The Zero Lower Bound: Frequency, Duration, and Determinacy*

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ABSTRACT

When monetary policy faces a zero lower bound (ZLB) constraint on the nominal interest rate, indeterminacy may occur even if the Taylor principle holds when the ZLB does not bind. This paper shows the boundary of the determinacy region imposes a clear tradeoff between the expected frequency and average duration of ZLB events. We show this tradeoff using a global solution to a nonlinear New Keynesian model with two alternative stochastic processes—one where monetary policy follows a 2-state Markov chain, which exogenously governs whether the ZLB binds, and the other where ZLB events are endogenous due to technology or discount factor shocks. In both cases, the household accounts for the possibility of going to and exiting the ZLB in expectation. We also show that small changes in the parameters of the stochastic process significantly impact the decision rules and the state at which the ZLB first binds.

Keywords: Monetary policy; zero lower bound; determinacy; global solution method
JEL Classifications: E31; E42; E58

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1 INTRODUCTION

Since the beginning of the Great Recession in late 2008, many central banks around the world have targeted a policy rate near zero and promised to maintain a low rate until economic conditions improve. Despite this policy and numerous unconventional policies, most of these countries face elevated unemployment levels and anemic growth five years later. This experience has ignited new research that studies the impacts of the zero lower bound (ZLB) on the nominal interest rate.

The ZLB constraint is equivalent to a pegged nominal interest rate rule (i.e., a rule that does not respond to inflation or output), with a truncated nominal interest rate distribution. In a fixed interest rate regime, it is well known that indeterminacy occurs when the Taylor (1993) principle—the principle that monetary policy pins down prices by adjusting the nominal interest rate more than one-for-one with inflation—does not hold. This means that if the household expects to stay at the ZLB forever, then the price level, and hence inflation, is not pinned down. If the household only expects to occasionally visit the ZLB, then the fraction of time spent in the regime that satisfies the Taylor principle *may* provide enough price stability to deliver a determinate equilibrium.

This paper shows the boundary of the determinacy region imposes a clear tradeoff between the expected frequency and average duration of ZLB events. We show this tradeoff using a global solution to a conventional nonlinear New Keynesian model with two alternative stochastic processes—one where monetary policy follows a 2-state Markov chain, which exogenously governs whether the ZLB binds, and the other where ZLB events are endogenous due to technology or discount factor shocks. In both cases, the household accounts for the complete distribution of shocks and the possibility of going to and exiting the ZLB in expectation. We also show that small changes in the parameters of the stochastic process significantly impact the decision rules and the state at which the ZLB first binds, which has important implications for estimation and policy analysis.

The ZLB constraint imposes a kink in the monetary policy rule. The literature has relied on several different techniques to deal with this challenge. One common technique is to break the problem into pre- and post-ZLB periods [e.g., Braun and Körber (2011); Braun and Waki (2006); Christiano et al. (2011); Eggertsson and Woodford (2003); Erceg and Linde (2010); Gertler and Karadi (2011)]. With this approach, a large unanticipated shock causes the ZLB to bind. Each period, there is a probability that the nominal interest rate exits the ZLB, but there is also a maximum duration of the ZLB event. Once the nominal rate exits the ZLB, there is no probability of returning to the ZLB. This approach simplifies the problem, but it also has several drawbacks.

First, assuming the ZLB is unanticipated is inconsistent with the Taylor rule. Since the household knows the Taylor rule, which implies a nonpositive interest rate for some shocks, they also know the conditions under which the ZLB binds. Second, it assumes households have perfect foresight about the sequence of shocks and the monetary policy response. Thus, they fully anticipate when the nominal interest rate will increase, which affects determinacy. If the duration of the ZLB event is stochastic, then there is a maximum expected duration of staying at the ZLB, which depends on the distribution of the shocks. However, if the household knows that at some future date the nominal interest rate will exit the ZLB, then the average expected duration is no longer relevant. Lastly, there is no chance of returning to the ZLB once the nominal interest rate exits the ZLB, which is logically inconsistent. If a shock causes the ZLB to bind in one period, there is no reason to expect that the same shock would not cause the ZLB to bind in some future period. Much of the ZLB literature also log-linearizes the equilibrium system, except the monetary policy rule, which causes large approximation errors that affect the qualitative properties of the model

[Braun et al. (2012); Fernández-Villaverde et al. (2012)]. These drawbacks motivate solving the fully nonlinear model to accurately account for the expectational effects of hitting the ZLB.

Recognizing the importance of expectational effects and the drawbacks with log-linearization, a recent segment of the ZLB literature uses global solution methods to solve fully nonlinear models with a ZLB constraint [e.g., Aruoba and Schorfheide (2013); Basu and Bundick (2012); Fernández-Villaverde et al. (2012); Gavin et al. (2013); Gust et al. (2013); Mertens and Ravn (2013); Richter et al. (2013); Wolman (2005)]. However, all of the work on determinacy uses a perfect foresight setup [e.g., Alstadheim and Henderson (2006); Benhabib et al. (2001a,b)], which faces the drawbacks discussed above. This paper contributes to the literature by providing the determinacy regions to the fully nonlinear model, which is important for estimation and policy analysis.

The paper is organized as follows. Section 2 describes our solution procedure, provides a definition of determinacy, and shows how the determinacy regions differ between the linear and nonlinear versions of a simple Fisherian economy with Markov switching in the monetary policy rule. Section 3 lays out the constrained nonlinear model and the calibration. Sections 4 and 5 define the two alternative stochastic processes that drive the economy to the ZLB and show the tradeoff between the expected frequency and average duration of ZLB events. Section 6 concludes.

2 SOLUTION PROCEDURE AND DEFINITION OF DETERMINACY

We solve the nonlinear model using the policy function iteration algorithm described in Richter et al. (2013), which is a numerical byproduct of using monotone operators to prove existence and uniqueness of equilibria.[1] This solution method discretizes the state space and uses time iteration to solve for the updated decision rules until the tolerance criterion is met. To account for the ZLB, we set the gross nominal interest rate equal to 1 on any node in the state space where the Taylor rule implies a value less than one. We obtain initial conjectures for the constrained nonlinear model using the solution to the log-linear model without the ZLB imposed. We find that this guess is very reliable and no evidence it affects convergence. For example, when we solve for the boundary of the determinacy region using this guess for each parameterization, it produces the same boundary as when we use the nonlinear solution to a model with a similar parameterization as our guess.

In the exogenous and endogenous ZLB setups, we classify the algorithm as *non-convergent* whenever the iteration step, defined as the maximum distance between decision rule values on successive iterations, increases at an increasing rate for more than 100 iterations or when all of the values in any decision rule consistently drift (e.g., negative consumption on any node or more than 50 percent deflation on every node). Additionally, when ZLB events are endogenous, we require that the ZLB binds on fewer than 50 percent of the nodes in the state space since it is infeasible for the ZLB to bind at the stochastic steady state. We classify the algorithm as *convergent* whenever the iteration step is less than 1^{-13} (the tolerance criterion) for 10 successive iterations, which prevents the algorithm from immediately converging when the tolerance criterion is first met. To provide evidence that the solution is unique, we randomly perturb the converged decision rules in multiple directions and check that the algorithm converges back to the same solution. To ensure that the

[1]Coleman (1991) proves existence and uniqueness of an equilibrium in a nonlinear stochastic production economy with an income tax. Greenwood and Huffman (1995) adapt this proof to a more general neoclassical model, including one with monopolistic competition. Coleman (1997) generalizes these proofs to allow for an endogenous labor supply and Datta et al. (2005, 2002); Mirman et al. (2008) extends them to more complex setups. The monotone mapping results in these papers are attractive because they serve as the theoretical foundations of our numerical algorithm.

solution is bounded, we simulate the model and check that it converges to a stochastic steady state. For a more formal description of the numerical algorithm and convergence, see appendix A.

