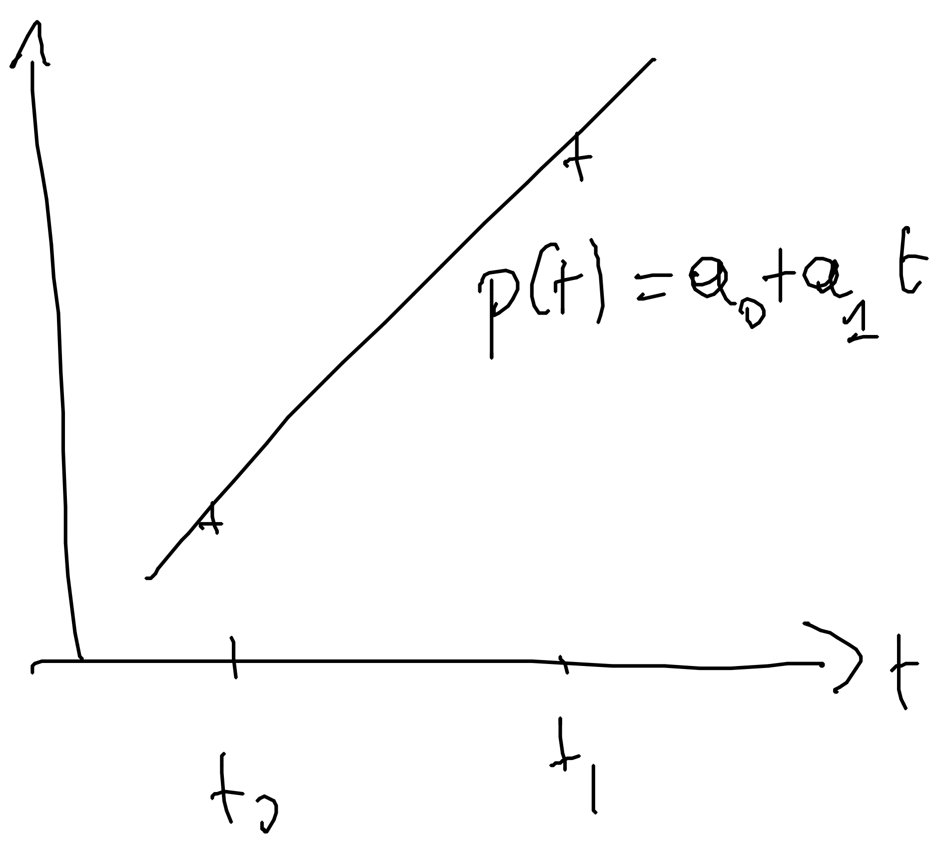
<u>Pbm:</u> données n entier pos.

