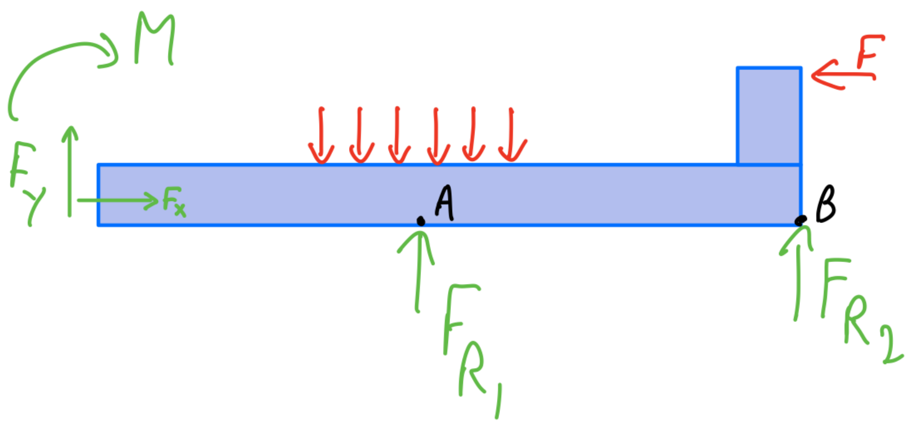
Semaine 9b

9b.1. Systèmes Hyperstatiques - Support Elastique – Charge distribuée

9b.1. Solution

i. Free Body Diagram drawing



ii. Calcul des redondants

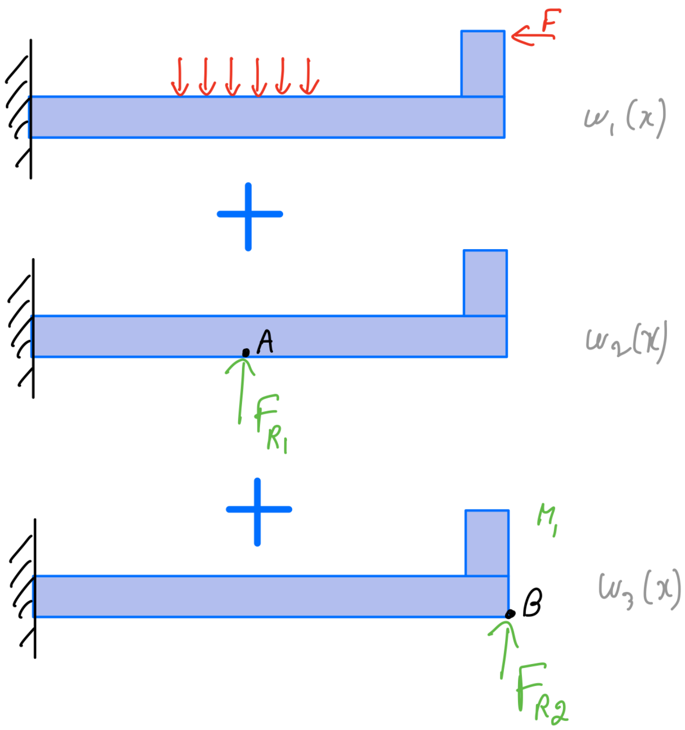
N. redondants = N. unknowns - N. available equations
In this case,
N. unknowns = 5 (M, F_x, F_y, F_{R1}, F_{R1}) ;
N. available equations = 3 (forces in x, forces in y, moment in z)
N. redondants = 2

iii. Choix des redondants redondants

F_{R1} and F_{R2} should be chosen as redundant

iv. Dessin des systèmes isostatiques nécessaires pour résoudre le problème.

The chosen redundant constraints are removed, thus obtaining an isostatic system.
The redundant reaction forces are treated like external loads.



v. Équations de compatibilité

The effects of the redundant constraints have to be expressed in equation form

$$F_{R1} = -k_1 w(A)$$
$$F_{R2} = -k_2 w(B)$$

Any form of the above equations is fine (specifying the superposition of the subsystem is not required)

9b.2. Support Elastique – Charge ponctuelle

Une poutre AC de longueur L est encastrée au point A et attaché par une articulation à un ressort BD au point B (voir Figure 9a.2.1). La section de la poutre AC est un carré de côté t = 0.1 L.

Le ressort a une constante $k = \frac{2Et^2}{L}$.

Déterminez :

(a) Les forces et moments de réactions sur la poutre AC
(b) Le diagramme de l'effort tranchant V(x) dans la poutre AC
(c) Le diagramme du moment de flexion M(x) dans la poutre AC
(d) w'(x) : La dérivée de la déflexion de AC en fonction de x
(e) w(x) : La déflexion de AC en fonction de x

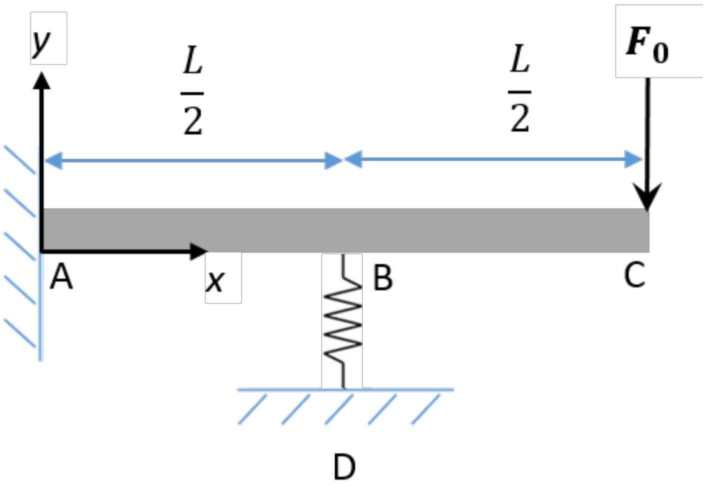


Figure 9a.2.1 | Poutre supportée par un ressort.

