CHAPTER 1: DIFFERENTIAL OPERATORS

1.1 Recalls, notation and terminology

- For $n \in \mathbb{N}$, s.t. $n > 1$ we denote $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ n-tuple

- For $n=2$ $\quad (x_1, x_2) = (x, y)$, for $n=3$ $\quad (x_1, x_2, x_3) = (x, y, z)$

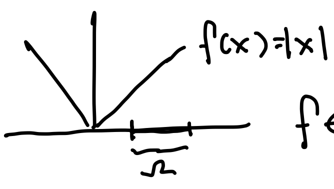
- Scalar field (champ) defined in $\Omega \subset \mathbb{R}^n$

$$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = f(x_1, x_2, \dots, x_n)$$

- Vector field

$$F : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad , \quad m \in \mathbb{N}, m > 1$$
$$x \longmapsto F(x) = (F_1(x_1, x_2, \dots, x_n), F_2(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n))$$

- For $k \in \mathbb{N}$ if $f \in C^k(\Omega)$ all the derivatives of f with order $\leq k$ exist and are continuous in Ω.



$f \in C^\infty(\Omega)$

- For $k \in \mathbb{N}$ we say that $F \in C^k(\Omega, \mathbb{R}^m)$ if $F_i \in C^k(\Omega)$, for $i = 1, 2, \dots, m$.

- Remark: frequently we write $f = f(x)$ or $F = F(x)$ for denoting the field and its image (abuse of notation)

- Differential operator (nabla) ∇, is defined by

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

1.2 The gradient operator

1.2.1 Definition

let be $\Omega \subset \mathbb{R}^n$ an open domain. and $f : \Omega \to \mathbb{R}$
$x \mapsto f(x) = f(x_1, x_2, \dots x_n)$

$f \in C^1(\Omega)$. The gradient of f is denoted as $\operatorname{grad} f(x)$ (or $\nabla f(x)$, $Df(x)$), is defined by

$$\operatorname{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$$

$\text{grad} f(x) : \Omega \longrightarrow \mathbb{R}^n$ (is a vector field)

- For $n=2$, $f = f(x,y)$ and $\text{grad} f(x) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right)$

- For $n=3$, $f(x,y,z)$ and $\text{grad} f(x) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \right)$

1.2.2 Examples:

- Example 1: let be $f : \mathbb{R}^3 \to \mathbb{R}$

$$(x,y,z) \mapsto f(x,y,z) = x^2 y^3 \sin(z^2)$$

compute the gradient.

$$\text{grad} f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \right)$$

$$= \left(2xy^3 \sin(z^2), 3x^2y^2 \sin(z^2), 2x^2y^3 z \cos(z^2) \right) \in \mathbb{R}^3$$

- Example 2: let's consider a particle with mass m placed at $P = (x,y,z) \in \mathbb{R}^3$ and another particle with mass M placed at $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$. The potential of the gravitational field (without considering the sign convention) is a scalar field.

$$f: \mathbb{R}^3 \setminus P_0 \longrightarrow \mathbb{R}$$

$$(x,y,z) \longmapsto f(x,y,z) = \frac{gmM}{r(x,y,z)}$$

$$r(x,y,z) \stackrel{\text{def}}{=} \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

$$\varphi(r) = \frac{c}{r}, \quad c = gmM \text{ (a constant)}$$

Compute grad $f(x,y,z)$.

$$\frac{\partial f}{\partial x} = \frac{d\varphi}{dr}\frac{\partial r}{\partial x} = -\frac{c}{r^2}\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{1/2}\right)$$

$$= \frac{1}{\cancel{2}}\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{-1/2} \cancel{2}(x-x_0) = \frac{x-x_0}{r}$$

$$\frac{\partial f}{\partial x} = -\frac{c}{r^2}\frac{x-x_0}{r}, \quad \frac{\partial f}{\partial y} = \frac{-c}{r^2}\frac{y-y_0}{r}, \quad \frac{\partial f}{\partial z} = \frac{-c}{r^2}\frac{z-z_0}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y-y_0}{r}, \quad \frac{\partial r}{\partial z} = \frac{z-z_0}{r}$$

$$\operatorname{grad} f(x,y,z) = -\frac{c}{r^3}(x-x_0, y-y_0, z-z_0)$$

This is the gravitational force that the particle M applies on m.

