CHAPTER 1: FOURIER SERIES

1.1 Introduction

1.1.1 Motivation and problematic

Fourier series approximate a periodic function as an infinite sum of sines and cosines

Problem: given $f: \mathbb{R} \to \mathbb{R}$ a T-periodic, can we write it as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right] \text{ ?}$$

Which are the coefficients $a_n, b_n \in \mathbb{R}$?

Answer: Yes, if we impose certain conditions to f.

1.1.2 Recalls and preliminary results

- Recall 1: A function $f: \mathbb{R} \to \mathbb{R}$ is T-periodic if $\exists\, T > 0$ s.t $f(x) = f(x+T)$ $\forall x \in \mathbb{R}$

  T is the period of f

Examples: $u_n(x) = \cos\left(\frac{2\pi n}{T} x\right)$

is $\frac{T}{n}$-periodic for $n \in \mathbb{N}^*$ ($n > 0$ and n is an integer)

$$u_n\left(x + \frac{T}{n}\right) = \cos\left[\frac{2\pi n}{T}\left(x + \frac{T}{n}\right)\right]$$

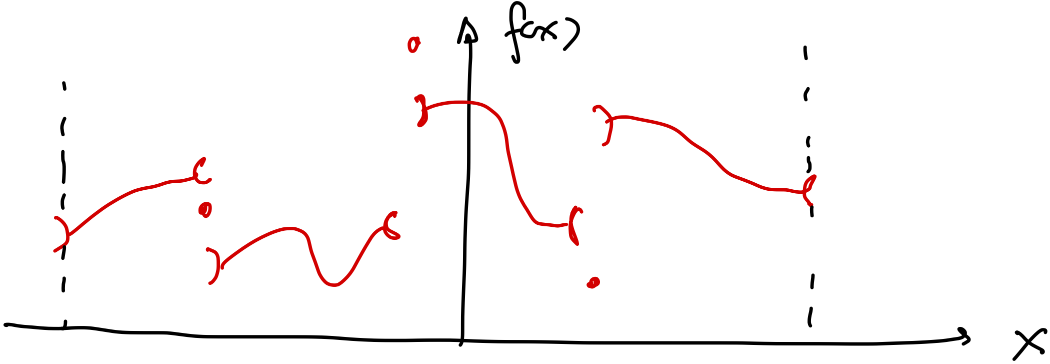
$$= \cos\left[\frac{2\pi n}{T} x + 2\pi\right] = \cos\left(\frac{2\pi n}{T} x\right) = u_n(x).$$

The same applies to $v_n(x) = \sin\left(\frac{2\pi n}{T} x\right)$

- Recall 2: A function $f: \mathbb{R} \to \mathbb{R}$ is <u>piecewise-defined</u> (continue par morceaux) if it has a finite number of discontinuities over every bounded interval and if at each discontinuity point x the limits:

$$\lim_{\substack{t \to x \\ t > x}} f(t) \doteq f(x+0) \quad \text{and} \quad \lim_{\substack{t \to x \\ t < x}} f(t) \doteq f(x-0)$$

exist and are finite.



- Result 1: let $m, n \in \mathbb{N}^*$ and $T > 0$ then

a) $$\frac{2}{T}\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx$$

$$= \frac{2}{T}\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi m}{T}x\right)dx$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

b) $$\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx = 0$$

1.2 Fourier series of a periodic function

1.2.1 Definitions:

Let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic piecewise-defined function. For $n \in \mathbb{N}^*$, the partial Fourier series of f and order N is

$$F_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right]$$

where the coefficients a_n and b_n (called Fourier coefficients) are given by

$$(a_0, a_1, a_2 \dots) \quad a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi n}{T} x\right) dx \quad n = 0, 1, \dots, N$$

$$(b_1, b_2, \dots) \quad b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n}{T} x\right) dx \quad n = 1, 2, \dots, N$$

We call the Fourier series of f to the limit of $F_N f(x)$, $(N \to \infty)$, when it exists.

$$Ff(x) = \lim_{N\to\infty} F_N f(x).$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)\right]$$

1.2.2 Heuristic justification of the definition

(next Thursday).

