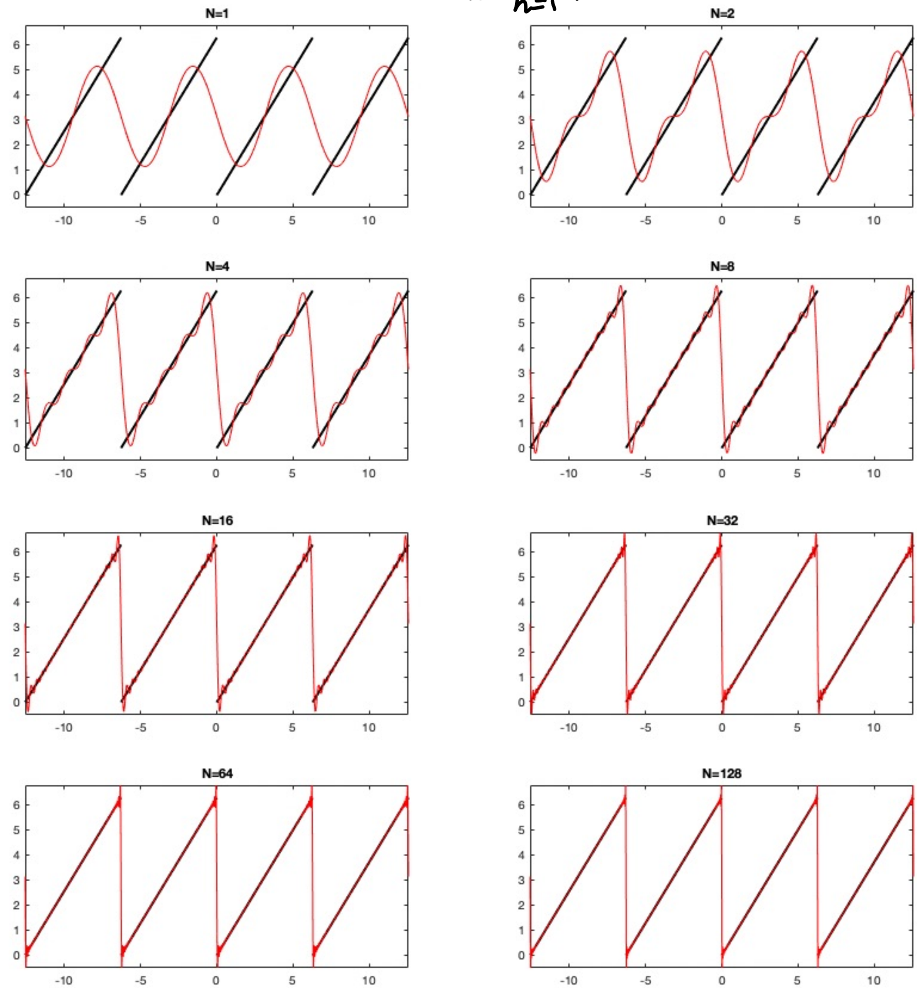
02/12/2021

About example 2 of section 1.2.4

$f : [0, 2\pi[\to \mathbb{R}$ defined by $f(x) = x$ be a function extended by 2π-periodicity to $\mathbb{R}$.

