Wave transmission and distributed systems

Research project

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In this project we will explore the wave dynamics along a tapered transmission line, and its possible use for impedance matching.

The Telegraphers equations, as learned in the class read

$$-\frac{dV}{dz} = j\omega L I(z), \qquad -\frac{dI}{dz} = j\omega C V(z)$$

Where C and L are the per-unit-length characteristic line capacitance and inductance. Strictly speaking, these are constant values along the line. In this project, however, we will allow a gradual and slow (adiabatic) variation of these parameters as a function of the length coordinate along the line, z.

Question 1

Consider a generator at $f = 10\text{GHz}$, and voltage amplitude $V_p = 1V$. The generator internal impedance is $R_g = 50\Omega$. The generator is connected to a line with characteristic impedance $Z_0 = 75\Omega$, and phase velocity $v_p = c/2$, where c is the speed of light in vacuum. The line length is $\ell = 10\text{cm}$, and it is terminated by a resistive load with impedance $Z_L = 150\Omega$.

1. Discretize the transmission line to N node points. So that the voltage at $z_n = n\Delta$ is $V(n)$. Then, approximate the derivatives in the Telegrapher's equations by 2nd order accuracy approximation

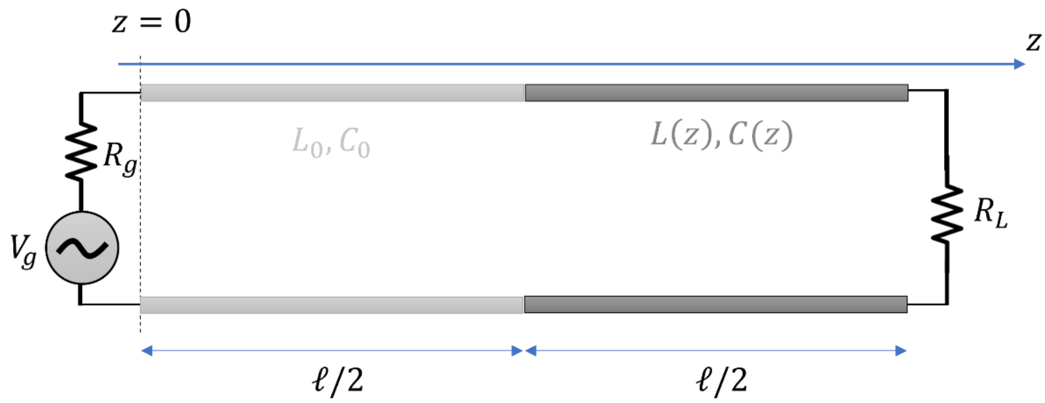
$$\frac{dV}{dz} = \frac{V[n+1] - V[n-1]}{2\Delta} + O(\Delta^2)$$

   and similarly for $I(z)$. Where $O(\Delta^2)$ means second order accuracy, namely – "error that is in the worst case proportional to Δ^2, and may be smaller".

2. Make sure that $n = 0$ is the point at the line input (right at the generator), and $n = N$ is the point at the load. Write equations for the boundary conditions. These equation should establish a connection between $V[n=0], I[n=0]$ and $V[n=N], I[n=N]$. See the hint below.
3. Write a matrix equation for the voltages and currents on the line. This matrix should contain both the discretized Telegraphers equations, and the boundary conditions.
4. For the line parameters that are given above, solve the matrix equation and draw the voltage and current on the line.
5. Derive analytically the solution for the voltage and current on the line, and compare with your numerical solution.

Question 2

Now, a different system. A transmission line of length ℓ is divided into two parts. The left part has characteristic capacitance and inductance C_0, L_0 (lighter gray color). The right half has L and C that vary along the line as a function of the location z. Specifically, assume that $L(z) = L_1 + \alpha_1(z - \ell/2)^2$, and $C(z) = C_1 + \alpha_2(z - \ell/2)^2$. L_0, C_0 are the same as the transmission line in Q1. Assume that the load is real. Our goal, using this system, is to achieve broadband matching – to match the load over the widest band of frequencies possible, around $f = 10\text{GHz}$. For a certain application, we consider the system as matched if $|\Gamma_{in}| < 0.1$.



1. How would you match this load for a single frequency, $f = 10\text{GHz}$, using the right transmission line segment? [solve analytically].
2. What is the bandwidth of your proposed matching? Plot $|\Gamma_{in}|$ as a function of the frequency f around 10GHz.
3. Write the matrix equation for the voltage and current along the line in this case.
4. Assume that the systems parameters are as given above, except that this time the right line parameters are unknown. Find $L_1, C_1, \alpha_1, \alpha_2$ to achieve matching over the <u>widest band you are able to achieve</u>. Plot the reflection coefficient as seen at the input $|\Gamma_{in}|$ vs. f and compare against the single-frequency matching.

Hint:

At the end points, besides the boundary conditions, there are two equations that are missing in order to have equal number of equations and unknowns. The reason is that the Telegrapher's equations in their discrete form cannot be applied there using central derivative. In order to address this lack of two equations, and get a voltage – current relation at the end points in addition to the boundary conditions we follow what is called "the ghost point procedure". Here, as an example I provide the derivation for the end point at the source.

The nodes are indexed by integer numbers $n = 0..N$. We need to connect V_0 to I_0, V_1 and I_1. To that end we use Taylor expansion about the "ghost point" $z = \frac{\Delta}{2}$ where Δ is the spacing between the nodes on the TL.

$$V\left(\frac{\Delta}{2} + dz\right) = V\left(\frac{\Delta}{2}\right) + \frac{dV}{dz} @ \left(z = \frac{\Delta}{2}\right) dz$$

we now take $dz = -\frac{\Delta}{2}$ and get

$$V(0) = V\left(\frac{\Delta}{2}\right) - \frac{dV}{dz} @ \left(z = \frac{\Delta}{2}\right) \frac{\Delta}{2}$$

Going back to the discrete form

$$V_0 = V_{0.5} - \frac{dV}{dz}_{0.5}\frac{\Delta}{2}$$

In light of the Telegraph equation for the voltage derivative, we get

$$V_0 = V_{0.5} + j\omega L_{0.5} I_{0.5}\frac{\Delta}{2} \quad (*)$$

Next, the voltage and current at the ghost point, $V_{0.5}$ and $I_{0.5}$ can be approximated as the mean value of the values at nodes 0 and 1, giving

$$V_{0.5} = \frac{1}{2}(V_0 + V_1)\ , \qquad I_{0.5} = \frac{1}{2}(I_0 + I_1)$$

The accuracy of this approximations (linear interpolation) is $O(\Delta^2)$ thus maintain the accuracy of the overall scheme.

We substitute these into equation (*) and get

$$V_0 = \frac{V_0 + V_1}{2} + j\omega\frac{L_0 I_0 + L_1 I_1}{2}\frac{\Delta}{2}$$

and finally,

$$\frac{V_1 - V_0}{\Delta} + j\omega\frac{L_0 I_0 + L_1 I_1}{2} = 0$$

This equation, and its counterpart equation at the other end (which you need to develop by your own), are the two missing equations in order to complete the numerical scheme.