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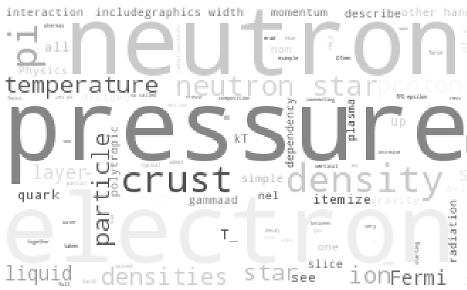

1 Physics of neutron star interiors

In this chapter, we will study the neutron star interiors in more detail. In practice, this means describing the behavior of the matter from a densities of $\sim 10^{-2}\,\mathrm{g\,cm^{-3}}$ to $\sim 10^{16}\,\mathrm{g\,cm^{-3}}$, an impressive 18 orders of magnitude range starting from a hot and rarefied electron corona to an ultra-dense neutron liquid.

The structure of the star can be roughly divided into three distinct sections: atmosphere, crust, and core. Neutron star atmosphere holds a negligible amount of matter in comparison to the whole star, but it is important for the radiative processes. It is the radiation from the atmosphere that we actually observe. The crust, like the name implies, can be understood as a solidified layer surrounding the liquid core. Physics describing the crust is relatively well known and same type of matter consisting of ions, protons, and electrons can be found inside white dwarf stars. Bulk of the mass, on the other hand, is located on the liquid neutron core. Detailed microphysics of such matter are still unknown and this is reflected in a large uncertainty in the actual size of the star that is still unconstrained.

We begin by giving an overview of the characteristics of each of the different layers. By combining this information, we can then build different models for the neutron stars and describe some more global aspects of them such as mass and radius. For this we need to solve the relativistic equations of hydrostatic equilibrium, that are also discussed. In the end, this enables us to build a mapping between the (un)known microphysics of the dense matter and the astrophysical observables.

1.1 Equation of state

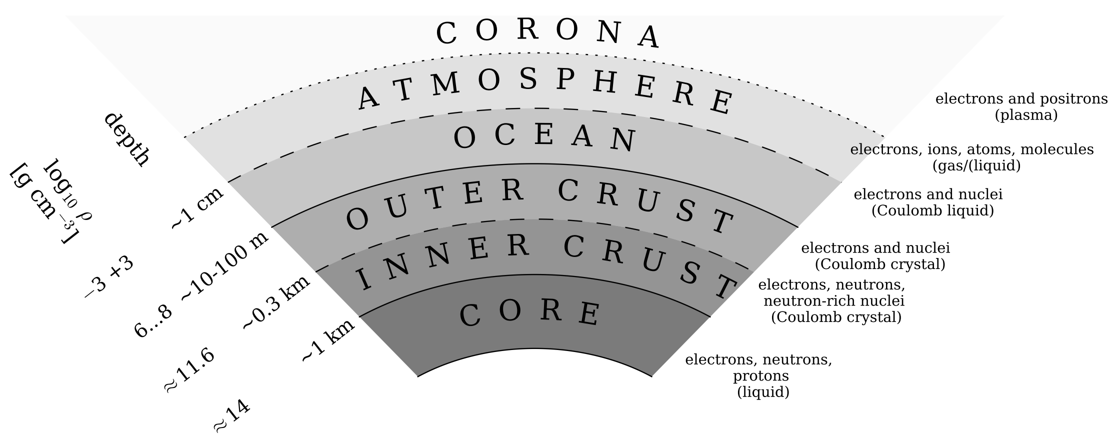

Figure 1.1: Schematic view of neutron star structure.

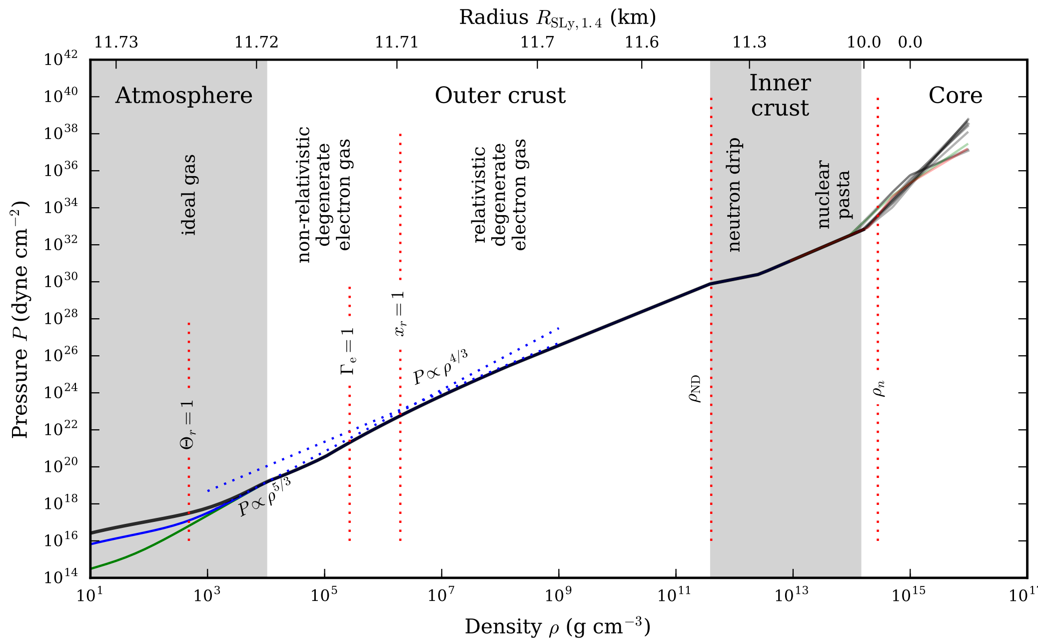

Figure 1.2: Overview of the pressure versus density relation for the full range of densities relevant for neutron stars. Here the evolution of the pressure is shown against the densities depicted in the bottom vertical axis. Green solid line shows the EoS for matter at $T = 10^6$ K, whereas blue line is for $T = 5 \times 10^6$ K, and black for $T = 10^7$ K. Additionally, the upper vertical axis shows the evolution of the radial coordinate computed for one particular EoS (SLy, see Sect. 1.4) and neutron star configuration (mass of $1.4\,M_{\odot}$). Different shaded vertical regions show the corresponding interior structures of the star. Additionally, some interesting densities are highlighted with dashed red lines and text labels (see Sects. 1.2–1.4).

In thermodynamics, we speak of *state variables* that describe a current state of the matter under a given physical conditions. These include, for example, the density ρ, pressure P, and temperature T of the matter. Equation of state is a thermodynamical equation connecting these states variables together. Often, when focusing on neutron stars, what we mean by EoS is a function connecting the pressure and the density of the matter only, $P(\rho)$.

Degenerate matter

The dependency on the rest of the variables such as temperature can be often forgotten because the matter is *degenerate*. In contrast to the "normal" matter where a statistical moments such as temperature can be used to describe a large ensemble of particles, the degenerate matter is dominated by quantum mechanical effects of single particles. Because of the immense densities, a free particle in a degenerate matter is actually bounded into a finite volume. Inside this small volume, the energy levels of the particle are restricted to take only a discrete set of values called quantum states, because of the underlying wave-nature of the quantum mechanical description. Hence, a notion of temperature, for example, does not make much sense.

