Complex Analysis

Complex number properties

Cartesian vs. Polar:

$x = \mathrm{Re} = r\cos(\theta)$ $\quad r = \sqrt{x^2+y^2} = |z|$

$y = \mathrm{Im} = r\sin(\theta)$ $\quad \theta = \arg(z)$

$\arg(z):\ \theta = \arctan\left(\frac{y}{x}\right)$ $\quad x>0$

$\theta = \pm\frac{\pi}{2}$ $\quad x=0$

$\theta = \pi + \arctan\left(\frac{y}{x}\right)$ $\quad x<0,\ y>0$

$\theta = \arctan\left(\frac{y}{x}\right) - \pi$ $\quad x<0,\ y<0$

Complex sum and multiplication:

- $z_1+z_2 = (x_1+x_2) + i(y_1+y_2)$
- $z_1\cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
- $\forall z \neq 0 \quad \frac{1}{z} = \frac{\bar z}{|z|^2}$

Basic equations:

$r^n e^{in\theta} = \rho\cdot e^{i\varphi} \Leftrightarrow \begin{cases} r^n = \rho \\ e^{in\theta} = e^{i\varphi}\end{cases}$

$\Leftrightarrow \begin{cases} r = \sqrt[n]{\rho} \\ \theta_k = \frac{\varphi}{n} + \frac{2k\pi}{n}\end{cases}$ $\quad k = 0,\dots,n-1$ $\quad \hookrightarrow n$ solutions

⚠ $z^2 = i \ \cancel{\Rightarrow}\ z = \pm\sqrt{i}$

Complex and holomorphic functions

$f: \Omega \to \mathbb{C}$

$x+iy = z \to f(x,y) = \underbrace{U(x,y)}_{\mathrm{Re}(f)} + i\underbrace{V(x,y)}_{\mathrm{Im}(f)}$

Basic functions:

$f(z) =$	$U + i\cdot$	V
$\bar z$	x	$-y$
z^2	(x^2-y^2)	$(2xy)$
$e^z = e^{x+iy}$	$e^x\cos(y)$	$e^x\sin(y)$
$\log(z)$	$\log(\|z\|)$	$\arg(z)$

$\sin(z) = \frac{e^{iz}-e^{-iz}}{2}$

$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$

$\sinh(z) = \frac{e^z - e^{-z}}{2}$

$\cosh(z) = \frac{e^z+e^{-z}}{2}$

Continuity:

Continuous at $z_0 \in \Omega$ if $\forall \varepsilon>0, \exists \delta>0$:

$|z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon$

$\Rightarrow \lim_{z\to z_0} f(z) = f(z_0)$ (all directions)

Holomorphic:

$f: \Omega \in \mathbb{C} \to \mathbb{C}$ holomorphic if $\forall z_0 \in \Omega$

$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists + is finite

Let $U(x,y) = \mathrm{Re}(f)$ and $V(x,y) = \mathrm{Im}(f)$

f holomorphic $\Leftrightarrow$ $U,V: \Omega \to \mathbb{R}$ are $C^1(\Omega)$ and satisfy C-R equ.

$$\boxed{\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \text{ and } \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}}$$

$$\boxed{f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} - i\frac{\partial U}{\partial y}}$$

Complex derivative properties

$(f+g)' = f'+g'$ $\quad (fg)' = f'g + g'f$

$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$ $\quad (f\circ g)' = f'(g)\cdot g'$

Complex logarithm properties

$\log(z\cdot w) = \begin{cases} \log(z)+\log(w) & \text{if } \arg(z)+\arg(w) \in]-\pi;\pi] \\ \log(z)+\log(w) - 2\pi i & \arg(z)+\arg(w) \in [\pi;2\pi] \\ \log(z)+\log(w)+2\pi i & \arg(z)+\arg(w) \in [-2\pi;-\pi] \end{cases}$

Complex function integral along curve $\Gamma \in \mathbb{C}$

Curves:

$\Gamma \in \mathbb{R}^n$ is image of $\gamma: I \in \mathbb{R} \to \mathbb{R}^n$

Param: $\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$ $\quad t \in [a,b]$

- Γ simple $\Rightarrow$ does not cross same point twice
- Γ closed $\Rightarrow$ $\gamma[a] = \gamma[b]$
- Γ regular $\Rightarrow$ $\|\gamma'(t)\| \neq 0\ \forall t$, i.e. only 1 direction
- Γ piecewise regular $\Rightarrow$ union of regular curves
- Γ positively oriented, simple, closed $\Rightarrow$ goes along border with domain to the left.