9b.2. Solution

2 Façons (également) valables de résoudre:

i) Superposition, trouver w(x) en fonction de F_B, utiliser $F_B = k\, w(B)$,pour trouver F_B. donc trouver $w(x)$, puis dériver pour $M(x)$

ou

ii) Méthodes Sections, trouver V(x) et M(x) et par intégration $w(x)$, tous en fonction de F_B. utiliser $F_B = k\, w(B)$ et puis on a donc une expression pour V(x) et M(x) et w(x).

What is given? **Ces infos sont utiles!**

- Length of beam L
- $t = 0.1 \cdot L$
- $k = 2\,Et^2/L$ k est donné! Ça nous permettra de simplifier par la suite

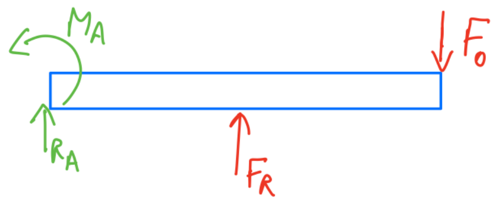
Assumptions

The material is homogeneous and isotropic.

What is asked?

(a) The reactions forces and moments
(b) Shear force diagram in the beam AC
(c) Bending moment diagram in the beam AC
(d) Derivative of the deflection along AC as a function of x
(e) Deflection along the beam AC as a function of x

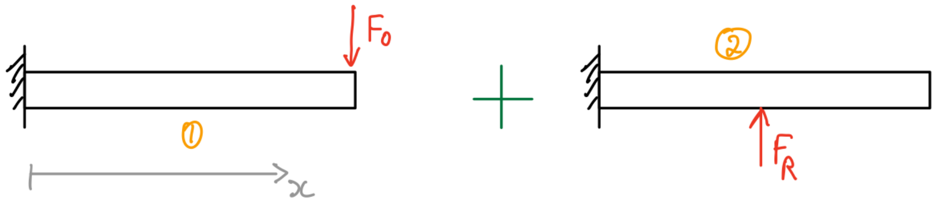
Methode i (formules de poutres)



1 redondant. on doit choisir F_R car l'équation de compatibilité est pour F_R

$$F_R = -k\, w(x = L/2)$$

SUPERPOSITION



$$w_1(x) = -\frac{F_0 x^2}{6EI}(3L - x) \qquad 0 \le x \le L$$

$$w_2(x) = \begin{cases} +F_R \dfrac{x^2}{6EI}\left(\dfrac{3L}{2} - x\right) & 0 \le x \le L/2 \\ +F_R \dfrac{L^2}{24EI}\left(3x - \dfrac{L}{2}\right) & L/2 \le x \le L \end{cases}$$

je m'intéresse en premier à $w(x=L/2) = w_1(x=L/2) + w_2(x=\frac{L}{2})$

$$w(x = L/2) = -\frac{F_0 L^2}{4 \cdot 6EI}\left(3L - \frac{L}{2}\right) + F_R \frac{L^2}{24EI}\left(\frac{3L}{2} - \frac{L}{2}\right)$$

$$= \frac{L^2}{24EI}\left[-\frac{5}{2}F_0 L + F_R L\right]$$

$$= -\frac{F_R}{k}$$

je trouve alors F_R

$$F_R = \frac{kL^3}{24EI}\,\frac{5}{2}F_0 - \frac{F_R L^3 k}{24EI}$$

$$F_R = \frac{5}{2}F_0 \quad \frac{1}{1+\frac{2Et^4}{kL^3}}$$

nous connaissons k : voir l'énnoncé

$$k = \frac{2Et^2}{L} \qquad \text{donc } F_R = \frac{5}{2}F_0 \quad \frac{1}{1+(t/L)^2}$$

$$\left(\frac{t}{L}\right)^2 = \frac{1}{100} \qquad \boxed{F_R \simeq \frac{5}{2}F_0}$$

$$w(x) = \begin{cases} -\frac{F_0 x^2}{6EI}(3L-x) + \frac{5}{2}F_0\frac{x^2}{6EI}\left(\frac{3L}{2}-x\right) & 0 \le x \le \frac{L}{2} \\ -\frac{F_0 x^2}{6EI}(3L-x) + \frac{5}{2}F_0\frac{L^2}{24EI}\left(3x-\frac{L}{2}\right) & \frac{L}{2} \le x \le L \end{cases}$$

$$\boxed{w(x) = \begin{cases} \frac{F_0}{6EI}\left(\frac{3}{4}Lx^2 - \frac{3}{2}x^3\right) & 0 \le x \le \frac{L}{2} \\ \frac{F_0}{2EI}\left(\frac{x^3}{3} - Lx^2 + \frac{5Lx}{8} - \frac{5L^3}{48}\right. & \frac{L}{2} \le x \le L \end{cases}}$$

on peut dériver pour trouver $w'(x)$ $M(x)$ $V(x)$

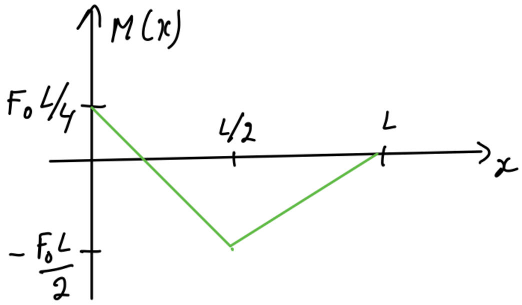
$$\boxed{\begin{aligned} &0 \le x \le L/2 \\ &w'(x) = \frac{F_0}{EI}\left(\frac{Lx}{4} - \frac{3x^2}{4}\right) \\ &M(x) = \frac{F_0 L}{4} - \frac{3}{2}F_0 x \\ &V(x) = -\frac{3}{2}F_0 \end{aligned}}$$

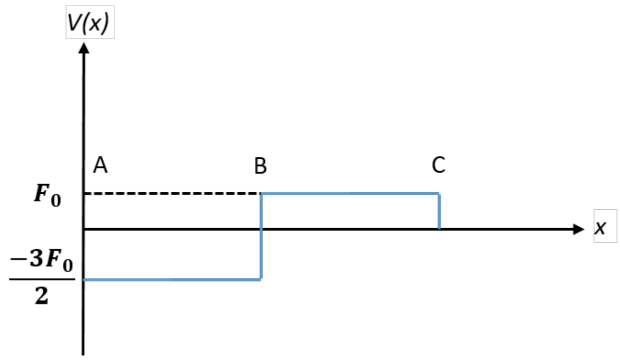
$\frac{L}{2} \leq x \leq L$

$$w'(x) = \frac{F_0}{2EI}\left(x^2 - 2Lx + \frac{5L}{8}\right)$$

$$M(x) = \frac{F_0}{2}(2x - 2L) = F_0(x - L)$$

$$V(x) = F_0$$





Methode ii (méthodes sections)

Principles and formula

(a) We calculate the reaction forces from the free body diagram of the beam, shown in Figure 9a.2.2.