Davig and Leeper (2007) also study determinacy, but with log-linear models that do not include a ZLB constraint. Their models contain two monetary policy rules—one that aggressively responds to inflation and one that reacts less aggressively to inflation—governed by a 2-state Markov chain. The special case where the central bank pegs the nominal interest rate in one regime and obeys the Taylor principle in the other regime is similar to a model with a ZLB constraint. Thus, we use their regime switching setup as a benchmark for our algorithm. When we adopt the models they use (log-linear Fisherian economy, log-linear New Keynesian economy), our algorithm produces the same determinacy regions they analytically derive. This means our algorithm is non-convergent whenever the monetary policy parameters are outside their analytical determinacy region and convergent whenever the Long-run Taylor Principle is met. Our numerical solutions to these models also equal the minimum state variable (MSV) solutions they derive. Thus, we define any convergent solution as a determinate rational expectations equilibrium and the set of convergent solutions as the determinacy region (i.e., the set of parameters that deliver a unique stable MSV solution). While this exercise does not constitute a formal proof, it provides strong evidence that our algorithm accurately captures when MSV solutions are unique and bounded.

Our finding that there exists a tradeoff between the expected frequency and average duration of ZLB events is similar to the conclusion in Davig and Leeper (2007). They prove that when there are distinct monetary policy regimes, the Taylor principle does not need to hold in both regimes to guarantee a unique bounded MSV solution. As long as one of the regimes satisfies the Taylor principle, the monetary authority can passively respond to inflation (i.e., adjust the nominal interest rate less than one-for-one with inflation) in the other regime and still deliver a determinate solution. However, there are two key differences between our setups. First, an occasionally binding ZLB constraint truncates the nominal interest rate distribution, which affects the household's expectations and their decision rules. Second, the parameters of the exogenous driving processes affect determinacy, since the log-linearized version of a nonlinear model with a discontinuity misses key interaction terms between exogenous variables and expected inflation.

To see how the exogenous driving processes matter for determinacy, consider the nonlinear analogue of the endowment economy Davig and Leeper (2007) study. A representative household chooses $\{c_t, b_t\}_{t=0}^{\infty}$ to maximize $E_0 \sum_{t=0}^{\infty} \tilde{\beta}_t \log c_t$, where c is consumption, $\tilde{\beta}_0 \equiv 1$, and $\tilde{\beta}_t = \prod_{i=1}^{t} \beta_i$ for $t > 0$. These choices are constrained by $c_t + b_t = y + i_{t-1} b_{t-1}/p_t$, where y is a constant endowment, b is a one-period nominal bond, i is the gross nominal interest rate set by the monetary authority, and $\pi_t = p_t/p_{t-1}$ is the gross inflation rate. The fiscal authority has a balanced budget so bonds are in zero-net supply. The equilibrium system is composed of

$$1 = i_t E_t[\beta_{t+1}/\pi_{t+1}], \tag{1}$$

$$i_t = \bar{i}(\pi_t/\bar{\pi})^{\phi(s_t)}, \tag{2}$$

$$\beta_t = \bar{\beta}(\beta_{t-1}/\bar{\beta})^{\rho} \exp(\varepsilon_t), \tag{3}$$

where β_t is the discount factor, which evolves according to (3) with $|\rho| < 1$ and $\varepsilon_t \sim \mathbb{N}(0, \sigma^2)$. $\phi(s_t)$ is the policy response to changes in inflation, which evolves according to a 2-state Markov chain with transition matrix $\Pr\{s_t = j | s_{t-1} = i\} = p_{ij}$, $i, j \in \{1, 2\}$. A bar denotes a stationary value.

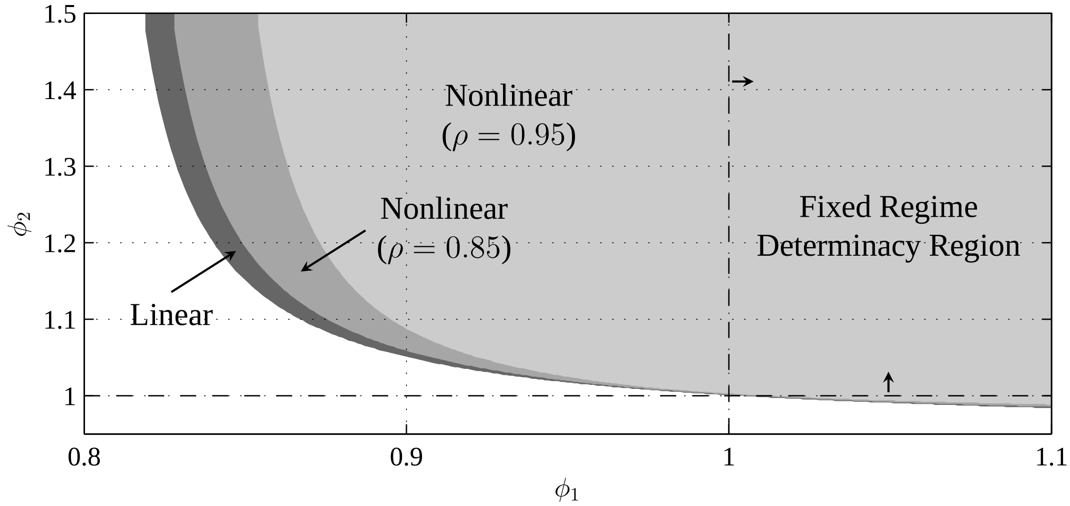


Figure 1: Fisherian economy determinacy (shaded) regions in (ϕ_1, ϕ_2)-space. We set $(p_{11}, p_{22}) = (0.8, 0.95)$.

A second-order approximation of (1) around the deterministic steady state implies

$$\hat{i}_t + \underbrace{(\hat{i}_t - E_t[\hat{\pi}_{t+1}] + E_t[\hat{\beta}_{t+1}])^2}_{=0 \text{ (First Order)}} = E_t[\hat{\pi}_{t+1}] - E_t[\hat{\beta}_{t+1}] - \underbrace{(E_t[(\hat{\pi}_{t+1} - \hat{\beta}_{t+1})^2] - (E_t[\hat{\pi}_{t+1} - \hat{\beta}_{t+1}])^2)}_{=0 \text{ (First Order, Jensen's Inequality)}}, \quad (4)$$

where a hat denotes log deviation from a steady state value. Up to a first order, this equation reduces to the standard log-linear Fisher equation, which, when combined with (2), reduces to

$$\phi(s_t)\hat{\pi}_t = E_t[\hat{\pi}_{t+1}] - E_t[\hat{\beta}_{t+1}].$$

If the monetary policy regime is fixed ($\phi(s_t) = \phi$), determinacy requires $\phi > 1$ (Taylor principle). If monetary policy is state-dependent ($\phi(s_t = i) = \phi_i$), determinacy requires

$$p_{11}(1 - \phi_2) + p_{22}(1 - \phi_1) + \phi_1\phi_2 > 1. \quad \text{(Long-run Taylor Principle)}$$

Neither of these conditions include parameters of the discount factor process. This is a byproduct of first-order approximations, which remove all interaction terms between the expected discount factor and expected inflation. With a higher order approximation, such as the second-order approximation in (4), these interaction terms appear and affect determinacy. When fluctuations in the discount factor are more persistent, it causes more persistent deviations of inflation from its steady state value, which shrinks the determinacy region. As an example, figure 1 shows the determinacy (shaded) regions for the state-dependent log-linear model and the fully nonlinear model with $\rho = 0.85$ and $\rho = 0.95$ in (ϕ_1, ϕ_2)-space.[2] The determinacy region is smaller in the fully nonlinear model and decreases with ρ. However, changes to σ do not influence the determinacy region since it only affects the magnitude of the shock and not the household's consumption/saving decision.

In models with a ZLB constraint, both the persistence and standard deviation of the exogenous driving processes affect the determinacy region. It is well known that these models contain two deterministic steady states [Benhabib et al. (2001a,b)]. Specifically, there are two nominal interest rate/inflation rate pairs consistent with the steady state equilibrium system. In one case the monetary authority meets its positive inflation target, while in the other, deflation occurs. Similar to

[2]For the purposes of this exercise, we fix $\bar{\beta} = 0.99$, $\bar{\pi} = 1.005$, $p_{11} = 0.8$, $p_{22} = 0.95$, and $\sigma = 0.0005$.

the sunspot shocks in Aruoba and Schorfheide (2013) and the confidence shocks in Mertens and Ravn (2013), exogenous switches in the monetary policy state that occur in our model cause the economy to switch between these two states, but it does *not* necessarily imply multiple equilibria in a stochastic economy.[3] As long as there is sufficient expectation on returning to a monetary policy rule that obeys the Taylor principle, we show that there is a unique bounded MSV solution.