Complex curve integrals

$f: \Omega \in \mathbb{C} \to \mathbb{C}$, $\Gamma \in \Omega$ simple regular curve

$$\boxed{\int_\Gamma F(z)\,dz := \int_a^b F(\gamma_1(t) + i\gamma_2(t)) \cdot \vec{\gamma}'(t)\,dt}$$

complex multiplication

Cauchy theorem

$D \in \mathbb{C}$ simply connected

$f: D \to \mathbb{C}$ holomorphic

$\Gamma \in D$ simple closed

$\oint_\Gamma F(z)\,dz = 0$

Cauchy corollary

$F: D \to \mathbb{C}$ holomorphic

D simply connected

$\gamma_1, \gamma_2: [a,b] \to D$ s.t.: $\gamma_1(a) = \gamma_2(b)$ and $\gamma_2(a) = \gamma_1(b)$

$\int_{\Gamma_1} F(z)\,dz = \int_{\Gamma_2} F(z)\,dz$

Cauchy integral formula

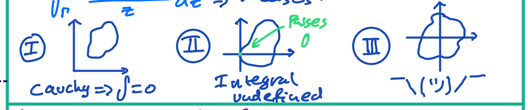
$D \in \mathbb{C}$ simply connected $\quad F: D \to \mathbb{C}$ holomorphic

$\gamma: [a,b] \to D$ simple closed curve

$$\boxed{\forall z_0 \in D: \quad F(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{F(\omega)}{\omega - z_0}\,d\omega \qquad F^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\Gamma \frac{F(\omega)}{(\omega-z_0)^{n+1}}\,d\omega}$$

USE: i.e. calculate all possible values for:

$\int_\Gamma \frac{\cos(2z)}{z}\,dz \to 3$ cases:



Laurent series in $\mathbb{C}$

$D \in \mathbb{C}$ simply connected, $F: D\setminus\{z_0\} \to \mathbb{C}$ holomorphic

$\Gamma \in D$ simply closed, $\quad z_0 \in \mathrm{Int}\,\Gamma$, $n \in \mathbb{Z}$

$R > 0$ s.t. $\{z \in \mathbb{C} \mid \|z-z_0\| < R\} \in D$

$$\boxed{C_n = \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{(z-z_0)^{n+1}}\,dz \qquad f(z) = Lf(z) = \sum_{-\infty}^{+\infty} C_n (z-z_0)^n \ \text{Laurent series}}$$

$L_N(f(z)) = \sum_{n=-N}^{N} C_n (z-z_0)^n$

If $F: D \to \mathbb{C}$ is holomorphic:

$Lf(z) = \sum_{n=0}^{+\infty} \frac{F^{(n)}(z_0)}{n!}(z-z_0)^n$ $\quad$ By Cauchy integral formula

Singular part, Regular part, Residue

$Lf(z) = \underbrace{\sum_{n=-\infty}^{-1} C_n (z-z_0)^n}_{\text{Singular/principal part}} + \underbrace{\sum_{n=0}^{+\infty} C_n (z-z_0)^n}_{\text{Regular part}}$

- z_0 is a regular point if Singular part of $Lf(z)$ is 0
- $f(z)$ has pole of order m in z_0 if: $C_{-m} \neq 0$, $C_{-k} = 0 \quad \forall k > m$
- z_0 is a essential isolated singularity if $C_{-k} \neq 0$ for an infinite number of different k.
- $C_{-1} = \mathrm{Res}_{z_0}(F)$ is the residue at z_0

Residues If z_0 is a pole of order n:

$$\boxed{\mathrm{Res}_{z_0}(f) = \frac{1}{(n-1)!}\lim_{z\to z_0}\left[\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n \cdot f(z)\right]}$$