1.3 The divergence operator

1.3.1 Definition

let be $\Omega \subset \mathbb{R}^n$ an open domain and $F: \Omega \longrightarrow \mathbb{R}^n$

$x \longmapsto F(x) = (F_1(x), \dots F_n(x))$

a vector field s.t. $F \in C^1(\Omega, \mathbb{R}^n)$

The divergence operator of F, denoted as $\operatorname{div} F(x)$ (or $\nabla \cdot F(x)$), defined by

$$\operatorname{div} F(x) = \frac{\partial F_1}{\partial x_1}(x) + \frac{\partial F_2}{\partial x_2}(x) + \dots + \frac{\partial F_n}{\partial x_n} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}(x)$$

Remark: $\operatorname{div} F(x) \in \mathbb{R}$, so $\operatorname{div} F: \Omega \longrightarrow \mathbb{R}$ is a scalar field.

1.3.2 Examples

- Example 1: let be

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x,y,z) \mapsto F(x,y,z) = \left(x^2 - e^y, \ \sin(xz), \ y^2 e^{2xz} \right)$$

Compute $\operatorname{div} F(x)$.

$$\operatorname{div} F(x) = \frac{\partial F_1}{\partial x}(x,y,z) + \frac{\partial F_2}{\partial y}(x,y,z) + \frac{\partial F_3}{\partial z}(x,y,z)$$

$$= \frac{\partial}{\partial x}\left(x^2 - e^y \right) + \frac{\partial}{\partial y}\left(\sin(xz) \right) + \frac{\partial}{\partial z}\left(y^2 e^{2xz} \right)$$

$$= 2x + 0 + 2xy^2 e^{2xz} = 2x\left(1 + y^2 e^{2xz} \right) \in \mathbb{R}$$

- Example 2: let us consider the gravitational field force of the Example 2 of § 1.2.1 with $P_0 = (0,0,0)$:

$$F : \mathbb{R}^3 \setminus (0,0,0) \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longmapsto F(x,y,z) = -\frac{c}{r^3}(x,y,z)$$

$(F = \operatorname{grad} f)$. Compute $\operatorname{div} F(x,y,z)$, $c = gMm$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}\left(-c \frac{x}{r^3} \right) = -c\left(\frac{1}{r^3} - 3\frac{x}{r^4}\frac{\partial r}{\partial x} \right)$$

$$\left(\frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

$$\frac{\partial F_1}{\partial x} = -c\left(\frac{1}{r^3} - 3\frac{x^2}{r^5}\right) = c\,\frac{3x^2 - r^2}{r^5}$$

$$\frac{\partial F_2}{\partial y} = c\,\frac{3y^2 - r^2}{r^5}, \quad \frac{\partial F_3}{\partial z} = c\,\frac{3z^2 - r^2}{r^5}$$

$$\operatorname{div} F(x,y,z) = c\,\frac{3\overbrace{(x^2+y^2+z^2)}^{r^2} - 3r^2}{r^5} = 0 \in \mathbb{R}$$

1.4 The curl operator (rotational)

1.4.1 Definition

let be $\Omega \subset \mathbb{R}^n$ an open domain and $F : \Omega \to \mathbb{R}^n$

$x \mapsto F(x) = (F_1(x), \ldots, F_n(x))$

a vector field s.t. $F \in C^1(\Omega, \mathbb{R}^n)$

The curl operator (or rotational), denoted $\operatorname{curl} F$ (or $\operatorname{rot} F$ or $\nabla \wedge F$), is defined by

a) when $n = 2$, $\operatorname{curl} F(x,y) = \frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) \in \mathbb{R}$

b) when $n=3$, $\operatorname{curl} F(x,y,z) =$

$$\left(\frac{\partial F_3}{\partial y}(x,y,z) - \frac{\partial F_2}{\partial z}(x,y,z),\ \frac{\partial F_1}{\partial z}(x,y,z) - \frac{\partial F_3}{\partial x}(x,y,z),\ \frac{\partial F_2}{\partial x}(x,y,z) - \frac{\partial F_1}{\partial y}(x,y,z) \right)$$

$$\underbrace{\operatorname{curl} F(x,y,z)}_{\nabla \wedge F} = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

- Remark: for $n=2$, $\operatorname{curl} F(x,y) \in \mathbb{R}$ is a scalar field.

for $n=3$, $\operatorname{curl} F(x,y,z) \in \mathbb{R}^3$ is a vector field.

