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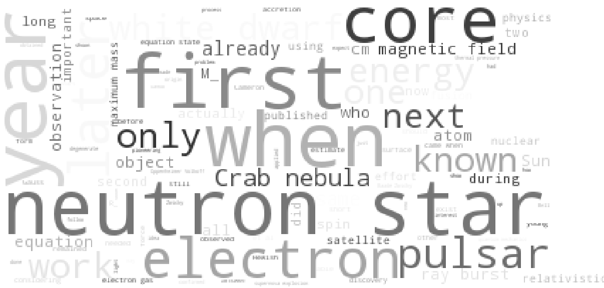

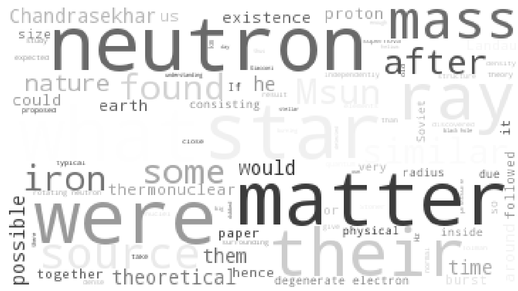

1 Introduction

Neutron stars are curious objects encompassing many problems of modern physics and astrophysics still unsolved. Their unique nature makes them ideal laboratories for many of the most energetic phenomena in space. Born from the ashes of a supernova, they begin their life only when a normal star fades away and dies in a spectacular supernova explosion.[*] From there on, they continue their life by slowly collecting the surrounding interstellar matter or by devouring an unlucky companion star floating next to them. It is not the impressive $\sim 10^{30}$ kilograms they weigh but the sphere mere $\sim 10\,\mathrm{km}$ in radius that they encapsulate this material into that is then able to bend spacetime itself. Such an impressive feat awards them a categorization of a special stellar group called *compact objects*. Along with white dwarfs and black holes, these strange compact objects have been under scientific scrutiny for almost nine decades. Still, some of the most fundamental questions remain: What are neutron stars made of? How big are they? How do we see them?

In the next few sections, we will discuss these peculiar objects habitating the Space around us in detail. First, a short history of their discovery is given, followed by simple physical arguments on why we actually expect them to exist in the first place. This will hopefully help us in building some intuition about physical phenomena possible near and inside these stars.

1.1 Short history

1.1.1 From imagination to reality

In 1908, Lev Landau was born in Baku, Azerbaijan. Landau turned out to become a brilliant Soviet scientist, who already at the age of 14 matriculated at the Baku State University studying simultaneously in both the Department of Physics and Mathematics and the Department of Chemistry. Soon after, he became disinterested in chemistry but continued to be fascinated by physics throughout his whole career. In 1931, at the age of 23, Landau published an exceptional paper where he speculated on the existence of stars containing matter at nuclear densities:[†]

> *...the density of matter becomes so great that atomic nuclei come in close contact, forming one gigantic nucleus.* (L. Landau, 1939)

What made this paper exceptional was that it was written before the existence of neutrons was confirmed. For contemporary science, this meant a violation of quantum mechanics, since atoms were thought to consist of protons and electrons only, and they certainly could not exist together in the same place inside such a hypothetical star. Nevertheless, it marked the first theoretical speculation on the existence of what we now know as neutron stars.

Landau needed not wait for long. Already next year, in 1932, James Chadwick confirmed that neutrons really were a fundamental part of our nuclear physics with his works dubbed *Possible Existence of a*

[*] For the pedantics, we specify that a supernova is actually an implosion followed by a subsequent explosion.

[†] [1] L. D. Landau. *Phys. Z. Sowjetunion.* (1932).

Neutron[*] and the follow-up *Existence of a Neutron*[†]. His experimental findings then confirmed the theoretical predictions his supervisor Ernest Rutherford made already in 1920[‡]. Later on, in 1935, Chadwick was awarded the Nobel Prize in Physics for his findings. Chadwick himself continued his career as part of the Manhattan project, as it was basically his groundbreaking work that inspired the U.S. government to begin serious research into the atomic bomb.

Now that the existence of the neutron was confirmed, it did not take long for others, independent of Landau, to propose similar stars. At the Meeting of the American Physical Society at Standford in December 1933, one year after the neutron discovery, Wilhelm Baade and Frank Zwicky made their famous proposal that Supernovae should be considered a new category of astronomical objects.[§] At the same time, they also stated:

> *...we advance the view that a super-nova represents the transition of an ordinary star into a neutron star, consisting mainly of neutrons. Such a star may possess a very small radius and an extremely high density.*

Such statements were, however, deemed a work of imagination by a bunch of weird theorists. Zwicky, on the other hand, kept on insisting that neutron stars really are out there. Much later on, A.G.W. Cameron, a former post-doc at Caltech (where Zwicky was also situated) during 1959−1969, recalls:

> *For years Fritz [Zwicky] had been pushing his ideas about neutron stars to anyone who would listen and had been universally ignored. I believe that the part of the problem was his personality, which implied strongly that people were idiots if they did not believe in neutron stars.* (A.G.W. Cameron, 1999)

Progress on the theoretical understanding of neutron stars was also tightly connected to understanding the interiors of white dwarfs. Unlike the mysterious nuclear forces related to neutrons, the physics of white dwarfs was more related to understanding the behavior of electrons. A breakthrough in this field came in 1925, when a young Paul Dirac formulated the quantum wave equations for the motion of electrons[¶]. What soon followed was a description of the pressure of degenerate electron gas by Ralph Fowler, Dirac's supervisor[‖]. The implications were severely against the previously known physics; even in zero temperature, there would be a degeneracy pressure preventing matter from collapsing due to the exclusion principle of quantum mechanics.

Using a simplified uniform density approximation, Edmund Stoner was then able to show that this implied a maximum mass limit for white dwarfs.[**] Thus, a surprising result was obtained: when the density of a white dwarf approaches infinity, the mass reaches a maximum value. The German-Estonian Wilhelm Anderson later realized that the electrons in this problem must actually be treated relativistically[††], something overlooked by Stoner. Anderson tried to correct the crude mistake by deriving the equation of the state of relativistic degenerate electron gas but ended up making severe mistakes. It was Stoner who corrected his equations based on communication with Anderson and re-derived his maximum mass limit. Regardless of Stoner's efforts, it was later named Chandrasekhar's mass limit for its importance in astrophysics.

This was to honor Subrahmanyan Chandrasekhar, a young and prolific Indian physicist and astrophysicist, who was working on the same topic after reading Fowler's paper on degenerate electron gas. Unlike

* [2] J. Chadwick. *Nature*. (1932).

† [3] J. Chadwick. *Proceedings of the Royal Society of London Series A*. (1932).

‡ [4] E. Rutherford. *Proceedings of the Royal Society of London Series A*. (1920).

§ [5] W. Baade and F. Zwicky. *Proceedings of the National Academy of Science*. (1934); [6] W. Baade and F. Zwicky. *Physical Review*. (1934).

¶ [7] P. A. M. Dirac. *Proceedings of the Royal Society of London Series A*. (1925).

‖ [8] R. H. Fowler. *MNRAS*. (1926).

** [9] E. C. Stoner. *Philos. Mag.* (1930).

†† [10] W. Anderson. *Zeitschrift für Physik*. (1929).

