Unit ONE

Solutions/Answers to Exercises of page 6

1. Which of the following sentences are propositions? For those that are, indicate the truth value.

	Solutions/Answers
a. 123 is a prime number.	*It is a proposition with truth value true T.*
b. 0 is an even number.	*It is a proposition with truth value true T.*
c. $x^2-4=0$.	*It is not a proposition.*
d. Multiply $5x+2$ by 3.	*It is not a proposition.*
e. What an impossible question!	*It is not a proposition.*

2. State the negation of each of the following statements.

	Solutions/Answers
a. $\sqrt{2}$ is a rational number.	*$\sqrt{2}$ is not a rational number.*
b. 0 is not a negative integer.	*0 is a negative integer.*
c. 111 is a prime number.	*111 is not a prime number.*

3. Let p: 15 is an odd number.
   q: 21 is a prime number.

		Solutions/Answers	
a.	$p \vee q$:	*15 is an odd number or 21 is a prime number.*	*Truth value True*
b.	$p \wedge q$:	*15 is an odd number and 21 is a prime number.*	*Truth value False*
c.	$\neg p \vee q$:	*15 is not an odd number or 21 is a prime number.*	*Truth value False*
d.	$p \wedge \neg q$:	*15 is an odd number and 21 is not a prime number.*	*Truth value True*
e.	$p \Rightarrow q$:	*If 15 is an odd number then 21 is a prime number.*	*Truth value False*
f.	$q \Rightarrow p$:	*If 21 is a prime number then 15 is an odd number.*	*Truth value True*
a.	$\neg p \Rightarrow \neg q$:	*If 15 is not an odd number then 21 is not a prime number.*	*Truth value True*
g.	$\neg q \Rightarrow \neg p$:	*If 21 is not a prime number then 15 is not an odd number.*	*Truth value False*

4. Complete the following truth table.

p	q	$\neg q$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Solutions/Answers to Exercises of pages 11-12

1. For statements p, and r, use a truth table to show that each of the following pairs of statements is logically equivalent.

a. $(p \wedge q) \Leftrightarrow p$ and $p \Longrightarrow q$.

P	Q	$p \wedge q$	$(p \wedge q) \Leftrightarrow p$	$p \Longrightarrow q$
T	T	T	T	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Therefore, $(p \wedge q) \Leftrightarrow p$ and $p \Longrightarrow q$ are logically equivalent, since the forth and the fifth columns have the same truth values .

b. $p \Longrightarrow (q \vee r)$ and $\neg q \Longrightarrow (\neg p \vee r)$.

P	Q	r	$q \vee r$	$\neg$P	$\neg q$	$\neg p \vee r$	$p \Longrightarrow (q \vee r)$	$\neg q \Longrightarrow (\neg p \vee r)$
T	T	T	T	F	F	T	T	T
T	T	F	T	F	F	F	T	T
T	F	T	T	F	T	T	T	T
T	F	F	F	F	T	F	F	F
F	T	T	T	T	F	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	F	T	T	T	T	T

Therefore, $p \Longrightarrow (q \vee r)$ and $\neg q \Longrightarrow (\neg p \vee r)$ are logically equivalent. since the eighth and the ninth columns have the same truth values .

c. $(p \vee q) \Longrightarrow r$ and $(p \Longrightarrow q) \wedge (q \Longrightarrow r)$.

P	q	r	$p \vee q$	$p \Longrightarrow q$	$q \Longrightarrow r$	$(p \vee q) \Longrightarrow r$	$(p \Longrightarrow q) \wedge (q \Longrightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	F
T	F	T	T	F	T	T	F
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

The truth values of the combinations differ in the third row of the seventh and the eighth columns where $(p \vee q) \Longrightarrow$r is T and $(p \Longrightarrow q) \wedge (q \Longrightarrow r)$ is F so that $p \vee q) \Longrightarrow r$ and $(p \Longrightarrow q) \wedge (q \Longrightarrow r)$ are not logically equivalent.

d. $p \Longrightarrow (q \vee r)$ and $(\neg r) \Longrightarrow (p \Longrightarrow q)$.
The truth values of the combinations differ in the second and forth rows of the seventh and the eighth columns so that $p \Longrightarrow (q \vee r)$ and $(\neg r) \Longrightarrow (p \Longrightarrow q)$ are not logically equivalent.

P	q	r	$\neg r$	$q \vee r$	$p \Longrightarrow q$	$p \Longrightarrow (q \vee r)$	$(\neg r) \Longrightarrow (p \Longrightarrow q)$
T	T	T	F	T	T	T	T
T	T	F	T	F	T	F	T
T	F	T	F	T	F	T	T
T	F	F	T	T	F	T	F
F	T	T	F	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	F	T	T	T	T
F	F	F	T	T	T	T	T

e. $p \Longrightarrow (q \vee r)$ and $((\neg r) \wedge p) \Longrightarrow q$.

p	q	r	$\neg r$	$q \vee r$	$(\neg r) \wedge p$	$p \Longrightarrow (q \vee r)$	$((\neg r) \wedge p) \Longrightarrow q$
T	T	T	F	T	F	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	F	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	F	T	T
F	T	F	T	T	F	T	T
F	F	T	F	T	F	T	T
F	F	F	T	F	F	T	T

The truth values of the combinations the propositions of the seventh and the eighth columns are the same so that $p \Longrightarrow (q \vee r)$ and $(\neg r) \Longrightarrow (p \Longrightarrow q)$ are logically equivalent.

2. For statements p, q, and r, show that the following compound statements are tautology.

a. $p \Longrightarrow (p \vee q)$.

p	q	$p \vee$q	$p \Longrightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

$p \Longrightarrow (p \vee q$ is a tautology since all the possible combinations in the last column are all T

b. $(p \wedge (p \Longrightarrow q)) \Longrightarrow q$.

p	q	$p \Longrightarrow$q	$p \wedge (p \Longrightarrow q)$	$(p \wedge (p \Longrightarrow q)) \Longrightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$(p \wedge (p \Longrightarrow q)) \Longrightarrow q$ is a tautology since all the possible combinations in the last column are all T

c. $((p \Longrightarrow q) \wedge (q \Longrightarrow r)) \Longrightarrow (p \Longrightarrow r)$.

p	q	r	$p \Longrightarrow$q	$q \Longrightarrow r$	$p \Longrightarrow r$	$(p \Longrightarrow q) \wedge (q \Longrightarrow r)$	$((p \Longrightarrow q) \wedge (q \Longrightarrow r)) \Longrightarrow (p \Longrightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$((p \Longrightarrow q) \wedge (q \Longrightarrow r)) \Longrightarrow (p \Longrightarrow r)$ is a tautology since all the possible combinations in the last column are all T

3. For statements p and q, show that $(p\wedge\neg q)\wedge(p\wedge q)$
is a contradiction.

P	q	$\neg q$	$p\wedge\neg$q	$p\wedge q$	$(p\wedge\neg q)\wedge(p\wedge q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	F	F
F	F	T	F	F	F

$(p\wedge\neg q)\wedge(p\wedge q)$ is a contradiction since the last column has all the truth values F

4. Write the contrapositive and the converse of the following conditional statements.

a. If it is cold, then the lake is frozen.
   Contrapositive: If the lake is not frozen then it is not cold.
   Converse: If the lake is frozen then it is cold.

b. If Solomon is healthy, then he is happy.
   Contrapositive: If he is not happy then Solomon is not healthy.
   Converse: If he is happy then Solomon is healthy.

c. If it rains, Tigist does not take a walk.
   Contrapositive: If Tigist takes a walk, it doesn't rain.
   Converse: If Tigist doesn't take a walk, it rains.

5. Let p and q be statements. Which of the following implies that $p\vee q$ is false?

$p\vee q$ is false means p is True and q is False so that:

a. $\neg p\vee\neg q$ is false.

$\neg p$ is F and $\neg q$ is T which implies F ∨ T which implies T so that $\neg p\vee\neg q$ is T

b. $\neg p\vee q$ is true.

This means F or F which is F

c. $\neg p\wedge\neg q$ is true.

This means F ∧ T which is F

d. $p\Longrightarrow q$ is true.

This means T implies F which is F

e. $p\wedge q$ is false.

This means T and F which is F

So for in general for question number 5 , the answers are b, c, d, e

6. Suppose that the statements p, q, r, and s are assigned the truth values T,,, and T, respectively. Find the truth value of each of the following statements.

a. $(p \vee q) \vee r$.

(T∨F)∨F which gives T∨F and gives T

b. $p \vee (q \vee r)$.

T∨(F∨F) which gives T∨F and gives T

c. $r \Longrightarrow (s \wedge p)$

F⟹(T∧T) which gives F⟹T which also gives T

d. $p \Longrightarrow (r \Longrightarrow s)$.

T⟹(F⟹T) which gives T⟹T which also gives T

e. $p \Longrightarrow (r \vee s)$.

T⟹(F∨T) which gives T⟹T which lso gives T

f. $(p \vee r) \Leftrightarrow (r \wedge \emptyset s)$.

(T∨F)⟺(F∧F) which gives T⟺F which also gives F

g. $(s \Leftrightarrow p) \Longrightarrow (\emptyset p \vee s)$.

(T⟺T)⟹(F∨T) which gives T⟺T which also gives T

h. $(q \wedge \emptyset s) \Longrightarrow (p \Leftrightarrow s)$.

(F∧F)⟹(T⟺T) which gives F⟹T which also gives T

i. $(r \wedge s) \Longrightarrow (p \Longrightarrow (\emptyset q \vee s))$.

(F∧T)⟹(T⟹(T∨T)) which gives F⟹(T⟹T) which also gives T⟹T and also gives T

j. $(p \vee \emptyset q) \vee r \Longrightarrow (s \wedge \emptyset s)$.

(T∨T)∨F⟹$(s \wedge \emptyset s)$ which gives F⟹(T⟹T) which also gives T⟹T and also gives T

7. Suppose the value of $p \Longrightarrow q$ is T; what can be said about the value of $\neg p \wedge q \Leftrightarrow p \vee q$?

$p \Longrightarrow q$ is T has three cases which are p is T and q is T; p is F and q is T; ; p is F and q is F

<u>**Case 1):-**</u> When p is T and q is T then $\neg p \wedge q \Leftrightarrow p \vee q$ means F∧T⟺T∨T which is F ⟺ T which is F

<u>**Case 2):-**</u> When p is F and q is T then $\neg p \wedge q \Leftrightarrow p \vee q$ means T∧T⟺T∨T which is T ⟺ T which is T

<u>**Case 3):-**</u> When p is F and q is F then $\neg p \wedge q \Leftrightarrow p \vee q$ means T∧F⟺F∨F which is F ⟺ T which is F

Therefore, $\neg p \wedge q \Leftrightarrow p \vee q$ has a truth value of either T or F

8. a. Suppose the value of $p \Leftrightarrow q$ is T; what can be said about the values of $p \Leftrightarrow \neg q$ and $\neg p \Leftrightarrow q$?

$p \Leftrightarrow q$ is T means p and q have the same truth value

Which means p is T and q is T or p is F and q is F

Therefore, when p is T and q is T then $p \Leftrightarrow \neg q$ means T⇔F and $\neg p \Leftrightarrow q$ means F⇔T

which means F

And, when p is F and q is F then $p \Leftrightarrow \neg q$ means T⇔T and $\neg p \Leftrightarrow q$ T⇔T which means T

b. Suppose the value of $p \Leftrightarrow q$ is F; what can be said about the values of $p \Leftrightarrow \neg q$ and $\neg p \Leftrightarrow q$?

$p \Leftrightarrow q$ is F means p and q have different truth values

which means p is T and q is F or p is F and q is T

Therefore, when p is T and q is F then $p \Leftrightarrow \neg q$ means T⇔T and $\neg p \Leftrightarrow q$ means F⇔F

which means T

9. Construct the truth table for each of the following statements.

a. $p \Rightarrow (p \Rightarrow q)$.

P	q	$p \Rightarrow q$	$p \Rightarrow (p \Rightarrow q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

b. $(p \vee q) \Leftrightarrow (q \vee p)$.

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \Leftrightarrow (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

c. $p \Rightarrow \neg(q \wedge r)$.

p	q	r	$q \wedge r$	$\neg(q \wedge r)$	$p \Rightarrow \neg(q \wedge r)$
T	T	T	T	F	F
T	T	F	F	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	F	T	T
F	F	T	F	T	T
F	F	F	F	T	T

d. $(p \Longrightarrow q) \Leftrightarrow (\neg p \vee q)$.

p	q	$\neg p$	$p \Longrightarrow q$	$\neg p \vee q$	$(p \Longrightarrow q) \Leftrightarrow (\neg p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

e. $(p \Longrightarrow (q \wedge r)) \vee (\neg p \wedge q)$.

p	q	r	$\neg$p	q∧r	$\neg$p∧q	p⟹(q∧r)	(p⟹(q∧r))∨($\neg$p∧q)
T	T	T	F	T	F	T	T
T	T	F	F	F	F	F	F
T	F	T	F	F	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	F	F	T	T
F	F	F	T	F	F	T	T

f. $(p \wedge q) \Longrightarrow ((q \wedge \neg q) \Longrightarrow (r \wedge q))$.

p	Q	r	$\neg$q	$p \wedge q$	$q \wedge \neg q$	$r \wedge q$	$((q \wedge \neg q) \Longrightarrow (r \wedge q))$	$(p \wedge q) \Longrightarrow ((q \wedge \neg q) \Longrightarrow (r \wedge q))$
T	T	T	F	T	F	T	T	T
T	T	F	F	T	F	F	T	T
T	F	T	T	F	F	T	T	T
T	F	F	T	F	F	F	T	T
F	T	T	F	T	F	T	T	T
F	T	F	F	T	F	F	T	T
F	F	T	T	F	F	T	T	T
F	F	F	T	F	F	F	T	T

10. For each of the following determine whether the information given is sufficient to decide the truth value of the statement. If the information is enough, state the truth value. If it is insufficient, show that both truth values are possible.

a. $(p \Longrightarrow q) \Longrightarrow r$, where $r = T$.

r =T is sufficient to determine the truth value of $(p \Longrightarrow q) \Longrightarrow r$

since when r =T, whatever the truth value of $p \Longrightarrow q$ is, r =T gives $(p \Longrightarrow q) \Longrightarrow r$ is T

b. $p\wedge(q\Longrightarrow r)$, where $q\Longrightarrow r=T$.

$q\Longrightarrow r=T$ is insufficient to determine the truth value of $p\wedge(q\Longrightarrow r)$

Since when $q\Longrightarrow r=T$, the truth value of is $p\wedge(q\Longrightarrow r)$ either T or F depending the truth value of P, i.e., when $q\Longrightarrow r=T$; p is T, then $p\wedge(q\Longrightarrow r)$ is T

when $q\Longrightarrow r=T$; p is F, then $p\wedge(q\Longrightarrow r)$ is F

c. $p\vee(q\Longrightarrow r)$, where $q\Longrightarrow r=T$.

$q\Longrightarrow r=T$ is sufficient to determine the truth value of $p\vee(q\Longrightarrow r)$

since when $q\Longrightarrow r=T$, whatever the truth value of p is, $\Longrightarrow r=T$ gives $p\vee(q\Longrightarrow r)$ is T

d. $\neg(p\vee q)\Longleftrightarrow(\neg p\wedge\neg q)$, where $p\vee q=T$.

$p\vee q=T$ is sufficient to determine the truth value of $\neg(p\vee q)\Longleftrightarrow(\neg p\wedge\neg q)$

Since $p\vee q=T$ means $\neg(p\vee q)$is F and $(\neg p\wedge\neg q)\equiv\neg(p\vee q)$ is F

so that $(\neg p\wedge\neg q)\equiv\neg(p\vee q)$ is T

e. $(p\Longrightarrow q)\Longrightarrow(\neg q\Longrightarrow\neg p)$, where $q=T$.

$q=T$ is sufficient to determine the truth value of $(p\Longrightarrow q)\Longrightarrow(\neg q\Longrightarrow\neg p)$

Since $q=T$ means $p\Longrightarrow q$ is T ; $\neg q$ is F and $\neg q\Longrightarrow\neg p$ is T

and hence $(p\Longrightarrow q)\Longrightarrow(\neg q\Longrightarrow\neg p)$ is T

f. $(p\wedge q)\Longrightarrow(p\vee s)$, where $p=T$ and $s=F$.

$p=T$ and $s=F$ are sufficient to determine the truth value of $(p\wedge q)\Longrightarrow(p\vee s)$

Since $p=T$ and $s=F$ means $p\vee s$ is T and $(p\wedge q)\Longrightarrow(p\vee s)$ is T whatever the truth

value of $p\wedge q$ is.

Even, only $p=T$ is sufficient to determine the truth value of $(p\wedge q)\Longrightarrow(p\vee s)$,

Since $p=T$ means $p\vee s$ is T for any truth value of s,

and $p\vee s$ is T means $(p\wedge q)\Longrightarrow(p\vee s)$ is T for any truth value of $p\wedge q$

Solutions/Answers to Exercises of pages 19-20

1. In each of the following, two open statements $P(x,y)$ and $Q(x,y)$ are given, where the domain of both x and y is Z. Determine the truth value of $(x,)\Longrightarrow Q(x,y)$ for the given values of x and y.

a. $(x,):x^2-y^2=0$. and $Q(x,y):x=y$. $(x,y)\in\{(1,-1),(3,4),(5,5)\}$.

$(x,):x^2-y^2=0$ is F since when $(x,)=(3,4)$; $x^2-y^2= 3^2-4^2=9-16=-7\neq 0$ and

$(x,):x=y$ is F since when $(x,)=(3,4)$; $3=4$ is F

Therefore, $(x,y)\Longrightarrow Q(x,y)$ means F⟹F which is T

b. $P(x,y):|x|=|y|$. and $Q(x,y):x=y$. $(x,y)\in\{(1,2),(2,-2),(6,6)\}$.

$(x,):|x|=|y|$ is F since when $(x,)=(1,2)$;$|1|\neq|2|$ and

$(x,):x=y$ is F since when $(x,)=(1,2)$;$1\neq 2$

Therefore, $(x,y)\Longrightarrow Q(x,y)$ means F⟹F which is T

c. $(x,):x^2+y^2=1$. and $Q(x,y):x+y=1$. $(x,y)\in\{(1,-1), (-3,4), (0,-1), (1,0)\}$.

$(x,)$: $x^2+y^2=1$ is F since when $(x,)= (-3,4)$; $(-3)^2+4^2=1$ is F and

$Q(x,y)$: $x+y=1$ is F since when $(x,y)= (1,0)$;$0=1$ is F

Therefore, $(x,y)\Longrightarrow Q(x,y)$ means F⟹F which is T

2. Let O denote the set of odd integers and let $(x):x^2+1$ is even, and $(x):x^2$ is even. be open statements over the domain O. State $(\forall x\in O)(x)$ and $(\exists y\in O)Q(x)$ in words.

For every odd integer x , x^2+1 is even

and there is a an odd integer y such that y^2 is even

3. State the negation of the following quantified statements.

a. For every rational number r, the number $1/r$ is rational.

Not for every rational number , the number $1/r$ is rational.

OR, ***For some rational number , the number $1/r$ is not rational.***

b. There exists a rational number r such that $r^2=2$.

Not there exists a rational number r such that $r^2=2$.

OR, ***For every a rational number , $r^2\neq 2$.***

4. Let (n): $\frac{5n-6}{3}$ is an integer. be an open sentence over the domain . Determine, with explanations, whether the following statements are true or false:

a. $(\forall n\in\mathbb{Z})(n)$.

($\forall n\in\mathbb{Z}$)(n) is False **since for** $n=1\in Z$ *such that* $\frac{5n-6}{3}=\frac{5x1-6}{3}=\frac{-1}{3}$ ***is not an integer.***

b. $(\exists n\in\mathbb{Z})(n)$.

($\exists n\in\mathbb{Z}$)(n) is True **since** $n=0\in Z$ *such that* $\frac{5n-6}{3}=\frac{5x0-6}{3}=\frac{-6}{3}=-2$ ***is an integer.***

5. Determine the truth value of the following statements.

a. $(\exists x\in\mathbb{R})(x^2-x=0)$.

True *(when we take x=1 ; $x^2-x=0$ which means $1^2-1=1-1=0$*

b. $(\forall x\in\mathbb{N})(x+1\geq 2)$.

False *(when we take x=−10 ; $x+1=-10+1=-9\geq 2$ is false)*

a. $(\forall x\in\mathbb{R})(\sqrt{x^2}=x)$.

False *(when we take x=−1 ;* $\sqrt{x^2}=\sqrt{(-1)^2}=\sqrt{1}=1\neq x=-1$*)*

b. $(\exists x\in\mathbb{Q})(3x^2-27=0)$.

True *(when we take x=3 ; $3x^2-27=3(3^2)-27=27-27=0$)*

c. $(\exists x\in\mathbb{R})(\exists y\in\mathbb{R})(x+y+3=8)$.

True *(when we take x=3 & y=2 ;* $x+y+3=3+2+3=8$)

d. $(\exists x\in\mathbb{R})(\exists y\in\mathbb{R})(x^2+y^2=9)$.

True *(when we take x=* $\sqrt{8}$ *& y=1 ;* $x^2+y^2=\left(\sqrt{8}\right)^2+1^2=8+1=9$)

e. $(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})(x+y=5)$.

True *(since x is chosen first, we can determine a single y=5-x so that x+y = x+5-x=5*)

f. $(\exists x\in\mathbb{R})(\forall y\in\mathbb{R})(x+y=5)$

True *(since y is chosen first, and then we vary x arbitrarily after y is determined,* $x+y=5$ can't be true like when we fix y=0, x could be any real number like 100, 0, -21 that makes $x+y=5$ false)

6. Consider the quantified statement

For every $x\in A$ and $y\in A$, $xy-2$ is prime.

where the domain of the variables x and y is $A=\{3,5,11\}$.

a. Express this quantified statement in symbols.

Let P(x, y): $xy-2$ is prime

$(\forall x\in A)\ (\forall y\in A)P(x, y)$ or $(\forall x\in A)\ (\forall y\in A)$ ($xy-2$ is prime).

b. Is the quantified statement in (a) true or false? Explain.

True

c. $\neg$ [($\forall x \in$A) ($\forall y \in$A)P(x, y)] or [($\forall x \in$A) ($\forall y \in$A) ($xy-2$ is prime].
OR [($\exists x \in$A) ($\exists y \in$A) $\neg$P(x, y)] or [($\exists x \in$A) ($\exists y \in$A) $\neg$($xy-2$ is prime].

d. Is the negation of the quantified in (a) true or false? Explain.

It is **False**

since ($\forall x \in$A) ($\forall y \in$A)P(x, y) or ($\forall x \in$A) ($\forall y \in$A) ($xy-2$ is prime). is **True**

and the **negation** of True is **False**

7. Consider the open statement $(x,)$:x/y<1. where the domain of x is A={2,3,5} and the domain of y is B = {2,4,6}.

a. State the quantified statement $(\forall x \in A)(\exists y \in B)P(x, y)$ in words.
For every $x \in A$ and for some $y \in$B, x/y<1.

b. Show quantified statement in (a) is true.
Since y is determined just after x is chosen like when x=5, we can determine y=6 so that x/y= 5/6<1 is true.

8. Consider the open statement (x, y): $x - y < 0$. where the domain of x is $A = \{3,5,8\}$ and the domain of y is $B = \{3,6,10\}$.

a. State the quantified statement $(\exists y \in B)(\forall x \in A)P(x, y)$ in words.
There is a y in B which stands to every x in A such that $x - y < 0$

b. Show quantified statement in (a) is true.

Let y=10, then for each x in A,

i.e., x=3, x=5, or x=8 , x−y = 8−10 = −2< 0, x−y =5−10=−5< 0, , x−y =3−10=−7< 0

Solutions/Answers to Exercises of pages 24-25

1. Use the truth table method to show that the following argument forms are valid.

i. $\neg p \Longrightarrow \neg q,\ q\ \vdash p$.

It is to check whether $((\neg p \Longrightarrow \neg q) \wedge q) \Rightarrow p$ is tautology or not

p	q	$\neg p$	$\neg q$	$\neg p \Longrightarrow \neg q$	$(\neg p \Longrightarrow \neg q) \wedge q$	$((\neg p \Longrightarrow \neg q) \wedge q) \Rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	F	T
F	F	T	T	T	F	T

Since $((\neg p \Longrightarrow \neg q) \wedge q) \Rightarrow p$ is a tautology, $\neg p \Longrightarrow \neg q, q\ \vdash p$ is valid

ii. $p \Longrightarrow \neg p,, \Longrightarrow q\ \vdash \neg r$.

It is to check whether $\{[(p \Longrightarrow \neg p) \wedge p] \wedge (r \Longrightarrow q)\} \Rightarrow \neg r$ is tautology or not

p	q	r	¬p	$p \Longrightarrow \neg p$	$(p \Longrightarrow \neg p) \wedge p$	$r \Longrightarrow q$	$[(p \Longrightarrow \neg p) \wedge p] \wedge (r \Longrightarrow q)$	$\{[(p \Longrightarrow \neg p) \wedge p] \wedge (r \Longrightarrow q)\} \Rightarrow \neg r$
T	T	T	F	F	F	T	F	T
T	T	F	F	F	F	T	F	T
T	F	T	F	F	F	F	F	T
T	F	F	F	F	F	T	F	T
F	T	T	T	T	F	T	F	T
F	T	F	T	T	F	T	F	T
F	F	T	T	T	F	F	F	T
F	F	F	T	T	F	T	F	T

Since $\{[(p \Longrightarrow \neg p) \wedge p] \wedge (r \Longrightarrow q)\} \Rightarrow \neg r$ is a tautology, $p \Longrightarrow \neg p, p, r \Longrightarrow q\ \vdash \neg r$ is valid

iii. $p \Longrightarrow q,\ \neg r \Longrightarrow \neg q\ \vdash \neg r \Longrightarrow \neg p$.

It is to check whether $\{[(p \Longrightarrow q) \wedge (\neg r \Longrightarrow \neg q)]\} \Longrightarrow (\neg r \Longrightarrow \neg p)$ is tautology or not

p	q	r	¬p	¬q	¬r	$p \Longrightarrow$q	¬r⟹¬q	¬r⟹¬p	$(p \Longrightarrow q) \wedge (\neg r \Longrightarrow \neg q)$	$\{[(p \Longrightarrow q) \wedge (\neg r \Longrightarrow \neg q)]\} \Longrightarrow (\neg r \Longrightarrow \neg p)$
T	T	T	F	F	F	T	T	T	T	T
T	T	F	F	F	T	T	F	F	F	T
T	F	T	F	T	F	F	T	T	F	T
T	F	F	F	T	T	F	T	F	F	T
F	T	T	T	F	F	T	T	T	T	T
F	T	F	T	F	T	T	F	T	F	T
F	F	T	T	T	F	T	T	T	T	T
F	F	F	T	T	T	T	T	T	T	T

Since $\{[(p \Longrightarrow q) \wedge (\neg r \Longrightarrow \neg q)]\} \Longrightarrow (\neg r \Longrightarrow \neg p)$is a tautology, $p \Longrightarrow q,\ \neg r \Longrightarrow \neg q\ \vdash \neg r \Longrightarrow \neg p$ is valid

iv. $\neg r \vee \neg s, (\neg s \Longrightarrow p) \Longrightarrow r\ \vdash \neg p$.

It is to check whether$\{(\neg r \vee \neg s) \wedge [(\neg s \Longrightarrow p) \Longrightarrow r]\} \Longrightarrow \neg p$ is tautology or not

p	r	s	¬p	¬r	¬s	$\neg r \vee \neg s$	$\neg s \Longrightarrow p$	$(\neg s \Longrightarrow p) \Longrightarrow r$	$(\neg r \vee \neg s) \wedge [(\neg s \Longrightarrow p) \Longrightarrow r]$	$\{(\neg r \vee \neg s) \wedge [(\neg s \Longrightarrow p) \Longrightarrow r]\} \Longrightarrow \neg p$
T	T	T	F	F	F	F	T	T	F	T
T	T	F	F	F	T	T	T	T	T	T
T	F	T	F	T	F	T	T	F	F	T
T	F	F	F	T	T	T	T	F	F	T
F	T	T	T	F	F	F	T	T	F	T
F	T	F	T	F	T	T	F	T	T	T
F	F	T	T	T	F	T	T	F	F	T
F	F	F	T	T	T	T	F	T	T	T

Since $\{(\neg r \vee \neg s) \wedge [(\neg s \Longrightarrow p) \Longrightarrow r]\} \Longrightarrow \neg p$ is a tautology, $\neg r \vee \neg s, (\neg s \Longrightarrow p) \Longrightarrow r\ \vdash \neg p$ is valid

v. $p \Rightarrow q$, $\neg p \Rightarrow r, \Rightarrow s \vdash \neg q \Rightarrow s$.

It is to check whether $[(p \Rightarrow q) \wedge (\neg p \Rightarrow r) \wedge (r \Rightarrow s)] \Rightarrow (\neg q \Rightarrow s)$ is tautology or not

p	Q	r	s	$p \Rightarrow q$	$\neg p \Rightarrow r$	$r \Rightarrow s$	$(p \Rightarrow q) \wedge (\neg p \Rightarrow r) \wedge (r \Rightarrow s)$	$\neg q \Rightarrow s$	$[(p \Rightarrow q) \wedge (\neg p \Rightarrow r) \wedge (r \Rightarrow s)] \Rightarrow (\neg q \Rightarrow s)$
T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	T	F	F	T	T
T	T	F	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T	T
T	F	T	T	F	T	T	F	T	T
T	F	T	F	F	T	F	F	F	T
T	F	F	T	F	T	T	F	T	T
T	F	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T	T
F	T	T	F	T	T	F	F	T	T
F	T	F	T	T	F	T	F	T	T
F	T	F	F	T	F	T	F	T	T
F	F	T	T	T	T	T	T	T	T
F	F	T	F	T	T	F	F	F	T
F	F	F	T	T	F	T	F	F	T
F	F	F	F	T	F	T	F	F	T

Since $[(p \Rightarrow q) \wedge (\neg p \Rightarrow r) \wedge (r \Rightarrow s)] \Rightarrow (\neg q \Rightarrow s)$ is a tautology, $p \Rightarrow q$, $\neg p \Rightarrow r, r \Rightarrow s \vdash \neg q \Rightarrow s$ is valid

2. For the following argument given a, b and c below:
i. Identify the premises.
ii. Write argument forms.
iii. . Check the validity.