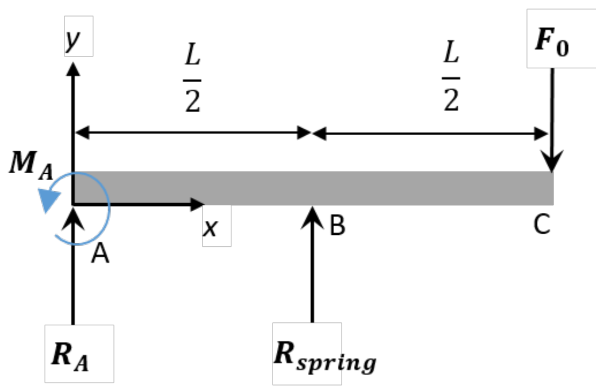


Figure 9a.2.2| Free body diagram

The equilibrium of forces in y yields:

$$\Sigma F_y = 0 \implies R_A + R_{spring} - F_0 = 0 \tag{9a.2.1}$$

$$R_A + R_{spring} = F_0 \tag{9a.2.2}$$

From the equilibrium of the moment M_z about A, we get:

$$\Sigma M_z = 0 \implies M_A + R_{spring}\frac{L}{2} - F_0 L = 0 \tag{9a.2.3}$$

$$R_{spring} L + 2M_A = 2F_0 L \tag{9a.2.4}$$

Note that this is a statically indeterminate problem. R_{spring} will remain unknown until the deflection is calculated, when we can then use the following formula:

$$R_{spring} = -k \cdot w(x = L/2)$$

We will express everything as a function of R_{spring} to be able to compute it using compatibility. (9a.2.5)

(b) Shear force

From A to B the shear force is defined as:

$$V(x) = V(x = 0) = R_A \tag{9a.2.6}$$

From B to C:

$$V(x) = F_0 = R_{spring} + R_A \tag{9a.2.7}$$

Therefore, at the middle of the beam:

$$V(L/2^+) - V(L/2^-) = R_{spring} \tag{9a.2.8}$$

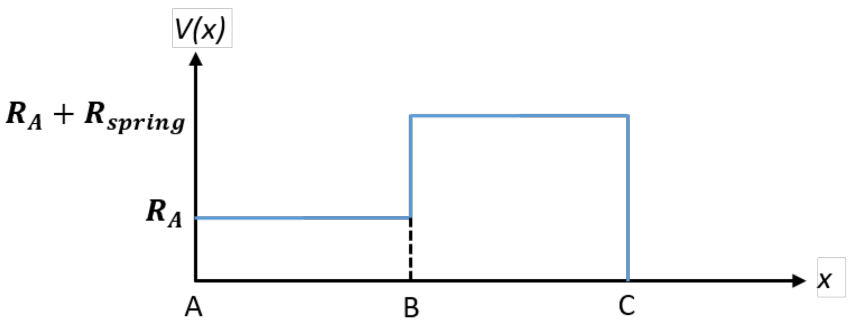


Figure 9a.2.3 | Shear Force Diagram

(c) Bending Moment

We know from theory that:

$$\frac{\partial M_z(x)}{\partial x} = V(x) \rightarrow \int_0^x dM_z = \int_0^x V(x')\,dx' \tag{9a.2.9}$$

Which in the section from A to B means:

$$M_z(x) = -M_A + \int_0^x (R_A)dx' = -M_A + R_A x \tag{9a.2.10}$$

$$M_z\left(x = \frac{L}{2}\right) = -M_A + \frac{R_A L}{2} \tag{9a.2.11}$$

In the section from B to C means:

$$M_z(x) = -M_A + R_A x + R_{spring}\left(x - \frac{L}{2}\right) = F_0(x - L) \tag{9a.2.12}$$

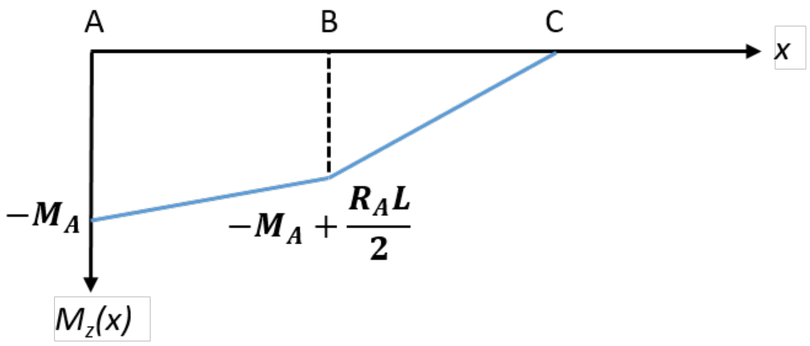


Figure 9a.2.4 | Bending Moment Diagram

(d) Derivative of deflection of the beam as a function of x

The curvature of the beam due to bending moments at any point is given by:

$$\frac{d^2 w}{dx^2} = \frac{M_z(x)}{EI_{z,y_0}} \tag{9a.2.13}$$

Hence we can calculate the derivative by integration of Eq. (9a.2.13):

$$w'(x') = \frac{1}{EI_{z,y_0}} \int_0^{x'} M_z(x'')dx'' \tag{9a.2.14}$$

Which in the section from A to B takes the form:

$$w'(x') = \frac{1}{EI_{z,y_0}} \int_0^{x'} (-M_A + R_A x'')dx'' = \frac{1}{EI_{z,y_0}} \left(\frac{R_A}{2} x'^2 - M_A x'\right) + w'(x' = 0) \tag{9a.2.15}$$

