CHAPTER 3 : SURFACES, SURFACE INTEGRALS, DIVERGENCE AND STOKES' THEOREMS

3.1 Recalls and preliminary notations

3.1.1 Change of variables in an integral with several variables

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous bounded function defined as $f(x) = f(x_1, x_2, \dots, x_n)$. Let $\Omega \subset \mathbb{R}^n$ an open domain.

A change of variables is described by a bijective function u defined by:

$$u : \Omega \longrightarrow u(\Omega) \subset \mathbb{R}^n$$

$$(y_1, \dots, y_n) \longmapsto u(y_1, \dots, y_n) = (x_1, \dots, x_n)$$

$x = (x_1, \dots, x_n)$ old variables

$y = (y_1, \dots, y_n)$ new variables

$$x_i = u_i(y_1, \dots, y_n) \qquad i = 1, \dots, n$$

$$u_i : \Omega \to \mathbb{R}.$$

$$\underbrace{\det \overbrace{\nabla u(y)}^{\text{Jacobian Matrix}}}_{\text{Jacobian}} = \det \begin{pmatrix} \frac{\partial u_1}{\partial y_1}(y) & \frac{\partial u_1}{\partial y_2}(y) & \cdots & \frac{\partial u_1}{\partial y_n}(y) \\ \vdots & & & \\ \vdots & & & \\ \frac{\partial u_n}{\partial y_1}(y) & \cdots & \cdots & \frac{\partial u_n}{\partial y_n}(y) \end{pmatrix}$$

$$\iint \cdots \int_{\Omega} f(x_1, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

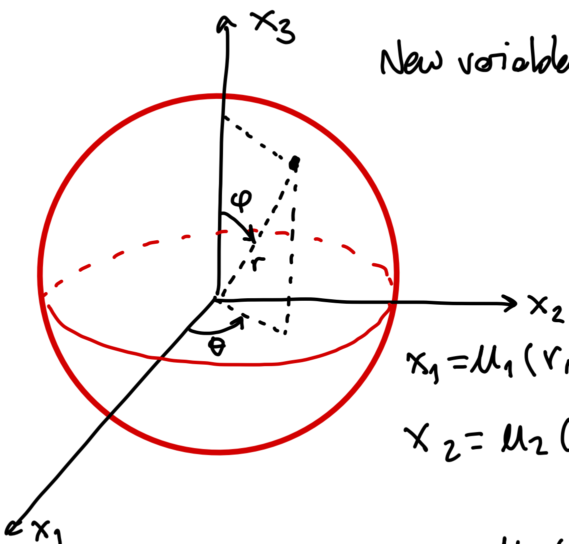
$$= \iint \cdots \int_{u^{-1}(\Omega)} f(u(y_1, \ldots, y_n))\, |\det \nabla u(y)|\, dy_1\, dy_2 \ldots dy_n$$

Example:

Old variables (x_1, x_2, x_3) : Cartesian coords $\mathbb{R}^3$

New variables: $(y_1, y_2, y_3) = (r, \theta, \varphi)$

spherical coordinates in $\mathbb{R}^3$



$x_1 = u_1(r, \theta, \varphi) = r \sin\varphi \cos\theta$ $\quad r \geq 0$

$x_2 = u_2(r, \theta, \varphi) = r \sin \sin\theta$ $\quad \theta \in [0, 2\pi)$

$\varphi \in [0, \pi]$

$x_3 = u_3(r, \theta, \varphi) = r \cos\varphi$

$$\det \nabla u(r,\theta,\varphi) = \det \begin{pmatrix} \frac{\partial u_1}{\partial r} & \frac{\partial u_1}{\partial \theta} & \frac{\partial u_1}{\partial \varphi} \\ \frac{\partial u_2}{\partial r} & \frac{\partial u_2}{\partial \theta} & \frac{\partial u_2}{\partial \varphi} \\ \frac{\partial u_3}{\partial r} & \frac{\partial u_3}{\partial \theta} & \frac{\partial u_3}{\partial \varphi} \end{pmatrix}$$

$$= \det \begin{pmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{pmatrix}$$

$$= -r^2\sin\varphi$$

$$|\det \nabla u(r,\theta,\varphi)| = |-r^2\sin\varphi| = \underline{r^2\sin\varphi}$$

(because $\varphi \in [0,\pi]$)

$$\iiint_\Omega f(x_1,x_2,x_3)\,dx_1dx_2dx_3 =$$

$$\iiint_{u^{-1}(\Omega)} f(r\sin\varphi\cos\theta,\, r\sin\varphi\sin\theta,\, r\cos\varphi)\, \overline{r^2\sin\varphi}\, dr\,d\theta\,d\varphi$$