Overview of the EoS for the full range of densities relevant to neutron stars is shown in Fig. 1.2. From here it is easy to see that temperature only plays a role in the very uppermost $\sim$ 10 meters of the stars interiors. Behavior of the matter is also quite well known all the way up to the crust-core interface, after which we start to see larger deviations because of the different EoS models. In the Earthly laboratories we can probe the matter somewhere close to $10^{14}\,\mathrm{g\,cm^{-3}}$, after which the densities becomes too great for us to handle in.* On the other hand, it is exactly starting from this density range that the bulk of the neutron star just starts. Another curious quirk of Nature is how all of the complicated microphysics gets reduced to simple line segments in the logarithmic scales, also known as polytropic pressure relations. In the following sections, we will focus on deriving these simple relations as it helps us in understanding the underlying physics.

1.2 Atmosphere

Atmosphere of a star is the first and uppermost layer responsible for the emergent radiation. It consists of a thin layer of plasma and ranges from a few millimeters to couple of centimeters in height. In most situations the plasma is in a gaseous state, but in some more rare cases when the magnetic field is extraordinarily strong and the temperature is low, the plasma can condensate into a liquid or a solid surface. Such condensed surfaces are, however, rare and usually the gaseous description is more than enough.†

Properties of the emergent thermal radiation strongly depend on the chemical composition of the atmosphere. In the atmospheres of normal stars the composition is a mixture of multiple elements. The most stable chemical element on the surface of a neutron star is iron. However, even a small accreted mass of $10^{-17}\,M_\odot$, originating from the surrounding interstellar medium, is enough to cover the whole star, and hence a variety of elements are also expected in the neutron star atmospheres. On the other hand, the enormous gravity results in an effective separation of elements leading to a strong sedimentation of the atmosphere where the lighter elements are expected to lay on top of the heavier ones.‡ Hence, the atmosphere is usually expected to consist of mainly hydrogen.

1.2.1 General relativistic effects

The effects from the gravity can be quantified by considering a so-called compactness parameter

$$u = \frac{R_\mathrm{S}}{R} \tag{1.1}$$

where R is the radius of the neutron star and the corresponding Schwarzchild radius is defined as

$$R_\mathrm{S} = \frac{2GM}{c^2} \approx 2.95 \frac{M}{M_\odot}\ \mathrm{km}, \tag{1.2}$$

where G is the gravitational constant, c is the speed of light, and M is the mass of the star. Hence, a neutrons star has a compactness parameter in the range of $u \approx 1/5$ to $1/2$ resulting in a considerable general relativistic corrections. In comparison, the Sun has $u \approx 4.24\times10^{-6}$. Gravitational acceleration under general relativistic theory is

$$g = \frac{GM}{R^2}\frac{1}{\sqrt{1-u}} = 1.38\times10^{14}\ \frac{1}{\sqrt{1-u}}\left(\frac{M}{M_\odot}\right)\left(\frac{10\,\mathrm{km}}{R}\right)^2\ \mathrm{cm\,s^{-2}}. \tag{1.3}$$

Hence, a surface gravitational accelerations g of $\sim 10^{14}$ to $\sim 10^{15}\,\mathrm{cm\,s^{-2}}$ are expected for neutron stars. By considering a barometric atmosphere we can also estimate the scale height as

$$H_\mathrm{a} = \frac{k_\mathrm{B}T}{m_i g} \approx \frac{0.83}{A}\left(\frac{T}{10^6\,\mathrm{K}}\right)\left(\frac{10^{14}\,\mathrm{cm\,s^{-2}}}{g}\right)\ \mathrm{cm} \tag{1.4}$$

*Maximum densities reached in the Earth are usually obtained by colliding heavy nucleons together, momentarily creating a core of even denser matter. The densest naturally occurring element found on top of planet Earth is osmium that has a density of "just" $\rho \approx 2.2\times10^1\,\mathrm{g\,cm^{-3}}$.

†for a review, see [1] V. E. Zavlin and G. G. Pavlov. 2002; [2] A. Y. Potekhin. *Physics Uspekhi*. (2014).

‡ [3] C. Alcock and A. Illarionov. *ApJ*. (1980).

where $k_{\rm B} = 1.38 \times 10^{-16}$ erg K^{-1} is the Boltzmann constant, T is the temperature of the atmosphere, $m_i = Am_u$, and $m_u \approx 1.66 \times 10^{-24}$ g is the atomic mass unit. From here, the typical scale height values of ~ 1 cm to ~ 10 cm are re-obtained for atmospheres of $T = 10^6$ and 10^7 K.* Strong gravitational field also bends the photon trajectories.† Hence, in addition to the radius R of the star as measured in the local reference frame, another *apparent* radius, as measured by an observer at infinity,

$$R_\infty = \frac{R}{\sqrt{1-u}}, \tag{1.5}$$

is usually needed when describing the observable features of the atmosphere. From here it is then clear that the atmosphere and the emerging radiation encodes information from the physical parameters of the star. More specifically information about the temperature, surface gravity, chemical composition, compactness can be obtained.

1.2.2 Basic equations of the atmosphere

The standard approach in describing the atmosphere structure includes solving three main equations of radiative transfer, hydrostatic balance, and energy conservation. First such a low-B field model of neutron star atmospheres were presented in the pioneering work by Romani.‡ Let us next see, walk through these equations, as they are rater simple. A more general description for the atmosphere model computations are given in Sect.XX§, where the full relativistic electron scattering is also taken into account.

Because the thickness of the atmosphere is much smaller than the radius of the star,¶ the atmosphere can be considered in plane-parallel approximation. Rather high densities, on the other hand, allow to consider the plasma of the atmosphere in local thermodynamical equilibrium.

Spectrum, beaming and polarization of emerging radiation can be determined from radiation transfer problem in atmospheric layers. Radiation can be understood as an energy flow, i.e., energy E per area A, time t, frequency ν, and solid angle Ω. This is known as the specific spectral intensity which we can mathematically formulate as

$$I_\nu = \frac{dE}{dA dt d\nu d\Omega}. \tag{1.6}$$

Radiation averaged over the solid angle, or the so-called mean specific intensity (zeroth moment of I_ν) is then

$$J_\nu = \frac{1}{4\pi}\int_\Omega I_\nu d\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_\nu \sin\theta d\theta d\phi = \frac{1}{2}\int_{-1}^{+1} I_\nu d\mu, \tag{1.7}$$

where we have assumed that the radiation does not depend on the azimuthal ϕ angle (as is typical for atmosphere calculations) and introduced $\mu = \cos\theta$. Net rate of energy flowing across an unit area (for example a photon detector) from *all directions* per time and frequency is known as physical flux.‖ It is proportional to the first-order moment of I_ν and is defined as

$$F_\nu = 2\pi \int_{-1}^{+1} I_\nu \mu d\mu. \tag{1.8}$$

* [1] V. E. Zavlin and G. G. Pavlov. 2002; [2] A. Y. Potekhin. *Physics Uspekhi*. (2014).