Given that the beam is clamped at $x = 0$, the slope of the beam at this point will be zero. Solving for M_A in Eq. (9a.2.12) and R_A in Eq. (9a.2.11) and substituting both into Eq. (9a.2.15), we find:

$$w'(x') = \frac{1}{EI_{z,y_0}} \left(x'^2 \frac{(F_0 - R_{spring})}{2} + x' \frac{L}{2} (R_{spring} - 2F_0) \right) \tag{9a.2.16}$$

From section B to C we get:

$$w'(x') = \frac{1}{EI_{z,y_0}} \int_{\frac{L}{2}}^{x'} (F_0(x'' - L))dx''$$
$$= \frac{1}{EI_{z,y_0}} \left[\frac{F_0}{2} \left(x'^2 - \frac{L^2}{4} \right) - F_0 L \left(x' - \frac{L}{2} \right) \right] + w'(x' = \frac{L}{2}) \tag{9a.2.17}$$

Solving for $w'\left(x' = \frac{L}{2}\right)$ from the derivative of the deflection for the AB section:

$$w'\left(x' = \frac{L}{2}\right) = \frac{1}{EI_{z,y_0}} (-3F_0 + R_{spring}) \frac{L^2}{8} \tag{9a.2.18}$$

Substituting $w'\left(x' = \frac{L}{2}\right)$ into Eq. (9a.2.17), we find that:

$$w'(x') = \frac{1}{EI_{z,y_0}} \left(\frac{F_0}{2} x'^2 - F_0 L x' + \frac{R_{spring} L^2}{8} \right) \tag{9a.2.19}$$

Note that we are leaving the derivative as a function of F_0 and R_{spring} which we will calculate once we have calculated the overall deflection.

(e) Deflection of the Beam as a function of x

Deflection of the beam from A to B:

$$w(x) - w(x = 0) = \int_0^{x} w'(x')\, dx' \tag{9a.2.20}$$

$$w(x) = \frac{1}{EI_{z,y_0}} \left(x^3 \left(\frac{F_0 - R_{spring}}{6} \right) + x^2 \frac{L}{4} (R_{spring} - 2F_0) \right) \tag{9a.2.21}$$

Now we can write the deflection for $x = \frac{L}{2}$:

$$w\left(x=\frac{L}{2}\right)=\frac{L^3}{EI_{z,y_0}}\left(\frac{R_{spring}}{24}-\frac{5F_0}{48}\right) \tag{9a.2.22}$$

Deflection of the beam from B to C:

$$w(x)-w\left(x=\frac{L}{2}\right)=\int_{L/2}^{x} w'(x')\,\mathrm{d}x' \tag{9a.2.23}$$

$$w(x)-w\left(x=\frac{L}{2}\right)=\frac{1}{EI_{z,y_0}}\left(F_0\left(\frac{5L^3}{48}-\frac{Lx^2}{2}+\frac{x^3}{6}\right)+R_{spring}\left(\frac{xL^2}{8}-\frac{L^3}{16}\right)\right) \tag{9a.2.24}$$

$$w(x)=\frac{1}{EI_{z,y_0}}\left(\frac{F_0}{6}x^3-\frac{F_0}{2}Lx^2+\frac{R_{spring}}{8}L^2x-\frac{R_{spring}}{48}L^3\right) \tag{9a.2.25}$$

Now we combine Eqs. (9a.2.5) and (9a.2.22) to estimate the reaction of the spring, taking into account that the cross section is squared and the value for $k = \frac{2Et^2}{l}$:

$$R_{spring}=-k\,w\left(\frac{L}{2}\right)=-k\cdot\frac{L^3}{EI_{z,y_0}}\left(\frac{R_{spring}}{24}-\frac{5F_0}{48}\right)=-\left(\frac{2Et^2}{L}\right)\frac{L^3}{E\left(t^4/_{12}\right)}\left(\frac{R_{spring}}{24}-\frac{5F_0}{48}\right) \tag{9a.2.26}$$

$$R_{spring}=-\left(\frac{L}{t}\right)^2\left(R_{spring}-\frac{5}{2}F_0\right)\rightarrow R_{spring}=\frac{\left(\frac{L}{t}\right)^2\frac{5}{2}F_0}{1+\left(\frac{L}{t}\right)^2}\approx\frac{5}{2}F_0 \tag{9a.2.27}$$

It is important to note that in Eq. (9a.2.27) we have used the fact that $\left(\frac{L}{t}\right)^2 = 100 \gg 1$, which allows us to perform the approximation with only around 1% error.

Now we can calculate all three reactions:

$$R_{spring}\approx\frac{5}{2}F_0 \;\;\&\&\;\; R_A\approx-\frac{3}{2}F_0 \;\;\&\&\;\; M_A\approx-\frac{F_0L}{4} \tag{9a.2.28}$$

We then use Eq. (9a.2.7) to properly plot the shear force diagram and the bending moment considering that F_0 is positive:

Shear force diagram

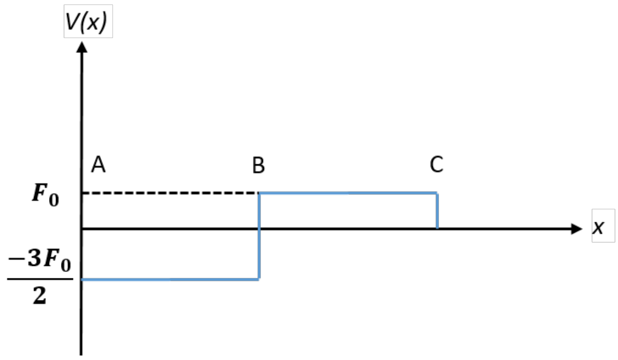


Figure 9a.2.5 | Shear Force Diagram

Bending Moment Diagram

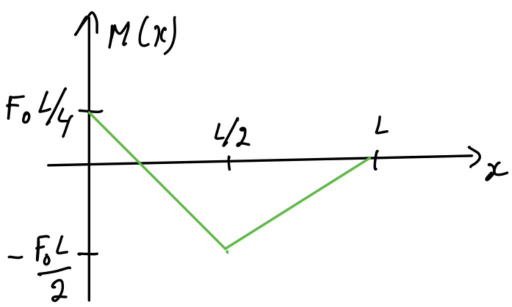


Figure 9a.2.6 | Bending Moment Diagram

Derivative of the Deflection

$$w'(x) \approx \begin{cases} \dfrac{F_0}{4EI_{z,y_0}}(-3x^2 + xL) & x < \dfrac{L}{2} \\ \dfrac{F_0}{2EI_{z,y_0}}\left(x^2 - 2xL + \dfrac{5L^2}{8}\right) & x > \dfrac{L}{2} \end{cases} \tag{9a.2.29}$$

Deflection

$$w(x) \approx \begin{cases} \dfrac{F_0}{4EI_{z,y_0}}x^2\left(\dfrac{L}{2} - x\right) & x < \dfrac{L}{2} \\ \dfrac{F_0}{2EI_{z,y_0}}\left(-\dfrac{5L^3}{48} + \dfrac{5}{8}xL^2 - Lx^2 + \dfrac{x^3}{3}\right) & x > \dfrac{L}{2} \end{cases} \tag{9a.2.30}$$

9b.3. Question courte – Question conceptuelle sur le problème 9b.2

Dans le problème précèdent (19.1), vous avez résolu un système qui incluait un ressort.

Comment modifieriez-vous vos calculs si le ressort était remplacé par une poutre déformable ?

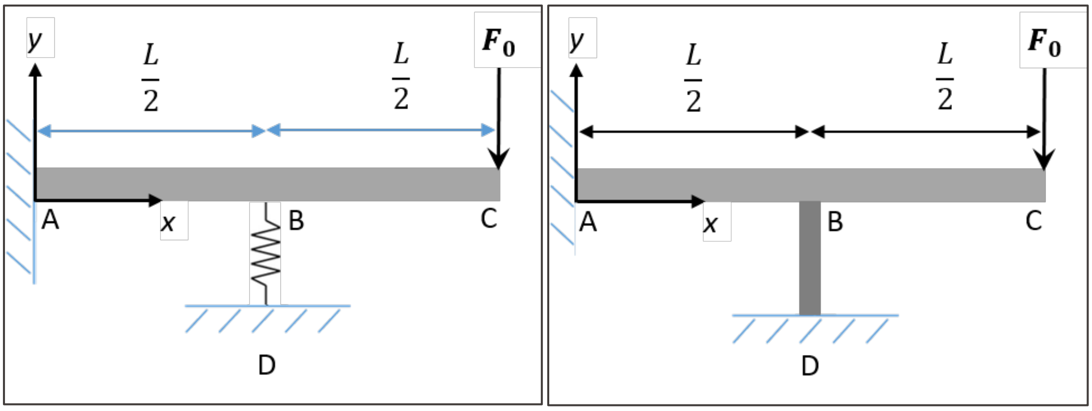


Figure 9a.3.1 | Première poutre avec le ressort (gauche); Poutre avec une barre (droite)

9b.3. Solution

Assumptions

The material is homogeneous and isotropic.

The cross-section of the beam in the YZ plane remains un-deformed along the length of the beam.

What is asked?

What would change in the deflection calculations if a beam replaced the spring?

Principles and formula

In many ways, the calculations in Problem 9a.1 would not be changed by replacing the spring with a beam. The system would still be indeterminate and one would still need to solve for the reaction force, this time for the beam. The force felt by the beam would be:

$$R_{bar} = k \cdot w(x = L/2) = \frac{AE}{l} \cdot w(x = L/2) \quad (9a.3.1)$$

Where l, A and E are the length, cross section and Young's Modulus of the support beam, respectively.

9b.4. Support Elastique – Charge uniforme

Une poutre AC est soumise à une charge uniformément répartie (voir Figure 9a.4.1). Sa longueur est 2L = 2m.

Quel est la constante du ressort k nécessaire pour avoir un moment de flexion en B : $M_B = -\frac{qL^2}{10}$?

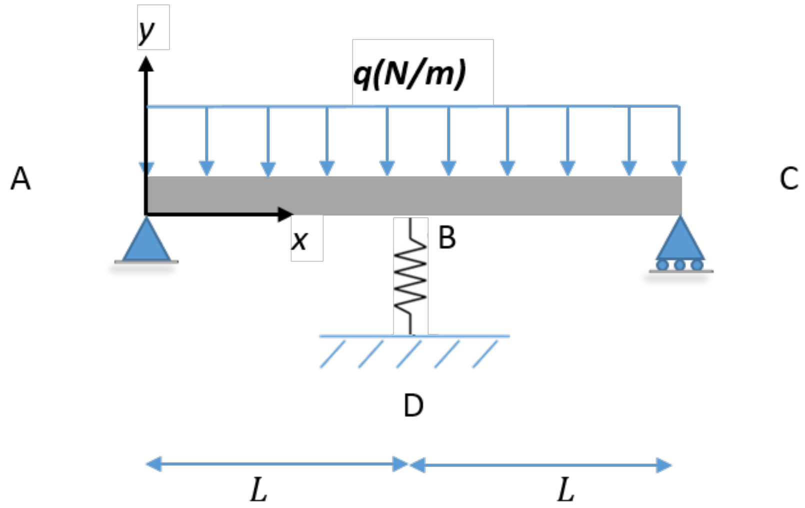


Figure 9a.4.1 | Poutre avec une charge uniforme répartie

9b.4. Solution

What is given?