Answers

i. Identify the premises.

a. The following are premises:
   If he studies medicine, he will get a good job.
   If he gets a good job, he will get a good wage.
   He did not get a good wage.
b. The following are premises:
   If the team is late, then it cannot play the game.
   If the referee is here, then the team is can play the game.
   The team is late.
c. The following are premises:
   If the professor offers chocolate for an answer, you answer the professor's question.
   The professor offers chocolate for an answer.

ii. Write argument forms.

a) Let
   p: he studies medicine
   q: he will get a good job
   r: he will get a good wage

Then

$$\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \\ \underline{\quad \neg r} \\ \quad \neg p \end{array}$$

b) Let

p: the team is late

q: the team can play the game

r: the referee is here

Then

$$\begin{array}{r} p\Rightarrow\neg q \\ \underline{\quad p} \\ \neg r \end{array}$$

C) Let

p: the professor offers chocolate for an answer

q: you answer the professor's question

Then

$$\begin{array}{r} p\Rightarrow q \\ \underline{\quad p} \\ q \end{array}$$

iii. Check the validity.

a)

$$\begin{array}{r} p\Rightarrow q \\ q\Rightarrow r \\ \underline{\quad \neg r} \\ \neg p \end{array}$$

p	q	r	¬p	¬r	$p\Rightarrow q$	$q\Rightarrow r$	$(p\Rightarrow q)\wedge(q\Rightarrow r)\wedge\neg r$	$[(p\Rightarrow q)\wedge(q\Rightarrow r)\wedge\neg r]\Rightarrow\neg p$
T	T	T	F	F	T	T	F	T
T	T	F	F	T	T	F	F	T
T	F	T	F	F	F	T	F	T
T	F	F	F	T	F	T	F	T
F	T	T	T	F	T	T	F	T
F	T	F	T	T	T	F	F	T
F	F	T	T	F	T	T	F	T
F	F	F	T	T	T	T	T	F

Since $[(p\Rightarrow q)\wedge(q\Rightarrow r)\wedge\neg r]\Rightarrow\neg p$ is not a tautology, the argument formed is not valid.

b)

$$\begin{array}{r} p\Rightarrow\neg q \\ \underline{\quad p} \\ \neg r \end{array}$$

p	q	r	¬q	¬r	$p\Rightarrow\neg q$	$(p\Rightarrow\neg q)\wedge p$	$[(p\Rightarrow\neg q)\wedge q]\Rightarrow\neg r$
T	T	T	F	F	F	F	T
T	T	F	F	T	F	F	T
T	F	T	T	F	T	T	F
T	F	F	T	T	T	T	T
F	T	T	F	F	F	F	T
F	T	F	F	T	F	F	T
F	F	T	T	F	F	F	T
F	F	F	T	T	F	F	T

Since $[(p\Rightarrow\neg q)\wedge q]\Rightarrow\neg r$ is not a tautology, the argument formed is not valid.

C)

$$\begin{array}{r} p\Rightarrow q \\ \underline{\quad p} \\ q \end{array}$$

p	q	$p\Rightarrow q$	$(p\Rightarrow q)\wedge p$	$[(p\Rightarrow q)\wedge q]\Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since $[(p\Rightarrow q)\wedge q]\Rightarrow q$ is a tautology, the argument formed is valid.

3. Give formal proof to show that the following argument forms are valid.

a. $\neg p \Longrightarrow \neg q,\ q \vdash p$.

We want to show the following is valid:

$$\begin{array}{c} \boldsymbol{q} \\ \neg \boldsymbol{p} \Rightarrow \neg \boldsymbol{q} \\ \hline \boldsymbol{p} \end{array}$$

(1) q is true — premise

(2) $\neg p \Longrightarrow \neg q$ is true — premise

(3) $q \Longrightarrow p$ is true — Contrapositive of (2)

(4) Therefore, $\begin{array}{c} q \\ q \Rightarrow p \\ \hline p \end{array}$ is valid by Modes Ponens and hence $\begin{array}{c} q \\ \neg p \Rightarrow \neg q \\ \hline p \end{array}$ is valid

b. $p \Longrightarrow \neg q, p, r \Longrightarrow q \vdash \neg r$.

We want to show the following is valid:

$$\begin{array}{c} \boldsymbol{p} \\ \boldsymbol{p} \Rightarrow \neg \boldsymbol{q} \\ \boldsymbol{r} \Rightarrow \boldsymbol{q} \\ \hline \neg \boldsymbol{r} \end{array}$$

(1) $p \Longrightarrow \neg q$ **is true** — **premise**

(2) p **is true** — **premise**

(3)

$\begin{array}{c} \boldsymbol{p} \\ \boldsymbol{p} \Rightarrow \neg \boldsymbol{q} \\ \hline \neg \boldsymbol{q} \end{array}$ is valid — Modes Ponens (Meaning $\neg \boldsymbol{q}$ is true)

(4)

$\begin{array}{c} \boldsymbol{p} \\ \boldsymbol{p} \Rightarrow \neg \boldsymbol{q} \\ \boldsymbol{r} \Rightarrow \boldsymbol{q} \\ \hline \neg \boldsymbol{r} \end{array} \equiv \begin{array}{c} \neg \boldsymbol{q} \\ \boldsymbol{r} \Rightarrow \boldsymbol{q} \\ \hline \neg \boldsymbol{r} \end{array}$ from (3) above

(5)

$\begin{array}{c} \neg \boldsymbol{q} \\ \boldsymbol{r} \Rightarrow \boldsymbol{q} \\ \hline \neg \boldsymbol{r} \end{array}$ is valid by Modes Tollens and hence $\begin{array}{c} \boldsymbol{p} \\ \boldsymbol{p} \Rightarrow \neg \boldsymbol{q} \\ \boldsymbol{r} \Rightarrow \boldsymbol{q} \\ \hline \neg \boldsymbol{r} \end{array}$ is valid

c. $p \Rightarrow q, \neg r \Rightarrow \neg q \vdash \neg r \Rightarrow \neg p$.

We want to show the following is valid:

$$\begin{array}{l} \neg r \Rightarrow \neg q \\ \underline{p \Rightarrow q} \\ \neg r \Rightarrow \neg p \end{array}$$

(1) $\neg r \Rightarrow \neg q$ is true Premise

(2) $p \Rightarrow q$ is true Premise

(3) $\neg q \Rightarrow \neg p$ is true Contrapositive of (2) above

(4) Therefore, to check the validity of:

$$\begin{array}{l} \neg r \Rightarrow \neg q \\ \underline{p \Rightarrow q} \\ \neg r \Rightarrow \neg p \end{array} \quad \text{is the same as checking the validity of} \quad \begin{array}{l} \neg r \Rightarrow \neg q \\ \underline{\neg q \Rightarrow \neg p} \\ \neg r \Rightarrow \neg p \end{array}$$

(5) So,

$$\begin{array}{l} \neg r \Rightarrow \neg q \\ \underline{\neg q \Rightarrow \neg p} \\ \neg r \Rightarrow \neg p \end{array} \quad \text{is valid by principle of Syllogism}$$

d. $\neg r \wedge \neg s, (\neg s \Rightarrow p) \Rightarrow r \vdash \neg p$.

We want to show the following is valid:

$$\begin{array}{l} \neg r \wedge \neg s \\ \underline{(\neg s \Rightarrow p) \Rightarrow r} \\ \neg p \end{array}$$

(1) $\neg r \wedge \neg s$ is true Premise

(2) $\neg r$ is true Principle of detachment from (1) above

(3) So,

$$\begin{array}{l} \neg r \wedge \neg s \\ \underline{(\neg s \Rightarrow p) \Rightarrow r} \\ \neg p \end{array} \quad \text{becomes} \quad \begin{array}{l} \neg r \\ \underline{(\neg s \Rightarrow p) \Rightarrow r} \\ \neg p \end{array} \quad \text{from (2) above}$$

(4)

$$\begin{array}{l} \neg r \\ \underline{(\neg s \Rightarrow p) \Rightarrow r} \\ \neg (\neg s \Rightarrow p) \end{array} \quad \text{by Modes Tollens}$$

(5)

$$\neg (\neg s \Rightarrow p) \equiv \neg (s \vee p) \equiv \neg s \wedge \neg p \quad \text{by equivalence property}$$

(6)

$\neg p$ is true Principle of detachment from (5) above

(7) Therefore, $$\begin{array}{l} \neg r \wedge \neg s \\ \underline{(\neg s \Rightarrow p) \Rightarrow r} \\ \neg p \end{array} \quad \text{is valid}$$

e. $p \Longrightarrow, \neg p \Longrightarrow r, r \Longrightarrow s \vdash \neg q \Longrightarrow s$.

We want to show the following is valid:

$$\begin{array}{l} p \Rightarrow \\ \neg p \Rightarrow r \\ r \Rightarrow s \\ \hline \neg q \Rightarrow s \end{array}$$

which is wrong (incomplete) question

f. $\neg p \vee q, r \Longrightarrow p, r \vdash q$.

We want to show the following is valid:

$$\begin{array}{l} \neg p \vee q \\ r \Rightarrow p \\ r \\ \hline q \end{array}$$

(1) r is true premise

(2) $r \Rightarrow p$ is true premise

(3)

$$\begin{array}{l} r \\ r \Rightarrow p \\ \hline p \end{array}$$

is valid Modes Ponens

(4) $\neg p \vee q \equiv p \Rightarrow q$ By principle of equivalence

(5)

$$\begin{array}{l} \neg p \vee q \\ r \Rightarrow p \\ r \\ \hline q \end{array} \quad \text{becomes} \quad \begin{array}{l} p \\ p \Rightarrow q \\ \hline q \end{array}$$

(6)

$$\begin{array}{l} p \\ p \Rightarrow q \\ \hline q \end{array}$$

is valid by Modes Ponens

(7) $\neg p \wedge \neg q, (q \vee r) \Longrightarrow p \vdash \neg r$.

(8) $p \Longrightarrow (q \vee r), \neg r, p \vdash q$.

i. $\neg q \Rightarrow \neg p, \Rightarrow p, \neg q \vdash r$.

We want to show the following is valid:

$$\begin{array}{l} \neg q \Rightarrow \neg p \\ r \Rightarrow p \\ \neg q \\ \hline \quad r \end{array}$$

(1) $\neg q$ is true Premise

(2) $\neg q \Rightarrow \neg p$ is true Premise

(3)

$$\begin{array}{l} \neg q \\ \neg q \Rightarrow \neg p \\ \hline \quad \neg p \end{array}$$ is valid Modes Ponens

(4)

$$\begin{array}{l} \neg q \Rightarrow \neg p \\ r \Rightarrow p \\ \neg q \\ \hline \quad r \end{array} \quad \text{becomes} \quad \begin{array}{l} \neg p \\ r \Rightarrow p \\ \hline \quad r \end{array}$$

(5)

$$\begin{array}{l} \neg p \\ r \Rightarrow p \\ \hline \quad r \end{array}$$ is valid by Modes Tollens

4. Prove the following are valid arguments by giving formal proof.

a. If the rain does not come, the crops are ruined and the people will starve. The crops are not ruined or the people will not starve. Therefore, the rain comes.

Let p: The rain comes

q: The crops are ruined

r: The people will starve

Hence:

$$\begin{array}{l} \neg p \Rightarrow (q \wedge r) \\ \neg q \vee \neg r \\ \hline \quad p \end{array}$$

(1) $\neg q \vee \neg r$ is true premise

(2) $\neg (q \wedge r)$ is true properties of equivalence

(3)

$$\begin{array}{r} \neg p \Rightarrow (q \wedge r) \\ \neg q \vee \neg r \\ \hline p \end{array} \text{ becomes } \begin{array}{r} \neg p \Rightarrow (q \wedge r) \\ \neg(q \wedge r) \\ \hline p \end{array} \text{ or } \begin{array}{r} \neg(q \wedge r) \\ \neg p \Rightarrow (q \wedge r) \\ \hline p \end{array}$$

(4)

$$\begin{array}{r} \neg(q \wedge r) \\ \neg p \Rightarrow (q \wedge r) \\ \hline p \end{array} \text{ is valid}$$ Modes Tollens

c. If the team is late, then it cannot play the game. If the referee is here then the team can play the game. The team is late. Therefore, the referee is not here.

Let p: The team is late

q: It can play the game

r: the referee is here

Hence:

$$\begin{array}{l} p \Rightarrow \neg q \\ r \Rightarrow q \\ p \\ \hline \neg r \end{array}$$

(1) p is true Premise

(2) $p \Rightarrow \neg q$ is true Premise

(3)

$$\begin{array}{l} p \\ p \Rightarrow \neg q \\ \hline \neg q \end{array} \text{ is valid}$$ Modes Ponens

(4)

$$\begin{array}{l} p \Rightarrow \neg q \\ r \Rightarrow q \\ p \\ \hline \neg r \end{array} \text{ becomes } \begin{array}{r} r \Rightarrow q \\ \neg q \\ \hline \neg r \end{array} \text{ or } \begin{array}{r} \neg q \\ r \Rightarrow q \\ \hline \neg r \end{array}$$

(5)

$$\begin{array}{r} \neg q \\ r \Rightarrow q \\ \hline \neg r \end{array} \text{ is valid}$$ Modes Tollens

Solutions/Answers to Exercises of pages 29-31

1. Which of the following are sets?

a. 1,2,3

Not a set since the elements are not contained by braces.

b. {1,2},3

Not a set since the element 3 is not contained by braces.

d. {{1},2},3

Not a set since the element 3 is not contained by braces.

e. {1,{2},3}

It is a set which has three elements 1,{2},3.

f. {1,2,a,b}.

It is a set which has four elements 1,2,a, b.

Generally, the objects in 1a, 1b, 1c are not sets but 1d and 1e are sets.

2. Which of the following sets can be described in complete listing, partial listing and/or set-builder methods? Describe each set by at least one of the three methods.

The sets can be described as followed by each question:

a. The set of the first 10 letters in the English alphabet.

(i) Complete listing method as: {a, b, c, d, e, f, g, h, i, j}

b. The set of all countries in the world.

(i) Set-builder method: {x: x is a country in the world}

c. The set of students of Addis Ababa University in the 2018/2019 academic year.

(i) Set-builder method:

{x: x is a student in Addis Ababa University in the 2018/2019 academic year}

d. The set of positive multiples of 5.

(i) Set-builder method: {x: x is a positive multiple of 5}

(ii) partial listing method: {5, 10, 15, …}

e. The set of all horses with six legs.

(i) Set-builder method: {x: x is a horse with six legs}

3. Write each of the following sets by listing its elements within braces.

a. $A=\{x\in\mathbb{Z}: -4<x\leq 4\}$ $A=\{-3, -2, -1, 0, 1, 2, 3, 4\}$

b. $B=\{x\in\mathbb{Z}: x^2<5\}$ B={0, 1, 4 }

c. $C=\{x\in\mathbb{N}: x^3<5\}$ C={1}

d. $=\{x\in\mathbb{R}: x^2-x=0\}$ D=$\{0, 1\}$

e. $=\{x\in\mathbb{R}: x^2+1=0\}$. E={ }

4. Let A be the set of positive even integers less than 15. Find the truth value of each of the following.
a. $15\in A$ False
b. $-16\in A$ False
c. $\phi\in A$ True
d. $12\subset A$ False
e. $\{2,8,14\}\in A$ False
f. $\{2,3,4\}\subseteq A$ False
g. $\{2,4\}\in A$ False
h. $\phi\subset A$ True
i. $\{246\}\subseteq A$ False

5. Find the truth value of each of the following and justify your conclusion.

a. $\phi\subseteq\phi$ **True** since empty set is the subset of any set.

b. $\{1,2\}\subseteq\{1,2\}$ **True** since any set is the subset of itself

c. $\phi\in A$ for any set A **False** since if A={1, 2} the elements of A are only 1 and 2 only but not ϕ

d. $\{\phi\}\subseteq A$, for any set A

False since if A={1} then subsets of A are written as: $\phi\subseteq A$ and $\{1\}\subseteq A$ only but not $\phi\}\subseteq A$

e. 5, 7⊆{5, 6, 7, 8} **False** since 5, 7∈{5, 6, 7, 8}but not 5, 7⊆{5, 6, 7, 8}

f. $\phi\in\{\{\phi\}\}$ **False** since $\phi\in\{\phi\}$but not $\phi\in\{\{\phi\}\}$

g. For any set A, $A\subset A$

False since for any two sets A & B, B ⊂A means (∀x)(x∈B⇒ x∈A and ∃y ∈A but y ∉B)

Or B ⊂A ⇒ B ⊆A but A≠ B

h. $\{\phi\}=\phi$ **False** since $\{\phi\}$ has one element which is ϕ but ϕ has no any element

6. For each of the following set, find its power set.
The proper subsets are described below each question:
a. $\{ab\}$ ϕ

b. {1, 1.5} ϕ, {1}, {1.5}

c. $\{a, b\}$ ϕ, $\{a\}$, $\{b\}$

d. $\{a, 0.5, x\}$ ϕ, $\{a\}$, {0.5}, { }

7. How many subsets and proper subsets do the sets that contain exactly 1,2,3,4,8,10 and 20 elements have?

To determine the number of subsets and proper subsets of the given set, first it is better to determine the number of elements of the set which is 7.

Therefore the number of subsets is found by $2^7 = 128$

and the number of proper subsets is found by $(2^7) - 1 = 128 - 1 = 127$

8. If n is a whole number, use your observation in Problems 6 and 7 to discover a formula for the number of subsets of a set with n elements. How many of these are proper subsets of the set?

Number of subsets a set with n elements is 2^n

Number of subsets a set with n elements is $(2^n) - 1$

9. Is there a set A with exactly the following indicated property?

a. Only one subset

Yes, A=ϕ has exactly one subset which is ϕ itself,

or $n(\phi) = n(A) = 0$ and number of subsets of A is $2^0=1$

b. Only one proper subset

Yes, A={1} has exactly one proper subset which is ϕ,

or $n(A)= 1$ and number of proper subsets of A = $(2^1) - 1 = 2 - 1 = 1$

c. Exactly 3 proper subsets

Yes, A={1, 5} has exactly three proper subsets which are ϕ, {1}, { 5}

or $n(A)= 2$ and number of proper subsets of A = $(2^2) - 1 = 4 - 1 = 3$

d. Exactly 4 subsets

Yes, A={1, 5} has exactly four subsets which are ϕ, {1}, { 5}, {1, 5}

or $n(A)= 2$ and number of subsets of A = $2^2 = 4$

e. Exactly 6 proper subsets

No, there is no set A such that n(A) = n and $(2^n) - 1 = 6$ which means $(2^n) = 7$

Since $\sqrt[n]{7} \neq 2$ ***for any n***$\in$***N***

f. Exactly 30 subsets

No, for n(A) = n, then $2^n = 30 \Rightarrow$ ***there is no such n***$\in$***N such that*** $\sqrt[n]{30} = 2$

g. Exactly 14 proper subsets

No, if n(A) = n, then $(2^n) - 1 = 14 \Rightarrow 2^n = 14+1=15 \Rightarrow$ ***there is no such n***$\in$***N such that*** $\sqrt[n]{15} = 2$

h. Exactly 15 proper subsets

Yes, A={1, 2, 3, 5} has exactly 15 proper subsets

Since $n(A) = 4 \Rightarrow (2^n) - 1 = 15 \Rightarrow 2^4 = 15 + 1 = 16 \Rightarrow 2^4 = 16$ is true

10. How many elements does A contain if it has:

a. 64 subsets?

$2^n = 64 \Rightarrow 2^n = 2^6 \Rightarrow$ n=6 is the number of elements of set A.

b. 31 proper subsets?

$(2^n) - 1 = 31 \Rightarrow \ 2^n = 31+1=32 \Rightarrow \ 2^n = 2^5 \Rightarrow$ n=5 is the number of elements of set A.

c. No proper subset?

$(2^n) - 1 = 0 \Rightarrow \ 2^n = 0+1=1 \Rightarrow \ 2^n = 2^0 \Rightarrow$ n=0 is the number of elements of set A={ }.

e. 255 proper subsets?

$(2^n) - 1 = 255 \Rightarrow \ 2^n = 255+1=256 \Rightarrow \ 2^n = 2^8 \Rightarrow$ n=8 is the number of elements of set A

11. Find the truth value of each of the following.

a. $\phi \in (\phi)$ **True**

b. $For\ any\ set\ A,\ \phi \subseteq (A)$ **True**

c. $For\ any\ set\ A, \in P(A)$ **True**

d. $For\ any\ set\ A, \subset P(A)$. **True**

12. For any three sets A, B and , prove that:

a. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

$$[(\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{A} \Rightarrow \boldsymbol{x} \in \boldsymbol{B}) \wedge (\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{B} \Rightarrow \boldsymbol{x} \in \boldsymbol{C})] \Rightarrow (\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{A} \Rightarrow \boldsymbol{x} \in \boldsymbol{C})$$

b. If $A \subset B$ and $B \subset C$, then $A \subset C$.

$$[\{(\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{A} \Rightarrow \boldsymbol{x} \in \boldsymbol{B}) \wedge \boldsymbol{A} \neq \boldsymbol{B}\} \wedge \{(\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{B} \Rightarrow \boldsymbol{x} \in \boldsymbol{C}) \wedge \boldsymbol{B} \neq \boldsymbol{C}\}] \Rightarrow [(\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{A} \Rightarrow \boldsymbol{x} \in \boldsymbol{C}) \wedge \boldsymbol{A} \neq \boldsymbol{C}]$$

Solutions/Answers to Exercises of pages 36-38

1. If $B \subseteq A$, $A \cap B' = \{1,4,5\}$ and $A \cup B = \{1,2,3,4,5,6\}$, find B.

n($A \cap B'$) = n(A) – n(B) = 3

n(A) = 3 + n(B)

since $B \subseteq A$ then $A \cap B = \mathrm{B}$ which means $\mathrm{n}(A \cap B) = \mathrm{n(B)}$

$\mathrm{n}(A \cup B) = 6$

n(A) + n(B) = $\mathrm{n}(A \cup B) + \mathrm{n}(A \cap B)$

3 + n(B) + n(B) = 6 + n(B)

n(B) + n(B) - n(B) = 6 – 3

n(B) = 3

2. Let $A = \{2,4,6,7,8,9\}$, $B = \{1,3,5,6,10\}$ and $C = \{x : 3x+6=0 \text{ or } 2x+6=0\}$. Find

a. $A \cup B$.

$A \cup B = \{2,4,6,7,8,9\} \cup \{1,3,5,6,10\}$= =$\{1, 2. 3, 4, 5, 6, 7, 8, 9, 10\}$

b. Is $(A \cup B) \cup C = A \cup (B \cup C)$?

Yes which holds by associative property of union, ∪

3. Suppose $U =$ The set of one digit numbers and

$A = \{x : x$ is an even natural number less than or equal to $9\}$

Describe each of the sets by complete listing method:

First let us determine U: U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} and A: A = {2, 4, 6, 8}

Then:

a. A'. $\boldsymbol{A'}$ **= {0, 1, 3, 5, 7, 9}**

b. $A \cap A'$. $\boldsymbol{A \cap A'}$ = { } = ϕ

c. $A \cup A'$. $\boldsymbol{A \cup A'}$ **= U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}**

d. $(A')'$ $\boldsymbol{(A')'}$ **= A = {2, 4, 6, 8}**

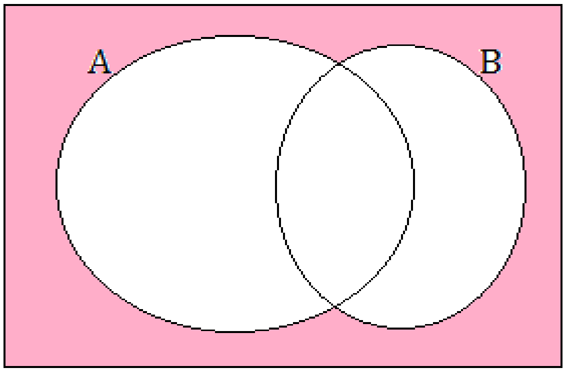
e. $\phi - U$. $\boldsymbol{\phi - U = \phi}$

f. ϕ' ϕ' = $\boldsymbol{U - \phi = U}$ **= {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}**

g. U' U' = $\boldsymbol{U}- U \boldsymbol{=\phi}$

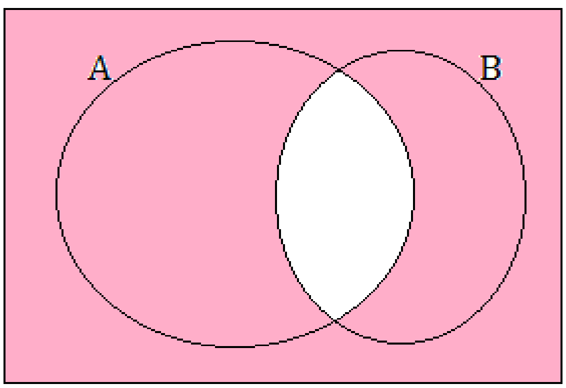
5. Use Venn diagram to illustrate the following statements:

a. $(A \cup B)' = A' \cap B'$.



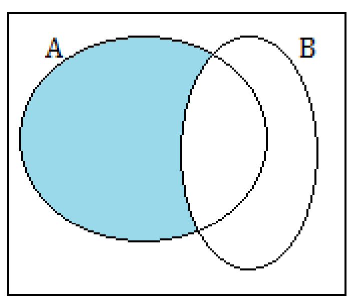
The shaded region is $(A \cup B)' = A' \cap B'$

b . $(A \cap B)' = A' \cup B'$.



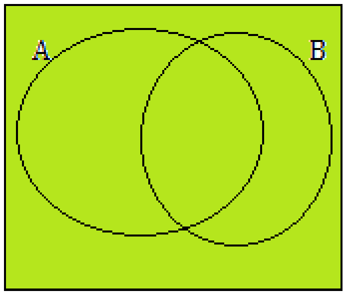
The shaded region is $(A \cap B)' = A' \cup B$

d. If $A \not\subseteq B$, then $A \setminus B \neq \phi$.



The shded region is:
$A \setminus B \neq \phi$ where $A \not\subseteq B$

d.$A\cup A'=U$.



The shaded region is:
$A\cup A'=U$

6. Let $A=\{5,7,8,9\}$ and $C=\{6,7,8\}$. Then show that $(A\backslash B)\backslash C=A(B\backslash C)$.
What is the operation between set A and $(B\backslash C)$ and what is B ? [It is wrong question]

Anyway, let us start from $(A\backslash B)\backslash$ and arrive at an equivalent conclusion using the properties

$(A\backslash B)\backslash c=(A\cap B')\cap C'$	definition of relative complement of sets
$\Rightarrow A\cap(B'\cap C')=A\cap(B\cup C)'$	property of complement of ∪, union
$\Rightarrow A\cap(B\cup C)'=A\backslash(B\cup C)$	definition of relative complement of sets

7. Perform each of the following operations.

a. $\phi\cap\{\phi\}$	$\phi\cap\{\phi\}=\phi$
b. $\{\phi,\{\phi\}\}-\{\{\phi\}\}$	$\{\phi,\{\phi\}\}-\{\{\phi\}\}=\{\phi\}$
c. $\{\phi,\{\phi\}\}-\{\phi\}$	$\{\phi,\{\phi\}\}-\{\phi\}=\{\{\phi\}\}$
d. $\{\{\{\phi\}\}\}-\phi$	$\{\{\{\phi\}\}\}-\phi=\{\{\{\phi\}\}\}$

8. Let $U=\{2,3,6,8,9,11,13,15\}$, $A=\{x|x$ is a positive prime factor of 66$\}$
$B=\{x\in U|\ x$ is composite number $\}$ and $C=\{x\in U|\ x-5\in U\}$. Then find each of the following.

$A\cap B$, $(A\cup B)\cap C$, $(A-B)\cup C$, $(A-B)-C$, $A-(B-C)$, $(A-C)-(B-A)$, $A'\cap B'\cap C'$

<u>**Given**</u>

$\boldsymbol{U=\{2,3,6,8,9,11,13,15\}}$ $\boldsymbol{A=\{2,3,11\}}$ $\boldsymbol{B=\{6,8,9,15\}}$ $\boldsymbol{C=\{8,11,13\}}$

$A\cap B=\phi$ $(A\cup B)\cap C=\{2,3,6,8,9,11,15\}\cap\{8,11,13\}=\{8,11\}$

$(A-B)\cup C=\{2,3,11\}\cup\{8,11,13\}=\{2,3,8,11,13\}$

$(A-B)-C=\{2,3,11\}-\{8,11,13\}=\{2,3\}$

$A-(B-C)=2,3,11\}-\{6,9,15\}=\{2,3,11\}$

$(A-C)-(B-A)=\{2,3\}-=\{6,8,9,15\}=\{2,3\}$

$A'\cap B'\cap C'=\{6,8,9,13,15\}\cap\{2,3,11,13\}\cap\{2,3,6,9,15\}=\phi$

9. Let $A\cup B = \{a, b, c, d, e, x, y, z\}$ and $A\cap B = \{b, e, y\}$.

<u>We recall the following:</u> $B - A = B\cap A' = (A\cup B) - A$ **and** $A - B = A\cap B' = (A\cup B) - B$

a. $If\ B - A = \{x, z\}$, then $A =$ ____________________

<u>$A = \{a, b, c, d, e, y\}$</u>

b. $If\ A - B = \phi$, then $B =$ ____________________

<u>$B = \{a, b, c, d, e, x, y, z\}$</u>

c. $If\ B = \{b, e, y, z\}$, then $A - B =$ ________________

<u>$A - B = \{a, c, d, x\}$</u>

10. Let $U = \{1, 2, ..., 10\}$, $A = \{3, 5, 6, 8, 10\}$, $B = \{1, 2, 4, 5, 8, 9\}$, $C = \{1, 2, 3, 4, 5, 6, 8\}$ and $D = \{2, 3, 5, 7, 8, 9\}$.

Verify each of the following.

a. $(A\cup B)\cup C = A\cup(B\cup C)$.

$(A\cup B)\cup C = A\cup(B\cup C)$={ $\{1, 2, 3, 4, 5, 6, 8, 9, 10\}$

[Associative property of $\cup$]

b. $A\cap(B\cup C\cup D) = (A\cap B)\cup(A\cap C)\cup(A\cap D) = \{3,5,6,8\}$

[Distributive property of $\cap$ over $\cup$]

c. $(A\cap B\cap C\cap D)' = A'\cup B'\cup C'\cup D' = \{5, 8\}$

[Distributive property of absolute complement over $\cap$]

d. $C - D = C\cap D' = \{1, 4, 6\}$

[Property relating relative complement and absolute complement]

e. $A\cap(B\cap C)' = (A - B)\cup(A - C) = \{3, 6, 10\}$

[Property relating distributive property of absolute complement over $\cap$ and relative complement]

11. Depending on question No. 10 find.

a. $A\ \Delta\ B$

$A\ \Delta\ B = (A - B)\cup(B - A) = \{1, 2, 3, 4, 6, 9, 10\}$

b. $C\ \Delta\ D = (C - D)\cup(D - C) = \{1, 4, 6, 7\}$

c. $(A\ \Delta\ C)\Delta\ D$

Let $M = A\ \Delta\ C = (A - C)\cup(C - A) = \{1, 2, 4, 10\}$ $D = D = \{2, 3, 5, 7, 8, 9\}$.

Then $(A\ \Delta\ C)\Delta\ D = M\ \Delta\ D = (M - D)\cup(D - M) = \{1, 3, 4, 5, 7, 8, 9, 10\}$

Therefore, $(A\ \Delta\ C)\Delta\ D = \{1, 3, 4, 5, 7, 8, 9, 10\}$

d. $(A \cup B) \setminus (A \,\Delta\, B)$

$A \cup B) \setminus (A \,\Delta\, B) = (A \cup B) \cap (A \,\Delta\, B)' = (A \cup B) \cap [(A \cup B) - (A \cap B)]' = (A \cup B) \cap [(A \cup B) \cap (A \cap B)']'$

$(A \cup B) \cap [(A \cup B)' \cup (A \cap B)] = [(A \cup B) \cap (A \cup B)'] \cup [(A \cup B) \cap (A \cap B)] = \phi \cup (A \cap B)$

$= A \cap B = \{5, 8\}$

12. For any two subsets A and B of a universal set U, prove that:

a. $\Delta\, B = B \,\Delta\, A$.

$A \,\Delta\, B = (A - B) \cup (B - A) = (B - A) \cup (A - B) = B \,\Delta\, A$

b. $A \,\Delta\, B = (A \cup B) - (A \cap B) = (B \cup A) - (B \cap A) = B \,\Delta\, A$.

$A \,\Delta\, B = (A - B) \cup (B - A) = (A \cap B') \cup (B \cap A') = (A \cup B) \cap (A \cap B)' = (A \cup B) - (A \cap B)$

c. $A \,\Delta\, \phi = A$.

