EPFL - Autumn 2021	Dr. Pablo Antolin
Analysis III SV MT	**Exercises**
Serie 2	October, 7

Note: several exercises are directly taken from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Exercise 1 (Ex 1.1 page 7)**.**
Let

$$F(x, y, z) = \big(y^2 \sin(xz), e^y \cos(x^2 + z), \ln(2 + \cos(xy))\big) = (F_1, F_2, F_3).$$

Compute:

1. $\operatorname{grad} F_1$, $\operatorname{grad} F_2$, $\operatorname{grad} F_3$
2. $\operatorname{div} F$
3. $\operatorname{rot} F$.

Exercise 2 (Ex 1.2 page 7)**.**
Which of the following expressions are correct in the case that $f : \mathbb{R}^3 \to \mathbb{R}$ lives in $C^1(\mathbb{R}^3)$ and $F : \mathbb{R}^3 \to \mathbb{R}^3$ lives in $C^1(\mathbb{R}^3; \mathbb{R}^3)$?

a) $\operatorname{grad} f$	*b*) $f \operatorname{grad} f$	*c*) $F \cdot \operatorname{grad} f$	*d*) $\operatorname{div} f$
e) $\operatorname{div}(fF)$	*f*) $\operatorname{rot}(fF)$	*g*) $\operatorname{rot} f$	*h*) $f \operatorname{rot} F$
i) $\operatorname{rot} \operatorname{div} F$.			

Exercise 3 (Exemple 1.3 page 5)**.**
Let $x = (x_1, \dots, x_n)$, $a = (a_1, \dots, a_n)$, and r such that $r = \sqrt{\sum_{i=1}^n (x_i - a_i)^2}$. Let f be a scalar field defined by $f(x) = 1/r$. Express Δf.

Exercise 4 (Ex 1.6 et 1.7 page 8)**.**
Let $\Omega \subset \mathbb{R}^3$ be an open domain. Verify that:

1. If $f \in C^1(\Omega)$ and $g \in C^2(\Omega)$, then:
$$\operatorname{div}(f \operatorname{grad} g) = f \Delta g + \operatorname{grad} f \cdot \operatorname{grad} g$$
2. If $f, g \in C^1(\Omega)$, then:
$$\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f$$

3. If $f \in C^1(\Omega)$ et $F \in C^1(\Omega, \mathbb{R}^3)$ then:

$$\operatorname{div}(fF) = f \operatorname{div} F + F \cdot \operatorname{grad} f$$

4. If $F \in C^2(\Omega, \mathbb{R}^3)$, then:

$$\operatorname{rot} \operatorname{rot} F = -\Delta F + \operatorname{grad} \operatorname{div} F,$$

   where $\Delta F = (\Delta F_1, \Delta F_2, \Delta F_3)$ for a given vector field $F = (F_1, F_2, F_3)$.

5. If $f \in C^1(\Omega)$ and $F \in C^1(\Omega, \mathbb{R}^3)$, then:

$$\operatorname{rot}(fF) = \operatorname{grad} f \wedge F + f \operatorname{rot} F$$

Exercise 5 (Ex 1.4 page 7)**.**
Let $f \in C^2(\Omega)$, where:

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : \ x, y > 0\right\}.$$

1. In the case where we have:

$$g\,(r, \theta) := f\,(r \cos\theta, r \sin\theta) = f(x, y),$$

   show that:

$$\frac{\partial^2 g(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial g(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g(r,\theta)}{\partial \theta^2} = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = \Delta f(x, y).$$

2. Compute Δf when:

$$f(x, y) := \sqrt{x^2 + y^2} + \left(\arctan \frac{y}{x}\right)^2.$$