Within the class of Markov-switching rational expectations models, Farmer et al. (2009, 2010), Barthélemy and Marx (2013), and Cho (2013) prove that non-MSV solutions with fundamental or non-fundamental components may still exist even when the MSV solution is determinate. To be clear, our numerical algorithm rules out many indeterminate equilibria that are subject to sunspot fluctuations (e.g., in fixed regime models without a ZLB constraint, our algorithm only converges when the Taylor principle is satisfied), but it cannot capture the types of non-MSV solutions that may exist when a determinate MSV solution exists.[4] Studying these types of solutions in models with a ZLB constraint is an important topic for future research, but we believe locating regions of the parameter space that deliver a determinate MSV solution while accurately capturing the ZLB is significant since most macroeconomic research (including estimation) is based on MSV solutions.

3 MODEL ECONOMY AND BASELINE CALIBRATION

At the ZLB, monetary policy cannot directly affect the real interest rate to stabilize inflation. Without price adjustment costs or sticky prices, such as in the Fisherian economy described above, no region of the parameter space delivers a determinate equilibrium even if ZLB events are infrequent. In a new Keynesian model, the nominal frictions anchor prices at the ZLB so that a strong enough expectation of leaving the ZLB delivers a determinate solution. Thus, we show the determinacy regions in a conventional nonlinear New Keynesian model commonly used for policy analysis.

A representative household chooses $\{c_t, n_t, b_t\}_{t=0}^{\infty}$ to maximize expected lifetime utility, given by, $E_0 \sum_{t=0}^{\infty} \tilde{\beta}_t \{c_t^{1-\sigma}/(1-\sigma) - \chi n_t^{1+\eta}/(1+\eta)\}$, where $1/\sigma$ is the intertemporal elasticity of substitution, $1/\eta$ is the Frisch elasticity of labor supply, c_t is consumption of the final good, n_t is labor hours, $\tilde{\beta}_0 \equiv 1$, and $\tilde{\beta}_t = \prod_{i=1}^{t} \beta_i$ for $t > 0$. β_i is the subjective discount factor in period i. These choices are constrained by $c_t + b_t + \tau_t = w_t n_t + r_{t-1} b_{t-1}/\pi_t + d_t$, where $\pi_t = p_t/p_{t-1}$ is the gross inflation rate, w_t is the real wage rate, τ_t is a lump-sum tax, b_t is a one-period real bond, r_t is the gross nominal interest rate, and d_t are profits from intermediate firms. The optimality conditions to the household's problem imply

$$w_t = \chi n_t^{\eta} c_t^{\sigma}, \tag{5}$$
$$1 = r_t E_t \{\beta_{t+1} (c_t/c_{t+1})^{\sigma}/\pi_{t+1}\}. \tag{6}$$

The production sector consists of a continuum of monopolistically competitive intermediate goods firms and a perfectly competitive final goods firm. Each firm $i \in [0,1]$ in the intermediate goods sector produces a differentiated good, $y_t(i)$, according to $y_t(i) = a_t n_t(i)$, where a_t is technology and $n_t(i)$ is the level of employment used by firm i. The final goods firm purchases $y_t(i)$ units from each intermediate firm to produce the final good, $y_t \equiv [\int_0^1 y_t(i)^{(\theta-1)/\theta} di]^{\theta/(\theta-1)}$,

[3] Aruoba and Schorfheide (2013) discuss sunspot equilibria, but these are *not* the same sunspots Farmer et al. (2009, 2010) and Cho (2013) emphasize, since they omit the non-MSV component from their solution.

[4] Barthélemy and Marx (2013) refer to unique bounded MSV solutions as bounded Markovian solutions (i.e., dependent on a finite number of past regimes). Our numerical algorithm rules out the possibility of multiple bounded Markovian solutions, but there may exist other bounded non-Markovian solutions. We call these non-MSV solutions.

according to a Dixit and Stiglitz (1977) aggregator, where $\theta > 1$ is the price elasticity of demand between intermediate goods. Profit maximization yields the demand function for good i, $y_t(i) = (p_t(i)/p_t)^{-\theta} y_t$, where $p_t = [\int_0^1 p_t(i)^{1-\theta} di]^{1/(1-\theta)}$ is the final good price. Each intermediate firm chooses its price level, $p_t(i)$, to maximize the expected present value of real profits, $E_t \sum_{k=t}^{\infty} q_{t,k} d_k(i)$, where $q_{t,t} \equiv 1$, $q_{t,t+1} = \beta_{t+1}(c_t/c_{t+1})^{\sigma}$ is the pricing kernel between periods t and $t+1$, and $q_{t,k} \equiv \prod_{j=t+1}^{k} q_{j-1,j}$. Following Rotemberg (1982), each firm faces a cost to adjusting its price, which emphasizes the potentially negative effect that price changes can have on customer-firm relationships. Using the functional form in Ireland (1997), firm i's real profits are

$$d_t(i) = \left[\left(\frac{p_t(i)}{p_t} \right)^{1-\theta} - mc_t \left(\frac{p_t(i)}{p_t} \right)^{-\theta} - \frac{\varphi}{2} \left(\frac{p_t(i)}{\bar{\pi} p_{t-1}(i)} - 1 \right)^2 \right] y_t,$$

where $\varphi \geq 0$ determines the magnitude of the adjustment cost, mc_t is the real marginal cost of producing a unit of output, and $\bar{\pi}$ is the steady-state gross inflation rate. In a symmetric equilibrium, all intermediate goods firms make the same decisions and the optimality condition reduces to

$$\varphi \left(\frac{\pi_t}{\bar{\pi}} - 1 \right) \frac{\pi_t}{\bar{\pi}} = (1-\theta) + \theta mc_t + \varphi E_t \left[q_{t,t+1} \left(\frac{\pi_{t+1}}{\bar{\pi}} - 1 \right) \frac{\pi_{t+1}}{\bar{\pi}} \frac{y_{t+1}}{y_t} \right]. \tag{7}$$

In the absence of price adjustment costs (i.e., $\varphi = 0$), the real marginal cost equals $(\theta - 1)/\theta$, which is the inverse of the firm's markup of price over marginal cost.

Each period the fiscal authority finances its spending, $\bar{g}$, by levying lump-sum taxes ($\tau_t = \bar{g}$). The resource constraint is $c_t + \bar{g} = [1 - \varphi(\pi_t/\bar{\pi} - 1)^2/2] y_t$. The household's and firm's optimality conditions, the government's budget constraint, the monetary policy rule (defined below), the bond market clearing condition ($b_t = 0$), and the resource constraint form the equilibrium system.

The model is calibrated at a quarterly frequency using values that are common in the literature. We set $\bar{\beta} = 0.99$ and $\sigma = 1$, implying log utility in consumption. The Frisch elasticity of labor supply, $1/\eta$, is set to 1 and the leisure preference parameter, χ, is set so that steady-state labor equals $1/3$ of the available time. The price elasticity of demand between intermediate goods, θ, is calibrated to 6, which corresponds to an average markup of price over marginal cost equal to 20 percent. The costly price adjustment parameter, φ, is set to 58.25, which is consistent with a Calvo (1983) price-setting specification where prices change on average once every four quarters ($\omega = 0.75$).[5] Steady-state technology, $\bar{a}$, is normalized to 1. In the policy sector, the steady-state gross inflation rate, $\bar{\pi}$, is calibrated to 1.005, which implies an annual (net) inflation rate target of 2 percent. The steady-state ratio of government spending to output is calibrated to 20 percent.