Let: $f(z) = \frac{p(z)}{q(z)}$ $\quad p(z_0) \neq 0$ $\quad q(z_0) = 0$, $q'(z_0) \neq 0$

$$\boxed{\mathrm{Res}_{z_0}(f) = \frac{p(z_0)}{q'(z_0)}}$$ $\quad p, q$ holomorphic

Residue theorem

$D \in \mathbb{C}$ simply connected, $\Gamma \in D$ p.w. regular simple closed

$z_1, \dots, z_n \in \mathrm{Int}(\Gamma)$

Let $F: D\setminus\{z_1,\dots,z_n\} \to \mathbb{C}$ holomorphic

$$\boxed{\frac{1}{2\pi i}\oint_\Gamma F(z)\,dz = \sum_{i=1}^{n} \mathrm{Res}_{z_i}(F(z))}$$

Solving real integrals with residue theorem

TYPE 1:

$\int_0^{2\pi} f(\cos(\theta), \sin(\theta))\,d\theta$ with: $f(x,y) = \frac{P(x,y)}{Q(x,y)}$

P, Q polynomials $\quad Q(\cos(\theta), \sin(\theta)) \neq 0\ \forall\theta$

RECIPE:

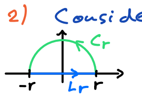
1) $\tilde F(z) = \frac{1}{iz} f\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)$
2) Find the singularities of $\tilde F(z)$
3) Check which ones are in $\mathrm{Int}(\gamma)$ $\quad \gamma = e^{i\theta}$ $\theta \in [0;2\pi]$
4) Compute the residues of $\tilde F(z)$ at z_i $\forall i$
5) Use residue theorem: $\int_0^{2\pi} f(\cos(\theta), \sin(\theta))\,d\theta = 2\pi i \sum \mathrm{Res}$

TYPE 2:

$\int_{-\infty}^{+\infty} R(x)\,e^{i\alpha x}\,dx$ $\quad$ - $\alpha \geq 0$ $\quad$ - $R(x) = \frac{P(x)}{Q(x)}$ P, Q polynomials $\quad$ - $Q(x) \neq 0\ \forall x \in \mathbb{R}$ $\quad$ - $\deg Q \geq \deg(P) + 2$

1) $\tilde R(z) = R(z)\,e^{i\varphi z}$

2) Consider $\Gamma_r = L_r \cup C_r$



$L_r = \{z \in \mathbb{C} \mid \mathrm{Im}(z) = 0, -r \leq \mathrm{Re}(z) \leq r\}$

$C_r = \{z \in \mathbb{C} \mid z = re^{i\theta}, 0 \leq \theta \leq 2\pi\}$

$\int_{-\infty}^{+\infty} R(x)\,e^{i\varphi x}\,dx = \lim_{r\to\infty}\int_{\Gamma_r} \tilde R(z)\,dz = 2\pi i\left(\sum_k \mathrm{Res}_{z_k}\right)$

With z_k singularities of $\tilde R(z)$ contained in Γ_r and r large enough that all z_k with $\mathrm{Im}>0$ are in Γ_r

3) Find singularities of $\tilde R(z)$ where $\mathrm{Im}(z_k) \geq 0$

4) Verify in specific case:

$\left|\int_{C_r} R(z)e^{i\varphi z}\,dz\right| = \left|\int_0^\pi \frac{P(re^{i\theta})}{Q(re^{i\theta})} e^{i\varphi r(\cos\theta + i\sin\theta)}\, i\,re^{i\theta}\,d\theta\right|$

$\leq \int_0^\pi \frac{|P|}{|Q|}\cdot r\,d\theta \xrightarrow[r\to\infty]{} 0$

5) $\int_{-\infty}^{+\infty} R(x)\,e^{i\varphi x}\,dx = 2\pi i \sum_k \mathrm{Res}_{z_k}$

Laplace transforms

Laplace transform: Definition

$f: [0,+\infty[\to \mathbb{R}$ piecewise continuous, $\gamma_0 \in \mathbb{R}$, $\int_0^{+\infty} |f(t)|\,e^{-\gamma_0 t}\,dt < \infty$,