$$F_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right)$$



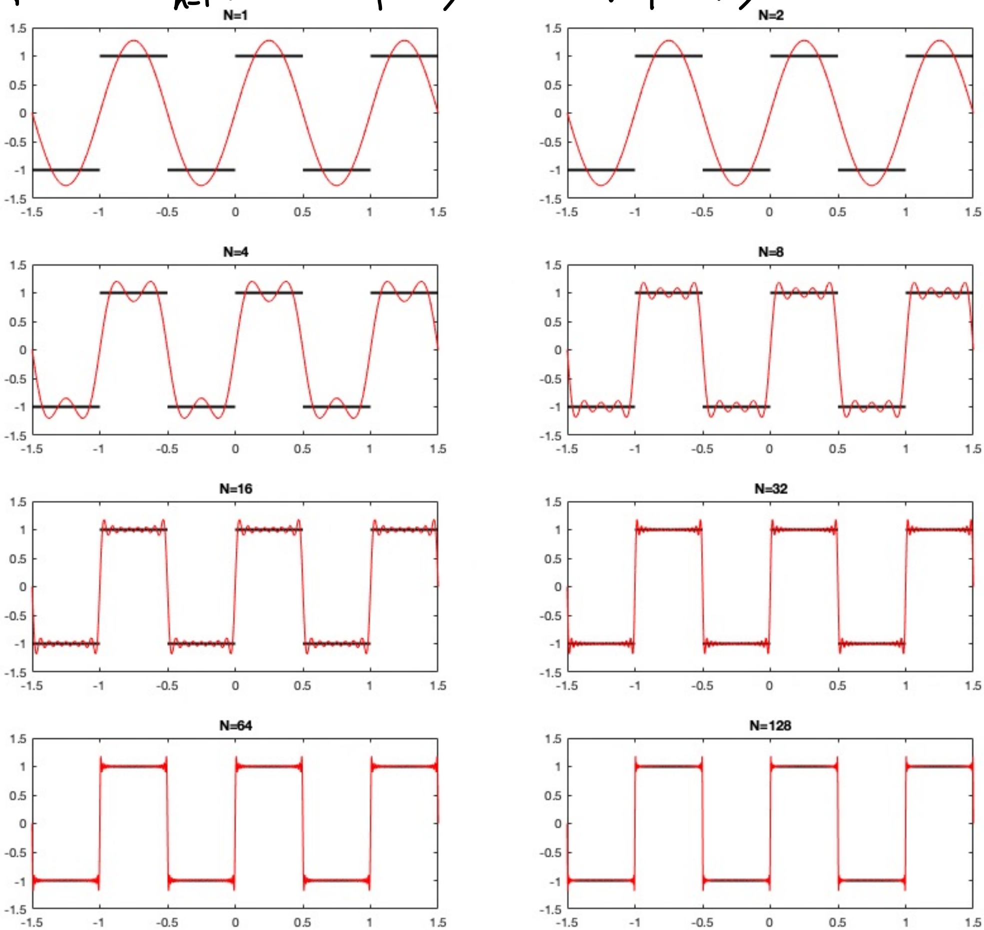
About example 3 of section 1.2.4

$f : [0, T[\to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{T}{2}[\\ -1 & \text{if } x \in [\frac{T}{2}, T[\end{cases}$$

be a function extended by T-periodicity to $\mathbb{R}$.

$$F_N f(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right)$$



1.3 Properties of Fourier series

1.3.1 Periodicity and parity

- **Theorem** $f: \mathbb{R} \to \mathbb{R}$ a T-periodic function s.t. f is piecewise-defined. Then:

  a) The Fourier series Ff is also T-periodic

  b) If f is an even function ($f(x) = f(-x)$ $\forall x \in \mathbb{R}$)

  we have $b_n = 0$ $\forall n \geq 1$

  and $$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T} x\right)$$

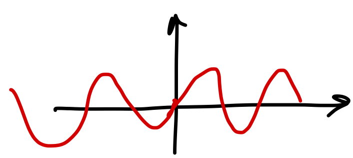
  is also an even function.

  c) If f is an odd function ($f(x) = -f(-x)$ $\forall x \in \mathbb{R}$)

  we have $a_n = 0$ $\forall n \geq 0$.

  Then $$Ff(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T} x\right)$$

  is also an odd function



1.3.2 Parseval identity

Theorem: let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic function s.t f and f' are piecewise-defined. Then:

$$\frac{2}{T}\int_0^T [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right)$$

Example: let $f: [0, 2\pi[\to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [\pi, 2\pi[\end{cases}$$ be a function

extended by 2π-periodicity to $\mathbb{R}$.

The Fourier coefficients are $a_0 = 1$, $a_n = 0 \ \forall n \geq 1$

and $b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$

$$\frac{2}{T}\int_0^T [f(x)]^2 dx = \frac{2}{2\pi}\int_0^{2\pi} (f(x))^2 dx = \frac{2}{2\pi}\int_0^{\pi} (1)^2 = 1$$

$$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{2} + \sum_{n \text{ odd}} \overbrace{\frac{4}{n^2\pi^2}}^{b_n^2}$$

$$= \frac{1}{2} + \sum_{k=0}^{+\infty} \frac{4}{(2k+1)^2\pi^2}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2} = 1 - \frac{1}{2} \Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

1.3.3 Differentiation and integration of Fourier series term by term

- Theorem 1: let $f : \mathbb{R} \to \mathbb{R}$ a continuous T-periodic s.t. f' and f'' are piecewise-defined.

  let $$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right]$$ its Fourier series.