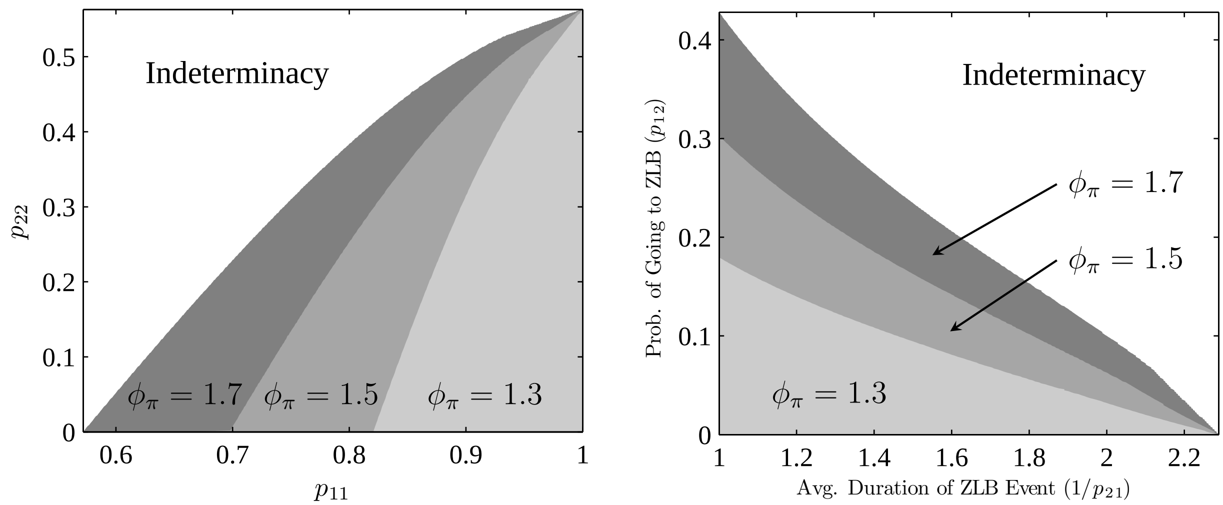
4 EXOGENOUS ZLB EVENTS: MONETARY POLICY SWITCHING

In this section, the monetary authority sets the gross nominal interest rate according to

$$r_t = \begin{cases} \bar{r}(\pi_t/\bar{\pi})^{\phi_\pi}(y_t/\bar{y})^{\phi_y} \exp(\varepsilon_t) & \text{for} \quad s_t = 1 \\ 1 & \text{for} \quad s_t = 2 \end{cases}, \tag{8}$$

where ϕ_π and ϕ_y are the policy responses to deviations in inflation and output from their steady-state values and $\varepsilon_t \sim \mathbb{N}(0, \sigma_\varepsilon^2)$ is a discretionary monetary policy shock. The monetary policy state

[5] If ω represents the fraction of firms that cannot adjust prices, $\varphi = \omega(\theta-1)/[(1-\omega)(1-\beta\omega)]$.



(a) Determinacy (shaded) regions in (p_{11},p_{22})-space

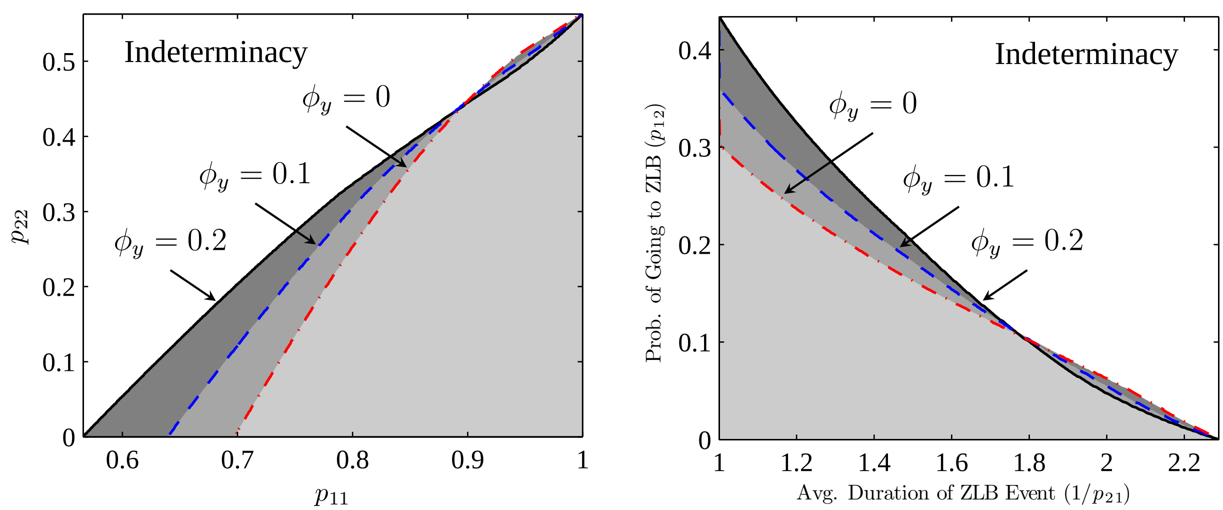
(b) Expected frequency/average duration of ZLB events tradeoff

Figure 2: Determinacy regions across alternative monetary policy responses to changes in inflation.

evolves according to a 2-state Markov chain with transition matrix $\Pr\{s_t = j|s_{t-1} = i\} = p_{ij}$, for $i, j \in \{1, 2\}$. When $s_t = 1$, the monetary authority obeys the Taylor principle and when $s_t = 2$, the monetary authority exogenously pegs the gross nominal interest rate at one. We set $a_t = \bar{a}$, $\beta_t = \bar{\beta}$, and $\sigma_\varepsilon = 0.003$, which is small enough that the ZLB never binds due to the policy shock. Thus, all ZLB events in this section are due to exogenous changes in the monetary policy state, s_t.

The exogenous switches between the two monetary policy states are equivalent to large discretionary shocks. When the nominal interest rate switches from state 1 to state 2 (state 2 to state 1), the nominal interest rate falls (rises) sharply. This means that expectations about the future state play a key role in determining inflation. To understand how inflation changes when ZLB events are exogenous, assume the state is fixed, but there is a probability it changes. When $s_t = 1$ and $p_{11} < 1$, the household expects a lower future nominal interest rate, which increases expected future consumption growth and drives up inflation. When $s_t = 2$, the household expects to leave the ZLB and the future nominal interest rate to rise. This reduces expected future consumption growth, which would normally reduce inflation, but since the nominal rate is stuck at 1, inflation rises to clear the bond market. Thus, the possibility of ZLB events increases inflation in both states.

Figure 2a plots the determinacy (shaded) regions in (p_{11}, p_{22})-space for $\phi_\pi \in \{1.3, 1.5, 1.7\}$. To isolate the impact of ϕ_π on the determinacy region, we initially set $\phi_y = 0$. The boundary of the shaded region for each ϕ_π represents the largest p_{22} value that yields a determinate solution for each p_{11} value. These results show a clear tradeoff between p_{11} and p_{22}. When there is a low probability of going to the ZLB (i.e., a high p_{11} value), it is possible to have a high probability of staying at the ZLB (i.e., a high p_{22} value) and still guarantee a determinate solution. This suggests that there is a tradeoff between the expected frequency and average duration of ZLB events. To see this more clearly, figure 2b plots the probability of going to the ZLB (i.e., p_{12}) as a function of the average duration of each ZLB event (i.e., $1/p_{21}$) for each value of ϕ_π. When the average duration of ZLB events is short, the determinacy region permits a high expected frequency of ZLB events. However, as the average duration of ZLB events increases, the maximum expected frequency of ZLB events must decrease to avoid the indeterminacy (non-shaded) region of the parameter space.



(a) Determinacy (shaded) regions in (p_{11},p_{22})-space

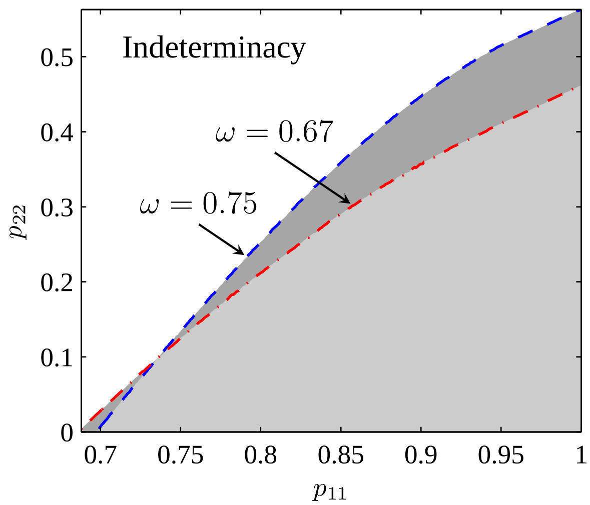
(b) Expected frequency/average duration of ZLB events tradeoff

Figure 3: Determinacy regions across alternative monetary policy responses to changes in output.