$n+1$ valeurs $t_0\ t_1\ t_2\ \dots t_n$ distinctes

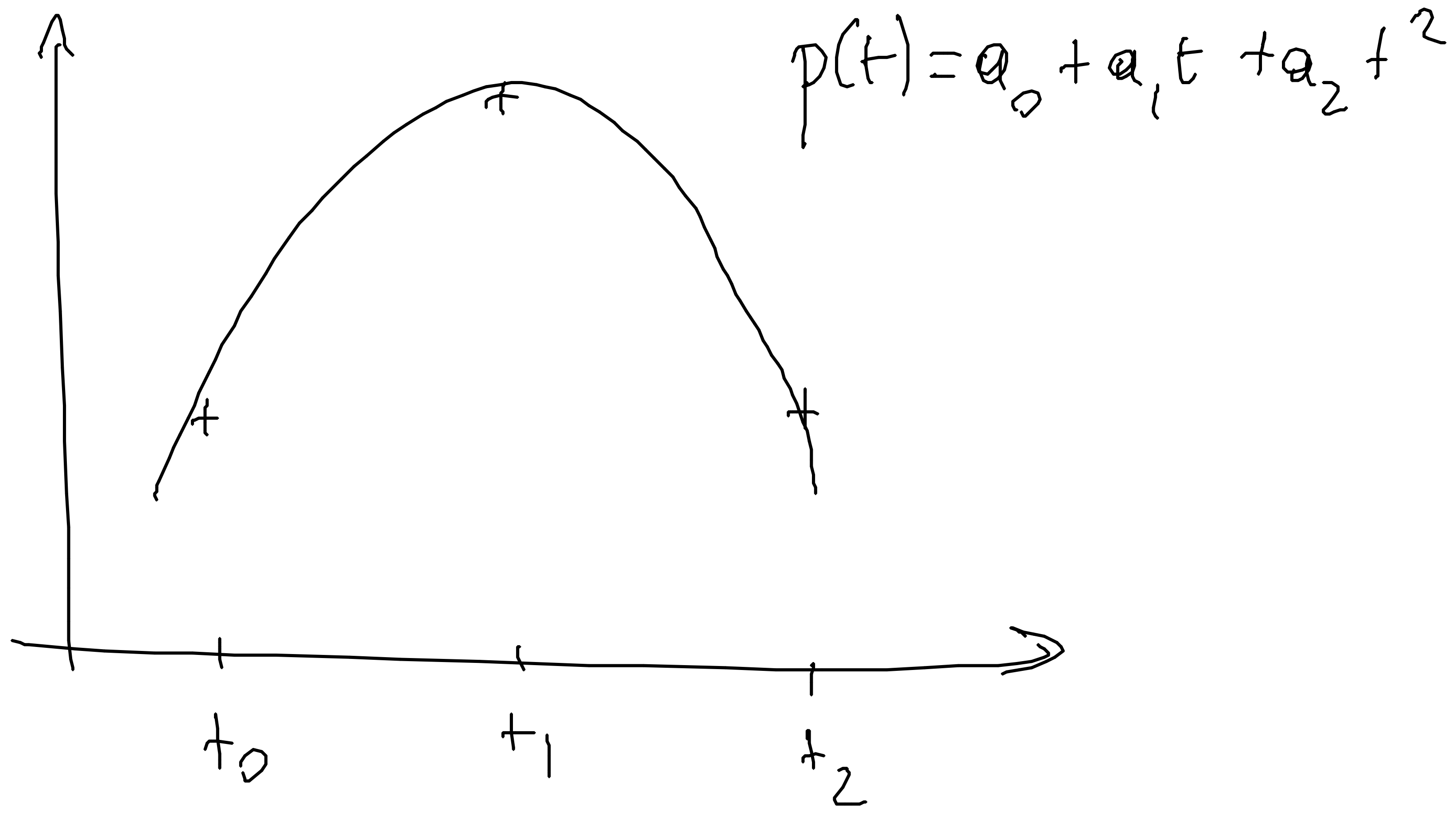
$n+1$ valeurs $p_0\ p_1\ p_2\ \dots p_n$

cherché $p \in \mathbb{P}_n$ tq $p(t_j) = p_j \quad j = 0, 1, 2, \dots, n$

<u>$n = 1$</u>



<u>$n = 2$</u>



Mauvaise méthode : $p \in \mathbb{P}_n$: $p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$

$n+1$ inconnues $\quad a_0 \quad a_1 \quad a_2 \quad \dots \quad a_n$

$n+1$ équations $\quad p(t_0) = p_0 = a_0 + a_1 t_0 + a_2 t_0^2 + \dots + a_n t_0^n$