$A \,\Delta\, \phi = A = (A \cup \phi) - (A \cap \phi) = A - \phi = A \cap \phi' = A \cap U = A$

d. $A \,\Delta\, A = \phi$

$A \,\Delta\, A = (A \cup A) - (A \cap A) = A - A = A \cap A' = \phi$

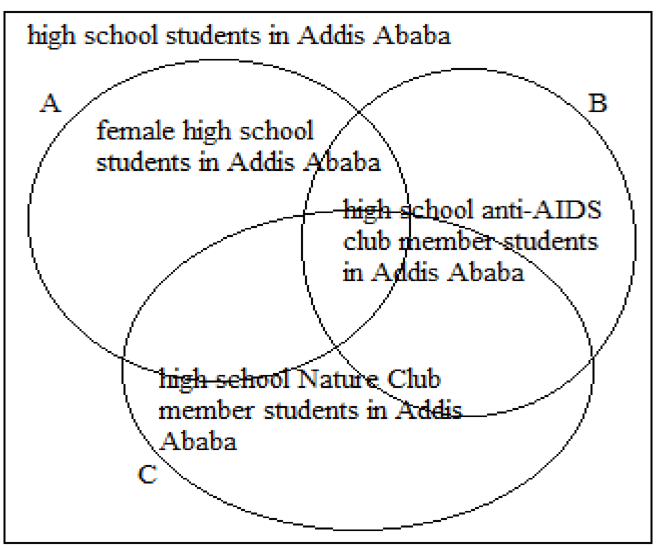
13. Draw an appropriate Venn diagram to depict each of the following sets.

a. U = The set of high school students in Addis Ababa.

A = The set of female high school students in Addis Ababa.

B = The set of high school anti-AIDS club member students in Addis Ababa.

C = The set of high school Nature Club member students in Addis Ababa.



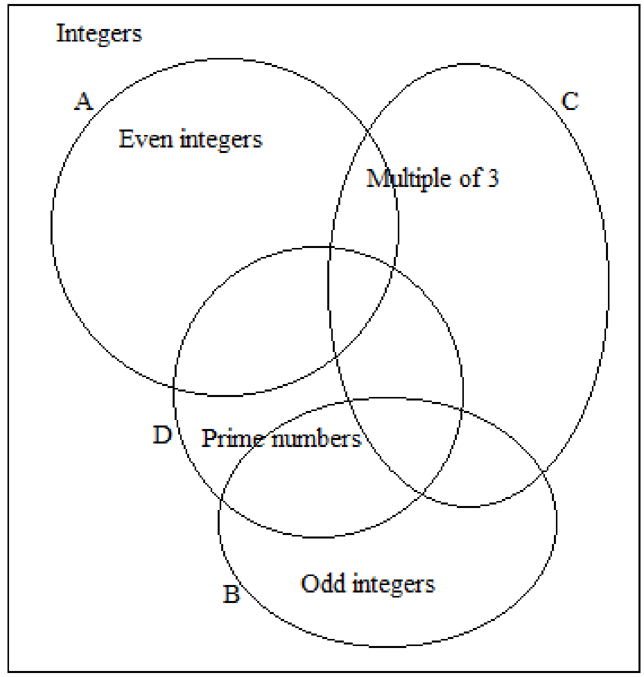
b. U = The set of integers.

A = The set of even integers.

B = The set of odd integers.

C = The set of multiples of 3.

D = The set of prime numbers.



Unit TWO

Solutions/Answers to Exercises 2.1 of page 47

1. Find an odd natural number x such that LCM (x, 40) = 1400.

Solution:- x is odd means x = 2n+1, $n \in N$

Let GCF(x, 40) = g

Therefore, 1400g= 40x = 40(2n+1)

$\Rightarrow \frac{1400}{40} = \frac{x}{2n+1} \Rightarrow 35 = \frac{x}{2n+1} \Rightarrow x = 35(2n+1)$

When n=1, x=35(2x1+1) = 35x3 = 105 which is odd

When n=2, x=35(2x2+1) = 35x5 = 175 which is odd

When n=1, x=35(2x3+1) = 35x7 = 245 which is odd

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In a similar way, the odd natural number x such that LCM (X, 40) = 1400 is the number

X = $35(2n+1)$, $n \in N$

2. There are between 50 and 60 number of eggs in a basket. When Loza counts by 3's, there are 2 eggs left over. When she counts by 5's, there are 4 left over. How many eggs are there in the basket?

Solution:- Let the number of eggs be n, 50 ≤n ≤60 , x & y are number of counts

Then n = 3x+2 = 5y+4

i.e., n = 3x+2 ⇒50 ≤3x+2 ≤60⇒48 ≤3x ≤58⇒48/3 ≤x≤58/3⇒16 ≤x+ ≤19.333 …⇒16 ≤x+ ≤19

n= 5y+4⇒50 ≤5y+4 ≤60⇒46 ≤5y ≤56⇒46/5≤y ≤56/5⇒46/5≤y ≤56/5⇒9.2≤y ≤11.2⇒9≤y ≤11

⇒3x+2=5y+4 ⇒3x-5y=4-2⇒3x-5y=2

When x=16, 3(16)-5y=2⇒48-2=5y⇒46=5y⇒46/5=y not counting number

Proceeding in a similar way, when x=19, 3(19)-5y=2⇒57-2=5y⇒55=5y⇒55/5=y=11

Therefore, there are n = 3x+2=3(19)+2=59 eggs

3. The GCF of two numbers is 3 and their LCM is 180. If one of the numbers is 45, then find the second number.

Solution:- Let the second number be y, then 3x180=45y ⇒ y= 540/45=12 ⇒ y=12

4. Using Mathematical Induction, prove the following:

a) 6^n - 1 is divisible by 5 for n ≥0

Proof:-

(1) For n=0, 6^0 - 1=1-1=0 is divisible by 5 is true and

for n=1, $6^1 - 1=6-1=5$ is divisible by 5 is true

(2) Assume for n=k, $6^k - 1$ is divisible by 5 is true i.e., $\exists m \in \mathbf{W}$ such that $6^k - 1 = 5m$

We should show that it is true for n = k+1,

Claim:- $6^{k+1} - 1$ is divisible by 5 or $\exists d \in \mathbf{W}$ such that $6^{k+1} - 1 = 5d$

Now $6^k - 1 = 5m \Rightarrow 6^k = 5m + 1$

$\Rightarrow 6.(6^k) = 6(5m+1) = 6(5m) + 6$

$\Rightarrow (6^{k+1}) - 1 = 6(5m) + 6 - 1$

$\Rightarrow (6^{k+1}) - 1 = 6(5m) + 5$

$\Rightarrow (6^{k+1}) - 1 = 5[(6m) + 1]$, $m \in$, $6m \in \mathbf{W}$, $(6m+1) \in \mathbf{W}$

$\Rightarrow (6^{k+1}) - 1 = 5d$, where $d = [(6m) + 1] \in \mathbf{W}$ is true

Therefore, $6^n - 1$ is divisible by 5 for any $n \geq 0$

(3) b) $2^n \leq (n+1)!$, for $n \geq 0$

Proof:- For n=0, $2^0 \leq (1+0)! \Rightarrow 1 \leq 1! \Rightarrow 1 \leq 1$ is true and for n=1, $2^1 \leq (1+1)! \Rightarrow 2 \leq 2! \Rightarrow 2 \leq 2$is true

Assume that it is true for n = k i.e., $2^k \leq (k+1)!$

We should show that it is true for n = k+1

Claim:- $2^{(k+1)} \leq [(k+1)+1]! = [(k+2)!$

Now, $2^k \leq (k+1)! \Rightarrow 2^k(2) \leq 2(k+1)! \leq (k+2)(k+1)!$ Since $2 \leq k+1 \leq k+2$

$\Rightarrow 2^k(2) = 2^{k+1} \leq (k+2)(k+1)! = (k+2)! \Rightarrow 2^{k+1} \leq (k+2)!$

c) $x^n + y^n$ is divisible by x+y , for odd natural number $n \geq 1$

Proof:- For n=1, $x^1 + y^1$ is divisible by x+y is true

Assume that it is true for an odd number n = k

i.e., $x^k + y^k$ is divisible by x+y is true ; since k is odd, $\exists m \in \mathbf{W}$ such that k = 2m+1

$x^k + y^k = t(x+y)$, for some $t \in N$, is true i.e., $x^{2m+1} + y^{2m+1} = t(x+y)$, for some $t \in N$, is true

We should show that it is true for the next odd number n = k+2=2m+1+2=2(m+1)+1

Claim:- $x^{(k+2)} + y^{(k+2)} = x^{(2m+2+1)} + y^{(2m+2+1)}$ is divisible by x+y is true

Or $x^{(2m+2+1)} + y^{(2m+2+1)} = x^{2(m+1)+1)} + y^{2(m+1)+1)} = d(x+y)$, for some $d \in N$

$\Rightarrow x^{2m+1} + y^{2m+1} = t(x+y)$

$\Rightarrow (x)(x^{2m}) + (y)(y^{2m}) = (x)(x^{2m}) + (y)(x^{2m}) - (y)(x^{2m}) + (y)(y^{2m}) + (x)(y^{2m}) - (x)(y^{2m})$

$= (x)(x^{2m}) + (y)(x^{2m}) - (y)(x^{2m}) + (y)(y^{2m}) + (x)(y^{2m}) - (x)(y^{2m})$

$= (x^{2m})(x+y) - (y)(x^{2m}) + (y^{2m})(y+x) - (x)(y^{2m})$

$= (x^{2m})(x+y) + (y^{2m})(y+x) - (y)(x^{2m}) - (x)(y^{2m})$

$= (x+y)(x^{2m} + y^{2m}) - (y)(x^{2m}) - (x)(y^{2m}) = (x+y)(x^{2m} + y^{2m}) - (xy)(x^{2m-1}) - (xy)(y^{2m-1})$

Here $= \underbrace{(x+ y) (x^{2m} +y^{2m})}_{\text{Divisible by x+y}} -(xy)[\underbrace{(x^{2m-1}) +(y^{2m-1})}_{\text{Divisible by x+y, since odd number2m-1<k}}]$

$= (x+ y) (x^{2m} +y^{2m}) -(xy)[k(x+ y)]$ because $(x^{2m-1}) +(y^{2m-1})$ & $x^{2m} +y^{2m}$ are is divisible by x+y

$\Rightarrow (x^{2m-1}) +(y^{2m-1}) = k(x+ y)$ and $x^{2m} +y^{2m} = (x^{2m}) +(y^{2m}) = r(x+ y)$ where $k, r \in N$

$\Rightarrow (x+ y) (x^{2m} +y^{2m}) -(xy)[k(x+ y)] == (x+ y) r(x+ y) -(xy)[k(x+ y)] =(x+ y)[r(x+ y) - k(xy)]$

which is divisible by x+y since $\{(x+ y)[r(x+ y) - k(xy)]\}/(x+y) = r(x+ y) - k(xy)$ We are done

d) $2+4+6+ \dots + 2n = n(n+1)$

Proof:- For n=1, $2 = 1(1+1)! \Rightarrow 2= 2$ is true

Assume that it is true for n = k i.e., $2+4+6+ \dots + 2k = k(k+1)$ is true

We should show that it is true for n = k+1

Claim:- $2+4+6+ \dots + 2(k+1) = (k+1)(k+1+1)$ is true

Now, $2+4+6+ \dots + 2k = k(k+1)$

$\Rightarrow 2+4+6+ \dots + 2k +2(k+1)= k(k+1)+2(k+1)$

$\Rightarrow 2+4+6+ \dots + 2k +2(k+1)= (k+1) (k+2)$

Therefore, $2+4+6+ \dots +2(k+1)= (k+1) (k+2)$ is true

Alternative Way

$$\sum_{i=1}^{n}(2i)=n(n+1)$$

Proof:- For n=1, $\sum_{i=1}^{1}(2x1)=1(1+1)\Rightarrow 2=2$ is true

Assume that it is true for n = k i.e., $\sum_{i=1}^{k}(2k)=k(k+1)$ is true

We should show that it is true for n = k+1

Claim:- $\sum_{i=1}^{k+1}(2k)=(k+1)(k+2)$ is true

Now, $\sum_{i=1}^{k}(2k)=k(k+1)$

$$\Rightarrow \left(\sum_{i=1}^{k}(2k)\right)+2(k+1)=k(k+1)+2(k+1)$$

$$\Rightarrow \left(\sum_{i=1}^{k+1}(2k)\right)=k(k+1)+2(k+1) \Rightarrow \left(\sum_{i=1}^{k+1}(2k)\right)=(k+1)(k+2)$$

Therefore, $\left(\sum_{i=1}^{k+1}(2k)\right)=(k+1)(k+2)$ is true (We are done)

e) $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

Proof:- For n=1, $1^2=\frac{1(1+1)(2x1+1)}{6}=\frac{6}{6}=1$ is true

Assume that it is true for n = k i.e., $1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$ is true is true

We should show that it is true for n = k+1

Claim:-

$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)(k+1+1)[2(k+1)+1]}{6}=\frac{(k+1)(k+2)(2k+3)}{6}$ is true

Now, $[1^1+2^2+3^2+\ldots+k^2]+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$

$\Rightarrow 1^1+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$\Rightarrow 1^1+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$

$\Rightarrow 1^1+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)(2k^2+7k+6)}{6}$

$\Rightarrow 1^1+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)(2k^2+7k+6)}{6}$

$\Rightarrow 1^1+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$

Therefore, $1^1+2^2+3^2+\ldots+k^2+(k+1)^2==\frac{(k+1)(k+2)(2k+3)}{6}$ is true

Alternative Way

$$\sum_{i=1}^{n}(i^2)=\frac{n(n+1)(2n+1)}{6}$$

Proof:- For n=1, $\sum_{i=1}^{n}(1^2)=\frac{1(1+1)(2x1+1)}{6}$

For n=1, $1=\frac{1(1+1)(2x1+1)}{6}=\frac{2(3)}{6}=\frac{6}{6}=1$ is true

Assume that it is true for n = k i.e., $\sum_{i=1}^{k}(i^2)=\frac{k(k+1)(2k+1)}{6}$ is true

We should show that it is true for n = k+1

Claim:- $\sum_{i=1}^{k+1}(i^2)=\frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$ is true

Simplifying RHS $\sum_{i=1}^{k+1}(i^2)=\frac{(k+1)(k+2)(2k+3)}{6}$ is true

Now, $\sum_{i=1}^{k}(i^2)=\frac{k(k+1)(2k+1)}{6}$

$$\Rightarrow \left(\sum_{i=1}^{k}(i^2)\right)+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$\Rightarrow \left(\sum_{i=1}^{k}(i^2)\right)+(k+1)^2=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$\Rightarrow \left(\sum_{i=1}^{k}(i^2)\right)+(k+1)^2=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$\Rightarrow \left(\sum_{i=1}^{k}(i^2)\right)+(k+1)^2=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$\Rightarrow \left(\sum_{i=1}^{k}(i^2)\right)+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$$

We are done

f) $1^3+2^3+3^3+\ldots+n^3=\frac{n^2(n+1)^2}{4}$

Proof:- For n=1, $1^3=1^3=\frac{1^2(1+1)^2}{4}\Rightarrow 1=\frac{4}{4}=1$ is true

Assume that it is true for n = k i.e., $1^3+2^3+3^3+\ldots+k^3=\frac{n^2(n+1)^2}{4}$ is true is true

We should show that it is true for n = k+1

Claim:- $1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{(k+1)^2(k+1+1)^2}{4}=\frac{(k+1)^2(k+2)^2}{4}$ is true

Now, $[1^3+2^3+3^3+\ldots+k^3]+(k+1)^3=\frac{(k)^2(k+1)^2}{4}+(k+1)^3$

$\Rightarrow 1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{k^2(k+1)^2+4(k+1)^3}{4}$

$\Rightarrow 1^3+2^3+3^3+ \ldots + k^3 + (k+1)^3= \dfrac{(k+1)^2\left[k^2+4(k+1)\right]}{4}$

$\Rightarrow 1^3+2^3+3^3+ \ldots + k^3 + (k+1)^3= \dfrac{(k+1)^2\left[k^2+4k+4\right]}{4}$

$\Rightarrow 1^1+2^2+3^2+ \ldots + k^2 + (k+1)^2= \dfrac{(k+1)^2(k+2)^2}{4}$

$\Rightarrow 1^1+2^2+3^2+ \ldots + k^2 + (k+1)^2= \dfrac{(k+1)(k+2)(2k+3)}{6}$

Therefore, $1^3+2^3+3^3+ \ldots + k^3 + (k+1)^3 = \dfrac{(k+1)^2(k+2)^2}{4}$ is true

<u>Alternative Way</u>

$$\sum_{i=1}^{n}\left(i^2\right)=\frac{n(n+1)(2n+1)}{6}$$

<u>Proof</u>:- For n=1, $\sum_{i=1}^{n}\left(1^2\right)=\dfrac{1(1+1)(2x1+1)}{6}$

For n=1, $1=\dfrac{1(1+1)(2x1+1)}{6}=\dfrac{2(3)}{6}=\dfrac{6}{6}=1$ is true

Assume that it is true for n = k i.e., $\sum_{i=1}^{k}\left(i^2\right)=\dfrac{k(k+1)(2k+1)}{6}$ is true

We should show that it is true for n = k+1

<u>Claim</u>:- $\sum_{i=1}^{k+1}\left(i^2\right)=\dfrac{(k+1)(k+1+1)(2(k+1)+1)}{6}$ is true

Simplifying RHS $\sum_{i=1}^{k+1}\left(i^2\right)=\dfrac{(k+1)(k+2)(2k+3)}{6}$ is true

Now, $\sum_{i=1}^{k}\left(i^2\right)=\dfrac{k(k+1)(2k+1)}{6}$

$$\Rightarrow \left(\sum_{i=1}^{k}\left(i^2\right)\right)+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$\Rightarrow \left(\sum_{i=1}^{k}\left(i^2\right)\right)+(k+1)^2=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$\Rightarrow \left(\sum_{i=1}^{k}\left(i^2\right)\right)+(k+1)^2=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$\Rightarrow \left(\sum_{i=1}^{k}\left(i^2\right)\right)+(k+1)^2=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$\Rightarrow \left(\sum_{i=1}^{k}\left(i^2\right)\right)+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$$ We are done

g) $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Proof:- For n=1, $\frac{1}{1\times 2}=\frac{1}{2+1}\Rightarrow\frac{1}{2}=\frac{1}{2}$ is true

Assume that it is true for n = k i.e., $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true is true

We should show that it is true for n = k+1

Claim:- $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}$ is true

Now, $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

$$\Rightarrow \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$\Rightarrow \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$\Rightarrow \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$\Rightarrow \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$\Rightarrow \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{(k+1)}{(k+2)}$$

$$\Rightarrow \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}$$ We are done

Solutions/Answers to Exercises 2.2 of page 55

2. Express each of the following rational numbers as decimal:

   a) $\frac{4}{9}$ **= 0.444 … =** $0.\overline{4}$

   b) $\frac{2}{25}$ **= 0.08**

   c) $\frac{11}{7}$ = 1.571428571428 … = $1.\overline{571428}$

   d) $-5\frac{2}{3}$ **=** $-5.\overline{6}$

   e) $\frac{2}{77}$ **=** 0.025974025974 … $0.\overline{025974}$

2. Write each of the following as decimal and then as a fraction:

   a) three tenths = $0.3 = \frac{3}{10}$

   b) four thousands = 4,000

3. Write each of the following in meters as a fraction and then as a decimal

   a) 4 mm = $\frac{4}{1000}$ m **=** 0.004 m

   b) 6 cm and 4 mm = $\frac{64}{1000}$ m **=** 0.064 m

   c) 56 cm and 4 mm = $\frac{564}{1000}$ m **=** 0.564 m

4. Classify each of the following as terminating or non-terminating periodic

   a) $\frac{5}{13}$ it is non-terminating periodic
   since the denominator has the prime factor, 13, different from 2 and 5 in its lowest term

   b) $\frac{7}{10}$ terminating
   since the only prime factors of the denominator are 2 and 5 in its lowest term

   c) $\frac{69}{64}$ terminating
   since the only prime factors of the denominator is 2 in its lowest term

   d) $\frac{11}{60}$ it is non-terminating periodic
   since the denominator has the prime factor, 3, different from 2 and 5 in its lowest term

   e) $\frac{5}{12}$ it is non-terminating periodic
   since the denominator has the prime factor, 3, different from 2 and 5 in its lowest term

5. Convert the following decimals to fractions:

a) $3.2\bar{5}=3.2\bar{5}\left(\frac{10^{1+1}-10^{1}}{10^{1+1}-10^{1}}\right)=3.2\bar{5}\left(\frac{100-10}{100-10}\right)=\frac{3.2\bar{5}x100-3.2\bar{5}x10}{100-10}=\frac{325.\bar{5}-32.\bar{5}}{90}$

$=\frac{325-32}{90}=\frac{325-32}{90}=\frac{293}{90}$

b) $0.3\overline{14}=0.3\overline{14}\left(\frac{10^{1+1}2-10^{1}}{10^{1+2}-10^{1}}\right)=0.3\overline{14}\left(\frac{1000-10}{1000-10}\right)=\frac{0.3\overline{14}x1000-0.3\overline{14}x10}{1000-10}$

$=\frac{314.\overline{14}-3.\overline{14}}{990}=\frac{314-3}{990}==\frac{311}{990}$

c) $0.\overline{275}=0.\overline{275}\left(\frac{10^{0+3}-10^{0}}{10^{0+3}-10^{0}}\right)=0.\overline{275}\left(\frac{1000-1}{1000-1}\right)=\frac{0.\overline{275}x1000-0.\overline{275}x1}{1000-1}$

$=\frac{275.\overline{275}-0.\overline{275}}{999}=\frac{275}{999}$

6. Determine whether the following are rational or irrational:

a) $2.7\bar{5}$ is rational number which is non-terminating periodic number

b) 0.272727 … is rational number which is non-terminating periodic number which is $0.\overline{27}$

c) $\sqrt{8}-\frac{1}{\sqrt{2}}$ is irrational number which is neither terminating nor non-terminating periodic number which is equal to:

$\sqrt{8}-\frac{1}{\sqrt{2}}=\frac{\sqrt{2}\sqrt{8}-1}{\sqrt{2}}=\frac{\sqrt{16}-1}{\sqrt{2}}=\frac{4-1}{\sqrt{2}}=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}=(1.5)(\sqrt{2})$

$=(1.5)(1.4142135..)=2.12132034..$

7. Which of the following statements are true and which of them are false?

a) The sum of any two rational numbers is rational
True since the set of rational is closed under addition

b) The sum of any two irrational numbers is irrational
False since for irrational number m and m+ –m =0 is rational number

c) The product of any two rational numbers is rational
True since the set of rational is closed under multiplication

d) The product of any two irrational numbers is irrational
False since for irrational number $\sqrt{2},(\sqrt{2})(\sqrt{2})=2$ is rational number

11. Find two rational numbers between $\frac{1}{3}$ and $\frac{1}{2}$

$\frac{\frac{1}{3}+\frac{1}{2}}{2}=\frac{\left(\frac{2+3}{6}\right)}{2}=\frac{5}{12}$ and $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\frac{2}{9}+\frac{1}{4}=\frac{8+9}{36}=\frac{17}{36}$

Note for any two rational numbers m & n : 0.5m+0.5n
and 0.6m+0.5n are rational numbers between m & n

Solutions/Answers to Exercises 2.3 of page 60-61

1. Verify that

a. $\left(\sqrt{2}-\boldsymbol{i}\right)-\boldsymbol{i}\left(1-\sqrt{2}\,\boldsymbol{i}\right)=-2\boldsymbol{i}$

$$\left(\sqrt{2}-\boldsymbol{i}\right)-\boldsymbol{i}\left(1-\sqrt{2}\,\boldsymbol{i}\right)=-2\,\boldsymbol{i}=\left(\sqrt{2}-\boldsymbol{i}\right)-\boldsymbol{i}+\sqrt{2}\,\boldsymbol{i}^2=\sqrt{2}-\boldsymbol{i}-\boldsymbol{i}+\sqrt{2}\,(-1)$$

$$=\sqrt{2}-\boldsymbol{i}-\boldsymbol{i}-\sqrt{2}\;=\sqrt{2}-\sqrt{2}-\boldsymbol{i}-\boldsymbol{i}\;=0-2\boldsymbol{i}=2\boldsymbol{i}$$

b. (2, -3)(-2, 1) = (-1, 8)

$$(2,-3)(-2,1)=\,(2+-3i)(-2+i)=(2)(-2)+(2)(i)+(-3i)(-2)+(-3i)(i)$$

$$=-4+2i+6i+-3(-1)=-4+8i+3=-1+8i=(-1,8)$$

c. $(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right)=(2,1)$

$$(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right)=(2,1)$$

$$\Rightarrow(3+i)(3-i)\left(\frac{1}{5}+\frac{1}{10}i\right)=[(3)(3)+(3)(-i)+(i)(3)+(i)(-i)=9-3i+3i-(-1)]\left(\frac{1}{5}+\frac{1}{10}i\right)$$

$$=(9+1)\left(\frac{1}{5}+\frac{1}{10}i\right)=10\left(\frac{1}{5}+\frac{1}{10}i\right)=\left(\frac{10}{5}+\frac{10}{10}i\right)=(2+i)=(2,1)$$

d. $(2+3\mathrm{i})^2-(3i-6)=1+9i$

$$(2+3\mathrm{i})^2-(3i-6)=4+2(2)(3i)+(3i)^2-(3i-6)=4+12i+9(-1)-3i+6$$

$$=4+6-9+12i-3i=1+9i$$

2. Show that

a. $\mathrm{Re}(iz)=-\mathrm{Im}(z)$

Let z = x + yi

$$\mathrm{Re}(iz)=\mathrm{Re}(i(x+yi))=\mathrm{Re}(xi+y(i)^2)=\mathrm{Re}(xi-y)=-y$$

$$and\quad -\mathrm{Im}(z)=-\mathrm{Im}(x+yi)=\mathrm{Im}(-x-yi)=-y$$

Therefore, $\mathrm{Re}(iz)=-\mathrm{Im}(z)=-y$

b. $\mathrm{Im}(iz)=\mathrm{Re}(z)=$

Let z = x + yi

$$\mathrm{Im}(iz)=\mathrm{Im}[i(x+yi)\,]=\mathrm{Im}\left(xi+y(i)^2\right)=\mathrm{Im}\left(xi+y(i)^2\right)=\mathrm{Im}(xi+-y)=x$$

$$\mathrm{Re}(z)=\mathrm{Re}(x+yi)=x$$

c. $(z+1)^2=z^2+2z+1$ Let z = x + yi

$$(z+1)^2=(x+yi+1)^2=((x+1)+yi)^2=(x+1)^2+2(x+1)(yi)+(yi)^2$$

$$=x^2+2x+1+2xyi+2yi+-y^2=\left(x^2+2x+1-y^2\right)+(2xy+2y)i$$

$$z^2+2z+1=(x+yi)^2+2(x+yi)+1=x^2+2xyi+(yi)^2+2x+2yi+1$$

$$=x^2+2xyi-y^2+2x+2yi\;+1=\left(x^2+2x+1-y^2\right)+(2xy+2y)i$$

Therefore, $(z+1)^2=z^2+2z+1=\left(x^2+2x+1-y^2\right)+(2xy+2y)i$

3. Do the following operations and simplify your answer.

a) $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$

$$\frac{1+2i}{3-4i}+\frac{2-i}{5i}=\frac{(5i)(1+2i)+(3-4i)(2-i)}{(3-4i)(5i)}=\frac{5i+(6i)(2i)+6-3i-8i+(-4i)(-i)}{15i-(4i)(5i)}$$

$$=\frac{5i-12+6-3i-8i-4}{15i+20}=\frac{5i-3i-8i-12+6-4}{15i+20}=\frac{-6i-10}{15i+20}=\left(\frac{-10-6i}{20+15i}\right)\left(\frac{20-15i}{20-15i}\right)$$

$$=\frac{-200+150i-120i+90i^2}{400+225}=\frac{-200-50i-90}{625}=\frac{-290}{625}-\frac{50i}{625}=\frac{-58}{125}-\frac{2}{25}i$$

b) $\dfrac{5i}{(1-i)(2-i)(3-i)}$

$$\frac{5\boldsymbol{i}}{(1-\boldsymbol{i})(2-\boldsymbol{i})(3-\boldsymbol{i})}=\frac{5\boldsymbol{i}}{(2-\boldsymbol{i}-2\boldsymbol{i}+\boldsymbol{i}^2)(3-\boldsymbol{i})}=\frac{5\boldsymbol{i}}{(2-3\boldsymbol{i}-1)(3-\boldsymbol{i})}=\frac{5\boldsymbol{i}}{(1-3\boldsymbol{i})(3-\boldsymbol{i})}$$

$$=\frac{5\boldsymbol{i}}{3-\boldsymbol{i}-9\boldsymbol{i}+3\boldsymbol{i}^2}=\frac{5\boldsymbol{i}}{3-10\boldsymbol{i}-3}=\frac{5\boldsymbol{i}}{-10\boldsymbol{i}}=-\frac{1}{2}$$

c) $(1-i)^3$

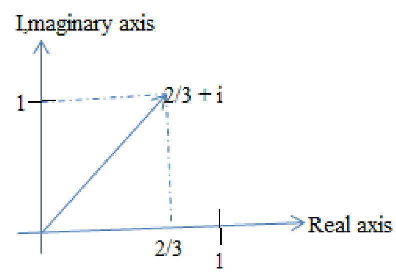
$$(1-\boldsymbol{i})^3=(1-\boldsymbol{i})(1-\boldsymbol{i})(1-\boldsymbol{i})=(1-\boldsymbol{i}-\boldsymbol{i}+\boldsymbol{i}^2)(1-\boldsymbol{i})=(1-2\boldsymbol{i}-1)(1-\boldsymbol{i})=(-2\boldsymbol{i})(1-\boldsymbol{i})$$

$$=-2\boldsymbol{i}+2\boldsymbol{i}^2=-2\boldsymbol{i}-2=-2-2\boldsymbol{i}$$

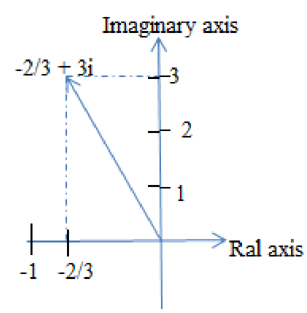
4. Locate the complex numbers z_1+z_2 and z_1-z_2, as vectors where

a) $\boldsymbol{z}_1=2\boldsymbol{i}$, $\boldsymbol{z}_2=\dfrac{2}{3}-\boldsymbol{i}$

$$\boldsymbol{z}_1+\boldsymbol{z}_2=2\boldsymbol{i}+\frac{2}{3}-\boldsymbol{i}=\frac{2}{3}-\boldsymbol{i}$$



$$\boldsymbol{z}_1-\boldsymbol{z}_2=2\boldsymbol{i}-\left(\frac{2}{3}-\boldsymbol{i}\right)=2\boldsymbol{i}-\frac{2}{3}+\boldsymbol{i}=3\boldsymbol{i}-\frac{2}{3}=-\frac{2}{3}+3\boldsymbol{i}$$



b) $\boldsymbol{z}_1=(-\sqrt{3},1)$, $\boldsymbol{z}_2=(\sqrt{3},0)$

$$\boldsymbol{z}_1+\boldsymbol{z}_2=(-\sqrt{3},1)+(\sqrt{3},0)=(-\sqrt{3}+\sqrt{3},1+0)=(0,1)$$

$$\boldsymbol{z}_1-\boldsymbol{z}_2=(-\sqrt{3},1)-(\sqrt{3},0)=(-\sqrt{3}-\sqrt{3},1-0)=(2\sqrt{3},1)$$

c) $\boldsymbol{z}_1=(-3,1)$, $\boldsymbol{z}_2=(1,4)$

$$\boldsymbol{z}_1+\boldsymbol{z}_2=(-3,1)+(1,4)=(-3+1,1+4)=(-2,5)$$

$$\boldsymbol{z}_1-\boldsymbol{z}_2=(-3,1)-(1,4)=(-3-1,1-4)=(-4,-3)$$

d) $z_1 = a + bi,\ z_2 = a - ib$

$$z_1 + z_2 = (a + bi) + (a - ib) = a + a + bi - bi = a + oi = (a, 0)$$
$$z_1 - z_2 = (a + bi) - (a - ib) = a - a + bi + bi = 0 + 2bi = (0, 2b)$$