Stoner's limit computed using the uniform density approximation, Chandrasekhar realized that a polytropic density profile is a more physical albeit mathematically more challenging formulation. Still, the 19-year-old Chandrasekhar, already known for his mathematical prowess, was able to integrate the equations numerically by hand and obtained a similar limiting mass*. Later on, however, it has been found that Chandrasekhar was not even the second person to derive the mass limit, but the third:† the Soviet physicist Yakov Frenkel published a similar derivation, independently and unknowingly of the progress in the west, in which he applied the relativistic degenerate electron gas results to white dwarfs and concluded that an upper limit on the mass must exist‡. However, due to the slow publishing pace in the Soviet journals, his work never became available to his western peers in time.

Nevertheless, the maximum mass for a white dwarf had been laid out, and in the end, after all the relevant physical inclusions, it turned out to be 1.44 $M_\odot$, or 1.44 times the mass of our Sun. What makes this limit important for us, is that the maximum mass for a white dwarf is just the minimum mass for a neutron star, an important connection first made by George Gamow in 1939§. The idea behind it is simple: if the degenerate electron gas pressure, quantum mechanical in nature, is what keeps the white dwarfs from collapsing, what happens when the maximum mass is reached and even this strange pressure is unable to resist the forces of gravity? At a conference in Paris in 1939, Chandrasekhar laid out the answer:

> *If the degenerate core attains sufficiently high densities, the protons and electrons will combine to form neutrons. This would cause a sudden diminution of pressure resulting in the collapse of the star to a neutron core.*

A neutron star should thus have a mass close to the Chandrasekhar limit, i.e. $M \sim 1.44\, M_\odot$, and consist of neutrons only, exactly like proposed by Landau eight years earlier without the knowledge of neutrons, or later on by Baade and Zwicky when they presented their theory of supernovae!

It was before the Second World War that a solid basis for a theory of neutron stars was established. This was, however, just the beginning. The next question would be the critical one that we are still trying to answer today: if they exist, how big are they? The problem was that because of the extremely dense nature of these objects, the classical stellar equilibrium equations were no longer valid, and thus it was not possible to even estimate the size of a neutron star. The problem was unwieldy due to its general relativistic nature; the immense mass of the neutron star was bending spacetime itself, and the more compact it was, the more it could bend it. On the other hand, the more curved spacetime was, the more the star would gain weight and the more compact it would become.

It was already during the same year as Gamow's remark, in 1939, that a theoretical framework for studying this problem was published. This was done independently by Richard Tolman¶ and Robert Oppenheimer together with his student George Volkoff‖. Both papers were even submitted on the same day, the 3rd of January, to Physical Review and were published on the same February issue. More importantly, they both described a hydrostatic equilibrium for a spherically symmetric object in general relativity, exactly what was needed to study neutron stars. Because of its great importance, the solution is now known as the Tolman-Oppenheimer-Volkoff equation. In addition, Oppenheimer and Volkoff applied their equation and numerically calculated the structure of a neutron star consisting of non-interacting strongly degenerate neutron gas. This marked the very first attempt in characterizing neutron stars. Similar to white dwarfs, they also obtained an upper limit for their mass. However, as a disappointment for everyone, it was calculated to be around 0.7 $M_\odot$, i.e. less than the Chandrasekhar limit of 1.44 $M_\odot$ for white dwarfs, indicating that neutron

* [11] S. Chandrasekhar. *MNRAS*. (1931).
† [12] D. G. Yakovlev. *Physics Uspekhi*. (1994).
‡ [13] J. Frenkel. *Zeitschrift für Physik*. (1928).
§ [14] G. Gamow. *Physical Review*. (1939).
¶ [15] R. C. Tolman. *Physical Review*. (1939).
‖ [16] J. R. Oppenheimer and G. M. Volkoff. *Physical Review*. (1939).

stars could not exist in nature. It took almost two more decades to show that it was actually the assumption of no interaction between the neutrons that was causing this hiccup.

Moreover, it was actually not Tolman nor Oppenheimer and Volkoff who first discovered the general relativistic hydrostatic equation. It was now Chandrasekhar's turn to avoid having an important result credited to him; together with John Von Neumann, Chandrasekhar extended his work on white dwarfs to also cover neutron stars and in the process derived exactly the same equilibrium equation in 1934, five years before the groundbreaking publication of Tolman, Oppenheimer and Volkoff.* It is, however, worth mentioning that later on, in 1983, Chandrasekhar received the Nobel Prize in Physics for his work on "theoretical studies of the physical processes of importance to the structure and evolution of the stars". So he certainly received at least some credit for his important work.

Around the same time, in 1937, Gamow and Landau also independently proposed that the accretion of matter onto a dense neutron star core could be the missing source of energy for stars. This increased the interest towards neutron stars, and the field flourished in the 1930s. However, it was soon shown that stars are powered not by accretion but by thermonuclear reactions as first suggested in the 1920s by Sir Arthur S. Eddington.† The interest in neutron stars then faded away and the research focused on weaponizing the nuclear forces.

The next big breakthrough came almost 20 years later in the 1950s, when John Wheeler and his collaborators constructed the first realistic equation of the state of dense matter‡. For the outer layers, known as the crust, they applied a semi-empirical mass formula together with the equation of the state of degenerate electrons. For the dense core, they assumed a mixture of three ideal Fermi gases composed of neutrons, protons, and electrons. This marked the first consistent formulation of neutron star structure. It was followed by Cameron, who applied the Skyrme equation of state for the high-density matter.§ This had important implications, as he was then able to show that the nuclear forces stiffen the matter considerably in comparison to the non-interacting free neutrons. Similar to Tolman and Volkoff, he then went to calculate the maximum possible mass of a neutron star and arrived at approximately $2\,M_\odot$. This was an important theoretical breakthrough as it implied that neutron stars can, after all, exist. A new wave of interest towards neutron stars was thus launched as everybody wanted to observe them.

1.1.2 Many observational faces of neutron stars

After Wheeler and Cameron had laid the modern foundation for studies on neutron star structure, everyone was eager to find these strange objects in the night sky. It did not take long before researchers realized that as neutron stars are born in the supernova explosions, we expect them to be hot. Most of the theoretical effort in the 60s was then focused on developing models for the cooling of neutron stars.¶ It was the potential thermal radiation from this cooling that could then be used to detect them, as was first shown by Hong-Yee Chiu‖. The first calculations predicted surface temperatures of $T \sim 10^6$ K for a neutron star of the age of around 1000 years. This had important implications for the observers as it meant that neutron stars would mainly radiate in the range of X-rays. The atmosphere of Earth, on the other hand, was impenetrable to the X-ray wavelengths. Luckily, the 60s also marked the beginning of a golden era for spaceborn observatories.

Since X-rays could not reach the surface of the Earth, humankind went into space to observe them. In the late 1950s and early 1960s, it was the pioneering experiments of the Italian astrophysicist Riccardo Giacconi

* [17] G. Baym. 1982.
† [18] A. S. Eddington. 1926.
‡ [19] J. A. Wheeler. *ARA&A*. (1966).
§ [20] A. G. Cameron. *ApJ*. (1959).
¶ [21] R. C. Stabler. 1960; [22] H.-Y. Chiu. *Annals of Physics*. (1964); [23] D. C. Morton. *Nature*. (1964); [24] H.-Y. Chiu and E. E. Salpeter. *Physical Review Letters*. (1964); [25] J. N. Bahcall and R. A. Wolf. *Physical Review*. (1965); [26] J. N. Bahcall and R. A. Wolf. *ApJ*. (1965); [27] S. Tsuruta and A. G. W. Cameron. *Canadian Journal of Physics*. (1966).
‖ [22] H.-Y. Chiu. *Annals of Physics*. (1964).

that opened this new window into the Universe. Giacconi first started with rocket-borne experiments and later continued by leading the development of the first orbiting X-ray satellite Uhuru, "*freedom*" in Swahili.*
After the first X-ray satellite, Giacconi continued with the Einstein Observatory, the first fully imaging X-ray satellite, and later with the Chandra X-ray observatory. For all of his efforts, he received the Nobel Prize in Physics in 2002 "for pioneering contributions to astrophysics, which have led to the discovery of cosmic X-ray sources".

During the starting boom, several extra-terrestrial X-ray sources were discovered. As is common in science, the first discovery actually came by accident. A team led by Giacconi launched an Aerobee 150 rocket to the skies in June 1962 with a payload of a highly sensitive soft X-ray detector meant to observe the X-rays from the Moon. Due to a slight change (or a mistake) in the planned trajectory, it ended up observing the constellation of Scorpius and caught a glimpse of what is now known as the first extraterrestrial X-ray source, Sco X−1. Little did they know that this was actually the first neutron star radiating towards us. Five years later, in 1967, Iosif Shklovsky was the first to propose that Scorpius X-1 is a neutron star†, but his work attracted little to no attention.

The first deliberate searches of neutron stars were aimed at the Crab Nebula, a well-known candidate for hosting a neutron star. The Crab Nebula, already known in the 1920s and 1930s to be a supernova remnant is known to have exploded exactly on the 4th of July, 1054.‡ In contemporary Chinese, Japanese, and Arab history writings, a "guest star" is described to appear in the constellation of Taurus and to persist even in broad daylight for 23 consecutive days. Even after that, it remained visible in the night sky for two years. For astronomers, this was a clear sign of a nearby supernova going off.

But it was not only the spectacular supernova but what was left behind that eluded astronomers. Already in 1942, our old friends Baade and Rudolf Minkowski correctly found that the center of the Crab Nebula contained an unusual star.§ In the following years, the mystery gained depth when a radio emission was also detected.¶ This gathered a lot of interest from the theorists, as they were trying to explain the origin of the energy powering the nebula. In 1953, Shklovsky was on the right track again when he predicted that the emission is due to synchrotron radiation from relativistic electrons spiraling along magnetic field lines. The next piece of the puzzle came in 1964, when Lodewijk Woltjer, who did his PhD on the Crab Nebula, argued, based on the conservation of magnetic flux, that neutron stars should have a strong magnetic field, enough to produce this synchrotron radiation.‖ Similar results were independently obtained in the East by Vitaly Ginzburg.**

Early X-ray telescopes of the time had a very poor angular resolution, so imaging the Crab Nebula to get an answer to the puzzle was difficult. The first observation in 1964 by S. Bowyer et al. used a clever method of partial lunar occultation to cover unwanted parts of the sky with the Moon, and what followed was the first X-ray observation of the neutron star candidate everybody was waiting for.†† It was, however, followed by a disappointment when a follow-up observation measured the source size to be about 1 light-year in size (10^{13} km) in comparison to the 11 light-years of the whole nebula.‡‡ The result was much larger than what was expected for a neutron star that should be a mere $\sim 10\,\mathrm{km}$ in radius. Ironically, what they did not know was that this was just as expected; for young neutron stars like the one in the Crab Nebula, a pulsar wind (consisting of charged particles similar to solar wind) is expected. This wind will then create a surrounding shell called a plerion, much bigger in size, around the neutron star, and this shell is the source of the X-rays.

* [28] R. Giacconi *et al*. *Physical Review Letters*. (1962).
† [29] I. S. Shklovsky. *ApJ*. (1967).
‡ [30] J. H. Oort. 1997; [31] K. Lundmark. *PASP*. (1921); [32] N. U. Mayall. *Leaflet of the Astronomical Society of the Pacific*. (1939); [33] D. A. Green and F. R. Stephenson. 2003.
§ [34] W. Baade. *ApJ*. (1942); [35] R. Minkowski. *ApJ*. (1942).
¶ [36] J. G. Bolton, G. J. Stanley, and O. B. Slee. *Nature*. (1949).
‖ [37] L. Woltjer. *ApJ*. (1964).
** [38] V. L. Ginzburg. *Soviet Physics Doklady*. (1964).
†† [39] S. Bowyer *et al*. *Nature*. (1964).
‡‡ [40] S. Bowyer *et al*. *Science*. (1964).

Hence, the mystery remained even though Nikolai Kardashev in the East and Franco Pacini in the West gave plausible pioneering explanations for the formation of the wind in 1964 and 1967, respectively.*

Despite all the efforts (and partly due to bad luck), no concrete observations supporting the existence of neutron stars still existed. This all changed in July 1967, in the farmlands near Cambridge. There, a pasture was filled with a primitive antenna consisting of wires hanging from stakes — a state-of-the-art radio antenna of those times. The idea was to use this newly build radio telescope to study interplanetary scintillation that could help in resolving quasars, another form of compact objects powered by black holes, from extended sources in the sky. Among several other students who were working for Anthony Hewish was a young post-doc named Jocelyn Bell. In addition to the signal from the scintillation, she discovered a deviation on her chart-recorded papers; an extremely regular signal of 1.3373012 seconds caught Bell's attention. Originally, this was dubbed (partially as a joke) Little Green Men 1 (LGM-1). In reality, what they were seeing, Bell quickly realized, was the first pulsar, a rapidly rotating neutron star whose radio emission beam sometimes points towards us, like a distant lighthouse. More Little Green Men quickly followed, and by the end of the year 1968, dozens of LGMs were known. The finding was later published in the Nature of 1968 by Hewish.† Hewish's announcement was quickly followed by more than 100 papers on pulsars, speculating the possible origin of the signal. The winning argument came from Timothy Gold, who showed that pulsars are strongly magnetized rapidly rotating neutron stars.‡ However, one should not forget the similar seminal theoretical paper already made in 1967, before the discovery, by Pacini.§ More proof came when our old friend the Crab Nebula was shown to host a pulsar rotating at a period of merely 33 milliseconds.¶ Anything but a neutron star would be destroyed by the centrifugal forces from such rotation.

The finding of Bell and Hewish was sensational and marked the first detection of a neutron star, almost 40 years after the theoretical speculation by Landau. Later on, Hewish was awarded the Nobel Prize in Physics in 1974 ”for the discovery of pulsars”, a somewhat unfair recognition taken into account that it was Bell who found them in practice. Hence, despite all the efforts in X-ray astronomy, the concluding evidence finally came from the radio wavelengths.

One should not, however, feel sorry for the X-ray astronomers, as they got their fair share of neutron-star-related revelations during the next decade. Important discoveries especially for studying the nature of accretion, or how matter infalls onto a compact object, came from the first long-duration observations done with the Dutch astronomy satellite ANS. As a direct competitor for the European ANS, the U.S. funded Los Alamos nuclear research center was also in the game of observing X-rays from compact objects. Their Vela satellites were sent to space mostly to monitor the compliance of the 1963 Partial Test Ban Treaty of nuclear weapons but they were used for science, too. In 1975, the ANS satellite was commissioned to study possible black holes in the center of globular clusters but happened to stumble upon something completely different; Short, $\sim$ 60-second-long X-ray flares were detected from the globular cluster NGC6624 by Grindlay and Heise.‖ The competing Los Alamos group found similar energetic bursts, but due to the poor angular resolution (collecting X-rays from the Earth was easy and hence no effort was put in for a good spatial accuracy), they could not pin point the exact location of the sources.** Later on, Clark et al. went through the existing SAS-3 data from May 1975 and found a series of ten similar bursts from the same location, NGC6624.†† Even more retrospectively, it turned out that these strange flares had already been observed in 1969 from Cen X-4‡‡ with another Vela satellite and in 1971 with the Soviet Kosmos 428 X-ray

* [41] N. S. Kardashev. *AZh*. (1964); [42] F. Pacini. *Nature*. (1967).
† [43] A. Hewish *et al*. *Nature*. (1968).
‡ [44] T. Gold. *Nature*. (1968).
§ [42] F. Pacini. *Nature*. (1967).
¶ [45] J. M. Comella *et al*. *Nature*. (1969).
‖ [46] J. Grindlay *et al*. *ApJ*. (1976).
** [47] R. D. Belian, J. P. Conner, and W. D. Evans. *ApJ*. (1976).
†† [48] G. W. Clark *et al*. *ApJ*. (1976).
‡‡ [49] R. D. Belian, J. P. Conner, and W. D. Evans. *ApJ*. (1972).

detector[*]. Their nature, however, remained elusive.

Pioneering theoretical work on thermonuclear instabilities on the surface layers of accreting neutron stars was initiated by Hansen and van Horn in 1975.[†] They constructed stationary burning shells to lay on top of neutron stars but instead found out that most of them were actually unstable. The choice of word, unstable, might not convey the full weight of the physical issue though; such a layer on top of the surface of a neutron star burning uncontrollably meant a spectacular firework. Shortly after the Los Alamos results came in, an Italian astrophysicist Laura Maraschi was able to connect the dots while visiting MIT in February 1976 and speculated that these recently observed X-ray bursts were due to thermonuclear flashes on the surface of accreting neutron stars.[‡] Woosley and Taam concluded similarly in their 1976 paper titled "Gamma-ray bursts from thermonuclear explosions on neutron stars". Observational evidence soon followed when van Parajids et al. and Thorstensen et al. were independently able to optically resolve the companions of two known bursting sources, Cen X-4[§] and Aql X-1[¶]. Not only did these observations confirm that there is a companion star close by but also that it must be within such a close distance of the neutron star that accretion, i.e. a constant flow of new fuel for the explosions, can exist.

All of the aforementioned discoveries were, however, nothing but a prelude to what was discovered in the years to follow. We will end this short historical review by listing some of the most important more modern findings. A big revelation came in 1979 when a very intense burst of gamma rays was detected by two Soviet satellites, Venera 11 and Venera 12.[‖] Later dubbed Soft Gamma Repeaters (SGRs), their energy source remained mysterious for decades. A theoretical breakthrough came in 1992 when Robert Duncan, Chistropher Thompson, and Bohdan Paczynski showed that the bursts, orders of magnitude stronger than the X-ray bursts, could originate from a neutron star with a magnetic field 100 to 1000 times more powerful than what was previously known.[**] Today, these neutron stars are more commonly known as magnetars, a subclass of young neutron stars where the initial magnetic field has been amplified by delicate dynamo processes during the supernova explosion. Another surprise came in 1982, when a team led by Backer changed how we look at pulsars when, using the world's largest radiotelescope in Arecibo, they found a pulsar spinning 641 times per second.[††] This new neutron star was dubbed a millisecond pulsar, and unlike its predesessors, we now know that instead of slowly decreasing in spin, it belongs to a class of old pulsars that have been spun up by the accretion. In 2000, our understanding of the thermonuclear X-ray bursts was also changed when Cornelisse observed a very long, not minutes but hours long, burst from a neutron star normally exhibiting regular short bursts.[‡‡] These were then dubbed as superbursts, in contrast to normal ones. The reason for this difference is, we think, the burning material; normal bursts use hydrogen and helium as their fuel but superbursts can devour a more rare carbon shell in a matter of hours if the conditions are just right.

1.2 From first principles to a neutron star

1.2.1 Background: Sun and stars

Stellar orders of magnitude

First, let us see what we can learn from neutron stars using simple estimates and conservation laws. Neutron stars are born from the death of a normal star. The most familiar one to us is our Sun, one Astronomical Unit

[*] [50] O. P. Babushkina *et al*. *Soviet Astronomy Letters*. (1975).

[†] [51] C. J. Hansen and H. M. van Horn. *ApJ*. (1975).

[‡] [52] L. Maraschi and A. Cavaliere. 1977; [53] W. H. G. Lewin, J. van Paradijs, and R. E. Taam. *SSRv*. (1993).

[§] [54] J. van Paradijs *et al*. *ApJ*. (1980).

[¶] [55] J. Thorstensen, P. Charles, and S. Bowyer. *ApJ*. (1978).

[‖] [56] E. P. Mazets *et al*. *Nature*. (1979).

[**] [57] R. C. Duncan and C. Thompson. *ApJ*. (1992).

[††] [58] D. C. Backer *et al*. *Nature*. (1982).

[‡‡] [59] R. Cornelisse *et al*. *A&A*. (2000).

or 1.496×10^{13} cm away from us.* With a mass of $M_\odot = 1.99 \times 10^{33}$ g and a radius of $R_\odot = 6.96 \times 10^{10}$ cm, our Sun gives us an idea of the typical stellar scale. Curiously, these numbers also mean that the mean density of the Sun is $\rho_\odot \approx 1.41\,\mathrm{g\,cm^{-3}}$, only 1.4× the density of water.

Like all stars, our Sun is held together by the inward-pulling gravity. Gravity does not prefer any direction more than another, and so a spherical object is expected to form. In addition to the inward-facing force, an outward-facing force is needed to balance the system. For normal stars this force originates from thermal pressure.

We observe stars in the night sky because they shine. This radiation, also the origin of the thermal pressure, is from the thermonuclear fusion reaction burning inside the star. *Thermo* here refers to the temperature and heat, *nuclear* to the atomic nuclei, and *fusion* to a process where elements are fused together. During the thermonuclear fusion process, the star's core fuses light elements such as protons into heavier ones like helium. The mass of two protons is less than a mass of one helium atom. This mass difference between the start and the end results is then transferred into energy in accordance to Einstein's famous $E = mc^2$ formula. A whole sequence of such fusion processes takes place inside the star, where lighter elements are merged together to build heavier and heavier elements. The energy release from this mass-to-energy conversion will then give the star a sufficient thermal pressure support to keep it from collapsing under the relentless gravity trying to squash it.

The fusion of elements does not continue forever. In the beginning, two protons collide to form helium. In the next stage, three helium nuclei collide to form carbon, and so on, until iron is created. Production of iron marks the end of the possible fusion chain, since the fusion of two iron cores no longer releases energy. On the contrary, it requires external energy to take place.† This iron produced will then sink to the center of the star, forming a dead core without any energy output.

Like all big furnaces, at some point the star will run out of fuel to burn. What is left behind is an inner core of iron with subsequent onion-like layers of lighter and lighter elements. The crucial question to ask next is: what is supporting this iron core now that the thermal pressure from the fusion process is lost? To answer this, we need to look inside the iron atom. An iron nucleus, consisting of 26 protons and 32 neutrons and surrounded by 26 electrons, repulses its neighbors because of the negatively charged electron cloud around it does not want to get in touch with its neighbors in the iron-atom lattice. ‡ It is this antisocial avoidance of neighboring particles, originating from electric charge repulsion, that will then give the internal support for the iron core to not collapse under its own gravitational pull. If enough iron is built up during the lifetime of the star so that even this repulsion force is not enough, we can continue our thought experiment and ask again: what will follow? This was the question that led scientists like Chandrasekhar to the realization of degenerate matter and white dwarf stars in the 1920s.

1.2.2 White dwarfs and quantum mechanics

The answer lies in the elusive quantum mechanics. When the atoms inside matter are packed close enough together, we need to apply wave-like characteristics for them instead of classical point-like thinking. Because of their smaller mass, the electrons orbiting the nuclei enter the realm of quantum mechanics first, in comparison to the heavier protons and neutrons in the atomic core. A freely moving electron confined into a small enough space because of its surrounding neighbors will start to attain only some fixed values

*Throughout this thesis, we will typically present our quantities only up to some fixed precision instead of the full litany of numbers. We will also adopt the centimeter-gram-second (cgs) unit system instead of the (maybe) more common SI-system. Such a selection is sure to disappoint some, but try to endure.

†This opens up another possibility of creating energy by splitting heavy elements, an inverse process to what is described here. Such a process is called fission and is familiarly taken advantage of in Earth's nuclear power plants.

‡A more precise consideration shows that nickel ($^{56}_{28}$Ni) is actually thermodynamically more favored in the core because of the lack of neutrons needed to synthesize iron-58 ($^{58}_{26}$Fe). The underlying idea presented here, however, remains the same.

of momenta. In physics, we speak about the quantization of energy levels. The reason for this is similar to a vibrating string of a guitar; a string fixed from both ends can only vibrate on some specific wave modes that are set by its length. An additional complication for the electrons is set by the Pauli exclusion principle, which forbids more than one electron to occupy the same wave mode or quantum state inside the same region.[*] This gives rise to a degeneracy pressure as electrons fill their quantum states from the lowest to the highest, and can thus not be packed any tighter together. A star held together by this degeneracy pressure of its electrons is known as a white dwarf. From this setup, it only takes a short step into realizing the existence of neutron stars, because we can, once again, push forward and ask: what next?[†]

1.2.3 Neutron stars, at last

What if, at some point, even these quantum effects of the electrons are not enough to support the star? One does not need to worry, since after the lightweight electrons have given all they can, it is the heavy neutrons that slowly start to enter the quantum mechanical realm. In practice, the matter will turn into a one big team of neutrons because when the positive (+) central proton and the surrounding negative (−) electron come in contact, a neutral neutron is created.[‡] The degeneracy pressure of such a neutron porridge is multiple orders of magnitude larger than what the electrons can offer, yielding an ultimate solution to the pressure support problem.

Let us consider the consequences of this thought experiment. More detailed calculations show that the resulting iron core sitting at the center of the star is weighing a maximum of around $\sim M_{\odot}$.[§] Hence, there are $M_{\rm core}/m_{\rm atom} \sim M_{\odot}/m_{\rm p} \approx 1.99 \times 10^{33}\,\mathrm{g}/1.67 \times 10^{-24}\,\mathrm{g} \sim 10^{57}$ atoms trapped inside the core. [¶] Here we are already considering not iron atoms but pure hydrogen atoms only to simplify the presentation. Working backwards from these numbers, we can estimate the size of the compressed core. Using a typical radius of $r_{\rm n} \approx 1.25 \times 10^{-13}$ cm for the nuclei, we would expect these particles to form an object of around $R \sim (10^{57})^{\frac{1}{3}} \times 1.25 \times 10^{-13}\,\mathrm{cm} \sim 10^{6}$ cm. Thus, we have ended up with a star consisting of only neutrons, with a size of ~ 10 km and a mass of $\sim 1\,M_{\odot}$: a neutron star!

By considering simple order-of-magnitude estimates we have now ended up characterizing the dimensions of a typical neutron star. A canonical neutron star is often taken to have $R = 10$ km and $M = 1.4\,M_{\odot}$, so let us also adopt these numbers for the following considerations. Such dimensions give us an impressive mean density of $\rho \sim 7 \times 10^{14}\,\mathrm{g\,cm^{-3}}$. In comparison, for a typical nucleon (such as a neutron or a proton) we had $m_p \approx m_n \approx 1.67 \times 10^{-24}$ g and $r_n \approx 1.25 \times 10^{-13}$ cm, yielding us a nuclear density of $\rho_{\rm n} \approx 2 \times 10^{14}\,\mathrm{g\,cm^{-3}}$. Not surprisingly, the densities are of similar magnitude. However, when comparing these numbers to our everyday matter, the difference is huge, almost 14 orders of magnitude; a cubic centimeter of water weighs 1 g, whereas the same volume of neutron star matter would weigh 100 000 000 000 000 g or 100 million metric tons.

Density

Matter compressed to such a small volume has an extreme impact even on the surrounding spacetime. Let us try to estimate, again, the order of magnitude of these effects by considering the escape velocity — a velocity needed to escape the local gravitational pull of an object. For us, on the surface of the Earth, it turns out to be $v_{\oplus} = \sqrt{2GM_{\oplus}/R_{\oplus}} = 1.12 \times 10^{6}\,\mathrm{cm\,s^{-1}}$, for $M_{\oplus} = 5.97 \times 10^{27}$ g and $R_{\oplus} = 6.37 \times 10^{8}$ cm. Similarly, for the Sun it is $v_{\odot} = 6.18 \times 10^{7}\,\mathrm{cm\,s^{-1}}$, or 0.002× the speed of light. On the other hand, for a

Escape velocity

[*] Explain Pauli repulsion for the layman

[†] The reader can be assured that the chain of thermal pressure → charge repulsion → electron degeneracy will come to a halt as the final neutron degeneracy really is the last possible supporting force in nature (maybe excluding quark matter, though...)

[‡] In reality, the beta decay formula is $e^{-} + p = n + \bar{\nu}_e$, where the additional electron anti-neutrino is needed to preserve the quark color neutrality.

[§] This is quite a reasonable-sounding assumption considering that the stars that explode are around $\sim 10\,M_{\odot}$ in size and we certainly do not expect everything to fall into the core.

[¶] The mass of the atom, $m_{\rm atom} = m_{\rm p} + m_{\rm e}$, is approximated (to an excellent accuracy) by only considering the central nuclei alone as the electron mass $m_{\rm e} \approx 9.11 \times 10^{-28}$ g is negligible in comparison to the proton mass.

neutron star, we obtain $v_{\mathrm{NS}} = 1.93 \times 10^{10}\,\mathrm{cm\,s^{-1}}$, which is already about half of the speed of light! Hence, relativistic effects become crucial to take into account when considering neutron stars, as one can not even escape from the surface of the star without velocities close to those of the light.

Spin

Let us next think about the possible spin rates that a neutron star can have. For our Sun, it takes about one month (or approximately 25.5 days, to be more exact) to revolve around itself, corresponding to a spin rate of 4.5×10^{-7} Hz. When compressed to the dimensions of a neutron star, the radius changes by a factor of $R_{\odot}/R_{\mathrm{NS}} \approx 6.96 \times 10^{10}\,\mathrm{cm}/10^{6}\,\mathrm{cm} \sim 7 \times 10^{4}$. It is important to notice that when a rotating object collapses, it preserves it's angular momentum, not the spin rate. Similar to an ice-figure skater pulling her arms inwards while spinning, we observe an increase in the spin in order to preserve the angular momentum. As the rotational inertia increases as a square from the distance to the axis, our Sun, when compressed to the scale of a neutron star, would obtain a spin of $4.5 \times 10^{-7}\,\mathrm{Hz} \times (7 \times 10^{4})^{2} \sim 2 \times 10^{3}$ Hz, 2000 revolutions per second. The young proto neutron star, however, quickly slows down after its birth, so more typically, spins of around 100 to 1000 Hz are observed, which is still about one revolution per 1 to 10 milliseconds.

B-field

One final characteristic we can try and estimate is the magnetic field. Here we can follow a similar chain of reasoning as with the spin and start from typical values such as those of our Sun. For the Sun, the slow rotation gives rise to a dynamo process that produces a magnetic field of around $B_{\odot} \approx 1\,\mathrm{G}$.* When considering magnetic field, it is the magnetic flux through the surface that conserves, hence we expect the field to scale also as a square of the radius. Using the same compression ratio of 7×10^{4} for the radius, we then obtain $B_{\mathrm{NS}} \approx 1 \times (7 \times 10^{4})^{2}\,\mathrm{G} \sim 10^{10}\,\mathrm{G}$. Comparing this to the value of 10^{6} G for the strongest non-destructive magnet on Earth, we start to grasp the level of energetics that neutron stars have to offer; even their original non-amplified magnetic field is $\times 10\,000$ stronger. In some cases, a dynamo effect originating from the rapid rotation of the star can amplify the magnetic field even by a factor of a million. This gives rise to neutron stars with immense magnetic field strengths of $B \sim 10^{16}$ G.

It is fair to conclude that neutron stars are dominating the record tables of physics in almost all of their aspects. They are *super*dense, *super*fast rotators, sources of *super*strong magnetic fields, and *super*rich in the range of physics involved. In short: they are the *super*stars of physics!†

1.3 Connection to this thesis

In this thesis we study many aspects of the neutron stars ranging from their astrophysical environments to the nuclear physical interiors. In the end, our main goal is to use this plethora of information to better constrain the behavior of the ultra-dense matter inside the core.

*A typical refrigerator magnet is about 50× stronger with a magnetic field of 50 G.

†This is (humorously) called the Pines theorem as everything is *super*- when considering neutron stars, as postulated by David Pines in a talk given at the conference on Neutron Stars: Theory and Observation (The NATO Advanced Study Institute, Crete, Greece, September 3–14, 1990).

2 Physics of neutron star interiors

In this chapter, we will study the neutron star interiors in more detail. In practice, this means describing the behavior of the matter from a densities of $\sim 10^{-2}\,\mathrm{g\,cm^{-3}}$ to $\sim 10^{16}\,\mathrm{g\,cm^{-3}}$, an impressive 18 orders of magnitude range starting from a hot and rarefied electron corona to an ultra-dense neutron liquid.

The structure of the star can be roughly divided into three distinct sections: atmosphere, crust, and core. Neutron star atmosphere holds a negligible amount of matter in comparison to the whole star, but it is important for the radiative processes. It is the radiation from the atmosphere that we actually observe. The crust, like the name implies, can be understood as a solidified layer surrounding the liquid core. Physics describing the crust is relatively well known and same type of matter consisting of ions, protons, and electrons can be found inside white dwarf stars. Bulk of the mass, on the other hand, is located on the liquid neutron core. Detailed microphysics of such matter are still unknown and this is reflected in a large uncertainty in the actual size of the star that is still unconstrained.

We begin by giving an overview of the characteristics of each of the different layers. By combining this information, we can then build different models for the neutron stars and describe some more global aspects of them such as mass and radius. For this we need to solve the relativistic equations of hydrostatic equilibrium, that are also discussed. In the end, this enables us to build a mapping between the (un)known microphysics of the dense matter and the astrophysical observables.

2.1 Equation of state

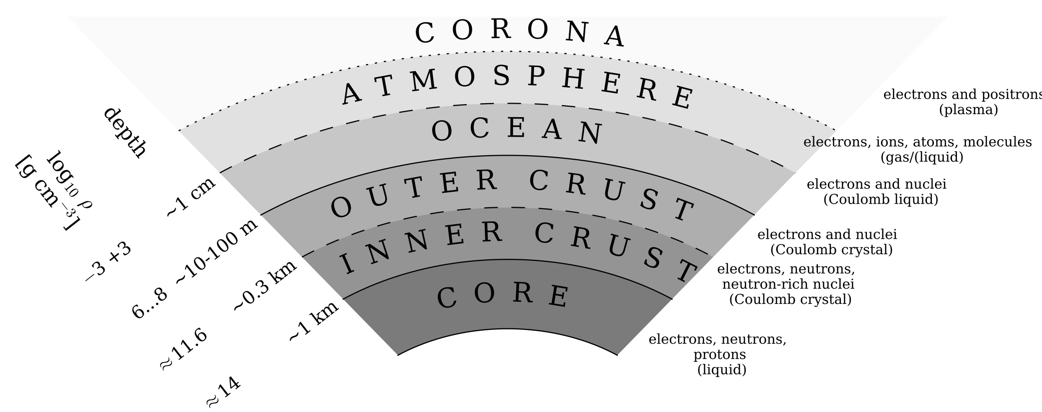

Figure 2.1: Schematic view of neutron star structure illustrating the different internal regions, related densities, and the compositions.

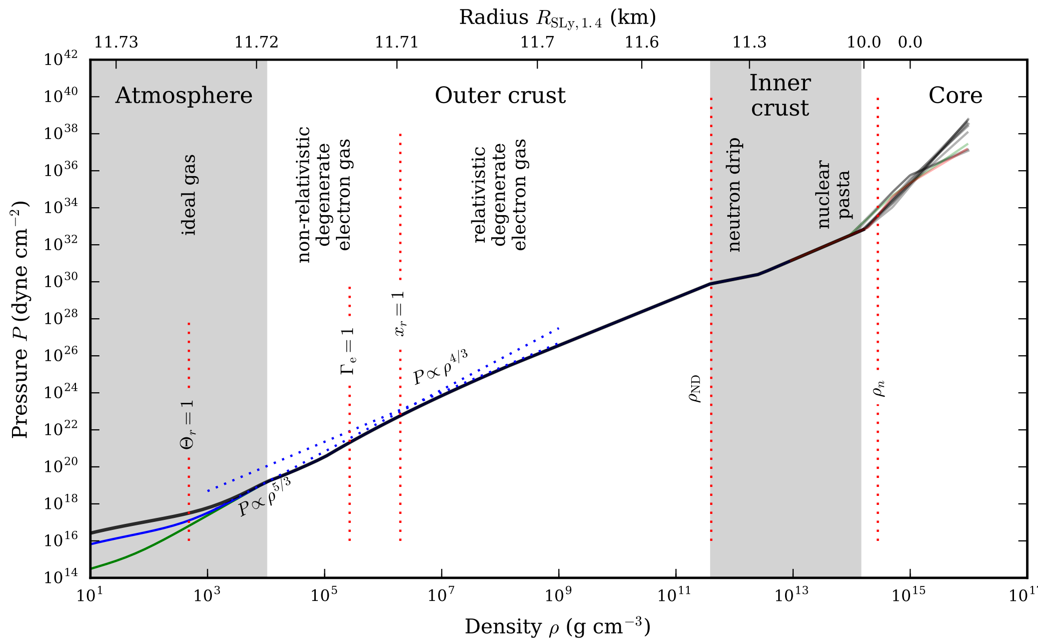

Figure 2.2: Overview of the pressure versus density relation for the full range of densities relevant for neutron stars. Here the evolution of the pressure is shown against the densities depicted in the bottom vertical axis. Green solid line shows the EoS for matter at $T = 10^6$ K, whereas blue line is for $T = 5 \times 10^6$ K, and black for $T = 10^7$ K. Additionally, the upper vertical axis shows the evolution of the radial coordinate computed for one particular EoS (SLy, see Sect. 2.4) and neutron star configuration (mass of $1.4\,M_\odot$). Different shaded vertical regions show the corresponding interior structures of the star. Additionally, some interesting densities are highlighted with dashed red lines and text labels (see Sects. 2.2–2.4).

In thermodynamics, we speak of *state variables* that describe a current state of the matter under a given physical conditions. These include, for example, the density ρ, pressure P, and temperature T of the matter. Equation of state is a thermodynamical equation connecting these states variables together. Often, when focusing on neutron stars, what we mean by EoS is a function connecting the pressure and the density of the matter only, $P(\rho)$.

Degenerate matter

The dependency on the rest of the variables such as temperature can be often forgotten because the matter is *degenerate*. In contrast to the "normal" matter where a statistical moments such as temperature can be used to describe a large ensemble of particles, the degenerate matter is dominated by quantum mechanical effects of single particles. Because of the immense densities, a free particle in a degenerate matter is actually bounded into a finite volume. Inside this small volume, the energy levels of the particle are restricted to take only a discrete set of values called quantum states, because of the underlying wave-nature of the quantum mechanical description. Hence, a notion of temperature, for example, does not make much sense.

Overview of the EoS for the full range of densities relevant to neutron stars is shown in Fig. 2.2. From here it is easy to see that temperature only plays a role in the very uppermost ~ 10 meters of the stars interiors. Behavior of the matter is also quite well known all the way up to the crust-core interface, after which we start to see larger deviations because of the different EoS models. In the Earthly laboratories we can probe the matter somewhere close to $10^{14}\,\mathrm{g\,cm^{-3}}$, after which the densities becomes too great for us to handle in.[*] On the other hand, it is exactly starting from this density range that the bulk of the neutron star just starts. Another curious quirk of Nature is how all of the complicated microphysics gets reduced to simple line segments in the logarithmic scales, also known as polytropic pressure relations. In the following sections, we will focus on deriving these simple relations as it helps us in understanding the underlying physics.

2.2 Atmosphere

Atmosphere of a star is the first and uppermost layer responsible for the emergent radiation. It consists of a thin layer of plasma and ranges from a few millimeters to couple of centimeters in height. In most situations the plasma is in a gaseous state, but in some more rare cases when the magnetic field is extraordinarily strong and the temperature is low, the plasma can condensate into a liquid or a solid surface. Such condensed surfaces are, however, rare and usually the gaseous description is more than enough.[†]

Properties of the emergent thermal radiation strongly depend on the chemical composition of the atmosphere. In the atmospheres of normal stars the composition is a mixture of multiple elements. The most stable chemical element on the surface of a neutron star is iron. However, even a small accreted mass of $10^{-17}\, M_\odot$, originating from the surrounding interstellar medium, is enough to cover the whole star, and hence a variety of elements are also expected in the neutron star atmospheres. On the other hand, the enormous gravity results in an effective separation of elements leading to a strong sedimentation of the atmosphere where the lighter elements are expected to lay on top of the heavier ones.[‡] Hence, the atmosphere is usually expected to consist of mainly hydrogen.

2.2.1 General relativistic effects

The effects from the gravity can be quantified by considering a so-called compactness parameter

$$u = \frac{R_\mathrm{S}}{R} \tag{2.1}$$

where R is the radius of the neutron star and the corresponding Schwarzchild radius is defined as

$$R_\mathrm{S} = \frac{2GM}{c^2} \approx 2.95 \frac{M}{M_\odot}\ \mathrm{km}, \tag{2.2}$$

where G is the gravitational constant, c is the speed of light, and M is the mass of the star. Hence, a neutrons star has a compactness parameter in the range of $u \approx 1/5$ to $1/2$ resulting in a considerable general relativistic corrections. In comparison, the Sun has $u \approx 4.24\times10^{-6}$. Gravitational acceleration under general relativistic theory is

$$g = \frac{GM}{R^2}\frac{1}{\sqrt{1-u}} = 1.38\times10^{14}\ \frac{1}{\sqrt{1-u}}\left(\frac{M}{M_\odot}\right)\left(\frac{10\,\mathrm{km}}{R}\right)^2\ \mathrm{cm\,s^{-2}}. \tag{2.3}$$

Hence, a surface gravitational accelerations g of $\sim 10^{14}$ to $\sim 10^{15}\,\mathrm{cm\,s^{-2}}$ are expected for neutron stars. By considering a barometric atmosphere we can also estimate the scale height as

$$H_\mathrm{a} = \frac{k_\mathrm{B}T}{m_i g} \approx \frac{0.83}{A}\left(\frac{T}{10^6\,\mathrm{K}}\right)\left(\frac{10^{14}\,\mathrm{cm\,s^{-2}}}{g}\right)\mathrm{cm} \tag{2.4}$$

[*] Maximum densities reached in the Earth are usually obtained by colliding heavy nucleons together, momentarily creating a core of even denser matter. The densest naturally occurring element found on top of planet Earth is osmium that has a density of "just" $\rho \approx 2.2\times10^1\,\mathrm{g\,cm^{-3}}$.

[†] for a review, see [60] V. E. Zavlin and G. G. Pavlov. 2002; [61] A. Y. Potekhin. *Physics Uspekhi*. (2014).

[‡] [62] C. Alcock and A. Illarionov. *ApJ*. (1980).

where $k_\mathrm{B} = 1.38 \times 10^{-16}$ erg K^{-1} is the Boltzmann constant, T is the temperature of the atmosphere, $m_i = Am_u$, and $m_u \approx 1.66 \times 10^{-24}$ g is the atomic mass unit. From here, the typical scale height values of ~ 1 cm to ~ 10 cm are re-obtained for atmospheres of $T = 10^6$ and 10^7 K.* Strong gravitational field also bends the photon trajectories.† Hence, in addition to the radius R of the star as measured in the local reference frame, another *apparent* radius, as measured by an observer at infinity,

$$R_\infty = \frac{R}{\sqrt{1-u}}, \tag{2.5}$$

is usually needed when describing the observable features of the atmosphere. From here it is then clear that the atmosphere and the emerging radiation encodes information from the physical parameters of the star. More specifically information about the temperature, surface gravity, chemical composition, compactness can be obtained.