$f=1$ and $\Omega = \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2+x_2^2+x_3^2 < R^2\}$

$$\iiint_\Omega 1 \, dx_1 dx_2 dx_3 = \int_0^R r^2 dr \int_0^{2\pi} d\theta \int_0^\pi \sin\varphi \, d\varphi = \frac{1}{3} R^3 \, 2\pi \left[-\cos\varphi \Big|_0^\pi \right]$$

$$= \frac{4}{3} \pi R^3 \quad \text{(volume of sphere of radius } R\text{)}.$$

3.1.2 Surfaces in $\mathbb{R}^3$

- New notations:

  a) for a function $f: \mathbb{R}^2 \to \mathbb{R}^3$ we denote

  $f(x,y) = (f^1(x,y), f^2(x,y), f^3(x,y))$

  where $f^i: \mathbb{R}^2 \to \mathbb{R}$ (super-indices for components)

  b) for a function $g(x,y)$ we denote

  $\frac{\partial g}{\partial x} = g_x$ and $\frac{\partial g}{\partial y} = g_y$ (sub-indices for indicating the variable respect to which we derive).

- Definition 1:

  $\Sigma \subset \mathbb{R}^3$ is called a <u>regular surface</u> if:

a) There exists $A \subset \mathbb{R}^2$, a bounded open domain, such that its boundary ∂A is a (piecewise) closed regular simple curve.

b) There exists a function $\sigma : \bar{A} \longrightarrow \mathbb{R}^3$
$$(u,v) \mapsto \sigma(u,v)$$
$$= (\sigma^1(u,v), \sigma^2(u,v), \sigma^3(u,v))$$

that satisfies the following properties:

- $\sigma \in C^1(\bar{A}, \mathbb{R}^3)$;

  $\sigma(\bar{A}) = \Sigma$ and σ is injective over A

- the vector $\sigma_u \wedge \sigma_v = \begin{vmatrix} e_1 & e_2 & e_3 \\ \sigma^1_u & \sigma^2_u & \sigma^3_u \\ \sigma^1_v & \sigma^2_v & \sigma^3_v \end{vmatrix}$

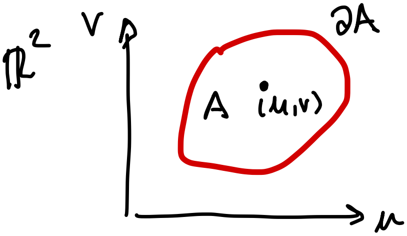
  s.t. $\|\sigma_u \wedge \sigma_v\| \neq 0 \quad \forall (u,v) \in A$

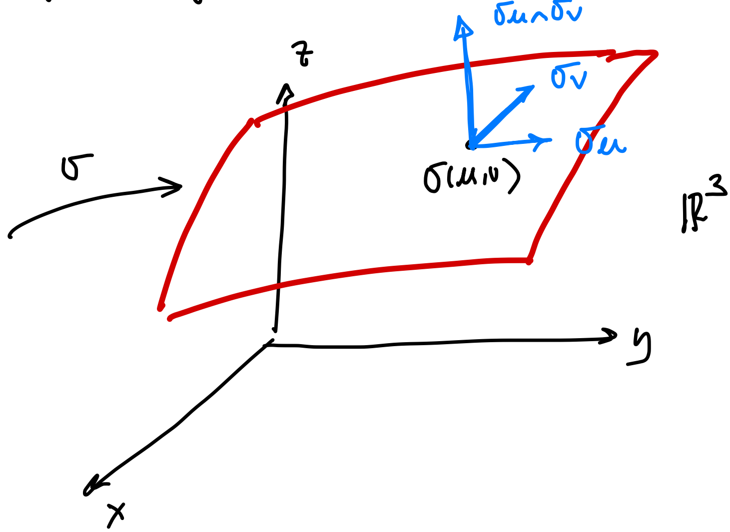
Remarks:

1) σ is called a regular parameterization of the surface Σ

2) The vector $\nu(u,v) = \dfrac{\sigma_u \wedge \sigma_v}{\| \sigma_u \wedge \sigma_v \|}$ is called the unit normal of the surface Σ at the point $\sigma(u,v)$

3) Illustration:





4) Analogy with curves in $\mathbb{R}^3$.

