7/10/21.

2) § 1.2.2 Chapter 1 → example gravitational field.

$$F(x,y,z) = -\frac{c}{r^3}(x,y,z)$$

with $c = gmM$ and $r = \sqrt{x^2+y^2+z^2}$

this field derives from a potential.

The potential is $f(x,y,z) = \frac{c}{r} + \alpha$, $\alpha \in \mathbb{R}$.

$$F = \text{grad}\left(\frac{c}{r} + \alpha\right) = \text{grad}\left(\frac{c}{r}\right)$$

2.3.2 Important results

- Theorem 1: let $\Omega \subset \mathbb{R}^n$ be an open domain and

$$F : \Omega \to \mathbb{R}^n$$
$$x \mapsto F(x) = (F_1(x), F_2(x), \dots, F_n(x))$$

a vector field s.t. $F \in C^1(\Omega, \mathbb{R}^n)$

a) Necessary condition: If F derives from a potential on Ω, then

$$(*) \quad \frac{\partial F_i}{\partial x_j}(x) = \frac{\partial F_j}{\partial x_i}(x), \ \forall i,j = 1,\dots,n \text{ and } \forall x \in \Omega$$

b) sufficient condition: If (*) holds and if Ω is <u>convex and/or simply connected</u> then F derives from a potential on Ω.

Remarks:

1- The condition (*) it is necessary but not sufficient.

2- The condition (*) is equivalent to curl $F=0$ ($\text{rot } F=0$). Indeed, $n=2$, $F=(F_1, F_2)$

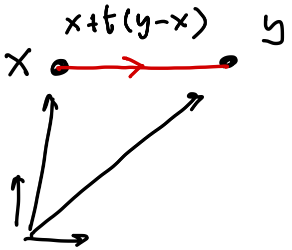
$$\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1} \iff \text{curl } F = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} = 0$$

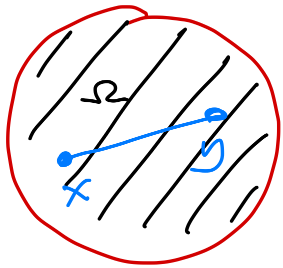
But the same applies to $n=3$.

Recalls:

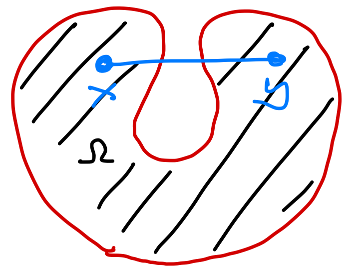
1) $\Omega \subset \mathbb{R}^n$ is convex if $\forall t \in [0,1]$ and $\forall x, y \in \Omega$ we have: $x + t(y-x) \in \Omega$

(i.e, the segment joining the points x and y is fully contained in Ω).

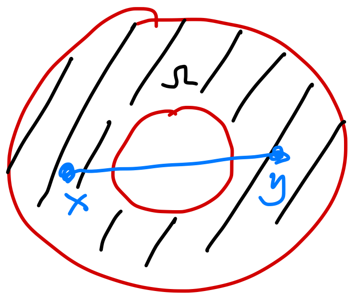




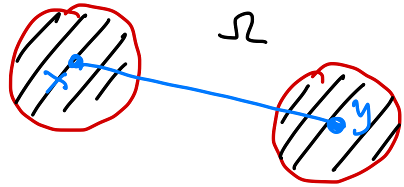
convex
simply connected



non-convex
simply connected



non-convex
non-simply connected

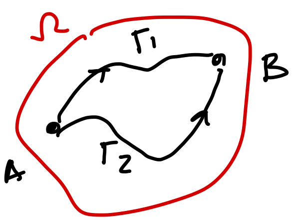


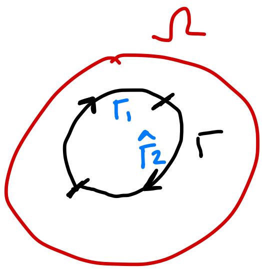
non-convex.
non-simply connected.