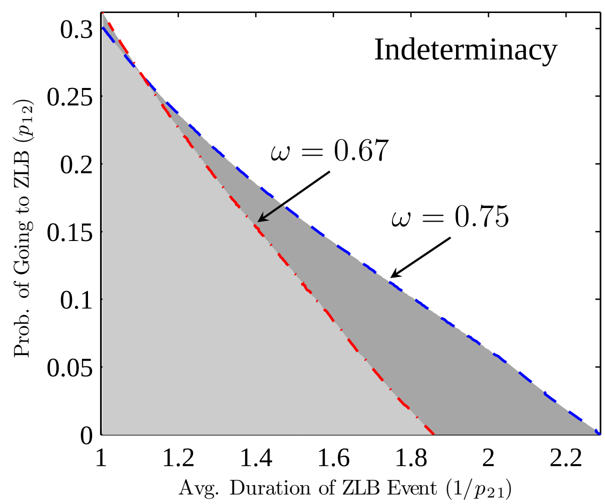
These results show that stochastic processes commonly embedded in dynamic models cannot generate *average* ZLB events that are consistent with observed ZLB events, which is similar to the points made in Chung et al. (2012) and Fernández-Villaverde et al. (2012); however, it is possible for longer ZLB events to occur and still deliver a determinate equilibrium, because the household places little weight on these outcomes in their expectations. For example, when $\phi_\pi = 1.5$, $p_{11} = 0.95$, and $p_{22} = 0.5$, the average ZLB event is only 2 quarters, but the maximum ZLB event in a $500{,}000$ quarter simulation is 15 quarters, which is closer to ZLB events observed in the data.

The determinacy region also critically depends on how strongly the monetary authority responds to inflation when the ZLB does not bind. The darker shaded regions represent the additional area of the parameter space that delivers a determinate solution when ϕ_π increases. If the monetary authority responds more aggressively to inflation when $s_t = 1$ (i.e., a higher ϕ_π) and $p_{11} < 1$, the determinacy region widens, since greater price stability when $s_t = 1$ helps offset the destabilizing influence of $s_t = 2$. This means the determinacy region permits longer and/or more frequent trips to the ZLB. However, it is interesting that regardless of the value of ϕ_π, the longest average ZLB event inside the determinacy region is the same (2.3 quarters). As p_{11} rises, the expected frequency of ZLB events declines. This implies that $s_t = 2$ has a decreasing effect on $s_t = 1$ and the stabilizing effect of additional price stability in $s_t = 1$ has a smaller effect on overall price stability. Thus, the additional area of the parameter space that delivers determinacy shrinks as p_{11} increases. When $p_{11} = 1$, any ZLB event is completely unexpected by the household. This means that $s_t = 2$ has no effect on the decision rules in $s_t = 1$ and increases in ϕ_π do not widen the determinacy region. In short, as $p_{11} \to 1$, the model approaches a fixed-regime setup where increases in ϕ_π beyond a minimum threshold have no effect on the determinacy region of the parameter space.

When the monetary authority adjusts the nominal interest rate to deviations of output from its steady state ($\phi_y > 0$), it also affects the determinacy region. Figure 3 plots these regions for $\phi_y \in \{0, 0.1, 0.2\}$. Since changes in the monetary policy state represent demand-side shocks, countercyclical monetary policy provides additional price stability in state 1. However, it is destabilizing in state 2, because the household expects a relatively higher future nominal interest rate



(a) Determinacy (shaded) regions in (p_{11},p_{22})-space



(b) Expected frequency/average duration of ZLB events tradeoff

Figure 4: Determinacy regions across alternative degrees of price stickiness.

than when $\phi_y = 0$, which increases inflation at the ZLB. The strengths of these two competing effects change along the edge of the determinacy region. When p_{11} is not too high, state 2 has a larger effect on state 1. This means the maximum value of p_{22} and the average duration of ZLB events is low. Thus, the stabilizing effect of state 1 dominates and the determinacy region expands. As p_{11} increases, the effect of state 2 on state 1 declines and the additional area of the determinacy region (i.e., the darker shaded region) shrinks. At high values of p_{11}, which permit high values of p_{22}, the destabilizing effect of state 2 dominates and the overall determinacy region shrinks with ϕ_y.[6] When $p_{11} = 1$, state 2 has no effect on state 1 and ϕ_y has no impact on the determinacy region.

A common theme in the determinacy regions we have shown thus far is that the degree of price stickiness plays a key role. Figure 4 shows the determinacy regions for $\omega \in \{0.67, 0.75\}$. With a lower degree of price stickiness (i.e., a lower ω), firms have a greater ability to adjust prices with the monetary policy state. When the average duration of ZLB events is high (i.e., a high p_{22}), lower price stickiness shrinks the determinacy region, because expected prices are less anchored by the Taylor rule in state 1. For example, if ω declines from 0.75 to 0.67, the maximum average duration of ZLB events declines from 2.3 quarters to 1.85 quarters. As p_{11} and p_{22} fall, the household expects to visit the ZLB more often but for fewer quarters on average. This means the benefit of additional price stability declines and the determinacy regions shrink. Once again, the determinacy regions twist. While the determinacy region generally shrinks for lower values of ω, at low enough values of p_{11} the region expands. This is because it is less costly for firms to adjust prices consistent with state 2 when the expected duration of staying in state 2 is very short.

The other deep parameters in the model (e.g., σ, η, $\bar{\beta}$) also affect the size of the determinacy region. When the degree of risk aversion, σ, is higher, the household is less willing to intertemporally substitute consumption goods. When the Frisch elasticity of labor supply, $1/\eta$, is larger, the household's willingness to supply labor is more sensitive to changes in the real wage rate. Both of these effects make hours worked, consumption, and the inflation rate less volatile when the ZLB binds, which expands the determinacy region. When the household is more patient (i.e., a

[6]Higher values of ϕ_π than are shown in figure 2 also cause the determinacy region to twist for high values of p_{11}.

higher $\bar{\beta}$), the steady state nominal interest rate is lower, which reduces the demand-side effects of switching states. This makes inflation less volatile and also expands the determinacy region. Many models also include a smoothing component in the monetary policy rule. An increase in this parameter shrinks the determinacy region because it reduces the response to the fundamentals.

5 ENDOGENOUS ZLB EVENTS: EXOGENOUS SHOCKS

This section replaces the exogenous Markov-switching process, given in (8), with a either an AR(1) technology or discount factor process that determines the frequency and duration of ZLB events.[7] We remove the monetary policy shock, so the gross nominal interest rate is set according to

$$r_t = \max\{1, \bar{r}(\pi_t/\bar{\pi})^{\phi_\pi}\}. \tag{9}$$

Section 4 makes clear that when episodes at the ZLB are exogenous, the boundary of the determinacy region imposes a clear tradeoff between the expected frequency and average duration of ZLB events. This same tradeoff exists when ZLB events are endogenous. We discretize the state (the value of technology, a_{t-1}, or the discount factor, β_{t-1}) into N elements such that $z_{t-1} \in \{z^1, \ldots, z^N\}$. Let $s_t \in \{1, 2\}$ indicate that the ZLB is either not binding or binding, respectively. Let n denote the index corresponding to the minimum value of the state variable where the ZLB binds, which partitions the state-space into two subsets. Denote the corresponding sets of indices as $\mathcal{I}_{1,t-1} = \{1, \ldots, n-1\}$ and $\mathcal{I}_{2,t-1} = \{n, \ldots, N\}$. The probability of going to the ZLB (the analog of $p_{12} = 1 - p_{11}$ in the transition matrix defined in section 4) is given by

$$\Pr\{s_t = 2 | s_{t-1} = 1\} = \frac{\sum_{i \in \mathcal{I}_{1,t-1}} \Pr\{s_t = 2 | z_{t-1} = z^i\} \phi(z^i | \bar{z}, \sigma_z)}{\sum_{i \in \mathcal{I}_{1,t-1}} \phi(z^i | \bar{z}, \sigma_z)},$$

where

$$\Pr\{s_t = 2 | z_{t-1} = z^i\} = \frac{\sum_{j \in \mathcal{I}_{2,t}} \phi(\varepsilon_j | 0, \sigma_\varepsilon)}{\sum_{j \in \mathcal{I}_{1,t} \cup \mathcal{I}_{2,t}} \phi(\varepsilon_j | 0, \sigma_\varepsilon)}, \tag{10}$$

$\phi(x|\mu, \sigma)$ is the normal probability density function, given mean μ and standard deviation σ. For each z_{t-1}, there is a vector of realizations of z_t, where each realization corresponds to a Gauss-Hermite quadrature node, ε_j (the roots of the Hermite polynomial).

