• Result 2: let $f:\mathbb{R}\to\mathbb{R}$ a T-periodic piecewise function

Then: $$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx \quad \forall\, a\in\mathbb{R}$$

Proof:

$$\int_a^{a+T} f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^T f(x)\,dx + \int_T^{a+T} f(x)\,dx$$

Where $\int_T^{a+T} f(x)\,dx \overset{y=x-T}{=} \int_0^a f(y+T)\,dy \overset{f \text{ is } T\text{-periodic}}{=} \int_0^a f(y)\,dy = -\int_a^0 f(y)\,dy$

Then $\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$ □