be a vector field $F \in C^1(\Omega, \mathbb{R}^2)$. Then

$$\iint_A \operatorname{curl} F(x,y)\, dx\, dy = \int_{\partial A} F \cdot dl$$

$$= \iint_A \left[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) \right] dx\, dy$$

21/10/2021

Remarks:

a) Green's theorem allows to replace the computation of a double integral of $\operatorname{curl} F$ on $A \subset \mathbb{R}^2$ with a curvilinear integral of F along ∂A.

(The flux of $\operatorname{curl} F$ through a surface is equal to the work of that force field along the boundary of ~~the~~ that surface).

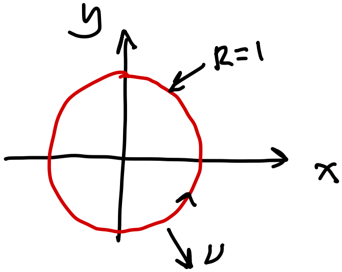
b) If F derives from a potential $\overset{\text{in } A}{\Rightarrow}$ $\operatorname{curl} F = 0$

$$\Rightarrow \int_{\partial A} F \cdot dl = 0$$

2.4.3 Examples

Example 1: Verify Green's theorem for $A = \{ (x,y) \in \mathbb{R}^2 :$

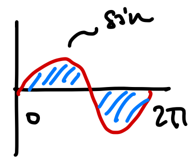
$x^2 + y^2 < 1 \}$ and $F = (y^2, x)$



$$\operatorname{curl} F(x,y) = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y^2)$$

$$= 1 - 2y$$

$$\iint_A \operatorname{curl} F(x,y)\,dx\,dy = \iint_A (1-2y)\,dx\,dy$$



$$= \int_0^1 \int_0^{2\pi} (1 - 2r\sin\theta)\, r\,dr\,d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$= \int_0^1 \Big[\int_0^{2\pi} d\theta - 2r \underbrace{\int_0^{2\pi} \sin\theta\,d\theta}_{= 0} \Big]\, r\,dr$$

$$= \int_0^1 2\pi r\,dr = 2\pi \frac{1}{2} r^2 \Big|_0^1 = \pi$$

- $\int_{\partial A} F \cdot d\ell$ being ∂A is <u>positively oriented</u>

$\partial A = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$

$\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

$$\int_{\partial A} F \cdot dl = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$\gamma'(t) = (-\sin t, \cos t)$$

$$\int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} (\sin^2 t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} (-\sin^3 t + \cos^2 t)\, dt = \int_0^{2\pi} (-\sin t\,(1-\cos^2 t) + \cos^2 t)\, dt$$

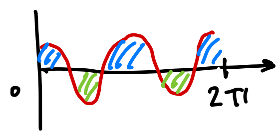
$$\sin^2 t = 1 - \cos^2 t$$

$$= \underbrace{-\int_0^{2\pi} \sin t\, dt}_{=0} + \underbrace{\int_0^{2\pi} \sin t \cos^2 t\, dt}_{I_1} + \underbrace{\int_0^{2\pi} \cos^2 t\, dt}_{I_2}$$

$$I_1 = \int_0^{2\pi} \sin \cos^2 t\, dt = -\frac{1}{3}\cos^3 t \Big|_0^{2\pi} = -\frac{1}{3}(1-1) = 0$$

$$I_2 = \int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{1}{2}(1 + \cos 2t)\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} dt + \underbrace{\frac{1}{2}\int_0^{2\pi} \cos 2t\, dt}_{=0}$$

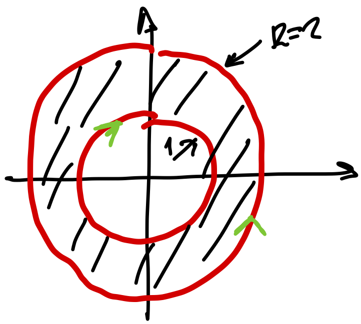


$$= \frac{1}{2}\, 2\pi = \pi$$

$$\iint_A \operatorname{curl} F\, dx\, dx = \pi = \int_{\partial A} F \cdot dl \quad \square$$

Example 2: Verify Green's theorem for

$B = \{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 4\}$ and $F(x,y) = (x^2y, 2xy)$



$B = B_0 \setminus \overline{B}_1$

with $B_0 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 4\}$

$B_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$

$\overline{B}_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$

$\Gamma_0 = \partial B_0 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 4\}$