1.4.2 Examples:

- Example 1: let be $F : \mathbb{R}^3 \to \mathbb{R}^3$

$$(x,y,z) \longmapsto F(x,y,z) = (\sin y, e^{xyz}, z^2)$$

$$\operatorname{curl} F(x,y,z) = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & e^{xyz} & z^2 \end{vmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(e^{xyz}) \\ \frac{\partial}{\partial z}(\sin y) - \frac{\partial}{\partial x}(z^2) \\ \frac{\partial}{\partial x}(e^{xyz}) - \frac{\partial}{\partial y}(\sin y) \end{pmatrix}$$

$$= \begin{pmatrix} 0 - xy\,e^{xyz} \\ 0 - 0 \\ yz\,e^{xyz} - \cos y \end{pmatrix} = \begin{pmatrix} -xy\,e^{xyz} \\ 0 \\ yz\,e^{xyz} - \cos y \end{pmatrix} \in \mathbb{R}^3$$

• <u>Example 2</u>: Let us consider the gravitational force of the Examp. 2 of § 1.2.1 with $P_0 = (0,0,0)$

$$F: \mathbb{R}^3 \setminus (0,0,0) \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longmapsto F(x,y,z) = \frac{-c}{r^3}(x,y,z)$$

$c = gMm$, $r = (x^2+y^2+z^2)^{1/2}$. Compute $\operatorname{curl} F(x,y,z)$.

$$\operatorname{curl} F(x,y,z) = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{cx}{r^3} & -\frac{cy}{r^3} & -\frac{cz}{r^3} \end{vmatrix} = c \begin{pmatrix} \frac{\partial}{\partial y}\left(\frac{-z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{-y}{r^3}\right) \\ \frac{\partial}{\partial z}\left(\frac{-x}{r^3}\right) - \frac{\partial}{\partial x}\left(\frac{-z}{r^3}\right) \\ \frac{\partial}{\partial x}\left(\frac{-y}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{-x}{r^3}\right) \end{pmatrix}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \ldots$$

$$= c \begin{pmatrix} \frac{3z}{r^4}\frac{\partial r}{\partial y} - \frac{3y}{r^4}\frac{\partial r}{\partial z} \\ \frac{3x}{r^4}\frac{\partial r}{\partial z} - \frac{3z}{r^4}\frac{\partial r}{\partial x} \\ \frac{3y}{r^4}\frac{\partial r}{\partial x} - \frac{3x}{r^4}\frac{\partial r}{\partial y} \end{pmatrix} \Big\downarrow = \begin{pmatrix} zy - yz \\ xz - zx \\ yx - xy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$$

1.5 The laplacian operator

1.5.1 Definition

let $\Omega \subset \mathbb{R}^n$ be an open domain and $f : \Omega \to \mathbb{R}$
$x \longmapsto f(x) = f(x_1, x_2, \ldots x_n)$

a scalar field s.t. $C^2(\Omega)$. The laplacian of f, denoted as Δf (or $\nabla^2 f$, or $\nabla \cdot \nabla f$), is defined as

$$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \in \mathbb{R}$$

Remark: as $\Delta f \in \mathbb{R}$, then $\Delta f : \Omega \to \mathbb{R}$ defineas a scalar field.

1.5.2 Examples

- Example 1: let be $f : \mathbb{R}^3 \to \mathbb{R}$

$$(x,y,z) \mapsto f(x,y,z) = x^2 y z^2 - z^3 + \sin 3x$$

Compute the Laplacian

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = \frac{\partial^2}{\partial x^2}\left(x^2 y z^2 - z^3 + \sin 3x\right)$$

$$= \frac{\partial}{\partial x}\left(2xyz^2 + 3\cos 3x\right) = 2yz^2 - 9\sin 3x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2}{\partial y}\left(x^2 y z^2 - z^3 + \sin 3x\right) =$$