$$p(t_1) = p_1 = a_0 + a_1 t_1 + a_2 t_1^2 + \dots + a_n t_1^n$$

$$\vdots$$

$$p(t_n) = p_n$$

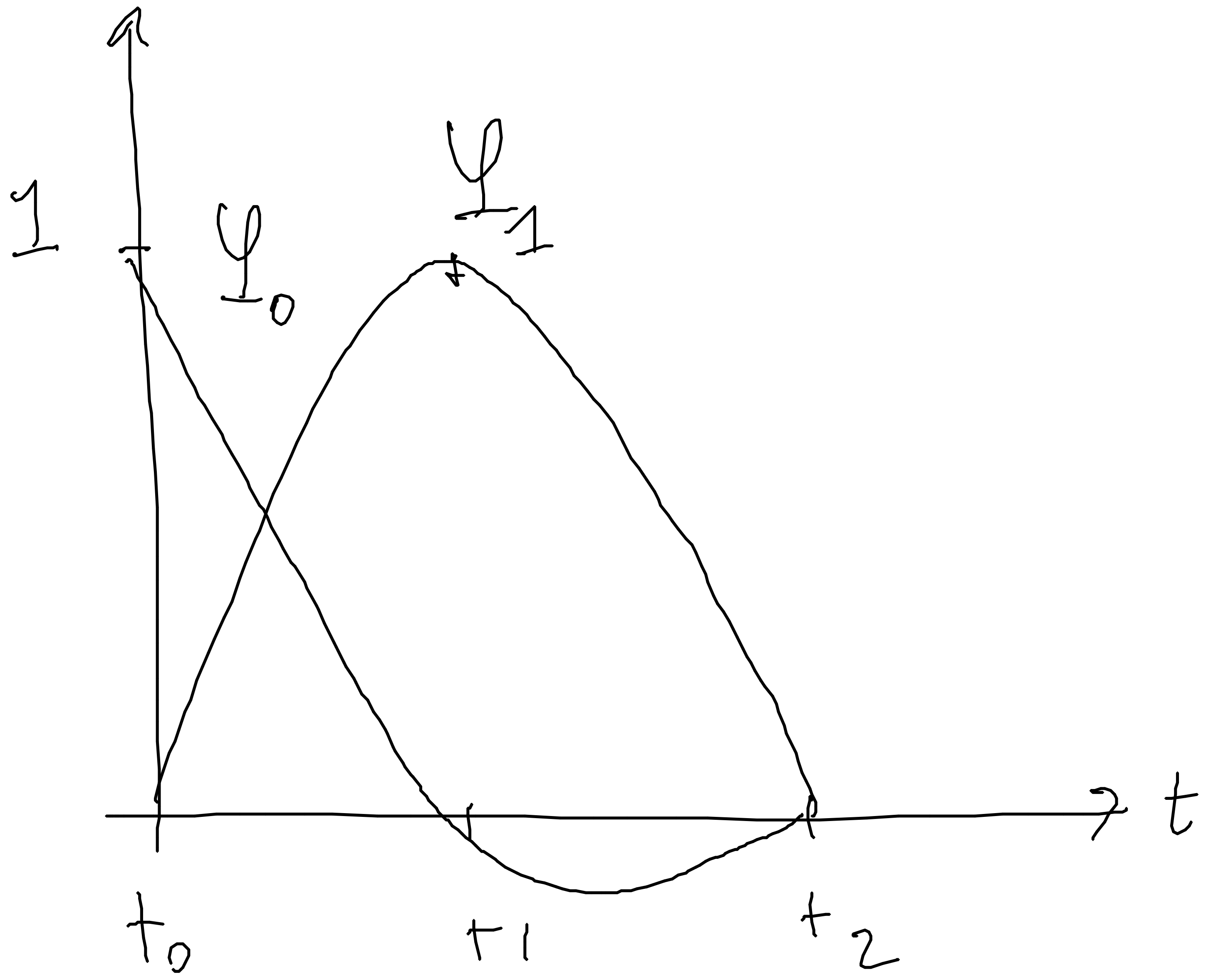
$$\underbrace{\begin{pmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^n \\ 1 & t_1 & t_1^2 & \cdots & t_1^n \\ \vdots & & & & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^n \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}_{\vec{a}} = \underbrace{\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}}_{\vec{P}}$$

nbre d'opération : $O(n^3)$

formule explicite : interpolation de Lagrange

$n=2$: $t_0 \; t_1 \; t_2$ $\quad \varphi_0 \varphi_1 \varphi_2$ base de Lagrange de $\mathbb{P}_2$ associée aux pts $t_0 \; t_1 \; t_2$