$L = 1\text{ m}$

$M_B = -\frac{qL^2}{10}$

Assumptions

The material is homogeneous and isotropic.

What is asked?

Spring constant k for given bending moment

Principles and formula

The basic procedure for this problem is to calculate first the value of the reaction that the spring is making on the beam, then calculate the deflection of the beam at the point where the spring is. The ratio between them would give us k through Equation $R_{spring} = -k \cdot w(x = L)$.

Reaction Forces:

We calculate the reaction forces based off the free body diagram shown in Figure 9a.4.2.

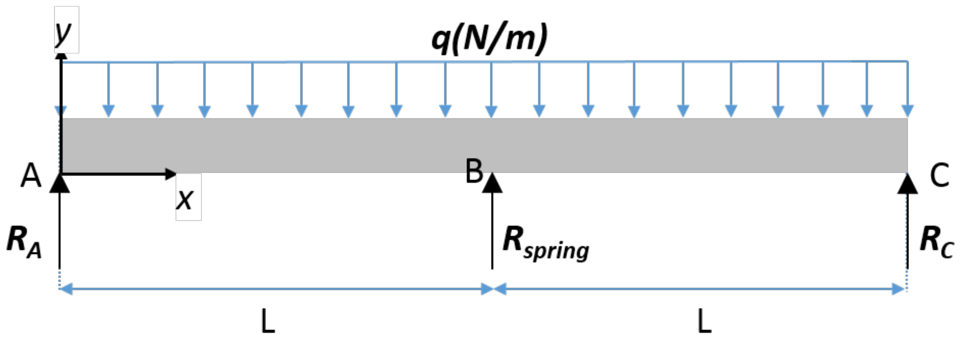


Figure 9a.4.2 | Free body diagram

The equilibrium of forces in y yields:

$$\Sigma F_y = 0 \implies R_A + R_C + R_{spring} - \int_0^{2L} q\,dx = R_A + R_C + R_{spring} - 2qL = 0 \tag{9a.4.2}$$

$$R_A + R_C + R_{spring} = 2qL \tag{9a.4.3}$$

From the equilibrium of the moment M_Z about A, we get:

$$\Sigma M_z = 0 \implies R_{spring}L + R_C(2L) - \int_0^{2L} q \cdot x\,dx = R_{spring}L + R_C(2L) - q\frac{(2L)^2}{2} = 0 \tag{9a.4.4}$$

$$R_{spring} + 2R_C = 2qL \tag{9a.4.5}$$

Comparing Eq. (9a.4.3) and (9a.4.5) gives us:

$$R_A = R_C \tag{9a.4.6}$$

Shear Force:

From A to B the shear force is defined as:

$$V(x) - V(x=0) = \int_0^x -q\,\mathrm{d}x' = -qx \tag{9a.4.7}$$

$$(x) = V(x=0) - qx = R_A - qx \tag{9a.4.8}$$

From B to C we have the same integral as in Eq. (9a.4.7) but we need to account for the discontinuity at $x = L$:

$$V(L^+) - V(L^-) = R_{spring} \tag{9a.4.9}$$

Then, from B to C:

$$V(x) = R_{spring} + R_A - qx \tag{9a.4.10}$$

Bending Moment:

$$M_z(\mathrm{x}) = \int_0^x V(x')\,\mathrm{d}x' \tag{9a.4.11}$$

Which in the section from A to B means:

$$M_z(\mathrm{x}) = \int_0^x (R_A - qx')dx' = R_A x - q\frac{x^2}{2} \tag{9a.4.12}$$

$$M_z(x=L) = R_A L - \frac{qL^2}{2} \tag{9a.4.13}$$

In the section from B to C means:

$$M_z(x) - M_z(x=L) = \int_L^x \left(R_{spring} + R_A - qx'\right)dx' \tag{9a.4.14}$$

$$M_z(x) = M_z(x=L) + \left(R_A + R_{spring}\right)(x-L) - q\frac{x^2 - L^2}{2} \tag{9a.4.15}$$

It is given in the problem that $M_z(x=L) = -\frac{qL^2}{10}$, therefore using Eq. (9a.4.13):

$$M_z(x=L) = R_A L - \frac{qL^2}{2} = -\frac{qL^2}{10} \rightarrow R_A = \frac{qL}{2} - \frac{qL}{10} = \frac{2}{5}qL \tag{9a.4.16}$$

Knowing $R_C = R_A$, we can substitute the value of R_C into Eq. (9a.4.5):

$$R_{spring} = 2qL - 2R_C = \frac{6}{5}qL \tag{9a.4.17}$$

<u>Deflection: (could also look up in a table)</u>

The curvature of the beam due to bending moments at any point is given by:

$$\frac{d^2w}{dx^2} = \frac{M_z(x)}{EI_{z,y_0}} \tag{9a.4.18}$$

Hence we can calculate the deflection of the beam by double integration of Eq. (9a.4.18).

$$w = \frac{1}{EI_{z,y_0}} \int_0^x \left(\int_0^{x'} M_z(x'')dx'' \right) dx' \tag{9a.4.19}$$

In the section from A to B:

$$w(x) = \frac{1}{EI} \int_0^x \left(\int_0^{x\prime} \left(R_A x'' - q\frac{x''^2}{2} \right) dx'' \right) dx' \tag{9a.4.20}$$

$$w(x) = \int_0^x \left[\frac{1}{EI_{z,y_0}} \left(\frac{qLx'^2}{5} - \frac{qx'^3}{6} \right) + w'(x=0) \right] dx' \tag{9a.4.21}$$

$$w(x) = \frac{1}{EI_{z,y_0}} \left(\frac{qLx^3}{15} - \frac{qx^4}{24} + \right) + x \cdot w'(x=0) \tag{9a.4.22}$$

