$\text{curl } F = 0 \quad \forall (x,y) \in \Omega_4 = \text{dom}(F)$

14/10/2021

Ω_1, Ω_2 are convex and simply connected, Ω_3 is simply connected but non convex

Ω_4 is non convex and non-simply connected.

- curl $F=0$ and $\left\{\begin{matrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{matrix}\right\}$ are $\left\{\begin{matrix}\text{convex}\\ \text{and/or}\\ \text{simply connected}\end{matrix}\right\} \Rightarrow$ F derives from a potential on $\Omega_1, \Omega_2, \Omega_3$.

- curl $F=0$ and Ω_4 is non-convex and non-simply connected

$\Rightarrow$ we don't know if F derives from a potential on Ω_4

- Ω_1: $\quad F = \text{grad } f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2} \qquad (1)$$

$$\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} \qquad (2)$$

$$(1) \longrightarrow f(x) = -\text{arctg}\left(\frac{x}{y}\right) + \alpha(y)$$

$\left[\text{Reminder } \frac{d}{dx}(\text{arctg}(x)) = \frac{1}{1+x^2}\right]$

$$(2) \longrightarrow \frac{\partial f}{\partial y} = -\frac{\frac{-x}{y^2}}{1+\left(\frac{x}{y}\right)^2} + \alpha'(y) = \frac{x}{x^2+y^2}$$

$$\frac{x}{\cancel{x^2+y^2}} + \alpha'(y) = \frac{x}{\cancel{x^2+y^2}} \longrightarrow \alpha'(y) = 0$$

$\alpha(y) = C_1 \in \mathbb{R}$, a constant.

The potential is $f(x,y) = -\operatorname{arctg}\left(\frac{x}{y}\right) + C_1$

- Ω_2: As the domain Ω_2 does not contain the line $y=0$, the previous result for Ω_1 is extendable to Ω_2.

$$f(x,y) = -\operatorname{arctg}\left(\frac{x}{y}\right) + C_2, \quad C_2 \in \mathbb{R}.$$

- Ω_3:

$$f(x,y) = \begin{cases} -\operatorname{arctg}\left(\frac{x}{y}\right) + C_1 & \text{if } y > 0 \\ -\operatorname{arctg}\left(\frac{x}{y}\right) + C_2 & \text{if } y < 0 \end{cases}$$

What happens in $y=0$, $x>0$.

$$\lim_{\substack{y \to 0^+ \\ x > 0}} f(x,y) = \lim_{\substack{y \to 0^+ \\ x > 0}} \underbrace{-\operatorname{arctg}\left(\overset{+\infty}{\frac{x}{y}}\right)}_{\pi/2} + C_1 = -\frac{\pi}{2} + C_1$$

$$\lim_{\substack{y\to 0^- \\ x>0}} f(x,y) = \lim_{\substack{y\to 0^- \\ x>0}} \underbrace{-\operatorname{arctg}\left(\overset{\to -\infty}{\frac{x}{y}}\right)}_{-\pi/2} + c_2 = \frac{\pi}{2} + c_2$$

We impose $\displaystyle\lim_{\substack{y\to 0^+ \\ x>0}} f(x,y) = \lim_{\substack{y\to 0^- \\ x>0}} f(x,y)$ (continuity)

$$-\frac{\pi}{2} + c_1 = \frac{\pi}{2} + c_2 \longrightarrow c_1 = \pi + c_2$$

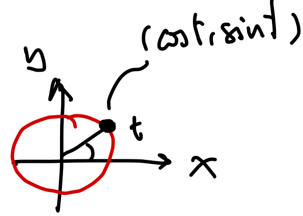
$$f(x,y) = \begin{cases} -\operatorname{arctg}\left(\frac{x}{y}\right) + \pi + c_2 & \text{if } y>0 \\ \pi/2 + c_2 & \text{if } y=0,\ x>0. \\ -\operatorname{arctg}\left(\frac{x}{y}\right) + c_2 & \text{if } y<0 \end{cases}$$

- Ω_4: we want to test if $F \neq \operatorname{grad} f$ in Ω_4.

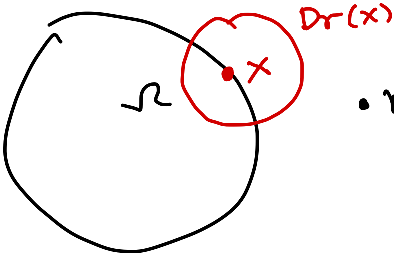
Then it is enough to find a closed curve Γ s.t. $\int_\Gamma F\cdot dl \neq 0$.

