## 2.1.2 Heuristic "discovery" of Fourier transform:

Let $f_T : \mathbb{R} \to \mathbb{R}$ a continuous $T$-periodic function such that $\lim_{T\to\infty} f_T(x) = f(x)$ and $f'$ is piecewise defined.

Then, the Fourier series of $f_T$ in complex notation is written as:

$$Ff_T(y) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{T}y} \quad \forall\, y \in \mathbb{R}$$

where $$c_n = \frac{1}{T}\int_0^T f_T(x)\, e^{-i\frac{2\pi n}{T}x}\, dx = \frac{1}{T}\int_{-T/2}^{T/2} f_T(x)\, e^{-i\frac{2\pi n}{T}x}\, dx$$

Let us introduce $\Delta\alpha = \frac{2\pi}{T}$ and $\alpha_n = n\Delta\alpha = \frac{2\pi n}{T}$.

Then $\frac{1}{T} = \frac{\Delta\alpha}{2\pi}$, $c_n = \frac{\Delta\alpha}{2\pi}\int_{-T/2}^{T/2} f_T(x)\, e^{-i\alpha_n x}\, dx$ and

$$Ff_T(y) = \sum_{n=-\infty}^{\infty}\left[\frac{\Delta\alpha}{2\pi}\int_{-T/2}^{T/2} f_T(x)\, e^{-i\alpha_n x}\, dx\right] e^{i\alpha_n y}$$

By exchanging the sum and the integral we can write.

$$Ff_T(y) = \frac{1}{2\pi}\int_{-T/2}^{T/2} f_T \left[\Delta\alpha \sum_{n=-\infty}^{\infty} e^{-i\alpha_n(x-y)}\right] dx$$

that is a Riemann sum that allows us to define an integral:

Ideed, $\Delta\alpha \sum_{n=-\infty}^{\infty} e^{-i\alpha_n(x-y)} = \sum_{n=-\infty}^{\infty} e^{-i\alpha_n(x-y)}(\alpha_n-\alpha_{n-1})$

$\Delta\alpha=\alpha_n-\alpha_{n-1}$

$= \int_{-\infty}^{\infty} e^{-i\alpha(x-y)}\,d\alpha$. Then, we obtain:

$$Ff_T(y) = \frac{1}{2\pi}\int_{-T/2}^{T/2} f_T(x)\left[\int_{-\infty}^{+\infty} e^{-i\alpha(x-y)}\,d\alpha\right]dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2} f_T(x)e^{-i\alpha x}\,dx\right]e^{i\alpha y}\,d\alpha$$

Because $f_T$ is continuous we have $f_T(y)=Ff_T(y)$ and when $T\longrightarrow\infty$:

$$\lim_{T\to\infty} f_T(y) = \lim_{T\to\infty} Ff_T(y) \iff f(y) = \lim_{T\to\infty} Ff_T(y)$$

$$\iff f(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,e^{-i\alpha x}\,dx\right]}_{\substack{\mathcal{F}(f) \text{ or } \hat{f} \\ \text{Fourier transform of } f.}}e^{i\alpha y}\,d\alpha$$

Then: $$\mathcal{F}(f)(\alpha) \equiv \hat{f}(\alpha) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx.$$

and: $$f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{i\alpha y}\, d\alpha$$

# Proofs of some properties of section 2.3

## 2.3.2 Fourier transform of a convolution product

Proof: let $(f*g)(x) = \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt$. Then:

$$\mathcal{F}(f*g)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{(f*g)(x)}\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \overbrace{\left[ \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt \right]}\, e^{-i\alpha x}\, dx$$

permuting the integration order

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{\infty} f(x-t)\, e^{-i\alpha x}\, dx \right] g(t)\, dt$$

$y = x - t$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} f(y)\, e^{-i\alpha(y+t)}\, dy \right] g(t)\, dt$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\alpha y}\, dy}_{\mathcal{F}(f)(\alpha)} \quad \underbrace{\int_{-\infty}^{+\infty} g(t)\, e^{-i\alpha t}\, dt}_{\sqrt{2\pi}\ \mathcal{F}(g)(\alpha)}$$

$$= \sqrt{2\pi}\ \mathcal{F}(f)(\alpha)\ \mathcal{F}(g)(\alpha) \quad \square$$

### 2.3.3 Fourier transform of the derivative

Proof: We have $\mathcal{F}(f)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx$

Then:

$$\mathcal{F}(f')(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(x) e^{-i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ f(x) e^{-i\alpha x} \Big|_{-\infty}^{+\infty} + i\alpha \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx \right]$$

$v = f,$
$u = e^{-i\alpha x}$

$$\lim_{x \to \pm\infty} |f(x) e^{-i\alpha x}| = \lim_{x \to \pm\infty} |f(x)| = 0 \text{ because } \int_{-\infty}^{+\infty} |f(x)| dx < \infty$$

$|e^{-i\alpha x}| = 1 \ \forall x \in \mathbb{R}$

Then: $$\mathcal{F}(f')(\alpha) = i\alpha \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx}_{\mathcal{F}(f)(\alpha)} = i\alpha \mathcal{F}(f)(\alpha)$$

$\square$

Note: For derivatives of order $k > 1$, it can be proven by iteration.

## 2.3.4 Shift

Proof for $b = -\alpha_0$ and $a = 1$. Let $g(x) = e^{i\alpha_0 x} f(x)$. Then

$$\mathcal{F}(g)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\alpha_0 x} f(x) e^{-i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i(\alpha-\alpha_0)x} dx = \mathcal{F}(f)(\alpha-\alpha_0)$$

$\square$

## 2.3.5 Plancherel identity

Proof: we remark that: ($t \in \mathbb{R}$)

$$\int_{-\infty}^{+\infty} f(x)\, g(x+t)\, dt = \int_{-\infty}^{+\infty} f(x) \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathcal{F}(g)(\alpha)\, e^{i\alpha(x+t)}\, d\alpha \right] dx$$

Fourier inversion theorem

$$= \int_{-\infty}^{+\infty} \mathcal{F}(g)(\alpha) \underbrace{\left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{i\alpha x}\, dx \right]}_{\overline{\mathcal{F}(f)(\alpha)}} e^{i\alpha t}\, d\alpha$$

permuting integration order

$$= \int_{-\infty}^{+\infty} \mathcal{F}(g)(\alpha)\, \overline{\mathcal{F}(f)(\alpha)}\, d\alpha .$$

where $\overline{\mathcal{F}(f)(\alpha)}$ is the complex conjugate

Setting $t=0$ and choosing $g=f$ we obtain:

$$\int_{-\infty}^{+\infty} [f(x)]^2\, dx = \int_{-\infty}^{+\infty} \mathcal{F}(f)(\alpha)\, \overline{\mathcal{F}(f)(\alpha)}\, d\alpha$$

$$= \int_{-\infty}^{+\infty} |\mathcal{F}(f)(\alpha)|^2\, dx \qquad \square$$