† [4] K. R. Pechenick, C. Ftaclas, and J. M. Cohen. *ApJ*. (1983).

‡ [5] R. W. Romani. *ApJ*. (1987).

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¶Recall the scale height of 1 to 10 cm in comparison to the radius of 10^6 cm.

‖Strictly speaking, the first-order moment of I_ν is known as the Eddington flux $H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu d\mu$. Physical flux is related to it as $F_\nu = 4\pi H_\nu$ and sometimes one also encounters the "astrophysical" flux defined as F_ν/π.

Similarly, the second-order moment of I_ν, or a so-called K-integral, is

$$K_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu^2 d\mu, \tag{1.9}$$

which is related to the radiation pressure, as we later on will see.

Now we can introduce the radiative transfer equation for I_ν as

$$\mu\frac{dI_\nu}{d\tau} = \frac{\mu}{\kappa_\nu}\frac{dI_\nu}{dy} = -\frac{\mu}{\rho\kappa_\nu}\frac{dI_\nu}{dz} = I_\nu - S_\nu \tag{1.10}$$

where τ is the optical depth, y is the column density (mass per area), z is the horizontal distance from the surface, $\kappa_\nu = \alpha_\nu + \sigma_\nu$ is the total radiative opacity including contributions from the "true" opacity α_ν and from the scattering opacity σ_ν. Here the connection between different independent variables is given as

$$d\tau = \kappa_\nu dy = -\kappa_\nu \rho dz, \tag{1.11}$$

relating the optical depth (distance as experienced by the radiation), column density (projected number density of matter along the path of the radiation), and the horizontal height. In addition we need the source function

$$S_\nu = (\sigma_\nu J_\nu + \alpha B_\nu)\kappa_\nu^{-1}, \tag{1.12}$$

where the scattering term is proportional to the mean spectral intensity J_ν and the "true" absorption term to the thermal Planck function

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left[h\nu/(k_{\mathrm{B}}T)\right] - 1}, \tag{1.13}$$

where $h = 6.63\times10^{-27}$ erg s is the Planck constant. As an boundary conditions for this equation, we can use $I_\nu = 0$ for $\mu < 0$ at $y = 0$ (i.e., the surface). The atmospheres are also usually considered to be in radiative and hydrostatic equilibrium, i.e., (quasi-)stationary. The first requirement can be formulated as

$$\int_0^\infty d\nu \int_{-1}^{+1} I_\nu \mu d\mu = \sigma_{\mathrm{SB}} T_{\mathrm{eff}}, \tag{1.14}$$

where $\sigma_{\mathrm{SB}} = 5.67\times10^{-5}$ g s^{-3} K^{-4} is the Stefan-Boltzmann constant and T_{eff} is the effective temperature of the atmosphere. The second hydrostatic equilibrium, demands that

$$\frac{dP}{dy} = g - g_{\mathrm{rad}}, \tag{1.15}$$

where, in addition to the gravitational acceleration g we need the opposing radiative acceleration g_{rad}. Finally, we need to supplement these equations with an equation connecting the pressure and density. For the rarefied atmosphere, the ideal gas law is an excellent approximation

$$P = nk_{\mathrm{B}}T, \tag{1.16}$$

where n is the number density of particles.

1.2.3 Eddington limit

Usually the atmospheres we calculate are dynamically stable and in hydrostatic balance because large gravity implies $g_{\rm rad} \ll g$. Sometimes, however, the radiation flux might increase to such a strengths that it is able to compete against even the enormous gravity of a neutron star. An important limit can then be defined for $g_{\rm rad} = g$, known as the Eddington limit after a renowned astrophysicist Sir Arthur Eddington. Let us now for completeness derive this limit.*

We can start by formulating the radiation pressure. This can be easily done when we realize that pressure is just momentum flux, and photons carry a momentum of E/c.† In terms of I_ν this is then

$$P_{{\rm rad},\nu} = \frac{1}{c}\int_0^{2\pi}\int_{-1}^{+1} I_\nu \mu^2 d\mu = \frac{4\pi}{c} K_\nu, \tag{1.17}$$

relating the pressure and the second-order moment K_ν together. Radiative acceleration is then

$$g_{\rm rad} = \frac{dP_{\rm rad}}{dy} = \frac{d}{dy}\int_0^\infty P_{{\rm rad},\nu}\, d\nu = \frac{4\pi}{c}\frac{d}{dy}\int_0^\infty K_\nu d\nu. \tag{1.18}$$

Let us refine this expression by inserting the definition of K_ν and applying the radiative transfer equation (1.10) without the source function S_ν. In the process we also have to take into account that we are only interested in the outgoing flux, hence the integration limits of μ need to be changed as $[-1, +1] \to [0, +1]$. We can then simplify the (1.18) to

$$\begin{aligned} g_{\rm rad} &= \frac{4\pi}{c}\frac{d}{dy}\int_0^\infty d\nu \frac{1}{2}\int_0^{+1} I_\nu\, \mu^2 d\mu \\ &= \frac{2\pi}{c}\int_0^\infty d\nu \int_0^{+1} \mu\, d\mu \left\{\mu \frac{d}{dy} I_\nu\right\} \\ &= \frac{2\pi}{c}\int_0^\infty d\nu \int_0^{+1} \mu\, d\mu\; \kappa_\nu I_\nu \\ &= \frac{2\pi}{c}\int_0^\infty d\nu\; \kappa_\nu \int_0^{+1} I_\nu \mu d\mu \\ &= \frac{1}{c}\int_0^\infty \kappa_\nu F_\nu d\nu. \end{aligned} \tag{1.19}$$

From the bottom line of Eq. (1.19) we see that the K-integral is related to the flux of the radiation F_ν. Not every photon, however, interacts and collides with the matter, and hence the opacity correction κ_ν is also needed, representing effectively the fraction of radiation interacting with the matter. If we now integrate over all the frequencies we obtain

$$g_{\rm rad} = \frac{1}{c}\,\kappa F \tag{1.20}$$

Setting it equal to g we can solve for the Eddington flux as

$$F_{\rm Edd} = \frac{gc}{\kappa}\frac{1}{\sqrt{1-u}} = \frac{GMc}{R^2\kappa}\frac{1}{\sqrt{1-u}}. \tag{1.21}$$

*for a more in depth discussion, see e.g., [6] G. B. Rybicki and A. P. Lightman. 1979; [7] J. Frank, A. King, and D. J. Raine. 2002.

†Energy of a photon is $E = h\nu = mc^2$, from which we obtain the photon rest mass of $h\nu/c^2 = E/c^2$. Momentum, on the other hand, is just velocity times the mass, so for a photon it is $E/c^2 \times c = E/c$.

