18/11/2021

3.4.3 Stokes theorem

Theorem: let $\Sigma \subset \mathbb{R}^3$ be a piecewise regular surface and orientable. Let $F : \Sigma \to \mathbb{R}^3$ be a vector field s.t. $F \in C^1(\Sigma, \mathbb{R}^3)$ defined by $(x,y,z) \mapsto (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$. Then

$$\iint_\Sigma \operatorname{curl} F \cdot ds = \int_{\partial\Sigma} F \cdot dl$$

- Remarks:

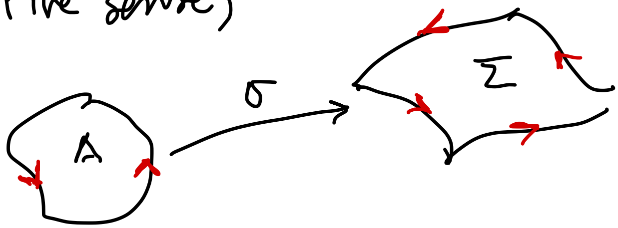
1) Stokes theorem is a generalization of Green theorem. for vector fields in $\mathbb{R}^3$

2) Once we have chosen a parameterization

$$\sigma : \bar{A} \to \Sigma \qquad \text{we } \sigma_u \wedge \sigma_v \text{ as normal vector}$$
$$(u,v) \mapsto \sigma(u,v)$$

$$\iint_\Sigma \operatorname{curl} F \cdot ds = \iint_A \operatorname{curl} F(\sigma(u,v)) \cdot (\sigma_u \wedge \sigma_v)\, du\, dv$$

3) The sense of circulation of $\partial\Sigma$ (for the curvilinear integral) is induced by the parameterization σ (i.e. this is obtained by circulating ∂A in positive sense)



3.4.4 Examples

- Example 1: verify Stokes theorem for

$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z^2 = x^2+y^2$ and $0<z<1$

and $F(x,y,z) = (z, x, y)$

- curl $F(x,y,z) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Parameterization of Σ : $A =]0, 2\pi[\times]0,1[$

$\sigma(\theta, z) = (z\cos\theta, z\sin\theta, z)$

$$\sigma_\theta \wedge \sigma_z = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ -z \end{pmatrix}$$

$$\iint_\Sigma \operatorname{curl} F \cdot dS = \iint_A (\operatorname{curl} F)(\sigma(\theta, z)) \cdot (\sigma_\theta \wedge \sigma_z)\, d\theta\, dz$$

$$= \int_0^{2\pi} \int_0^1 (1, 1, 1) \cdot (z\cos\theta, z\sin\theta, -z)\, d\theta\, dz$$

$$= \int_0^1 z\, dz \int_0^{2\pi} (\cos\theta + \sin\theta - 1)\, d\theta$$

$$= \frac{1}{2} \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{=0} + \frac{1}{2} \underbrace{\int_0^{2\pi} \sin\theta\, d\theta}_{=0} - \frac{1}{2}\int_0^{2\pi} d\theta = -\pi$$

- Calculation of $\int_{\partial\Sigma} F \cdot dl$

$\partial\Sigma = \{ \gamma(\theta) = (\cos\theta, \sin\theta, 1)$ with $\theta : 2\pi \to 0 \}$

$\gamma'(\theta) = (-\sin\theta, \cos\theta, 0)$.

$$\int_{\partial\Sigma} F \cdot dl = \int_{2\pi}^{0} F(\gamma(\theta)) \cdot \gamma'(\theta)\, d\theta$$

$F = (z, x, y)$

$$= \int_{2\pi}^{0} (1, \cos\theta, \sin\theta) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta$$

$$= \int_{2\pi}^{0} (-\sin\theta + \cos^2\theta)\, d\theta = \underbrace{-\int_{2\pi}^{0} \sin\theta\, d\theta}_{=0} + \int_{2\pi}^{0} \cos^2\theta\, d\theta$$

$$= \int_{2\pi}^{0} \cos^2\theta\, d\theta = \int_{2\pi}^{0} \frac{1}{2}(1 + \cos 2\theta)\, d\theta =$$

$$\frac{1}{2}\Big[-2\pi + \underbrace{\int_{2\pi}^{0} \cos 2\theta\, d\theta}_{=0} \Big] = -\pi = \iint_{\Sigma} \operatorname{curl} F \cdot dS \quad \checkmark$$

- Example 2: Verify Stokes theorem for

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$$

and $F(x,y,z) = (z, x, y^2)$

- Calculation of $\iint_{\Sigma} \operatorname{curl} F \cdot dS$

$$\operatorname{curl} F = \begin{pmatrix} 2y \\ 1 \\ 1 \end{pmatrix}$$

Parameterization of Σ: $A =]0, 2\pi[\times]0, \pi/2[$

$$\sigma(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$\sigma_\theta \wedge \sigma_\varphi = \begin{pmatrix} -\sin^2\varphi\cos\theta \\ -\sin^2\varphi\sin\theta \\ -\sin\varphi\cos\varphi \end{pmatrix}$$

$$\iint_\Sigma \operatorname{curl} F \cdot dS = \iint_A (\operatorname{curl} F)(\sigma(\theta,\varphi)) \cdot (\sigma_\theta \wedge \sigma_\varphi) \, d\theta \, d\varphi$$

$$= -\int_0^{2\pi}\int_0^{\pi/2} (2\sin\varphi\sin\theta, 1, 1) \cdot (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi) \, d\theta \, d\varphi$$

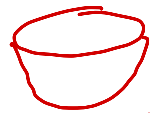
$$= -\int_0^{2\pi}\int_0^{\pi/2} (2\sin^3\varphi\sin\theta\cos\theta + \sin^2\varphi\sin\theta + \sin\varphi\cos\varphi) \, d\theta \, d\varphi$$

$$= -2\underbrace{\int_0^{2\pi} \sin\theta\cos\theta \, d\theta}_{=0} \int_0^{\pi/2} \sin^3\varphi \, d\varphi - \underbrace{\int_0^{2\pi} \sin\theta \, d\theta}_{=0} \int_0^{\pi/2} \sin^2\varphi \, d\varphi$$

$$- \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi\cos\varphi \, d\varphi$$

$$= -2\pi \int_0^{\pi/2} \sin\varphi \cos\varphi \, d\varphi = -2\pi \left[\frac{1}{2} \sin^2\varphi \right] \Big|_0^{\pi/2} = -\pi$$

- Calculation of $\int_{\partial\Sigma} F \cdot dl$

(similar reasoning as for this example 

$z<0$

$$\partial\Sigma = \{ \gamma_3(\theta) = (\cos\theta, \sin\theta, 0) \text{ with } \theta: 2\pi \to 0 \}$$

$$\gamma_3'(\theta) = (-\sin\theta, \cos\theta, 0)$$

$$\int_{\partial\Sigma} F \cdot dl = \int_{2\pi}^{0} F(\gamma_3(\theta)) \cdot \gamma_3'(\theta) \, d\theta$$

$$F = (z, x, y^2)$$

$$= \int_{2\pi}^{0} (0, \cos\theta, \sin^2\theta) \cdot (-\sin\theta, \cos\theta, 0) \, d\theta$$

$$= \int_{2\pi}^{0} \cos^2\theta \, d\theta = -\pi = \iint_\Sigma \operatorname{curl} F \cdot dS$$