$\Gamma_1 = \partial B_1 = \{(x,y) \in \mathbb{R} : x^2+y^2 = 1\}$

$\partial B = \Gamma_0 \cup \Gamma_1$

$$\operatorname{curl} F(x,y) = \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2y)$$

$$= 2y - x^2$$

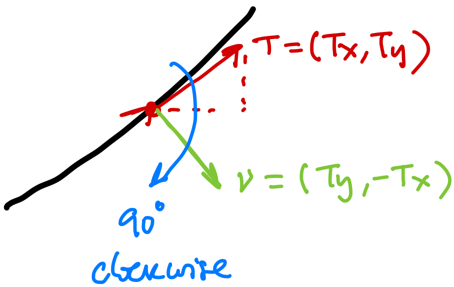
$$\iint_B \operatorname{curl} F(x,y)\,dx\,dy = \iint_B (2y - x^2)\,dx\,dy \underset{\text{polar coordin.}}{=}$$

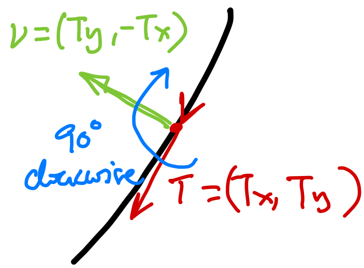
$$= \int_1^2 \int_0^{2\pi} (2r\sin\theta - r^2\cos^2\theta)\, r\,dr\,d\theta$$

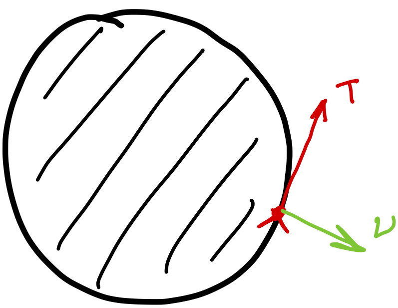
$$= \int_1^2 2r^2\,dr \underbrace{\int_0^{2\pi} \sin\theta \, d\theta}_{=0} - \int_1^2 r^3\,dr \underbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}_{= I_2 = \pi}$$

$$= -\int_1^2 r^3\,dr\,\pi = -\pi\,\frac{1}{4} r^4 \Big|_1^2 = \frac{-15\pi}{4}$$

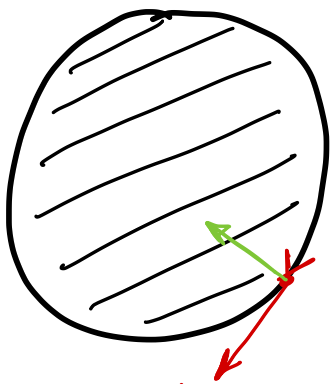
Note:





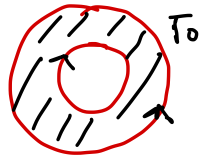


positive orientation



negative orientation

$\cdot \displaystyle\int_{\partial B} F \cdot dl$, <u>∂B is positively oriented</u>



$$\int_{\partial B} F \cdot dl = \int_{\Gamma_0} F \cdot dl + \int_{\Gamma_1} F \cdot dl \qquad \partial B = \Gamma_0 \cup \Gamma_1$$

$$\Gamma_0 = \{ \gamma_0(t) = (2\cos t, 2\sin t) \text{ for } t \in [0, 2\pi] \}$$

$$\Gamma_1 = \{ \gamma_1(t) = (\cos t, -\sin t) \text{ for } t \in [0, 2\pi] \}$$

(note the sense of circulation of Γ_1).

$$\gamma_0' = (-2\sin t, 2\cos t), \quad \gamma_1' = (-\sin t, -\cos t)$$

$$\int_{\Gamma_0} F \cdot dl = \int_0^{2\pi} F(\gamma_0(t)) \cdot \gamma_0'(t)\, dt$$

$$= \int_0^{2\pi} (8\cos^2 t \sin t, 8 \cos t \sin t) \cdot (-2\sin t, 2\cos t)\, dt$$

$$= -16 \underbrace{\int_0^{2\pi} \cos^2 t \sin^2 t\, dt}_{I_1} + 16 \underbrace{\int_0^{2\pi} \cos^2 t \sin t\, dt}_{I_2}$$

$$I_1 = \int_0^{2\pi} \cos^2 t \sin^2 t\, dt = \int_0^{2\pi} \left(\frac{1}{2} \sin 2t \right)^2 dt =$$

