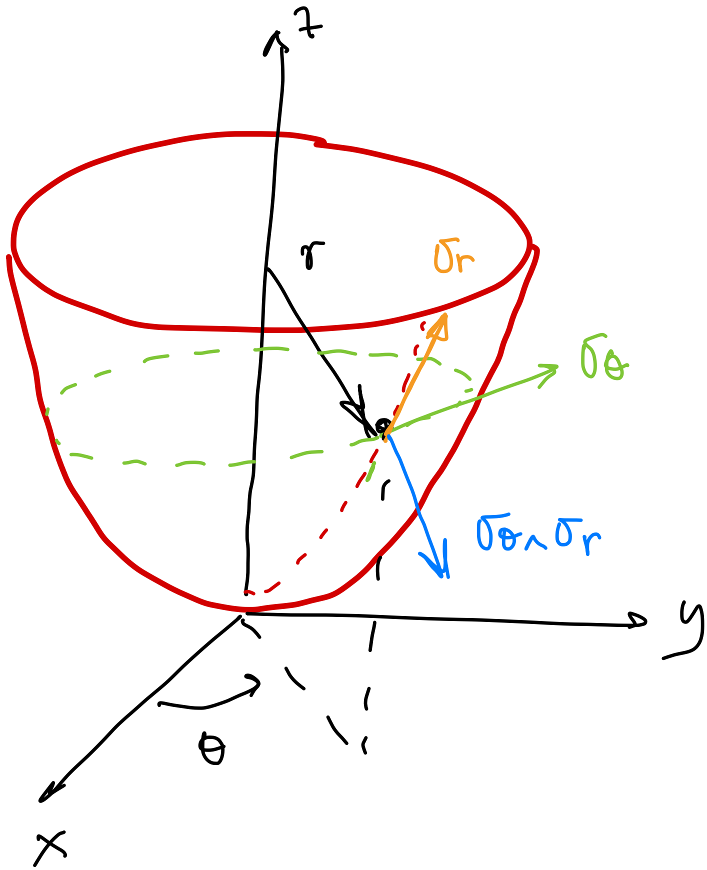
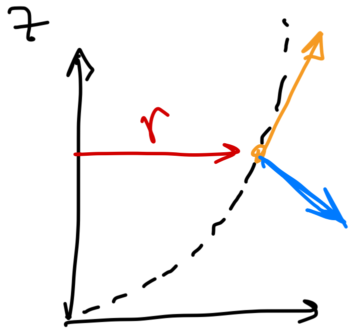




04/11/2021

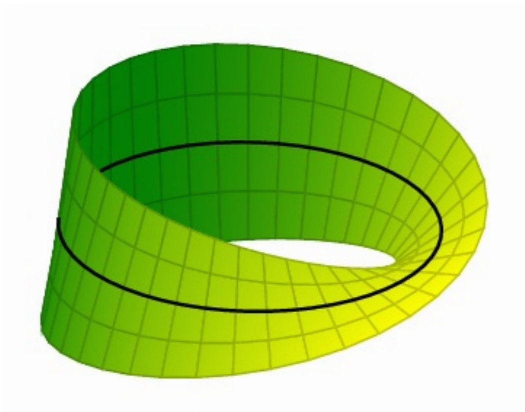
- Definition 3:

A (piecewise) regular surface is said to be orientable if $\exists$ a field of unit normals $\nu : \Sigma \to \mathbb{R}^3$ continuous.

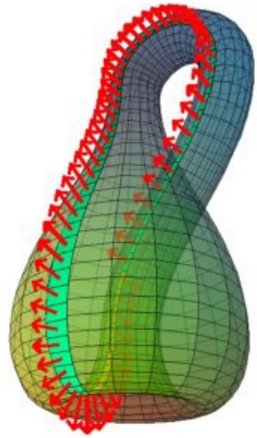
Such field of normals is called a orientation of Σ

Examples of non-orientable surfaces:

Möbius strip



Klein bottle



Remember...

3.2 Surface integrals

3.2.1 Scalar fields integrals

- Definition: let $\Sigma \subset \mathbb{R}^3$ be a regular surface parameterized by $\sigma : \bar{A} \to \Sigma$
$$(u,v) \mapsto \sigma(u,v)$$

and let be $f : \Sigma \to \mathbb{R}$ be a continuous
$$x \mapsto f(x)$$

scalar field.

