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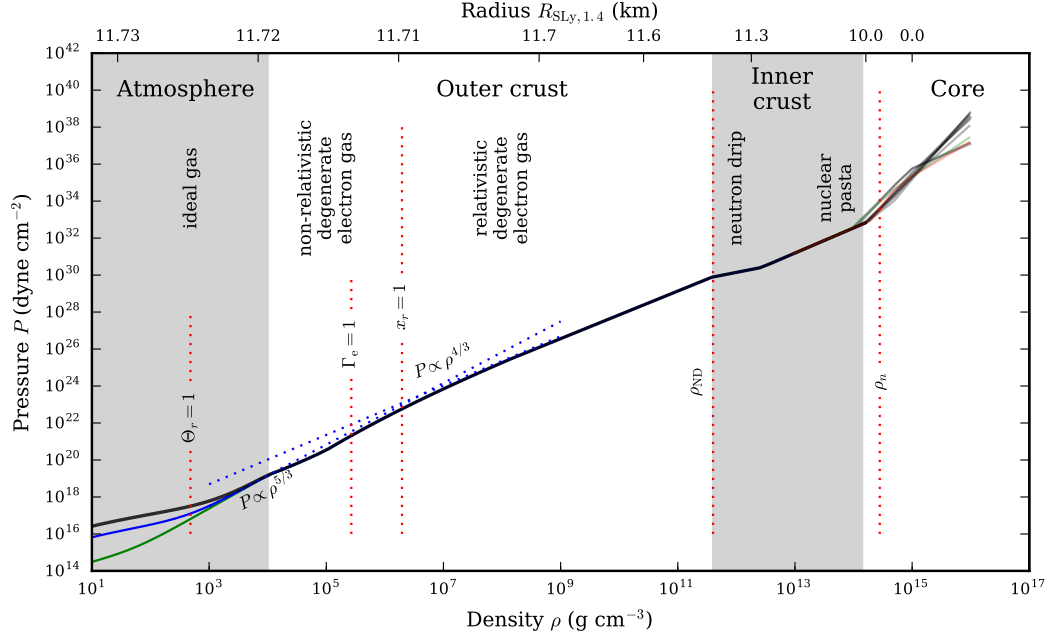


Figure 1.2: Overview of the pressure versus density relation for the full range of densities relevant for neutron stars. Here the evolution of the pressure is shown against the densities depicted in the bottom vertical axis. Green solid line shows the EoS for matter at $T = 10^6$ K, whereas blue line is for $T = 5 \times 10^6$ K, and black for $T = 10^7$ K. Additionally, the upper vertical axis shows the evolution of the radial coordinate computed for one particular EoS (SLy, see Sect. 1.4) and neutron star configuration (mass of $1.4 M_\odot$). Different shaded vertical regions show the corresponding interior structures of the star. Additionally, some interesting densities are highlighted with dashed red lines and text labels (see Sects. 1.2–1.4).

In thermodynamics, we speak of *state variables* that describe a current state of the matter under a given physical conditions. These include, for example, the density ρ , pressure P , and temperature T of the matter. Equation of state is a thermodynamical equation connecting these states variables together. Often, when focusing on neutron stars, what we mean by EoS is a function connecting the pressure and the density of the matter only, $P(\rho)$.

The dependency on the rest of the variables such as temperature can be often forgotten because the matter is *degenerate*. In contrast to the “normal” matter where a statistical moments such as temperature can be used to describe a large ensemble of particles, the degenerate matter is dominated by quantum mechanical effects of single particles. Because of the immense densities, a free particle in a degenerate matter is actually bounded into a finite volume. Inside this small volume, the energy levels of the particle are restricted to take only a discrete set of values called quantum states, because of the underlying wave-nature of the quantum mechanical description. Hence, a notion of temperature, for example, does not make much sense.

Overview of the EoS for the full range of densities relevant to neutron stars is shown in Fig. 1.2. From here it is easy to see that temperature only plays a role in the very uppermost ~ 10 meters of the stars interiors. Behavior of the matter is also quite well known all the way up to the crust-core interface, after which we start to see larger deviations because of the different EoS models. In the Earthly laboratories we can probe the matter somewhere close to $10^{14} \text{ g cm}^{-3}$, after which the densities becomes too great for us

to handle in.* On the other hand, it is exactly starting from this density range that the bulk of the neutron star just starts. Another curious quirk of Nature is how all of the complicated microphysics gets reduced to simple line segments in the logarithmic scales, also known as polytropic pressure relations. In the following sections, we will focus on deriving these simple relations as it helps us in understanding the underlying physics.

1.2 Atmosphere

Atmosphere of a star is the first and uppermost layer responsible for the emergent radiation. It consists of a thin layer of plasma and ranges from a few millimeters to couple of centimeters in height. In most situations the plasma is in a gaseous state, but in some more rare cases when the magnetic field is extraordinarily strong and the temperature is low, the plasma can condensate into a liquid or a solid surface. Such condensed surfaces are, however, rare and usually the gaseous description is more than enough.†

Properties of the emergent thermal radiation strongly depend on the chemical composition of the atmosphere. In the atmospheres of normal stars the composition is a mixture of multiple elements. The most stable chemical element on the surface of a neutron star is iron. However, even a small accreted mass of $10^{-17} M_\odot$, originating from the surrounding interstellar medium, is enough to cover the whole star, and hence a variety of elements are also expected in the neutron star atmospheres. On the other hand, the enormous gravity results in an effective separation of elements leading to a strong sedimentation of the atmosphere where the lighter elements are expected to lay on top of the heavier ones.‡ Hence, the atmosphere is usually expected to consist of mainly hydrogen.

1.2.1 General relativistic effects

The effects from the gravity can be quantified by considering a so-called compactness parameter

$$u = \frac{R_S}{R} \quad (1.1)$$

where R is the radius of the neutron star and the corresponding Schwarzschild radius is defined as

$$R_S = \frac{2GM}{c^2} \approx 2.95 \frac{M}{M_\odot} \text{ km}, \quad (1.2)$$

where G is the gravitational constant, c is the speed of light, and M is the mass of the star. Hence, a neutrons star has a compactness parameter in the range of $u \approx 1/5$ to $1/2$ resulting in a considerable general relativistic corrections. In comparison, the Sun has $u \approx 4.24 \times 10^{-6}$. Gravitational acceleration under general relativistic theory is

$$g = \frac{GM}{R^2} \frac{1}{\sqrt{1-u}} = 1.38 \times 10^{14} \frac{1}{\sqrt{1-u}} \left(\frac{M}{M_\odot} \right) \left(\frac{10 \text{ km}}{R} \right)^2 \text{ cm s}^{-2}. \quad (1.3)$$

Hence, a surface gravitational accelerations g of $\sim 10^{14}$ to $\sim 10^{15} \text{ cm s}^{-2}$ are expected for neutron stars. By considering a barometric atmosphere we can also estimate the scale height as

$$H_a = \frac{k_B T}{m_i g} \approx \frac{0.83}{A} \left(\frac{T}{10^6 \text{ K}} \right) \left(\frac{10^{14} \text{ cm s}^{-2}}{g} \right) \text{ cm} \quad (1.4)$$

*Maximum densities reached in the Earth are usually obtained by colliding heavy nucleons together, momentarily creating a core of even denser matter. The densest naturally occurring element found on top of planet Earth is osmium that has a density of

“just” $\rho \approx 2.2 \times 10^4 \text{ g cm}^{-3}$.

†for a review, see [1] V. E. Zavlin and G. G. Pavlov. 2002;

[2] A. Y. Potekhin. *Physics Uspekhi*. (2014).

‡ [3] C. Alcock and A. Illarionov. *ApJ*. (1980).

where $k_B = 1.38 \times 10^{-16}$ erg K⁻¹ is the Boltzmann constant, T is the temperature of the atmosphere, $m_i = Am_u$, and $m_u \approx 1.66 \times 10^{-24}$ g is the atomic mass unit. From here, the typical scale height values of ~ 1 cm to ~ 10 cm are re-obtained for atmospheres of $T = 10^6$ and 10^7 K.* Strong gravitational field also bends the photon trajectories.† Hence, in addition to the radius R of the star as measured in the local reference frame, another *apparent* radius, as measured by an observer at infinity,

$$R_\infty = \frac{R}{\sqrt{1-u}}, \quad (1.5)$$

is usually needed when describing the observable features of the atmosphere. From here it is then clear that the atmosphere and the emerging radiation encodes information from the physical parameters of the star. More specifically information about the temperature, surface gravity, chemical composition, compactness can be obtained.

1.2.2 Basic equations of the atmosphere

The standard approach in describing the atmosphere structure includes solving three main equations of radiative transfer, hydrostatic balance, and energy conservation. First such a low- B field model of neutron star atmospheres were presented in the pioneering work by Romani.‡ Let us next see, walk through these equations, as they are rather simple. A more general description for the atmosphere model computations are given in Sect.XX§, where the full relativistic electron scattering is also taken into account.

Because the thickness of the atmosphere is much smaller than the radius of the star,¶ the atmosphere can be considered in plane-parallel approximation. Rather high densities, on the other hand, allow to consider the plasma of the atmosphere in local thermodynamical equilibrium.

Spectrum, beaming and polarization of emerging radiation can be determined from radiation transfer problem in atmospheric layers. Radiation can be understood as an energy flow, i.e., energy E per area A , time t , frequency ν , and solid angle Ω . This is known as the specific spectral intensity which we can mathematically formulate as

$$I_\nu = \frac{dE}{dAdtd\nu d\Omega}. \quad (1.6)$$

Radiation averaged over the solid angle, or the so-called mean specific intensity (zeroth moment of I_ν) is then

$$J_\nu = \frac{1}{4\pi} \int_\Omega I_\nu d\Omega = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi I_\nu \sin\theta d\theta d\phi = \frac{1}{2} \int_{-1}^{+1} I_\nu d\mu, \quad (1.7)$$

where we have assumed that the radiation does not depend on the azimuthal ϕ angle (as is typical for atmosphere calculations) and introduced $\mu = \cos\theta$. Net rate of energy flowing across an unit area (for example a photon detector) from *all directions* per time and frequency is known as physical flux.‖ It is proportional to the first-order moment of I_ν and is defined as

$$F_\nu = 2\pi \int_{-1}^{+1} I_\nu \mu d\mu. \quad (1.8)$$

* [1] V. E. Zavlin and G. G. Pavlov. 2002; [2] A. Y. Potekhin. *Physics Uspekhi*. (2014).

† [4] K. R. Pechenick, C. Ftaclas, and J. M. Cohen. *ApJ*. (1983).

‡ [5] R. W. Romani. *ApJ*. (1987).

§NKS15.

¶Recall the scale height of 1 to 10 cm in comparison to the radius of 10^6 cm.

‖Strictly speaking, the first-order moment of I_ν is known as the Eddington flux $H_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu \mu d\mu$. Physical flux is related to it as $F_\nu = 4\pi H_\nu$ and sometimes one also encounters the “astro-physical” flux defined as F_ν/π .

Similarly, the second-order moment of I_ν , or a so-called K-integral, is

$$K_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu \mu^2 d\mu, \quad (1.9)$$

which is related to the radiation pressure, as we later on will see.

Now we can introduce the radiative transfer equation for I_ν as

$$\mu \frac{dI_\nu}{d\tau} = \frac{\mu}{\kappa_\nu} \frac{dI_\nu}{dy} = -\frac{\mu}{\rho \kappa_\nu} \frac{dI_\nu}{dz} = I_\nu - S_\nu \quad (1.10)$$

where τ is the optical depth, y is the column density (mass per area), z is the horizontal distance from the surface, $\kappa_\nu = \alpha_\nu + \sigma_\nu$ is the total radiative opacity including contributions from the “true” opacity α_ν and from the scattering opacity σ_ν . Here the connection between different independent variables is given as

$$d\tau = \kappa_\nu dy = -\kappa_\nu \rho dz, \quad (1.11)$$

relating the optical depth (distance as experienced by the radiation), column density (projected number density of matter along the path of the radiation), and the horizontal height. In addition we need the source function

$$S_\nu = (\sigma_\nu J_\nu + \alpha B_\nu) \kappa_\nu^{-1}, \quad (1.12)$$

where the scattering term is proportional to the mean spectral intensity J_ν and the “true” absorption term to the thermal Planck function

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp[h\nu/(k_B T)] - 1}, \quad (1.13)$$

where $h = 6.63 \times 10^{-27}$ erg s is the Planck constant. As an boundary conditions for this equation, we can use $I_\nu = 0$ for $\mu < 0$ at $y = 0$ (i.e., the surface). The atmospheres are also usually considered to be in radiative and hydrostatic equilibrium, i.e., (quasi-)stationary. The first requirement can be formulated as

$$\int_0^\infty d\nu \int_{-1}^{+1} I_\nu \mu d\mu = \sigma_{\text{SB}} T_{\text{eff}}, \quad (1.14)$$

where $\sigma_{\text{SB}} = 5.67 \times 10^{-5}$ g s⁻³ K⁻⁴ is the Stefan-Boltzmann constant and T_{eff} is the effective temperature of the atmosphere. The second hydrostatic equilibrium, demands that

$$\frac{dP}{dy} = g - g_{\text{rad}}, \quad (1.15)$$

where, in addition to the gravitational acceleration g we need the opposing radiative acceleration g_{rad} . Finally, we need to supplement these equations with an equation connecting the pressure and density. For the rarefied atmosphere, the ideal gas law is an excellent approximation

$$P = nk_B T, \quad (1.16)$$

where n is the number density of particles.

1.2.3 Eddington limit

Usually the atmospheres we calculate are dynamically stable and in hydrostatic balance because large gravity implies $g_{\text{rad}} \ll g$. Sometimes, however, the radiation flux might increase to such a strength that it is able to compete against even the enormous gravity of a neutron star. An important limit can then be defined for $g_{\text{rad}} = g$, known as the Eddington limit after a renowned astrophysicist Sir Arthur Eddington. Let us now for completeness derive this limit.*

We can start by formulating the radiation pressure. This can be easily done when we realize that pressure is just momentum flux, and photons carry a momentum of E/c .[†] In terms of I_ν this is then

$$P_{\text{rad},\nu} = \frac{1}{c} \int_0^{2\pi} \int_{-1}^{+1} I_\nu \mu^2 d\mu = \frac{4\pi}{c} K_\nu, \quad (1.17)$$

relating the pressure and the second-order moment K_ν together. Radiative acceleration is then

$$g_{\text{rad}} = \frac{dP_{\text{rad}}}{dy} = \frac{d}{dy} \int_0^\infty P_{\text{rad},\nu} d\nu = \frac{4\pi}{c} \frac{d}{dy} \int_0^\infty K_\nu d\nu. \quad (1.18)$$

Let us refine this expression by inserting the definition of K_ν and applying the radiative transfer equation (1.10) without the source function S_ν . In the process we also have to take into account that we are only interested in the outgoing flux, hence the integration limits of μ need to be changed as $[-1, +1] \rightarrow [0, +1]$. We can then simplify the (1.18) to

$$\begin{aligned} g_{\text{rad}} &= \frac{4\pi}{c} \frac{d}{dy} \int_0^\infty d\nu \frac{1}{2} \int_0^{+1} I_\nu \mu^2 d\mu \\ &= \frac{2\pi}{c} \int_0^\infty d\nu \int_0^{+1} \mu d\mu \left\{ \mu \frac{d}{dy} I_\nu \right\} \\ &= \frac{2\pi}{c} \int_0^\infty d\nu \int_0^{+1} \mu d\mu \kappa_\nu I_\nu \\ &= \frac{2\pi}{c} \int_0^\infty d\nu \kappa_\nu \int_0^{+1} I_\nu \mu d\mu \\ &= \frac{1}{c} \int_0^\infty \kappa_\nu F_\nu d\nu. \end{aligned} \quad (1.19)$$

From the bottom line of Eq. (1.19) we see that the K-integral is related to the flux of the radiation F_ν . Not every photon, however, interacts and collides with the matter, and hence the opacity correction κ_ν is also needed, representing effectively the fraction of radiation interacting with the matter. If we now integrate over all the frequencies we obtain

$$g_{\text{rad}} = \frac{1}{c} \kappa F \quad (1.20)$$

Setting it equal to g we can solve for the Eddington flux as

$$F_{\text{Edd}} = \frac{gc}{\kappa} \frac{1}{\sqrt{1-u}} = \frac{GMc}{R^2 \kappa} \frac{1}{\sqrt{1-u}}. \quad (1.21)$$

*for a more in depth discussion, see e.g., [6] G. B. Rybicki and A. P. Lightman. 1979; [7] J. Frank, A. King, and D. J. Raine. 2002.

[†]Energy of a photon is $E = h\nu = mc^2$, from which we obtain

the photon rest mass of $h\nu/c^2 = E/c^2$. Momentum, on the other hand, is just velocity times the mass, so for a photon it is $E/c^2 \times c = E/c$.

Usually in many astrophysical scenarios the opacity is dominated by the electron Thomson scattering opacity, $\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$. Mass for a hydrogen plasma is, on the other hand, mainly set by the protons, hence, opacity per mass is then

$$\kappa \approx \frac{\sigma_T}{m_p}. \quad (1.22)$$

Finally, by calculating the total radiation flux through the stellar surface, we can define a quantity called luminosity as

$$L = 4\pi R^2 F(R), \quad (1.23)$$

where $F(R)$ is the outgoing radiation flux at the surface. Using the aforementioned equation, we can then define the Eddington luminosity for a star as

$$L_{\text{Edd}} = \frac{4\pi G M c m_p}{\sigma_T} \approx 1.3 \times 10^{38} \left(\frac{M}{M_\odot} \right) \text{ erg s}^{-1}. \quad (1.24)$$

For neutron stars close to $T \sim 10^7 \text{ K}$ this limit can be reached, after which the atmosphere is momentarily blown away.

1.3 Crust

Below the gaseous atmosphere, a solidified layer of matter exists, called crust.* Between the atmosphere and crust, a liquid ocean of ions also exists, but the interface is not very strict and the matter is smoothly evolving from one state to another. The solidified crust is also typically divided into an outer and inner layers, but the interface is again ambiguous. The pressure here is almost fully given by the degenerate electrons, and hence the matter is familiar already from white dwarfs. In the beginning, the electrons can be taken to be non-relativistic but after about $\rho \sim 10^6 \text{ g cm}^{-3}$ they turn into ultra-relativistic because of the increasing density.

By definition, the outer crust is a layer in the neutron star interiors where the plasma consists of electrons and nuclei, whereas the inner crust is characterized by an additional appearance of neutrons that start to drip out from the extremely neutron-rich nuclei. The density this occurs is called the neutron drip density and is of around $\rho_{\text{ND}} \sim 4 \times 10^{11} \text{ g cm}^{-3}$. The outer crust, when defined to begin from the atmosphere at $\rho \sim 10^3 \text{ g cm}^{-3}$ and continue to about ρ_{ND} is only about some hundred meters in thickness. The characteristics of the matter are strongly dependent on the Coulomb interactions of charged particles that form a solid Coulomb crystal.

The inner crust is taken to continue all the way down to the crust-core interface, where the matter turns liquid again. This layer is about one kilometer thick. Here, at the bottom of the crust, the density is already close to the nuclear density of $\rho_n = 2.8 \times 10^{14} \text{ g cm}^{-3}$ but the exact location of the transition depends on the detailed microphysics of the core. The fraction of free neutrons grows with the increasing density. Because the normal nuclei here are immersed into free neutron gas, the nuclear interactions play a crucial role in defining the matter. Finally, the nuclei disappear totally when we enter the core. Before that, however, the nuclei form complicated structures that evolve together with the density. This region is also known as the nuclear pasta phase, as the different molecular structures are named after the pasta types that they resemble.†

1.3.1 Fermi gases

The matter in the crust is degenerate.‡ Let us discuss the physics behind degenerate matter and the related important concept of Fermi energy now. Elementary 6-dimensional phase space cell of any particle is bounded

*for a review, see, e.g., [8] P. Haensel, A. Y. Potekhin, and D. G. Yakovlev. 2007; [9] N. Chamel and P. Haensel. *Living Reviews in Relativity*. (2008); [10] C. Bertulani and J. Piekarewicz. 2012.

†see e.g. [11] M. E. Caplan and C. J. Horowitz. *ArXiv e-prints*. (2016).

‡for a more in depth discussion of the topic, see e.g., [12] L. D. Landau and E. M. Lifshitz. 1980.

by the Heisenberg uncertainty principle as

$$\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3, \quad (1.25)$$

where $\Delta x \Delta y \Delta z$ is the volume in ordinary space, $\Delta p_x \Delta p_y \Delta p_z$ the volume in the momentum space, and h is the previously defined Planck constant. In accordance with the quantum mechanics, there is room for only one particle of any kind inside the elementary cell. In general the number density is given as

$$n_{\text{av}} = \frac{g}{\exp[(E - \mu)/k_B T] \pm 1}, \quad (1.26)$$

where $+$ is used for fermions (such as electrons and protons), and $-$ for bosons (such as photons). Additionally, the g is the number of different quantum states a particle may have inside the cell, E is the total particle energy, and μ is the chemical potential explained more carefully later on. Number of particles with momentum between p and $p + dp$ is

$$n(p)dp = n_{\text{av}} \frac{4\pi p^2}{h^3} dp, \quad (1.27)$$

i.e., particles inside a spherical shell with a surface $4\pi p^2$ and thickness dp . Number density of particles with all momenta is then

$$n = \int_0^\infty n(p)dp \quad (1.28)$$

and the density is then simply

$$\rho = nm, \quad (1.29)$$

where m is the mass of the particle. The energy in equation (1.26) is the total energy $E \equiv E_{\text{tot}} = E_0 + E_k$, composed of the rest-mass energy $E_0 = mc^2$ and kinetic energy E_k . In special relativity, there exists a relation for the total energy as $E_{\text{tot}}^2 = (mc^2)^2 + (pc)^2$, so

$$E = mc^2 \left[1 + \left(\frac{p}{mc} \right)^2 \right]^{1/2}, \quad (1.30)$$

and the kinetic energy is then simply

$$E_k = mc^2 \left[\left(1 + \left(\frac{p}{mc} \right)^2 \right)^{1/2} - 1 \right]. \quad (1.31)$$

Simple asymptotic limits can be obtained for the kinetic energy as

$$\begin{aligned} E &\approx \frac{p^2}{2m} & p \ll mc & \text{(non-relativistic case)} \\ E &\approx pc & p \gg mc & \text{(ultra-relativistic case).} \end{aligned} \quad (1.32)$$

Let us now focus on fermions only, and hence select the $+$ -sign (Fermi-Dirac distribution) from the Eq. (1.26). Fermions are spin $\frac{1}{2}$ particles so we can also set $g = 2$. The chemical potential μ in Eq. (1.26) is expressed as

$$\mu = mc^2 + \epsilon_F \quad (1.33)$$

where ϵ_F is now the so-called Fermi energy. On the other hand, it can also be defined as

$$E - \mu = E_k - \epsilon_F \quad (1.34)$$

i.e., difference between the kinetic and Fermi energy. Hence, the number density of fermions in general is

$$n_f = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp(\frac{E_k - \epsilon_F}{k_B T}) + 1}. \quad (1.35)$$

If Fermi energy $\epsilon_F \ll 0$ the distribution will be Maxwellian even for small kinetic energies. On the other hand, if $\epsilon_F \gg k_B T$ we can divide the integrand of Eq. (1.35) into two distinct regions of

$$n_f = \begin{cases} p^2 & \text{if } E_k \ll \epsilon_F \\ 0 & \text{if } E_k \gg \epsilon_F \end{cases} \quad (1.36)$$

where the transition occurs rather sharply at $E_k = \epsilon_F$. This allows us to define a characteristic momentum related to the transition, called Fermi momentum p_F , so that

$$n_f \approx \frac{8\pi}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi}{3} \left(\frac{p_F}{h} \right)^3. \quad (1.37)$$

Physically this can be interpreted such that when the temperature decreases, the fermions start to occupy all the quantum states starting from the one with lowest energy all the way up to the Fermi energy. Because of the Pauli exclusion principle, no more than one fermion can exist in the same quantum state so the levels are filled in order, and all the higher states will remain empty. Hence, the highest momenta possible in the degenerate matter is the Fermi momentum

$$p_F = \left(\frac{3n_f}{8\pi} \right)^{1/3} h. \quad (1.38)$$

1.3.2 Why neutrons then?

Let us first consider an ideal gas of degenerate electron-proton-neutron plasma to understand the basic composition of the crust.* In a degenerate plasma all the quantum states are filled up all the way to the Fermi energy, as we just learned. Normal beta-decay mode for the neutrons, on the other hand, is

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad (1.39)$$

that describes the possible path of how a neutron n will decay into a proton p , electron e^- , and electron neutrino $\bar{\nu}_e$. It is because of this decay, that we do not expect to see any free neutron flying around. Such a decay is, however, might be blocked because there is no room for an emission of an extra electron e^- or a proton p .

Let us then only focus on the decay of the most energetic neutrons with an energy equal to the Fermi energy $\epsilon_F(n)$, where the related particle species is defined inside the parentheses. Co-existence of neutrons, protons, and electrons is then guaranteed (at zero temperature) if

$$\epsilon_F(n) = \epsilon_F(p) + \epsilon_F(e^-). \quad (1.40)$$

Massive neutrons and protons are to a good approximation non-relativistic up to a densities of $10^{14} \text{ g cm}^{-3}$, and hence energy is simply a sum of their rest mass energy and kinetic energy

$$\epsilon_F(n) \approx m_n c^2 + \frac{p_F(n)^2}{2m_n}, \quad (1.41)$$

*see e.g. [13] A. C. Phillips. 1994.

and

$$\epsilon_F(p) \approx m_p c^2 + \frac{p_F(p)^2}{2m_p}. \quad (1.42)$$

Electrons, on the other hand, are already ultra-relativistic, and so

$$\epsilon_F(e^-) \approx p_F(e^-)c^2. \quad (1.43)$$

Also note that $n_p = n_e$, as the star is electrically neutral.

From this we find relation of the $n_n/n_p \sim 1/200$ by taking into account the rest mass difference $m_p - m_n = 2.6 \text{ MeV } c^2$, for example at $\rho \sim 10^{14} \text{ g cm}^{-3}$. Thus, we conclude that the matter inside is expected to be neutron rich, even though normally the neutrons would β -decay back to protons and electrons. Not only is the degeneracy then responsible for the pressure but it is also the source of the neutron enrichment.

1.3.3 Degenerate electron gas

Let us next consider the equation of state for the crust. As we have seen, the pressure in the crust originates from the degenerate electron gas. Physics behind this are quite simple and we repeat the calculations here to introduce the reader to the topic. The result also bears some historical value as these are exactly the equations that were introduced by Dirac^{*}, Fowler[†], Frenkel[‡], Anderson[§], Stoner[¶], and Chandrasekhar^{||}. More thorough discussion of the electron thermodynamics is given, for example, in Refs.^{**}

The behavior of the matter in the crust is dominated mainly by the electrons. For this reason, it can be characterized by the electron number density n_e and temperature T_e , hereafter just T in this section. Instead of n_e , let us use the electron Fermi momentum p_F (1.38) as a measure of the number density. It is convenient to describe it in the units of electron rest mass, as

$$x_r \equiv \frac{p_F}{m_e c}, \quad (1.44)$$

also known as the relativity parameter.^{††} We will also need the more general relativistic form of the Fermi energy

$$\epsilon_F = c^2 \sqrt{(m_e c)^2 + p_F^2}, \quad (1.45)$$

that for a strongly degenerate gas, has the meaning of the chemical potential μ . Finally, the electron Fermi temperature is

$$T_F = \frac{m_e c^2}{k_B} \left(\sqrt{1 + \left(\frac{p_F}{m_e c} \right)^2} - 1 \right) = T_r (\gamma_r - 1), \quad (1.46)$$

where a typical temperature is

$$T_r = \frac{m_e c^2}{k_B} \sim 6 \times 10^9 \text{ K}, \quad (1.47)$$

^{*} [14] P. A. M. Dirac. *Proceedings of the Royal Society of London Series A*. (1925).

[†] [15] R. H. Fowler. *MNRAS*. (1926).

[‡] [16] J. Frenkel. *Zeitschrift für Physik*. (1928).

[§] [17] W. Anderson. *Zeitschrift für Physik*. (1929).

[¶] [18] E. C. Stoner. *Philos. Mag.* (1930).

^{||} [19] S. Chandrasekhar. *MNRAS*. (1931).

^{**} [20] S. Chandrasekhar. 1939; [21] E. L. Schatzman. 1958; [22] E. E. Salpeter. *ApJ*. (1961); [23] R. F. Tooper. *ApJ*. (1969); [24] Y. B. Zeldovich and I. D. Novikov. 1971; [12] L. D. Landau and E. M. Lifshitz. 1980; [25] S. I. Blinnikov. *Soviet Astronomy Letters*. (1987); [26] D. G. Yakovlev and D. A. Shalybkov. *Astrophysics and Space Physics Reviews*. (1989); [8] P. Haensel, A. Y. Potekhin, and D. G. Yakovlev. 2007.

^{††} [22] E. E. Salpeter. *ApJ*. (1961).

and a relativistic scaling factor is defined as

$$\gamma_r \equiv \sqrt{1 + x_r^2}. \quad (1.48)$$

Then, the temperature can also be expressed in units of T_r as

$$t_r \equiv \frac{T}{T_r}. \quad (1.49)$$

Using these definitions, it is easy to characterize how relativistic the electron gas is. We can divide it into three regions of

- non-relativistic, for which $t_r \ll 1$ and/or $x_r \ll 1$,
- mildly-relativistic, $t_r \sim 1$ and/or $x_r \sim 1$,
- ultra-relativistic, $t_r \gg 1$ and/or $x_r \gg 1$.

Similarly, we can scale the temperature with the Fermi temperature to get a so-called degeneracy parameter

$$\Theta_F \equiv \frac{T}{T_F}. \quad (1.50)$$

It can be then used to characterize the degeneracy of the plasma to regions of

- non-degenerate, for which $\Theta_F \gg 1$,
- mildly degenerate, $\Theta_F \sim 1$,
- and strongly degenerate $\Theta_F \ll 1$.

Moving on from an ideal gaseous plasma in the atmosphere, we can start by introducing corrections produced by the closely packed charged particles. In practice we use the so-called ion-sphere model to describe our Coulomb liquid of ions. We now assume that our ions are emerged into a sea of rigid electron background that takes care of the charge neutrality. In order to couple the number density of electrons and the mass of the plasma, let us first define the mean charge per mass ratio of the plasma as

$$\mu_{Z,A} = \frac{\langle Z \rangle}{\langle A \rangle}, \quad (1.51)$$

where $\langle Z \rangle$ is the mean charge number of the atomic nuclei (for an one-component plasma it is simply Z), and $\langle A \rangle$ is the average number of nucleons bound by one nucleus. For most plasmas, $\mu_{Z,A} \approx \frac{1}{2}$.

Let us now begin by defining a so-called electron sphere radius as

$$r_e = \left(\frac{4\pi n_e}{3} \right)^{-1/3}. \quad (1.52)$$

We can also parameterize the strength between Coulomb (charge) interactions by considering a ratio of potential energy to the kinetic energy with

$$\Gamma_e = \frac{e^2}{r_e k_B T} \approx 22.75 \left(\frac{10^6 \text{ K}}{T} \right) \left(\frac{\rho}{10^6 \text{ g cm}^{-3}} \right)^{1/3} \mu_{Z,A}^{1/3}, \quad (1.53)$$

where $e = 4.80 \times 10^{-10}$ esu is the electron charge. Similarly, for ion with a charge number of Z_i , we can define the ion-sphere radius

$$r_i = r_e Z_i^{1/3}, \quad (1.54)$$

that encapsulates enough area to be charge neutral, when considering a static electron-induced background from n_e . Ion Coulomb coupling factor is similarly

$$\Gamma_i = \Gamma_e Z_i^{5/3} = \frac{(Z_i e)^2}{r_i k_B T}. \quad (1.55)$$

At high temperatures, the electrons form a classical Boltzmann gas. When the temperature is decreased, the plasma will then, without a phase-transition, become a strongly coupled Coulomb liquid corresponding to the neutron star ocean. If the temperature is decreased further, the plasma will then transform into a Coulomb crystal with a phase transition. The gaseous regime can be constrained to be at $\Gamma_i \ll 1$, or $T \gg T_B$, where the T_B is given as

$$T_B = \frac{Z^2 e^2}{a_i k_B} \approx 2.28 \times 10^7 \left(\frac{\rho}{10^6 \text{ g cm}^{-3}} \right)^{1/3} \mu_{Z,A}^{1/3} \text{ K}. \quad (1.56)$$

The pressure for such a system can be obtained via the standard thermodynamical relation

$$P = \left(\frac{\partial F}{\partial V} \right)_T, \quad (1.57)$$

where F is the Helmholtz free energy and V is the volume of the system.* It is useful to divide the free energy into an ideal part F_{id} corresponding to non-interacting particles, and to the excess part F_{ex} , leading to a splitting of

$$F(V, T) = F_{\text{id}} + F_{\text{ex}}. \quad (1.58)$$

Similarly, we could decompose the ideal effects from ions to $F_{\text{id}}^{(i)}$, and from electrons to $F_{\text{id}}^{(e)}$, giving

$$F(V, T) = F_{\text{id}}^{(i)} + F_{\text{id}}^{(e)} + F_{\text{ex}}. \quad (1.59)$$

Most important non-ideal deviation for the plasma is from the Coulomb interaction between ions and electrons, and between electrons in the rigid background $F_{\text{ex}} \approx F_{\text{ii}}$. This free energy decomposition then induces similar splitting for the pressure

$$P \approx P_{\text{id}}^{(i)} + P_{\text{id}}^{(e)} + P_{\text{ii}}. \quad (1.60)$$

A simple order-of-magnitude estimate for the relative strength of the different terms yields that $P_{\text{id}}^{(e)}$ is the main contributor for the pressure. Other terms, given as $|P_{\text{part}}|/P_{\text{id}}^{(e)}$, result in leading order terms of[†]

$$\frac{P_{\text{id}}^{(i)}}{P_{\text{id}}^{(e)}} \approx \frac{\Theta_r^{(i)}}{Z} \quad (1.61)$$

$$\frac{P_{\text{ii}}}{P_{\text{id}}^{(e)}} \approx \alpha_f \frac{\gamma_r}{x_r}, \quad (1.62)$$

where $\Theta_i^{(i)}$ is the temperature in units of nucleon Fermi temperature (see Eq. 1.50), $\alpha_f = e^2/\hbar c \approx 1/137$ is the fine-structure constant, and $\hbar = h/2\pi$. From here it is easy to see that only the Coulomb interaction

* [12] L. D. Landau and E. M. Lifshitz. 1980.

and *Space Physics Reviews*. (1989).

† [26] D. G. Yakovlev and D. A. Shalybkov. *Astrophysics*

term P_{ii} , should give a measurable corrections to the pressure at low temperatures. This suggests that for the pressure P we can use the approximation

$$P \approx P_{id}^{(e)} + P_{ii}. \quad (1.63)$$

In the weak-coupling limit ($\Gamma_i \ll 1$) the Debye-Hückel result for the excess free energy is a sufficiently good approximation, expressed as^{*†}

$$\frac{F_{ex}}{V} = \frac{1}{\sqrt{3}} n_i k T \Gamma_i^{3/2}, \quad (1.64)$$

where n_i is the ion number density. Hence, the pressure correction due to the Coulomb interactions can be simply presented as

$$P_{ii} \approx -0.3 n_i \frac{Z^2 e^2}{r_i}. \quad (1.65)$$

by using Eq. (1.57). Note also that $\Gamma_i = \Gamma_i(V)$.

Let us, as a final task, look into the main degenerate electron pressure term $P_{id}^{(e)}$. Pressure is defined as a flux of momentum through a surface, hence for one particular wall it is

$$P = \int_0^\infty \langle v_x p_x \rangle n(p) dp, \quad (1.66)$$

where the number density $n(p)$ is described by the Fermi-Dirac distribution, given by Eq. (1.26). Velocity can be obtained from the standard expression

$$v = \frac{dE}{dp} = \frac{p}{m_e} \left[1 + \left(\frac{p}{m_e c} \right)^2 \right]^{-1/2}, \quad (1.67)$$

using the energy defined by Eq. (1.30). Here we have already inserted the electron mass $m = m_e = 9.11 \times 10^{-28}$ g into the equations. Hence, in the three-dimensional description the pressure is given as one third of what we would get by applying Eq. (1.27),

$$P = \frac{1}{3} \int_0^\infty v(p) p n(p) dp = \frac{8\pi}{3m_e h^3} \int_0^\infty \frac{p}{\left[1 + \left(\frac{p}{m_e c} \right)^2 \right]^{1/2}} p \frac{p^2}{\exp[(E - \mu)/k_B T] + 1} dp \quad (1.68)$$

Free energy of the system is then equal to what is left from the chemical potential after subtracting the electron rest-mass energy and the pressure contribution,

$$F = (\mu - m_e c^2) n_e - \frac{8\pi}{3m_e h^3} \int_0^\infty \frac{p^4 dp}{\left[1 + \left(\frac{p}{m_e c} \right)^2 \right]^{1/2}} \frac{1}{\exp[(E - \mu)/k_B T] + 1}. \quad (1.69)$$

Sommerfield expanding the free energy expression in powers of temperature, i.e., t_r , we finally obtain[‡]

$$\frac{F}{V} = \frac{m_e c^2}{\lambda_C^3} \frac{1}{8\pi^2} \left(x_r (1 + 2x_r^2) \gamma_r - \ln(x_r + \gamma_r) + \frac{4\pi^2}{3} t_r^2 x_r \gamma_r \right) + O(t_r^4) \quad (1.70)$$

^{*} [27] P. Debye and E. Hückel. *Physikalische Zeitschrift*. (1923); [12] L. D. Landau and E. M. Lifshitz. 1980; [28] S. L. Shapiro and S. A. Teukolsky. 1983; [29] D. R. Dewitt *et al.* *Phys. Rev. A*. (1996).

[†]Debye-Hückel approximation relies on the assumption that

the charge density surrounding an ion is described by electrostatics (Poisson's equation) and the distribution of the charge around the ion itself by thermal motions of electrons (Boltzmann's equation).

[‡]see e.g., [26] D. G. Yakovlev and D. A. Shalybkov. *Astrophysics and Space Physics Reviews*. (1989).

from which it is easy to obtain an expression for the pressure by applying Eq. (1.57),

$$P_{\text{id}}^{(\text{e})} \approx \frac{P_{\text{r}}}{8\pi^2} \left(\left(\frac{2}{3} x_{\text{r}}^2 - 1 \right) \gamma_{\text{r}} + \ln(x_{\text{r}} + \gamma_{\text{r}}) + \frac{4\pi^2}{9} x_{\text{r}}^2 (\gamma_{\text{r}} + \gamma_{\text{r}}^{-1}) \right) \quad (1.71)$$

where again a typical pressure is given as

$$P_{\text{r}} = \frac{m_{\text{e}} c^2}{\bar{\lambda}_{\text{C}}} \approx 1.4 \times 10^{25} \text{ dyn cm}^{-2}, \quad (1.72)$$

where $\bar{\lambda}_{\text{C}} = \hbar/m_{\text{e}}c = 3.86 \times 10^{-11} \text{ cm}$ is the reduced electron Compton wavelength.* Omitting the (small) temperature related term ($\propto t_{\text{r}}^2$) leads to the well-known Chandrasekhar equation of state of for a perfect, completely degenerate electron gas.†

In the non-relativistic and ultra-relativistic regimes we can apply the asymptotic limits of Eq. (1.32). Then, the pressure given in Eq. (1.71) takes a simple polytropic form of

$$P_{\text{id}}^{(\text{e})} \approx \frac{P_{\text{r}}}{9\pi^2 \gamma_{\text{AD}}} x_{\text{r}}^{3\gamma_{\text{AD}}}, \quad (1.73)$$

where the polytropic index is given as $\gamma_{\text{AD}} = \frac{5}{3}$ or $\gamma_{\text{AD}} = \frac{4}{3}$ for the non-relativistic ($x_{\text{r}} \ll 1$) and the ultra-relativistic ($x_{\text{r}} \gg 1$) cases, respectively. Recall also that $x_{\text{r}} \propto n_{\text{e}}^{1/3} \propto \rho^{1/3}$. The transition occurs at $x_{\text{r}} \approx 1$, corresponding to about $\rho \sim 10^6 \text{ g cm}^{-3}$.

Finally, as suggested by our analysis given in Eq.(1.61), the ideal degenerate electron gas pressure accompanied with the ion Coulomb correction will give us a rather good approximation for the equation of state as

$$P(x_{\text{r}}) \approx P_{\text{id}}^{(\text{e})} + P_{\text{ii}} = \frac{P_{\text{r}}}{8\pi^2} \left(\left(\frac{2}{3} x_{\text{r}}^2 - 1 \right) \gamma_{\text{r}} + \ln(x_{\text{r}} + \gamma_{\text{r}}) \right) - 0.3 n_{\text{i}} \frac{Z^2 e^2}{r_{\text{i}}}, \quad (1.74)$$

remaining valid in a large density range of $10^4 < \rho < 10^{10} \text{ g cm}^{-3}$. Below $\rho < 10^4 \text{ g cm}^{-3}$ the plasma is gaseous and the ideal gas law description (1.16) is better suited in modeling the equation of state. Beyond $\rho > 10^{10} \text{ g cm}^{-3}$ the densities become so high that nuclear degeneracy pressure and more importantly, the mutual nuclear interactions start to play an important role. This is the topic of our next section.

1.4 Core

After the relatively well-known crust, a physical no-man's land begins when we enter the core. At about half of the nuclear density, $\rho_{\text{n}} \approx 2.8 \times 10^{14} \text{ g cm}^{-3}$, the nuclei become so neutron-rich that more neutrons will leak out than what is left bounded. Hence, normal atoms can no longer exist, and the matter becomes an uniform plasma of neutrons, protons, and electrons. In the end, we will have a liquid sea of free neutrons with a several per cent admixture of protons and electrons. The neutrons and protons, interacting via the nuclear forces, constitute a strongly interacting Fermi liquid, whereas electrons form an almost ideal Fermi gas. Instead of purely degenerate non-interacting matter, like the electrons in the crust, the nucleons become close enough to interact with each other. When in close contact, the repulsive short-range neutron-neutron interaction introduces a considerable stiffening of the equation of state. Hence, a many-body theory is needed in order to describe the matter in the core. If we take the effective range of strong nuclear forces to be of around $r_{\text{sn}} \sim 10^{-13} \text{ cm}$, we can easily estimate the order-of-magnitude of the density when they become important by equating the mean distance of nucleons with r_{sn} . In this case we obtain $\rho_{\text{sn}} \sim m_{\text{n}}/V = 3 \times 1.66 \times 10^{-24} \text{ g} / 4\pi r_{\text{sn}}^3 \sim 4 \times 10^{14} \text{ g cm}^{-3}$, a density range surprisingly close to more sophisticated calculations giving $\rho \sim (1.5 - 2) \times 10^{14} \text{ g cm}^{-3}$.‡

*Normal Compton wavelength is simply $\lambda_{\text{e}} = 2\pi \times \bar{\lambda}_{\text{C}} = \hbar/m_{\text{e}}c = 2.43 \times 10^{-10} \text{ cm}$. drasekhar. *MNRAS*. (1935); [20] S. Chandrasekhar. 1939.

† [18] E. C. Stoner. *Philos. Mag.* (1930); [30] S. Chandrasekhar. *Zeitschrift für Physik*. (1936).

‡such estimates were first presented by Hund [31] F. Hund.

A density range between $0.5\rho_n < \rho < 2\rho_n$ ($1.4 - 5.6 \times 10^{14} \text{ g cm}^{-3}$) is named the *outer core*. For neutron stars it is already several kilometers thick and constitutes a substantial part of the total mass of the star. The matter consists of so-called *npeμ* composition, referring to neutrons *n*, protons *p*, electrons *e*, and in some models to muons *μ*. The basic physics of this matter is determined by charge neutrality, β -equilibrium, and many-body nuclear interactions. Beyond $\rho > 2\rho_n \approx 5.6 \times 10^{14} \text{ g cm}^{-3}$, we have the *inner core*. Its composition is even more unknown and the results here become heavily model-dependent. This is mainly because, in addition to the many-body forces, the exact particle composition is unclear. In addition to the *npeμ* matter, we might have hyperonization (Σ^- , Λ , or more generally sometimes labeled as *H*), pion condensation (π), kaon condensation (*K*), or even a phase-transition to a (pure) quark matter (*q*). Inner core of a neutron star can be many kilometers thick and the central densities of the most massive stars can go up to $\rho \approx (10 - 15) \times \approx (2.8 - 4.2) \times 10^{15} \text{ g cm}^{-3}$.

1.4.1 Polytropes

Instead of trying to obtain a very uncertain nuclear physics description of the equation of state, let us take another route. As the strongly degenerate equation of state relevant for neutron stars is mainly just the relation between pressure and the density, we can try and phenomenologically parameterize this relation in contrast to describing the accurate but complicated microphysics. One such a parameterization is a polytropic presentation of the $P(\rho)$ function defined as

$$P(\rho) = K\rho^\gamma, \quad (1.75)$$

where γ is a polytropic index and is a measure of the *stiffness* of the matter, and K is just a normalization factor. Instead of settling for only one polytrope (actually called a monotrope then), we can link them together to form a piecewise polytropic description of the equation of state. Such a phenomenological description was pioneer by Read et al.* In practice, we can then generalize Eq. (1.75) such that

$$P(\rho) = K_i \rho^{\gamma_i}, \quad \text{for } \rho_{i-1} \leq \rho \leq \rho_i, \quad (1.76)$$

for transition densities $\rho_0 < \rho_1 < \rho_2 < \dots < \rho_{i-1} < \rho_i < \rho_{i+1}$, and so on.

Note that ρ here is the rest-mass density. The full energy density can be obtained from the first law of thermodynamics as

$$d\frac{\epsilon}{\rho} = -\frac{P}{c^2} d\frac{1}{\rho}, \quad (1.77)$$

that has an immediate integral of

$$\frac{\epsilon}{\rho} = (1 + a) + \frac{1}{c^2} \frac{1}{\gamma - 1} K \rho^{\gamma-1}, \quad (1.78)$$

where a is a continuity constant. Now

$$\lim_{\rho \rightarrow 0} \frac{\epsilon}{\rho} = \frac{1}{c^2}, \quad (1.79)$$

implying $a = 0$ for $\rho = 0$. For the piecewise description we then have (for $\gamma_i \neq 0$)

$$\epsilon(\rho) = (1 + a_i)\rho + \frac{1}{c^2} \frac{K_i}{\gamma_i - 1} \rho_i^{\gamma_i} \quad (1.80)$$

where

$$a_i = \frac{\epsilon(\rho_{i-1})}{\rho_{i-1}} - 1 - \frac{1}{c^2} \frac{K_i}{\gamma_i - 1} \rho_{i-1}^{\gamma_i-1}, \quad (1.81)$$

defines the energy density continuous.

* [32] J. S. Read et al. *Phys. Rev. D.* (2009).

1.4.2 Library for the equation of state of the core

As noticed by Read et al.^{*}, in most cases, just three piecewise polytropes are enough to give a faithful description of the accurately calculated equations of states, up to a rms error of about 2% in pressure. Here we consider few (relatively) modern descriptions for the dense matter equation of state and parameterize them with polytropes. We follow the naming convention presented in the previous section and divide the compositions into

- normal nuclear matter ($npe\mu$),
- normal nuclear matter spiced up with hyperons ($npe\mu + H$),
- normal nuclear matter together with more exotic particles like pion and kaon condensates ($npe\mu + \pi + K$), and
- matter consisting of (or normal matter spiced up with) quarks (q or $npe\mu + q$).

The different equations of states we describe here are mainly presented and used in Refs.[†] and fitted with polytropes by Ref.[‡] Additionally, we require that the maximum mass of the equation of state fulfills the $2 M_{\odot}$ mass limit as set by astrophysical measurements of double pulsars.[§] For $npe\mu$ composition we include models computed with

- potential method using SLy effective nuclear interaction that is of Skyrme-type[¶],
- four variational method EoSs, APR3/4^{||} and WFF1/2^{**},
- two relativistic Brueckner-Hartree-Fock calculations, ENG^{††} and MPA1^{‡‡}, and
- two relativistic mean-field theory models, MS1 and MS1b (same as MS1 but with lower symmetry energy)^{§§}.

For hyperon models ($npe\mu + H$) we include

- one variant of relativistic mean-field theory model, H4^{¶¶}.

Here the introduction of hyperons into the matter considerably softens the equation of state and the stars constructed with these models do not end up reaching the $2 M_{\odot}$ limit. Similarly, models where mesons, like pion and kaon condensates ($npe\mu + \pi$ and/or $npe\mu + K$) are taken into account, end up not being stiff enough. Finally, for the hybrid nuclear matter and quark matter compositions ($npe\mu + q$) we include

- mixed APR nuclear matter and color-flavor-locked quark matter EoS ALF2.^{***}

Again, for pure quark matter composition, the equation of state ends up being too soft. Even in the case of the ALF2, the transition density from the nuclear matter to quark matter is defined at a relatively late phase corresponding to $\rho = 3 \rho_n$.

^{*} [32] J. S. Read et al. *Phys. Rev. D.* (2009).

[†] [33] J. M. Lattimer and M. Prakash. *ApJ.* (2001); [34] M. Bejger, T. Bulik, and P. Haensel. *MNRAS.* (2005); [35] B. D. Lackey, M. Nayyar, and B. J. Owen. *Phys. Rev. D.* (2006).

[‡] [32] J. S. Read et al. *Phys. Rev. D.* (2009).

[§] [36] P. B. Demorest et al. *Nature.* (2010); [37] J. Antoniadis et al. *Science.* (2013); [38] E. Fonseca et al. *ApJ.* (2016).

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^{||} [40] A. Akmal, V. R. Pandharipande, and D. G. Ravenhall. *Phys. Rev. C.* (1998).

^{**} [41] R. B. Wiringa, V. Fiks, and A. Fabrocini. *Phys. Rev. C.* (1988).

^{††} [42] L. Engvik et al. *ApJ.* (1996).

^{‡‡} [43] H. Mütter, M. Prakash, and T. L. Ainsworth. *Physics Letters B.* (1987).

^{§§} [44] H. Müller and B. D. Serot. *Nuclear Physics A.* (1996).

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^{***} [45] M. Alford et al. *ApJ.* (2005).

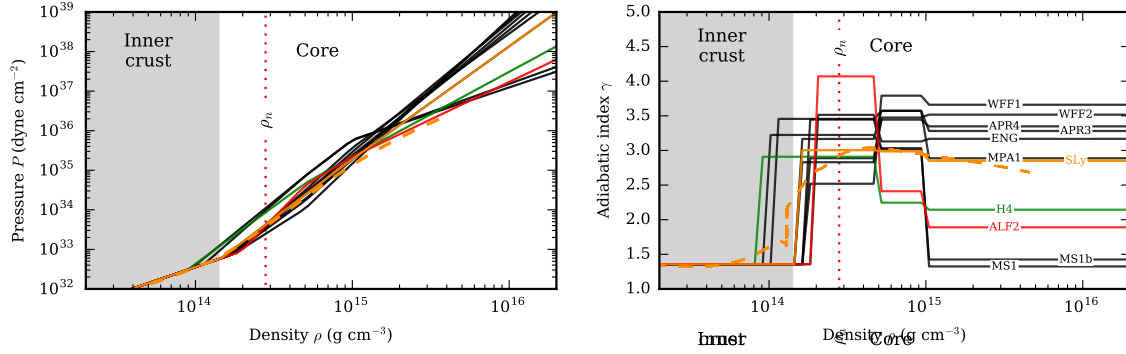


Figure 1.3: Equation of states for the core. *Left side:* Pressure versus density relation, $P(\rho)$. *Right side:* Corresponding adiabatic or the polytropic index evolution versus the density, $\gamma(\rho)$. Plain nuclear matter models are colored in black, hyperonic ($npe\mu + H$) with green, and quark (q) matter in red. Additionally, the exact SLy model (shown exceptionally in dark orange) is shown with dashed lines. Grey region highlights the crust densities and the red dotted vertical line shows the nuclear density of $\rho_n = 2.8 \times 10^{14} \text{ g cm}^{-3}$.

These 11 equations of states end up giving quite good overview of different models present in the literature. We, however, stress that by no means is it a complete list. The corresponding $P(\rho)$ evolution of these models is shown in Fig. 1.3. Unlike the previously well-defined crust, here the theoretical uncertainties become obvious as there is quite a large scatter in between different models.

1.5 Tolman-Volkoff-Oppenheimer equations

Let us, as a final task, try and construct an actual model of a star from the equation of state. For this, we require the star to be in hydrostatic equilibrium so that the pressure gradient from opposes the gravity. In classical Newtonian form this can be expressed simply as

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}, \quad (1.82)$$

where r is the radial coordinate. In addition, we will need a connection for the mass m and radius r , that in spherical symmetry is

$$\frac{dm}{dr} = 4\pi r^2 \rho. \quad (1.83)$$

Taking into account the general relativistic corrections we get

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \times \frac{(1 + P/\rho c^2)(1 + 4\pi r^3 P/mc^2)}{1 - 2Gm/rc^2}. \quad (1.84)$$

This is the relativistic hydrostatic equilibrium equation as first derived by Tolman*, Oppenheimer and Volkoff†.‡ Difference in respect to the classical formulation originates from the source of gravity: in the Newtonian case it is the mass m , whereas in the general relativity it is the energy momentum tensor that

* [46] R. C. Tolman. *Physical Review*. (1939).

‡for an introduction, see, e.g., [48] R. M. Wald. 1984;

† [47] J. R. Oppenheimer and G. M. Volkoff. *Physical Review*. (1939).

[49] C. W. Misner, K. S. Thorne, and J. A. Wheeler. 1973.

depend both on the energy density and the pressure. As a result, both, energy and pressure, give rise to a gravitational field.

This has an important consequence for the stability of neutron stars: Successive increase in the pressure to counter the gravity is ultimately self-defeating. We can get an idea of this, by using an unphysical but analytically easier special case of constant density, i.e. incompressible equation of state, as our model. By solving the Eq. (1.82) for constant density $\rho(r) = \rho_0$, we obtain

$$P(r) = G \frac{2\pi}{3} \rho_0^2 (R^2 - r^2) \quad (1.85)$$

whereas the general relativistic form described by Eq. (1.83) gives, after some tedious algebra,

$$P(r) = \rho_0^2 c^2 \frac{\sqrt{1 - u \left(\frac{r}{R}\right)^2} - \sqrt{1 - u}}{3 \sqrt{1 - u} - \sqrt{1 - u \left(\frac{r}{R}\right)^2}}, \quad (1.86)$$

where again the compactness parameter $u = 2GM/Rc^2$, is used. Unlike the classical counter-part, the central pressure $P(0)$ here attains a maximum for $u_{\max} = \frac{8}{9}$ equal to

$$M_{\max} = \frac{4}{9} R c^2 / G \approx 3.01 \left(\frac{R}{10 \text{ km}} \right) M_{\odot}. \quad (1.87)$$

Even though the assumption of constant density is unphysical, the solution shows that the relativistic equations of hydrostatic balance have a maximum mass after which the star becomes unstable. This is an important feature of the equations as it can be used to, for example, rule out certain equations of states, given that we have measured some (large) mass to exist in Nature. It was also the source of confusion in the 1930s because the degenerate neutron gas equation of state gives $M_{\max} \approx 0.7 M_{\odot}$, while the Chandrasekhar limit, i.e., the maximum mass for white dwarfs with degenerate electrons, give $M_{\text{Ch}} \approx 1.44 M_{\odot}$. From these arguments, it would seem that neutron stars can not exist because the maximum mass is less than that of the white dwarfs. Now we, of course, already know that it is the nuclear many-body interactions that have a huge impact on the equations of states and subsequently alter the maximum mass to be of around $M_{\max} \approx (1.5 - 3) M_{\odot}$. Similarly, the measurements of $M \approx 2.0 M_{\odot}$ neutron stars, that we already used to rule out some equations of states in the previous section, put stringent constraints on the possible behavior of the matter.

Instead of looking at the different quantities, such as the pressure, as a function of the density, let us now finally solve the structure of the star given the derived equations of states. The results are obtained by numerically solving the Tolman-Oppenheimer-Volkoff equations using the previously presented crust model and the different equations of states for the core.* In Fig. 1.4 the dependency of the pressure and cumulative mass of the star against the radial coordinate (as measured starting from the core) is visualized for the SLy equation of state for a central density of $P_c = 8.9 \times 10^{14} \text{ g cm}^{-3}$, corresponding to $M = 1.4 M_{\odot}$. This visualizes the real dimensions of the neutron stars internal structure: Inner core spans about 6 km in height from the core while the outer core extends from 6 to 11 km. The full crust is only about ~ 1 km thick, and the atmosphere is too thin to be even visible. The core also consists of more than 99% of the total mass of the star.

Instead of fixed central pressure, we can let it span a large range, starting from some small value and ending to the pressure corresponding to the maximum mass, to obtain mass–radius, $M - R$ curves for the

*TOV-solver that is used is available from [https:// github.com/natj/tov](https://github.com/natj/tov).

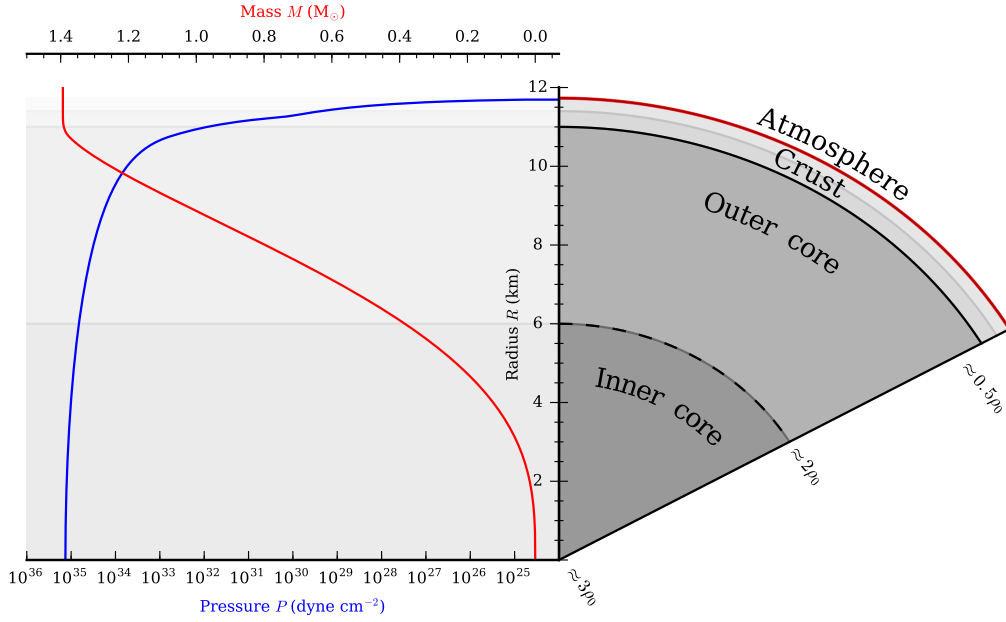


Figure 1.4: Overview of the neutron star structure for the SLy equation of state and core density of $P_c = 8.9 \times 10^{14} \text{ g cm}^{-3}$, corresponding to $M = 1.4 M_\odot$. Right side of the figure shows a schematic presentation of the star's interiors against the radial coordinate, whereas the left side shows the pressure (blue; bottom axis) and cumulative mass (red; top axis) evolution from the core to the surface.

equations of states. These are visualized in Fig. 1.5 along with the pressure–density relations that yield them. From here it is obvious how the large uncertainty in the nuclear physics of the core then translates into a large possible allowed radius range, from about 10 to 15 km. It also opens up a pathway of probing the nuclear physics with astrophysics because mass and radius measurements of real neutron stars can set constraints to the equation of state.

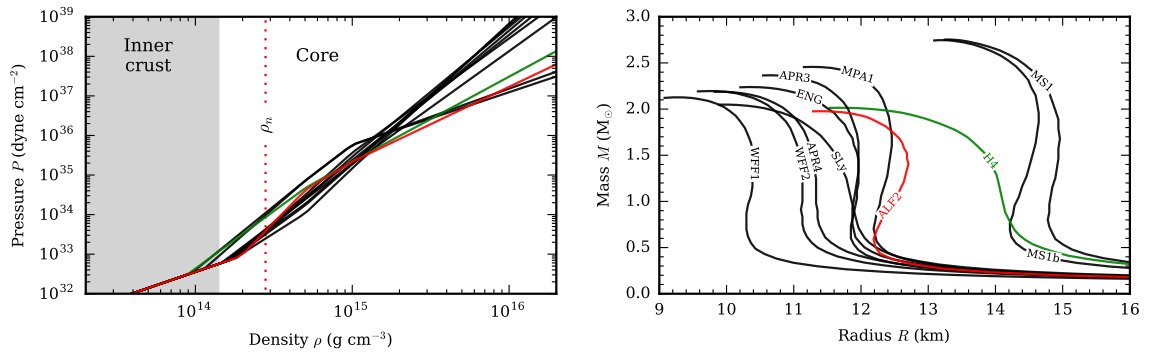


Figure 1.5: Equation of states and the resulting neutron star mass-radius curves for the different core models. *Left side:* Pressure versus density relation, $P(\rho)$. *Right side:* Corresponding $M - R$ relations. Symbols and colors are the same as in Fig. 1.3.

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