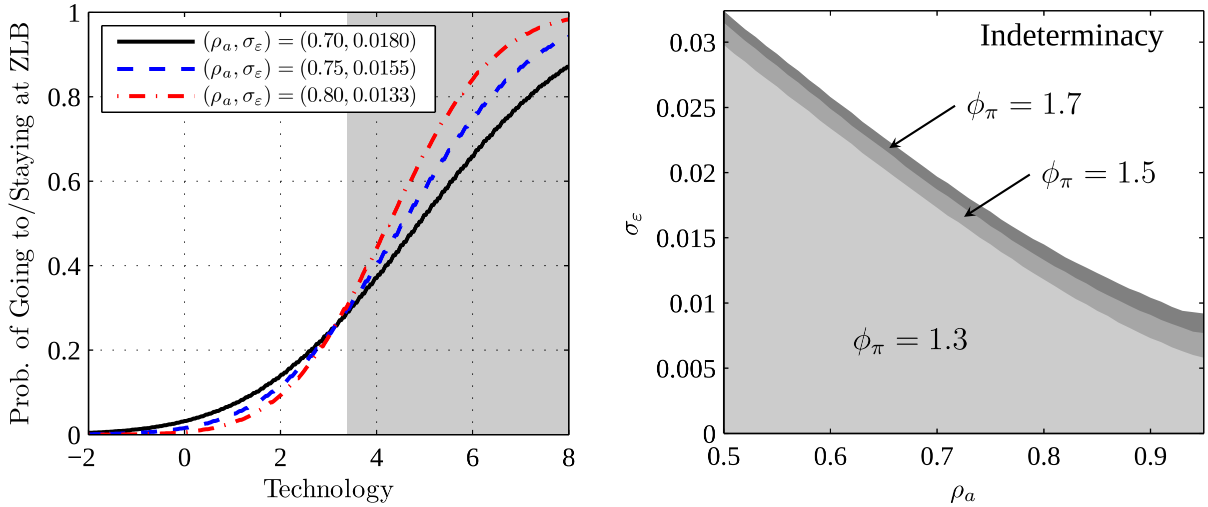
5.1 TECHNOLOGY SHOCKS

In this section, technology evolves according to

$$a_t = \bar{a}(a_{t-1}/\bar{a})^{\rho_a} \exp(\varepsilon_t), \tag{11}$$

where $0 \leq \rho_a < 1$ and $\varepsilon_t \sim \mathbb{N}(0, \sigma_\varepsilon^2)$. The discount factor is constant ($\beta_t = \bar{\beta}$ for all t). We define $\sigma_a = \sigma_\varepsilon/(1-\rho_a^2)^{1/2}$ as the standard deviation of (11). Positive technology shocks act as positive aggregate supply shocks. At high technology levels, firms' per unit marginal cost of production is low. Firms react by lowering their prices and raising their production. This causes deflation and, given a sufficiently high level of technology, the (net) nominal interest rate falls to zero according to the Taylor rule in (9). Thus, ZLB events are endogenous due to technology shocks.

[7] For a complete picture of the solution to New Keynesian models with and without capital see Gavin et al. (2013).



(a) Probabilities of going to the ZLB, given the technology state and $\phi_\pi = 1.5$. Technology is in percent deviations from steady state. The shaded region corresponds to the states where the ZLB binds.

(b) Tradeoff between the technology process persistence and shock standard deviation across alternative monetary policy responses to inflation. The shaded regions correspond to the determinacy regions.

Figure 5: Properties of the model where ZLB events arise endogenously due to technology shocks.

Figure 5a plots (10) as a function of the technology state for three alternative parameterizations of (11). The shaded region corresponds to technology states where the ZLB binds, which begins when technology is 3.5 percent above its steady-state value. The three combinations of $(\rho_a, \sigma_\varepsilon)$ are chosen to keep the boundary of the ZLB region unchanged. In technology states below the boundary, the probability on the vertical axis is the probability of going to the ZLB in the next quarter. In technology states above the boundary, it is the probability of staying at the ZLB. This figure demonstrates the tradeoff between the probability of hitting the ZLB and the average duration of ZLB events. As ρ_a increases and σ_ε decreases, it is less likely the ZLB will bind in technology states below the boundary and more likely the ZLB will continue to bind once the ZLB is hit.

The combinations of $(\rho_a, \sigma_\varepsilon)$ shown in figure 5a are *not* on the boundary of the determinacy region in $(\rho_a, \sigma_\varepsilon)$-space. The boundary of the ZLB region is a function of $(\rho_a, \sigma_\varepsilon)$, which affects the probabilities of going to and staying at the ZLB. Since ZLB events are endogenous due to (11), there is no way to map $(\rho_a, \sigma_\varepsilon)$ into equivalent (p_{11}, p_{22}) values and generate a picture equivalent to figure 2 (i.e., we cannot increase p_{22} by changing $(\rho_a, \sigma_\varepsilon)$ without altering p_{11}). Thus, fixing the boundary of the ZLB region offers the closest comparison to the Markov chain process in section 4.

Figure 5b shows that along the boundary of the determinacy (shaded) region, there is a clear tradeoff between the persistence of the technology process, ρ_a, and the standard deviation of the shock, σ_ε. As the persistence of the process increases, the standard deviation of the shock must decline to avoid an indeterminacy region. This tradeoff reflects that ρ_a and σ_ε both impact the expected frequency and average duration of ZLB events, as figure 5a shows. Once again, the monetary policy response to inflation, ϕ_π, affects the size of the determinacy region. For a given ρ_a, an increase in ϕ_π permits a larger σ_ε, as prices are more stable when the ZLB does not bind.

The fact that the parameters of the stochastic process impact the determinacy region is significant, because these parameters do not affect determinacy in linearized models, regardless of whether the ZLB is imposed. In models that impose a ZLB, it is common to linearize every equation in the equilibrium system, except for the Taylor rule, and assume ZLB events last for a

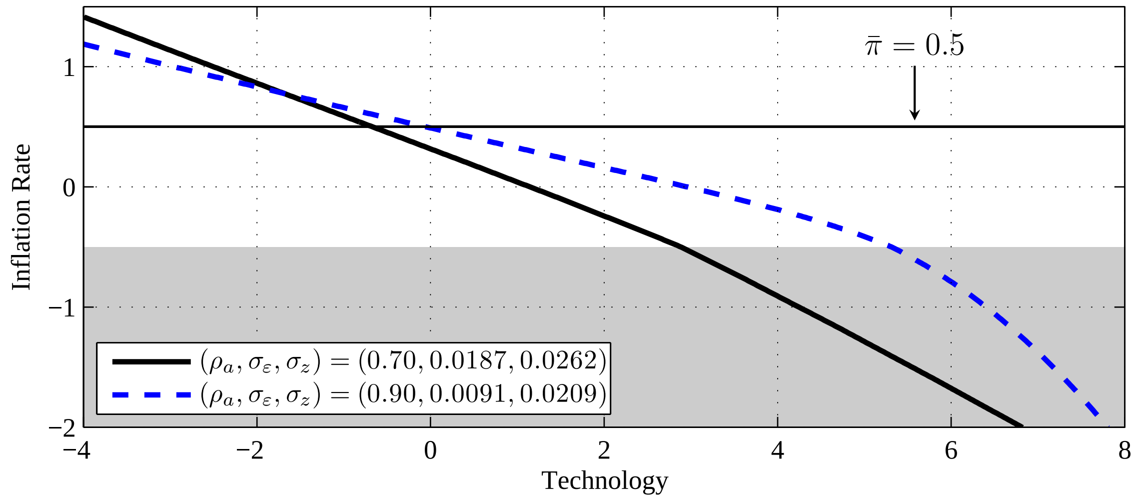


Figure 6: Comparison of the decision rules for $(\rho_a, \sigma_\varepsilon)$ combinations on the boundary of the determinacy region ($\phi_\pi = 1.5$). Technology is in percent deviations from steady state. The solid horizontal line represents the steady-state gross inflation rate. The shaded region corresponds to the technology states where the ZLB binds.

predetermined duration with no probability of recurrence. This approach does not account for the expectational effects of going to and exiting the ZLB, which are critical for determinacy.