$$= \frac{\partial}{\partial y}\left(x^2 z^2\right) = 0$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left(x^2 y z^2 - z^3 + \sin 3x\right) = \frac{\partial}{\partial z}\left(2x^2 yz - 3z^2\right)$$

$$= 2x^2 y - 6z$$

$$\Delta f(x,y,z) = 2yz^2 - 9\sin 3x + 2x^2 y - 6z$$

• Example 2: let us consider the gravitational potential of Ex 2. § 1.2.1 with $P_0 = (0,0,0)$

$$f: \mathbb{R}^3 \setminus (0,0,0) \longrightarrow \mathbb{R}$$
$$(x,y,z) \longmapsto \frac{c}{r}, \quad c = gmM$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\Big(\overbrace{\frac{\partial f}{\partial x}}^{F_1}\Big) = \frac{\partial}{\partial x}\Big(-c\frac{x}{r^3}\Big) = -c\Big(\frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x}\Big)$$

$$\frac{\partial f}{\partial x} = -\frac{cx}{r^3}$$

$$= -c\Big(\frac{1}{r^3} - \frac{3x^2}{r^5}\Big)$$

$$\frac{\partial^2 f}{\partial y^2} = -c\Big(\frac{1}{r^3} - \frac{3y^2}{r^5}\Big) = -c\Big(\frac{r^2 - 3y^2}{r^5}\Big)$$

$$\frac{\partial^2 f}{\partial z^2} = -c\Big(\frac{r^2 - 3z^2}{r^5}\Big)$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -c\left(\frac{\cancel{3r^2} - 3\cancel{\overbrace{(x^2+y^2+z^2)}^{r^2}}}{r^5}\right) = 0.$$

1.6 Differentiation formulas

1.6.1 Important results

• Theorem: let $\Omega \subset \mathbb{R}^n$ be an open domain, let $f: \Omega \to \mathbb{R}$ be a scalar field s.t. $f \in C^2(\Omega)$, and let $F: \Omega \to \mathbb{R}^n$ be a vector field s.t. $F \in C^2(\Omega, \mathbb{R}^n)$, then:

1) $\operatorname{div}(\operatorname{grad} f) = \Delta f$
2) if $n=2$ we have $\operatorname{curl}(\operatorname{grad} f) = 0 \quad (\in \mathbb{R})$
3) if $n=3$ we have $\operatorname{curl}(\operatorname{grad} f) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (\in \mathbb{R}^3)$
4) if $n=3$ we have $\operatorname{div}(\operatorname{curl} F) = 0 \quad (\in \mathbb{R})$

• Proof for $n=3$

1) $\operatorname{grad} f: \Omega \to \mathbb{R}^3$ is a vector field G, with $G_1 = \frac{\partial f}{\partial x}$, $G_2 = \frac{\partial f}{\partial y}$, $G_3 = \frac{\partial f}{\partial z}$. Thus $\operatorname{div}(G) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f \qquad \square$$

3) $$\operatorname{curl}(\overbrace{\operatorname{grad} f}^{G}) = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\square$

4) $\operatorname{div}(\operatorname{curl} F) = 0$. $\operatorname{curl} F : \Omega \to \mathbb{R}^3$ is a vector field G, with

$$G_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad G_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \quad G_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Thus
$$\operatorname{div}(G) = \cancel{\frac{\partial^2 F_3}{\partial y \partial x}} - \cancel{\frac{\partial^2 F_2}{\partial z \partial x}} + \cancel{\frac{\partial^2 F_1}{\partial z \partial y}} - \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} + \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 F_1}{\partial y \partial z}}$$

$= 0$ $\square$ (because $F_i \in C^2(\Omega)$ for $i = 1, 2, 3$).

- Remark: in Examples 2 of § 1.4.2 and § 1.5.2 could be deduced directly from the previous theorem

Indeed, the gravitational force $F = \operatorname{grad} f$, with $f = \frac{c}{r}$, the gravitational potential.

Thus
- $\operatorname{curl} F = \operatorname{curl}(\operatorname{grad} f) \underset{\text{Th (3)}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

- $\Delta f \underset{\text{Thm (1)}}{=} \operatorname{div}(\operatorname{grad} f) = \operatorname{div}(F) \underset{\text{Because of Ex 2. § 1.3.2}}{=} 0$