Usually in many astrophysical scenarios the opacity is dominated by the electron Thomson scattering opacity, $\sigma_{\rm T} = 6.65 \times 10^{-25}\,{\rm cm}^{-2}$. Mass for a hydrogen plasma is, on the other hand, mainly set by the protons, hence, opacity per mass is then

$$\kappa \approx \frac{\sigma_{\rm T}}{m_{\rm p}}. \tag{1.22}$$

Finally, be calculating the total radiation flux through the stellar surface, we can define a quantity called luminosity as

$$L = 4\pi R^2 F(R), \tag{1.23}$$

where $F(R)$ is the outcoming radiation flux at the surface. Using the aforementioned equation, we can then define the Eddington luminosity for a star as

$$L_{\rm Edd} = \frac{4\pi G M c m_{\rm p}}{\sigma_{\rm T}} \approx 1.3 \times 10^{38} \left(\frac{M}{M_\odot}\right) {\rm erg\,s^{-1}}. \tag{1.24}$$

For neutron stars close to $T \sim 10^7$ K this limit can be reached, after which the atmosphere is momentarily blown away.

1.3 Crust

Below the gaseous atmosphere, a solidified layer of matter exists, called crust. Between the atmosphere and crust, a liquid ocean of ions also exists, but the interface is not very strict and the matter is smoothly evolving from an one state to another. The solidified crust is also typically divided into an outer and inner layers, but the interface is again ambiguous. The pressure here is almost fully given by the degenerate electrons, and hence the matter is familiar already from white dwarfs. In the beginning, the electrons can be taken to be non-relativistic but after about $\rho \sim 10^6\ {\rm g\,cm^{-3}}$ they turn into ultra-relativistic because of the increasing density.

By definition, the outer crust is a layer in the neutron star interiors where the plasma consists of electrons and nuclei, whereas the inner crust is characterized by an additional appearance of neutrons that start to drip out from the extremely neutron-rich nuclei. The density this occurs is called the neutron drip density and is of around $\rho_{\rm ND} \sim 4 \times 10^{11}\ {\rm g\,cm^{-3}}$. The outer crust, when defined to begin from the atmosphere at $\rho \sim 10^3\ {\rm g\,cm^{-3}}$ and continue to about $\rho_{\rm ND}$ is only about some hundred meters in thickness. The characteristics of the matter are strongly dependent on the Coulomb interactions of charged particles that form a solid Coulomb crystal.

The inner crust is taken to continue all the way down to the crust-core interface, where the matter turns liquid again. This layer is about one kilometer thick. Here, at the bottom of the crust, the density is already close to the nuclear density of $\rho_{\rm n} = 2.8 \times 10^{14}\ {\rm g\,cm^{-3}}$ but the exact location of the transition depends on the detailed microphysics of the core. The fraction of free neutrons grows with the increasing density. Because the normal nuclei nuclei here are immersed into free neutron gas, the nuclear interactions play a crucial role in defining the matter. Finally, the nuclei disappear totally when we enter the core. Before that, however, the nuclei form complicated structures that evolve together with the density. This region is also known as the nuclear pasta phase, as the different molecular structures are named after the pasta types that they resemble.

1.3.1 Degenerate gas and Fermi momentum

The matter in the crust is degenerate. Let us discuss the physics behind degenerate matter and the related important concept of Fermi energy now. Elementary 6-dimensional phase space cell of any particle is bounded by the Heisenberg uncertainty principle as

$$\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3, \tag{1.25}$$

where $\Delta x \Delta y \Delta z$ is the volume in ordinary space, $\Delta p_x \Delta p_y \Delta p_z$ the volume in the momentum space, and h is the previously defined Planck constant. In accordance with the quantum mechanics, there is room for only one particle of any kind inside the elementary cell. In general the number density is given as

$$n_{\mathrm{av}} = \frac{g}{\exp[(E-\mu)/k_{\mathrm{B}}T] \pm 1}, \tag{1.26}$$

where + is used for fermions (such as electrons and protons), and − for bosons (such as photons). Additionally, the g is the number of different quantum states a particle may have inside the cell, E is the total particle energy, and μ is the chemical potential explained more carefully later on. Number of particles with momentum between p and $p + dp$ is

$$n(p)dp = n_{\mathrm{av}} \frac{4\pi p^2}{h^3} dp, \tag{1.27}$$

i.e., particles inside a spherical shell with a surface $4\pi p^2$ and thickness dp. Number density of particles with all momenta is then

$$n = \int_0^\infty n(p)dp \tag{1.28}$$

and the density is then simply

$$\rho = nm, \tag{1.29}$$

where m is the mass of the particle. The energy in equation (1.26) is the total energy $E \equiv E_{\mathrm{tot}} = E_0 + E_{\mathrm{k}}$, composed of the rest-mass energy $E_0 = mc^2$ and kinetic energy E_{k}. In special relativity, there exists a relation for the total energy as $E_{\mathrm{tot}}^2 = (mc^2)^2 + (pc)^2$, so

$$E = mc^2 \left[1 + \left(\frac{p}{mc} \right)^2 \right]^{1/2}, \tag{1.30}$$

and the kinetic energy is then simply

$$E_{\mathrm{k}} = mc^2 \left[\left(1 + \left(\frac{p}{mc} \right)^2 \right)^{1/2} - 1 \right]. \tag{1.31}$$

Simple asymptotic limits can be obtained for the kinetic energy as

$$\begin{aligned} E &\approx \frac{p^2}{2m} \quad p \ll mc \text{ (non-relativistic case)} \\ E &\approx pc \quad p \gg mc \text{ (ultra-relativistic case).} \end{aligned} \tag{1.32}$$

Let us now focus on fermions only, and hence select the +-sign (Fermi-Dirac distribution) from the Eq. (1.26). Fermions are spin $\frac{1}{2}$ particles so we can also set $g = 2$. The chemical potential μ in Eq. (1.26) is expressed as

$$\mu = mc^2 + \epsilon_{\mathrm{F}} \tag{1.33}$$

where ϵ_{F} is now the so-called Fermi energy. On the other hand, it can also be defined as

$$E - \mu = E_{\mathrm{k}} - \epsilon_{\mathrm{F}} \tag{1.34}$$

i.e., difference between the kinetic and Fermi energy. Hence, the number density of fermions in general is

$$n_{\mathrm{f}} = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp(\frac{E_{\mathrm{k}} - \epsilon_{\mathrm{F}}}{k_{\mathrm{B}}T}) + 1}. \tag{1.35}$$

If Fermi energy $\epsilon_{\rm F} \ll 0$ the distribution will be Maxwellian even for small kinetic energies. On the other hand, if $\epsilon_{\rm F} \gg k_{\rm B}T$ we can divide the integrand of Eq. (1.35) into two distinct regions of

$$n_{\rm f} = \begin{cases} p^2 & \text{if } E_{\rm k} \ll \epsilon_{\rm F} \\ 0 & \text{if } E_{\rm k} \gg \epsilon_{\rm F} \end{cases} \tag{1.36}$$

where the transition occurs rather sharply at $E_{\rm k} = \epsilon_{\rm F}$. This allows us to define a characteristic momentum related to the transition, called Fermi momentum $p_{\rm F}$, so that

$$n_{\rm f} \approx \frac{8\pi}{h^3}\int_0^{p_{\rm F}} p^2 dp = \frac{8\pi}{3}\left(\frac{p_{\rm F}}{h}\right)^3. \tag{1.37}$$