$\forall z \in \bar{\mathcal{O}} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$

$$\boxed{\mathcal{L}(f)(z) = \int_0^{+\infty} f(z)\,e^{-tz}\,dz}$$

Laplace transform properties

1) If $\int_0^{+\infty} |t\cdot f(t)|\,e^{-\gamma_0 t}\,dt < +\infty$

$\Rightarrow \mathcal{L}(f)(z)$ holomorphic $\forall z \in \mathcal{O}$

$\boxed{(\mathcal{L}(f)(z))' = -\mathcal{L}(t f(t))(z)}$

2) $\mathcal{L}$ is linear $\forall a, b \in \mathbb{R}$

$\boxed{\mathcal{L}(af+bg)(z) = a\,\mathcal{L}(f)(z) + b\,\mathcal{L}(g)(z)}$

3) If $\int_0^{+\infty} |f^k(t)|\,e^{-\gamma_0 t}\,dt < \infty$ $\quad \forall k = 0,\dots,n$

$\boxed{\mathcal{L}(f^{(n)})(z) = z^n\,\mathcal{L}(f)(z) - \sum_{k=0}^{n-1} z^k f^{(n-k-1)}(0)}$

4) Let $\varphi(s) = \int_0^t f(s)\,ds$

$\boxed{\mathcal{L}(\varphi)(z) = \mathcal{L}(f)(z)/z}$

5) Scaling: $\forall a, b \in \mathbb{R}$, $a > 0$

$\varphi(s) = e^{-bt}\,F(at)$ $\quad \boxed{\mathcal{L}(\varphi)(z) = \frac{1}{a}\mathcal{L}(f)\left(\frac{z+b}{a}\right)}$

6) Convolution of functions, $f, g: \mathbb{R}^+ \to \mathbb{R}$ pw continuous.

$(f * g)(t) = \int_{-\infty}^{+\infty} f(t-s)\,g(s)\,ds = \int_0^t f(t-s)\,g(s)\,ds$ $\quad (f, g = 0 \text{ for } t < 0)$

$\boxed{\mathcal{L}(f*g)(z) = \mathcal{L}(f)(z)\cdot\mathcal{L}(g)(z)}$

Solving ODEs with Laplace transforms

TYPE 1:

$y: \mathbb{R}^+ \to \mathbb{R}$ s.t. $\begin{cases} y'(t) + a y(t) = f(t) \\ y(0) = y_0 \end{cases}$

1) Apply Laplace:

$y(z) = \mathcal{L}(y)(z)$ $\quad f(z) = \mathcal{L}(f)(z)$

$\mathcal{L}(y')(z) = z\,y(z) - \underbrace{y(0)}_{=y_0}$

2) Solve for Laplace transform:

$z\,y(z) - y_0 + a\,y(z) = f(z)$

$\Rightarrow (z+a)\,y(z) = y_0 + f(z)$

$\Rightarrow y(z) = \frac{y_0}{z+a} + \frac{f(z)}{z+a}$ $\quad$ Solution for Laplace transform

3) Transform back

$y(t) = y_0\,e^{-at} + \underbrace{\int_0^t f(s)\,e^{-a(t-s)}\,ds}$

$\mathcal{L}(e^{-at})(z) = \frac{1}{z+a}$ $\quad \mathcal{L}(f * e^{-at}) = \mathcal{L}(f)(z)\cdot\mathcal{L}(e^{-at})(z)$

TYPE 2:

$\begin{cases} m y''(t) = -k\,y(t) - \beta\,y'(t) \\ y(0) = y_0, \quad y'(0) = V_0 \end{cases} \Leftrightarrow \begin{cases} y''(t) + \omega^2 y(t) - 2\alpha y'(t) = 0 \\ y(0) = y_0;\ y'(0) = V_0 \end{cases}$

$\omega^2 = \frac{k}{m}$, $2\alpha = \frac{\beta}{m}$, $\alpha > 0$

1) Laplace transform:

$y(z) = \mathcal{L}(y)(z)$

$\to z^2 y(z) - z\,y(0) - y'(0) + \omega^2 y(z) + 2\alpha(z\,y(z) - y_0) = 0$

2) Solve

$(z^2 + \omega^2 + 2\alpha z)\,y(z) - z\,y_0 - 2\alpha y_0 - V_0 = 0$

$y(z) = y_0\frac{z}{z^2+\omega^2+2\alpha z} + \frac{2\alpha y_0 + V_0}{z^2+\omega^2+2\alpha z}$

3) Reverse transform

3.1) If possible use properties to reduce to known transforms

3.2) Use inverse transform formula.