A possible curve is $\Gamma = \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$.

$$\gamma(t) : [0,2\pi] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (\cos t, \sin t)$$



$$\int_\Gamma F\cdot dl = \int_0^{2\pi} F(\gamma(t))\cdot\gamma'(t)\,dt$$



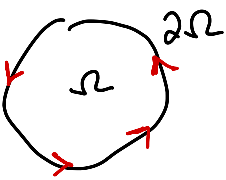
- we denote as $\overline{\Omega} = \Omega \cup \partial\Omega$ the <u>closure</u> of Ω.

(the set of points that don't belong to the interior of $\mathbb{R}^2 \setminus \Omega$)

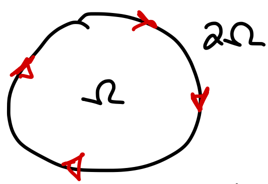
Definitions:

1) let $\Omega \subset \mathbb{R}^2$ be an open domain bounded such that $\partial\Omega$ is a closed (piecewise) regular curve.

we say that $\partial\Omega$ is <u>positively oriented</u> if when we tour along the curve, the domain is <u>on the left</u>.



positive orientation

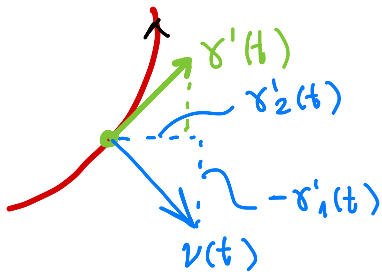


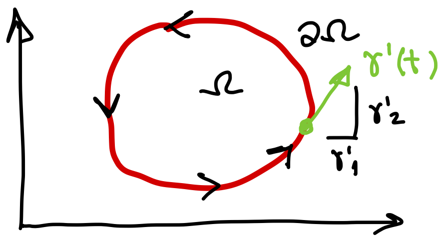
negative orientation

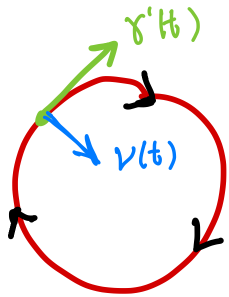
For a parameterization $\gamma : [a,b] \longrightarrow \partial\Omega$

$$t \longmapsto \gamma(t) = (\gamma_1(t), \gamma_2(t))$$

the normal vector of $\partial\Omega$ at x is given by $\nu(x) = (\gamma_2'(t), -\gamma_1'(t))$. It is an external normal if $\partial\Omega$ is positively oriented.







2) We say that a bounded open domain $A \subset \mathbb{R}^2$ is <u>a regular domain</u> if there exist bounded open domains $A_0, A_1, A_2, \dots, A_n \subset \mathbb{R}^2$ s.t.

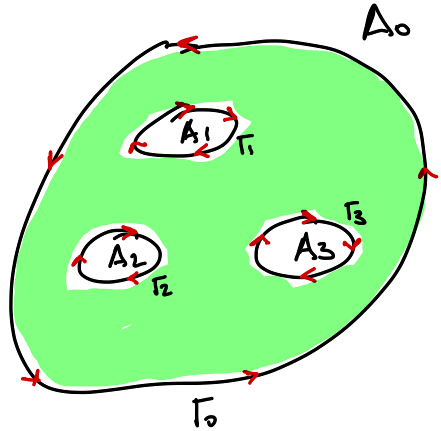
- $\bar{A}_j \subset A_0 \quad \forall\, j = 1, 2, \dots, n.$
- $\bar{A}_i \cap \bar{A}_j = \phi \quad \forall\, i, j = 1, \dots, n$ and $i \neq j$.

- $A = A_0 \setminus \left(\bigcup_{j=1}^{n} \bar{A}_j \right)$

- $\partial A_j = \Gamma_j$ for $j = 0, \ldots, n$

  are (piecewise) simple regular curves

- $\partial A = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n$



∂A is positively oriented if the circulation sense for each Γ_j, $j = 0, 1, \ldots, n$, is s.t. the domain A is on the left. I.e, the boundary Γ_0 is positively oriented, and the boundaries $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, are negatively oriented.

2.4.2 Green's theorem

- Theorem:

  Let $A \subset \mathbb{R}^2$ be a regular domain whose boundary ∂A is positively oriented. Let $F : \bar{A} \longrightarrow \mathbb{R}^2$

$$(x, y) \longmapsto F(x, y)$$

be a vector field $F \in C^1(\Omega, \mathbb{R}^2)$. Then

$$\iint_A \operatorname{curl} F(x,y)\,dx\,dy = \int_{\partial A} F \cdot d\ell$$

$$= \iint_A \left[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) \right] dx\,dy$$