Physically this can be interpreted such that when the temperature decreases, the fermions start to occupy all the quantum states starting from the one with lowest energy all the way up to the Fermi energy. Because of the Pauli exclusion principle, no more than one fermion can exists in the same quantum state so the levels are filled in order, and all the higher states will remain empty. Hence, the highest momenta possible in the degenerate matter is the Fermi momentum

$$p_{\rm F} = \left(\frac{3n_{\rm f}}{8\pi}\right)^{1/3} h. \tag{1.38}$$

1.3.2 Why neutrons then?

Let us first consider an ideal gas of degenerate electron-proton-neutron plasma to understand the basic composition of the crust.* In a degenerate plasma all the quantum states are filled up all the way to the Fermi energy, as we just learned. Normal beta-decay mode for the neutrons, on the other hand, is

$$n \to p + e^- + \bar{\nu}_e, \tag{1.39}$$

that describes the possible path of how a neutron n will decay into a proton p, electron e^-, and electron neutrino $\bar{\nu}_e$. It is because of this decay, that we do not expect to see any free neutron flying around. Such a decay is, however, might be blocked because there is no room for an emission of an extra electron e^- or a proton p.

Let us then only focus on the decay of the most energetic neutrons with an energy equal to the Fermi energy $\epsilon_{\rm F}(n)$, where the related particle species is defined inside the parentheses. Co-existence of neutrons, protons, and electrons is then guaranteed (at zero temperature) if

$$\epsilon_{\rm F}(n) = \epsilon_{\rm F}(p) + \epsilon_{\rm F}(e^-). \tag{1.40}$$

Massive neutrons and protons are to a good approximation non-relativistic up to a densities of $\rho_{\rm n}$, and hence energy is simply a sum of their rest mass energy and kinetic energy

$$\epsilon_{\rm F}(n) \approx m_n c^2 + \frac{p_{\rm F}(n)^2}{2m_n}, \tag{1.41}$$

and

$$\epsilon_{\rm F}(p) \approx m_p c^2 + \frac{p_{\rm F}(p)^2}{2m_p}. \tag{1.42}$$

*see e.g. [8] A. C. Phillips. 1994.

Electrons, on the other hand, are already ultra-relativistic, and so

$$\epsilon_{\mathrm{F}}(e^-) \approx p_{\mathrm{F}}(e^-)c^2. \tag{1.43}$$

Also note that $n_p = n_e$, as the star is electrically neutral.

From this we find relation of the $n_n/n_p \sim 1/200$ by taking into account the rest mass difference $m_p - m_n = 2.6\,\mathrm{MeV}\,c^2$ at $\rho \sim \rho_{\mathrm{n}}$. Thus, we conclude that the matter inside is expected to be neutron rich, even though normally the neutrons would β-decay back to protons and electrons. Not only is the degeneracy then responsible for the pressure but it is also the source of the neutron enrichment.

1.3.3 Degenerate electron gas

Let us next consider the equation of state for the crust. As we have seen, the pressure in the crust originates from the degenerate electron gas. Physics behind this are quite simple and we repeat the calculations here to introduce the reader to the topic. The result also bears some historical value as these are exactly the equations that were introduced by Dirac*, Fowler†, Frenkel‡, Anderson§, Stoner¶, and Chandrasekhar‖.More thorough discussion of the electron thermodynamics is given, for example, in Refs.**

The behavior of the matter in the crust is dominated mainly by the electrons. For this reason, it can be characterized by the electron number density n_{e} and temperature T_{e}, hereafter just T in this section. Instead of n_{e}, let us use the electron Fermi momentum p_{F} (1.38) as a measure of the number density. It is convenient to describe it in the units of electron rest mass, as

$$x_{\mathrm{r}} \equiv \frac{p_{\mathrm{F}}}{m_{\mathrm{e}}c}, \tag{1.44}$$

also known as the relativity parameter.†† We will also need the more general relativistic form of the Fermi energy

$$\epsilon_{\mathrm{F}} = c^2 \sqrt{(m_{\mathrm{e}}c)^2 + p_{\mathrm{F}}^2}, \tag{1.45}$$

that for a strongly degenerate gas, has the meaning of the chemical potential μ. Finally, the electron Fermi temperature is

$$T_{\mathrm{F}} = \frac{m_{\mathrm{e}}c^2}{k_{\mathrm{B}}}\left(\sqrt{1+\left(\frac{p_{\mathrm{F}}}{m_{\mathrm{e}}c}\right)^2} - 1\right) = T_{\mathrm{r}}(\gamma_{\mathrm{r}} - 1), \tag{1.46}$$

where a typical temperature is

$$T_{\mathrm{r}} = \frac{m_{\mathrm{e}}c^2}{k_{\mathrm{B}}} \sim 6\times 10^9\,\mathrm{K}, \tag{1.47}$$

and a relativistic scaling factor is defined as

$$\gamma_{\mathrm{r}} \equiv \sqrt{1 + x_{\mathrm{r}}^2}. \tag{1.48}$$

* [9] P. A. M. Dirac. *Proceedings of the Royal Society of London Series A*. (1925).

† [10] R. H. Fowler. *MNRAS*. (1926).

‡ [11] J. Frenkel. *Zeitschrift für Physik*. (1928).

§ [12] W. Anderson. *Zeitschrift für Physik*. (1929).

¶ [13] E. C. Stoner. *Philos. Mag.* (1930).

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** [15] S. Chandrasekhar. 1939; [16] E. L. Schatzman. 1958; [17] E. E. Salpeter. *ApJ*. (1961); [18] R. F. Tooper. *ApJ*. (1969); [19] Y. B. Zeldovich and I. D. Novikov. 1971; [20] L. D. Landau and E. M. Lifshitz. 1980; [21] S. I. Blinnikov. *Soviet Astronomy Letters*. (1987); [22] D. G. Yakovlev and D. A. Shalybkov. *Astrophysics and Space Physics Reviews*. (1989); [23] P. Haensel, A. Y. Potekhin, and D. G. Yakovlev. 2007.

†† [17] E. E. Salpeter. *ApJ*. (1961).

Then, the temperature can also be expressed in units of T_r as

$$t_\mathrm{r} \equiv \frac{T}{T_\mathrm{r}}. \tag{1.49}$$

Using these definitions, it is easy to characterize how relativistic the electron gas is. We can divide it into three regions of

- non-relativistic, for which $t_\mathrm{r} \ll 1$ and/or $x_\mathrm{r} \ll 1$,
- mildly-relativistic, $t_\mathrm{r} \sim 1$ and/or $x_\mathrm{r} \sim 1$,
- ultra-relativistic, $t_\mathrm{r} \gg 1$ and/or $x_\mathrm{r} \gg 1$.

Similarly, we can scale the temperature with the Fermi temperature to get a so-called degeneracy parameter

$$\Theta_\mathrm{F} \equiv \frac{T}{T_\mathrm{F}}. \tag{1.50}$$

It can be then used to characterize the degeneracy of the plasma to regions of

- non-degenerate, for which $\Theta_\mathrm{F} \gg 1$,
- mildly degenerate, $\Theta_\mathrm{F} \sim 1$,
- and strongly degenerate $\Theta_\mathrm{F} \ll 1$.

