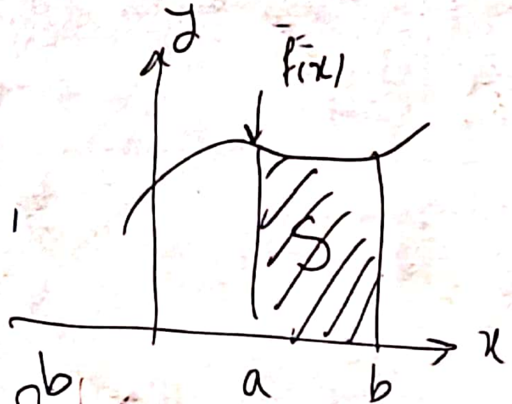
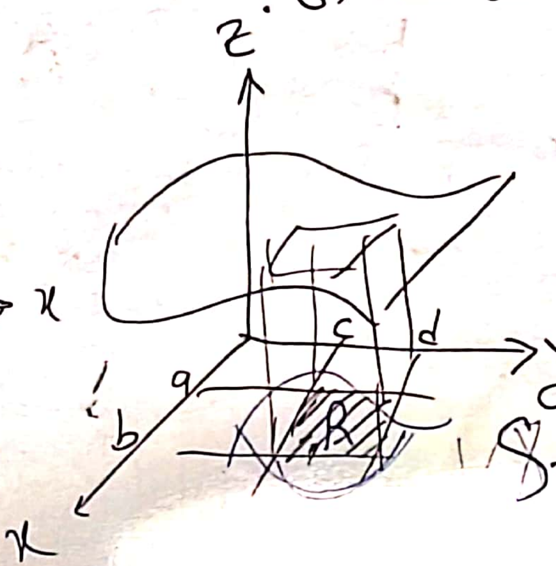
انتگرال های دوگانه در دستگاه مختصات دکارتی :



$$S=\int_a^b f(x)\,dx$$

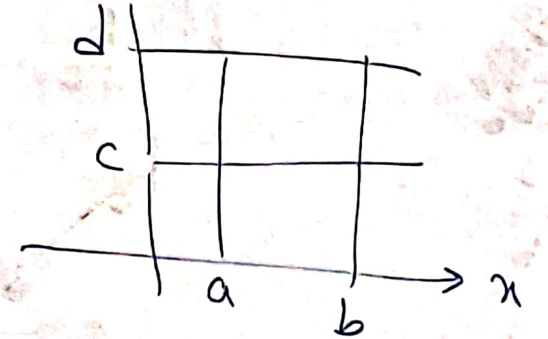


$z=f(x,y)$

$$V=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

$$V=\iint_R f(x,y)\,dA$$

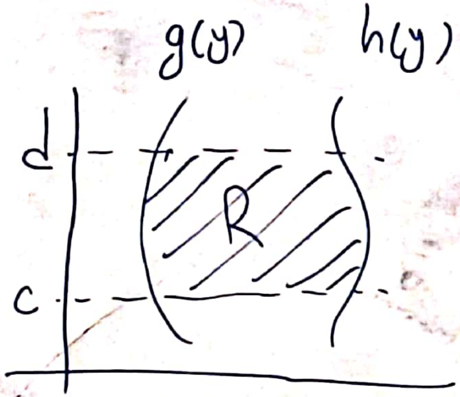
حالت اول: ناحیه R مستطیل باشد



$$R: \begin{cases} a \leq x \leq b \\ c \leq y \leq d \end{cases}$$

$$V = \iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

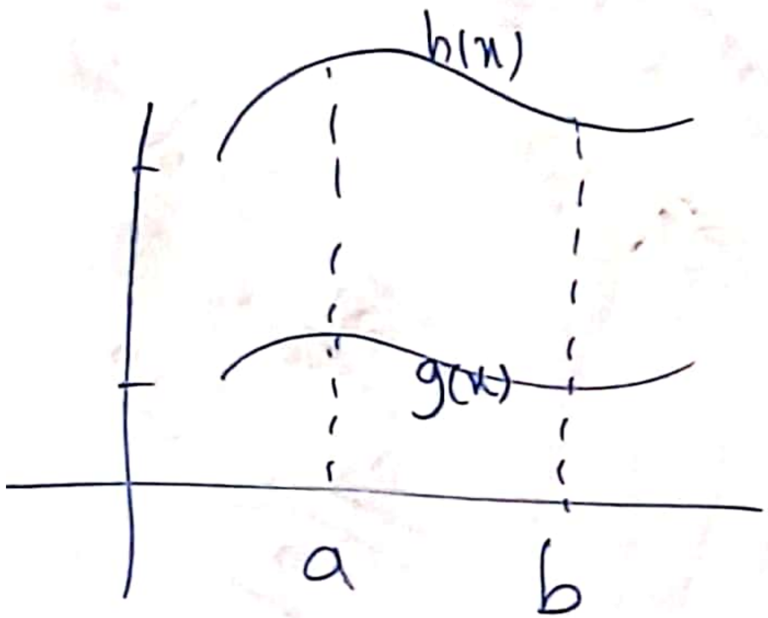
حالت سوم: غیر مستطیلی



$$R:\begin{cases} g(y)\le x\le h(y)\\ c\le y\le d\end{cases}$$

$$V=\iint_R f(x,y)\,dA=\int_c^d\int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

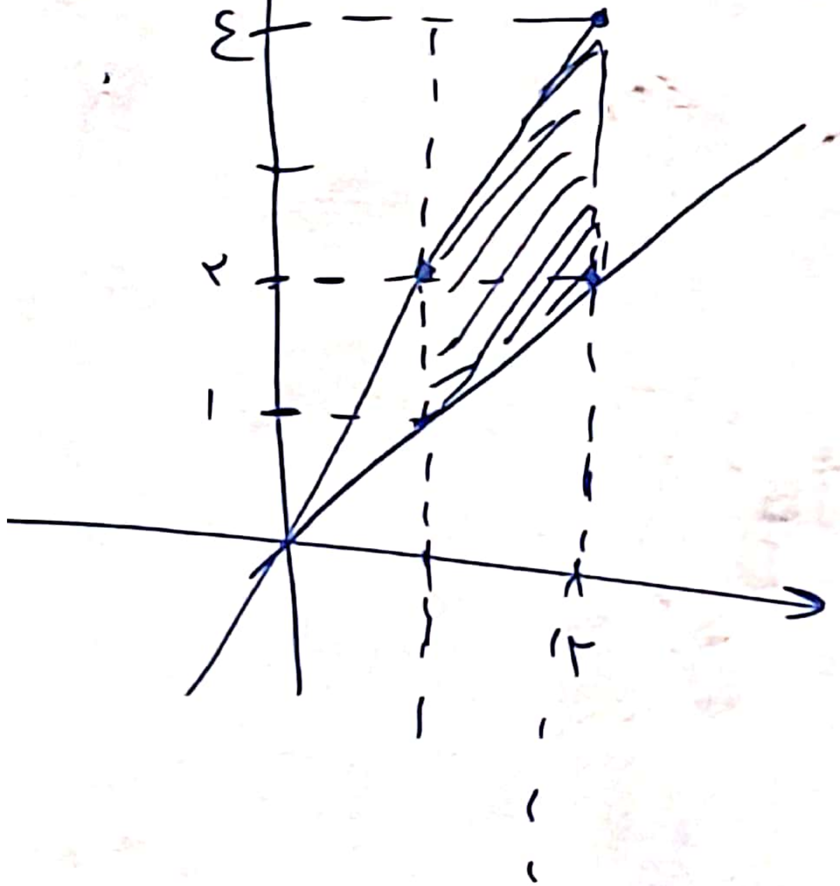
② ناحیه R غیر مستطیل



$$R: \begin{cases} a \leq x \leq b \\ g(x) \leq y \leq h(x) \end{cases}$$

$$V = \iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

ناحیه محدود به خط های $y=x$ و $y=2x$, $x=1$, $x=2$



x	۱	۲
y	۲	۴

$$\begin{cases} 1 \leq x \leq 2 \\ x \leq y \leq 2x \end{cases}$$

$$\int_1^2 \int_x^{2x} f(x,y)\,dy\,dx$$

ناحیه محدود به منحنی های $x=y^2$ و $y=x+1$, $y=1$, $y=-1$

$x=y-1$

$$R \begin{cases} -1 \leq y \leq 1 \\ y-1 \leq x \leq y^2 \end{cases}$$

$$\int_{-1}^{1} \int_{y-1}^{y^2} f(x,y)\, dx\, dy$$

$$\int_0^{\pi}\int_0^{\pi} (\sin x + \cos y)\, dy\, dx = \int_0^{\pi} \Big(y \sin x + \sin y \Big)_0^{\pi} dx$$

$$\int_0^{\pi} \Big(\pi \sin x + \overset{0}{\cancel{\sin \pi}} - (0 \times \overset{0}{\cancel{\sin x}} + \overset{0}{\cancel{\sin 0}}) \Big)\, dx = \Big(-\pi \cos x \Big)_0^{\pi}$$

