EPFL

1

**Teacher : Teachers of Analysis III**
**Analysis III - Mock exam - Student**
**December 2021**
**Duration : 120 minutes**

# Student One

SCIPER : **111111**

**Do not turn the page before the start of the exam. This document is double-sided, has 30 pages, the last ones possibly blank. Do not unstaple.**



- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  +2 points if your answer is correct,
  0 points if your answer is incorrect, you give no answer, or more than one answer is marked.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

| choisir une réponse \| select an answer Antwort auswählen | ne PAS choisir une réponse \| NOT select an answer NICHT Antwort auswählen | Corriger une réponse \| Correct an answer Antwort korrigieren |
|---|---|---|

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

## First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

**Question 1** All the non-zero real Fourier coefficients (sine-cosine form) of the function $f$ defined by:

$$f(x) = 5 + \frac{1}{2}(5 - i\pi)e^{3ix} + \frac{1}{2}(5 + i\pi)e^{-3ix} - \frac{3}{2}e^{7ix} - \frac{3}{2}e^{-7ix}$$

are

- ☐ $a_0, a_3, a_7$
- ☐ $a_3, a_7, b_3$
- ☐ $a_0, a_3, b_3, b_7$
- ☐ $a_0,\ a_3,\ a_7,\ b_7$

**Question 2** Let $f$ be the scalar field defined by:

$$f : \mathbb{R}^2 \to \mathbb{R};\ (x, y) \mapsto xy + x + 1,$$

and let $R \in \mathbb{R}, R > 0$, and $\Gamma$ the curved defined by:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2\right\}.$$

The integral $\displaystyle\int_\Gamma f \,\mathrm{d}l$ is equal to:

- ☐ $0$
- ☐ $2\pi R$
- ☐ $2\pi R^2 + \pi R + 1$
- ☐ $2\pi R^3 + \pi R^2 + R$

**Question 3** Let $F$ be the vector field defined by:

$$F : \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto \overset{(y, x)}{\cancel{(x, y)}},$$

and let $R \in \mathbb{R}, R > 0$ and $A$ be the domain defined by:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\right\}.$$

We also denote the boundary of $A$ by $\partial A$, and the outer unit normal of $\partial A$ by $\nu : \partial A \to \mathbb{R}^2$.
The integral $\displaystyle\int_{\partial A} F \cdot \nu \, dl$ is equal to:

- ☐ $0$
- ☐ $\pi R^2$
- ☐ $\frac{3}{4}\pi R$
- ☐ $\frac{3}{5}\pi^2 R$

**Question 4** Let $F$ be the vector field defined by:

$$F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right).$$

$F$ is conservative (*i.e.* it derives from a potential)

- ☐ over $\Omega = \{(x, y) :\ 1 \le x \le 3, 2 \le y \le 10\}$.
- ☐ over $\Omega = \{(x, y) :\ 2 \le x^2 + y^2 \le 4\}$.
- ☐ over $\Omega = \{(x, y) :\ x^2 + y^2 \le 10\}$.
- ☐ for any domain $\Omega$.

## Second, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

**Question 4:** *This question is worth 9 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ■ 9

(i) Let $\Gamma$ be the curve defined by

$$\Gamma = \left\{ \left( \frac{1}{3}t^3, 3t, \frac{\sqrt{6}t^2}{2} \right) \,\middle|\, t \in [-1,1] \right\}.$$

Compute the length of $\Gamma$.

(ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$F(x,y) = \left(x^2, y\cos(x^2)\right)$$

and $\Omega$ the triangle whose vertices are $(0,0)$, $(\sqrt{\pi/2}, 0)$, and $(\sqrt{\pi/2}, \sqrt{\pi/2})$. Compute

$$\int_{\partial\Omega} F \cdot \nu \mathrm{d}l$$

where $\nu : \partial\Omega \to \mathbb{R}^2$ is outer unit normal field of the boundary of $\Omega$.

**Question 5:** *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ■6

Let $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,y) = \left( \frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right).$$

(i) Compute curl $F$.