2) $\Omega \subset \mathbb{R}^n$ is simply connected if $\forall\, x, y \in \Omega$ there exists a family of curves Γ that joins x and y that is fully contained in Ω, and, being Γ_1 and Γ_2 two of those curves they can be deformed without leaving Ω.

- Theorem 2: let $\Omega \subset \mathbb{R}^n$ be an open domain and let $F : \Omega \to \mathbb{R}^n$ a continuous vector field. $(F \in C^0(\Omega, \mathbb{R}^n))$. Then the 3 statements below are equivalent:

  1) F derives from a potential.
  2) $\int_{\Gamma_1} F \cdot dl = \int_{\Gamma_2} F \cdot dl$ for <u>all</u> the (piecewise) regular simple curves Γ_1 and $\Gamma_2 \subset \Omega$ joining every pair of points $A, B \subset \Omega$
  3) $\int_{\Gamma} F \cdot dl = 0$ if Γ (piecewise) regular simple and <u>close</u> curve.





$$\int_{\Gamma} F \cdot dl$$
$$= \int_{\Gamma_1} F \cdot dl + \int_{\hat{\Gamma}_2} F \cdot dl$$
$$= \int_{\Gamma_1} F \cdot dl - \int_{\Gamma_2} F \cdot dl$$

- Summary of the use of Theorems 1 and 2 :

Let $\Omega \subset \mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^n$, $F \in C^1(\Omega, \mathbb{R}^n)$

a) If $\operatorname{curl} F \neq 0 \Rightarrow F$ does not derive from a potential

b) If $\operatorname{curl} F = 0$ on Ω convex and/or simply connected

$\Rightarrow F$ derives from a potential.

c) If $\operatorname{curl} F = 0$ on Ω is not convex and is not simply connected $\Rightarrow$ Theorem 1 does not provide any inform.

d) If we find <u>at least one</u> closed curve $\Gamma \subset \Omega$ such that $\int_\Gamma F \cdot dl \neq 0 \Rightarrow F$ does not derive from a potential.

e) If we find <u>one</u> closed curve $\Gamma \subset \Omega$ s.t.

$\int_\Gamma F \cdot dl = 0 \not\Rightarrow F$ derives from a potential.

2.3.2 Examples:

- Example 1: study if F derives from a potential. If so, find the potential.

a) $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto F(x,y) = \left(4x^3y^2 ,\ 2x^4y + y\right)$

$\underline{\Omega = \mathbb{R}^2 \text{ is convex and simply connected}}$

$\operatorname{curl} F \overset{?}{=} 0 \quad , \quad \operatorname{curl} F = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$

$= \dfrac{\partial}{\partial x}\left(2x^4y + y\right) - \dfrac{\partial}{\partial y}\left(4x^3y^2\right) = 0$

<u>F derives from a potential</u>

We want to find $f : \Omega \to \mathbb{R}$ s.t. $\operatorname{grad} f = F$

$\operatorname{grad} f = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) = (F_1, F_2)$

$$\frac{\partial f}{\partial x} = 4x^3y^2 \qquad (1)$$

$$\frac{\partial f}{\partial y} = 2x^4y + y \qquad (2)$$

$(1) \rightarrow \int \frac{\partial f}{\partial x} dx \rightarrow f(x,y) = \int 4x^3y^2 dx + \alpha(y)$

$f(x,y) = x^4y^2 + \alpha(y)$

$(2) \rightarrow \frac{\partial f}{\partial y} = \cancel{2x^4y} + \alpha'(y) = F_2 = \cancel{2x^4y} + \underline{y}$

$\alpha'(y) = \underline{y} \longrightarrow \alpha(y) = \frac{1}{2}y^2 + \beta \ ; \ \beta \in \mathbb{R}$

$f(x,y) = x^4y^2 + \frac{1}{2}y^2 + \beta$

b) $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \mapsto F = \left(ye^x \sin z,\ 1 + e^x \sin z,\ ye^x \cos z + z\right)$

$\underline{\Omega = \mathbb{R}^3 \text{ is convex and simply connected}}$

$\operatorname{curl} F \stackrel{?}{=} 0 \longrightarrow$ yes, $\operatorname{curl} F = 0$

<u>F derives from a potential</u>

We want to find $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ s.t. $\operatorname{grad} f = F$.