$\varphi_0 \; \varphi_1 \; \varphi_2 \in \mathbb{P}_2$



$\varphi_0 \in \mathbb{P}_2 \quad \varphi_0(t_0)=1 \quad \varphi_0(t_1)=0 \quad \varphi_0(t_2)=0$

$$\varphi_0(t) = \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)}$$

$\varphi_1 \in \mathbb{P}_2 \quad \varphi_1(t_0)=0 \quad \varphi_1(t_1)=1 \quad \varphi_1(t_2)=0$

$$\varphi_1(t) = \frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)}$$

$\varphi_2 \in \mathbb{P}_2 \quad \varphi_2(t_0)=0 \quad \varphi_2(t_1)=0 \quad \varphi_2(t_2)=1$

$\underline{\varphi_0 \; \varphi_1 \; \varphi_2}$ base de $\mathbb{P}_2$: $\dim \mathbb{P}_2 = 3 \quad p(t) \in \mathbb{P}_2 \quad p(t) = a_0 + a_1 t + a_2 t^2$

linéairement indép : $\left(\alpha_0 \varphi_0(t) + \alpha_1 \varphi_1(t) + \alpha_2 \varphi_2(t) = 0 \quad \forall t \in \mathbb{R}\right) \Rightarrow \left(\alpha_0 = \alpha_1 = \alpha_2 = 0\right)$

$p \in \mathbb{P}_2$: $p(t) = \alpha_0 \varphi_0(t) + \alpha_1 \varphi_1(t) + \alpha_2 \varphi_2(t)$

Solution du pbm : $p(t) = p_0 \varphi_0(t) + p_1 \varphi_1(t) + p_2 \varphi_2(t) \quad \in \mathbb{P}_2$

$p(t_0) = p_0 \cdot 1 \quad + p_1 \cdot 0 \quad + p_2 \cdot 0$

m qque : $\varphi_0\, \varphi_1\, \varphi_2 \dots \varphi_n$ base de lagrange de $\mathbb{P}_n$ ass. aux pts $t_0\ t_1\ t_2 \dots t_n$

$0 \leq k \leq n$ fixé $\quad \varphi_k \in \mathbb{P}_n \quad \varphi_k(t_k) = 1 \quad \varphi_k(t_j) = 0 \quad j \neq k$

$$\varphi_k(t) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{t - t_j}{t_k - t_j}$$

$\varphi_0\, \varphi_1\, \varphi_2 \dots \varphi_n$ base de $\mathbb{P}_n$: $\dim \mathbb{P}_n = n+1$

linéairement indep.

$$\alpha_0 \varphi_0(t) + \alpha_1 \varphi_1(t) + \dots + \alpha_n \varphi_n(t) = 0 \qquad \forall t \in \mathbb{R}$$

