EPFL - Autumn 2021	Dr. Pablo Antolin
Analysis III SV MT	**Exercises**
Serie 7	September, 28

Note: several exercises are directly taken from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Exercise 1.
Let $\Sigma \subset \mathbb{R}^3$ be a regular orientable surface with a field of unit normal vectors ν. Let $F : \Sigma \mapsto \mathbb{R}^3$ be a continuous vector field. Show that the flux of F through the surface Σ in the direction ν is equal to the integral of the scalar field $F \cdot \nu$ on Σ.

Exercise 2 (Ex 5.1 page 56)**.**
Let $f(x, y, z) = xy + z^2$ and

$$\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = z^2 \text{ and } 0 \leq z \leq 1\right\}.$$

Compute $\displaystyle\iint_\Sigma f ds$.

Exercise 3 (Ex 5.3 page 57)**.**
Let $\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 - z^2 = 0 \text{ et } 0 \leq z \leq 1\right\}$. Compute the mass of the surface Σ knowing that the density is $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

Exercise 4 (Ex 5.5 page 57)**.**
Let $\Omega = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z \leq 1\right\}$. Compute the area of $\partial\Omega$.

Exercise 5 (Ex 5.2 page 57)**.**
Let $F(x, y, z) = \left(x^2, y^2, z^2\right)$ et

$$\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2 \text{ and } 0 \leq z \leq 1\right\}.$$

Compute the flux through Σ in the upward direction (i.e. in the direction of $z > 0$).

Exercise 6 (Ex 5.4 page 57).
Let $F(x, y, z) = (0, z, z)$ et

$$\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 : \ z = 6 - 3x - 2y \text{ and } x \geq 0,\, y \geq 0,\, z \geq 0\right\}.$$

Compute the flux that passes through this surface and away from the origin.

Exercise 7 (Ex 5.7 page 57).
Let $\Omega \subset \mathbb{R}^2$ be an open set and $f : \bar{\Omega} \to \mathbb{R}$ be a continuous scalar field such that $f(x, y) > 0$ for all $(x, y) \in \bar{\Omega}$. Consider a simple and regular curve $\Gamma \subset \Omega$ and the surface $\Sigma \subset \mathbb{R}^3$ defined by

$$\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \Gamma \text{ et } 0 \leq z \leq f(x, y)\right\}.$$

Show that Area $(\Sigma) = \displaystyle\int_\Gamma f dl$.

Exercise 8.
Let $\alpha, \beta > 0$ and $F_{\alpha,\beta} \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$F_{\alpha,\beta}(x, y, z) = \left(\frac{x}{(y^2 + z^2)^\alpha}, y, \frac{z}{|x|^\beta}\right).$$

Identify for which values of α and β, the following integral is well-defined:

$$\left|\iint_S F_{\alpha,\beta} \cdot dS\right| < +\infty,$$

where S is the unit sphere around the origin, *i.e.*:

$$S = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}.$$