To calculate the value of the slope at the origin ($w'(x=0)$), we need to use another kinematic boundary condition. This can be for example the fact that the deflection at the end is zero. Another possibility is to consider that the system is symmetric with respect to $x = L$, which means that the slope at that point must be zero. Therefore:

$$w'(x=L) = 0 = \frac{1}{EI_{z,y_0}} \left(\frac{qL^3}{5} - \frac{qL^3}{6} \right) + w'(x=0) \tag{9a.4.23}$$

$$w'(x=0) = -\frac{1}{EI_{z,y_0}} \frac{q}{30} L^3 \tag{9a.4.24}$$

Now we can write the deflection for $x = L$:

$$w(x=L) = \frac{1}{EI_{z,y_0}} \left(\frac{qL^4}{15} - \frac{qL^4}{24} - \frac{q}{30}L^4 \right) \tag{9a.4.25}$$

$$w(x = L) = -\frac{qL^4}{120EI_{z,y_0}} \tag{9a.4.26}$$

The deflection calculated for length L is equal to the deflection of the spring δ. Therefore:

$$R_{spring} = -k \cdot w(x = L) \tag{9a.4.27}$$

$$k = \frac{6}{5}qL \cdot \frac{120EI_{z,y_0}}{qL^4} = 144\frac{EI_{z,y_0}}{L^3} \tag{9a.4.28}$$

9b.5. Support elastique – poutres croisées

Deux poutres AB et CD, disposées dans le plan horizontal se croisent à angle droit et supportent ensemble une charge P en leur milieu (voir Figure 9a.5.1). Avant que la charge P soit appliquée, les poutres ne font que se toucher (sans appliquer de charge l'une sur l'autre). Les matériaux des deux poutres ainsi que leurs largeurs sont les mêmes. Les deux poutres sont en appui simple à leurs extrémités.

La longueur de AB et CD sont L_{AB} et L_{CD} respectivement.

Quel doit être le ratio t_{AB}/t_{CD} des épaisseurs des poutres pour que les quatre réactions (en A,B, C, et D) soient les mêmes ?

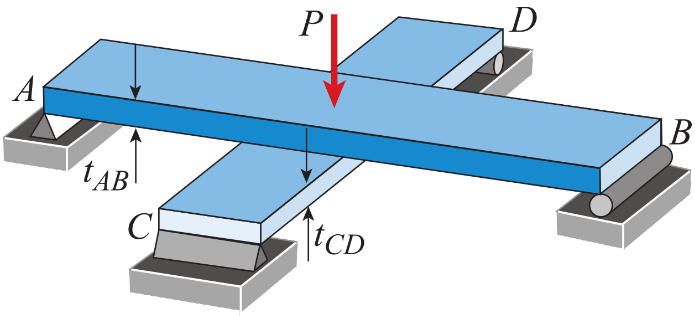


Figure 9a.5.1 | Poutres croisées, en appui simples à leurs extrémités.

9b.5. Solution

<u>What is given?</u>

- Beams are simply-supported
- Beams have the same width
- P is applied in the middle of the two beams
- The two beams are in contact before applying the load
- Materials are the same

<u>Assumptions</u>

The material is homogeneous and isotropic.

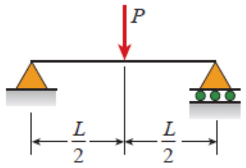
<u>What is asked?</u>

Ratio t_{AB}/t_{CD} in order to get the same reactions.

<u>Principles and formula</u>

If we want the four reactions to be the same, each beam must carry exactly one-half of the load P. Given that the beams are in contact at the start, their deflection at their central point is going to be the same.

We can use the formula for a simply-supported beam (see Figure 9a.5.2).



$$v = -\frac{Px}{48EI}(3L^2 - 4x^2) \quad v' = -\frac{P}{16EI}(L^2 - 4x^2) \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\delta_C = \delta_{max} = \frac{PL^3}{48EI} \quad \theta_A = \theta_B = \frac{PL^2}{16EI}$$

Figure 9a.5.2 | Formulas for the deflexion of simply-supported beams with a load in their center.

$$\delta_{AB}\left(\frac{L_{AB}}{2}\right) = \frac{\left(\frac{P}{2}\right)L_{AB}^3}{48EI_{AB}}$$

$$\delta_{CD}\left(\frac{L_{CD}}{2}\right) = \frac{\left(\frac{P}{2}\right)L_{CD}^3}{48EI_{CD}}$$

Given that the deflections are equal :

$$\frac{L_{AB}^3}{I_{AB}} = \frac{L_{CD}^3}{I_{CD}}$$ note that I_{AB} scales at t^3

$$\boldsymbol{\frac{t_{AB}}{t_{CD}} = \frac{L_{AB}}{L_{CD}}}$$

9b.6. Poutres avec support élastique

Deux poutres CE et FD sont assemblées en un point D (voir Figure 9a.6.1). La poutre CE est encastrée à une extrémité et libre à son autre extrémité, elle est soumise à une charge uniforme q. La poutre FD joue le rôle d'un support élastique au point D. On suppose qu'elle n'exerce que des efforts verticaux (dans la direction y).

Le moment d'inertie, la surface de la section et le module de Young de la poutre CE sont notés I_P, A_P et E_P. Ceux de la poutre FD sont notés I_R, A_R et E_R. Les distances CD, DE et FD sont respectivement L_1, L_2 et L_3.

Donnez l'expression de la flèche au point D.

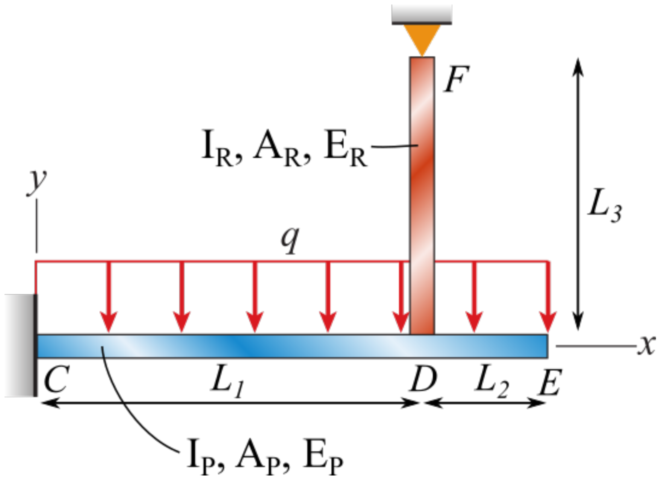


Figure 9a.6.1 | Assemblage de poutres. CE est soumise à une charge uniforme q. FD joue le rôle de support élastique.