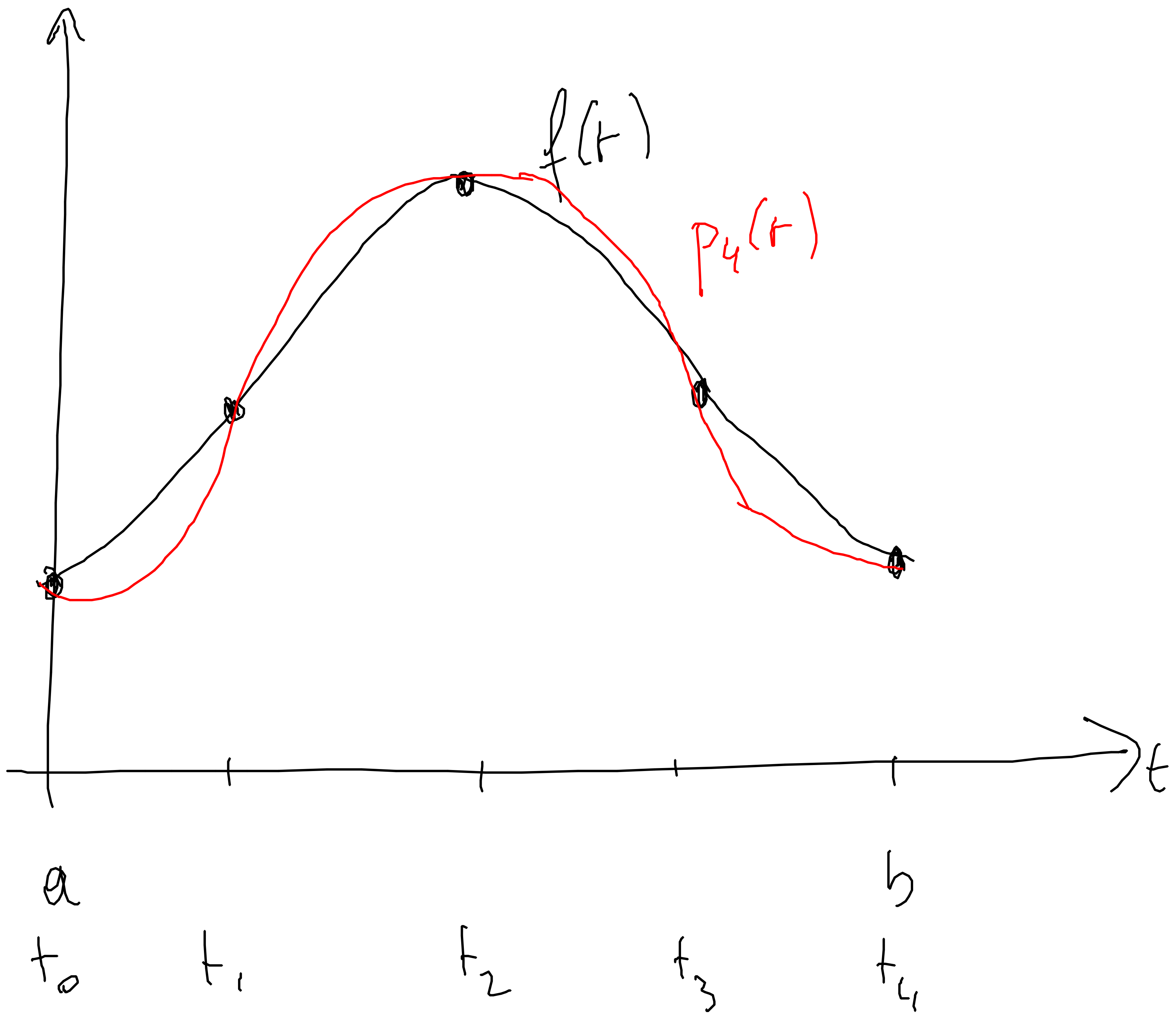
$$\alpha_0 \cdot 1 + \alpha_1 \cdot 0 + \dots + \alpha_n \cdot 0 = 0 \qquad t = t_0$$

Solution du pbm : $\quad p(t) = p_0\, \varphi_0(t) + p_1\, \varphi_1(t) + \dots + p_n\, \varphi_n(t) \quad \in \mathbb{P}_n$

$$p(t_0) = p_0 \cdot 1 + p_1 \cdot 0 + \dots + p_n \cdot 0$$

$\vdots$

Interpolation d'une fct continue par un polyn. (1.4 livre)



$f : [a,b] \to \mathbb{R}$ cont.

$t_j = a + \frac{b-a}{n} j \qquad j = 0, 1, \dots, n$

$p_n \in \mathbb{P}_n$ tq $p_n(t_j) = f(t_j)$ $j = 0, 1, \dots, n$

Question: $p_n \xrightarrow{n \to \infty} f$?

Réponse: ça dépend de f ....

