## 1.1.2 Recalls and preliminary results

- Result 1: let $m, n \in \mathbb{N}^*$ and $T > 0$ then

a) $$\frac{2}{T}\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx$$

$$= \frac{2}{T}\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi m}{T}x\right)dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

b) $$\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx = 0.$$

Proof: for the sake of simplicity $T = 2\pi$. Then

a)
$$I = \frac{1}{\pi}\int_0^{2\pi}\cos(nx)\cos(mx)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos((n-m)x)\,dx + \frac{1}{2\pi}\int_0^{2\pi}\cos((n+m)x)\,dx$$

$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

If $n \neq m$ $$I = \frac{1}{2\pi}\underbrace{\left.\frac{\sin((n-m)x)}{n-m}\right|_0^{2\pi}}_{=0} + \frac{1}{2\pi}\underbrace{\left.\frac{\sin((n+m)x)}{n+m}\right|_0^{2\pi}}_{=0} = 0$$

If $n = m$ $$I = \frac{1}{2\pi}\int_0^{2\pi}dx + \frac{1}{2\pi}\int_0^{2\pi}\cos(2mx)\,dx$$

$$= 1 + \underbrace{\left.\frac{\sin(2mx)}{2m}\right|_0^{2\pi}}_{=0} = 1$$

$$J = \frac{1}{\pi}\int_0^{2\pi} \sin(nx)\sin(mx)\,dx =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\cos((n-m)x) - \cos((n+m)x)\right]dx$$

$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right).$

If $n \neq m$

$$J = \frac{1}{2\pi}\left[\underbrace{\frac{\sin((n-m)x)}{n-m}\Big|_0^{2\pi}}_{=0} - \underbrace{\frac{\sin((n+m)x)}{n+m}\Big|_0^{2\pi}}_{=0}\right] = 0$$

If $n = m$

$$J = \frac{1}{2\pi}\int_0^{2\pi}\left[\cos(0) - \cos(2mx)\right]dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} dx - \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\cos(2mx)\,dx}_{=0} = 1 \qquad \square$$

b) $$K = \int_0^{2\pi} \sin(nx)\cos(mx)\,dx$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\sin((n-m)x) + \sin((n+m)x)\right]dx$$

$\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$

If $n \neq m$:

$$K = \frac{1}{2}\left[-\frac{\cos((n-m)x)}{n-m}\Bigg|_0^{2\pi} - \frac{\cos((n+m)x)}{n+m}\Bigg|_0^{2\pi}\right] = 0$$

If $n = m$:

$$K = \frac{1}{2}\int_0^{2\pi} \sin(0)\,dx + \int^{2\pi} \sin(2mx)\,dx = 0 + 0 = 0$$

$\square$