(ii) Determine if $F$ derives from a potential in $\Omega$. If it does, find a potential of $F$, otherwise, justify why it does not derive from a potential in $\Omega$.

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**Question 6:** *This question is worth 3 points.*

☐ 0 ☐ 1 ☐ 2 ■ 3

Let $F : \mathbb{R}^2 \to \mathbb{R}^2; F(x, y) = (F_1(x, y), F_2(x, y))$, be a vector field such that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $\operatorname{div} F = 0$. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined by:

$$G(x, y) = (F_2(-x, y), F_1(-x, y))\,.$$

Show that $G$ derives from a potential in $\mathbb{R}^2$.

**Question 7:** *This question is worth 14 points.*

0 1 2 3 4 5 6 7
8 9 10 11 12 13 14

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as $F(x, y, z) = (0, x, 0)$ and let $\Sigma$ be the surface defined by

$$\Sigma = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = \left(\sqrt{x^2 + y^2} + 1\right)\left(3 - \sqrt{x^2 + y^2}\right), y \geq 0, z \geq 0 \right\}.$$

Verify the Stokes theorem for $F$ and $\Sigma$.
*Note*: if necessary, use the following formulas:

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

**Question 8:** *This question is worth 9 points.*

0 1 2 3 4 5 6 7 8 9

Let $f : [0, \pi] \to \mathbb{R}$ be the function $f(x) = -x^2 + 2\pi x$.

(i) Compute $F_s f$, the Fourier series in sines of $f$.

(ii) Using the course's results, compare $F_s f$ and $f$ in the interval $[0, \pi]$.

**Question 9:** *This question is worth 4 points.*

☐0 ☐1 ☐2 ☐3 ■4

Let $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} -x & \text{if} \quad -\pi \leq x \leq 0 \\ \pi & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{extended by } 2\pi\text{-periodicity.}$$

The real Fourier coefficients of $g$ are

$$a_0 = \frac{3\pi}{2};$$

$$a_n = \begin{cases} -\frac{2}{n^2\pi} & \text{if} \quad n \text{ is odd} \\ 0 & \text{if} \quad n \text{ is even} \end{cases} \qquad \text{for } n \geq 1;$$

$$b_n = \frac{1}{n} \qquad \text{for } n \geq 1.$$

Using those coefficients and one result of the course, compute the sum

$$\sum_{k=1}^{+\infty} \frac{1}{\left(k - \frac{1}{2}\right)^2}.$$

**Question 10:** *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ■8

(i) Write the definition of the Fourier transform of a function detailing its hypotheses

(ii) Using the properties of the Fourier transform, find $u : \mathbb{R} \to \mathbb{R}$, the solution of

$$-10u(x) + \int_{-\infty}^{+\infty} \Big(9u(t) - 4u''(t)\Big) e^{-\frac{3}{2}|x-t|} \mathrm{d}t = \frac{4x^2}{(2\pi + x^2)^2}.$$

If needed, use the Fourier transforms of the table below.

| | $f(y)$ | $\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$ |
|---|---|---|
| 1 | $f(y) = \begin{cases} 1, & \text{si } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{sinon} \end{cases}$ | $\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert\alpha)}{\alpha}$ |
| 2 | $f(y) = \begin{cases} 1, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$ | $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$ |
| 3 | $f(y) = \begin{cases} e^{-wy}, & \text{si } y > 0 \\ 0, & \text{sinon} \end{cases} \quad (w > 0)$ | $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$ |
| 4 | $f(y) = \begin{cases} e^{-wy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$ | $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$ |
| 5 | $f(y) = \begin{cases} e^{-iwy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$ | $\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$ |
| 6 | $f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$ | $\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$ |
| 7 | $f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$ | $\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$ |
| 8 | $f(y) = e^{-w^2y^2} \quad (w \neq 0)$ | $\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$ |
| 9 | $f(y) = ye^{-w^2y^2} \quad (w \neq 0)$ | $\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$ |
| 10 | $f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$ | $\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert\right) e^{-\lvert w\alpha\rvert}$ |