CHAPTER 2 : FOURIER TRANSFORM

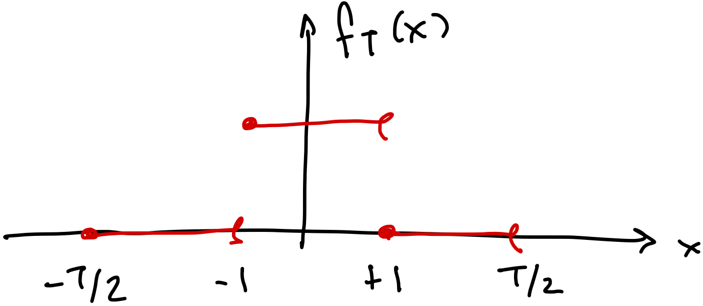
2.1 Introduction

2.1.1 Motivation

Fourier series develop periodic functions as an infinite series of sines and cosines.
Fourier transforms allow to study general functions (not necessarily periodic).

- Idea: let $T > 0$ and f_T be a T-periodic defined by

$$f_T(x) = \begin{cases} 0 & \text{if } x \in [-\frac{T}{2}, -1[\\ 1 & \text{if } x \in [-1, 1[\\ 0 & \text{if } x \in [1, \frac{T}{2}[\end{cases}$$



What if $T \to \infty$? Then:

$$\lim_{T \to \infty} f_T(x) = \begin{cases} 1 & \text{if } x \in [-1, +1[\\ 0 & \text{if } x \notin [-1, +1[\end{cases}$$

and this not a periodic function.

2.1.2 Heuristic "discovery" of Fourier transform

(See attached document on Moodle).

2.2 Fourier transform of a function

2.2.1 Definitions

let $f: \mathbb{R} \longrightarrow \mathbb{R}$ and $g: \mathbb{R} \longrightarrow \mathbb{C}$

$x \longmapsto f(x)$ $\qquad$ $\alpha \longmapsto g(\alpha)$

be piecewise-defined functions s.t.

$$\int_{-\infty}^{+\infty} |f(x)|\,dx < \infty \text{ and } \int_{-\infty}^{\infty} |g(\alpha)|\,d\alpha < \infty.$$

1- Then the Fourier transform maps a given function f (input) into a new function $\mathcal{F}(f)$ (output)

that is:

$$\mathcal{F}(f): \mathbb{R} \longrightarrow \mathbb{C}$$

$$\alpha \longmapsto \mathcal{F}(f)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

Note: $\alpha \in \mathbb{R}$ but (in principle) $\mathcal{F}(f)(\alpha) \in \mathbb{C}$.

Note: $\mathcal{F}(f)(\alpha)$ can also be denoted as $\hat{f}(\alpha)$.

2- The inverse Fourier transform maps a given function g (input) into a new function $\mathcal{F}^{-1}(g)$ (output)

$$\mathcal{F}^{-1}(g) : \mathbb{R} \longrightarrow \mathbb{C}$$
$$x \longmapsto \mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha x}\, d\alpha$$

Note: this new function $\mathcal{F}^{-1}(g) : \mathbb{R} \longrightarrow \mathbb{C}$ takes values $x \in \mathbb{R}$ and transforms them into values $\mathcal{F}^{-1}(g)(x)$ that can be complex.

Important note: the Fourier transform and the inverse Fourier transform are not function compositions $\left(\mathcal{F}(f)(\alpha) \neq \mathcal{F} \circ f\,(\alpha) \right)$.
Instead, they transform functions into new functions. And, in both cases, the newly created functions are like $h : \mathbb{R} \to \mathbb{C}$.

2.2.2 Fourier inversion theorem (reciprocity)

- Theorem

let $f : \mathbb{R} \to \mathbb{R}$ be a function s.t.
$x \mapsto f(x)$

f and f' are piecewise-defined and

$$\int_{-\infty}^{+\infty} |\hat{f}(\alpha)| \, d\alpha < \infty \quad \text{then } \forall x \in \mathbb{R}$$

Note that $\hat{f} : \mathbb{R} \to \mathbb{C}$. Then, we have

$$\mathcal{F}^{-1}(\hat{f})(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} = \frac{1}{2}\left(f(x+0) + f(x-0) \right)$$

In particular, if f is continuous, then

$$\frac{1}{2}\left(f(x+0) + f(x-0) \right) = f(x) \quad \text{and then}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha .$$

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = f(x)$$

$$f \xrightarrow{\mathcal{F}} \hat{f} \xrightarrow{\mathcal{F}^{-1}} f$$

2.2.3 Examples :

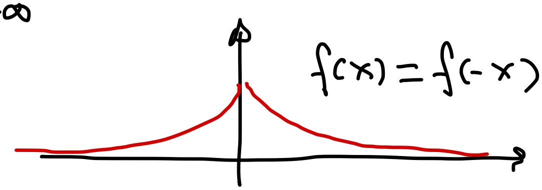
- Example 1: Compute the Fourier transform of

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \longmapsto f(x) = e^{-|x|} = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ e^{x} & \text{if } x < 0 \end{cases}$$