$$= -\pi \Big(\underbrace{\cos \pi}_{-1} - \underbrace{\cos 0}_{+1} \Big) = -\pi \times -2 = 2\pi$$

$$\int_0^{\pi}\underbrace{\int_0^{\pi}(\sin x+\cos y)\,dy}\,dx=\int_0^{\pi}\left(y\sin x+\sin y\right)\Big|_0^{\pi}dx$$

$$\int_0^{\pi}\left(\pi\sin x+\cancel{\overset{0}{\sin\pi}}-\left(\cancel{0\times\sin x}+\cancel{\overset{0}{\sin 0}}\right)\right)dx=\left(-\pi\cos x\right)\Big|_0^{\pi}$$

$$=-\pi\left(\underbrace{\cos\pi}_{-1}-\underbrace{\cos 0}_{+1}\right)=-\pi\times-2=2\pi$$

$$\int_0^1 \underbrace{\int_x^{x^2} (x-2y)\,dy}\,dx = \int_0^1 \left(xy - \cancel{2}\frac{y^2}{\cancel{2}} \right)\Bigg|_x^{x^2} dx$$

جای y

$$= \int_0^1 \left(x^3 - x^4 - (\cancel{x^2} - \cancel{x^2}) \right) dx = \left(\frac{x^4}{4} - \frac{x^5}{5} \right)\Bigg|_0^1 = \frac{1\times 5}{4\times 5} - \frac{1\times 4}{5\times 4} - (0)$$

$$= \frac{1}{20}$$

$$\int_0^1 \left[\int_{y^2}^{y} x^1 y^2 \, dx\right] dy = \int_0^1 \left(\frac{x^2 y^2}{2}\right)\Bigg|_{y^2}^{y} dy = \int_0^1 \left(\frac{y^4}{2} - \frac{y^6}{2}\right)\Bigg|_0^1 dy$$

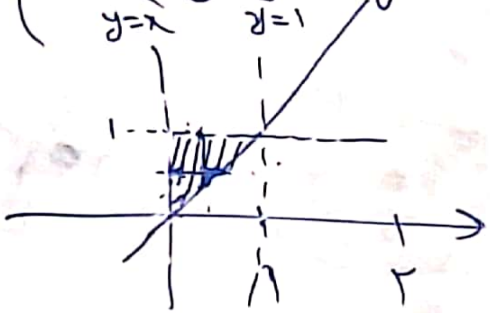
$$= \left(\frac{y^5}{10} - \frac{y^7}{14}\right)\Bigg|_0^1 = \frac{1}{\underset{\times 7}{10}} - \frac{1}{\underset{\times 5}{14}} = \frac{7-5}{70} = \frac{2}{70} = \frac{1}{35}$$

$y^2 = u \rightarrow 2y\,dy = du$

$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$$

تعویض ترتیب انتگرال گیری

$$\begin{cases} 0 \leq x \leq 1 \\ x \leq y \leq 1 \end{cases}$$



$$\int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 \left(x e^{y^2}\right)_0^y dy$$

$y^2 = u$ $\quad$ $y = 0 \rightarrow u = 0$

$2y\,dy = du$ $\quad$ $y = 1 \rightarrow u = 1$

$$\frac{1}{2}\int_0^1 2y e^{y^2}\,dy = \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}\left(e^u\right)_0^1 = \frac{1}{2}\left(e^1 - e^0\right) = \frac{1}{2}(e-1)$$

$$\begin{cases} 0 \leq y \leq 1 \\ 0 \leq x \leq y \end{cases}$$

$$\int_0^3 \int_x^0 e^{x^2}\,dy\,dx = \int_0^3 \left(y e^{x^2}\right)$$

$$= -\frac{1}{2}\int_0^9 e^u\,du = -\frac{1}{2}\left(e^u\right)_0^9 = -\frac{1}{2}$$

$x^2 = u \quad du = 2x\,dx$

$\begin{cases} x = 3 \to u = 9 \\ x = 0 \to u = 0 \end{cases}$

$$\int_0^3 \int_x^0 e^{x^2}\,dy\,dx = \int_0^3 \left(y e^{x^2}\right)_x^0 dx = \frac{1}{2}\int_0^3 \left(0 - 2x e^{x^2}\right) dx$$

$$= -\frac{1}{2}\int_0^9 e^u\,du = -\frac{1}{2}\left(e^u\right)_0^9 = -\frac{1}{2}\left(e^9 - e^0\right) = -\frac{1}{2}\left(e^9 - 1\right)$$