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| EPFL - Autumn 2021 | Dr. Pablo Antolin |
| **Analysis III** SV MT | **Exercises** |
| Serie 11 | December, 9 |

**Note:** several exercises are directly taken from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

*Hint: For the following exercises, we suggest to :*

1. *Start by sketching the graph of $f$ and the graph of $f'$,over at least two periods*
2. *Verify that the function $f$ is a piecewise $C^1$ function.*

**Exercise 1** (Ex 14.5 page 220)**.**

1. Compute the Fourier series of the $2\pi$-periodic and odd function $f$ which is defined by $f(x) = x(\pi - x)$ for $x \in [0, \pi]$.
2. With the help of question 1 and Parseval's identity, deduce the sum of the series :
$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^6}.$$

**Exercise 2** (Ex 14.6 page 220)**.**
With the help of Parseval's identity, show that :

$$\int_{-\pi}^{\pi} \cos^4 x \, dx = \frac{3}{4}\pi.$$

**Exercise 3** (Ex 14.7 page 220)**.**

1. Compute the Fourier series of the following $2\pi$-periodic function defined by :
$$f(x) = |x| \quad \text{for} \quad x \in [-\pi, \pi[.$$
2. Deduce the sums of the following series :
$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4} \ .$$

**Exercise 4.**
Let $f : [0, \pi] \to \mathbb{R}$ be the function defined by $f(x) = x$.

1. Compute the Fourier sine and cosine series denoted here $F_c f$ for the cosine form and $F_s f$ for the sinus form.
2. Compare $F_c f$, $F_s f$, and $f$ over $[0, \pi]$.
3. Deduce the values for the two following sums :

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} \qquad \text{and} \qquad \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}.$$

4. With the help of Parseval's identity, determine the value of the following sum :

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4}.$$

**Exercise 5.**
Let $f$ be the $2\pi$-periodic function defined in the exercise 1 (14.5). Let's define

$$F(x) = \int_0^x f(y)\, dy, \quad x \in \mathbb{R}.$$

1. Is $F$ a $2\pi$-periodic function ? Justify your answer.
2. Find the Fourier series of $F$ with the use of the Fourier series of $f$ (express also the constant term $a_0$ of the series of $F$ explicitly).