$f \in C^0(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_{-\infty}^{\infty} e^{-|x|}\, dx = 2\int_0^{\infty} e^{-|x|}\, dx = 2\int_0^{\infty} e^{-x}\, dx$$



$= 2 < \infty \Rightarrow$ the Fourier transform is well defined:

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x} e^{-i\alpha x}\, dx + \int_{0}^{\infty} e^{-x} e^{-i\alpha x}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x(1-i\alpha)}\, dx + \int_{0}^{\infty} e^{-x(1+i\alpha)}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1+i\alpha+1-i\alpha}{(1-i\alpha)(1+i\alpha)} = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

$$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

$$\hat{f}(\alpha) \in C^0(\mathbb{R})$$

$$\int_{-\infty}^{+\infty} |\hat{f}(\alpha)|\, d\alpha = \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}\, d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \arctan(\alpha)\Big|_{-\infty}^{+\infty} = \sqrt{\frac{2}{\pi}}\,\pi < \infty$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha$$

$$e^{-|x|} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+\alpha^2}\, d\alpha \quad \forall x \in \mathbb{R}$$

(because f is continuous)

For $x=0$

$$e^{-|0|} = 1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha 0}}{1+\alpha^2}\, d\alpha = \frac{1}{\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{1+\alpha^2}\, d\alpha}_{=\pi} = 1$$

For $x=1$

$$e^{-|1|} = \frac{1}{e} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha 1}}{1+\alpha^2}\, d\alpha =$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\cos\alpha}{1+\alpha^2}\, d\alpha + i\, \underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha}{1+\alpha^2}\, d\alpha}_{=0}$$

$\left(\frac{\sin\alpha}{1+\alpha^2} \text{ is odd} \right)$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\cos\alpha}{1+\alpha^2} = \frac{1}{e}$$

2.3 Properties of Fourier transform

f and $g : \mathbb{R} \to \mathbb{R}$, $\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty$

$\int_{-\infty}^{+\infty} |g(x)|\, dx < \infty$, f and g are piecewise-defined

$\mathcal{F}(f) \equiv \hat{f}$, $\mathcal{F}(g) \equiv \hat{g}$.

2.3.1 Continuity and linearity

- $\hat{f}$ is continuous $\forall\, \alpha \in \mathbb{R}$ and $\lim_{\alpha \to \pm\infty} \hat{f}(\alpha) = 0$.

- $\mathcal{F}(af+bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g)$, $a, b \in \mathbb{R}$

2.3.2 Convolution product

- Definition: the convolution (product) of two functions f and g is a function denoted as $f * g$ defined by

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt$$

$$(f*g)(x) = (g*f)(x)$$

(Hint: do a change of variable $t \to t' = x-t$)

$$\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g)$$

$$\widehat{f*g} = \sqrt{2\pi}\, \hat{f}\, \hat{g}$$

2.3.3 Fourier transform of the derivative:

If in addition $f \in C^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} |f'(x)|\, dx < \infty$

$$\mathcal{F}(f')(\alpha) = i\alpha\, \mathcal{F}(f)(\alpha)$$

If $f \in C^{(n)}(\mathbb{R})$ and $\int_{-\infty}^{+\infty} |f^{(k)}(x)|\, dx < \infty$

$k = 1, 2, \dots, n$.

$$\mathcal{F}(f^{(k)})(\alpha) = (i\alpha)^k\, \mathcal{F}(f)(\alpha)$$

(Ex: $m \dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + k\,x = f$)

2.3.4 shift (décalage)

If $a \in \mathbb{R}^*$, $b \in \mathbb{R}$ and $g(x) = e^{-ibx} f(ax)$

then, $\mathcal{F}(g)(\alpha) = \dfrac{1}{|a|}\, \mathcal{F}(f)\left(\dfrac{\alpha + b}{a}\right) \quad \forall \alpha \in \mathbb{R}$

2.3.5 Plancherel identity

- If in addition $\int_{-\infty}^{+\infty} (f(x))^2 dx < \infty$ then:

$$\int_{-\infty}^{+\infty} (f(x))^2 = \int_{-\infty}^{+\infty} |\mathcal{F}(f)(\alpha)|^2\, d\alpha.$$

2.3.6 Fourier transform in sines and cosines

- If the $f : \mathbb{R} \to \mathbb{R}$ is an even function ($f(x) = f(-x)$)

then:
$$\mathcal{F}(f)(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\alpha x)\, dx.$$

this is the <u>Fourier transform is cosines</u>

- If $f : \mathbb{R} \to \mathbb{R}$ is odd ($f(x) = -f(-x)$)

then:
$$\mathcal{F}(f)(\alpha) = -i \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\alpha x)\, dx.$$

this is <u>the Fourier transform in sines</u>.

- Remark: if in addition f' is piecewise-defined and $\int_{-\infty}^{+\infty} |\hat{f}(\alpha)|\, d\alpha < \infty$ then, using the Fourier

inversion theorem: $\forall x$ where $f(x)$ is continuous

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\alpha) \cos(\alpha x) \, d\alpha \quad \text{when } f \text{ is even.}$$

$$f(x) = i \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\alpha) \sin(\alpha x) \, d\alpha \quad \text{when } f \text{ is odd.}$$