Moving on from an ideal gaseous plasma in the atmosphere, we can start by introducing corrections produced by the closely packed charged particles. In practice we use the so-called ion-sphere model to describe our Coulomb liquid of ions. We now assume that our ions are emerged into a sea of rigid electron background that takes care of the charge neutrality. In order to couple the number density of electrons and the mass of the plasma, let us first define the mean charge per mass ratio of the plasma as

$$\mu_{Z,A} = \frac{\langle Z \rangle}{\langle A \rangle}, \tag{1.51}$$

where $\langle Z \rangle$ is the mean charge number of the atomic nuclei (for an one-component plasma it is simply Z), and $\langle A \rangle$ is the average number of nucleons bound by one nucleus. For most plasmas, $\mu_{Z,A} \approx \frac{1}{2}$.

Let us now begin by defining a so-called electron sphere radius as

$$r_\mathrm{e} = \left(\frac{4\pi n_\mathrm{e}}{3} \right)^{-1/3}. \tag{1.52}$$

We can also parameterize the strength between Coulomb (charge) interactions by considering a ratio of potential energy to the kinetic energy with

$$\Gamma_\mathrm{e} = \frac{e^2}{r_\mathrm{e} k_\mathrm{B} T} \approx 22.75 \left(\frac{10^6\,\mathrm{K}}{T} \right) \left(\frac{\rho}{10^6\,\mathrm{g\,cm^{-3}}} \right)^{1/3} \mu_{Z,A}^{1/3}, \tag{1.53}$$

where $e = 4.80 \times 10^{-10}$ esu is the electron charge. Similarly, for ion with a charge number of Z_i, we can define the ion-sphere radius

$$r_\mathrm{i} = r_\mathrm{e} Z_\mathrm{i}^{1/3}, \tag{1.54}$$

that encapsulates enough area to be charge neutral, when considering a static electron-induced background from n_e. Ion Coulomb coupling factor is similarly

$$\Gamma_\mathrm{i} = \Gamma_\mathrm{e} Z_\mathrm{i}^{5/3} = \frac{(Z_\mathrm{i} e)^2}{r_\mathrm{i} k_\mathrm{B} T}. \tag{1.55}$$

At high temperatures, the electrons form a classical Boltzmann gas. When the temperature is decreased, the plasma will then, without a phase-transition, become a strongly coupled Coulomb liquid corresponding to the neutron star ocean. If the temperature is decreased further, the plasma will then transform into a Coulomb crystal with a phase transition. The gaseous regime can be constrained to be at $\Gamma_\mathrm{i} \ll 1$, or $T \gg T_\mathrm{B}$, where the T_B is given as

$$T_\mathrm{B} = \frac{Z^2 e^2}{a_i k_\mathrm{B}} \approx 2.28 \times 10^7 \left(\frac{\rho}{10^6\,\mathrm{g\,cm^{-3}}} \right)^{1/3} \mu_{Z,A}^{1/3}\,\mathrm{K}. \tag{1.56}$$

The pressure for such a system can be obtained via the standard thermodynamical relation

$$P = \left(\frac{\partial F}{\partial V} \right)_T, \tag{1.57}$$

where F is the Helmholtz free energy and V is the volume of the system.* It is useful to divide the free energy into an ideal part F_id corresponding to non-interacting particles, and to the excess part F_ex, leading to a splitting of

$$F(V,T) = F_\mathrm{id} + F_\mathrm{ex}. \tag{1.58}$$

Similarly, we could decompose the ideal effects from ions to $F_\mathrm{id}^{(i)}$, and from electrons to $F_\mathrm{id}^{(e)}$, giving

$$F(V,T) = F_\mathrm{id}^{(i)} + F_\mathrm{id}^{(e)} + F_\mathrm{ex}. \tag{1.59}$$

Most important non-ideal deviation for the plasma is from the Coulomb interaction between ions and electrons, and between electrons in the rigid background $F_\mathrm{ex} \approx F_\mathrm{ii}$. This free energy decomposition then induces similar splitting for the pressure

$$P \approx P_\mathrm{id}^{(i)} + P_\mathrm{id}^{(e)} + P_\mathrm{ii}. \tag{1.60}$$

A simple order-of-magnitude estimate for the relative strength of the different terms yields that $P_\mathrm{id}^{(e)}$ is the main contributor for the pressure. Other terms, given as $|P_\mathrm{part}|/P_\mathrm{id}^{(e)}$, result in leading order terms of†

$$\frac{P_\mathrm{id}^{(i)}}{P_\mathrm{id}^{(e)}} \approx \frac{\Theta_\mathrm{r}^{(i)}}{Z} \tag{1.61}$$

$$\frac{P_\mathrm{ii}}{P_\mathrm{id}^{(e)}} \approx \alpha_\mathrm{f} \frac{\gamma_\mathrm{r}}{x_\mathrm{r}}, \tag{1.62}$$

where $\Theta_\mathrm{i}^{(i)}$ is the temperature in units of nucleon Fermi temperature (see Eq. 1.50), $\alpha_\mathrm{f} = e^2/\hbar c \approx 1/137$ is the fine-structure constant, and $\hbar = h/2\pi$. From here it is easy to see that only the Coulomb interaction term P_ii, should give a measurable corrections to the pressure at low temperatures. This suggests that for the pressure P we can use the approximation

$$P \approx P_\mathrm{id}^{(e)} + P_\mathrm{ii}. \tag{1.63}$$

* [20] L. D. Landau and E. M. Lifshitz. 1980.

† [22] D. G. Yakovlev and D. A. Shalybkov. *Astrophysics and Space Physics Reviews*. (1989).

In the weak-coupling limit ($\Gamma_i \ll 1$) the Debye-Hückel result for the excess free energy is a sufficiently good approximation, expressed as[*†]

$$\frac{F_{ex}}{V} = \frac{1}{\sqrt{3}} n_i k T \Gamma_i^{3/2}, \tag{1.64}$$

where n_i is the ion number density. Hence, the pressure correction due to the Coulomb interactions can be simply presented as

$$P_{ii} \approx -0.3 n_i \frac{Z^2 e^2}{r_i}. \tag{1.65}$$

by using Eq. (1.57). Note also that $\Gamma_i = \Gamma_i(V)$.