9b.6. Solution

First, we can separate the system into two subsystems and draw their free body diagram as shown in Figure 9a.6.2.

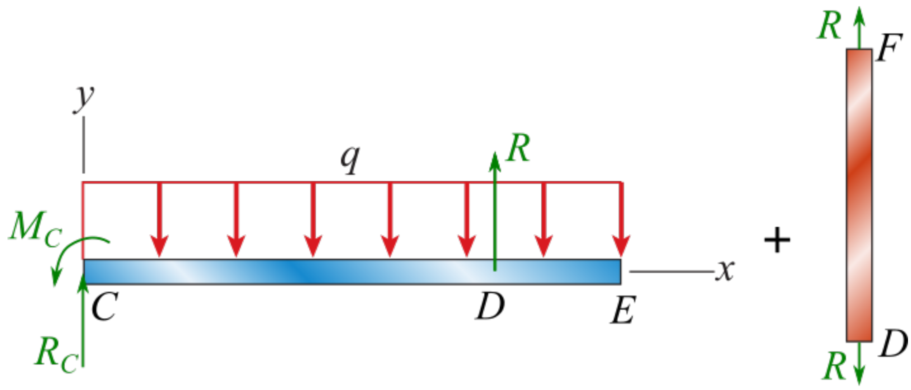


Figure 9a.6.2 | Free body diagrams of the two subsystems.

The subsystem FD is in tension (ie the spring is stretched down and pulls upward), so we can write the deflection of point D as :

$$w_D = -\frac{R}{k} \qquad (9a.6.1)$$

Where k is the equivalent stiffness of the bar FD.

The deflection at point D can also be written by applying flexion theory on beam CE. For this, we can separate the load into two different loading conditions (see Figure 9a.6.3) and sum up the corresponding beam deflection equations.

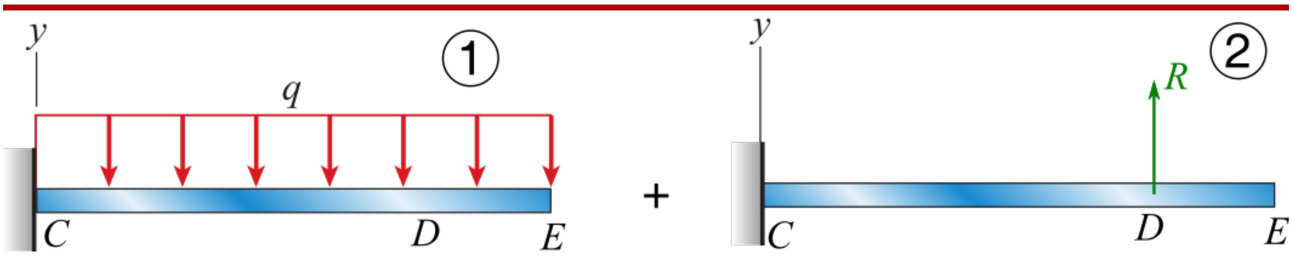


Figure 9a.6.3 | Loading is divided into two different simple configurations that can be added.

Using the formulas for simple loading case of clamped beams, we can write :

$$w_1(x) = -\frac{qx^2}{24E_pI_p}[6(L_1+L_2)^2 - 4(L_1+L_2)x + x^2] \tag{9a.6.2}$$

$$w_2(x) = \frac{Rx^2}{6E_pI_p}[3L_1 - x] \qquad x \le L_1 \tag{9a.6.3}$$

We can then add these two equations to get the deflection at point D.

$$w_D = w_1(L_1) + w_2(L_1) \tag{9a.6.4}$$

$$w_D = -\frac{qL_1^2}{24E_pI_p}[6(L_1+L_2)^2 - 4(L_1+L_2)L_1 + L_1^2] + \frac{RL_1^2}{6E_pI_p}[3L_1 - L_1] \tag{9a.6.5}$$

$$w_D = -\frac{qL_1^2}{24E_pI_p}[3L_1^2 + 8L_1L_2 + 6L_2^2] + \frac{RL_1^3}{3E_pI_p} \tag{9a.6.6}$$

Using compatibility of (9a.6.1) and (9a.6.4), we can write:

$$-\frac{R}{k} = -\frac{q(L_1)^2}{24E_pI_p}[3L_1^2 + 8L_1L_2 + 6L_2^2] + \frac{RL_1^3}{3E_pI_p} \tag{9a.6.7}$$

$$R\left[-\frac{1}{k} - \frac{L_1^3}{3E_pI_p}\right] = -\frac{q(L_1)^2}{24E_pI_p}[3L_1^2 + 8L_1L_2 + 6L_2^2] \tag{9a.6.8}$$

Given that the stiffness k can be computed as:

$$k = \frac{E_RA_R}{L_3} \tag{9a.6.9}$$

It is possible to write the reaction force R at point D as:

$$R = \frac{q(L_1)^2}{24E_pI_p}\frac{[3L_1^2 + 8L_1L_2 + 6L_2^2]}{\left[\frac{L_3}{E_RA_R} + \frac{L_1^3}{3E_pI_p}\right]} \tag{9a.6.10}$$

And the deflection as:

$$w_D = -\frac{R}{k} = -\frac{q(L_1)^2}{24E_pI_p}\frac{[3L_1^2 + 8L_1L_2 + 6L_2^2]}{\left[\frac{L_3}{E_RA_R} + \frac{L_1^3}{3E_pI_p}\right]}\frac{L_3}{E_RA_R} \tag{9a.6.11}$$

$$= -\frac{qL_3(L_1)^2}{8}\frac{[3L_1^2 + 8L_1L_2 + 6L_2^2]}{[3L_3E_pI_p + E_RA_RL_1^3]} \tag{9a.6.12}$$