MMDSSP Final Project - Convolutional Wasserstein Distances: Efficient Optimal Transportation on Geometric Domains

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Abstract

This paper explores the application of Convolutional Wasserstein Distances for color transfer in images. Although optimal transport is effective for distribution alignment, its high computational cost limits practical use. Using entropic regularization and heat-kernel approximations, Solomon et al. provide an efficient solution. We extend their approach by optimizing histogram binning and introducing a K-Nearest Neighbors (KNN) post-processing step, enhancing both computational efficiency and detail preservation. The method ensures smooth and accurate color transitions while maintaining scalability.

1 Introduction

Optimal transport is a powerful tool for color transfer, but is computationally expensive. Solomon et al. introduced Convolutional Wasserstein distances, using entropic regularization and heat-kernel approximations to improve scalability [2]. We implement this approach for efficient color transfer, leveraging Wasserstein barycenters to interpolate distributions while preserving spatial coherence. To further optimize efficiency, we reduce the complexity of the histogram from $O(n \log^2 n)$ to $O(n' \log^2 n')$ (where $n' \ll n$) and introduce a post-processing step K-Nearest Neighbors (KNN) for the preservation of detail. Our method achieves high-quality results with significantly reduced computation.

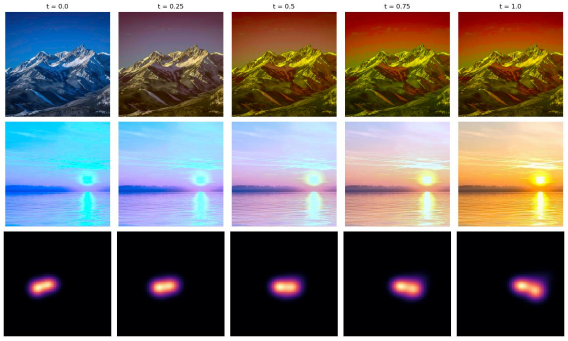

Figure 1. Color transfer interpolation between two images using the Wasserstein distance technique. The first two rows show the transferred images for different weight parameters t, where $t = 0$ corresponds to the first image and $t = 1$ to the second. The third row presents the corresponding 2D histograms in the color space, illustrating the gradual transition in color distributions.

2 Preliminaries

2.1 Mathematical Framework

A **Riemannian manifold** defines distances and angles. Here, M represents a surface or

image plane that is **compact** (finite), **connected** (any two points are path-connected) and has **volume of units** (normalized for simplicity).

The **geodesic distance function** $d(x, y)$ gives the shortest path between the points x and y on M. To describe **probability distributions**, we define:

- $Prob(M)$: Space of probability measures, where each μ_i represents a mass distribution.
- $Prob(M \times M)$: Space of joint probability measures, where each π_i represents a transport plan moving mass between distributions.

2.2 Optimal Transport and Wasserstein Distance

The goal of optimal transportation is to transform a source marginal μ_0 into a target marginal μ_1 using a transportation plan π that describes the amount of mass moving from each point x to y. The mass conservation constraint ensures that all mass is correctly redistributed:

$$\prod(\mu_0, \mu_1) = \Big\{ \pi \in \text{Prob}(M \times M) \mid \pi(\cdot, M) = \mu_0, \quad \pi(M, \cdot) = \mu_1 \Big\} \tag{1}$$

where $\pi(\cdot, M)$ and $\pi(M, \cdot)$ indicate the marginal distributions of π. To quantify how "costly" it is to transport μ_0 to μ_1, we define the **2-Wasserstein distance as:**

$$W_2(\mu_0, \mu_1) = \sqrt{\inf_{\pi \in \prod(\mu_0, \mu_1)} \int_{M \times M} d(x, y)^2 d\pi(x, y)} \tag{2}$$

This measures the minimum effort required to transform one probability distribution into another considering the squared geodesic distances $d^2(x, y)$.