5. Sketch the following set of points determined by the condition given below:

a) $|Z - 1 + i| = 1$

Let z = x+yi

$$|Z - 1 + i| = 1 \Rightarrow |(x + yi) - 1 + i| = 1 \Rightarrow |(x - 1) + yi + i| = 1 \Rightarrow |(x - 1) + (y + 1)i| = 1$$
$$\Rightarrow \sqrt{(x - 1)^2 + (y + 1)^2} = 1 \Rightarrow (x - 1)^2 + (y + 1)^2 = 1$$

Which represents points on the circle with center (1, 1) and radius 1.

b) $|z + i| \le 3$

Let z = x+yi

$$|z + i| \le 3 \Rightarrow |x + yi + i| \le 3 \Rightarrow |x + (y + 1)i| \le 3 \Rightarrow \sqrt{(x)^2 + (y + 1)^2} \le 3 \Rightarrow (x)^2 + (y + 1)^2 \le 9$$

Which represents points inside the circle with center (0, -1) and radius 3

c) $|z - 4i| \ge 4$

Let z = x+yi

$$|z - 4i| \ge 4 \Rightarrow |x + yi - 4i| \ge 4 \Rightarrow |x + (y - 4)i| \ge 4 \Rightarrow \sqrt{x^2 + (y - 4)^2} \ge 4 \Rightarrow x^2 + (y - 4)^2 \ge 16$$

Which represents points outside the circle with center (0, 4) and radius 4

6. Using properties of conjugate and modulus, show that

a) $\overline{\bar{z} + 3i} = z - 3i$

$$\overline{\bar{z} + 3i} = \bar{\bar{z}} + \overline{3i} = z + \overline{(0 + 3i)} = z + (0 - 3i) = z - 3i$$

b) $\overline{iz} = -i\,\bar{z}$

$$\overline{iz} = (\bar{i})(\bar{z}) = (\overline{0 + i})\bar{z} = (0 - i)\bar{z} = -i\,\bar{z}$$

c) $\overline{(2 + i)^2} = 3 - 4i$

$$\overline{(2 + i)^2} = \overline{4 + 4i + i^2} = \overline{4 + 4i - 1} = \overline{3 + 4i} = 3 - 4i$$

7. Show that

$$(-1 + i)^7 = 8(-1 - i)$$
$$(-1 + i)^7 = \left((-1 + i)^2\right)^3(-1 + i) = (1 - 2i - 1)^3(-1 + i) = (-2i)^3(-1 + i)$$
$$= (-8)(-i)(-1 + i) = 8i(-1 + i) = -8i + 8i^2 = -8i - 8 = -8 - 8i$$

8. Using mathematical induction, show that (when n = 2, 3, . . . ,)

$$\overline{z_1 + z_2 + \dots + z_n} = \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n$$

For n=1 : $\overline{z_1} = \bar{z}_1$ is true

For n=2: $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ by property of congugate

Assume, it is true for n=k , i.e., $\overline{z_1 + z_2 + \dots + z_k} = \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_k$

We want to prove that it is true for n=k+1 , that $\overline{z_1 + z_2 + \dots + z_k + z_{k+1}} = \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_k + \bar{z}_{k+1}$

Let $M = z_1 + z_2 + \dots + z_k$. Then $\overline{M} = \overline{z_1 + z_2 + \dots + z_k}$

And $\overline{M} + \overline{z_{k+1}} = \overline{M + z_{K+1}}$

$\Rightarrow \overline{z_1 + z_2 + \ldots + z_k} + \overline{z_{k+1}} = \overline{M + z_{K+1}} = \overline{z_1 + z_2 + \ldots + z_k + z_{k+1}}$

Hence $\overline{z_1 + z_2 + \ldots + z_n} = \overline{z_1} + \overline{z_2} + \ldots + \overline{z_n}$

9. Show that the equation $|z - z_0| = r$ which is a circle of radius r centered at z_o can be written as $|z|^2 - 2\operatorname{Re}(z\overline{z}) + |z_0|^2 = r^2$

Correction:-The question must be corrected as $|z|^2 - 2\operatorname{Re}(z\overline{z_0}) + |z_0|^2 = r^2$

Let $z = (x, y)$ *and* $z_0 = (x_0, y_0)$

$|z - z_0| = r \Rightarrow |(x + yi) - (x_0 + y_0 i)| = |(x - x_0) + (y - y_0 i)| = r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$

$\Rightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(x - x_0)^2 + (y - y_0)^2}$

$\Rightarrow (x - x_0)^2 + (y - y_0)^2 = (x - x_0)^2 + (y - y_0)^2 \quad , \quad |z_0|^2 = x_0^2 + y_0^2$

And $|z|^2 = x^2 + y^2$, $z\overline{z}_0 = (x + yi)(x_0 - y_0 i) = xx_0 - xy_0 i + x_0 yi - yy_0 (i)^2 = xx_0 - xy_0 i + x_0 yi + yy_0$

$\Rightarrow -2\operatorname{Re}(z\overline{z}_0) = \operatorname{Re}(-2xx_0 + 2xy_0 i - 2x_0 yi - 2yy_0) = -2xx_0 - 2yy_0$,

$r^2 = (x - x_0)^2 + (y - y_0)^2 = x^2 - 2xx_0 + x_0^2 + y^2 - 2yy_0 + y_0^2 = x^2 + y^2 - 2xx_0 - 2yy_0 + x_0^2 + y_0^2$

$$\left.\begin{array}{l} Now\ |z|^2 - 2\operatorname{Re}(z\overline{z_0}) + |z_0|^2 = x^2 + y^2 - 2xx_0 - 2yy_0 + x_0^2 + y_0^2 \\ And\ r^2 = x^2 + y^2 - 2xx_0 - 2yy_0 + x_0^2 + y_0^2 \end{array}\right\} \text{the same which means that we are done}$$

Solutions/Answers to Exercises 2.4 of page 67

1. Find the argument of the following complex numbers:

   a) $z=\dfrac{3i}{-1-i}$

   $$\left(\frac{3i}{-1-i}\right)\left(\frac{-1+i}{-1+i}\right)=\frac{-3i+3(i^2)}{1-i^2}=\frac{-3i-3}{2}=\frac{-3}{2}i-\frac{3}{2}$$

   $$\arg(z)=\arg\left(\frac{-3/2}{-3/2}\right)=\arg(1)=\frac{\pi}{4}$$

   b) $z=\left(\sqrt{3}-i\right)^6$

   Note :- $\arg(z^n)=n\arg(z)+2k\pi\ , k \text{ is an integer and } z^n=r^n(\cos n\theta+i\sin n\theta)=r^n e^{i(n\theta)}$

   $$\textit{Therefore, } \arg(z)=\arg\left(\left(\sqrt{3}-i\right)^6\right)=6\arg\left(\sqrt{3}-i\right)=6\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=6(-\pi/6)=-\pi$$

2. Show that:

   a) $\left|e^{i\theta}\right|=1$

   $$\textit{From } e^{i\theta}\ ,\quad r=1, n=1, \textit{ and } \left|e^{i\theta}\right|=\left|\cos\theta+i\sin\theta\right|=\sqrt{(\cos\theta)^2+(\sin)^2}=\sqrt{1}=1$$

   b) $\overline{e^{i\theta}}=e^{-i\theta}$

   $\textit{From } e^{i\theta}\ ,\quad r=1, n=1 \textit{ and from } e^{-i\theta}, r=1, n=-1$

   $\Rightarrow e^{i\theta}=(\cos\theta+i\sin\theta)\Rightarrow \overline{e^{i\theta}}=(\cos\theta-i\sin\theta)$

   $\textit{and } e^{-i\theta}=1^{-1}(\cos(-\theta)+i\sin(-\theta))=(\cos\theta-i\sin\theta)$

   $\textit{Therefore,}\qquad \overline{e^{i\theta}}=e^{-i\theta}=(\cos\theta-i\sin\theta)$

3. Using Mathematical induction, show that:

   $e^{i\theta_1}.e^{i\theta_2}.\cdots.e^{i\theta_n}=e^{i(\theta+\theta_2+\ldots+\theta_n)}$,n=2, 3, …

   Proof:-

   $\textit{For } n=2, e^{i\theta_1}.e^{i\theta_2}=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)=e^{i(\theta+\theta_2)}$

   is True

   $\textit{Assume it is True for } n=k, \textit{ i.e., } e^{i\theta_1}.e^{i\theta_2}.\cdots.e^{i\theta_k}=e^{i(\theta+\theta_2+\ldots+\theta_k)}$

   $\textit{We wan to prove that it is True for } n=k+1,\quad \textit{i.e., } e^{i\theta_1}.e^{i\theta_2}.\cdots.e^{i\theta_{k+1}}=e^{i(\theta+\theta_2+\ldots+\theta_{k+1})}$

   $e^{i\theta_1}.e^{i\theta_2}.\cdots.e^{i\theta_k}=e^{i(\theta+\theta_2+\ldots+\theta_k)}\Rightarrow e^{i\theta_1}.e^{i\theta_2}.\cdots.e^{i\theta_k}.e^{i\theta_{k+1}}$

   $=(\cos(\theta_1+\theta_2+\ldots+\theta_k)+i\sin(\theta_1+\theta_2+\ldots+\theta_k))(\cos(\theta_{k+1})+i\sin(\theta_{k+1}))$

   $=(\cos(\theta_1+\theta_2+\ldots+\theta_k+\theta_{k+1})+i\sin(\theta_1+\theta_2+\ldots+\theta_k+\theta_{k+1}))=e^{i(\theta+\theta_2+\ldots+\theta_{k+1})}$

   $\Rightarrow e^{i\theta_1}.e^{i\theta_2}.\cdots.e^{i\theta_k}.e^{i\theta_{k+1}}=e^{i(\theta+\theta_2+\ldots+\theta_n)}$

   is True

4. show that:

$a)\ \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$a)\ \cos 3\theta = \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

$= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

5. show that:

$$1+z+z^2+\ldots+z^n = \frac{1-z^{n+1}}{1-z}\ ,\ for\, z \neq 1$$

$$1+z+z^2+\ldots+z^n = \frac{(1+z+z^2+\ldots+z^n)(1-z)}{1-z} =$$

$$\frac{1+z+z^2+\ldots+z^n - z - z^2 - \ldots - z^n - z^{n+1}}{1-z} = \frac{1-z^{n+1}}{1-z}\quad ,\ for\, z \neq 1$$

6. Find the square root of:

$z = 9i$

$$z = 9i \Rightarrow r = 9,\ \theta = \frac{\pi}{2},\quad n = 2\ ,k = 0{,}1$$

$$C_k = (r)^{\frac{1}{n}}\left(e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}\right) = (9)^{\frac{1}{n}}\left(e^{i\left(\frac{\theta}{2}+\frac{2k\pi}{2}\right)}\right) = 3\left(e^{i\left(\frac{\pi}{4}+k\pi\right)}\right)\ \sin ce, \theta = \frac{\pi}{2}\ \ , n = 2$$

$$If\ \ k = 0\ , C_0 = 3\left(e^{i\left(\frac{\pi}{4}\right)}\right) = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 3\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}\,i}{2}\right)$$

$$If\ \ k = 1\ , C_1 = 3\left(e^{i\left(\frac{\pi}{4}+\pi\right)}\right) = 3\left(e^{i\left(\frac{5\pi}{4}\right)}\right) = 3\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 3\left(\frac{-\sqrt{2}}{2} + \frac{-\sqrt{2}}{2}i\right)$$

$$= \left(\frac{-3\sqrt{2}}{2} + \frac{3\sqrt{2}\,i}{2}\right)$$

$$Therefore, the\ square\ roots\ of\ \ z = 9i\ \ are: C_0 = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\quad and\ \ C_1 = \frac{-3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

7. Find the cube root of:

$z = -8i$

$$z = -8i \Rightarrow r = 9,\ \theta = \frac{3\pi}{2},\quad n = 3\ ,k = 0{,}1,\ 2$$

$$C_k = (r)^{\frac{1}{n}}\left(e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}\right) = (8)^{\frac{1}{n}}\left(e^{i\left(\frac{\theta}{3}+\frac{2k\pi}{3}\right)}\right) = 2\left(e^{i\left(\frac{\pi}{2}+\frac{2k\pi}{3}\right)}\right)\ \sin ce, \theta = \frac{3\pi}{2}\ \ , n = 3$$

$$If\ \ k=0\ ,C_0=2\left(e^{i\left(\frac{\pi}{2}\right)}\right)=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2(0+i)=2i$$

$$If\ \ k=1\ ,C_1=2\left(e^{i\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)}\right)=2\left(e^{i\left(\frac{7\pi}{6}\right)}\right)=2\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=2\left(\frac{-\sqrt{3}}{2}+\frac{1}{2}i\right)=-\sqrt{3}+2i$$

$$If\ \ k=2,\ C_2=2\left(e^{i\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)}\right)=2\left(e^{i\left(\frac{11\pi}{6}\right)}\right)=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)=2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=\sqrt{3}-2i$$

$Therefore,\ the\ cube\ roots\ of\ \ z=-8i\ \ are:$

$C_0=2i\ \ ,\ C_1=-\sqrt{3}+2i\ \ \ and\ \ C_2=\sqrt{3}-2i$

8. Solve the following equations:

$a)\ z^{3/2}=8i$

$Let\ z=x+yi\Rightarrow z^3=(8i)^2\Rightarrow(x+yi)^3=(8i)^2$

$\Rightarrow z^3=x^3+3x^2yi-3xy^2-y^3i=(x^3-3xy^2)+(3x^2y-y^3)i=-64$

$\Rightarrow x^3-3xy^2=-64\ \ and\ 3x^2y-y^3=0$

$\Rightarrow x^3-3xy^2=-64\ \ and\ \ y(3x^2-y^2)=0$

$\Rightarrow x^3-3xy^2=-64\ \ and\ \ y=0\ or\ \ 3x^2-y^2=0$

$\Rightarrow x^3-3xy^2=-64\ \ and\ \ y=0\ or\ \ 3x^2=y^2$

$\Rightarrow x^3-3xy^2=-64\ \ and\ \ y=0\ or\ \pm\sqrt{3}x=y$ $\boxed{y=0\ or\ \pm\sqrt{3}x\ =y}$

$y=0,x^3-3x(0)^2=-64\Rightarrow y=0,x^3=-64\Rightarrow y=0,x=-4$ $\boxed{y=0,x=-4}$

$y=\sqrt{3}x,x^3-3x(\sqrt{3}x)^2=-64\Rightarrow\ y=\sqrt{3}x,x^3-9x^3=-64\Rightarrow\ y=\sqrt{3}x,-8x^3=-64$

$\Rightarrow y=\sqrt{3}x,8x^3=64\Rightarrow y=\sqrt{3}x,x^3=8\Rightarrow y=\sqrt{3}x,x=2\Rightarrow y=2\sqrt{3},x=2$

$y=-\sqrt{3}x,x^3-3x(-\sqrt{3}x)^2=-64\Rightarrow\ y=-\sqrt{3}x,x^3-9x^3=-64\Rightarrow\ y=-\sqrt{3}x,-8x^3=-64$

$\Rightarrow y=-\sqrt{3}x,8x^3=64\Rightarrow y=-\sqrt{3}x,x^3=8\Rightarrow y=-\sqrt{3}x,x=2\Rightarrow y=-2\sqrt{3},x=2$

$S.S.\{(-4,0),(2,2\sqrt{3}),(2,-2\sqrt{3})\}$

$b)\ \ z^2+4i=0$

$Let\ z=x+yi\ \ and\ \ z^2+4i=0\Rightarrow(x+yi)^2=-4i\Rightarrow x^2+2xyi-y^2=-4i$

$\Rightarrow x^2-y^2+2xyi=-4i\Rightarrow\ x^2-y^2+2xyi=-4i\Rightarrow x^2-y^2=0\ \ and\ 2xy=-4$

$\Rightarrow x^2-y^2=0\ \ and\ \ xy=-2\Rightarrow x^2-y^2=0\ \ and\ \ y=-2/x$

$\Rightarrow x^2-(-2/x)^2=0\ \ and\ \ y=\frac{-2}{x}\Rightarrow x^2-\left(\frac{4}{x^2}\right)=0\ \ and\ \ y=\frac{-2}{x}$

$\Rightarrow\frac{x^4-4}{x^2}=0\ \ and\ \ y=\frac{-2}{x}\Rightarrow x^4-4=0\ and\ \ y=\frac{-2}{x}\Rightarrow x=\pm\sqrt{2}\ and\ \ y=\pm\sqrt{2}$

$S.S.=\{(\sqrt{2},-\sqrt{2}),(-\sqrt{2},\sqrt{2})\}$

$c)\ z^2 - 4i = 0$

$Let\ z = x + yi\ \ and\ z^2 = 4i \Rightarrow (x + yi)^2 = 4i \Rightarrow x^2 + 2xyi - y^2 = 4i$

$\Rightarrow x^2 - y^2 + 2xyi = 4i \Rightarrow x^2 - y^2 = 0\ and\ 2xy = 4 \Rightarrow x = \pm y\ and\ 2xy = 4$

$\Rightarrow x = y\ and\ \ 2x^2 = 4 \Rightarrow x = y\ and\ \ x^2 = 2 \Rightarrow x = y\ and\ \ x = \pm\sqrt{2}$

$\Rightarrow x = y = \sqrt{2}\,,\ x = y = -\sqrt{2}$

$Again\, x = -y\ and\ \ 2x^2 = 4 \Rightarrow x = -y\ and\ \ x^2 = 2 \Rightarrow x = -y\ and\ \ x = \pm\sqrt{2}$

$\Rightarrow x = -y = \sqrt{2}\,,\ x = -y = -\sqrt{2} \Rightarrow x = \sqrt{2}\, and\ y = -\sqrt{2}\,,\ x = -\sqrt{2}\ and\, y = \sqrt{2}$

$S.S. = \left\{\left(\sqrt{2}\,, \sqrt{2}\right), \left(-\sqrt{2}, -\sqrt{2}\right), \left(\sqrt{2}\,, -\sqrt{2}\right), \left(-\sqrt{2}, \sqrt{2}\right)\right\}$

Unit Three
Solutions/Answers to Exercises 3.1 of page 75

1. Let *R* be a relation on the set $A = \{1,2,3,4,5,6\}$ defined by $R = \{(a,b): a+b \le 9\}$.

i) List the elements of R

$R = \{(1,1),(1,2),(1,3),(1,4),(1,5)(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5)(3,6)$
$(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,3)\ \}$

ii) Is $R = R^{-1}$ Yes since addition is commutative

2. Let *R* be a relation on the set $A = \{1,2,3,4,5,6,7\}$ defined by $R = \{(a,b): 4\ divides\ a-b\}$

i) List the elements of R

$R = \{\ (5,1),(6,2),(7,3)\ \}$

ii) Find Dom(R) & Range(R)

Dom(R)= $\{5,6,7\}$ and Range(R)= $\{1,2,3\}$

iii) Find the elements of R^{-1}

$R^{-1} = \{\ (1,5),(2,6),(3,7)\ \}$

iv) Find Dom(R^{-1}) & Range(R^{-1})

Dom(R^{-1})= $\{1,2,3\}$ and Range(R^{-1})= $\{5,6,7\}$

3. Let $A = \{1,2,3,4,5,6\}$. Define a relation on A by $R = \{(x,y): y = x+1\}$. Write down the domain, codomain and range of R . Find R^{-1}

Dom(R) $= \{1,2,3,4,5\}$

Codomain $= \{1,2,3,4,5,6\}$

range of R $= \{1,2,3,4,5,6\}$

R^{-1}= $\{(2,1),(3,2),(4,3),(5,4),(6,5)\}$

4. Find the domain and range of the relation $\{(x,y): |x| + y \ge 2\}$.

Let G = $\{(x,y): |x| + y \ge 2\}$.

Then Dom(G) = $\{x : x\ is\ a\ real\ number\}$.

Range(G) = $\{y : y\ is\ a\ real\ number\}$.

5. Let $\{1,2,3\}$ *and* $B = \{3,5,6,8\}$. Which of the following are functions from A to B?

a) $f = \{(1,3),(2,3),(3,3)\}$ Function Dom(f) = A and Range(f) = $\{3\}$

b) $f = \{(1,3),(2,5),(1,6)\}$ Not a function since Dom(f) ≠ A

c) $f = \{(1,8),(2,5)\}$ Not a function since Dom(f) ≠ A

d) $f = \{(1,6),(2,5),(3,3)\}$ Function Dom(f) = A and Range(f) = $\{3,5,6\}$

e)

6. Determine the domain and range of the given relation. Is the relation a function?

a) $\{(-4,-3),(2,-5),(4,6),(2,0)\}$

Domain = $\{-4,2,4,2\}$ Range = $\{-3,-5,6,0\}$

It is not a function since the element 2 in the domain maps to more than one element (to 2 different elements: to -5 and to 0) of the range

b) $\left\{(8,-2),\left(6,-\frac{3}{2}\right),(-1,5)\right\}$

Domain = $\{8,6,-1\}$ Range = $\left\{-2,-\frac{3}{2},5\right\}$

It is a function since no element of the domain maps to more than one element of the range

c) $\{(-\sqrt{3},3),(-1,1),(0,0),(1,1),(\sqrt{3},3)\}$

Domain = $\{-\sqrt{3},-1,0,1,\sqrt{3}\}$ Range = $\{3,1,0\}$

It is a function since no element of the domain maps to more than one element of the range

d) $\left\{\left(-\frac{1}{2},\frac{1}{6}\right),(-1,1),\left(\frac{1}{3},\frac{1}{8}\right),(1,1),(\sqrt{3},3)\right\}$

Domain = $\left\{-\frac{1}{2},-1,\frac{1}{3},1,\sqrt{3}\right\}$ Range = $\left\{\frac{1}{6},1,\frac{1}{8},1,3\right\}$

It is a function since no element of the domain maps to more than one element of the range

e) $\{(0,5),(1,5),(2,5),(3,5),(4,5),(5,5)\}$

Domain = $\{0,1,2,3,4,5\}$ Range = $\{5\}$

It is a function since no element of the domain maps to more than one element of the range

f) $\{(5,0),(5,1),(5,2),(5,3),(5,4),(5,5)\}$

Domain = $\{5\}$ Range = $\{0,1,2,3,4,5\}$

It is not a function since the element 5 in the domain maps to more than one element (to 6 different elements: to 0, to 1, to 2, to 3, to 4 and to 5) of the range

7. Find the domain and the range of the following functions.

a) $f(x)=1+8x-2x^2$

Dom(f) is the set of all real numbers

b) $f(x)=\dfrac{1}{x^2-5x+6}$

$Dom(f)=\{x: x\neq 2\, and\ x\neq 3\}\ (-\infty,2)\bigcup(2,3)\bigcup(3,\infty)$

c) $f(x)=\sqrt{x^2-6x+8}$

$Dom(f)=\{x: x\leq 2\ or\ x\geq 4\}=(-\infty,2]\bigcup[4,\infty)$

d) $f(x)=\begin{cases}3x+4,-1\leq x<2\\ 1+x\ ,\ 2\leq x\leq 5\end{cases}$

$Dom(f)=\{x:-1\leq x<5\}=[-1,5)$

8. Given $f(x)=\begin{cases}3x-5, x<1\\ x^2-1\ ,\ x\geq 1\end{cases}$

Find a) f(-3) b) f(1) c) f(6)

a) f(-3) = 3x-5 = 3(-3)-5 = -9-5 = -14

b) f(1) = $x^2-1=1^2-1=1-1=0$

c) f(6) = $x^2-1=6^2-1=36-1=35$

Solutions/Answers to Exercises 3.2 of page 81

1. $For\ f(x) = x^2 + x\ and\ g(x) = \frac{2}{x+3}\ ,\ find\ each\ value:$

a) $(f-g)(2)$ b) $\left(\frac{f}{g}\right)(1)$ c) $(g)^2(3)$ d) $(fog)(1)$ e) $(gof)(1)$ f) $(gog)(3)$

a) $(f-g)(2) = f(2) - g(2) = 2^2 + 2 - \frac{2}{2+3} = 6 - \frac{2}{5} = \frac{30-2}{5} = \frac{28}{5}$

b) $\left(\frac{f}{g}\right)(1) = \frac{f(1)}{g(1)} = \frac{1^2+1}{\left(\frac{2}{1+3}\right)} = \frac{2}{\left(\frac{2}{4}\right)} = 2(2) = 4$

c) $(g)^2(3) = (g(3))^2 = \left(\frac{2}{3+3}\right)^2 = \left(\frac{2}{6}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

d) $(fog)(1) = f(g(1)) = f\left(\frac{2}{1+3}\right) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{1+2}{4} = \frac{3}{4}$

e) $(gof)(1) = g(f(1)) = g(1^2+1) = g(2) = \frac{2}{2+3} = \frac{2}{5}$

f) $(gog)(3) = g(g(3)) = g\left(\frac{2}{3+3}\right) = g\left(\frac{2}{6}\right) = g\left(\frac{1}{3}\right) = \frac{2}{\left(\frac{1}{3}+3\right)} = \frac{2}{\left(\frac{1+9}{3}\right)} = 2\left(\frac{3}{10}\right) = \frac{3}{5}$

2. If $f(x) = x^2 + 2\ and\ g(x) = \frac{2}{x-1},\ find\ a\ formula\ for\ each\ of\ the\ following\ and\ state\ its\ domian$

a) $(f+g)(x)$ b) $(fog)(x)$ c) $\left(\frac{g}{f}\right)(x)$ d) $(gof)(x)$

a) $(f+g)(x) = f(x) + g(x) = x^2 + 2 + \frac{2}{x-1} = \frac{(x^2+2)(x-1)+2}{x-1} = \frac{x^3 - x^2 + 2x - 2}{x-1}$

b) $(fog)(x) = f(g(x)) = f\left(\frac{2}{x-1}\right) = \left(\frac{2}{x-1}\right)^2 + 2 = \frac{4}{(x-1)^2} + 2 = \frac{4 + 2(x-1)^2}{(x-1)^2}$

$= \frac{4 + 2(x^2 - 2x + 1)}{(x-1)^2} = \frac{4 + 2x^2 - 4x + 2}{(x-1)^2} = \frac{2x^2 - 4x + 6}{(x-1)^2}$

c) $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\left(\frac{2}{x-1}\right)}{x^2+2} = \left(\frac{2}{x-1}\right) \div (x^2+2) = \left(\frac{2}{x-1}\right)\left(\frac{1}{x^2+2}\right) = \frac{2}{x^3 - x^2 + 2x - 2}$

d) $(gof)(x) = g(f(x)) = g(x^2+2) = \frac{2}{x^2+2-1} = \frac{2}{x^2+1}$

3. $Let\ f(x) = x^2\ g(x) = \sqrt{x}$

a) Find $(fog)(x)\ and\ its\ domain$ b) Find $(gof)(x)\ and\ its\ domain$

c)

$Are\ (fog)(x)\ and\ and\ (gof)(x)\ the\ same\ functions?\ Explain$

$a)\ (fog)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x\ and\ its\ domain\ is\ the\ set\ of\ all\ real\ numbers$

$b)\ (gof)(x) = g(f(x)) = g(x^2) = \sqrt{(x^2)} = |x|\ \ and\ its\ domain\ is\ the\ set\ of\ real\ njumbers$

$c)\ (fog)(x)\ and\ (gof)(x)\ are\ NOT\ the\ same\ functions$

$For\ example\ if\ we\ see\ a)\ and\ b)\ above\ f(g(x)) = x\ and\ g(f(x)) = |x|\ which\ are\ different$

4. $Let\ f(x)=5x-3.\ Fimd\ g(x)\ so\ that\ f(g(x))=2x+7$

$$f(g(x))=2x+7$$

$$\Rightarrow 5(g(x))-3=2x+7 \Rightarrow 5(g(x))=2x+7+3 \Rightarrow 5(g(x))=2x+10 \Rightarrow g(x)=\frac{2x+10}{5}$$

5. $Let\ f(x)=2x+1.\ Fimd\ g(x)\ so\ that\ f(g(x))=3x-1$

$$f(g(x))=3x-1 \Rightarrow 2(g(x))+1=3x+1 \Rightarrow 2(g(x))=3x \Rightarrow g(x)=\frac{3x}{2}$$

6. $If\ \ f\ \ is\ real\ \ function\ defined\ by\ \ f(x)=\frac{x-1}{x+1}.\ Show\ that\ \ f(2x)=\frac{3f(x)+1}{f(x)+3}$

$$f(2x)=\frac{2x-1}{2x+1} \quad and \quad \frac{3f(x)+1}{f(x)+3}=\frac{3\left(\frac{x-1}{x+1}\right)+1}{\frac{x-1}{x+1}+3}=\frac{\frac{3x-3}{x+1}+1}{\frac{x-1+3(x+1)}{x+1}}=\frac{\frac{3x-3+x+1}{x+1}}{\frac{x-1+3x+3}{x+1}}$$

$$=\frac{\left(\frac{4x-2}{x+1}\right)}{\left(\frac{4x+2}{x+1}\right)}=\left(\frac{4x-2}{x+1}\right)\div\left(\frac{4x+2}{x+1}\right)=\left(\frac{2(2x-2)}{x+1}\right)\left(\frac{x+1}{2(2x-2)}\right)=\frac{2x-2}{2x+2} \qquad \text{and}$$

$$Therefore,\ f(2x)=\frac{3f(x)+1}{f(x)+3}=\frac{2x-2}{2x+2}$$

7. Find tow functions f and g so that the given function $h(x)=(fog)(x)$ where :

a) $h(x)=(x+3)^3$ b) $h(x)=\sqrt{5x-3}$ c) $h(x)=\frac{1}{x}+1$ d) $h(x)=\frac{1}{x+6}$

$a)\ Let\ g(x)=x+3\ \ and\ f(x)=x^3.\ \ Then\ h(x)=(fog)(x)=f(g(x))=f(x+3)=(x+3)^3$

$b)\ Let\ g(x)=5x-3\ and\ f(x)=\sqrt{x}.\ Then\ h(x)=(fog)(x)=f(g(x))=f(5x-3)=\sqrt{5x-3}$

$c)\ Let\ g(x)=\frac{1}{x}\ and\ f(x)=x+1.\ Then\ h(x)=(fog)(x)=f(g(x))=f\left(\frac{1}{x}\right)=\frac{1}{x}+1$

$d)\ Let\ g(x)=x+6\ and\ f(x)=\frac{1}{x}.Then\ h(x)=(fog)(x)=f(g(x))=f(x+6)=\frac{1}{x+6}$

8. $Let\ f(x)=4x-3\ ,\ g(x)=\frac{1}{x}\ \ and\ \ h(x)=x^2-x.\ Find:$

$a)\ f(5x+7)\quad b)\ 5f(x)+7\quad c)\ f(g(h(3)))\quad d)\ f(1).g(2).h(3)\quad e)\ f(x+a)\quad f)\ f(x)+a$

$a)\ f(5x+7)=4(5x+7)-3=20x+28-3=20x-25$

$b)\ 5f(x)+7=5(4x-3)+7=20x-15+7=20x-8$

$$c)\ f(g(h(3)))=f(g(x^2-x))=f\left(\frac{1}{x^2-x}\right)=4\left(\frac{1}{x^2-x}\right)-3=\frac{4}{x^2-x}-3=\frac{4-3(x^2-x)}{x^2-x}=\frac{4-3x^2-3x}{x^2-x}$$

$$d)\ f(1).g(2).h(3)=(4(1)-3)\left(\frac{1}{2}\right)(3^3-3)=(1)\left(\frac{1}{2}\right)(6)=\frac{6}{2}=3$$

$e)\ f(x+a)=4(x+a)-3=4x+4a-3$

$f)\ f(x)+a=4x-3+a=4x+a-3$

Solutions/Answers to Exercises 3.3 of page 85

1. *Consider the function* $f = \{(x, x^2) : x \in S\}$ *from* $S = \{-3, -2, -1, 0, 1, 2, 3\}$ *into* Z.
Is f *one to one? Is it onto*

The function is of the form $f(x) = x^2$

Let $x_1, x_2 \in S \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$ *hence* f *is not one to one*

$Range(f) = \{0, 1, 4, 9\} \neq Z$ *so that* f *is not onto*

2. *Let* $A = \{1, 2, 3\}$. *List all one to one functions from* A *onto* A

$f_1 = \{(1,1), (2,2)(3,3)\}, \quad f_2 = \{(1,1), (2,3)(3,2)\}, \quad f_3 = \{(1,2), (2,1)(3,3)\},$

$f_4 = \{(1,2), (2,3)(3,1)\}, \quad f_5 = \{(1,3), (2,2)(3,1)\}, \quad f_6 = \{(1,3), (2,1)(3,2)\}$

3. *Let* $f : A \to B$. *Let* f^* *be the inverse relation, i.e.,* $f^* = \{(y, x) \in B \times A : f(x) = y\}$

a) Show by an example that f^* *need not be a function*

Let $f = \{(1,2), (2,2)\}$ *then* $f^* = \{(2,1), (2,2)\}$

where 2 *maps to two different images which means* f^* *is NOT a function.*

b) Show that f^* *is a function from* $range(f)$ *into* A *if and only if* f *is* $1-1$

Supose f^* *is a function from* $range(f)$ *into* A *and let* $f(x_1) = y_1$ *or* $f^*(y_1) = x_1$, $f(x_2) = y_{12}$ *or* $f^*(y_2) = x_2$.