  Then the series obtained by differentiating $Ff(x)$ term by term converges $\forall x \in \mathbb{R}$ and we have

$$\frac{d\,Ff(x)}{dx} = \sum_{n=1}^{\infty} \frac{2\pi n}{T}\left[-a_n \sin\left(\frac{2\pi n}{T}x\right) + b_n \cos\left(\frac{2\pi n}{T}x\right)\right]$$

$$= \frac{1}{2}\left[f'(x+0) + f'(x-0)\right] \text{ where}$$

$$f'(x+0) = \lim_{\substack{t\to x\\ t>x}} f'(t) \quad \text{and} \quad f'(x-0) = \lim_{\substack{t\to x\\ t<x}} f'(t).$$

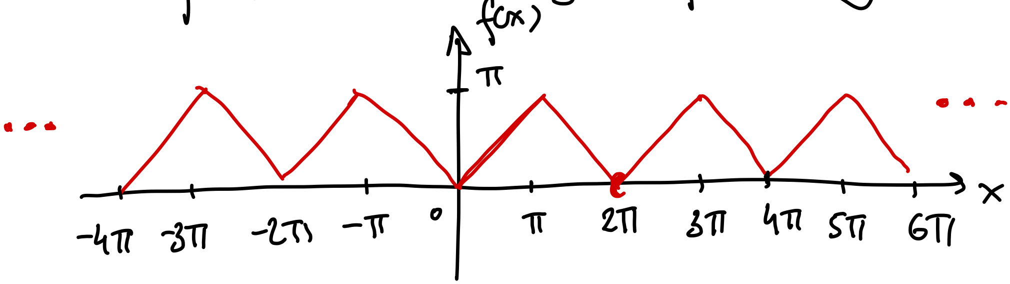
Because f is continuous $\longrightarrow$ $Ff(x) = f(x)$

If f' is continuous $\longrightarrow$ $\dfrac{d\,Ff(x)}{dx} = f'(x)$

Examples:

a) let $f : [0, 2\pi[\longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} x & \text{if } x \in [0, \pi[\\ 2\pi - x & \text{if } x \in [\pi, 2\pi[\end{cases}$

be a function extended by 2π-periodicity to $\mathbb{R}$.



The Fourier coefficients are : $b_n = 0 \quad \forall n \geq 1$

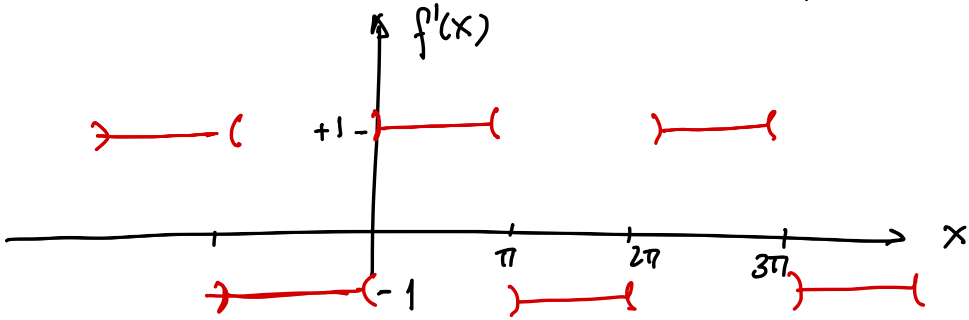
$$a_0 = \pi \quad \text{and} \quad a_n = \begin{cases} 0 & \text{if } n \geq 1 \text{ is even} \\ \dfrac{-4}{\pi n^2} & \text{if } n \geq 1 \text{ is odd} \end{cases}$$

Then $\forall x \in \mathbb{R}$ we have

$$f(x) = \mathcal{F}f(x) = \frac{\pi}{2} + \sum_{n \text{ odd}} \left(\frac{-4}{\pi n^2} \right) \cos(nx)$$

f is continuous

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)x]}{(2k+1)^2}$$



$$\frac{d\mathcal{F}f(x)}{dx} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)x]}{2k+1} = \frac{1}{2}\left[f'(x+0) + f'(x-0) \right]$$

$$\frac{1}{2}(1-1) = 0 \quad \text{if } x = 0$$

$$\frac{1}{2}(1+1) = 1 = f'(x) \quad \text{if } x \in]0, \pi[$$

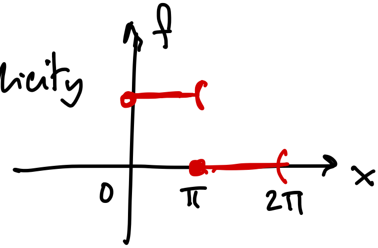
$$\frac{1}{2}(-1+1) = 0 \quad \text{if } x = \pi$$