The integral of f over Σ is defined by

$$\iint_\Sigma f \, ds = \iint_A f(\sigma(u,v)) \, \|\sigma_u \wedge \sigma_v\| \, du\,dv$$

- Remark : a) analogy with curves

$$\gamma : [a,b] \to \Gamma \qquad\qquad g : \Gamma \to \mathbb{R}$$
$$t \mapsto \gamma(t) \qquad\qquad x \mapsto g(x)$$

$$\int_\Gamma f \, dl = \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \, dt$$

b) Computing the area of Σ:

$$f=1 \longrightarrow \int_\Sigma 1 \, ds = \iint_A \|\sigma_u \wedge \sigma_v\| \, du dv$$

length of curve Γ: $\int_\Gamma 1 \, dl = \int_a^b \|\gamma'(t)\| \, dt$

c) If ρ is the surface's density, then we compute the mass as:

$$\iint_\Sigma \rho \, ds = \iint_A \rho(\sigma(u,v)) \, \|\sigma_u \wedge \sigma_v\| \, du dv$$

d) If $\Sigma = \bigcup_{i=1}^{K} \Sigma_i$ (piecewise surface) then:

$$\iint_\Sigma f \, ds = \sum_{i=1}^{K} \iint_{\Sigma_i} f \, ds$$

3.2.2 Examples

- Example 1: let be $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=R^2\}$

  Compute the area of Σ. (Sphere of radius R).

Parameterization of Σ (see Example 1 of § 3.1.2)

$A = \,]0, 2\pi[\times]0, \pi[$

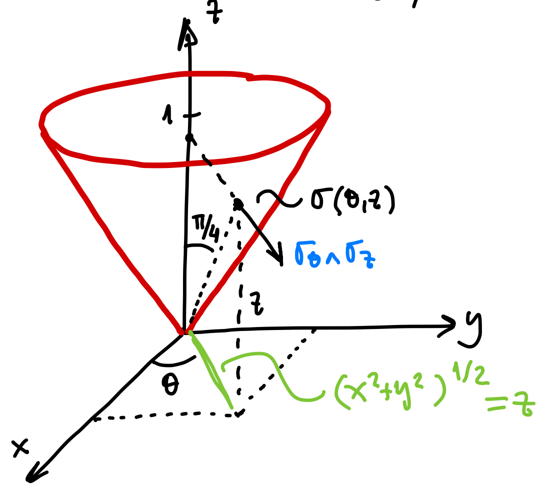
$$\sigma : \bar{A} \longrightarrow \Sigma$$
$$(\theta,\varphi) \longmapsto \sigma(\theta,\varphi) = (R\sin\varphi\cos\theta,\ R\sin\varphi\sin\theta,\ R\cos\varphi)$$

$$\|\sigma_\theta \wedge \sigma_\varphi\| = R|\sin\varphi|\ \|\sigma(\theta,\varphi)\| \underset{\varphi\in\,]0,\pi[}{=} R\sin\varphi\ \|\sigma(\theta,\varphi)\|$$

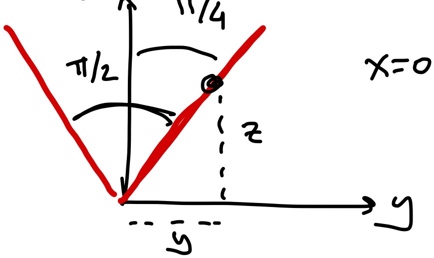
$$\left(\ \|\sigma(\theta,\varphi)\| = R\ \right) \qquad = R^2\sin\varphi$$

$$\text{Area}(\Sigma) = \iint_\Sigma 1\,ds = \iint_A 1\,\|\sigma_\theta\wedge\sigma_\varphi\|\,d\theta\,d\varphi$$
$$= \int_0^{2\pi}\int_0^{\pi} R^2\sin\varphi\,d\theta\,d\varphi = R^2\,2\pi\int_0^\pi \sin\varphi\,d\varphi$$
$$= 2\pi R^2\left[-\cos\varphi\right]\Big|_0^\pi = 4\pi R^2$$

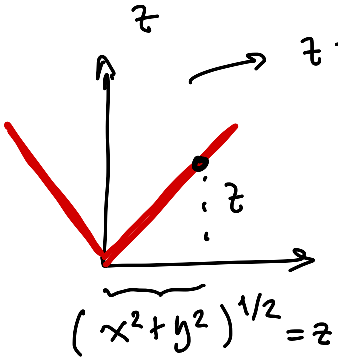
- Example 2: let $\Sigma = \{(x,y,z)\in\mathbb{R}^3 : z^2 = x^2+y^2,\ 0\le z\le 1\}$.