Curve Γ	Surface Σ
$\gamma : [a,b] \subset \mathbb{R} \to \Gamma \subset \mathbb{R}^{2/3}$ $t \mapsto \gamma(t)$	$\sigma : \bar{A} \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^3$ $(u,v) \mapsto \sigma(u,v)$
$\gamma([a,b]) = \Gamma$ and injective over $[a,b)$	$\sigma(\bar{A}) = \Sigma$ and injective over A

$\|\gamma'(t)\| \neq 0$
(non-null tangent vector)

- length

$d\ell = \|\gamma'(t)\|\,dt$

$\|\sigma_u \wedge \sigma_v\| \neq 0$
(non-null normal vector

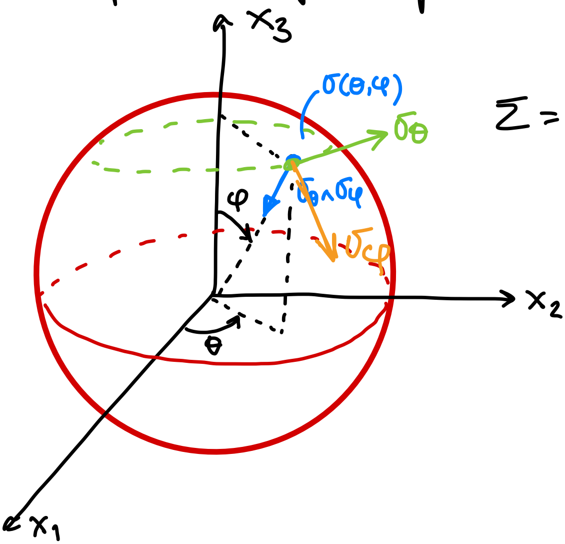
$dA = \|\sigma_u \wedge \sigma_v\|\,du\,dv$

- Definition 2:

We say that $\Sigma \subset \mathbb{R}^3$ is a piecewise regular surface if there exist regular surfaces $\Sigma_1, \Sigma_2, \dots, \Sigma_k$ s.t.

$$\Sigma = \bigcup_{i=1}^{k} \Sigma_i$$

- Example 1: Sphere of radius R (orange's peel)



$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = R^2\}$$

$$\sigma : \bar{A} \to \Sigma \qquad A = \overbrace{(0, 2\pi)}^{\theta} \times \overbrace{(0, \pi)}^{\varphi}$$

$$(\theta, \varphi) \longmapsto \sigma(\theta, \varphi) = (\sigma^1(\theta,\varphi), \sigma^2(\theta,\varphi), \sigma^3(\theta,\varphi))$$

$$= (R \sin\varphi \cos\theta, R \sin\varphi \sin\theta, R \cos\varphi)$$

Normal vector:

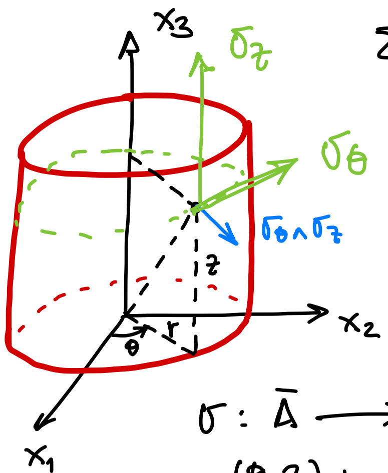
$$\sigma_\theta \wedge \sigma_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ \sigma^1_\theta & \sigma^2_\theta & \sigma^3_\theta \\ \sigma^1_\varphi & \sigma^2_\varphi & \sigma^3_\varphi \end{vmatrix}$$

$$= \begin{array}{c} \begin{vmatrix} e_1 & e_2 & e_3 \\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0 \\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi \end{vmatrix} \end{array} \begin{array}{l} \\ ] \ \sigma_\theta \\ ] \ \sigma_\varphi \end{array}$$

$$= \begin{pmatrix} -R^2 \sin^2\varphi \cos\theta \\ -R^2 \sin^2\varphi \sin\theta \\ -R^2 \sin\varphi \cos\varphi \end{pmatrix} = -R \sin\varphi \, \sigma(\theta, \varphi)$$

$\sigma_\theta \wedge \sigma_\varphi$ is an <u>internal normal</u> vector of the surface

Example 2: Cylinder ($R=1$, height $=1$)



$$\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2=1 \text{ and } 0 \le z \le 1 \}$$

$$\sigma : \bar{A} \longrightarrow \Sigma$$

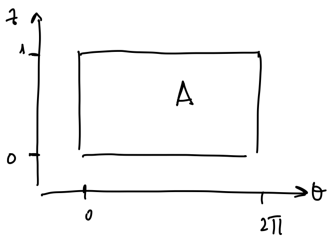
$$(\theta, z) \longmapsto \sigma(\theta,z) = \left(\sigma^1(\theta,z), \sigma^2(\theta,z), \sigma^3(\theta,z) \right)$$

$$= (\cos\theta, \sin\theta, z)$$

$$A = \underbrace{(0, 2\pi)}_{\theta} \times \underbrace{(0, 1)}_{z}$$

Normal vector:

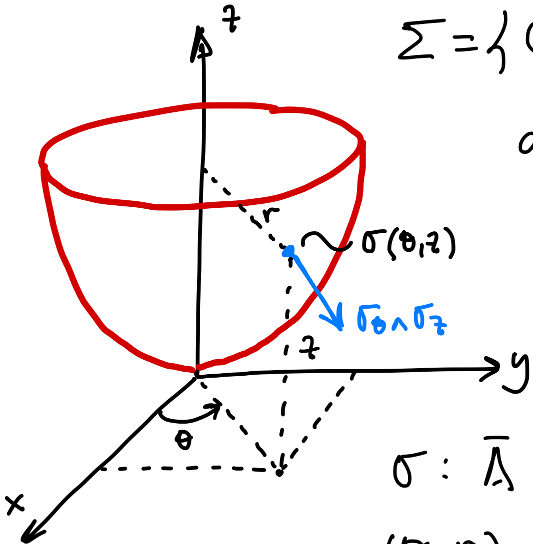
$$\begin{array}{c c} \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} & \begin{array}{l} \\ ]\ \sigma_\theta \\ ]\ \sigma_z \end{array} \end{array}$$



$$= \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \sigma_\theta \wedge \sigma_z \text{ is and}$$

<u>extend</u> normal vector

Example 3: symmetric paraboloid (glass of wine)



$$\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2 \text{ and } 0 \le z \le 1 \}$$

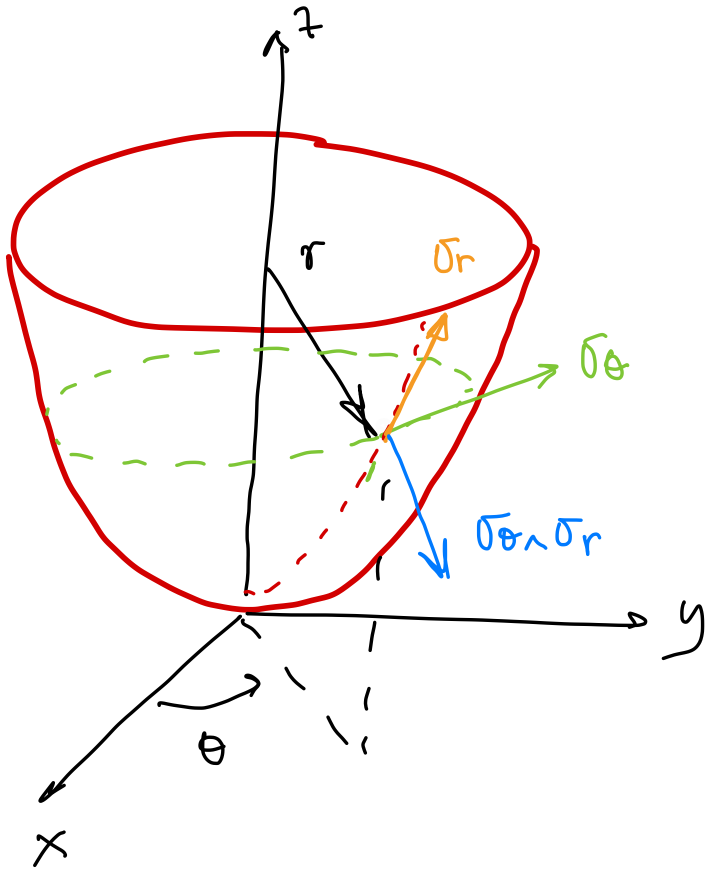
$$\sigma : \bar{A} \longrightarrow \Sigma$$

$$(\theta, r) \longmapsto (\sigma^1(\theta,z), \sigma^2(\theta,z), \sigma^3(\theta,z))$$

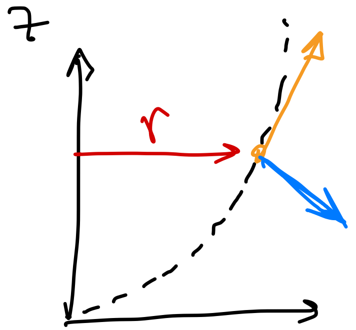
$$= (r\cos\theta, r\sin\theta, r^2)$$

Normal vector

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix} = \begin{pmatrix} 2r^2\cos\theta \\ 2r^2\sin\theta \\ -r \end{pmatrix}$$



z
r
σr
σθ
σθ∧σr
y
θ
x



z
r
for a fixed θ