Then $Dom(f^*) = range(f)$ *and* $y_1 = y_2 \Rightarrow f^*(y_1) = f^*(y_2) \Rightarrow x_1 = x \Rightarrow y_1 = y_2 \Rightarrow x_1 = x_2$

$\Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ *which means* f *is* $1-1$

Again, suppose f *is* $1-1$

Then $Dom(f^*) = Ramge(f)$ *and* $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ *which means* $y_1 = y_2 \Rightarrow x_1 = x_2$

that means $y_1 = y_2 \Rightarrow f(f^*(y_1)) = f(f^*(y_2)) \Rightarrow x_1 = x_2$ *which means* $y_1 = y_2 \Rightarrow x_1 = x_2$

i.e., $y_1 = y_2 \Rightarrow f^*(y_1) = f^*(x_2)$ *whic means* f^* *is* f^* *is a function*

c) Show that f^* *is a function from* B *into* A *if and only if* f *is* $1-1$ *and onto*

Since f *is onto* $Range(f) = B$

Then f^* *is a function from* $range(f) = B$ *into* A

Therefore, by b) above we are done

d) Show that if f^* *is a function from* B *into* A, *then* $f^{-1} = f^*$

f^* *is a function from* B *into* A

Therefore, f *is* $1-1$ *and onto by c) above.*

Therefore, f^* *is an inverse of* f, *i.e.,* $f^{-1} = f^*$

4. *Let* $A = \{x \in R : 0 \leq x \leq 1\}$ *and* $B = \{x \in R : 5 \leq x \leq 8\}$. *Show that* $f : A \to B$ *defined by*

$f(x) = 5 + (8-5)x$ *is a* $1-1$ *function from* A *onto* B

Let $x_1, x_2 \in A$ *such that* $f(x_1) = f(x_2) \Rightarrow 5 + (8-5)x_1 = 5 + (8-5)x_2 \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2$

f *is a* $1-1$ *function from* A *onto* B

5. Which of the following functions are one to one?

$a)\ f: R \to R$ *defined by* $f(x)=4,\ x \in R$

Let $x, y \in R$ *such that* $f(x)=f(y) \Rightarrow 4=4$ *that we can say nothing about* $x \,\&\, y$.
Therefore, f *is NOT* $1-1$
In other words every element of the domian maps to one element of the range, 4
which means it is onto but NOT $1-1$

$b)\ f: R \to R$ *defined by* $f(x)=6x-1,\ x \in R$

Let $x, y \in R$ *such that* $f(x)=f(y) \Rightarrow 6x-1=6y-1 \Rightarrow 6x=6y \Rightarrow x=y \quad \therefore\ f$ *is* $1-1$

$c)\ f: R \to R$ *defined by* $f(x)=x^2+7,\ x \in R$

Let $x, y \in R$ *such that* $f(x)=f(y) \Rightarrow x^2+7=y^2+7 \Rightarrow x^2=y^2 \Rightarrow x=\pm y$
which means f *is NOT* $1-1$
As a counterexample $f(3)=f(-3) \Rightarrow (3)^2+7=(-3)^2+7=9+7=16$ *but* $3 \neq -3$

$d)\ f: R \to R$ *defined by* $f(x)=x^3,\ x \in R$

Let $x, y \in R$ *such that* $f(x)=f(y) \Rightarrow x^3=y^3 \Rightarrow x=y \quad \therefore\ f$ *is* $1-1$

$e)\ f: R\setminus\{7\} \to R$ *defined by* $f(x)=\dfrac{2x+1}{x-7},\ x \in R\setminus\{7\}$

Let $x, y \in R\setminus\{7\}$ *such that* $f(x)=f(y) \Rightarrow \dfrac{2x+1}{x-7}=\dfrac{2y+1}{y-7} \Rightarrow (y-7)(2x+1)=(x-7)(2y+1)$

$2xy-14x+y-7=2xy+x-14y-7 \Rightarrow 2xy-2xy-14x-x=-14y-y-7+7 \Rightarrow -15x=-15y \Rightarrow x=y$
$\therefore\ f$ *is* $1-1$

6.Which of the following functions are onto?

$a)\ f: R \to R$ *defined by* $f(x)=115x+49,\ x \in R$
Range$(f)=R$ *, therefore,* $\quad \therefore\ f$ *is onto*

$b)\ f: R \to R$ *defined by* $f(x)=|x|,\ x \in R$
Range$(f)=[0,\infty) \neq R$*, therefore,* f *is NOT onto*

$c)\ f: R \to R$ *defined by* $f(x)=\sqrt{x^2},\ x \in R$
Range$(f)=[0,\infty) \neq R$*, therefore,* f *is NOT onto*

$d)\ f: R \to R$ *defined by* $f(x)=x^2+4,\ x \in R$
Range$(f)=[4,\infty) \neq R$*, therefore,* f *is NOT onto*

7. *Find* $f^{-1}(x)$ *if*

$a)\ f(x)=7x-6$

$$f^{-1}(y)=x=7y-6\Rightarrow x-6=7y\Rightarrow \frac{x-6}{7}=y\Rightarrow f^{-1}(x)=\frac{x-6}{7}$$

$b)\ f(x)=\dfrac{2x-9}{4}$

$$f^{-1}(y)=x=\frac{2y-9}{4}\Rightarrow 4x=2y-9\Rightarrow 4x+9=2y\Rightarrow y=\frac{4x+9}{2}\Rightarrow f^{-1}(x)=\frac{4x+9}{2}$$

$c)\ f(x)=1-\dfrac{1}{x}$

$$f^{-1}(y)=x=1-\frac{1}{y}\Rightarrow x-1=-\frac{1}{y}\Rightarrow -x+1=\frac{1}{y}\Rightarrow \frac{1}{1-x}=y\Rightarrow f^{-1}(x)=\frac{1}{1-x}$$

$d)\ f(x)=\dfrac{4-x}{3x}$

$$f^{-1}(y)=x=\frac{4-y}{3y}=\frac{4}{3y}-\frac{y}{3y}=\frac{4}{3y}-\frac{1}{3}\Rightarrow x+\frac{1}{3}=\frac{4}{3y}\Rightarrow \frac{3x+1}{3}=\frac{4}{3y}\Rightarrow \frac{3}{3x+1}=\frac{3y}{4}$$

$$\Rightarrow\left(\frac{3}{3x+1}\right)\left(\frac{4}{3}\right)=y\Rightarrow \frac{4}{3x+1}=y\Rightarrow f^{-1}(x)=\frac{4}{3x+1}$$

$e)\ f(x)=\dfrac{5x+3}{1-2x}$

$$f^{-1}(y)=x=\frac{5y+3}{1-2y}\Rightarrow x(1-2y)=5y+3\Rightarrow x-2xy=5y+3\Rightarrow -5y-2xy=3-x$$

$$\Rightarrow 5y+2xy=x-3\Rightarrow y(5+2x)=x-3\Rightarrow y=\frac{x-3}{5+2x}\Rightarrow f^{-1}(x)=\frac{x-3}{5+2x}$$

$f)\ f(x)=\sqrt[3]{x+1}$

$$f^{-1}(y)=x=\sqrt[3]{y+1}\Rightarrow x^3=y+1\Rightarrow x^3-1=y\Uparrow f^{-1}(x)=x^3-1$$

$g)\ f(x)=-(x+2)^2-1$

$$f^{-1}(y)=x=-(y+2)^2-1\Rightarrow x+1=-(y+2)^2\Rightarrow -x-1=(y+2)^2$$

$$\Rightarrow \pm\sqrt{-x-1}=y+2\Rightarrow y=-2\pm\sqrt{-x-1}\Rightarrow f^{-1}(x)=-2\pm\sqrt{-x-1}$$

$h)\ f(x)=\dfrac{2x}{1+x}$

$$f^{-1}(y)=x=\frac{2y}{1+y}\Rightarrow x(1+y)=2y\Rightarrow x+xy=2y\Rightarrow x=y(2-x)\Rightarrow \frac{x}{2-x}=y\Rightarrow f^{-1}(x)=\frac{x}{2-x}$$

Solutions/Answers to Exercises 3.4 of page 97 – 98

1. Perform the requested divisions. Find the quotient and the remainder and verify the Remainder Theorem by computing p(a).

$a)\ Divide\ p(x) = x^2 - 5x + 8\ \ by\ x+4$

$$
\begin{array}{r|l}
 & x - 9 \\
\hline
x+4 & x^2 - 5x + 8 \\
 & \underline{x^2 + 4x} \\
 & -9x + 8 \\
 & \underline{-9x - 36} \\
 & 44
\end{array}
$$

Therefore, the quotient is x-9 and the remainder is 44

$p(-4) = (-4)^2 - 5(-4) + 8 = 16 + 20 + 8 = 44\ \ (Verified\ Remainder Theorem)$

$b)\ Divide\ p(x) = 2x^3 - 7x^2 + x + 4\ \ by\ x - 4$

$$
\begin{array}{r|l}
 & 2x^2 + x + 5 \\
\hline
x-4 & 2x^3 - 7x^2 + x + 4 \\
 & \underline{2x^3 - 8x^2} \\
 & x^2 + x + 4 \\
 & \underline{x^2 - 4x} \\
 & 5x + 4 \\
 & \underline{5x - 20} \\
 & 24
\end{array}
$$

$Therefore,\ the\ quotient\ is\ 2x^2 + x + 5\ the\ remainder\ is\ 24$

$p(4) = 2(4)^3 - 7(4)^2 + 4 + 4 = 128 - 112 + 8 = 24\ (Verified\ by\ Remainder Theorem)$

$c)\ Divide\ p(x) = 1 - x^4\ \ by\ x - 1$

$$
\begin{array}{r|l}
 & -x^3 - x^2 - x - 1 \\
\hline
x-1 & 1 - x^4 \\
 & \underline{x^3 - x^4} \\
 & -x^3 + 1 \\
 & \underline{-x^3 + x^2} \\
 & -x^2 + 1 \\
 & \underline{-x^2 + x} \\
 & -x + 1 \\
 & \underline{-x + 1} \\
 & 0
\end{array}
$$

Therefore, the quotient is $-x^3 - x^2 - x - 1$
and no remainder (the remainder is 0)

$p(1) = 1 - 1^4 = 0\ \ \ (Verified\ by\ the\ Remainder Theorem)$

$d)$ *Divide* $p(x)=x^5-2x^2-3$ *by* $x+1$

$$\begin{array}{r|l} & x^4-x^3+x^2-3x+3 \\ \hline x+1 & x^5-2x^2-3 \\ & \underline{x^5+x^4} \\ & -x^4-2x^2-3 \\ & \underline{-x^4-x^3} \\ & x^3-2x^2-3 \\ & \underline{x^3+x^2} \\ & -3x^2-3 \\ & \underline{-3x^2-3x} \\ & 3x-3 \\ & \underline{3x+3} \\ & -6 \end{array}$$

Therefore, the quotient is $x^4-x^3+x^2-3x+3$ *the remainder is* -6

$p(-1)=(-1)^5-2(1)^2-3=-1-2-3=-6$ (*Verified by the* Re*mainder Theorem*)

2. *Given that* $p(4)=0$, *factor* $p(x)=2x^3-11x^2+10x+8$ *as completely as possible*

$p(4)=0 \Rightarrow x-4$ *is factor of* p

$$\begin{array}{r|l} & 2x^2-3x-2 \\ \hline x-4 & 2x^3-11x^2+10x+8 \\ & \underline{2x^3-8x^2} \\ & -3x^2+10x+8 \\ & \underline{-3x^2+12x} \\ & -2x+8 \\ & \underline{-2x+8} \\ & 0 \end{array}$$

Therefore, $2x^3-11x^2+10x+8=(2x^2-3x-2)(x-4)=(x-2)(2x+1)(x-4)$

3. *Given that* $r(x)=4x^3-x^2-36x+9$, $r\left(\frac{1}{4}\right)=0$ *find the remaining zeros of* $r(x)$.

$r\left(\frac{1}{4}\right)=0 \Rightarrow x-\frac{1}{4}$ *or* $4x-1$ *is a factor of* r

$$\begin{array}{r|l} & 4x^2-36 \\ \hline x-\frac{1}{4} & 4x^3-x^2-36x+9 \\ & \underline{4x^3-x^2} \\ & -36x+9 \\ & \underline{-36x+9} \\ & 0 \end{array}$$

Now $r(x)=4x^3-x^2-36x+9=(4x-1)(4x^2-36)=4(4x-1)(x^2-9)=4(4x-1)(x-3)(x+3)$

Therefore, $x=\frac{1}{4}$, $x=3$, $x=-3$ *are the zeros or roots of* r

4. *Given that* 3 *a double zero of* $p(x)=x^4-3x^3-19x^2+87x-90$. *Find all the zeros of* $p(x)$.

3 *a double zero of* $p \Rightarrow (x-3)^2$ *or* x^2-6x+9 *is a factor of* p

$$
\begin{array}{r|l}
 & x^2+3x-10 \\
\hline
x^2-6x+9 & x^4-3x^3-19x^2+87x-90 \\
 & \underline{x^4-6x^3+9x^2} \\
 & 3x^3-28x^2+87x-90 \\
 & \underline{3x^3-18x^2+27x} \\
 & -10x^2+60x-90 \\
 & \underline{-10x^2+60x-90} \\
 & 0
\end{array}
$$

$$\text{Therefore, } p(x)=x^4-3x^3-19x^2+87x-90=(x-3)^2(x^2+3x-10)$$
$$=(x-3)^2(x+5)(x-2)$$

Hence, $x=3$, $x=-5$ *and* $x=2$ *are the zeros f* p

5. a) Write the general polynomial p(x) whose only zeros are 1, 2, and 3, with multiplicity 3, 2 and 1 respectively. What is its degree?

$$p(x) = k(x-1)^3(x-2)^2(x-3)$$

b) Find p(x) described in part (a) if p(0) = 6

$$p(x) = k(x-1)^3(x-2)^2(x-3)$$

$$\text{and } p(o) = 6 \Rightarrow k(0-1)^3(0-2)^2(0-3) = 6$$
$$\Rightarrow k(-1)^3(-2)^2(-3) = 6$$
$$\Rightarrow k(-1)(4)(\cancel{-}3) = 6$$
$$\Rightarrow 12k = 6 \Rightarrow k = \frac{1}{2}$$

$$\text{Therefore, } p(x) = \frac{1}{2}(x-1)^3(x-2)^2(x-3)$$

And the degree is 6

6. If $2-3i$ is a root of $p(x) = 2x^3 - 5x^2 + 14x + 39$, the remainig zeros of $p(x)$

$$\begin{array}{r|l} & 2x^2-(1+6i)x-(6+9i) \\ \hline x-(2-3i) & 2x^3-5x^2+14x+39 \\ & 2x^3+(-4+6i)x^2 \\ \hline & (-1-6i)x^2++14x+ \quad 39 \\ & (-1-6i)x^2+(20+9i)x \\ \hline & (-6-9i)x+39 \\ & (-6-9i)x+39 \\ \hline & 0 \end{array}$$

To factorize $2x^2-(1+6i)x-(6+9i)$ *we use the quadiratic formula*:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x = \frac{(1+6i) \pm \sqrt{(1+6i)^2-(4)(2)(-(6+9i))}}{(2)(2)} = \frac{(1+6i) \pm \sqrt{(12i-35)+(48+72i)}}{4}$$

$$\Rightarrow x = \frac{(1+6i) \pm \sqrt{(12i+72i)+(48-35)}}{4} = \frac{(1+6i) \pm \sqrt{(84i)+13}}{4}$$

$$\Rightarrow x = \frac{(1+6i)+\sqrt{84i+13}}{4} \text{ or } x = \frac{(1+6i)-\sqrt{84i+13}}{4} \text{ are the other roots of } p(x)$$

7. Determine the rational zeros of the polynomials:

a) $p(x) = x^3 - 4x^2 - 7x + 10$

$x = 1$: $1^3 - 4(1^2) - 7(1) + 10 = 1 - 4 - 7 + 10 = 0 \Rightarrow x-1$ *is one factor of* p

$$\begin{array}{r|l} & x^2-3x-10 \\ \hline x-1 & x^3-4x^2-7x+10 \\ & x^3-x^2 \\ \hline & -3x^2-7x+10 \\ & -3x^2+3x \\ \hline & -10x+10 \\ & -10x+10 \\ \hline & 0 \end{array}$$

Therefore, $p(x) = x^3 - 4x^2 - 7x + 10 = (x-1)(x^2-3x-10)$

Again, to factorize $x^2 - 3x - 10$, *we use quadratic formula*:

$$x = \frac{3 \pm \sqrt{(-3)^2-(4)(1)(-10)}}{2} = \frac{3 \pm \sqrt{9+40}}{2} = \frac{3 \pm 7}{2} \Rightarrow x = 5, x = -2$$

$p(x) = x^3 - 4x^2 - 7x + 10 = (x-1)(x-5)(x+2)$

So that the rational zeros are $x = 1, x = 5$ *and* $x = -2$

$b)\ \ p(x) = 2x^3 - 5x^2 - 28x + 15$

$x = -3: \quad 2\left((-3)^3\right) - 5(-3)^2 - 28(-3) + 15 = -54 - 45 + 84 + 15 = -99 + 99 = 0$

which means $x = -3$ *is one root of* p.

$$
\begin{array}{r@{}l}
 & 2x^2 - 11x + 5 \\
x+3\,\big) & \overline{2x^3 - 5x^2 - 28x + 15} \\
 & \underline{2x^3 + 6x^2} \\
 & \quad -11x^2 - 28x + 15 \\
 & \quad \underline{-11x^2 - 33x} \\
 & \qquad\qquad 5x + 15 \\
 & \qquad\qquad \underline{5x + 15} \\
 & \qquad\qquad\quad 0
\end{array}
$$

Hence, $2x^3 - 5x^2 - 28x + 15 = (x+3)\left(2x^2 - 11x + 5\right)$ *and therefore we need to factorize* $2x^2 - 11x + 5$ *again using quadratic formula:*

$$x = \frac{11 \pm \sqrt{(11)^2 - (4)(2)(5)}}{4} = \frac{11 \pm \sqrt{121 - 40}}{4} = \frac{11 \pm \sqrt{81}}{4} = \frac{11 \pm 9}{4} \Rightarrow x = 5, or\ x = \frac{1}{2}$$

Therefore, $2x^3 - 5x^2 - 28x + 15 = (x+3)(x-5)(2x-1)$ *and then* $x = -3, x = 5, x = \frac{1}{2}$ *are the rationd zeros*

$c)\ \ p(x) = 6x^3 + x^2 - 4x + 1$

$x = -1: 6(-1)^3 + (-1)^2 - 4(-1) + 1 = -6 + 1 + 4 + 1 = 0$

which means $x = -1$ *is one root of* p

$$
\begin{array}{r@{}l}
 & 6x^2 - 5x + 1 \\
x+1\,\big) & \overline{6x^3 + x^2 - 4x + 1} \\
 & \underline{6x^3 + 6x^2} \\
 & \quad -5x^2 - 4x + 1 \\
 & \quad \underline{-5x^2 - 5x} \\
 & \qquad\qquad x + 1 \\
 & \qquad\qquad \underline{x + 1} \\
 & \qquad\qquad\ \ 0
\end{array}
$$

$Hence, 6x^3 + x^2 - 4x + 1 = (x+1)(6x^2 - 5x + 1)$ *and therefore we need to factorize* $6x^2 - 5x + 1$ *again using quadratic formula*:

$$x = \frac{5 \pm \sqrt{(-5)^2 - (4)(6)(1)}}{12} = \frac{5 \pm \sqrt{25-24}}{12} = \frac{5 \pm \sqrt{1}}{12} = \frac{5 \pm 1}{12} \Rightarrow x = \frac{6}{12} = \frac{1}{2}, or\ x = \frac{4}{12} = \frac{1}{3}$$

Therefore, $6x^3 + x^2 - 4x + 1 = (x+1)(2x-1)(3x-1)$ *and then* $x = 1, x = \frac{1}{2}, x = \frac{1}{3}$ *are therationd zeros*

8. Find the domain and the real zeros of the given function.

a) $f(x) = \dfrac{3}{x^2 - 25}$

$Dom(f) = \{x : x^2 - 25 \neq 0\} = \{all\ real\ nimbers\} - \{-5,5\} = R - \{-5,5\}$

$Dom(f): x^2 - 25 \neq 0 \Rightarrow x^2 \neq 25 \Rightarrow x \neq \pm\sqrt{25} \Rightarrow x \neq \pm 5$

Real zeros: $f(x) = 0 \Rightarrow \dfrac{3}{x^2 - 25} = 0 \Rightarrow 0(x^2 - 25) = 3$ *which is false sothat there is no real zero.*

b) b) $g(x) = \dfrac{x-3}{x^2 + 4x - 12}$ $[Correcting\ x^2 4x - 12\ as\ x^2 + 4x - 12]$

$Dom(f) = \{x : x^2 + 4x - 12 \neq 0\}$

$$Dom(f): x^2 + 4x - 12 \neq 0 \Rightarrow x \neq \frac{-4 \pm \sqrt{16 - 4(-12)}}{2} \Rightarrow x \neq \frac{-4 \pm \sqrt{16+48}}{2} \Rightarrow x \neq \frac{-4 \pm \sqrt{64}}{2}$$

$$\Rightarrow x \neq \frac{-4 \pm 8}{2} \Rightarrow x \neq 2\ \ or\ x \neq -6$$

$Dom(f) = \{x : x \neq 2,\ x \neq -6\} = R - \{2, -6\}$

Real zero: $x - 3 = 0 \Rightarrow x = 3$ *which means* 3 *is the real zero.*

or $g(x) = \dfrac{x-3}{x^2 - 4x - 12}$ $[Correcting\ x^2 4x - 12\ as\ x^2 - 4x - 12]$

$$Dom(f): x^2 - 4x - 12 \neq 0 \Rightarrow x \neq \frac{4 \pm \sqrt{16 - 4(-12)}}{2} \Rightarrow x \neq \frac{4 \pm \sqrt{16+48}}{2} \Rightarrow x \neq \frac{4 \pm \sqrt{64}}{2}$$

$$\Rightarrow x \neq \frac{4 \pm 8}{2} \Rightarrow x \neq -2\ \ or\ x \neq 6$$

$Dom(f) = \{x : x \neq -2,\ x \neq 6\} = R - \{-2, 6\}$

Real zero: $x - 3 = 0 \Rightarrow x = 3$ *which means* 3 *is the real zero.*

c) $f(x) = \dfrac{(x-3)^2}{x^3 - 3x^2 + 2x}$

$Dom(f) = \{x : x^3 - 3x^2 + 2x \neq 0\}$

$$Dom(f): x^3-3x^2+2x\neq 0 \Rightarrow x(x^2-3x+2)\neq 0 \Rightarrow x\neq 0 \ or \ x\neq \frac{3\pm\sqrt{9-4(2)}}{2}$$

$$\Rightarrow x\neq 0 \ or \ x\neq \frac{4\pm\sqrt{1}}{2} \Rightarrow x\neq 0 \ or \ x\neq \frac{4\pm 1}{2} \Rightarrow x\neq 0 \ or \ x\neq \frac{5}{2} \ or \ x\neq \frac{3}{2}$$

$$\therefore \ Dom(f)=R-\left\{0, \frac{5}{2}, \frac{3}{2,}\right\}$$

Real Zero : $(x-3)^2=0 \Rightarrow x-3=0 \Rightarrow x=3 \ which \ means \ 3 \ is \ the \ real \ zero$

d) $f(x)=\dfrac{x^2-16}{x^2+4}$

$Dom(f): x^2+4\neq 0 \Rightarrow x^2\neq -4 \Rightarrow x\neq \pm\sqrt{-4} \quad x\neq \pm 2i \ Not \ real \ number$

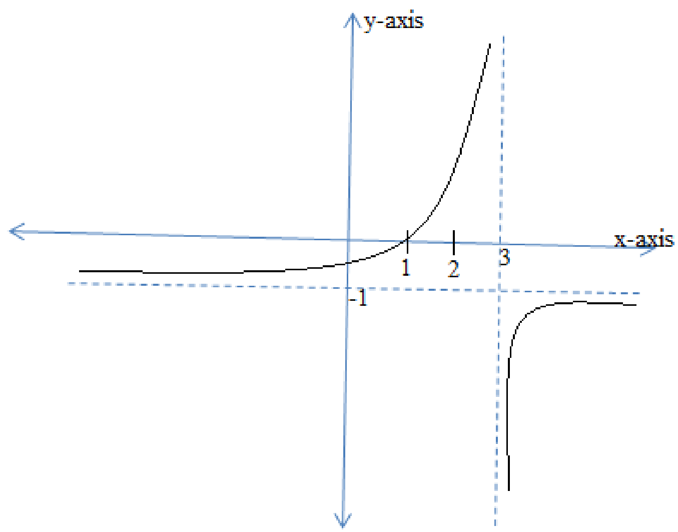
$Dom(f)=R$

$\mathrm{Re}\,al \ Zero: x^2-16=0 \Rightarrow x=\pm\sqrt{16} \Rightarrow x=\pm 4$

$Therefore \ 4 \ and \ -4 \ are \ the \ real \ zeros$

9. Sketch the graph of

a) $f(x)=\dfrac{1-x}{x-3}$



$x-\text{int}\,ercept \ (y=0):$

$\Rightarrow f(x)=\dfrac{1-x}{x-3}=0 \Rightarrow 1-x=0 \Rightarrow x=1$

$\Rightarrow (1,0) is \ x-\text{int}\,ercept$

$y-\text{int}\,ercept \ (x=0):$

$f(0)=\dfrac{1-x}{x-3} \Rightarrow y=\dfrac{1-0}{0-3} \Rightarrow y=-\dfrac{1}{3}$

$\Rightarrow (0,1/3) is \ the \ y-\text{int}\,ercept$

$Since \ the \ \deg reeof \ the numerator$

$is \ equal to the \ \deg reeof \ the denomin\,ator,$

$then \ y=-1 \ is \ the \ horizontal \ asymptote$

$and \ the denomin\,ator is \ zero when$

$x=3 \ and$

$as \ x\to 3^-, f(x)\to\infty$

$and \ As \ x\to 3^+, f(x)\to -\infty$

$x=3 \ is \ the \ vertical asymptote$

b) $f(x)=\dfrac{x^2+1}{x}$

$x-$intercept $(y=0)$:

$\Rightarrow f(x)=\dfrac{x^2+1}{x}=0$

$\Rightarrow x^2+1=0 \Rightarrow x^2=-1,$ *No real number*

so that no $x-$intercept *is* $(-1,0)$

$y-$intercept $(x=0)$:

$f(0)=\dfrac{0^2+1}{0}$ *which doesn't exist*

so that no $y-$intercept

As the degree of the numerator is one more than the degree of the denominator then

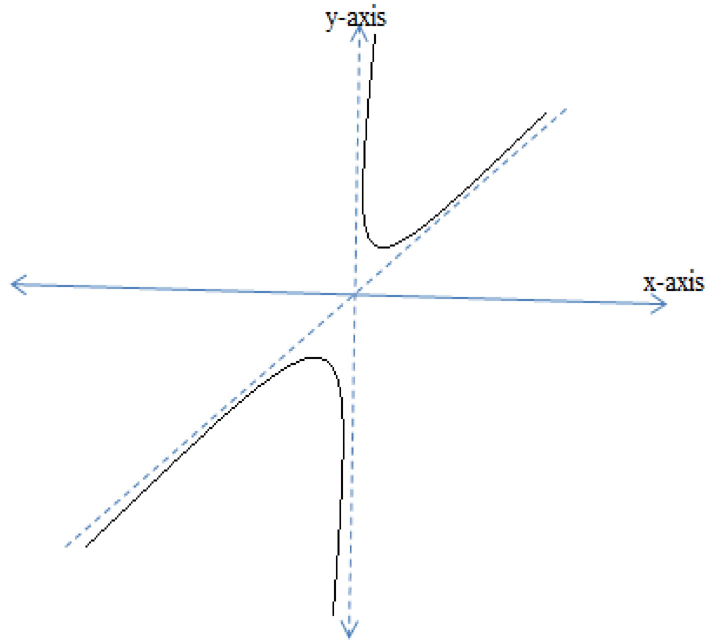
$y=x$ *is the Oblique asymptote*

and the denominator is zero when $x=0$ *and*

as $x\to 0^-,\ f(x)\to -\infty$

and As $x\to 0^+,\ f(x)\to \infty$

hence $x=0$ *is the vertical asymptote*



c) $f(x)=\dfrac{1}{x}+2$

$x-$intercept $(y=0)$:

$\Rightarrow f(x)=\dfrac{1}{x}+2=\dfrac{1+2x}{x}$

$\Rightarrow f(x)=\dfrac{1}{x}+2=0 \Rightarrow \dfrac{1}{x}=-2 \Rightarrow x=-\dfrac{1}{2},$