Cone of angle $\pi/2$



$x=0$



$$z = (x^2+y^2)^{1/2} \longrightarrow z^2 = x^2 + y^2$$

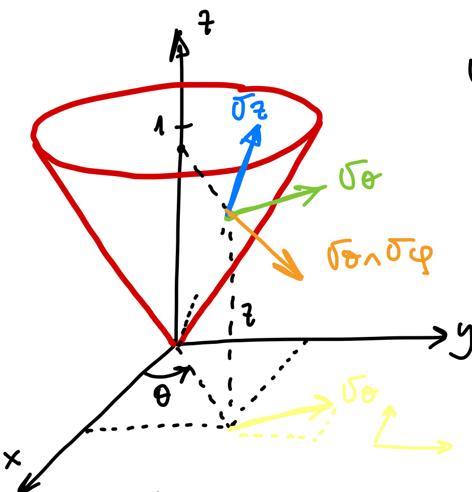
$$A =]0, 2\pi[\times]0, 1[$$

$$\sigma : A \longrightarrow \mathbb{R}^3$$

$$(\theta, z) \longmapsto (z\cos\theta, z\sin\theta, z)$$

$$\sigma_\theta = \frac{\partial \sigma}{\partial \theta} = (-z\sin\theta, z\cos\theta, 0)$$

$$\sigma_z = \frac{\partial \sigma}{\partial z} = (\cos\theta, \sin\theta, 1)$$



$$\sigma_\theta \wedge \sigma_z = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ -z \end{pmatrix}$$

$$\|\sigma_\theta \wedge \sigma_z\| = \sqrt{2}\, z$$

Compute: $\iint_\Sigma f \, dS$ s.t. $f : \Sigma \longrightarrow \mathbb{R}$

$$(x, y, z) \longmapsto xy + z^2$$

$$\iint_\Sigma f \, dS = \iint_A f(\sigma(\theta, z)) \, \|\sigma_\theta \wedge \sigma_z\| \, d\theta \, dz$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 \underbrace{(z^2 \sin\theta\cos\theta + z^2)}_{f(\sigma(\theta,z))} \, z \, d\theta \, dz$$

$$(z\cos\theta, z\sin\theta, z)$$

$$= \sqrt{2} \int_0^{2\pi} (\sin\theta \cos\theta + 1)\, d\theta \int_0^1 z^3 dz = \frac{\sqrt{2}}{4} \int_0^{2\pi} (\sin\theta \cos\theta + 1)\, d\theta$$

$$\int_0^{2\pi} \sin\theta \cos\theta \, d\theta = \left[\frac{1}{2} \sin^2\theta \right] \Big|_0^{2\pi} = 0$$

$$\iint_\Sigma f \, ds = \frac{\sqrt{2}}{4} \int_0^{2\pi} 1 \, d\theta = \frac{\sqrt{2}}{2} \pi$$

3.2.3 Vector fields integrals

- Definition: let $\Sigma \subset \mathbb{R}^3$ be orientable and regular

parameterized by $\sigma : \bar{A} \to \Sigma \subset \mathbb{R}^3$ and
$(u,v) \mapsto \sigma(u,v) \in \mathbb{R}^3$

let $F : \Sigma \to \mathbb{R}^3$
$x \mapsto F(x) = (F_1(x), F_2(x), F_3(x))$ be a

continuous vector field.

The integral of F over Σ in the direction $\sigma_u \wedge \sigma_v$ is

defined by:

$$\iint_\Sigma F \cdot ds = \iint_A F(\sigma(u,v)) \cdot (\sigma_u \wedge \sigma_v) \, du \, dv$$

$$\nu(u,v) = \frac{\sigma_u(u,v) \wedge \sigma_v(u,v)}{\| \sigma_u(u,v) \wedge \sigma_v(u,v) \|}$$

$$\iint_\Sigma F\cdot ds = \iint_A F(\sigma(u,v))\cdot \nu(u,v)\ \|\sigma_u\wedge\sigma_v\|\ du\,dv$$

Remarks: a) Analogy with curves:

$$\gamma:[a,b]\to\Gamma \qquad\qquad G:\Gamma\to\mathbb{R}^3$$
$$t\mapsto\gamma(t) \qquad\qquad x\mapsto G(x)\in\mathbb{R}^3$$

$$\int_\Gamma G\cdot dl = \int_a^b G(\gamma(t))\cdot\gamma'(t)\,dt$$

b) The integral $\iint_\Sigma F\cdot ds$ computes the flux of F through the surface Σ in the direction normal to Σ.

c) For a piecewise regular surface $\Sigma = \bigcup_{i=1}^{k}\Sigma_i$

$$\iint_\Sigma F\cdot ds = \sum_{i=1}^{k}\iint_{\Sigma_i} F\cdot ds$$

3.2.4 Examples:

- Example 1: let $\Sigma = \{(x,y,z)\in\mathbb{R}^3 : z^2 = x^2+y^2,\ 0\le z\le 1\}$ and the vector field $F:\Sigma\to\mathbb{R}^2$ s.t. $F(x,y,z) = (y,-x,z^2)$

Compute the flux of F through Σ along the ascending direction

(Parameterization of Example 2 of § 3.2.2)