2.3 Wasserstein Barycenters

A Wasserstein barycenter computes the geometric mean of multiple probability distributions under the Wasserstein metric. Given distributions $\mu_1, ..., \mu_k$ with weights $\alpha_1, ..., \alpha_k$, the barycenter minimizes:

$$\mu^* = \arg\min_{\mu} \sum_{i=1}^{k} \alpha_i W_2^2(\mu, \mu_i) \tag{3}$$

This approach is crucial for color transfer, ensuring a smooth interpolation between color distributions.

2.4 Entropy-Regularized Optimal Transport

To improve efficiency, an entropy term $H(\pi)$ is added to the transport problem:

$$W_{2,\gamma}^2(\mu_0, \mu_1) = \inf_{\pi \in \Pi} \left\{ \int d(x, y)^2 \pi(x, y) dx dy \quad -\gamma H(\pi) \right\} \tag{4}$$

where $\gamma > 0$ controls smoothness. Using the heat kernel $H_t(x, y)$, geodesic distances are approximated as:

$$d(x, y)^2 \approx -2t \ln H_t(x, y) \tag{5}$$

This enables fast computation via diffusion processes instead of solving large transport matrices.

2.5 Entropy Sharpening in Optimal Transport

Entropy regularization improves efficiency but can produce overly diffuse Wasserstein barycenters. To counteract this, entropy sharpening enforces an entropy constraint $H(\mu) \leq H_0$, where H_0 is typically the maximum entropy of the input distributions $\mu_1, \ldots, \mu_k$. This preserves sharp features and geometric structures. Computationally, entropy sharpening is applied via iterative projection of entropy, ensuring that the barycenter meets the constraint while minimizing the transport cost.

2.6 Computational Advantages of the Convolutional Wasserstein Distance

The Convolutional Wasserstein Distance (CWD) framework computes optimal transport using iterative convolution with diffusion kernels instead of explicitly solving the transport problem. Instead of solving for the full $n \times n$ transport matrix, we exploit: $\pi = D_v H_t D_w$ where $v, w \in \mathbb{R}^n$ satisfy $D_v H_t D_w a = \mu_0$ and $D_w H_t D_v a = \mu_1$ (where a is an area weight vector). This reduces the number of unknowns from n^2 to $2n$, significantly improving scalability.

3 Implementation - Color Transfer

We apply the CWD for color transfer by computing an optimal transport map between two color histograms.

Steps: 1. Compute normalized histograms μ_0 and μ_1 from source and target images. 2. Discretize the color space and compute the heat kernel. 3. Solve the entropy-regularized transport problem using Sinkhorn iterations:

$$v \leftarrow \mu_0 \oslash K_\gamma(a \otimes w), \quad w \leftarrow \mu_1 \oslash K_\gamma(a \otimes v) \quad (6)$$

4. Extract the transport plan π and apply it to the source image. This method efficiently transfers color while preserving spatial structure. See Figure 1 for results.

4 Efficient Color Transfer: Balance of quality and complexity

In color transfer, the number of bins n significantly affects both computational complexity and visual quality. Reducing the number of bins too aggressively leads to a loss of fine color details, causing visible artifacts and less accurate color distributions. This trade-off arises because fewer bins provide a coarser representation of the color space, limiting the precision of the color mapping and introducing undesirable distortions. To optimize efficiency while preserving high-quality results, we aim to reduce the number of bins while maintaining the accuracy and detail achievable with a higher bin count. The original Wasserstein distance computation scales as $O(n^2)$, but entropic regularization and CWD reduce it to $O(n)$ per iteration. The Wasserstein barycenter computation further involves iterating over the entropic transport problem, yielding a final complexity of $O(n \log^2 n)$. This complexity results from $O(\log n)$ heat kernel convolutions, which efficiently approximate transport costs, and $O(\log n)$ Sinkhorn iterations, which refine the transport plan. To improve efficiency, we reduce the bins from n to n' (where $n' \ll n$), lowering computational complexity to $O(n' \log^2 n')$. However, reducing bins too much degrades the quality of the color trans-