$\Rightarrow (-1/2,0)$ *is* $x-$intercept

$y-$intercept $(x=0)$:

$f(0)=\dfrac{1}{0}+2=0$ *which doesn't exist*

meaning no $y-$intercept

As the degree of the numerator is equal to that of the denominator,

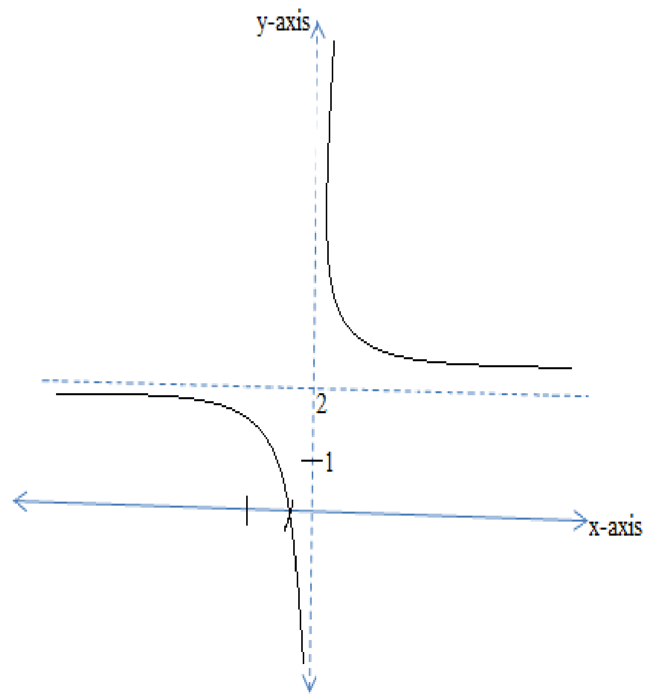
$y=2$ *is the Horizontal asymptote*

and and the denominator is zero when $x=0$

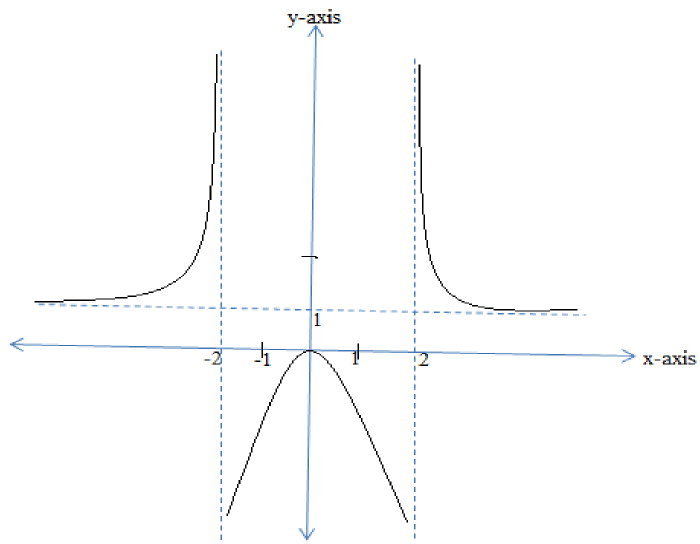
and as $x\to 0^-,\ f(x)\to -\infty$

and As $x\to 0^+,\ f(x)\to \infty$

$x=0$ *is the vertical asymptote*



d) $f(x)=\dfrac{x^2}{x^2-4}$



$x-\text{intercept}\ (y=0)$:

$\Rightarrow f(x)=\dfrac{x^2}{x^2-4}=0$

$\Rightarrow f(x)=0(x^2-4)=x^2$

$\Rightarrow x=0 \Rightarrow (0,0)$ *is the* $-$ *intercept*

similarly, $y-\text{intercept}\ (x=0)$:

$f(0)=\dfrac{0^2}{0^2-4}=0 \Rightarrow y=0,$

As the degrees of the numarator and

the denominator are equal,

$y=1$ *is the Horizontal asymptote, and*

the denominator is zero when and $x^2-4 \Rightarrow x=\pm 2,$

$\Rightarrow x=2\ , x=-2$ *are the vertical asymptotes and*

s $x\to 2^-$, $f(x)\to -\infty$, *and As* $x\to 2^+$, $f(x)\to \infty$;

As $x\to -2^-$, $f(x)\to \infty$ *and As* $x\to -2^+$, $f(x)\to -\infty$

10. Determine the behavior of $f(x)=\dfrac{x^3-8x-3}{x-3}$ when x is near 3

$x^3-8x-3=(x-3)(x^2+3x+1),$

hence $f(x)=\dfrac{x^3-8x-3}{x-3}=\dfrac{(x-3)(x^2+3x+1)}{x-3}$

$\Rightarrow f(x)=x^2+3x+1, x\neq 3$

As $x\to 3^+$, $f(x)\to 19$ *and*

as $x\to 3^-$, $f(x)\to 19$

The graph of f *has a whole at* $(3,19)$

11. The graph of any rational function in which the degree of the numerator is exactly one more than the degree of the denominator will have an oblique (or slant) asymptote.

a) Use long division to show that $f(x)=\dfrac{x^2-x+6}{x-2}=x+1+\dfrac{8}{x-2}$ since

$$\begin{array}{r|l} & x+1 \\ \hline x-2 & x^2-x+6 \\ & \underline{x^2-2x} \\ & \quad x+6 \\ & \quad \underline{x-2} \\ & \qquad 8 \end{array}$$

b) Show that this means that the line y = x+1 is a slant asymptote for the graph and sketch the graph of $y = f(x)$

$As\ x \to \infty \Rightarrow f(x) = x+1+\frac{8}{x-2} \to x+1\ and$

$as\ x \to .-\infty \Rightarrow f(x) = x+1+\frac{8}{x-2} \to x+1\ so\ that\ y = x+1\ is\ slant\ asymptote$

Again, the degree of the numeratoreis one greater than the degree of the denominatore.

$As\ x \to 2^{+} \Rightarrow f(x) \to \infty\ and\ as\ x \to 2^{-} \Rightarrow f(x) \to -\infty$

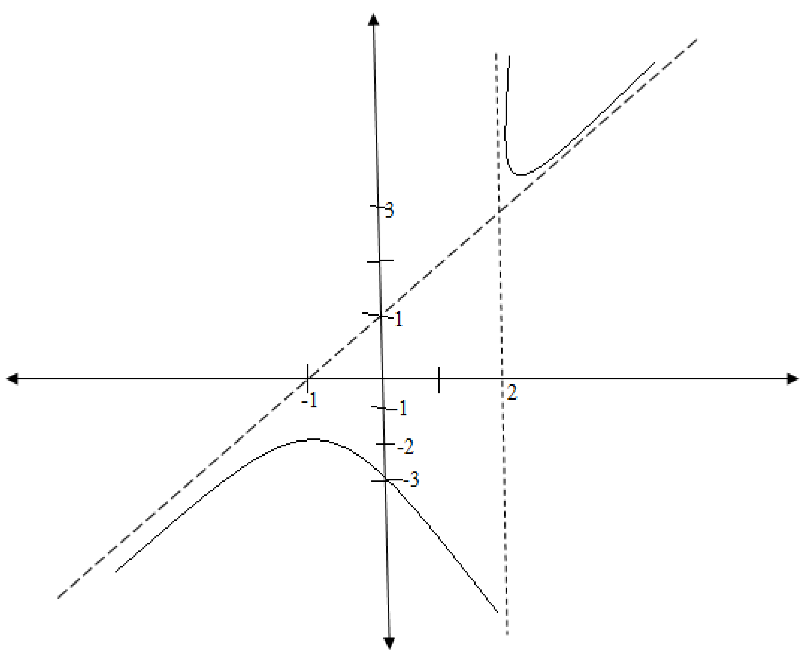
$Hence, x = 2\ is\ vertical\ asymptote$

$x-\text{int}ercept(y=0): f(x) = 0 \Rightarrow x+1+\frac{8}{x-2} = 0 \Rightarrow \frac{x^2-x-2+8}{x-2} = 0 \Rightarrow x^2 - x + 6 = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{1-4(6)}}{2} = \frac{1 \pm \sqrt{-23}}{2} \Rightarrow Not\ real\ number\ sothat\ No\ x-\text{int}ercept$

$y-\text{int}ercept(x=0): f(0) = x+1+\frac{8}{x-2} = 0+1+\frac{8}{0-2} = 1-4 = -3$

$sotaht\ (0,-3) is\ the\ y-\text{int}ercept$



<u>Solutions/Answers to Exercises 3.5 of page 122-123</u>

1. Find the domain of the given function

   a) $f(x)=\dfrac{1}{6^x}$

      $Dom(f)=R$

   b) $g(x)=\sqrt{3^x+1}$

      $Dom(f)=R$

   c) $h(x)=\sqrt{2^x-8}$

      $Dom(f)=[3,\infty)$

   d) $f(x)=\dfrac{1}{2^{3x}-2}$

      $Dom(f)=R-\{1/3\}$

2. Sketch the graph of the given function. Identify the domain, the range, intercepts and asymptotes.

   a) $y=5^{-x}$

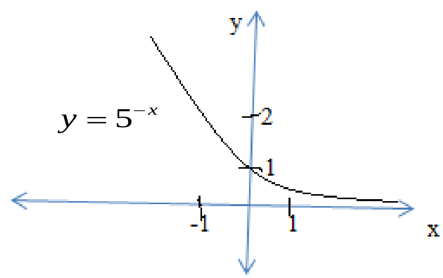
      $y=5^{-x}$

      $Domf(f)=R,$

      $Range(f)=(0,\infty)$

      $y-\text{int}\,ercept=(0,-1)$

      $y=0\,is\,horizontal asymptote$



   b) $y=9-3^x$

$y=9-3^x$

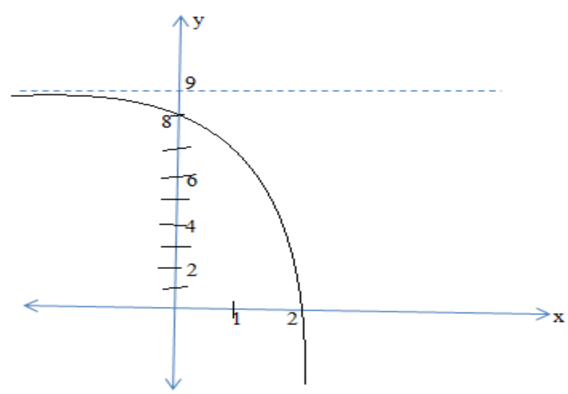
$Domf(f)=R,$

$Range(f)=(-\infty,9)$

$y-\text{int}\,ercept=(0,8)$

$x-\text{int}\,ercept=(2,0)$

$y=9\,is\,horizontal asymptote$



   c) $y=1-e^{-x}$

$y=1-e^{-x}$

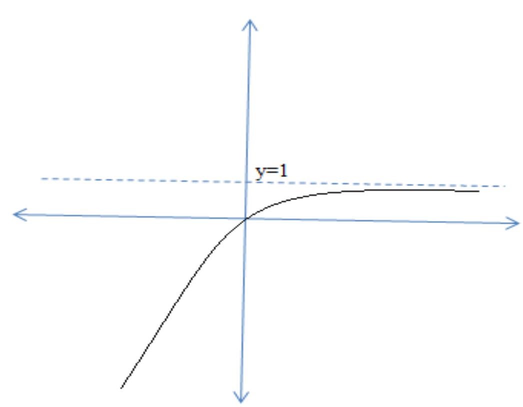
$Domf(f)=R,$

$Range(f)=(-\infty,1)$

$y-\text{int}\,ercept=(0,0)$

$x-\text{int}\,ercept=(0,0)$

$y=1\,is\,horizontal asymptote$



d) $y=e^{x-2}$

$y=e^{x-2}$

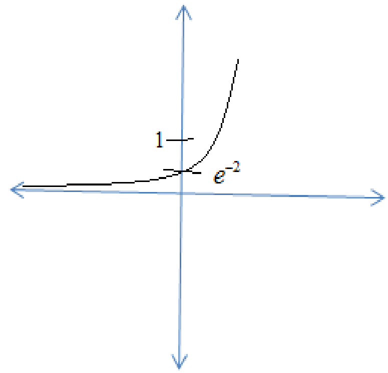
$Domf(f)=R,$

$Range(f)=(-\infty,e^{-2})$

$y-\text{int}\,ercept=(0,e^{-2})$

$Not\ \ x-\text{int}\,ercept$

$y=0\,is\,horizontal\,asymptote$



3. Solve the given exponential ec

a) $2^{x-1}=8$

$2^{x-1}=8\Leftrightarrow 2^{x-1}=2^3\Rightarrow x-1=3\Leftrightarrow x=4$

b) $3^{2x}=243$

$3^{2x}=243\Leftrightarrow 3^{2x}=3^5\Rightarrow 2x=5\Leftrightarrow x=5/2$

c) $8^x=\sqrt{2}$

$8^x=\sqrt{2}\Leftrightarrow 2^{3x}=2^{1/2}\Rightarrow 3x=1/2\Leftrightarrow x=1/6$

d) $16^{3a-2}=\dfrac{1}{4}$

$16^{3a-2}=\dfrac{1}{4}\Leftrightarrow 2^{4(3a-2)}=2^{-2}\Rightarrow 4(3a-2)=-2\Leftrightarrow 12a-8=-2\Leftrightarrow 12a=6\Leftrightarrow a=1/2$

4. $Let\ \ f(x)=2^x\,.Show\,that\ \ f(x+3)=8\,f(x)$

$f(x+3)=2^{x+3}=(2^3)(2^x)=8(2^x)=8\,f(x)$

5. $Let\ \ g(x)=5^x\,.Show\,that\ \ g(x-2)=\dfrac{1}{25}g(x)$

$g(x-2)=5^{x-2}=\dfrac{5^x}{5^2}=\dfrac{5^x}{25}=\dfrac{1}{25}(5^x)=\dfrac{1}{25}g(x)$

6. $Let\ \ f(x)=3^x\,.Show\,that\ \dfrac{f(x+2)-f(2)}{2}=4(3^x)$

$\dfrac{f(x+2)-f(2)}{2}=\dfrac{3^{x+2}-3^x}{2}=\dfrac{3^x(3^2)-3^x}{2}=\dfrac{3^x(9-1)}{2}=\dfrac{8(3^x)}{2}=4(3^x)$

7. Evaluate the given logarithmic expression (where it is defined).

a) $\log 2^{32}$

$\log 2^{32}=\log 2^{2^4}=4\log 2^2=4$

b) $\log\dfrac{1}{3}^{9}$

$\log\dfrac{1}{3}^{9}=\dfrac{\log 3^9}{\log 3^{\frac{1}{3}}}=\dfrac{\log 3^{3^2}}{\log 3^{3^{-1}}}=\dfrac{2og3^3}{-\log 3^3}=-2$

c) $\log 3^{(-9)}$

$\log 3^{(-9)}$ *doesn't exist since the domian is the set of positive real numbers*

d) $\log 6^{\frac{1}{\sqrt{6}}}$

$\log 6^{\frac{1}{\sqrt{6}}} = \log 6^{6^{-1/2}} = -\frac{1}{2}\log 6^{6} = -\frac{1}{2}$

e) $\log 5^{\left(\log 3^{243}\right)}$

$\log 5^{\left(\log 3^{243}\right)} = \log 5^{\left(\log 3^{3^5}\right)} = \log 5^{5\left(\log 3^{3}\right)} = \log 5^{5} = 1$

f) $2^{\log 2^{\sqrt{5}}}$

$2^{\log 2^{\sqrt{5}}} = \sqrt{5}$

8. *If* $f(x) = \log 2^{(x^2-4)}$, *find* $f(6)$ *and the domain of* f.

If $f(6) = \log 2^{(6^2-4)} = \log 2^{(36-4)} \log 2^{(32)} = \log 2^{2^5} = 5$

$Dom(f): x^2 - 4 > 0 \Rightarrow x^2 > 4 \Rightarrow x > 2$ *or* $x < -2$ *so that the domain is* $(-\infty, -2)\bigcup(2,\infty)$

9. *If* $g(x) = \log 3^{(x^2-4x+3)}$, *find* $g(4)$ *and the domian of* g.

$g(4) = \log 3^{(4^2-4(4)+3)} = \log 3^{(16-16+3)} = \log 3^{3} = 3$

$Dom(f): x^2 - 4x + 3 > 0 \Rightarrow (x-3)(x-1) > 0$

		1	3
x − 3	- -	- - -	0 + +
x-1	- - -	0+ + +	+ +
f	+ +	0 - - -	0 + +

From the Sign chart to the right,

$Dom(f) = (-\infty, 1)\bigcup(3,\infty)$

10. *Show that* $\log \frac{1}{6}^{x} = -\log 6^{x}$

$$\log \frac{1}{6}^{x} = \frac{\log 6^{x}}{\log 6^{1/6}} = \frac{\log 6^{x}}{\log 6^{6^{-1}}} = \frac{\log 6^{x}}{-\log 6^{6}} = \frac{\log 6^{x}}{-1} = -\log 6^{x}$$

11. Sketch the graph of the given function and identify the domain, the range, intercepts and asymptotes.

a) $f(x) = \log 2^{(x-3)}$

$f(x) = \log 2^{(x-3)}$

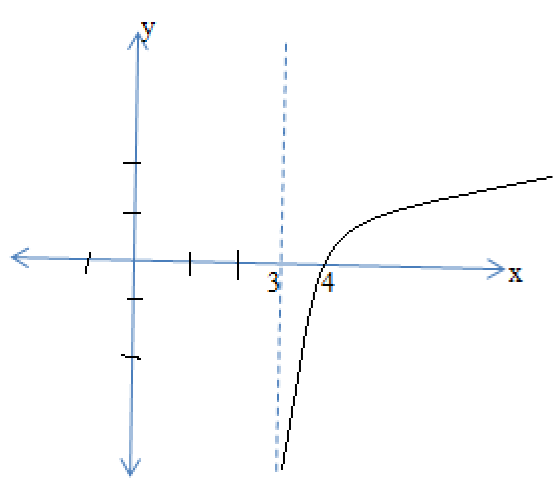
$x = 3$, *is vertical asymptote*

$\log 2^{(x-3)} = 0 \Rightarrow 2^{0} = x - 3$

$\Rightarrow 1 = x - 3 \Rightarrow x = 4$

$(4,0)$ *is* x – *intercept*

$Dom(f) = (3,\infty)$, $Range(f) = R$



b) $f(x) = -3 + \log 2^x$

$f(x) = -3 + \log 2^x$

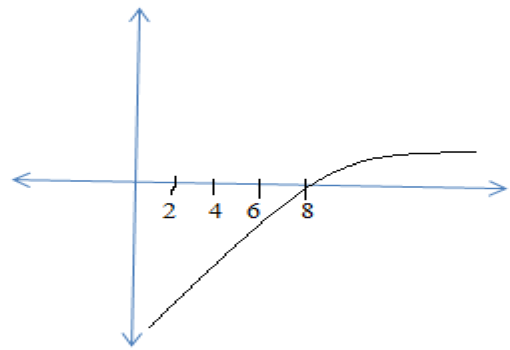
$x = 0$, *is vertical asymptote*

$-3 + \log 2^x = 0 \Rightarrow \log 2^x = 3$

$\Rightarrow 2^3 = x \Rightarrow 8 = x$

$(8, 0)$ *is* x*-intercept*

$Dom(f) = (0, \infty), Range(f) = R$



c) $f(x) = -\log 3^{(-x)}$

$f(x) = -\log 3^{(-x)}$

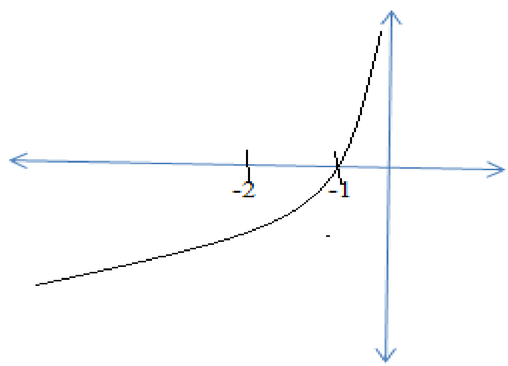
$x = 0$, *is vertical asymptote*

$-\log 2^x = 0 \Rightarrow \log 2^x = 0$

$\Rightarrow 2^0 = x \Rightarrow 1 = x$

$(1, 0)$ *is* x*-intercept*

$Dom(f) = (-\infty, 0), Range(f) = R$



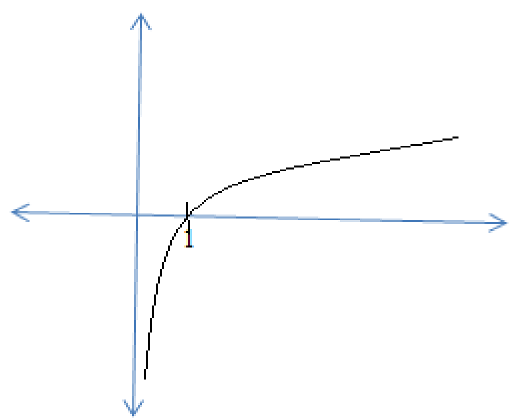
d) $f(x) = 3\log 5^x$

$f(x) = 3\log 5^x \Rightarrow x = 0$, *is vertical asymptote*

$3\log 5^x = 0 \Rightarrow \log 2^x = 0 \Rightarrow 2^0 = x \Rightarrow 1 = x$

$(1, 0)$ *is* x*-intercept*

$Dom(f) = (0, \infty), Range(f) = R$



12. *Find the inverse of* $f(x) = e^{(3x-1)}$

$f(x) = e^{(3x-1)} \Leftrightarrow y = e^{(3x-1)}$ *so that we interchange* x & y *and solve for* y *to find the inverse*:

$$x == e^{(3y-1)} \Leftrightarrow \log e^x = 3y - 1 \Leftrightarrow \ln x = 3y - 1 \Leftrightarrow 1 + \ln x = 3y \Leftrightarrow \frac{1 + \ln x}{3} = y$$

is the inverse

13. *Let* $f(x) = e^{\sqrt{x}}$. *Find a function so that* $(fog)(x) = (gof)(x) = x$

$(fog)(x) = (gof)(x) = x$ *means one is the inverse function of the other*

Hence to find the inverse of $y = e^{\sqrt{x}}$: $x = e^{\sqrt{y}} \Leftrightarrow \log e^x = \sqrt{y} = y^{1/2} \Rightarrow (\ln x)^2 = y = g(x)$

is the inverse of f

Check! $(fog)(x) = f(g(x)) = f\left((\log e^x)^2\right) = e^{\sqrt{(\ln x)^2}} = e^{\ln x} = x$

$$= (gof)(x) = g(f(x)) = g\left(e^{\sqrt{x}}\right) = \left(\ln\left(e^{\sqrt{x}}\right)\right)^2 = \left(\sqrt{x}\right)^2 = x$$

14. Convert the given angle from radians to degrees

a) $\frac{\pi}{3}$ $\quad \Rightarrow \frac{\pi}{3} = \frac{180^0}{3} = 60^0$ or $\frac{\theta}{180^0} = \frac{\frac{\pi}{3}}{\pi} \Rightarrow \theta = \frac{180^0}{3} = 60^0$

b) $\frac{-5\pi}{2}$ $\quad \Rightarrow \frac{-5\pi}{2} = \frac{-5(180^0)}{2} = -5(90^0) = -450^0$ or $\frac{\theta}{180^0} = \frac{\frac{-5\pi}{2}}{\pi} \Rightarrow \theta = \frac{-5\pi}{2}(180^0) = -450^0$

c) $\frac{-4\pi}{3}$ $\quad \Rightarrow \frac{-4\pi}{3} = \frac{-4(180^0)}{3} = -4(60^0) = -240^0$ or $\frac{\theta}{180^0} = \frac{\frac{-4\pi}{3}}{\pi} \Rightarrow \theta = \frac{-4}{3}(180^0) = -240^0$

15. Convert the given angle from degrees to radians

a) 315^0 $\quad \Rightarrow \frac{\theta}{\pi} = \frac{315^0}{180^o} = \frac{35}{20} = \frac{7}{4} \Rightarrow \theta = \frac{7}{4}\pi$

b) -40^0 $\quad \Rightarrow \frac{\theta}{\pi} = \frac{-40^0}{180^o} = \frac{-4}{18} = \frac{-2}{9} \Rightarrow \theta = \frac{-2}{9}\pi$

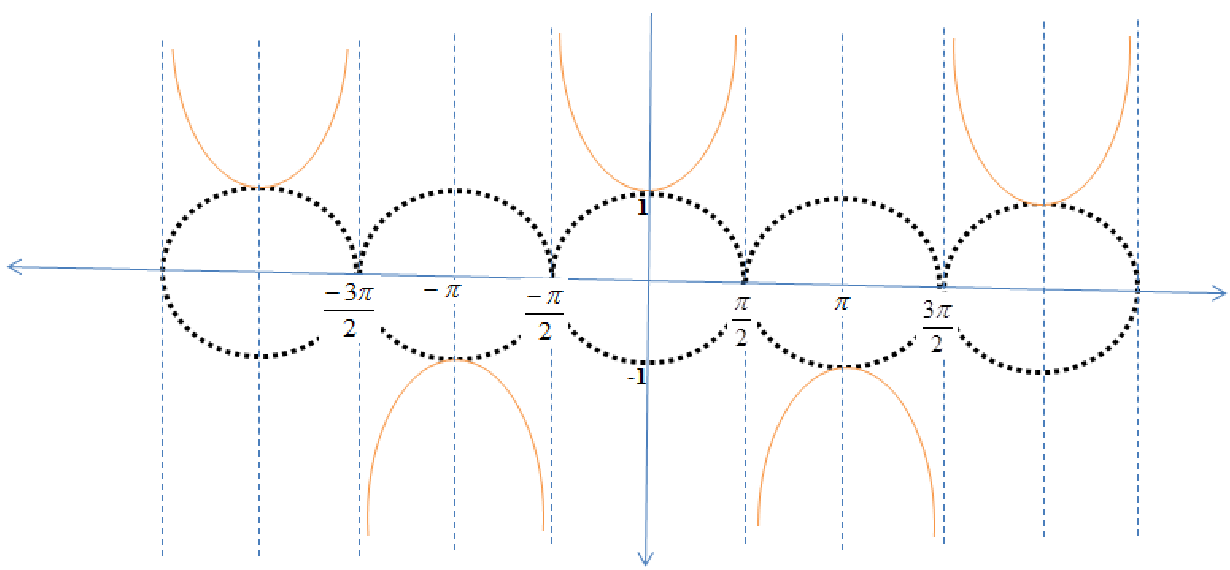
c) 330^0 $\quad \Rightarrow \frac{\theta}{\pi} = \frac{330^0}{180^o} = \frac{33}{18} = \frac{11}{6} \Rightarrow \theta = \frac{11}{6}\pi$

16. Sketch the graph of:

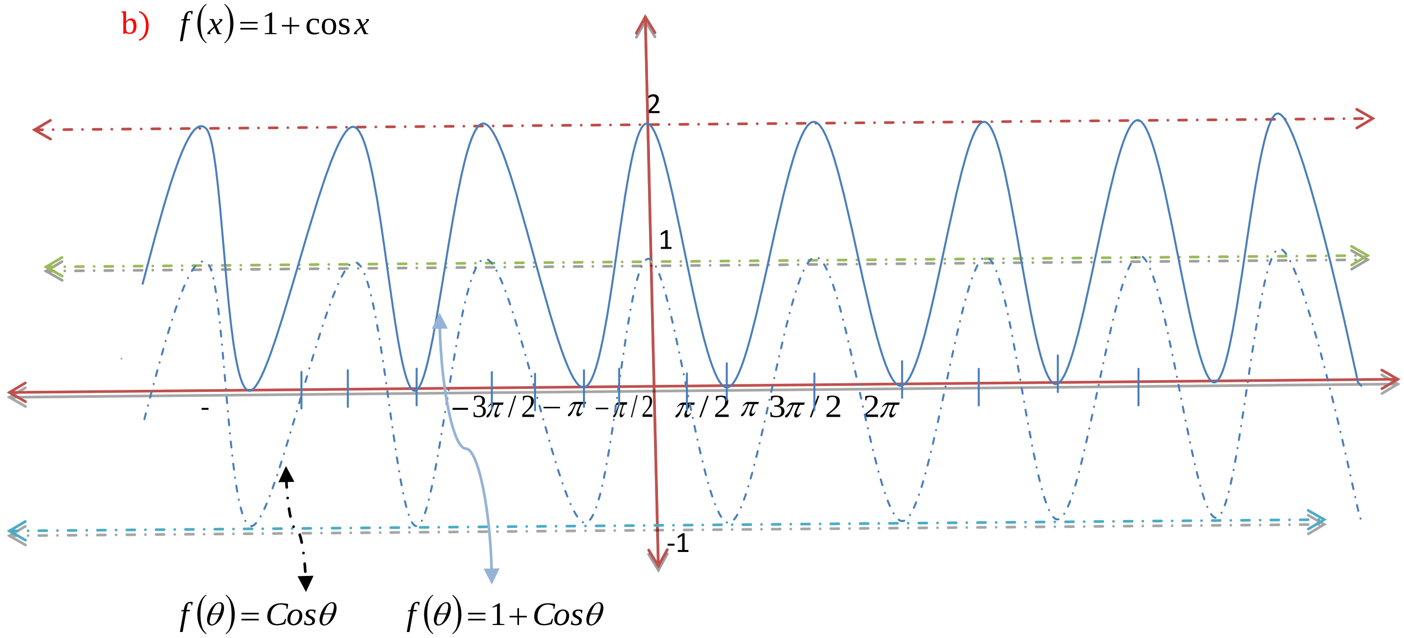
a) $f(\theta) = \sec\theta$

$f(\theta) = \sec\theta = \frac{1}{\cos\theta}$, *all the values of* θ *such that* $\cos\theta = 0$ *are the vertical asymptotes so that*

$\theta = \left(\frac{\pi}{2} \pm n\pi\right)$ *are vertical adymptotes* $(0,1)$ *is the* $y-$*intercept, No* $x-$*intercept*



b) $f(x)=1+\cos x$



d) $f(\theta)=\csc\theta$

e) $f(x)=\sin\left(x+\dfrac{\pi}{2}\right)$

f) $f(\theta)=\cot\theta$

g) $f(x)=\tan 2x$

17. Verify the following identities:

a) $(\sin x-\cos x)(\csc x+\sec x)=\tan x-\cot x$

$$(\sin x-\cos x)(\csc x+\sec x)=(\sin x-\cos x)\left(\frac{1}{\sin x}+\frac{1}{\cos}\right)=(\sin x-\cos x)\left(\frac{\sin x+\cos x}{\sin x\cos x}\right)$$

$$=\left(\frac{(\sin x-\cos x)((\sin x+\cos x))}{\sin x\cos x}\right)=\frac{\sin^2 x-\cos^2 x}{\sin x\cos x}=\frac{\sin^2 x}{\sin x\cos x}-\frac{\cos^2 x}{\sin x\cos x}$$

$$=\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}=\tan x-\cot x$$

b) $\sec^2 x-\csc^2 x=\tan^2 x-\cot^2 x$

From the trigonometric identities, we have:

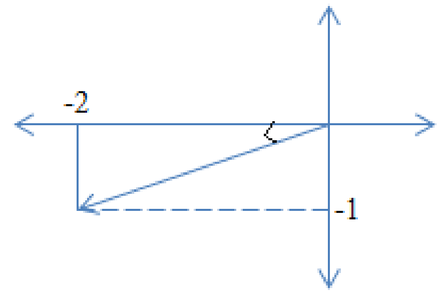
$\sec^2 x=\tan^2 x+1$ *and* $\csc^2 x=\cot^2 x+1$

from these equations we have:

$$-\begin{cases}\sec^2 x=\tan^2 x+1\\ \csc^2 x=\cot^2 x+1\end{cases}$$

$$\overline{\sec^2 x-\csc^2 x=\tan^2 x-\cot^2 x}$$

18. *Given* $\tan\theta = \frac{1}{2}$ *and* $\sin\theta < 0$, *find* $\cos\theta$