Using the inverse transform formula

$f: \mathbb{R}^+ \to \mathbb{R}$ continuous with $F(0) = 0$

$\tilde f(z) = \mathcal{L}(f)(z)$

Assume that $\exists \gamma \in \mathbb{R}$ s.b.:

$\int_0^{+\infty} |f(t)|\,e^{-\gamma t}\,dt < +\infty$; $\int_{-\infty}^{+\infty} |\tilde f(\gamma + is)|\,ds < +\infty$

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde f(\gamma+is)\,e^{(\gamma+is)t}\,ds}$$

Using the formula:

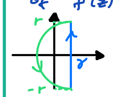
$\to$ Use residue theorem.

1) What we have:

complex function: $\tilde f(z) := f(z)\,e^{tz}$

2) Domain choice:

We take $\gamma \in \mathbb{R}$, $r \in \mathbb{R}^+$ s.t. all singularities of $\tilde f(z)$ are in $\Gamma_{\gamma,r}$



$\Gamma_{\gamma,r} = L_{\gamma,r} \cup C_{\gamma,r}$

$L_{\gamma,r} = \left\{z \in \mathbb{C} \,\middle|\, \begin{matrix}\mathrm{Re}(z) = \gamma \\ \mathrm{Im}(z) \in [-r,r]\end{matrix}\right\}$

$C_{\gamma,r} = \{z \in \mathbb{C} \mid |z-\gamma| = r, \mathrm{Re}(z) < \gamma\}$

3) Show that $\lim_{r\to\infty}\int_{C_r} \tilde f(z)\,dz = 0$

We will need to show that:

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tilde f(z)\,d\theta \xrightarrow[r\to\infty]{} 0$

showing the following is sufficient:

$\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tilde f(z)\,dz\right| \leq \int_{\pi/2}^{\frac{3\pi}{2}} |\tilde f(z)|\,dz$

4) Residue theorem gives result

$f(t) = \sum_{j=1}^{n} \mathrm{Res}_{z_j}(\tilde f)$

Partial differential equations

3 methods to solve PDES

- Separation of variables
- Fourier series
- Method of propagation of the fundamental solution
  ↳ Distributions Theory

METHOD 1: Separation of variables

→ Reduces a PDE to a system of ODEs

→ Useful when equations are on an interval

I.E. Heat equation on interval

$$(H)\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2} = f(x,t) & x\in[0,1],\ t>0 \\ u(0,t) = u(1,t) = 0 & \forall t>0 \\ u(x,0) = u_0(x) \end{cases}$$

I.E. Wave equation on interval

$$(W)\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} - c^2\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t) & x\in[0,1],\ t>0 \\ u(x,0) = u_0(x) \qquad u(0,t) = u(1,t) \\ \frac{\partial}{\partial t}u(x,0) = v_0(x) \end{cases}$$

Francesca Carocci method

1) Extend $u(x,t)$, $f(x,t)$, $u_0(x)$ ($, v_0(x)$ if (W)) to odd, 2 periodic functions

2) Take fourier series expansions

$$u(x,t) = \sum_{n\geq 1} b_n(t)\sin(\pi n x)$$

↳ $b_n(t) = 2\int_0^1 u(x,t)\sin(\pi n x)\,dx$ → Unknown

$$F_f(f(x,t)) = \sum_{n\geq 1}\beta_n(t)\sin(\pi n x)\,dx$$

↳ $\beta_n(t) = 2\int_0^1 f(x,t)\sin(\pi n x)\,dx$ → known

$$F_f(u_0(x)) = \sum_{n\geq 1} b_{0n}\sin(\pi n x) \to b_{0n} \to \text{known}$$

if (W): $F_f(v_0(x)) = \sum_{n\geq 1} c_{0n}\sin(\pi n x) \to c_{0n}$ → known

4) (H) and (W) become:

$$(H)\begin{cases} b_n'(t) + \pi^2 n^2 b_n(t) = \beta_n(t) \\ b_n(0) = b_{0n} \end{cases} \quad \forall n$$

$$(W)\begin{cases} b_n''(t) + c^2\pi^2 n^2 b_n(t) = \beta_n(t) \\ b_n(0) = b_{0n} \\ c_n(0) = c_{0n} \end{cases} \quad \forall n$$

5) Solve ODEs with i.e. Laplace transform
→ Find values of $b_n(t)$

$$u(x,t) = \sum b_n(t)\sin(\pi n x)$$

6) Try to convert back to function
→ If Fourier series recognized → give fct.
→ If not answer is Fourier series.