$A =]0, 2\pi[\times]0, 1[\,, \quad \sigma(\theta, z) = (z\cos\theta, z\sin\theta, z)$

$$\sigma_\theta \wedge \sigma_z = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ -z \end{pmatrix}$$ is an "outer" normal is descending

$$\iint_\Sigma F \cdot dS = -\iint_A F(\sigma(\theta, z)) \cdot (\sigma_\theta \wedge \sigma_z)\, d\theta\, dz$$

$$= \iint_A F(\sigma(\theta, z)) \cdot (-\sigma_\theta \wedge \sigma_z)\, d\theta\, dz$$

$$= \iint_A F(\sigma(\theta, z)) \cdot (\sigma_z \wedge \sigma_\theta)$$

$$= -\int_0^{2\pi} \int_0^1 \underbrace{(z\sin\theta, -z\cos\theta, z^2)}_{F(\sigma(\theta, z))} \cdot (z\cos\theta, z\sin\theta, -z)\, dz\, d\theta$$

$$= -\int_0^{2\pi} \int_0^1 (\cancel{\sin\theta\cos\theta} - \cancel{\cos\theta\sin\theta} - z)\, z^2 = \int_0^{2\pi} \int_0^1 z^3\, dz\, d\theta$$

$$= 2\pi \frac{1}{4} = \frac{\pi}{2}$$

3.3 Divergence Theorem

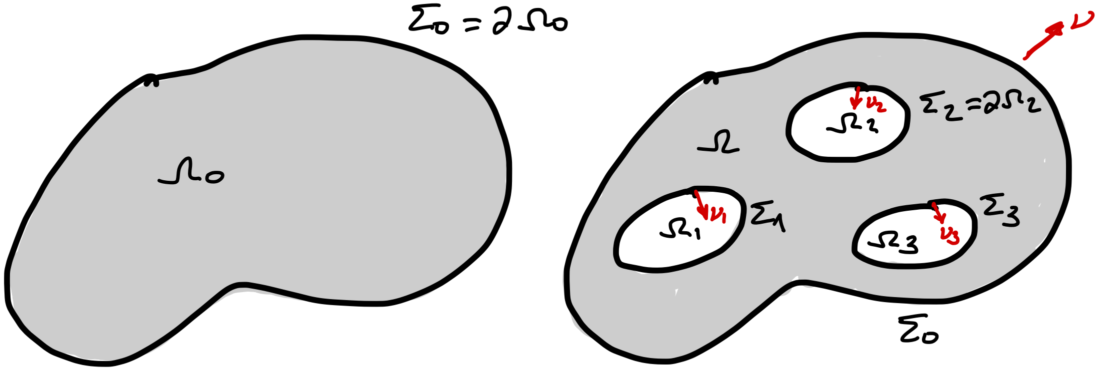
3.3.1 Notation and preliminary results

- Definition: We say that $\Omega \subset \mathbb{R}^3$ is a regular domain if there exist bounded open domains such that $\Omega_0, \Omega_1, \dots, \Omega_m \subset \mathbb{R}^3$
  - $\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \bar{\Omega}_j$
  - $\bar{\Omega}_j \subset \Omega_0, \ \forall\ j = 1, 2, \dots, m$
  - $\bar{\Omega}_i \cap \bar{\Omega}_j = 0$ if $i \neq j$, $i, j = 1, \dots, m$
  - $\partial \Omega_j = \Sigma_j$, $j = 0, 1, \dots, m$ where Σ_j are (piecewise) orientable surfaces and such that $\partial \Sigma_j = \emptyset$

$$\partial(\partial \Omega_j) = \phi$$

In addition, $\exists$ a (piecewise) continuous vector field of <u>unit outer normals</u> ν of Ω.

This is a 2D design, but you must think on 3D bodies.



This is a generalization of the analogous definition given for a regular domain A in the plane $\mathbb{R}^2$. (§ 2.4.1)

3.3.2 Divergence theorem

- Theorem: let $\Omega \subset \mathbb{R}^3$ be a regular domain and $\nu : \partial\Omega \to \mathbb{R}^3$ an outer unit normal vector field of Ω defined by $\nu = (\nu_1, \nu_2, \nu_3)$. let $F : \bar{\Omega} \to \mathbb{R}^3$ be a vector field such that $F \in C^1(\bar{\Omega}, \mathbb{R}^3)$ defined by $F = (F_1, F_2, F_3)$. Then:

$$\iiint_\Omega \operatorname{div} F(x,y,z)\, dx\, dy\, dz = \iint_{\partial\Omega} F \cdot \nu \, dS$$

in other words

$$\iiint_{\Omega} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

$$= \iint_{\partial\Omega} \left(F_1 \nu_1 + F_2 \nu_2 + F_3 \nu_3 \right) dS$$