EPFL - Autumn 2021	Dr. Pablo Antolin
Analysis III SV MT	**Exercises**
Serie 4	October, 21

Note: several exercises are directly taken from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Exercise 1 (Ex 3.1 page 27).
Let $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector fields defined as :

$$F_1(x, y) = (y, xy - x), \quad F_2(x, y) = (3x^2y + 2x, x^3), \quad F_3(x, y) = (3x^2y, x^2).$$

Does the vector field F_i derive from a potential on $\mathbb{R}^2$?
If yes, find a potential function from which F_i is deriving. If not find a closed path Γ such that: $\int_\Gamma F_i \cdot dl \neq 0$.

Exercise 2 (Ex 3.3 page 28).
Let $F(x, y, z) = \left(2xy + \dfrac{z}{1 + x^2},\, x^2 + 2yz,\, y^2 + \arctan x\right)$.

Does the vector field F derive from a potential on $\mathbb{R}^3$? If yes, find this potential.

Exercise 3 (Ex 3.8 page 29).
Let:

$$F(x, y) = \left(\frac{-x}{(x^2 + y^2)^2},\, \frac{-y}{(x^2 + y^2)^2}\right),$$

and

$$G(x, y) = \left(\frac{y^3}{(x^2 + y^2)^2},\, \frac{-xy^2}{(x^2 + y^2)^2}\right),$$

defined on $\Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$.
Do they derive from a potential on Ω? (If yes find a potential function, if no justify your answer.)

Exercise 4 (Ex 3.6 page 28).
Let us define the differential equation :

$$F_2\left(t, u(t)\right) u'(t) + F_1\left(t, u(t)\right) = 0 \quad \text{for } t \in \mathbb{R}.$$

Let $F(x, y) = (F_1(x, y),\ F_2(x, y))$ be a vector field which derives from a potential f on $\mathbb{R}^2$.

1. Show that a solution $u(t)$, in implicit form, of this differential equation is given by :

   $$f\left(t, u(t)\right) = \text{constant} \qquad \text{for all } \ t \in \mathbb{R}.$$

   Indication: Calculate $\dfrac{d}{dt} f\left(t, u(t)\right)$.

2. Deduce a solution of:

   $$u^2(t)u'(t) + \sin t = 0, \text{ for } t \in \mathbb{R} \text{ with the initial condition } u(0) = 3.$$

Exercise 5 (Ex 3.2 page 27).
Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field such that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, defined by $F(u, v) = (f(u, v), g(u, v))$.
Let:

$$\varphi(x, y) = \int_0^1 \left[x f(tx, ty) + y g(tx, ty)\right] dt.$$

1. Show that if $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial v}$, then $F(x, y) = \operatorname{grad} \varphi(x, y)$.
2. Deduce a potential for $F(x, y) = (2xy, x^2 + y)$.
3. Generalize the result shown in question 1 to $\mathbb{R}^n$, *i.e.* in the case that:

   $$F : \mathbb{R}^n \to \mathbb{R}^n;\ u \mapsto F(u) = (F_1(u), \ldots, F_n(u)).$$