Figure 6 compares the inflation rate decision rules across two parameterizations of (11), both of which are on the boundary of the determinacy region in $(\rho_a, \sigma_\varepsilon)$-space. The horizontal dashed line is the steady-state inflation rate ($\bar{\pi} = 1.005$). When the technology state equals the steady-state technology level ($\bar{a} = 1$), the deviations of the inflation rate from its steady-state value provide a measure of the expectational effect of hitting the ZLB. The shaded region represents values of the inflation rate where the ZLB binds. When σ_ε is relatively small (dashed line), the expectational effect is small because the likelihood of hitting the ZLB in expectation is also small. As σ_ε increases, and σ_a increases with it, the expectational effect of hitting the ZLB also increases.

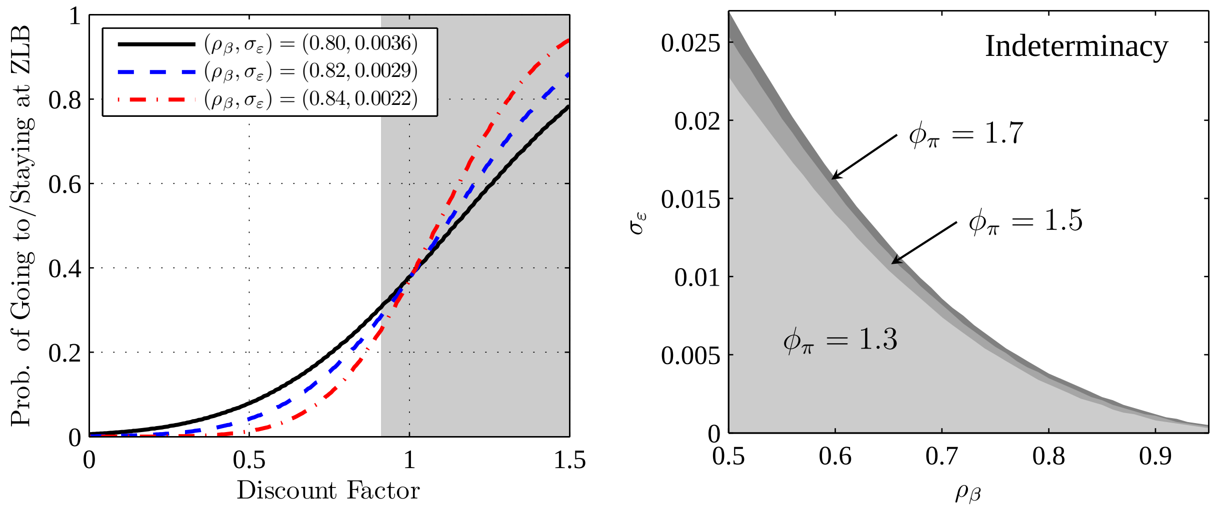
When the ZLB binds, higher real interest rates reduce consumption and put downward pressure on inflation as firms respond to the lower demand. Thus, when there is a higher probability of going to the ZLB (solid line), the slope of the inflation rate policy function is steeper. Since the downward pressure on inflation happens across the entire state space, it also influences where the ZLB first binds in the state space. For smaller standard deviations of (11), the probability of hitting the ZLB in expectation is smaller and the boundary of the ZLB region lies at a higher technology state. Unlike log-linearized models, where the calibration of the stochastic process has a much smaller effect on the decision rules, these results imply that changes in the calibration of the stochastic process can significantly impact the quantitative properties of the model.

5.2 DISCOUNT FACTOR SHOCKS

In this section, the discount factor evolves according to

$$\beta_t = \bar{\beta}(\beta_{t-1}/\bar{\beta})^{\rho_\beta} \exp(\varepsilon_t), \tag{12}$$

where $0 \leq \rho_\beta < 1$ and $\varepsilon_t \sim \mathbb{N}(0, \sigma_\varepsilon^2)$. Technology is constant ($a_t = \bar{a}$ for all t). We define $\sigma_\beta = \sigma_\varepsilon/(1 - \rho_\beta^2)^{1/2}$ as the standard deviation of (12). Positive discount factor shocks act as negative aggregate demand shocks. A high discount factor means that the household is more patient and elects to defer consumption to future periods. Firms respond to the lower demand



(a) Probabilities of going to the ZLB, given the discount factor state and $\phi_\pi = 1.5$. The discount factor is in percent deviations from steady state. The shaded region corresponds to the states where the ZLB binds.

(b) Tradeoff between the discount factor process persistence and shock standard deviation across alternative monetary policy responses to inflation. The shaded regions correspond to the determinacy regions.

Figure 7: Properties of the model where ZLB events arise endogenously due to discount factor shocks.

by cutting output and reducing their prices. This causes deflation and, given a sufficiently high discount factor, the (net) nominal interest rate falls to zero according to the Taylor rule in (9). Thus, ZLB events are endogenous due to discount factor shocks.

Figure 7a reproduces figure 5 for three alternative parameterizations of the discount factor process given in (12). The shaded region corresponds to discount factor states where the ZLB binds, which begins when the discount factor is 0.9 percent above its steady-state value. Once again, there is a clear tradeoff between the expected frequency and average duration of ZLB events.

Figure 7b shows the determinacy regions in $(\rho_\beta, \sigma_\varepsilon)$-space. For a given persistence value, the discount factor process permits a much smaller shock size than the technology process. This is because the discount factor directly affects the household's willingness to intertemporally substitute, which is critical for determinacy since it affects expected inflation. As an example, the maximum shock size is only 0.0003 when $\rho_\beta = 0.95$. This is significant because estimates of this parameter using a log-linear model without a ZLB constraint are outside of this region. The data prefers highly persistent shocks (i.e., $\rho_\beta > 0.95$) with a standard deviation that is over four times the maximum value inside the determinacy region. While the model is slightly different, the estimates of the constrained nonlinear model in Gust et al. (2013) are also well outside of the determinacy region. They estimate that $\rho_\beta = 0.88$ and $\sigma_\varepsilon = 0.0025$. The data also prefers highly persistent technology shocks, but it does not pose as serious a problem for estimation because the constrained model permits large shocks. When $\rho_a = 0.95$ and $\phi_\pi = 1.5$, the maximum shock size 0.75 percent.

Figure 8 plots the decision rules for inflation. The slope is steeper (more negative) when the discount factor is more persistent. In discount factor states where the ZLB does (does not) bind, inflation is lower (higher). At the ZLB, higher persistence means the household expects relatively higher consumption growth. Since the nominal interest rate does not respond to inflation, the only way for the real interest rate to rise and for the bond market to clear is if inflation falls sharply. The expectational effect of the ZLB in discount factor states where the ZLB does not bind also drives down inflation. In states further from ZLB region, the expectational effect is weaker and the

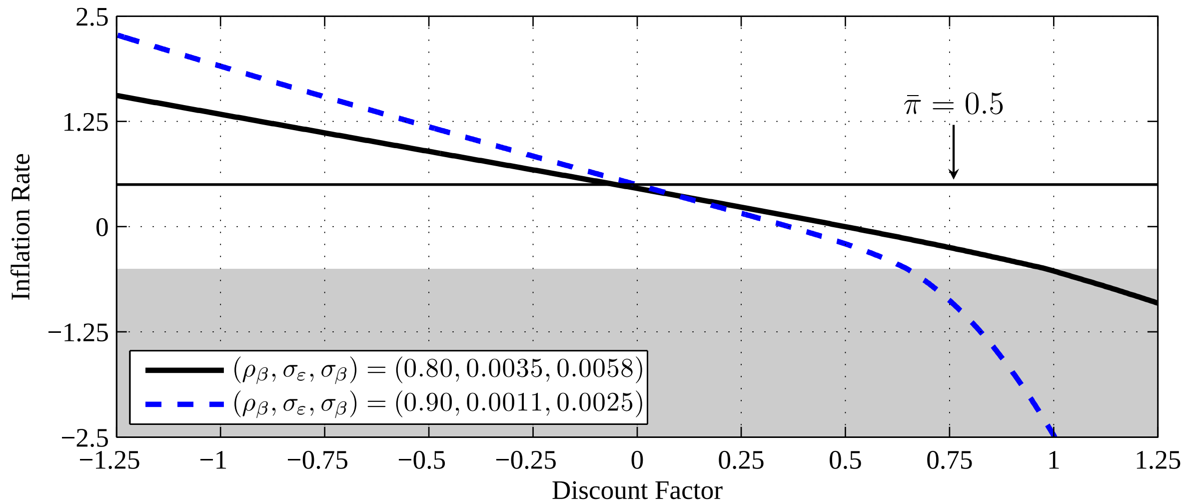


Figure 8: Comparison of the decision rules for $(\rho_\beta, \sigma_\varepsilon)$ combinations on the boundary of the determinacy region ($\phi_\pi = 1.5$). The discount factor is in percent deviations from steady state. The solid horizontal line represents the steady-state gross inflation rate. The shaded region corresponds to the discount factor states where the ZLB binds.

higher demand associated with a lower, more persistent discount factor increases inflation. Once again, these results show that even small changes in the parameterization of the exogenous driving process significantly affect the decision rules, and hence the quantitative properties of the model.