David Strutt method

1) On pose $u(x,t) = \varphi(x)\cdot\Psi(t)$

2) On obtient: (H)

$$\begin{cases} \varphi(x)\cdot\Psi'(t) = c^2\varphi''(x)\cdot\Psi(t) \\ \varphi(0)\cdot\Psi(t) = \varphi(L)\cdot\Psi(t) = 0 \\ \varphi(x)\cdot\Psi(0) = f(x) \end{cases}$$

$$\begin{cases} \frac{1}{c^2}\frac{\Psi'(t)}{\Psi(t)} = \frac{\varphi''(x)}{\varphi(x)} & (1) \\ \varphi(0) = \varphi(L) & (2) \\ \varphi(x)\Psi(0) = f(x) & (3) \end{cases}$$

→ L'équation n'est que valide si les fonctions sont des constantes

3) On a ensuite: (1) devient:

$$\frac{1}{c^2}\frac{\Psi'(t)}{\Psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda \quad \text{une constante}$$

$$\Psi'(t) - c^2\lambda\Psi(t) = 0 \qquad \varphi''(x) - \lambda\varphi(x) = 0$$

↳ 1 EDO par variable!

4) On regroupe ce qui va ensemble:

ce qui va avec x:

$$\varphi''(x) - \lambda\varphi(x) = 0 \quad \text{et} \quad \varphi(0) = \varphi(L) = 0$$

ce qui va avec t:

$$\Psi'(t) - c^2\lambda\Psi(t) = 0$$

ce qu'on ne peut pas séparer: (sera laissé de côté)

$$\varphi(x)\cdot\Psi(0) = f(x)$$

5) On retrouve un problème de Sturm-Liouville

$$\begin{cases} \varphi''(x) - \lambda\varphi(x) \\ \varphi(0) = \varphi(L) = 0 \end{cases} \quad (P_1^x) \to \text{Sturm-Liouville 1}$$

$$\Psi'(t) - c^2\lambda\Psi(t) = 0 \quad (P_1^t)$$

6) Sturm-Liouville nous donne:

$$\varphi_n(x) = \beta_n\sin\left(\frac{n\pi}{L}x\right), \quad \lambda = -\left(\frac{n\pi}{L}\right)^2 \quad n\in\mathbb{N}_{\geq 1}$$

7) On reprend (P_1^t)

$$\Psi_n'(t) + c^2\left(\frac{n\pi}{L}\right)^2\Psi_n(t) = 0 \Rightarrow \Psi_n(t) = \gamma_n e^{-c^2\left(\frac{\pi n}{L}\right)^2 t}$$

8) On a la solution générale:

$$u_n(x,t) = \varphi_n(x)\cdot\Psi_n(t) = \beta_n\gamma_n\sin\left(\frac{n\pi}{L}x\right)e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}$$

9) On finit de résoudre:

$$u(x,t) = \sum_{n=1}^{+\infty}\tilde{\beta}_n\sin\left(\frac{n\pi}{L}x\right)e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

où on choisit les $\tilde{\beta}_n$ tels que $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{+\infty}\tilde{\beta}_n\sin\left(\frac{n\pi}{L}x\right)$$

↳ Avec $\tilde{\beta}_n$ les coeffs en sinus de $f(x)$

METHOD 2: Fourier transform

1) Fourier transform
Find the fourier transforms of all relevant fcts.

for (H):

$$\hat{u}(\alpha,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}u(x,t)e^{-i\alpha x}dx \quad \text{(unknown)}$$

$$\hat{f}(\alpha,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x,t)e^{-i\alpha x}dx$$

$$\hat{u}_0(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}u_0(x)e^{-i\alpha x}dx$$

2) Rewrite diff. eq in terms of F.T.