It is in the third quadrant that $\tan\theta$ *is positive*

and $\sin\theta$ *is negative*

From the diagram $r = \sqrt{(-1)^1 + (-2)^2} = \sqrt{5}$

Hence $\cos\theta = \frac{-2}{\sqrt{5}}$

19. Prove the identities (2) and (3)

From the text on page 118, we have the following identities:

$(1).\ \sin^2 x + \cos^2 x = 1$

$(2).\ \tan^2 x + 1 = \sec^2 x$

$(3).\ \cot^2 x + 1 = \csc^2 x$

To prove (2) *and* (3), *we use* (1) *and hence from the first*:

$\sin^2 x + \cos^2 x = 1$ *dividing both sides of this equationby* $\cos^2 x$, *we get*:

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x\,[(2)\,is\ provede]$$

Again, dividing both sides of this equationby $\sin^2 x$, *we get*:

$$\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 1 + \cot^2 x = \csc^2 x\,[(3)\,is\ provede]$$

20. Find the exact numerical value of

a) $\sinh(\ln 2)$

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{e^{\ln 2} - e^{\ln 2^{-1}}}{2} = \frac{e^{\ln 2} - e^{\ln\frac{1}{2}}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{\left(\frac{3}{2}\right)}{2} = \frac{3}{2} \div 2 = \frac{3}{4}$$

b) $\cosh(-\ln 3)$

$$\cosh(-\ln 3) = \frac{e^{-\ln 3} + e^{\ln 3}}{2} = \frac{e^{\ln 3^{-1}} + e^{\ln 3}}{2} = \frac{e^{\ln\frac{1}{3}} + e^{\ln 3}}{2} = \frac{1/3 + 3}{2} = \frac{\left(\frac{10}{3}\right)}{2} = \frac{10}{6}$$

c) $\tanh(2\ln 3)$

$$\tanh(2\ln 3) = \frac{e^{2\ln 3} - e^{-2\ln 3}}{e^{2\ln 3} + e^{-2\ln 3}} = \frac{e^{\ln 9} - e^{\ln 1/9}}{e^{\ln 9} + e^{\ln 1/9}} = \frac{9 - 1/9}{9 + 1/9} = \frac{80}{82}$$

21. Prove the following identities:

a) $\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y$

$$\sinh(x-y)=\sinh(x+(-y))=\sinh x\cosh(-y)+\cosh x\sinh(-y)$$
$$=\sinh x\cosh y+\cosh x(-\sinh y)$$
$$=\sinh x\cosh y-\cosh x\sinh y$$

$\sin ce\ \sinh y\, is\ odd\ and \cosh y\ is\ even,$

$we\ have \sinh(-y)=-\sinh y\, and \cosh(-y)=\cosh y$

b) $\cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y$

$$\cosh(x-y)=\cosh(x+(-y))=\cosh x\cosh(-y)+\sinh x\sinh(-y)$$
$$=\cosh x\cosh y+\sinh x(-\sinh y)$$
$$=\cosh x\cosh y-\sinh x\sinh y$$

UNIT 4

<u>**Solutions/Answers to Exercises 4.1.1 of page 126**</u>

1. Find the distance between the following pair of points.

(a) (-1,0) and (3, 0)

Distance=d= $\sqrt{(-1-3)^2+(0-0)^2} = \sqrt{(-4)^2+(0)^2} = \sqrt{16} = 4$

(d) The origin and $\left(-\sqrt{3}, \sqrt{6}\right)$

Distance=d= $\sqrt{\left(-\sqrt{3}-0\right)^2+\left(\sqrt{6}-0\right)^2} = \sqrt{\left(-\sqrt{3}\right)^2+\left(\sqrt{6}\right)^2} = \sqrt{3+6} = \sqrt{9} = 3$

(e) (a,a) and (-a, -a)

Distance=d= $\sqrt{(a--a)^2+(a--a)^2} = \sqrt{(a+a)^2+(a+a)^2} = \sqrt{(2a)^2+(2a)^2}$

$= \sqrt{(2a)^2+(2a)^2} = \sqrt{4(a)^2+4(a)^2} = \sqrt{8(a)^2} = 2a\sqrt{2}$

(f) (a,b) and (-a, -b)

Distance=d= $\sqrt{(a--a)^2+(b--b)^2} = \sqrt{(a+a)^2+(b+b)^2} = \sqrt{(2a)^2+(2b)^2}$

$= \sqrt{4(a)^2+4(b)^2} = \sqrt{4\left(a^2+b^2\right)} = 2\sqrt{a^2+b^2}$

3. Let P=(-3,0) and Q be a point on the positive y-axis. Find the coordinates of Q if $|PQ| = 5$

Let (0, y), since Q is on the positive y-axis, and then $|PQ| = 5 = \sqrt{(-3-0)^2+(0-y)^2} = \sqrt{9+y^2}$

$\Rightarrow 9+y^2 = 25 \Rightarrow y^2 = 16 \Rightarrow y = 4$ [since Q is on the positive y-axis]

4. Suppose the end points of a line segment AB are A(-1,1) and B(5,10). Find the coordinates of point P and Q if

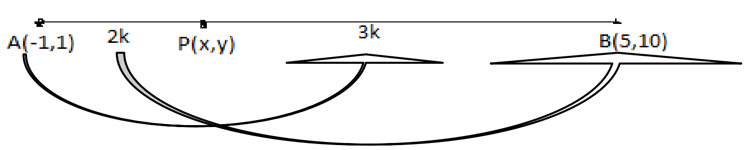
(a) P is midpoint of AB

Let P=(a, b)

Then P=(a, b)= $\left(\frac{-1+5}{2}, \frac{1+10}{2}\right) = \left(\frac{4}{2}, \frac{11}{2}\right) = (2, 5.5)$

(b) P divides AB in the ratio 2:3 (That is , $|AP|:|PB| = 2:3$)

Let P= (x, y)



Then P= (x, y)= $\frac{3A+2B}{2+3}$

$= \left(\frac{-1\times 3+5\times 2}{2+3}, \frac{1\times 3+10\times 2}{2+3}\right) = \left(\frac{-3+10}{5}, \frac{3+20}{5}\right) = \left(\frac{7}{5}, \frac{23}{5}\right)$

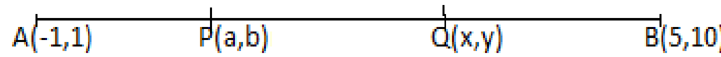
(c) Q divides AB in the ratio 3:2

Similar to C above (i.e. interchanging 2 and 3 only, which gives P= $\left(\frac{13}{5}, \frac{32}{5}\right)$

(d) P and Q trisect AB (i.e., divide it in to three equal parts)

Let P= (a,b) and Q=(x,y)

P= $\frac{2\times A+1\times B}{1+2}$ and Q= $\frac{1\times A+2\times B}{1+2}$



$$P=\frac{2\times(-1,1)+1\times(5,10)}{1+2}=\left(\frac{2\times-1+1\times5}{3},\frac{2\times1+1\times10}{3}\right)=\left(\frac{-2+5}{3},\frac{2+10}{3}\right)=(1,4)$$

Similarly $P=\frac{1\times(-1,1)+2\times(5,10)}{1+2}=\left(\frac{1\times-1+2\times5}{3},\frac{1\times1+2\times10}{3}\right)=\left(\frac{-1+20}{3},\frac{1+20}{3}\right)=\left(\frac{1}{19},7\right)$

5. Let M(-1,3) be the midpoint of a line segment PQ. If the coordinates of P is (-5,-7) then what are the coordinates of Q?

Let Q=(a,b), then M= $\frac{P+Q}{2}\Rightarrow(-1,3)=\frac{(-5,-7)+(a,b)}{2}=\left(\frac{-5+a}{2},\frac{-7+b}{2}\right)$

$$\Rightarrow -1=\frac{-5+a}{2}\ \ and\ \ 3=\frac{-7+b}{2}\Rightarrow -2=-5+a\ \ and\ \ 6=-7+b\Rightarrow a=3\ \ and\ \ b=13$$

$Therefore, Q=(3,13)$

6. Let A(a,0), B(0,b) and O(0,0) be the vertices of a right triangle. Show that midpoint of AB is equidistant from the vertices of the triangle.

We do have four cases here,

(i) a is positive and b is positive, or

(ii) a is positive and b is negative, or

(iii) a is negative and b is positive, or

(iv) a is negative and b is negative

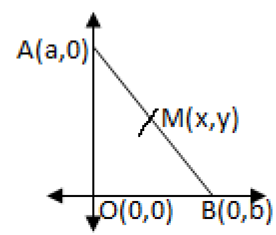


fig.1

(i) Let a be positive and b be positive as in fig.1

Where M(x,y) is the midpoint of AB

Then $M=\left(\frac{a+0}{2},\frac{0+b}{2}\right)=\left(\frac{a}{2},\frac{b}{2}\right)$

Then we want to show that M is equidistant from A, B, O

$$MA=\sqrt{\left(a-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2}=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\frac{1}{2}\sqrt{a^2+b^2}$$

$$MB=\sqrt{\left(0-\frac{a}{2}\right)^2+\left(b-\frac{b}{2}\right)^2}=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\frac{1}{2}\sqrt{a^2+b^2}$$

$$M0=\sqrt{\left(0-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2}=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\frac{1}{2}\sqrt{a^2+b^2}$$

Which means MA = MB = MO = $\frac{1}{2}\sqrt{a^2+b^2}$ all are equidistant

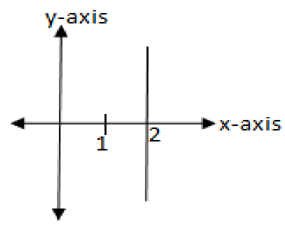
Similarly, for the cases (ii), (iii) and (iv), we get similar result.

Solutions/Answers to Exercises 4.1.2 of page 129-130

1. Find the slope and equation of the line determined by the following pair of points. Also find the y- and x-intercepts, if any, and draw each line. ONLY SELECTED QUESTIONS:

(b) (2,0) and (2,3)

Slope=m=$\frac{3-0}{2-2}=\frac{3}{0}$



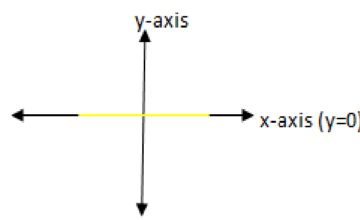
which is not defined, meaning the line is vertical

The equation of the line is x=2

x-intercept is 2 and there is no y-intercept

(c) the origin and (1,0)

m=$\frac{0-0}{1-0}=0$ where the line is horizontal

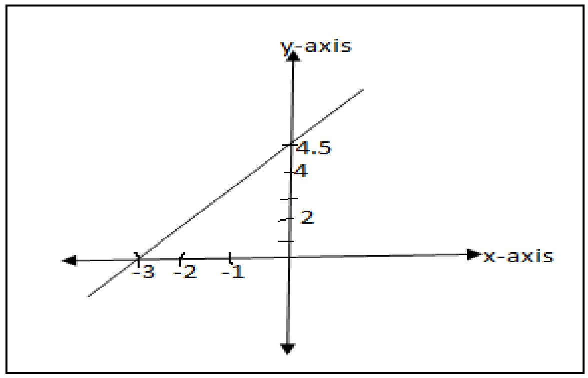


The equation of the line is y=0 (the x-axis)

x-intercept are all real numbers and y-intercept is 0

(i) (-1,3) and (1,6)

m=$\frac{6-3}{1--1}=\frac{3}{2}$



The equation is

$$\frac{y-6}{x-1}=\frac{3}{2}\Rightarrow y-6=\frac{3}{2}(x-1)\Rightarrow y=\frac{3}{2}x-\frac{3}{2}+6$$

$$\Rightarrow \text{y}=\frac{3}{2}\text{x}+\frac{9}{2}\ [\textit{Equatiuon of the line}]$$

x-intercept is – 3 and y-intercept is $\frac{9}{2}$=4.5

2. Find the slope and equation of the line whose angle of inclination is θ and passes through the point P, if ONLY SELECTED QUESTIONS:

(a) $\theta=\frac{1}{4}\pi$, P=(1,1)

Slope =m=tanθ=tan$\left(\frac{1}{4}\pi\right)$= tan45^0=1

Equation is $\frac{y-1}{x-1}=1\Rightarrow y-1=(x-1)\Rightarrow y=x-1+1\Rightarrow y=x$

(c) $\theta=\frac{3}{4}\pi$, P=(0,1)

Slope =m=tanθ=tan$\left(\frac{3}{4}\pi\right)$=tan(135^0)=-1 (second quadrant angle supplementary to 45^0)

Equation is $\frac{y-1}{x-0}=-1\Rightarrow y-1=(-1)(x)\Rightarrow y=-x+1$

(d) $\theta = 0$, P=(1,1)

Slope =m=tanθ=tan0=0 [It is horizontal line]

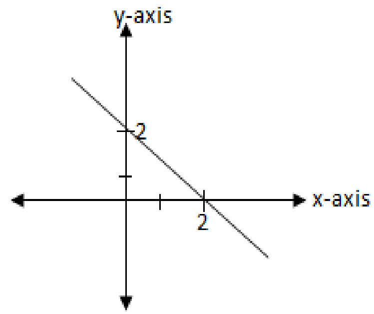
Equation is $\frac{y-1}{x-1} = 0 \Rightarrow y-1=0 \Rightarrow y=1$

(f) $\theta = \frac{1}{3}\pi$, P=(1,1)

Slope =m=tanθ=tan$\left(\frac{1}{3}\pi\right)$=tan($60^0$)=$\sqrt{3}$

Equation is $\frac{y-1}{x-1} = \sqrt{3} \Rightarrow y-1=\sqrt{3}(x-1) \Rightarrow y=\sqrt{3}\,x-\sqrt{3}+1$

3. Find the x-and y-intercepts and slope of the line given by $\frac{x}{2}+\frac{y}{2}=1$ and draw the line.



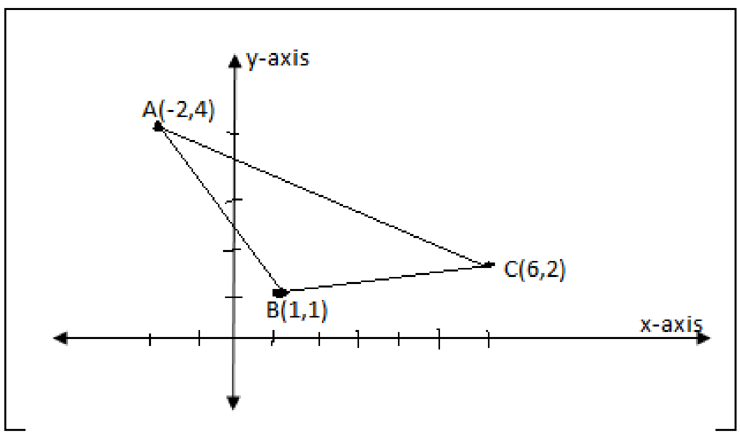
$\frac{x}{2}+\frac{y}{2}=1 \Rightarrow 2\left(\frac{x}{2}+\frac{y}{2}\right)=2(1) \Rightarrow x+y=2$

$\Rightarrow y=-x+2$

x-intercept is 2, y-intercept is 2

and the slope is – 1

4. Draw the triangle with vertices A(-2,4) , B(1,-1) and C(6,2) and find the following.



(a) Equations of the sides

Here we need to find equations of AB, AC, and BC, so that as an example we find equation of AB, others can be found in a similar way

Slope=m=$\frac{4-1}{-2-1}=\frac{3}{-3}=-1$

Equation is $\frac{y-1}{x-1}=-1 \Rightarrow y-1=(-1)(x-1) \Rightarrow y-1=-x+1 \Rightarrow y=-x+2$

Similarly we can find equations of AC and BC

(b) Equations of the medians

Median is line that passes through a vertex and midpoint of the opposite side, i.e., in this case lines that pass through A and midpoint of BC, B and midpoint of AC, and C and mud point of AB

Therefore, as an example let us find the equation of the line that passes through A and midpoint of BC, the others can be found in a similar way

(c) Equations of the perpendicular bisectors of the sides

Perpendicular bisector of a line is a perpendicular line that passes through the midpoint of the line, i.e., in this case perpendicular bisector AB or BC or AC,

Therefore, as an example let us find the equation of the perpendicular bisector AB , the others can be found in a similar way

the perpendicular bisector AB is the line whose slope is the negative reciprocal of the slope of AB that passes through the midpoint of AB

Slope of AB = - 1 (from a above)

so that slope of the perpendicular line to AB = 1

Midpoint of AB= $\frac{A+B}{2}=\frac{(-2,4)+(1,1)}{2}=\left(\frac{-2+1}{2},\frac{4+1}{2}\right)=\left(\frac{-1}{2},\frac{5}{2}\right)$

Therefore, the equation of the perpendicular bisector of AB is the equation with slope =1 and passing through $\left(\frac{-1}{2},\frac{5}{2}\right)$

$$Equation\, is\ \frac{y-\frac{5}{2}}{x-\frac{-1}{2}}=1\Rightarrow\frac{\left(\frac{2y-5}{2}\right)}{\left(\frac{2x-1}{2}\right)}=\frac{2y-5}{3x-1}=1\Rightarrow 2y-5=3x-1\Rightarrow 2y=3x+4$$

(d) Equations of the line through the vertices parallel to the opposite sides

The line through the vertices parallel to the opposite sides are the lines passing through the points A, B and C which are parallel BC, AC and AB respectively

Therefore, as an example let us find the equation of the line through the vertex A parallel to the opposite side BC, the others can be found in a similar way

The slope of BC = $\frac{-1-2}{1-6}=\frac{-3}{-5}=\frac{3}{5}$ which is also the slope of the parallel line

The required parallel line passes through point A(-2,4) which has slope $\frac{3}{5}$

$$So,\ its\ equation\, is\ \frac{y-4}{x--2}=\frac{3}{5}=\frac{y-4}{x+2}\Rightarrow 5y-20=3x+6\Rightarrow 5y=3x+26$$

5. Find the equation of the line that passes through (2,-1) and perpendicular to 3x+4y=6

   To determine the slope of 3x+4y=6, $4y = -3x + 6 \Rightarrow y = \frac{-3}{4}x + \frac{6}{4}$

   So the slope is $\frac{-3}{4}$ and the line passes through (2,-1)

   $$\textit{The equation is } \frac{y--1}{x-2} = \frac{-3}{4} \Rightarrow \frac{y+1}{x-2} = \frac{-3}{4} \Rightarrow y+1 = \frac{-3}{4}(x-2)$$

   $$y+1 = \frac{-3}{4}x - \frac{-3}{4}(-2) \Rightarrow y = \frac{-3}{4}x + \frac{3}{2} - 1 \Rightarrow y = \frac{-3}{4}x + \frac{1}{2}$$

6. Suppose ℓ_1 and ℓ_2 are perpendicular lines intersecting at (-1, 2).

   If the angle of inclination of ℓ_1 is 45^0, then find the equation of ℓ_2.

   Let the slopes of ℓ_1 and ℓ_2 be m_1 and m_2 respectively

   $m_1 = \tan(45^0) = 1 \Rightarrow m_2 = -1$

   Since the two perpendicular lines intersect at (-1, 2), we can find equation of ℓ_2 as follows:

   ℓ_2 has $m_2=1$ and it passes through (-1, 2)

   $$\textit{The equation is } \frac{y-2}{x--1} = 1 \Rightarrow y-2 = x+1 \Rightarrow y = x+3$$

7. This question is Simple! that can be determined based on the slopes of the two lines
8. Let L_1 be the line passing through P(a, b) and Q(b, a) such that $a \neq b$.
   Find an equation of the line L_2 in terms of a and b if

   (a) L_2 passes through P and perpendicular to L_1

$$\textit{Slope of } L_1 = m_1 = \frac{a-b}{b-a} = \frac{a-b}{-(a-b)} = -1 \quad \textit{and} \quad \textit{Slope of } L_2 = m_2 = 1$$

$$\textit{Equation of } L_2 \textit{ is } \frac{y-b}{x-a} = 1 \Rightarrow y-b = x-a \Rightarrow y = x-a+b$$

   (b) L_2 passes through (a, a) and perpendicular to L_1

Similar to a above, $\textit{Equation of } L_2 \textit{ is } \frac{y-a}{x-a} = 1 \Rightarrow y-a = x-a \Rightarrow y = x-a+a \Rightarrow y = x$

9. Let L_1 and L_2 be given by 2x+3y – 4 =0 and x+3y – 5 =0, respectively. A third line L_3 is perpendicular to L_1. Find the equation of L_3 if the three lines intersect at the same point.

   The slope of L_1: $2x + 3y - 4 = 0 \Rightarrow 3y = -2x + 4 \Rightarrow y = \frac{-2}{3}x + \frac{4}{3} \Rightarrow \textit{Slope of } L_1 = \frac{-2}{3}$

   Since L_3 is perpendicular to L_1, slope of L_3 $= = \frac{3}{2}$

   And since L_1, L_2 and L_3 intersect at a point, we can determine the intersection point by L_1 and L_2

Equation of L_1 is simplified above as $y=\frac{-2}{3}x+\frac{4}{3}$ which we can substitute it in equation of L_2: x+3y – 5 =0 as $x+3\left(\frac{-2}{3}x+\frac{4}{3}\right)=5 \Rightarrow x-2x+4=5 \Rightarrow -x=5-4 \Rightarrow x=-1$

$And\ \ y=\frac{-2}{3}x+\frac{4}{3} \Rightarrow y=\frac{-2}{3}(-1)+\frac{4}{3}=\frac{2}{3}+\frac{4}{3} \Rightarrow y=2$

L_1 , L_2 and L_3 intersect at (-1, 2) and slope of $L_3=\frac{3}{2}$

$$Equation\, of\ \ L_3=\frac{y-2}{x--1}=\frac{3}{2} \Rightarrow 2(y-2)=3(x+1) \Rightarrow 2y-4=3x+3 \Rightarrow 2y=3x+7$$

10. Determine the value(s) of k for which the line $(k+2)x+(k^2-9)y+3k^2-8k+5=0$

In each case write the equation of the line

(a) is parallel to the x-axis

$(k+2)x+(k^2-9)y+3k^2-8k+5=0$ is parallel to the x-axis is to mean the slope is 0

$$S0,(k+2)x+(k^2-9)y+3k^2-8k+5=0 \Rightarrow (k^2-9)y=-(k+2)x-+3k^2-8k+5$$

$$\Rightarrow y=\frac{-(k+2)}{(k^2-9)}x-+\frac{3k^2-8k+5}{(k^2-9)} \Rightarrow the\ slope\ is\ m=\frac{-(k+2)}{(k^2-9)}=0 \Rightarrow k+2=0 \Rightarrow k=-2$$

$$The\ equation\, of\ \ the\ line\ is\ (2+2)x+(2^2-9)y+3(2^2)-8(2)+5=0 \Rightarrow 4x-5y+12-16+5=0$$

$$\Rightarrow 4x-5y+1=0$$

(b) is parallel to the y-axis

$(k+2)x+(k^2-9)y+3k^2-8k+5=0$ is parallel to the y-axis is to mean the slope is undefined

$$S0,(k+2)x+(k^2-9)y+3k^2-8k+5=0 \Rightarrow (k^2-9)y=-(k+2)x-+3k^2-8k+5$$

$$\Rightarrow y=\frac{-(k+2)}{(k^2-9)}x-+\frac{3k^2-8k+5}{(k^2-9)} \Rightarrow the\ slope\ is\ m=\frac{-(k+2)}{(k^2-9)}=undefined \Rightarrow k^2-9=0$$

$$\Rightarrow k^2=9 \Rightarrow k=3\ or\, k=-3$$

$$The\ equation\, of\ \ the\ line\, when\, k=-3\ is\ (-3+2)x+((-3)^2-9)y+3((-3)^2)-8(-3)+5=0$$

$$\Rightarrow -x-(0)y+27+24+5=0 \Rightarrow -x+56=0\, and\ similarly\, it\ can\ be\ done\ when\ k=3$$

(c) passes through the origin

$(k+2)x+(k^2-9)y+3k^2-8k+5=0$ passes through the origin is to mean $(k+2)x+(k^2-9)y+3k^2-8k+5=0$ contains (0,0)

$$S0,(k+2)(0)+(k^2-9)(0)+3k^2-8k+5=0 \Rightarrow 3k^2-8k+5=0 \Rightarrow k=\frac{8\pm\sqrt{(-8)^2-4(3)(5)}}{2(3)}$$

$$\Rightarrow k=\frac{8\pm\sqrt{64-60}}{6}=\frac{8\pm\sqrt{4}}{6}=\frac{8\pm 2}{6} \Rightarrow k=\frac{5}{3}\, or\, k=1$$

The equation can be found in similar ways as a and b above

(d) passes through the point (1, 1)

$(k+2)x+(k^2-9)y+3k^2-8k+5=0$ passes through (1,1) is to mean

$(k+2)x+(k^2-9)y+3k^2-8k+5=0$ contains (1, 1)

$$S0,(k+2)(1)+(k^2-9)(1)+3k^2-8k+5=0 \Rightarrow k+2+k^2-9+3k^2-8k+5=0 \Rightarrow$$

$$4k^2+-7k+7=0 \Rightarrow k=\frac{7\pm\sqrt{(-7)^2-4(4)7}}{2(4)} \Rightarrow k=\frac{7\pm\sqrt{49-112}}{8}=\frac{7\pm\sqrt{-63}}{8}$$

$$\Rightarrow k=\frac{7\pm\sqrt{63}i}{8} \text{ which means no value of } k \text{ in the system real numbers but}$$

$$\Rightarrow k=\frac{7\pm\sqrt{63}i}{8} \text{ in the system Complex numbers}$$

11. Determine the values of a and b for which the two lines ax - 2y =1 and 6x - 4y=b

(a) have exactly one intersection point

ax-2y=1 and 6x-4y=b intersect at one point

$$ax-2y=1 \Rightarrow ax=2y+1 \Rightarrow y=\frac{a}{2}x-\frac{1}{2} \text{ where the slope is } \frac{a}{2} \text{ and } y-\text{intercept is } \frac{-1}{2}$$

$$\Rightarrow 6x-4y=b \Rightarrow -4y=-6x+b \Rightarrow y=\frac{3}{2}x-\frac{b}{4} \text{ where the slope is } \frac{3}{2} \text{ and } y-\text{intercept is } \frac{-b}{4}$$

$$\text{the two lines intersect at one point means they have different slopes which means } \frac{a}{2}\neq\frac{3}{2} \Rightarrow a\neq 3$$

Therefore for all real number $a\neq 3$ and for all real number b, the two lines intersect at exactly one point

(b) are distinct parallel lines

The two lines are distinct parallel lines when they have the same slope but different y-intercepts

Hence from (a) above, $\frac{a}{2}=\frac{3}{2} \text{ and } \frac{-1}{2}\neq\frac{-b}{4} \Rightarrow a=3 \text{ and } b\neq 2$

(c) coincide

The two lines coincide when they have the same slope the same y-intercepts

Hence from (a) above, $\frac{a}{2}=\frac{3}{2} \text{ and } \frac{-1}{2}=\frac{-b}{4} \Rightarrow a=3 \text{ and } b=2$

(d) are perpendicular

The two lines are perpendicular lines when the product of the slopes is -1

Hence from (a) above, $\left(\frac{a}{2}\right)\left(\frac{3}{2}\right)=-1 \Rightarrow \frac{3a}{4}=-1 \Rightarrow 3a=-4 \Rightarrow a=\frac{-4}{3}$

Solutions/Answers to Exercises 4.1.3 of page 132

1. Find the distance between the line L given by y=2x+3 and each of the following

   NOTE:- Distance from a line L: ax+by+c=0 to the point P(x_0, y_0) is given by $d(P,L)=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$

   The origin (b) (2, 3) (c) (1, 5) (d) (-1, 1)

   (c) (1, 5) $L: y=2x+3 \Leftrightarrow -2x+y-3$ where $a=-2, b=1, c=-3$ and $P(x_o, y_o)=(1,5)$ so that

$$d(P,L)=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}=\frac{|-2(1)+1(5)+(-3)|}{\sqrt{(-2)^2+(1)^2}}=\frac{|-2+5-3|}{\sqrt{5}}=0\text{ , which means the point is on the line}$$

2. Suppose L is the line through (1, 2) and (3, 2). What is the distance between L and

   (a) The origin (b) (2, -3) (c) (a, 0) (d) (a, b) (e) (a, 2)

   It is similar to question number 1 but here we have to determine the equation of the line

$$m=\frac{2-2}{3-1}=0 \text{ which means the line is horizontal line } L: y-2=, \text{ where } a=0, b=1 \text{ and } c=-2$$

(d) P(a, b), $d(P,L)=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}=\frac{|0(a)+1(b)+(-2)|}{\sqrt{(0)^2+(1)^2}}=\frac{|b-2|}{\sqrt{1}}=|b-2|$

3. Suppose L is vertical line that crosses the x-axis at (5, 0. Find d(P, L), when P is

   (a) The origin (b) (2, -4) (c) (0, b) (d) (5, b) (e) (a, b)

   In a similar way to question number 2, the equation is x-5=0 where $a=1, b=0$ and $c=-5$

$$(e)\ P(a,b): d(P,L)=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}=\frac{|1(a)+0(b)+(-5)|}{\sqrt{(1)^2+(0)^2}}=\frac{|a-5|}{\sqrt{1}}=|a-5|$$

4. Suppose L is the line that passes through (0, -3) and (4, 0). Find the distance between L and each of the following points.