6 CONCLUSION

This paper demonstrates that the boundary of the determinacy region imposes a clear tradeoff between the expected frequency and average duration of episodes at the ZLB, regardless of whether ZLB events arise exogenously or endogenously. This tradeoff is critical for at least three reasons. First, even though the Taylor principle does not hold at the ZLB, it shows that central banks can still pin down prices when the nominal interest rate is pegged at its ZLB, so long as households have a strong enough expectation of returning to a regime where the central bank aggressively responds to inflation. Second, it imposes an important constraint on the parameter space that the econometrician must account for when estimating the fully nonlinear model. Third, it implies that small changes in the parameters of stochastic processes significantly impact the decision rules and the state at which the ZLB first binds. This means accurately calibrating or estimating the parameters of the exogenous driving processes is important for policy analysis at the ZLB.

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A NUMERICAL ALGORITHM

A formal description of the numerical algorithm begins by writing the model compactly as

$$\mathbb{E}[f(\mathbf{v}_{t+1}, \mathbf{w}_{t+1}, \mathbf{v}_t, \mathbf{w}_t)|\Omega_t] = 0,$$

where f is vector-valued function that contains the equilibrium system, $\mathbf{v}$ is a vector of exogenous variables, $\mathbf{w}$ is a vector of endogenous variables, and $\Omega_t = \{M, P, \mathbf{z}_t\}$ is the household's information set in period t, which contains the structural model, M, its parameters, P, and the state vector, $\mathbf{z}$. In the model where ZLB events are exogenous, $\mathbf{v} = \mathbf{z} = (\varepsilon, s)$. When ZLB events are endogenous due to technology shocks $\mathbf{v} = (a, \varepsilon)$ and $\mathbf{z} = a$ and when ZLB events are endogenous due to discount factor shocks $\mathbf{v} = (\beta, \varepsilon)$ and $\mathbf{z} = \beta$. In all models, $\mathbf{w} = (c, \pi, y, n, w, mc, r)$.

Policy function iteration approximates the vector of decision rules, Φ, as a function of the state vector, $\mathbf{z}$. The time-invariant decision rules for the exogenous model are

$$\underbrace{\Phi(\mathbf{z}_t)}_{\text{True RE Solution}} \approx \underbrace{\hat{\Phi}(\mathbf{z}_t)}_{\substack{\text{Approximating}\\ \text{Function}}}.$$

We choose to iterate on $\Phi = (c, \pi)$ so that we can easily solve for future variables that enter the household's expectations using f. Each continuous state variable in $\mathbf{z}$ is discretized into N^d points, where $d \in \{1, \ldots, D\}$ and D is the dimension of the state space. The discretized state space is represented by a set of unique D-dimensional coordinates (nodes). In general, we set the bounds of continuous stochastic state variables to encompass 99.999 percent of the probability mass of the distribution. We specify 101 grid points for each continuous state variable and use the maximum number of Gauss-Hermite weights (66) for each continuous shock. These techniques minimize extrapolation and ensure that the location of the kink in the decision rules is accurate.

The following outline summarizes the policy function algorithm we employ. Let $i \in \{0, \ldots, I\}$ index the iterations of the algorithm and $n \in \{1, \ldots, \Pi_{d=1}^{D} N^d\}$ index the nodes.

1. Obtain initial conjectures for the approximating functions, $\hat{c}_0$ and $\hat{\pi}_0$, on each node, from the log-linear model without the ZLB imposed. We use `gensys.m` to obtain these conjectures.

2. For $i \in \{1, ..., I\}$, implement the following steps:

   (a) On each node, solve for $\{r_t, mc_t\}$ given $\hat{c}_{i-1}(\mathbf{z}_t^n)$ and $\hat{\pi}_{i-1}(\mathbf{z}_t^n)$ with the ZLB imposed.

   (b) Linearly interpolate $\{c_{t+1}, \pi_{t+1}\}$ given $\{\varepsilon_{t+1}^m\}_{m=1}^M$ (exogenous and endogenous ZLB models) and $s_{t+1} \in \{1, 2\}$ (exogenous ZLB model only). Each of the M values ε_{t+1}^m are Gauss-Hermite quadrature nodes. We use Gauss-Hermite quadrature to numerically integrate, since it is very accurate for normally distributed shocks. We use piecewise linear interpolation to approximate future variables that show up in expectation, since this approach more accurately captures the kink in the decision rules than continuous functions such as cubic splines or Chebyshev polynomials.[8]

[8] Aruoba and Schorfheide (2013) use a linear combination of two Chebyshev polynominals—one that captures the dynamics when the ZLB binds and one that captures the dynamics when the Taylor principle holds. While this approach is more accurate than using one Chebyshev polynomial, there is no guarantee that it will accurately locate the kink. Moreover, Chebyshev polynomials can lead to large approximation errors due to extrapolation. With linear interpolation, a dense state space will lead to more predictable extrapolation and more accurately locate the kink.

(c) We use the nonlinear solver, `csolve.m`, to minimize the Euler equation errors. On each node, numerically integrate to approximate the expectation operators,

$$\mathbb{E}\left[f(\mathbf{x}_{t+1}^{k,m}, \mathbf{x}_t^n)|\Omega_t\right] \approx \frac{1}{\sqrt{\pi}} \sum_{j=1}^{2} p_{jk} \sum_{m=1}^{M} f(\mathbf{x}_{t+1}^{k,m}, \mathbf{x}_t^n)\phi(\varepsilon_{t+1}^m), \qquad \text{(Exogenous ZLB Model)}$$

$$\mathbb{E}\left[f(\mathbf{x}_{t+1}^{m}, \mathbf{x}_t^n)|\Omega_t\right] \approx \frac{1}{\sqrt{\pi}} \sum_{m=1}^{M} f(\hat{\mathbf{x}}_{t+1}^{m}, \hat{\mathbf{x}}_t^n)\phi(\varepsilon_{t+1}^m), \qquad \text{(Endogenous ZLB Model)}$$

where $\mathbf{x} \equiv (\mathbf{v}, \mathbf{w})$, ϕ are the respective Gauss-Hermite weights, and $p_{jk} = \Pr(s_{t+1} = k|s_t^n = j)$. The superscripts on $\mathbf{x}$ indicate which realizations of the state variables are used to compute expectations. The nonlinear solver searches for $\hat{c}_i(\mathbf{z}_t^n)$ and $\hat{\pi}_i(\mathbf{z}_t^n)$ so that the Euler equation errors are less than 1^{-4} on each node.

3. Define $\mathrm{maxdist}_i \equiv \max\{|\hat{c}_i - \hat{c}_{i-1}|, |\hat{\pi}_i - \hat{\pi}_{i-1}|\}$. Repeat the steps in item 2 until one of the following conditions is satisfied.

   - If for all n, $\mathrm{maxdist}_i < 1^{-13}$ for 10 consecutive iterations, then the algorithm converged to the unique bounded MSV solution. Since the state is composed of only exogenous variables, the solution is bounded so long as the decisions rules are positive and finite.
   - Otherwise, we say the algorithm is non-convergent for one of the following reasons:
     - $i = I = 500{,}000$ (Algorithm times out)
     - For all n and any i, $\hat{\pi}_i < .5$, or for any n, $\hat{c}_i < 0$ (Approximating functions drift)
     - Define $\mathrm{dir}_i = \mathrm{maxdist}_i - \mathrm{maxdist}_{i-1}$. For all n, $\mathrm{dir}_i \geq 0$ and $\mathrm{dir}_i - \mathrm{dir}_{i-1} \geq 0$ for 100 consecutive iterations (Algorithm diverges)

To provide evidence that the solution is unique, we randomly perturb the converged decision rules and check that the algorithm converges back to the same solution.