Use: $\mathcal{F}(f^{(n)})(\alpha) = (i\alpha)^n\mathcal{F}(f)(\alpha)$

$$\begin{cases} \frac{\partial\hat{u}}{\partial t}(\alpha,t) + \alpha^2\hat{u}(\alpha,t) = \hat{f}(\alpha,t) \\ \hat{u}(\alpha,0) = \hat{u}_0(\alpha) \end{cases}$$

3) Solve the ODE in t (i.e. with Laplace transform)

Here: $\hat{u}(\alpha,t) = \hat{u}_0(\alpha)e^{-\alpha^2 t} + \int_0^t\hat{f}(\alpha,s)e^{-\alpha^2(t-s)}ds$

4) Invert fourier transform

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty}u_0(y)e^{-\frac{(x-y)^2}{4t}}dy + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\int_0^t\hat{f}(\alpha,s)e^{-\alpha^2(t-s)}ds\right)e^{i\alpha x}dx$$

Solving wave equations with distributions

Problem:

$$(1)\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} = c^2\frac{\partial^2 u(x,t)}{\partial x^2} & t\geq 0 \\ u(x,0) = g(x), \quad \frac{\partial u}{\partial t}(x,0) = h(x) \end{cases}$$

Generalized problem in distributional sense

$$(2)\quad \left\langle\frac{\partial^2 k}{\partial t^2} - c^2\frac{\partial^2 k}{\partial x^2}, \varphi\right\rangle = 0 \qquad \forall\varphi\in\mathcal{D}_{\mathbb{R}\times\mathbb{R}^+}$$

$$\left\langle\frac{\partial k}{\partial t}, \varphi(x,t)\right\rangle \xrightarrow{t\to 0} 0$$

We want: $k\in\mathcal{D}_{\mathbb{R}\times\mathbb{R}^+}$

$$\left\langle\frac{\partial k}{\partial t}, \varphi(x,t)\right\rangle \xrightarrow{t\to 0} \langle\delta_{(0,0)}, \varphi\rangle$$

If k is a solution to (2):

$$\langle u, -\rangle := \left\langle\frac{\partial k}{\partial t} * g(x), -\right\rangle + \langle k * h(x), -\rangle$$

is a solution to (1) in the distributional sense

(2) is solved by the distribution associated with the p.w. cont. fct.:

$$k(x,t) = \frac{1}{2c}\left(H(x+ct) - H(x-ct)\right)$$

$$H(y) = \begin{cases} 0 & y<0 \\ \frac{1}{2} & y=0 \\ 1 & y>0 \end{cases}$$

$$\Rightarrow u(x,t) = \frac{g(x+ct)+g(x-ct)}{2} + \int_{h-ct}^{h+ct}h(y)\,dy$$

Theory of distributions

Test function

A test function is such that:

$\varphi\in C^\infty_{ct}\,\mathbb{R}$ i.e. φ infinitely differentiable with compact support

- $\forall\alpha\in\mathbb{N}$ $\varphi^{(\alpha)}$ exists and is continuous
- $\text{supp}(\varphi) = \{x\in\mathbb{R} \mid \varphi(x)\neq 0\}$
- Compact support ⇒ closed + bounded

Dirac mass

We take F_ε s.t.:

$$\langle F_\varepsilon, \varphi\rangle := \int_{-\infty}^{+\infty}F_\varepsilon(x)\varphi(x)dx = \frac{1}{\varepsilon}\int_0^\varepsilon\varphi(x)dx$$

$$\langle F_\varepsilon, -\rangle : C^\infty_{ct}(\mathbb{R}) \to \mathbb{R}, \quad \varphi \to \langle F_\varepsilon, \varphi\rangle$$

⚠ F_ε is not a function, $\lim_{\varepsilon\to 0}F_\varepsilon(x)$ makes no sense

We can however take:

assuming φ has a primitive Φ

$$\lim_{\varepsilon\to 0}\langle F_\varepsilon, \varphi\rangle = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon\varphi(x)dx = \varphi(0)$$