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (F_1, F_2, F_3)$

$$\frac{\partial f}{\partial x} = F_1 = ye^x \sin z \qquad (1)$$

$$\frac{\partial f}{\partial y} = F_2 = 1 + e^x \sin z \qquad (2)$$

$$\frac{\partial f}{\partial z} = F_3 = ye^x \cos z + z \qquad (3)$$

$$(1) \longrightarrow \quad f(x,y,z) = \int ye^x \sin z \, dx + \alpha(y,z)$$

$$= ye^x \sin z + \alpha(y,z)$$

$$(2) \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(ye^x \sin z + \alpha(y,z) \right) = F_2 = 1 + e^x \sin z$$

$$\cancel{e^x \sin z} + \frac{\partial \alpha}{\partial y}(y,z) = 1 + \cancel{e^x \sin z}$$

$$\frac{\partial \alpha}{\partial y}(y,z) = 1 \longrightarrow \quad \alpha(y,z) = y + \beta(z)$$

$$(3) \Rightarrow \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(ye^x \sin z + y + \beta(z) \right) = F_3 = ye^x \cos z + z$$

$$\cancel{ye^x \cos z} + 0 + \beta'(z) = \cancel{ye^x \cos z} + z$$

$$\beta'(z) = z \longrightarrow \quad \beta(z) = \frac{1}{2} z^2 + c, \qquad c \in \mathbb{R}$$

$$\alpha(y,z) = y + \frac{1}{2} z^2 + c$$

$$f(x,y) = y e^{x} \sin z + y + \frac{1}{2} z^2 + c$$

c) $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$(x,y,z) \longmapsto F = \left(3x^2, \ 2xz - y, \ z \right)$$

- Example 2: Let be a vector field

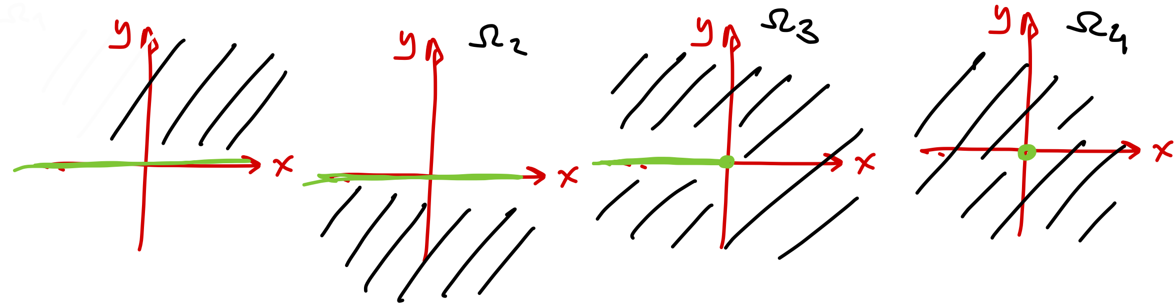
$$F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$$
$$(x,y) \longmapsto F(x,y) = \left(\frac{-y}{x^2+y^2}, \ \frac{x}{x^2+y^2} \right)$$

and consider the following sets:

$\Omega_1 = \{ (x,y) \in \mathbb{R}^2 : y > 0 \}$; $\Omega_2 = \{ (x,y) \in \mathbb{R}^2 : y < 0 \}$

$\Omega_3 = \mathbb{R}^2 \setminus \{ (x,y) \in \mathbb{R}^2 : x \leq 0 \text{ and } y = 0 \}$

$\Omega_4 = \mathbb{R}^2 \setminus \{(0,0)\} = \operatorname{dom}(F)$



$$\Omega_1 \subset \Omega_3 \subset \Omega_4 \quad , \quad \Omega_2 \subset \Omega_3 \subset \Omega_4$$

$$\text{curl } F = 0 \quad \forall\, (x,y) \in \Omega_4 = \text{dom}\,(F)$$