$\sin 2t = 2 \sin t \cos t$ $\qquad\qquad$ $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$

$$= \frac{1}{4}\int_0^{2\pi} \frac{1-\cos 4t}{2}\,dt = \frac{1}{8}\int_0^{2\pi} dt - \frac{1}{8}\underbrace{\int_0^{2\pi} \cos 4t\,dt}_{\overset{\shortparallel}{0}}$$

$$= \frac{\pi}{4}$$

$$I_2 = \int_0^{2\pi} \cos^2 t \sin t = -\frac{1}{3}\cos^3 t \Big|_0^{2\pi} = -\frac{1}{3}(1-1) = 0$$

$$\int_{\Gamma_0} F\cdot dl = 16\left(-\frac{\pi}{4}+0\right) = -4\pi$$

$$\int_{\Gamma_1} F\cdot dl = \int_0^{2\pi} F(\gamma_1(t))\cdot\gamma_1'(t)\,dt$$

$$= \int_0^{2\pi} (-\cos^2 t \sin t, -2\cos t \sin t)\cdot(-\sin t, -\cos t)\,dt$$

$$= \int_0^{2\pi} (\cos^2 t \sin^2 t + 2\cos^2 t \sin t)\,dt$$

$$= \underbrace{\int_0^{2\pi} \cos^2 t \sin^2 t\,dt}_{I_1 = \pi/4} + 2\underbrace{\int_0^{2\pi} \cos^2 t \sin t\,dt}_{I_2 = 0} = \frac{\pi}{4}$$

$$\int_{\partial B} F\cdot dl = \int_{\Gamma_0} F\cdot dl + \int_{\Gamma_1} F\cdot dl = -4\pi + \frac{\pi}{4} = -\frac{15}{4}\pi$$

$$= \iint_B \operatorname{curl} F\,dx\,dy \qquad \square$$

2.4.5. Corollaries of Green's theorem.

- Corollary 1: Divergence Theorem in the plane

Let $A \subset \mathbb{R}^2$ be a regular domain whose boundary ∂A is positively oriented. Let $\nu : \partial A \longrightarrow \mathbb{R}^2$ the field of outer <u>unit</u> normal vectors defined by $\nu = (\nu_1, \nu_2)$.

Let $F : \bar{A} \to \mathbb{R}^2$ be a vector field s.t. $F \in C^1(\bar{A}, \mathbb{R}^2)$ defined as $F = (F_1, F_2)$. Then:

$$\iint_A \operatorname{div} F(x,y)\, dx\, dy = \int_{\partial A} F \cdot \nu \, dl$$

In other words:

$$\iint_A \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dx\, dy = \int_{\partial A} (F_1 \nu_1 + F_2 \nu_2)\, dl$$

(Proof on video).

- Corollary 2:

Let $A \subset \mathbb{R}^2$ be a regular domain, ∂A is positively oriented.

Let's consider the vector fields F, G_1, and $G_2 : \bar{A} \to \mathbb{R}^2$ defined by $F(x,y) = (-y, x)$, $G_1(x,y) = (0, x)$ and $G_2(x,y) = (-y, 0)$

$$\text{Area}(A) = \frac{1}{2} \int_{\partial A} F \cdot dl = \int_{\partial A} G_1 \cdot dl = \int_{\partial A} G_2 \cdot dl$$

(Hint: $\text{Area}(A) = \iint_A 1 \, dx\, dy$)

$$\iint_A \operatorname{div} F \, dx\, dy = \int_{\partial A} F \cdot \nu \, dl$$

$$\iint_A \operatorname{curl} F \, dx\, dy = \int_{\partial A} F \cdot dl$$

$\operatorname{curl} F = 1 \longrightarrow$ Find F s.t. $\operatorname{curl} F = 1$

$\longrightarrow$ $\operatorname{curl} F = \frac{\partial}{\partial x}(F_y) - \frac{\partial}{\partial y}(F_x)$

- Corollary 3:

Let $A \subset \mathbb{R}^2$ be a regular domain whose boundary ∂A is positively oriented. Let $\nu : \partial A \to \mathbb{R}^2$ the unit outer normal field of ∂A and let $f : \bar{A} \to \mathbb{R}$ be a scalar field s.t. $f \in C^2(\bar{A})$. Then:

$$\iint_A (\Delta f)\, dx\, dy = \iint_A [\operatorname{div}(\operatorname{grad} f)]\, dx\, dy$$

$$\underset{\text{Div. Thm}}{=} \int_{\partial A} \operatorname{grad} f \cdot \nu \, dl$$