Thm 1.1 : données $f:[a,b]\to\mathbb{R}$

$t_j = a + \frac{b-a}{n} j \quad j=0,1,2,\dots,n$ équidistants

$p_n \in \mathbb{P}_n \quad p_n(t_j) = f(t_j) \quad j=0,1,\dots,n$

$p_n(t) = f(t_0)\varphi_0(t) + f(t_1)\varphi_1(t) + \dots + f(t_n)\varphi_n(t)$

hyp : $f \in \mathcal{C}^{n+1}([a,b])$

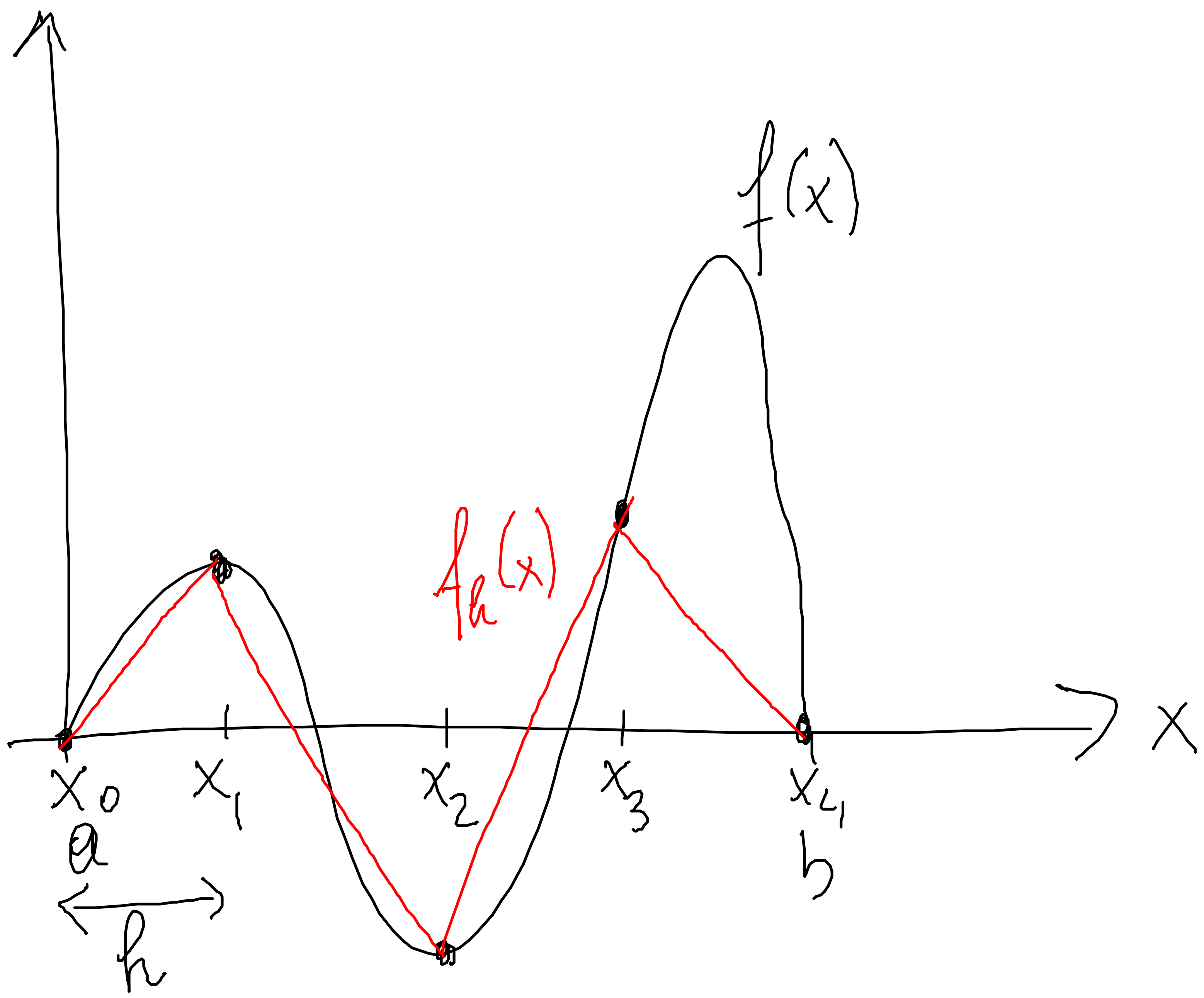
Conclusion : $\max\limits_{a\le t\le b} |f(t)-p_n(t)| \le \frac{1}{2(n+1)}\left(\frac{b-a}{n}\right)^{n+1} \max\limits_{a\le t\le b} |f^{(n+1)}(t)|$

