Electromagnetic Fields

Material for Computer Project

In this text, we will present how to find the charge distribution function on complex geometric bodies, by formulating and solving an integral equation. For a distribution of charges in free-space (without any additional boundary conditions), the electric potential in space is given by:

$$\Phi(x,y,z) = \iiint_V \frac{\rho(x',y',z')}{4\pi\epsilon_0|\boldsymbol{r}-\boldsymbol{r}'|}dx'dy'dz' \qquad \textbf{(1)}$$

This equation can be treated in two different ways:

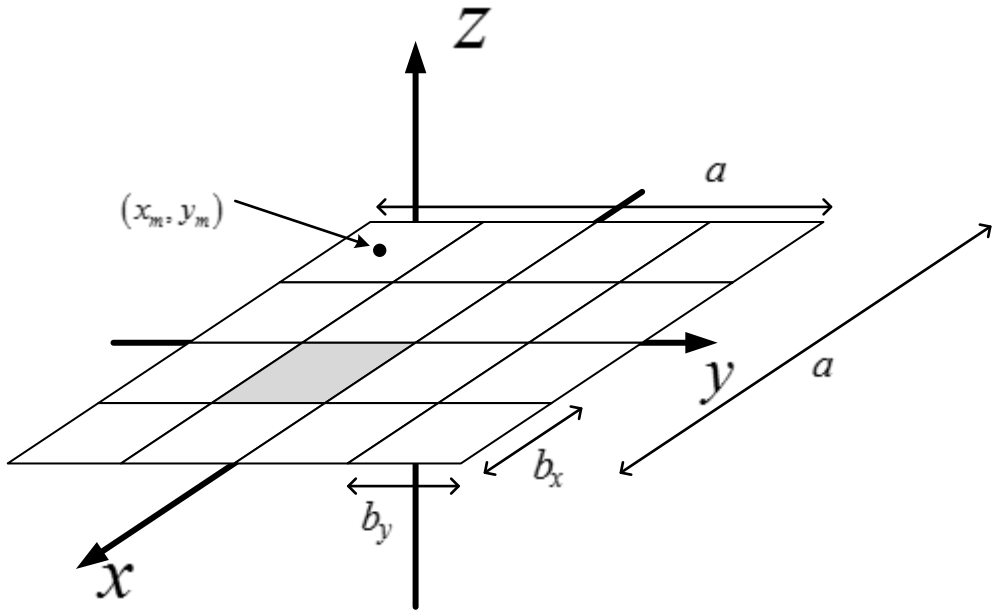
1) If the charge distribution is given, it may be used to calculate the potential in the entire space.

2) If the potential in different points in space is given, it may be used to calculate the charge distribution.

We will focus on the second point. Since we want to calculate the charge distribution, and this function is integrated in the equation, this equation is called an **integral equation**.

Through an example, we will understand how to solve such equations numerically, under certain approximations.

A plate with dimensions $a \times a$, carrying surface charge density $\sigma(x,y)$, is placed on the xy plane.



The potential everywhere in space can be calculated using the integral:

$$\Phi(x,y,z) = \int_{-a/2}^{a/2} dx' \int_{-a/2}^{a/2} dy' \frac{\sigma(x',y')}{4\pi\epsilon_0|\boldsymbol{r}-\boldsymbol{r}'|} \quad (2)$$

Where in this example:

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} \ , \ r' = x'\hat{\boldsymbol{x}} + y'\hat{\boldsymbol{y}} \quad (3)$$

And therefore:

$$|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + z^2} \quad (4)$$

Now let's assume that the plate is connected to a voltage source, that creates a constant potential V on the plate. This condition allows us to write integral equation for the potential on the plate, where the unknown function in this equation is the surface charge distribution σ.

$$\Phi(x,y,0) = \int_{-a/2}^{a/2} dx' \int_{-a/2}^{-a/2} dy' \frac{\sigma(x',y')}{\sqrt{(x-x')^2 + (y-y')^2}} = V \ ; |x|,|y| < \frac{a}{2} \quad (5)$$

We shall divide the plate into N rectangular elements, with the dimensions given in the illustration above. The elements may be square, but not necessarily. In addition, we will assume that the centers of the elements are denoted by the series:

$$(x_m, y_m) \ , \ 1 \le m \le N \quad (6)$$

Now we will assume, that the charge distribution function σ is constant on each element, meaning, it changes only when we go from one element to another. Now we can express the potential at a certain point by:

$$\Phi(x,y,0) \approx \sum_{n=1}^{N} l_n(x,y)\sigma_n \quad (7)$$

Where σ_n is the value of the distribution function on the $n'th$ element. $l_n(x,y)$ is a function that represents the geometry of the element. For example, for a rectangular element with dimensions $b_x \times b_y$, centered at the origin, and the charge distribution on it is constant, the potential contributed by this element at some point (x,y) is:

$$\Phi_{element}(x,y) = \int_{-b_x/2}^{b_x/2} dx' \int_{-b_y/2}^{b_y/2} dy' \frac{\sigma_n}{4\pi\varepsilon_0} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2}} =$$
$$= \left(\int_{-b_x/2}^{b_x/2} dx' \int_{-b_y/2}^{b_y/2} dy' \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2}} \right) \sigma_n \quad (8)$$

Now it is clear that the function in parentheses is $l(x,y)$. To calculate the contribution of an element that is not centered at the origin, we will simply adjust the limits of integration to the location of the element.

Now, assuming that the potential on each element is represented by calculating its value in the center of the element, the integral equation (**8**) will be:

$$\Phi_{element}\left(x_m, y_m\right) = V = \\ = \sum_{n=1}^{N}\left(\int_{x_n - b_x/2}^{x_n + b_x/2} dx' \int_{y_n - b_y/2}^{y_n + b_y/2} dy' \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{\left(x_m - x'\right)^2 + \left(y_m - y'\right)^2}} \right)\sigma_n \qquad (9)$$

Here we have changed the limits of integration, to match an element that is not centered at the origin.

Formulating equation (**9**) a bit differently:

$$\begin{cases} \sum_{n=1}^{N} l_{mn}\sigma_n = V \\ l_{mn} = \left(\int_{x_n - b_x/2}^{x_n + b_x/2} dx' \int_{y_n - b_y/2}^{y_n + b_y/2} dy' \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{\left(x_m - x'\right)^2 + \left(y_m - y'\right)^2}} \right) \end{cases} \qquad (10)$$

We note that for N elements we may formulate N such equations, so that these equations have a form of a matrix equation, of dimension $N \times N$. The unknowns in this equation are the values of the charge distribution function on the elements - σ_n.

In a matrix for, this equation will be written:

$$[l][\sigma] = \begin{bmatrix} V \\ \vdots \\ V \end{bmatrix} \qquad (11)$$

Now we should calculate the elements in matrix $[l_{mn}]$, substitute them in equation (10), and the solution will be given by:

$$[\sigma] = [l]^{-1} \begin{bmatrix} V \\ \vdots \\ V \end{bmatrix} \qquad (12)$$

Calculating the elements of l_{mn}

The off-diagonal elements of matrix l, is the potential contributed by element n in the center of element m. The diagonal elements define the contribution of each element to the potential in its own center.

To simplify the calculation, we shall assume that when the mutual contribution of two different elements is considered, the charge element can be approximated by a point charge, thus allowing us to write:

$$\Phi_{n-element}\left(x_m, y_m\right) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\overbrace{b_x b_y \sigma_n}^{Q_n}}{\sqrt{\left(x_n - x_m\right)^2 + \left(y_n - y_m\right)^2}} \qquad (13)$$

Then the off-diagonal elements will be:

$$l_{mn} \approx \frac{1}{4\pi\varepsilon_0} \cdot \frac{b_x b_y}{\sqrt{\left(x_n - x_m\right)^2 + \left(y_n - y_m\right)^2}} \qquad (14)$$

The condition for the validity of this approximation is that the distance between elements will not be small compared to the dimensions of the element itself. Therefore, it is not recommended to choose "elongated" elements when we use this approximation, but square or nearly-square elements. When square elements are considered, the approximation in equations (**13**) and (**14**) adds an error of 3.8% for adjacent elements, compared to full calculation of the integral in equation (**10**). This error decreases as the elements are further apart.

Now we shall calculate the elements on the diagonal, that describe the contribution of the element to the potential on the element itself. Here we perform the integration, to obtain a more accurate result.

For the elements of the diagonal we shall assume, without loss of generality, that every element is centered at the origin, then calculating:

$$l_{nn} = \int_{-b_x/2}^{b_x/2} dx' \int_{-b_y/2}^{b_y/2} dy' \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{x'^2 + y'^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{-b_x/2}^{b_x/2} dx' \left[\ln\left(y' + \sqrt{x'^2 + y'^2} \right) \right]_{-b_y/2}^{b_y/2}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{-b_x/2}^{b_x/2} dx' \ln\left(\frac{\frac{b_y}{2} + \sqrt{x'^2 + \frac{b_y^2}{4}}}{-\frac{b_y}{2} + \sqrt{x'^2 + \frac{b_y^2}{4}}} \right)$$

$$= \frac{1}{4\pi\varepsilon_0} \left[y' \ln\left(\frac{\frac{b_y}{2} + \sqrt{x'^2 + \frac{b_y^2}{4}}}{-\frac{b_y}{2} + \sqrt{x'^2 + \frac{b_y^2}{4}}} \right) + b_y \ln\left(x' + \sqrt{x'^2 + \frac{b_y^2}{4}} \right) \right]_{-b_x/2}^{b_x/2} \quad (15)$$

$$= \frac{1}{4\pi\varepsilon_0} b_x \ln\left[\frac{\sqrt{b_x^2 + b_y^2} + b_y}{\sqrt{b_x^2 + b_y^2} - b_y} \right] + \frac{1}{4\pi\varepsilon_0} b_y \ln\left[\frac{\sqrt{b_x^2 + b_y^2} + b_x}{\sqrt{b_x^2 + b_y^2} - b_x} \right]$$

The integrals required for this calculation – in the appendix.

Defining $\alpha = \frac{b_y}{b_x}$ – we obtain a more compact way to represent this relation:

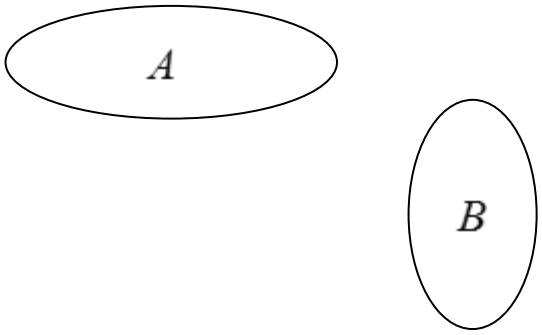
$$l_{nn} = \frac{1}{4\pi\varepsilon_0} b_x \ln\left[\frac{\sqrt{\alpha^2 + 1} + \alpha}{\sqrt{\alpha^2 + 1} - \alpha} \right] + \frac{1}{4\pi\varepsilon_0} b_y \ln\left[\frac{\sqrt{\alpha^2 + 1} + 1}{\sqrt{\alpha^2 + 1} - 1} \right] \quad (16)$$

And in the case that the elements are square - $b_x = b_y = b$:

$$l_{nn} = \frac{1}{2\pi\varepsilon_0} b \ln\left[\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right] = \frac{b}{\pi\varepsilon_0} \cdot 0.8814 \quad (17)$$

Problems with multiple Bodies

If more that one conductor is given, and for each one of them a boundary condition for the potential is given, the method is the same.



We formulate the matrix $[l_{mn}]$, where we note that now the matrix has terms that describe the interaction between a body and itself (elements that relate different elements on the same body, as well as diagonal elements that describe the effect of a certain element on the potential at its own center), as well as terms that describe the effect of elements located in different bodies on each other. Qualitatively, the matrix can be defined:

$$[l]=\begin{bmatrix} l^{AA} & l^{AB} \\ l^{BA} & l^{BB} \end{bmatrix} \qquad (18)$$

Where l^{AA}, l^{BB} are square elements, identical to the matrix $[l]$ that was formulated for a single body. We should match the number of elements in each body, and this number doesn't need to be the same in each one. l^{AA} and l^{BB} are matrices that describe the effect of one body on the potential of the second body, and clearly: $l^{AB} = (l^{BA})^T$. Formulation of the problem in a matrix form:

$$\begin{bmatrix} l^{AA} & l^{AB} \\ l^{BA} & l^{BB} \end{bmatrix}[\sigma_n]=\begin{bmatrix} \mathbf{V}_A \\ \mathbf{V}_B \end{bmatrix} \qquad (19)$$

If body A is divided into N_A elements and body B is divided into N_B elements (obviously $N_A + N_B = N$), then $\boldsymbol{V}_A$ is a column vector with N_A elements, all equal to the given potential on it - V_A, similarly to vector $\boldsymbol{V}_B$.

Additional Options for Given Conditions

So far, we have dealt with problems in which the potential is given explicitly on each one of the bodies. However, there are other cases as well. For example: if there is information on the sum of charges on each of the bodies, but not the potential. Let's assume a problem with two bodies, as shown in the figure, but this time, it is known that the total charge on body A is Q_A and the total charge on body B is Q_B. We will add to our problem two variables, that will describe (the unknown) potential on each of the bodies. For the points on body A:

$$\Phi = \sum_{n=1}^{N} l_{mn}\sigma_n = V_A \;\Rightarrow\; \sum_{n=1}^{N} l_{mn}\sigma_n - V_A = 0 \quad ;\{1 < m < N_A\} \qquad (20)$$

And on body B:

$$\sum_{n=1}^{N} l_{mn}\sigma_n = V_B \;\Rightarrow\; \sum_{n=1}^{N} l_{mn}\sigma_n - V_B = 0 \quad ;\{N_A + 1 < m < N_A + N_B\} \qquad (21)$$

For each value of m, equations (**20**) and (**21**) describe a single row of the matrix system we construct. Writing this equation explicitly we get:

$$\begin{aligned} &l_{m1}\sigma_1 + l_{m2}\sigma_2 + \ldots + l_{mN}\sigma_n + (-1)V_A + 0\cdot V_B = 0 \quad ;\{1 < m < N_A\} \\ &l_{m1}\sigma_1 + l_{m2}\sigma_2 + \ldots + l_{mN}\sigma_n + 0\cdot V_A + (-1)V_B = 0 \quad ;\{N_A + 1 < m < N_A + N_B\} \end{aligned} \qquad (22)$$

The vector of unknowns also changes, and now it will be:

$$\begin{bmatrix} \boldsymbol{\sigma}_n \\ V_A \\ V_B \end{bmatrix} \qquad (23)$$

Where $\boldsymbol{\sigma}_n$ in this notation is a column vector with N elements: $\sigma_1, \ldots, \sigma_N$. Since we added two additional unknowns, we need to additional equations that stem from the conditions on the total charges. If we denote the area of the area of element n with Δ_n, we obtain that:

$$\begin{aligned} &\sum_{n=1}^{N_A} \sigma_n \Delta_n = Q_A \\ &\sum_{n=N_A+1}^{N_B} \sigma_n \Delta_n = Q_B \end{aligned} \qquad (24)$$

When Q_A and Q_B are known.

The matrix system will now have the following form:

$$\underbrace{\begin{bmatrix} \left(l^{AA}\right)_{N_A\times N_A} & \left(l^{AB}\right)_{N_A\times N_B} & \left(-1\right)_{N_A\times 1} & \left(0\right)_{N_A\times 1} \\ \left(l^{BA}\right)_{N_B\times N_A} & \left(l^{BB}\right)_{N_B\times N_B} & \left(0\right)_{N_B\times 1} & \left(-1\right)_{N_B\times 1} \\ \left(\Delta_{n,A}\right)_{1\times N_A} & \left(0\right)_{1\times N_B} & \left(0\right)_{1\times 1} & \left(0\right)_{1\times 1} \\ \left(0\right)_{1\times N_A} & \left(\Delta_{n,B}\right)_{1\times N_B} & \left(0\right)_{1\times 1} & \left(0\right)_{1\times 1} \end{bmatrix}}_{(N+2)\times(N+2)\ matrix} \underbrace{\begin{bmatrix} \left(\boldsymbol{\sigma}_n\right)_{N_A+N_B} \\ \left(V_A\right)_1 \\ \left(V_B\right)_1 \end{bmatrix}}_{N+2\ vector} = \underbrace{\begin{bmatrix} \left(0\right)_{N_A+N_B} \\ \left(Q_A\right)_1 \\ \left(Q_B\right)_1 \end{bmatrix}}_{N+2\ vector} \quad \textbf{(25)}$$

Next to each block in the matrix (in subscript) is its dimensions. Element $\Delta_{n,A}$ describes the area of the elements in body A, and similarly for body B.

Appendix – Integrals for Estimation of l_{nn}

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right) \quad \textbf{(26)}$$

$$\int \ln\left(\frac{\sqrt{x^2+a^2}+a}{\sqrt{x^2+a^2}-a}\right)dx = x\ln\left(\frac{\sqrt{x^2+a^2}+a}{\sqrt{x^2+a^2}-a}\right)+2a\ln\left(x+\sqrt{a^2+x^2}\right) \quad \textbf{(27)}$$