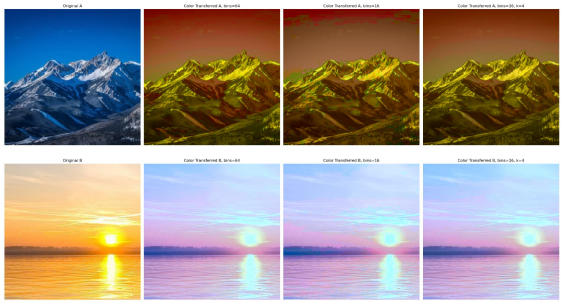

Figure 2. Effect of reducing the number of bins and introducing K-Nearest Neighbors (KNN) in color transfer using the Wasserstein distance. The first row shows results for Image A, while the second row shows results for Image B. From left to right: the original image, color transfer with $n = 64$ bins, color transfer with $n' = 16$ bins, and color transfer with $n' = 16$ bins with $k = 4$ using KNN. Reducing the number of bins significantly lowers computational complexity from $O(n \log^2 n)$ to $O(n' \log^2 n')$, at the cost of loss of detail. The introduction of KNN mitigates these artifacts while preserving the speed-up.

fer, leading to loss of detail and unwanted artifacts. To counteract this, we incorporate **K-Nearest Neighbors (KNN)**, which introduces an additional $O(n'k)$ term. Using KNN, we effectively restore finer color variations, achieving results comparable to those obtained using bins n without KNN. Since k remains small, the overall complexity is still dominated by $O(n' \log^2 n')$, ensuring a significant speed-up while preserving high-quality color transfer. See Figure 2 for results.

5 Further Research Proposal

Regularized convolutional Wasserstein transport provides computational scalability but at the cost of smoothing effects [2], which can compromise edge preservation in imaging applications. A small entropic regularization parameter γ improves feature sharpness but risks numerical instability. To mitigate this trade-off, we propose a research direction focused on developing a spatially adaptive regularization framework where $\gamma(x)$ dynamically adjusts based on local feature importance.

A promising approach is to define a feature importance function $F(x)$ and modulate transport cost as follows:

$$d^f(x, y) = d(x, y)(1 + \lambda |F(x) - F(y)|), \quad (7)$$

which discourages transport across sharp boundaries. Investigating methods for defining $F(x)$ in a robust and computationally efficient manner is a key research challenge. Studies on anisotropic wavelets [3] provide a potential foundation, demonstrating how localized wavelets can preserve both global and local structures. Furthermore, research on data-specific wavelet design [4] suggests that leveraging image-dependent features could enhance the adaptivity of our regularization strategy, guiding the selection of $F(x)$ to capture essential structures while minimizing computational overhead.

Another potential scheme is represented in the transport-diffusion model introduced by Bürkle et al. [1]. Their approach adapts diffusion coefficients based on local structure. Their "one-sided" diffusion mechanism could be extended to spatially adaptive Wasserstein transport, preventing excessive smoothing in critical regions. Additionally, modifying the heat kernel construction to incorporate curvature-adaptive diffusivity could enable structured regularization. Unlike the Gaussian kernel, which naturally arises from the isotropic heat equation, the anisotropic case introduces additional complexity [1]. Investigating whether such a construction can improve transport accuracy without increasing computational cost is a crucial aspect of our research.

In summary, this research proposal aims to advance Wasserstein transport by incorporating adaptive regularization and leveraging insights from anisotropic diffusion, wavelet theory, and transport-diffusion models.

References

[1] Daniel Bürkle, Tobias Preußer and Martin Rumpf. "Transport and Anisotropic Diffusion in Time-Dependent Flow". In: *Journal of Mathematical Imaging and Vision* 64 (2022), pp. 359–382.

[2] Justin Solomon et al. "Convolutional Wasserstein Distances: Efficient Optimal Transportation on Geometric Domains". In: *ACM Transactions on Graphics (TOG)* 34.4 (2015), 66:1–66:11.

[3] Qingzheng Wang. "Anisotropic Localized Wavelets for Image Processing". In: *Pattern Recognition and Image Analysis* 32.2 (2022), pp. 259–284. DOI: 10 . 1134 / S1054661822040149.

[4] Qingzheng Wang et al. "Data-Specific Anisotropic Mexican Hat Wavelets for Structure-Preserving Image Processing". In: *Open Access Journal* (2022).