   (a) The origin (b) (1, 4) (c) (-1, 0) (d) (8, 3) (e) (0, 1) (f) (4, -2) (g) (1, -9/4) (h) (7, -4)

   Similar to question numbers 1 and 2

5. The vertices of ΔABC are given below. Find the length of the side BC, the height of the altitude from the vertex A to BC, and the area of the triangle when its vertices are

   (a) A(3, 4), B(2, 1) and C(6, 1) (b) A(3, 4), B(1, 1) and C(5, 2)

   (b) Since (a) and (b) are similar, let us work out (b) A(3, 4), B(1, 1) and C(5, 2)

   The length of BC is the distance from B to C

$$d=\sqrt{(5-1)^2+(2-1)^2}=\sqrt{4^2+1^2}=\sqrt{17}$$

   the height of the altitude from the vertex A to BC is the distance from point A to line BC, so that we have to determine the equation of line BC.

$$m=\frac{2-1}{5-1}=\frac{1}{4}, \text{ equation: } \frac{y-1}{x-1}=\frac{1}{4}\Rightarrow 4y-4=x-1\Rightarrow L: x-4y+3=0 \text{ is equation of line BC}$$

$$A(3,4), d(A,L)=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}=\frac{|1(3)+-4(4)+(3)|}{\sqrt{(1)^2+(-4)^2}}=\frac{|3-16+3|}{\sqrt{17}}=\frac{10}{\sqrt{17}}$$

To find the area of the triangle we have to determine the lengths of each side:

BC= $\sqrt{17}$ done above

AB= $\sqrt{(3-1)^2+(4-1)^2} = \sqrt{2^2+3^2} = \sqrt{13}$

AC= $\sqrt{(3-5)^2+(4-2)^2} = \sqrt{(-2)^2+2^2} = \sqrt{8}$

Let BC=a= $\sqrt{17}$, AB=b= $\sqrt{13}$, AC=c= $\sqrt{8}$ and, $s=\frac{a+b+c}{2}=\frac{\sqrt{17}+\sqrt{13}+\sqrt{8}}{2}$,

then the area of the triangle is given by:

$$\sqrt{(s-a)(s-b)(s-c)} = \sqrt{\left(\frac{\sqrt{17}+\sqrt{13}+\sqrt{8}}{2}-\sqrt{17}\right)\left(\frac{\sqrt{17}+\sqrt{13}+\sqrt{8}}{2}-\sqrt{13}\right)\left(\frac{\sqrt{17}+\sqrt{13}+\sqrt{8}}{2}-\sqrt{8}\right)}$$

$$=\sqrt{\left(\frac{\sqrt{17}+\sqrt{13}+\sqrt{8}-2\sqrt{17}}{2}\right)\left(\frac{\sqrt{17}+\sqrt{13}+\sqrt{8}-2\sqrt{13}}{2}\right)\left(\frac{\sqrt{17}+\sqrt{13}+\sqrt{8}-2\sqrt{8}}{2}\right)}$$

$$=\sqrt{\left(\frac{\sqrt{13}+\sqrt{8}-\sqrt{17}}{2}\right)\left(\frac{\sqrt{17}+\sqrt{8}-\sqrt{13}}{2}\right)\left(\frac{\sqrt{17}+\sqrt{13}-\sqrt{8}}{2}\right)}$$

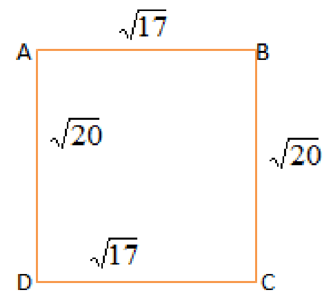
6. Consider the quadrilateral whose vertices are A(1, 2), B(2, 6), C(6, 8) and D(5, 4). Then

(a) Show that the quadrilateral is parallelogram

First let us find the lengths of the sides, AB, BC, CD, AD

$AB=\sqrt{(1-2)^2+(2-6)^2}=\sqrt{17}$, $BC=\sqrt{(2-6)^2+(6-8)^2}=\sqrt{20}$,

$CD=\sqrt{(6-5)^2+(8-4)^2}=\sqrt{17}$, $AD=\sqrt{(1-5)^2+(2-4)^2}=\sqrt{20}$



Since the lengths of the two pairs of the opposite sides of the quadrilateral are equal, it is a parallelogram.

(b) How long is the side AD? $AD=\sqrt{(1-5)^2+(2-4)^2}=\sqrt{20}$

(c) What is the height of the altitude of the quadrilateral from vertex A to side AD?
Does it want to say from vertex A to side CD or side BC? [since A is on AD, no distance]
Now, anyway to determine the type of the parallelogram, let us determine Angle D.
Consider ΔADC. To find the altitude from Point A to side DC, we have to determine the distance from point A to line DC just after determining the equation of line DC

$\overleftrightarrow{DC}: m=\frac{8-4}{6-5}=4 : Equation: \frac{y-4}{x-5}=4 \Rightarrow y-4=4x-20 \Rightarrow L: 4x-y-16=0$ *is the eqaution*

$d(A,L)=\frac{|4(1)+(-1)(2)-16|}{\sqrt{4^2+(-1)^2}}=\frac{|4-2-16|}{\sqrt{17}}=\frac{14}{\sqrt{17}}=\frac{14\sqrt{17}}{17}$ *is the altitudeto the base* $DC=\sqrt{17}$

(b) Determine the area of the quadrilateral

From 6(*c*) *above we have* $Area=\left(\sqrt{17}\right)\left(\frac{14}{\sqrt{17}}\right)=14$ *square unit*

<u>Solutions/Answers to Exercises 4.2.1. of page 134</u>

1. Suppose the center of a circle is C(1,-2) and P(7, 6) is a point on the circle. What is the radius of the circle?

Since radius of a circle is the distance from the center of the circle to any point of the circle:

$$r = CP = \sqrt{(7-1)^2 + (6--2)^2} = \sqrt{36+64} = 10$$

2. Let A(1, 2) and B(5, -2) are endpoints of a diameter of a circle. Find the center and radius of the circle.

Center is the midpoint of end points of the diameter, and radius is half the length of the diameter,

$$C = \frac{A+B}{2} = \left(\frac{1+5}{2}, \frac{2+-2}{2}\right) = (3,0) \text{ is the center}$$

$$d = AB = \sqrt{(5-1)^2 + (2--2)^2} = \sqrt{16+16} = 4\sqrt{2} \text{ is the length of the diameter and hence}$$

$$r = \frac{d}{2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2} \text{ is the length of the radius}$$

3. Consider a circle whose center is the origin and radius is $\sqrt{5}$. Determine whether or not the circle contains the following point.

Here we need to find the distance from the given point to the center and hence if this distance is less than the radius, the point is inside the circle, if this distance is equal to the radius the point is on the circle, if this distance is greater than the radius the point is outside the circle. Let us workout (g)

(a) (1, 2) (b) (0,0) (c) (0, -$\sqrt{5}$) (d) (3/2, 3/2) (e) (5,0) (f) (-1, -2), (g) ($\sqrt{3}$, $\sqrt{2}$) (h) (5/2, 5/2)

Here we need to find the distance from the given point to the center and hence if this distance is less than the radius, the point is inside the circle, if this distance is equal to the radius the point is on the circle, if this distance is greater than the radius the point is outside the circle. Let us workout (g), (h)

(g) ($\sqrt{3}$, $\sqrt{2}$), Radius = $\sqrt{5}$, Center=C(0, 0), P($\sqrt{3}$, $\sqrt{2}$)

$$CP = \sqrt{(\sqrt{3}-0)^2 + (\sqrt{2}-0)^2} = \sqrt{3+2} = \sqrt{5} = r, \text{then the point } (\sqrt{3}, \sqrt{2}) \text{ is on the circle}$$

(h) (5/2, 5/2), Radius = $\sqrt{5}$, Center=C(0, 0), P(5/2, 5/2)

$$CP = \sqrt{(5/2-0)^2 + (5/2-0)^2} = \sqrt{\frac{25}{4}+\frac{25}{4}} = \sqrt{\frac{50}{4}} = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}} \approx 3.54, \text{ but } r = \sqrt{5} \approx 1.58$$

$$\text{hence } CP > r \text{ so that } p \text{ is out side the cercle}$$

4. Consider a circle of radius 5 whose center is at C(-3, 4). Determine whether each of the following points is on the circle, inside the circle or outside the circle:

(a) (0, 9), (b) (0,0), (c) (1,6), (d) (1, 0), (e) (-7, 1), (f) (-1, -1), (g) (2,4), (h) (5/2, 5/2)

Similar to question number 3

Let us work (b), (c) and (f)

r = 5 , C(-3, 4)

$$(b)\ \text{P}(0,0),\ CP = \sqrt{(-3-0)^2 + (4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5 = r, \text{so } P \text{ is on the circle}$$

$$(c)\ \text{P}(1,6),\ CP = \sqrt{(-3-1)^2 + (4-6)^2} = \sqrt{16+4} = \sqrt{20} < 5 = r, \text{so } P \text{ is inside the circle}$$

$$(f)\ \text{P}(-1,-1),\ CP = \sqrt{(-3--1)^2 + (4--1)^2} = \sqrt{4+25} = \sqrt{29} > 5 = r, \text{so } P \text{ is outside the circle}$$

<u>Solutions/Answers to Exercises 4.2.2. of page 136-137</u>

1. Determine whether each of the following points is inside, outside or on the circle with equation $x^2 + y^2 = 5$

   (a) (-1, 2), (b) (3/2, 2) (c) $\left(0, -\sqrt{5}\right)$ (d) (-1, 3/2)

   Similar to questions 3 and 4 in **<u>Exercises 4.2.1. of page 134</u>**

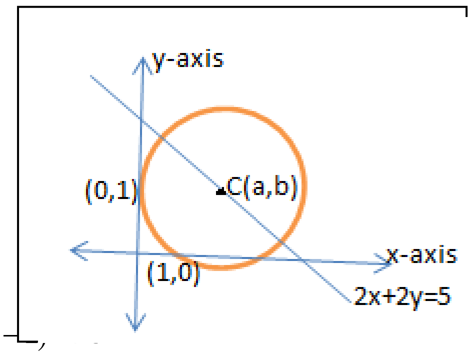
2. Find an equation of the circle whose endpoints of a diameter are (0, -3) and (3, 3).

   Similar to questions 2 in **<u>Exercises 4.2.1. of page 134</u>**

3. Determine an equation of a circle whose center is on *y*-axis and radius is 2.

   Let the center be (0,b). hen the equation is

   $(x-0)^2 + (y-b)^2 = 2^2 \Leftrightarrow x^2 + (y-b)^2 = 4$



4. Find an equation of the circle passing through (1, 0) and (0, 1) which has its center on the line $2x + 2y = 5$.

   Let the center be C = (a, b)

   Since C(a, b) is on the line $2x + 2y = 5$, 2a+2b=5 ...... [eqn. 1]

   And the equation of the circle is $(x-a)^2 + (y-b)^2 = r^2$

   Hence sine (0, 1) and (1, 0) are on the circle,

   $(a-0)^2 + (b-1)^2 = r^2 = (a-1)^2 + (b-0)^2 \Leftrightarrow a^2 + (b-1)^2 = (a -$

   $\Leftrightarrow a^2 + b^2 - 2b + 1 = a^2 - 2a + 1 + b^2 \Leftrightarrow a^2 + b^2 - 2b + 1 - a^2 + 2a - 1 - b^2 = 0$

   $\Leftrightarrow (a^2 - a^2) + (b^2 - b^2) - 2b + 2a + (1-1) = 0 \Leftrightarrow 2a = 2b \Leftrightarrow a = b \text{..........}[eqn2]$

   Using eqn. 1 and 2 above, we get, $2a + 2b = 5 \Rightarrow 2a + 2a = 5 \Rightarrow a = b = 5/4 \Rightarrow C = (5/4, 5/4)$

   Using point (1, 0) on the circle, and the center C = (5/4, 5/4) , we can find the radius r:

   $$r = \sqrt{\left(\frac{5}{4} - 0\right)^2 + \left(\frac{5}{4} - 1\right)^2} = \sqrt{\frac{25}{16} + \frac{1}{16}} = \frac{\sqrt{26}}{4}$$

   Therefore, the equation is $(x - 5/4)^2 + (y - 5/4)^2 = \dfrac{26}{16}$

5. Find the value(s) of *k* for which the equation $2x^2 + 2y^2 + 6x - 4y + k = 0$ represent a circle.

   $2x^2 + 2y^2 + 6x - 4y + k = 0 \Leftrightarrow x^2 + y^2 + 3x - 2y + k/2 = 0$ [dividing both sides of the equation by 2]

   $\Leftrightarrow (x + 3/2)^2 + (y-1)^2 = -k/2 + 9/4 + 1$ [*completing the square*]

   which is a circle with $C = (-3/2, 1)$ *and radius for some value of* $r = \sqrt{-k/2 + 9/4 + 1}$

   But the radius $r = \sqrt{-k/2 + 9/4 + 1}$ should be some positive real number

   Then $r = \sqrt{-k/2 + 9/4 + 1} > 0 \Rightarrow \dfrac{9 + 4 - 2k}{4} > 0 \Rightarrow 13 - 2k > 0 \Rightarrow 13 > 2k \Rightarrow k < 6.5$

6. An equation of a circle is $2x^2 + 2y^2 + 6x - 6y + k = 0$. If the radius of the circle is 2, .then what is the coordinates of its center?

$2x^2 + 2y^2 + 6x - 6y + k = 0 \Leftrightarrow x^2 + y^2 + 3x - 3y = -k/2 \Leftrightarrow (x + 3/2)^2 + (y - 3/2)^2 = -k/2 + 9/4 + 9/4$

$\Leftrightarrow (x + 3/2)^2 + (y - 3/2)^2 = -k/2 + 9/2 \Rightarrow \textit{The corditates of the center} = (-3/2, 3/2)$

If it is neccessary to determine $k :\Rightarrow r = 2 = \sqrt{-k/2 + 9/2} = \sqrt{(9-k)/2} \Rightarrow 9 - k = 8 \Rightarrow k = 1$

7. Find equation of the circle passing through (0,0), (4, 0) and (2, 2).

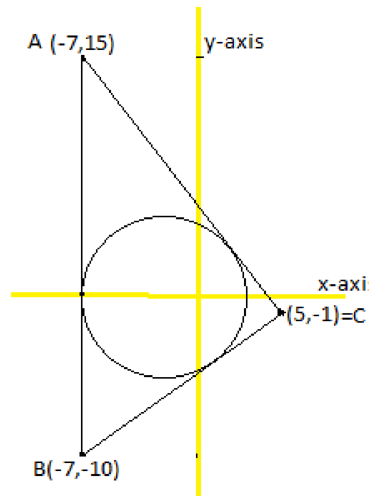
   Consider the equation of the circle to be: $x^2 + y^2 + Dx + Ey + F = 0$, then

$$\begin{cases} when(x,y)=(0,0):\ 0^2+0^2+D(0)+E(0)+F=0 \\ when(x,y)=(4,0):\ 4^2+0^2+D(4)+E(0)+F=0 \\ when(x,y)=(2,2):\ 2^2+2^2+D(2)+E(2)+F=0 \end{cases} \Rightarrow \begin{cases} F=0 \\ 16+4D+F=0 \\ 4+4+2D+2E+F=0 \end{cases} \Rightarrow \begin{cases} F=0 \\ 16+4D=0 \\ 2D+2E=-8 \end{cases}$$

$$\Rightarrow \begin{cases} F=0 \\ D=-4 \\ 2D+2E=-8 \end{cases} \Rightarrow \begin{cases} F=0 \\ D=-4 \\ 2(-4)+2E=-8 \end{cases} \Rightarrow \begin{cases} F=0 \\ D=-4 \\ E=0 \end{cases}$$

$$Therefore,\ the\ equation\ is\ x^2+y^2-4x=0 \Rightarrow (x-2)^2+y^2=4$$
$$which\ is\ equation\ of\ a\ circle\ with\ center\ C=(2,0)\ and\ radius\ r=2$$

8. Find equation of the circle inscribed in the triangle with vertices (-7, -10), (-7, 15), and (5,-1).



As seen in the fig or concept of circle inscribed in the triangle,
The sides of the triangle are tangents to the circle.
Therefore, distance from the center (a, b) to each of the sides of the triangle are equal to the radius, r.

$$d(C,\overline{AB})=d(C,\overline{AC})=d(C,\overline{BC})=r$$

$$(i)\ \overline{AB}\ has\ the\ same\ first\ cordinate\ so\ that\ the\ equation\ is$$
$$a\ vertical\ line,\ i.e.,\ x=-7 \Rightarrow x+7=0\ is\ the\ equation\ of\ \overline{AB}$$

$$(ii)\ \overline{AC}:\ \frac{y-15}{x--7}=\frac{-1-15}{5--7} \Rightarrow \frac{y-15}{x+7}=\frac{-16}{12}=\frac{-4}{3}$$
$$\Rightarrow (3)(y-15)=(-4)(x+7)$$
$$\Rightarrow 4x+3y-17=0\ is\ the\ equation\ of\ \overline{AC}$$

$$(iii)\ \overline{BC}:\ \frac{y--1}{x-5}=\frac{-10--1}{-7-5} \Rightarrow \frac{y+1}{x-5}=\frac{-9}{-12}=\frac{3}{4}$$
$$\Rightarrow 4(y+1)=3(x-5) \Rightarrow 3x-4y-19=0\ is\ the\ equation\ of\ \overline{BC}$$

$$d(C,\overline{AB})=\frac{|(1)(a)+7|}{\sqrt{4^2+3^2}}=|a+7|\ ,\ d(C,\overline{AC})=\frac{|(4)(a)+(3)(b)+17|}{\sqrt{3^2+(-4)^2}}=\frac{|4a+3b+17|}{5}$$

$$d(C,\overline{BC})=\frac{|(3)(a)+(-4)(b)-19|}{\sqrt{3^2+(-4)^2}}=\frac{|3a-4b-19|}{5}\quad where\ a>-7\ and\ b\ near\ to\ zero$$

$$r=|a+7|=\frac{|4a+3b-17|}{5}=\frac{|3a-4b-19|}{5} \Rightarrow a+7=-\left(\frac{4a+3b-17}{5}\right)=-\left(\frac{3a-4b-19}{5}\right)$$

$$=a+7=\frac{-4a-3b+17}{5}=\frac{-3a+4b+19}{5}$$

$$a+7=\frac{-4a-3b+17}{5}\quad and\quad \frac{-4a-3b+17}{5}=\frac{-3a+4b+19}{5}$$

$$\Rightarrow 5a+35=-4a-3b+17 \ and \quad 4a+3b-17=3a-4b-19$$
$$\Rightarrow 9a=-3b-18 \qquad and \quad a+7b=-2$$
$$\Rightarrow a=-\frac{1}{3}b-2 \qquad and \left(-\frac{1}{3}b-2\right)+7b=-2$$
$$\Rightarrow a=-\frac{1}{3}b-2 \qquad and \ (20/3)b=0 \Rightarrow b=0$$
$$\Rightarrow a=-\frac{1}{3}b-2 \Rightarrow a=a=-2 \quad which \ means \ C=(-2,0) \ and \ r=-2+7=5$$
$$Therefore, (x+2)^2+y^2=25 \ is \ the \ equation \ of \ the \ circle$$

9. In each of the following, check whether or not the given equation represents a circle. If the equation represents a circle, then identify its center and the length of its diameter.

(a) $x^2+y^2-18x+24y=0$ (b) $x^2+y^2-2x+4y+5=0$ (c) $x^2+y^2-4x-2y+11=0$

(d) $5x^2+5y^2+125x+60y-100=0$ (e) $36x^2+36y^2+12x+24y-139=0$ (f) $3x^2+3y^2+2x+4y+6=0$

For the equation $x^2+y^2+Dx+Ey+F=0$, if $D^2+E^2-4F>0$, then the equation represents a circle of

$$with \ center \ C=\left(\frac{-D}{2},\frac{-E}{2}\right) \ and \ radius \ r=\frac{1}{2}\sqrt{D^2+E^2-4F}$$

$$but \ when \ D^2+E^2-4F=0 \ OR \ D^2+E^2-4F<0, \ the \ equation \ does \ NOT \ represent \ a \ circle$$

$$(a)\ x^2+y^2-18x+24y=0 \Rightarrow D^2+E^2-4F=(-18)^2+(24)^2+-4(0)=324+576=900>0$$
$$\Rightarrow The \ equation \ represents \ a \ circle, \ C=\left(\frac{--18}{2},\frac{-24}{2}\right)=(9,-12) \ and \ r=\frac{1}{2}\sqrt{900}=15$$

$$(c)\ x^2+y^2-4x-2y+11=0 \Rightarrow D^2+E^2-4F=(-4)^2+(-2)^2+-4(11)=16+4-44=-24<0$$
$$\Rightarrow The \ equation \ does \ not \ represents \ a \ circle.$$

(d) $5x^2+5y^2+125x+60y-100=0$ which should be reduced to the form

whose coficients of x^2 *and* y^2 *should be* 1, *so that*:

$5x^2+5y^2+125x+60y-100=0 \Rightarrow x^2+y^2+25x+12y-20=0$ [dividing both sides by 5]

$$\Rightarrow D^2+E^2-4F=(25)^2+(12)^2+-4(-20)=625+144+80=849>0$$
$$\Rightarrow The \ equation \ represents \ a \ circle, \ C=\left(\frac{-25}{2},\frac{-12}{2}\right)=\left(\frac{-25}{2},-6\right) \ and \ r=\frac{1}{2}\sqrt{849}$$

All the others can be worked out in a similar way.

10. Show that $x^2+y^2+Dx+Ey+F=0$ represents a circle of positive radius iff $D^2+E^2-4F>0$.

$$x^2+y^2+Dx+Ey+F=0 \Rightarrow \left(x+\frac{D}{2}\right)^2+\left(Y+\frac{E}{2}\right)^2=-F+\frac{D^2}{4}+\frac{E^2}{4} \Rightarrow \left(x+\frac{D}{2}\right)^2+\left(y+\frac{E}{2}\right)^2=\frac{D^2+E^2-4F}{4}$$

If such a form is to be equation of circle, $C=\left(\frac{-D}{2},\frac{-E}{2}\right)$ *is the center and*

$$r=\sqrt{\frac{D^2+E^2-4F}{4}}=\frac{1}{2}\sqrt{D^2+E^2-4F}>0 \ is \ the \ radius$$

Solutions/Answers to Exercises 4.2.3. of page 138

1. Find the equation of the line tangent to the circle with the center at (-1, 1) and point of tangency at (-1, 3).

   The distance between the center and the point of tangency is the radius of the circle so that:

   $$r = \sqrt{(-1--1)^2 + (1-3)^2} = \sqrt{0+4} = 2$$

   $$\textit{therefore, equation is } (x--1)^2 + (y-1)^2 = 2^2 \Leftrightarrow (x+1)^2 + (y-1)^2 = 4$$

2. The center of a circle is on the line $y = 2x$ and the line $x=1$ is tangent to the circle at (1, 6). Find the center and radius of the circle.

   Let the center be C = (a, b) then the radius is:

   $$r = \frac{|(1)(a) + (0)(b) + -1|}{\sqrt{1^2 + 0^2}} = |a-1| \ \textit{which is the distance from the center } C = (a,b) \textit{ and the tangent line } x = 1$$

   $$\textit{therefore, equation is } (x--1)^2 + (y-1)^2 = 2^2 \Leftrightarrow (x+1)^2 + (y-1)^2 = 4 \ \textit{and olternatively}$$

   $$r = \sqrt{(a-1)^2 + (b-6)^2} \ \textit{ which is the distance from the center to the point of tangency} (1,6)$$

   $$\textit{Therefe, } r = \sqrt{(a-1)^2 + (b-6)^2} = |a-1| \Rightarrow (a-1)^2 + (b-6)^2 = |a-1|^2 = (b-6)^2 = 0 \Rightarrow b-6 = 0 \Rightarrow b = 6$$

   $$\textit{but since the center is on } y = 2x \Rightarrow b = 2a \Rightarrow 6 = 2a \Rightarrow a = 3$$

   $$\textit{and } r = 3-1 = 2$$

   $$\textit{therefore, } C = (3,6) \textit{ and } r = 2 \Rightarrow (x-3)^2 + (y-6)^2 = 4 \textit{ is the rquation}$$

3. Suppose two lines y = x and y = x – 4 are tangent to a circle at (2, 2) and (4, 0), respectively.

   Find equation of the circle.

   Let the center be C = (a, b) then the radius is:

$$r = \sqrt{(a-2)^2 + (b-2)^2} = \sqrt{(a-4)^2 + (b-0)^2} \Rightarrow (a-2)^2 + (b-2)^2 = (a-4)^2 + (b)^2$$

$$\Rightarrow a^2 - 4a + 4 + b^2 - 4b + 4 = a^2 - 8a + 16 + b^2 \Rightarrow a^2 - a^2 - 4a + 8a + b^2 - b^2 - 4b = 16 - 8$$

$$\Rightarrow 4a - 4b = 8 \Rightarrow a - b = 2 \Rightarrow a = b + 2$$

$$\Rightarrow r = \sqrt{(a-2)^2 + (b-2)^2} \Rightarrow \sqrt{((b+2)-2)^2 + (b-2)^2} = \sqrt{(b-2)^2 + (b)^2} = \sqrt{b^2 - 4b + 4 + b^2} = 2\sqrt{2-b}$$

$$\textit{and using the center and one of the tangent lines, say, } y = x \Rightarrow x - y = 0, \textit{ we determine the radius,}$$
$$\textit{distance from the center to the line,}$$

$$r = \frac{|(1)(a) + (-1)(b)|}{\sqrt{1^2 + (-1)^2}} = \frac{|a-b|}{\sqrt{2}} = 2\sqrt{2-b} \Rightarrow |a-b| = \sqrt{2}\left(2\sqrt{2-b}\right) \Rightarrow (a-b)^2 = 8(2-b)$$

$$\Rightarrow ((b+2)-b)^2 = 8(2-b) \Rightarrow 4 = 16 - 8b \Rightarrow 8b = 12 \Rightarrow b = 3/2$$

$$\textit{But } a = b + 2 \Rightarrow a = 3/2 + 2 \Rightarrow a = 7/2 \Rightarrow \textit{the center } C = (a,b) \Rightarrow C = (7/2, 3/2)$$

$$\Rightarrow r = \sqrt{(a-2)^2 + (b-2)^2} = r = \sqrt{(7/2-2)^2 + (3/2-2)^2} = r = \sqrt{(3/2)^2 + (-1/2)^2} = \frac{1}{2}\sqrt{10} \Rightarrow r = \frac{1}{2}\sqrt{10}$$

$$\textit{therefore, the equation is } (x - 7/2)^2 + (y - 3/2)^2 = \left(\frac{1}{2}\sqrt{10}\right)^2 \Rightarrow (x-7/2)^2 + (y-3/2)^2 = \frac{10}{4}$$

4. Find an equation of the line tangent to the circle $x^2 + y^2 + 2x - 2y = 2$ at (1,1).

$$x^2 + y^2 + 2x - 2y = 2 \Rightarrow (x+1)^2 + (y-1)^2 = 2+1+1 \Rightarrow (x+1)^2 + (y-1)^2 = 4$$

$\Rightarrow$ Center $C = (-1,1)$, *radius* $r = 2$

$\Rightarrow$ *the line* L_c *passing through the cnter* $(-1,1)$ *and point of tangency* $(1,1)$ *is perpendicular to the tangent line*

Then slope of $L_{c:}$ *is* $m_c = \dfrac{1-1}{-1-1} = 0 \Rightarrow$ *this line is horizontal and the tangent line is vertical passing throgh* $(1,1)$ *so that the equation is* $x = 1)$

5. Find equation of the line through $(\sqrt{32}, 0)$ and tangent to the circle with equation $x^2 + y^2 = 16$

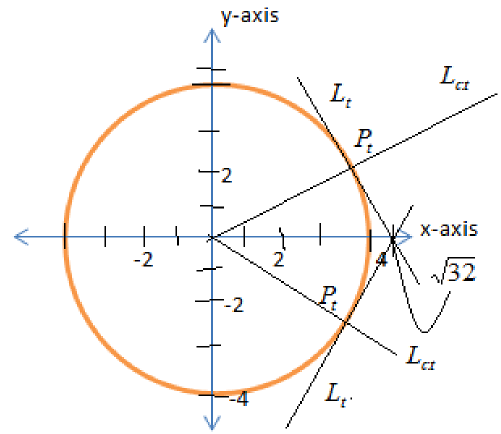
$x^2 + y^2 = 16 \Rightarrow C = (0,0)$ *and* $r = 4$

Let the point of tangency be $P_t(m, n)$

then $r = 4 = \sqrt{m^2 + n^2}$

$\Rightarrow m^2 + n^2 = 16$ ....(*eqn6.a*)

the line L_{ct} *passing through the cnter* $C(0,0)$ *and point of tangency* $P_t(m,n)$ *is perpendicular to the tangent line,* L_t



Then slope of L_{ct} *is* $m_{ct} = \dfrac{n-0}{m-0} = \dfrac{n}{m}$

$\Rightarrow$ *the slope of the tangent line,* L_t *is* $\dfrac{-m}{n} = \dfrac{n-0}{m-\sqrt{32}}$

$$n^2 = -m(m - \sqrt{32}) \Rightarrow n^2 = -m^2 + \sqrt{32}\,m \Rightarrow n^2 + m^2 = \sqrt{32}\,m \quad and \quad m^2 + n^2 = 16 \ \ from(eqn6.a)$$

$$\Rightarrow \sqrt{32}\,m = 16 \Rightarrow m = \frac{16}{\sqrt{32}} \Rightarrow m = \frac{1}{2}\sqrt{32}$$

$$But \ \ m^2 + n^2 = 16 \Rightarrow \left(\frac{1}{2}\sqrt{32}\right)^2 + n^2 = 16 \Rightarrow n^2 = 16 - 8 \Rightarrow n = \pm 2\sqrt{2}$$

$P_t = \left(\frac{1}{2}\sqrt{32}, 2\sqrt{2}\right)$ *or* $P_t = \left(\frac{1}{2}\sqrt{32}, -2\sqrt{2}\right)$ *which are two lines whose slopes are:*

$$m_1 = \frac{2\sqrt{2} - 0}{\frac{1}{2}\sqrt{32} - \sqrt{32}} = \frac{4\sqrt{2}}{-4\sqrt{2}} = -1 \ \ OR \ \ m_2 = \frac{-2\sqrt{2} - 0}{\frac{1}{2}\sqrt{32} - \sqrt{32}} = \frac{-4\sqrt{2}}{-4\sqrt{2}} = 1 \ \ and\ the\ equatins\ are:$$

$$L_1 : \frac{y-0}{x-\sqrt{32}} = -1 \ \ OR \ \ L_2 : \frac{y-0}{x-\sqrt{32}} = 1$$

$$\Rightarrow L_1 : y + x - \sqrt{32} = 0 \ \ OR \ \ L_2 : y - x + \sqrt{32} = 0$$

6. Suppose P(1,2) and Q(3, 0) are the endpoints of a diameter of a circle and L is the line tangent to the circle at Q.

(a) Show that R(5, 2) is on L.

(b) Find the area of △PQR, when R is the point given in (a).

(a) Center, $Center, C = \frac{P+Q}{2} = \frac{(1,2)+(3,0)}{2} = \left(\frac{1+3}{2}, \frac{2+0}{2}\right) = (2,1)$

Slope of the line passing through C & Q is $\frac{1-0}{2-3} = -1$ which is perpendicular to the tangent line Passing through Q.

So, the slope of the tangent line passing through Q is 1,

And its equation is L : $\frac{y-0}{x-3} = 1 \Rightarrow y - x + 3 = 0$ *is the equation*

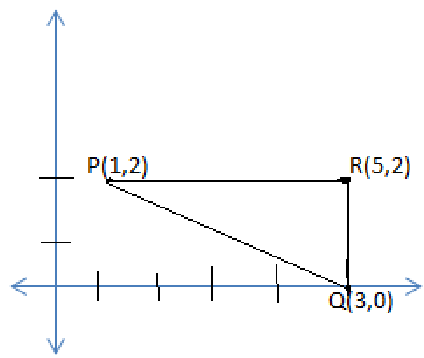
Take R(5,2) to substitute in

$y - x + 3 = 0 \Rightarrow 2 - 5 + 3 = 0 \Rightarrow 0 = 0$ *which is true*

meaning R is on L .

(b) To find area of ΔPQR,

first let us determine PQ, PR & QR



$PR = \sqrt{(5-1)^2 + (2-2)^2} = \sqrt{4^2} = 4$

$PQ = \sqrt{(3-1)^2 + (2-2)0^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$QR = \sqrt{(5-3)^2 + (2-0)^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$And\ (PQ)^2 + (QR)^2 = (PR)^2 \sin ce \left(2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2 = 4^2$

$\Rightarrow 8 + 8 = 16$ *is true*

$which\ means \Delta PQR\ is\, a\, right - angled triangel, the right\ angle\ at\ Q.$

$Then\ the area of\ \Delta PQR = \frac{1}{2}(PQ)(QR) = \frac{1}{2}\left(2\sqrt{2}\right)\left(2\sqrt{2}\right) = \frac{1}{2}(8) = 4\, sq.unit$