Let us, as a final task, look into the main degenerate electron pressure term $P_{id}^{(e)}$. Pressure is defined as a flux of momentum through a surface, hence for one particular wall it is

$$P = \int_0^\infty \langle v_x p_x \rangle n(p) dp, \tag{1.66}$$

where the number density $n(p)$ is desribed by the Fermi-Dirac distribution, given by Eq. (1.26). Velocity can be obtained from the standard expression

$$v = \frac{dE}{dp} = \frac{p}{m_e}\left[1 + \left(\frac{p}{m_e c}\right)^2\right]^{-1/2}, \tag{1.67}$$

using the energy defined by Eq. (1.30). Here we have already inserted the electron mass $m = m_e = 9.11 \times 10^{-28}$ g into the equations. Hence, in the three-dimensional description the pressure is given as one third of what we would get by applying Eq. (1.27),

$$P = \frac{1}{3}\int_0^\infty v(p)\, p\, n(p)\, dp = \frac{8\pi}{3 m_e h^3}\int_0^\infty \frac{p}{\left[1 + \left(\frac{p}{m_e c}\right)^2\right]^{1/2}}\, p \,\frac{p^2}{\exp[(E-\mu)/k_B T] + 1}\, dp \tag{1.68}$$

Free energy of the system is then equal to what is left from the chemical potential after substracting the electron rest-mass energy and the pressure contribution,

$$F = (\mu - m_e c^2) n_e - \frac{8\pi}{3 m_e h^3}\int_0^\infty \frac{p^4 dp}{\left[1 + \left(\frac{p}{m_e c}\right)^2\right]^{1/2}} \frac{1}{\exp[(E-\mu)/k_B T] + 1}. \tag{1.69}$$

Sommerfield expanding the free energy expression in powers of temperature, i.e., t_r, we finally obtain[‡]

$$\frac{F}{V} = \frac{m_e c^2}{\bar{\lambda}_C^3}\frac{1}{8\pi^2}\left(x_r(1 + 2x_r^2)\gamma_r - \ln(x_r + \gamma_r) + \frac{4\pi^2}{3} t_r^2 x_r \gamma_r\right) + O(t_r^4) \tag{1.70}$$

from which it is easy to obtain an expression for the pressure by applying Eq. (1.57),

$$P_{id}^{(e)} \approx \frac{P_r}{8\pi^2}\left(\left(\frac{2}{3}x_r^2 - 1\right)\gamma_r + \ln(x_r + \gamma_r) + \frac{4\pi^2}{9} t_r^2 x_r (\gamma_r + \gamma_r^{-1})\right) \tag{1.71}$$

* [24] P. Debye and E. Hückel. *Physikalische Zeitschrift*. (1923); [20] L. D. Landau and E. M. Lifshitz. 1980; [25] S. L. Shapiro and S. A. Teukolsky. 1983; [26] D. R. Dewitt *et al*. *Phys. Rev. A*. (1996).

† Debye-Hückel approximation relays on the assumption that the charge density surrounding an ion is described by electrostatics (Poisson's equation) and the distribution of the charge around the ion itself by thermal motions of electrons (Boltzmann's equation).

‡ see e.g., [22] D. G. Yakovlev and D. A. Shalybkov. *Astrophysics and Space Physics Reviews*. (1989).

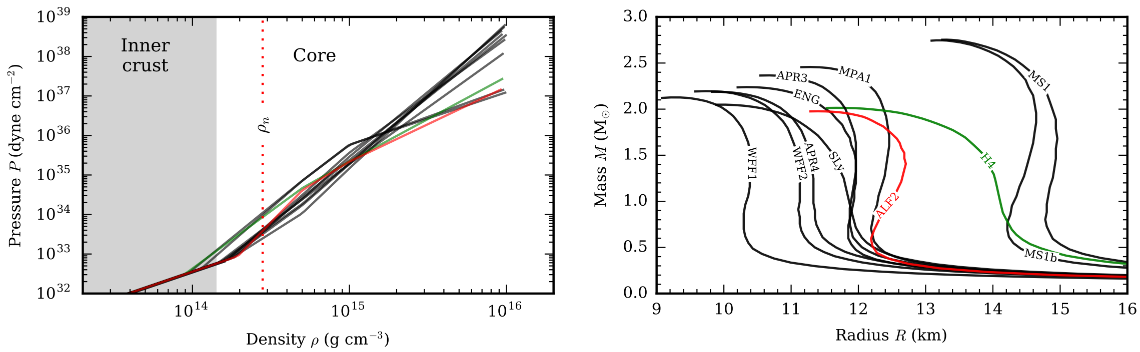

Figure 1.3: Core EoS.

where again a typical pressure is given as

$$P_\mathrm{r} = \frac{m_\mathrm{e} c^2}{\bar{\lambda}_\mathrm{C}} \approx 1.4 \times 10^{25}\,\mathrm{dyn\,cm^{-2}}, \tag{1.72}$$

where $\bar{\lambda}_\mathrm{C} = \hbar/m_\mathrm{e} c = 3.86 \times 10^{-11}$ cm is the reduced electron Compton wavelength.* Omitting the (small) temperature related term ($\propto t_\mathrm{r}^2$) leads to the well-known Chandrasekhar equation of state of for a perfect, completely degenerate electron gas.†

In the non-relativistic and ultra-relativistic regimes we can apply the asymptotic limits of Eq. (1.32). Then, the pressure given in Eq. (1.71) takes a simple polytropic form of

$$P_\mathrm{id}^{(\mathrm{e})} \approx \frac{P_\mathrm{r}}{9\pi^2 \gamma_\mathrm{AD}} x_\mathrm{r}^{3\gamma_\mathrm{AD}}, \tag{1.73}$$

where the polytropic index is given as $\gamma_\mathrm{AD} = \frac{5}{3}$ or $\gamma_\mathrm{AD} = \frac{4}{3}$ for the non-relativistic ($x_\mathrm{r} \ll 1$) and the ultra-relativistic ($x_\mathrm{r} \gg 1$) cases, respectively. Recall also that $x_\mathrm{r} \propto n_\mathrm{e}^{1/3} \propto \rho^{1/3}$. The transition occures at $x_\mathrm{r} \approx 1$, corresponding to about $\rho \sim 10^6\,\mathrm{g\,cm^{-3}}$.

Finally, as suggested by our analysis given in Eq.(1.61), the ideal degenerate electron gas pressure accompanied with the ion Coulomb correction will give us a rather good approximation for the equation of state as

$$P(x_\mathrm{r}) \approx P_\mathrm{id}^{(\mathrm{e})} + P_\mathrm{ii} = \frac{P_\mathrm{r}}{8\pi^2}\left((\frac{2}{3}x_\mathrm{r}^2 - 1)\gamma_\mathrm{r} + \ln(x_\mathrm{r} + \gamma_\mathrm{r}) \right) - 0.3 n_\mathrm{i} \frac{Z^2 e^2}{r_\mathrm{i}}, \tag{1.74}$$

remaining valid in a large density range of $10^4 < \rho < 10^{10}\,\mathrm{g\,cm^{-3}}$. Below $\rho < 10^4\,\mathrm{g\,cm^{-3}}$ the plasma is gaseous and the ideal gas law description (1.16) is better suited in modeling the equation of state. Beyond $\rho > 10^{10}\,\mathrm{g\,cm^{-3}}$ the densities become so high that nuclear degeneracy pressure and more importantly, the mutual nuclear intearactions start to play an important role. This is the topic of our next section.

1.4 Liquid core

Outer core. Density ranges $0.5\rho_\mathrm{n} < \rho < 2\rho_\mathrm{n}$. Several kilometers. Neutrons with several per cent admixture of protons p and electrons e^-. Strongly degenerate. Electrons form almost ideal Fermi gas. Neutrons and protons, interacting via nuclear forces, constitute a strongly interacting Fermi liquid.