Ex : $f(t) = \sin t \quad |f^{(n+1)}(t)| \le 1 \quad \forall t\in\mathbb{R} \quad \lim\limits_{n\to\infty} \max\limits_{0\le t\le b} |f(t)-p_n(t)| = 0$

$f(t) = \frac{1}{1+25t^2} \quad |f^{(n+1)}(t)| \xrightarrow{n\to\infty} +\infty$ on ne peut pas conclure.

Conclusion :
- pas souhaitable pts équidistants $n\to\infty$
- points distribués de manière adéquate sur $[a,b]$
- interpolation par intervalles

Interpol degré 1 par intervalle :



$f : [a,b] \to \mathbb{R}$

$x_i = a + \underbrace{\frac{b-a}{N}}_{h} i \qquad i = 0, 1, \dots, N$

$f_h \in \mathcal{C}^0[a,b]$

$f_h(x_i) = f(x_i) \qquad i = 0, 1, \dots, N$

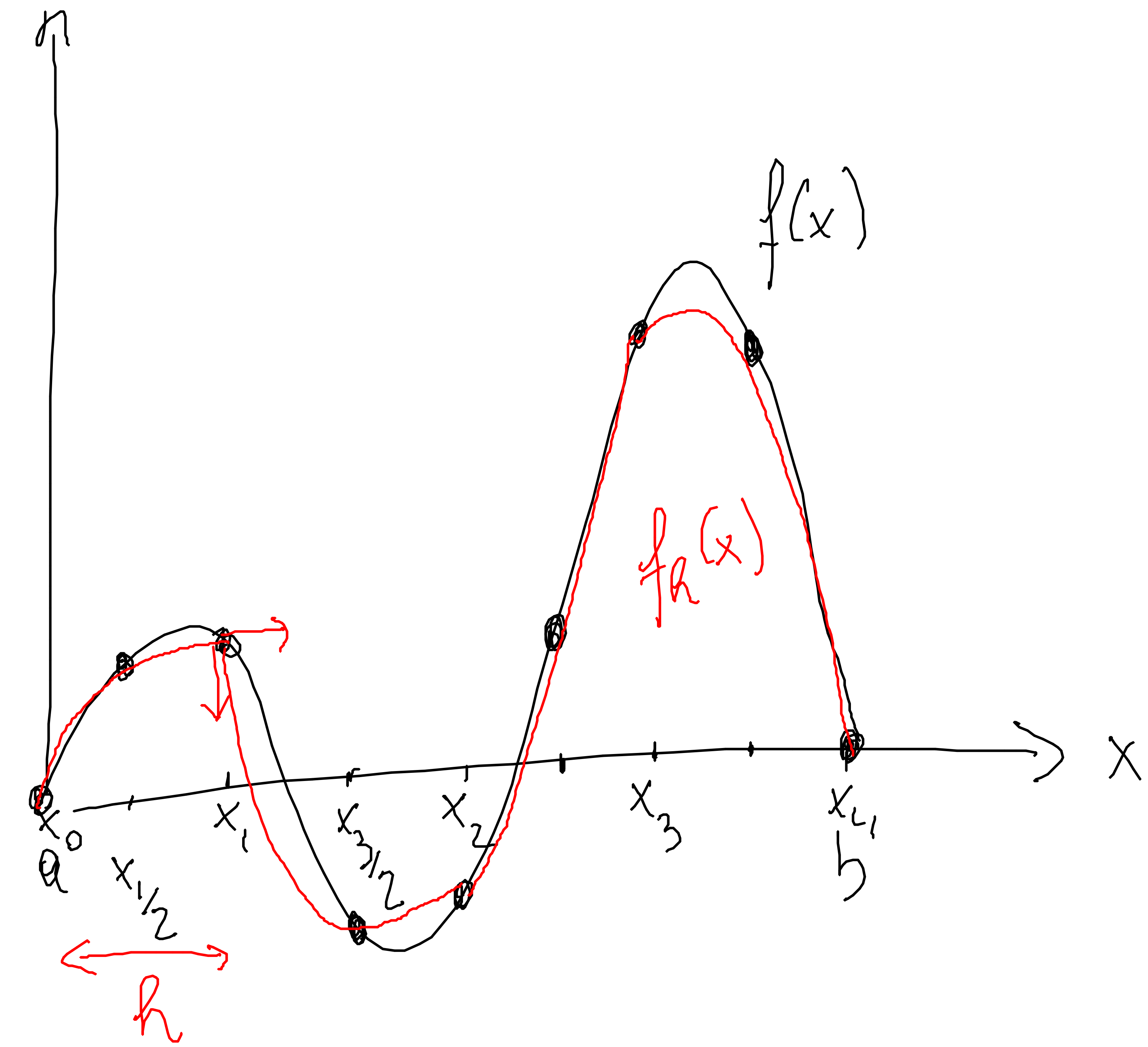
$f_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_1 \qquad i = 0, 1, \dots, N-1$