What is the retionship between $Ff(x)$ and $f(x)$?

1.2.3 Fourier series convergence

Dirichlet theorem

Let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic s.t. f and f' are piecewise-defined. Then:

$Ff(x) = \lim_{N\to\infty} F_N f(x)$ exists $\forall x \in \mathbb{R}$ and

$$Ff(x) = \frac{1}{2}\left[f(x+0) + f(x-0)\right] \quad \forall x \in \mathbb{R}.$$

$$\lim_{\substack{t \to x \\ t > x}} f(t) \doteq f(x+0) \qquad \lim_{\substack{t \to x \\ t < x}} f(t) \doteq f(x-0)$$

If f is continuous in x then

$f(x+0) = f(x-0) = f(x)$ and $(Ff)(x) = f(x)$.

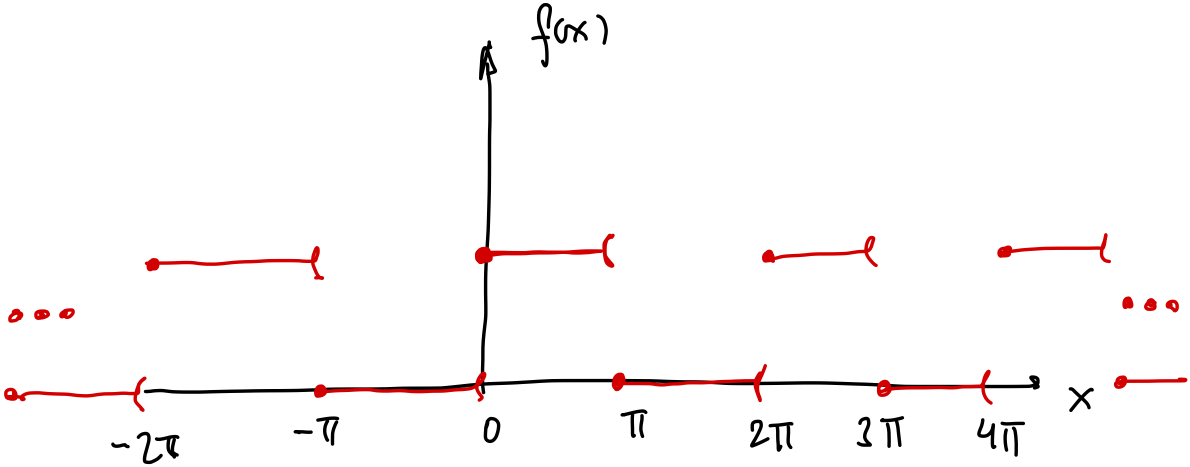
$$\frac{1}{2}\left(f(x+0) + f(x-0)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right)\right]$$

1.2.4 Examples:

- Example 1: let $f : [0, 2\pi[\to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [\pi, 2\pi[\end{cases}$$

extended by 2π-periodicity to $\mathbb{R}$.



Compute the Fourier series Ff and compare Ff and f on $[0, 2\pi]$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi = T} f(x) = \frac{1}{\pi}\int_0^{\pi} dx + \int_{\pi}^{2\pi} 0\, dx = 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \cos(nx)\,dx$$

$\left(\frac{2}{T}\right)$ $\qquad T = 2\pi$

$$= \frac{1}{\pi}\left.\frac{\sin(nx)}{n}\right|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

$$= -\frac{1}{\pi}\left.\frac{\cos(nx)}{n}\right|_0^{\pi} = -\frac{1}{n\pi}\left[\cos(n\pi) - \cos(0)\right] =$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$= \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi}\sin(nx) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi}\frac{\sin((2k+1)x)}{2k+1}$$

$n = 2k+1$, n is an odd number for $k = 0, 1, 2, \ldots$