We call $\lim_{\varepsilon\to 0}F_\varepsilon$ the distribution δ_0

Dirac mass is given by:

$$\langle\delta_0, \varphi\rangle := \varphi(0) \quad \forall\varphi\in C^\infty_{ct}(\mathbb{R})$$

Definition of a distribution

A distribution is a continuous linear form on the set $C^\infty_{ct}(\mathbb{R}) := D$ − set of test functions

i.e.: $\langle\Psi, -\rangle : D \to \mathbb{R}$, $\varphi \to \langle\Psi, \varphi\rangle$

1) $\langle\Psi, \varphi\rangle < +\infty$ 2) $\langle\Psi, a_1\varphi_1 + a_2\varphi_2\rangle = a_1\langle\Psi, \varphi_1\rangle + a_2\langle\Psi, \varphi_2\rangle$

Set of continuous linear forms is called set of distributions denoted D'.

Distribution derivatives

For any $\Psi\in D'$ a distribution

$$\boxed{\langle\Psi^{(n)}, \varphi\rangle = \langle D^{(n)}\Psi, \varphi\rangle := (-1)^n\langle\Psi, \varphi^{(n)}\rangle}$$

Laplace transform of a distribution:

Take $\langle\Psi, -\rangle\in D'$ s.t.

$$\langle\Psi, -\rangle = \langle D^{(k)}h, -\rangle \quad \text{for } h:\mathbb{R}\to\mathbb{R}$$

$$\boxed{\mathcal{L}(\Psi)(z) := z^k\mathcal{L}(h)(z)}$$

Convolution of functions in distributions

Take $\langle\Psi_1, -\rangle\in\mathcal{D}'$ and $\langle\Psi_2, -\rangle\in\mathcal{D}'$

with: $\langle\Psi_1, -\rangle = \langle D^{(k_1)}h_1, -\rangle$ and $\langle\Psi_2, -\rangle = \langle D^{(k_2)}h_2, -\rangle$

$$\Psi_1 * \Psi_2 := D^{(k_1+k_2)}(h_1 * h_2)$$

↳ Distributional derivative

$$\langle\Psi_1 * \Psi_2, -\rangle := \langle D^{k_1+k_2}(h_1 * h_2), -\rangle$$

Generalization to multiple variables

Test functions in $\mathbb{R}^n$:

$$\mathcal{D}_{\mathbb{R}^n} = \left\{\varphi:\mathbb{R}^n\to\mathbb{R} \;\middle|\; \begin{array}{l}(1)\ \varphi\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) \\ (2)\ \text{supp}(\varphi) \text{ is closed+bounded}\end{array}\right\}$$

Distributions in $\mathbb{R}^n$:

$$\mathcal{D}'_{\mathbb{R}^n} = \left\{\varphi:\mathbb{R}^n\to\mathbb{R} \;\middle|\; \begin{array}{l}(1)\ \varphi\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) \\ (2)\ \text{supp}(\varphi) \text{ is closed+bounded}\end{array}\right\}$$

Distributional partial derivatives:

let $\alpha = (\alpha_1, \dots, \alpha_n)$ $|\alpha| = \sum\alpha_i$ → Total order of deriv.

$$\langle D^\alpha\Psi, \varphi\rangle := (-1)^{|\alpha|}\left\langle\Psi, \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\right\rangle$$

Convolutions in $\mathcal{D}'_{\mathbb{R}^n}$:

$$g_1 * g_2(x_1,\dots,x_n) = \int_{\mathbb{R}^n}g_1(x_1-t_1,\dots,x_n-t_n)\cdot g_2(t_1,\dots,t_n)\,dt_1\cdots dt_n$$

$$\langle\Psi_1, -\rangle = \langle D^\alpha g_1, -\rangle \quad \text{and} \quad \langle\Psi_2, -\rangle = \langle D^\beta g_2, -\rangle$$

$$\langle\Psi_1 * \Psi_2, -\rangle := D^{\alpha+\beta}(g_1 * g_2)$$

→ Component

Also:

$$D^\alpha(g_1 * g_2(x)) = D^\alpha g_1(x) * g_2(x)$$

— Laplace transform of distribution (Lecture 10)

— Diff. eq. in distributions (Lecture 10)