$\frac{1}{2}(-1-1) = -1 = f'(x)$ if $x \in]\pi, 2\pi[$

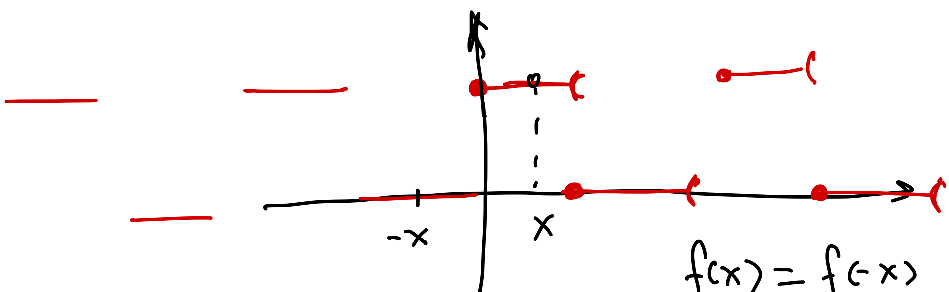
$\frac{1}{2}(1-1) = 0$ if $x = 2\pi$

b) let $f : [0, 2\pi[\to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [\pi, 2\pi[\end{cases}$

be a function extended by 2π-periodicity

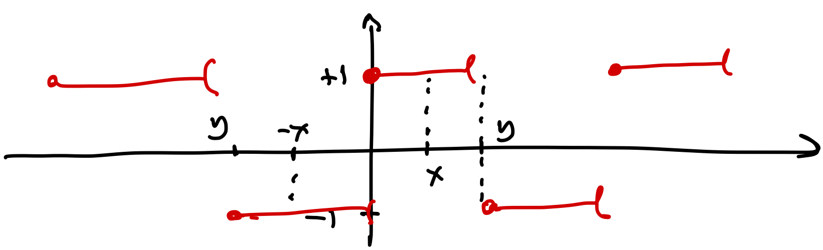


$$Ff(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{+\infty} \frac{\sin[(2k+1)x]}{2k+1}$$



$f(x) = f(-x)$ even

$f(x) = -f(-x)$ odd



Differentiating term by term $Ff(x)$

$$(Ff)'(x) = \frac{2}{\pi}\sum_{k=0}^{+\infty} \cos\big((2k+1)x\big) \quad \text{for } x \neq \frac{\pi}{2} \text{ diverges.}$$

This result does not contradict the theory because f is not continuous at $x = \pi, 2\pi, \dots$

- Theorem 2: let $f : \mathbb{R} \to \mathbb{R}$ be a T-periodic function s.t. f and f' are piecewise-defined. Let

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right)\right]$$

be its Fourier series. Then $\forall\, x_0$ and $x \in [0, T]$ we have:

$$\int_{x_0}^{x} f(t)\,dt = \int_{x_0}^{x} \frac{a_0}{2}\,dt + \sum_{n=1}^{\infty}\int_{x_0}^{x}\left[a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right]dt$$

1.4 Other Fourier series formulations

1.4.1 Fourier cosines series

- Theorem:

Let $f : [0, L] \to \mathbb{R}$ be a $\overbrace{\text{continuous}}^{\text{for the sake of simplicity.}}$ function s.t.

f' is piecewise-defined. Then, the series

$$F_c f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L} x\right) \text{ with}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\pi n}{L} x\right) dx \quad \text{for } n = 0, 1, 2, \dots$$

it is called a Fourier series of cosines of f and

it converges to f in the interval $[0, L]$.

We have $f(x) = F_c f(x) \ \forall \ x \in [0, L]$

1.4.2 Fourier series of sines

- Theorem:

Let $f:[0,L]\to\mathbb{R}$ be a continuous function s.t.

$\underline{f(0)=f(L)=0}$ and f' is piecewise defined. Then,

the Fourier series of sines is:

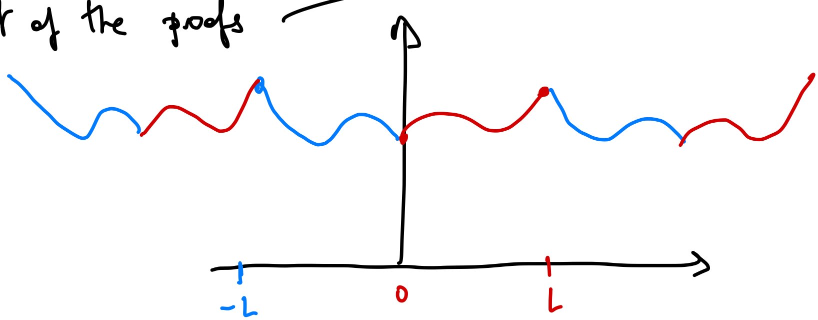
$$F_s f(x) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{\pi n}{L} x\right) \text{ with}$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) dx \quad \text{for } n=1,2,3,\dots$$

and it converges to f in the interval $[0,L]$.

We have that $F_s f(x) = f(x)$ $\forall$ $x\in[0,L]$.

Hint of the proofs $\longrightarrow$ for cosines.



For sines: