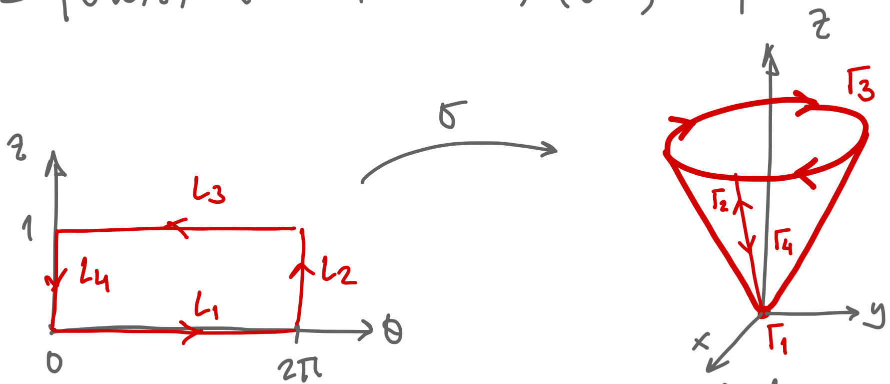
semi-arc Γ_2 circulated upwards.

Applying the procedure detailed above the curves Γ_2, Γ_3, and Γ_4 are eliminated from $\sigma(\partial A)$ and we obtain $\partial\Sigma = \Gamma_1$, circulated in a counter-clockwise sense (seen from above).

- Example 3: cone $\Sigma = \{ (x,y,z) \in \mathbb{R}^3 \; z^2 = x^2+y^2$ and $0 \le z \le 1 \}$.

Parameterization: $A =]0, 2\pi[\times]0,1[$ and

$\Sigma = \{ \sigma(\theta, z) = (z\cos\theta, z\sin\theta, z) : (\theta, z) \in \bar{A} \}$.



$\sigma(\partial A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ with $\Gamma_i = \sigma(L_i)$ for $i = 1, \dots, 4$.

$\Gamma_1 = \{ r_1(\theta) = \sigma(\theta, 0) = (0,0,0)$ with $\theta : 0 \to 2\pi \}$

a single point at the origin.

$\Gamma_2 = \{ r_2(\varphi) = \sigma(2\pi, z) = (z, 0, z)$ with $z: 0 \to 1 \}$

intersection of the cone with the plane (xz), circulated upwards.

$\Gamma_3 = \{ r_3(\theta) = \sigma(\theta, 1) = (\cos\theta, \sin\theta, 1)$ with $\theta: 2\pi \to 0 \}$

circulated in clockwise way (seen from above)

$\Gamma_4 = \{ r_4(\varphi) = \sigma(0, z) = (z, 0, z)$ with $z: 1 \to 0 \}$.

same as Γ_2, but circulated downwards

Applying the procedure above, we have to eliminate Γ_1, Γ_2, and Γ_4 and we obtain $\partial\Sigma = \Gamma_3$ that is oriented in a clockwise sense (seen from above).

↓ end of the extra material