*Normal Compton wavelength is simply $\lambda_\mathrm{e} = 2\pi \times \bar{\lambda}_\mathrm{C} = h/m_\mathrm{e} c = 2.43 \times 10^{-10}$ cm.

† [13] E. C. Stoner. *Philos. Mag.* (1930); [27] S. Chandrasekhar. *MNRAS*. (1935); [15] S. Chandrasekhar. 1939.

Inner core. Where $\rho > 2\rho_n$. Appearance of muon. Central density can be around $(10-15)\rho_n$. Very model dependent. Main problem.

Here we consider few (relatively) modern descriptions for the dense matter equation of state. The EoSs can be divided into few different classes based on the particles that they consist of. Here they are divided into matter consisting of

- plain $(npe\mu)$nuclear matter,
- normal nuclear matter spiced up with hyperons (H),
- normal nuclear matter together with more exotic particles like pion and kaon condensates (πK),
- and matter consisting of (or normal matter spiced up with) quarks (q).

For $(npe\mu)$we include models computed with

- potential method using SLy effective nuclear interaction that is of Skyrme-type,*,
- four variational method EoSs, APR3/4† and WFF1/2‡,
- two relativistic Brueckner-Hartree-Fock calculations, ENG§ and MPA1¶,
- two relativistic mean-field theory, MS1 and MS1b (same as MS1 but with lower symmetry energy)‖.

For hyperon models (H)we include

- one variant of relativistic mean-field theory EoS H4**.

EoSs where mesons, like pion and kaon condensates (πK), are taken into account end up not being stiff enough.

Finally, for the hybrid nuclear matter and quark matter compositions (q)we include

- mixed APR nuclear matter and color-flavor-locked quark matter EoS ALF2.††

1.4.1 Polytropes

Parameterize everything with polytropes.

$P(\rho) = K\rho^\gamma$, γ is a the polytropic index and is a measure of *stiffness* of the EoS. how strongly the matter responds to an increase in density with an increase in pressure

1.5 Tolman-Volkoff-Oppenheimer equations

Newtonian pressure gradient needed to oppose the gravity is

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}. \tag{1.75}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho, \tag{1.76}$$

* [28] F. Douchin and P. Haensel. *A&A*. (2001).

† [29] A. Akmal, V. R. Pandharipande, and D. G. Ravenhall. *Phys. Rev. C*. (1998).

‡ [30] R. B. Wiringa, V. Fiks, and A. Fabrocini. *Phys. Rev. C*. (1988).

§ [31] L. Engvik *et al*. *ApJ*. (1996).

¶ [32] H. Müther, M. Prakash, and T. L. Ainsworth. *Physics Letters B*. (1987).

‖ [33] H. Müller and B. D. Serot. *Nuclear Physics A*. (1996).

** [34] B. D. Lackey, M. Nayyar, and B. J. Owen. *Phys. Rev. D*. (2006).

†† [35] M. Alford *et al*. *ApJ*. (2005).

Taking into account the general relativistic corrections we get

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \times \frac{(1 + P/\rho c^2)(1 + 4\pi r^3 P/mc^2)}{1 - 2Gm/rc^2}. \tag{1.77}$$

Difference originates from the source of gravity: in the Newtonian case it is the mass m, whereas in the General relativity it is the energy momentum tensor that depend both on the energy density and the pressure. As a result, energy and pressure give rise to a gravitational fields.

Severness of the GR corrections can be estimated from the so called compactness parameter

$$x = \frac{GM}{Rc^2} \approx 2.95\ M/\ M_\odot\ \mathrm{km} \tag{1.78}$$

It has an important consequence to the stability of neutron stars: Successive increase in the pressure to counter the gravity is ultimately self-defeating.

Solution for a constant density ρ_0 gives

$$P(r) = G\frac{2\pi}{3}\rho_0^2(R^2 - r^2) \tag{1.79}$$

whereas the GR gives

$$P(r) = \rho_0^2 c^2 \left[\frac{(1 - u\left(\frac{r}{R}\right)^2)^{1/2} - (1 - u)^{1/2}}{3(1 - u)^{1/2} - (1 - u\left(\frac{r}{R}\right)^2)^{1/2}}\right], \tag{1.80}$$

where $u = 2GM/Rc^2$.

Another perspective is shown in Fig. 1.4 that visualizes the dependency of the pressure and cumulative mass of the star against the radial coordinate as measured starting from the core.

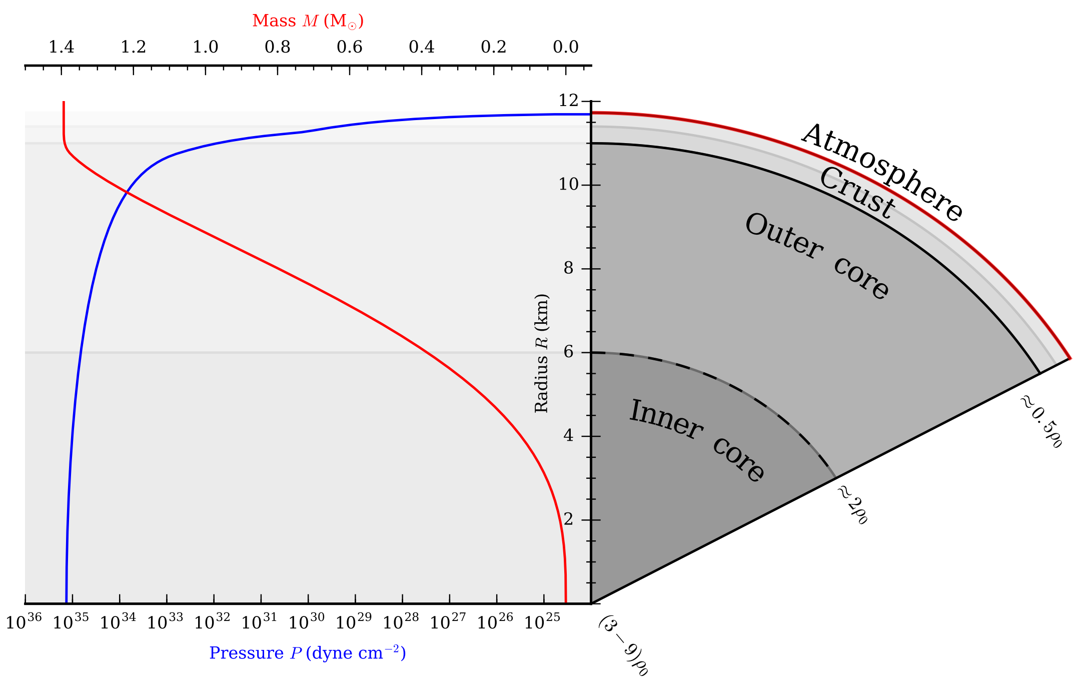

Figure 1.4: Overview of the neutron star structure. Right side of the figure shows a schematic presentation of the star's interiors against the radial coordinate, whereas the left side shows the pressure (blue; bottom axis) and cumulative mass (red; top axis) evolution from the core to the surface.

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