$f_h \xrightarrow[N \to +\infty]{h \to 0} f$?

Thm 1.2 : $\exists C > 0 \ \forall f \in \mathcal{C}^2[a,b] \ \forall h > 0 \qquad \max_{a \leq x \leq b} |f_h(x) - f(x)| \leq C h^2 \max_{a \leq x \leq b} |f''(x)|$

Interprét : $f \in \mathcal{C}^2[a,b]$ l'erreur est au moins divisée par 2^2 chaque fois que h est divisé par 2.

Interpol degré 2 par intervalle :



$f : [a,b] \to \mathbb{R}$

$$x_i = a + \underbrace{\frac{b-a}{N}}_{h} i \qquad i = 0, 1, \dots, N$$

$f_h \in \mathcal{C}^0[a,b]$

$f_h(x_i) = f(x_i) \quad i = 0, 1, \dots, N$

$f_h(x_{i+1/2}) = f(x_{i+1/2}) \quad i = 0, 1, \dots, N-1$

$f_h|[x_i, x_{i+1}] \in \mathbb{P}_2 \quad i = 0, 1, \dots, N-1$

$$f_h \xrightarrow[N \to +\infty]{h \to 0} f \; ?$$

Thm 1.2 $\exists C > 0 \; \forall f \in \mathcal{C}^3[a,b] \; \forall h > 0$
$$\max_{a \le x \le b} |f_h(x) - f(x)| \le C h^3 \max_{a \le x \le b} |f'''(x)|$$

Interprét : $f \in \mathcal{C}^3[a,b]$ l'erreur divisée par 2^3 chaque fois que h divisé par 2

Résumé Chap 1 interpolation

- $p \in \mathbb{P}_n$ tq $p(t_j) = p_j \quad j = 0, 1, \dots, n$

$$p(t) = \sum_{j=0}^{n} p_j \, \varphi_j(t)$$

$\varphi_0 \varphi_1 \varphi_2 \dots \varphi_n$ base de Lagr. de $\mathbb{P}_n$ ass. aux pts $t_0 t_1 t_2 \dots t_n$

$$\varphi_k(t) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{t - t_j}{t_k - t_j}$$

- $f : [a,b] \to \mathbb{R}$

$$p_n(t) = \sum_{j=0}^{n} f(t_j) \, \varphi_j(t)$$

t_j équidist $\quad t_j = a + \frac{b-a}{n} j \quad j = 0, 1, \dots, n$

$p_n \xrightarrow{n \to \infty} f$? dépend de f

- interpol. intervalles $\quad f : [a,b] \to \mathbb{R} \quad f_h \in \mathcal{C}^0[a,b]$

degré 1 $\quad |f - f_h| = O(h^2)$

— 2 $\quad |f - f_h| = O(h^3)$.