2.2.2 Basic equations of the atmosphere

The standard approach in describing the atmosphere structure includes solving three main equations of radiative transfer, hydrostatic balance, and energy conservation. First such a low-B field model of neutron star atmospheres were presented in the pioneering work by Romani.‡ Let us next see, walk through these equations, as they are rater simple. A more general description for the atmosphere model computations are given in Sect.XX§, where the full relativistic electron scattering is also taken into account.

Because the thickness of the atmosphere is much smaller than the radius of the star,¶ the atmosphere can be considered in plane-parallel approximation. Rather high densities, on the other hand, allow to consider the plasma of the atmosphere in local thermodynamical equilibrium.

Spectrum, beaming and polarization of emerging radiation can be determined from radiation transfer problem in atmospheric layers. Radiation can be understood as an energy flow, i.e., energy E per area A, time t, frequency ν, and solid angle Ω. This is known as the specific spectral intensity which we can mathematically formulate as

$$I_\nu = \frac{dE}{dAdtd\nu d\Omega}. \tag{2.6}$$

Radiation averaged over the solid angle, or the so-called mean specific intensity (zeroth moment of I_ν) is then

$$J_\nu = \frac{1}{4\pi}\int_\Omega I_\nu d\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_\nu \sin\theta d\theta d\phi = \frac{1}{2}\int_{-1}^{+1} I_\nu d\mu, \tag{2.7}$$

where we have assumed that the radiation does not depend on the azimuthal ϕ angle (as is typical for atmosphere calculations) and introduced $\mu = \cos\theta$. Net rate of energy flowing across an unit area (for example a photon detector) from *all directions* per time and frequency is known as physical flux.‖ It is proportional to the first-order moment of I_ν and is defined as

$$F_\nu = 2\pi \int_{-1}^{+1} I_\nu \mu d\mu. \tag{2.8}$$

* [60] V. E. Zavlin and G. G. Pavlov. 2002; [61] A. Y. Potekhin. *Physics Uspekhi*. (2014).

† [63] K. R. Pechenick, C. Ftaclas, and J. M. Cohen. *ApJ*. (1983).

‡ [64] R. W. Romani. *ApJ*. (1987).

§ **NKS15**.

¶ Recall the scale height of 1 to 10 cm in comparison to the radius of 10^6 cm.

‖ Strictly speaking, the first-order moment of I_ν is known as the Eddington flux $H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu d\mu$. Physical flux is related to it as $F_\nu = 4\pi H_\nu$ and sometimes one also encounters the "astrophysical" flux defined as F_ν/π.

Similarly, the second-order moment of I_ν, or a so-called K-integral, is

$$K_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu \mu^2 d\mu, \tag{2.9}$$

which is related to the radiation pressure, as we later on will see.

Now we can introduce the radiative transfer equation for I_ν as

$$\mu \frac{dI_\nu}{d\tau} = \frac{\mu}{\kappa_\nu} \frac{dI_\nu}{dy} = -\frac{\mu}{\rho\kappa_\nu} \frac{dI_\nu}{dz} = I_\nu - S_\nu \tag{2.10}$$

where τ is the optical depth, y is the column density (mass per area), z is the horizontal distance from the surface, $\kappa_\nu = \alpha_\nu + \sigma_\nu$ is the total radiative opacity including contributions from the "true" opacity α_ν and from the scattering opacity σ_ν. Here the connection between different independent variables is given as

$$d\tau = \kappa_\nu dy = -\kappa_\nu \rho dz, \tag{2.11}$$

relating the optical depth (distance as experienced by the radiation), column density (projected number density of matter along the path of the radiation), and the horizontal height. In addition we need the source function

$$S_\nu = (\sigma_\nu J_\nu + \alpha B_\nu)\kappa_\nu^{-1}, \tag{2.12}$$

where the scattering term is proportional to the mean spectral intensity J_ν and the "true" absorption term to the thermal Planck function

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left[h\nu/(k_\mathrm{B}T)\right] - 1}, \tag{2.13}$$

where $h = 6.63 \times 10^{-27}$ erg s is the Planck constant. As an boundary conditions for this equation, we can use $I_\nu = 0$ for $\mu < 0$ at $y = 0$ (i.e., the surface). The atmospheres are also usually considered to be in radiative and hydrostatic equilibrium, i.e., (quasi-)stationary. The first requirement can be formulated as

$$\int_0^\infty d\nu \int_{-1}^{+1} I_\nu \mu d\mu = \sigma_\mathrm{SB} T_\mathrm{eff}, \tag{2.14}$$

where $\sigma_\mathrm{SB} = 5.67 \times 10^{-5}$ g s^{-3} K^{-4} is the Stefan-Boltzmann constant and T_eff is the effective temperature of the atmosphere. The second hydrostatic equilibrium, demands that

$$\frac{dP}{dy} = g - g_\mathrm{rad}, \tag{2.15}$$

where, in addition to the gravitational acceleration g we need the opposing radiative acceleration g_rad. Finally, we need to supplement these equations with an equation connecting the pressure and density. For the rarefied atmosphere, the ideal gas law is an excellent approximation

$$P = nk_\mathrm{B}T, \tag{2.16}$$

where n is the number density of particles.

2.2.3 Eddington limit

Usually the atmospheres we calculate are dynamically stable and in hydrostatic balance because large gravity implies $g_{\rm rad} \ll g$. Sometimes, however, the radiation flux might increase to such a strengths that it is able to compete against even the enormous gravity of a neutron star. An important limit can then be defined for $g_{\rm rad} = g$, known as the Eddington limit after a renowned astrophysicist Sir Arthur Eddington. Let us now for completeness derive this limit.*

We can start by formulating the radiation pressure. This can be easily done when we realize that pressure is just momentum flux, and photons carry a momentum of E/c.† In terms of I_ν this is then

$$P_{{\rm rad},\nu} = \frac{1}{c}\int_0^{2\pi}\int_{-1}^{+1} I_\nu \mu^2 d\mu = \frac{4\pi}{c}K_\nu, \tag{2.17}$$

relating the pressure and the second-order moment K_ν together. Radiative acceleration is then

$$g_{\rm rad} = \frac{dP_{\rm rad}}{dy} = \frac{d}{dy}\int_0^\infty P_{{\rm rad},\nu}\, d\nu = \frac{4\pi}{c}\frac{d}{dy}\int_0^\infty K_\nu d\nu. \tag{2.18}$$

Let us refine this expression by inserting the definition of K_ν and applying the radiative transfer equation (2.10) without the source function S_ν. In the process we also have to take into account that we are only interested in the outgoing flux, hence the integration limits of μ need to be changed as $[-1,+1] \to [0,+1]$. We can then simplify the (2.18) to

$$\begin{aligned} g_{\rm rad} &= \frac{4\pi}{c}\frac{d}{dy}\int_0^\infty d\nu \frac{1}{2}\int_0^{+1} I_\nu\, \mu^2 d\mu \\ &= \frac{2\pi}{c}\int_0^\infty d\nu \int_0^{+1} \mu\, d\mu\, \left\{\mu \frac{d}{dy} I_\nu\right\} \\ &= \frac{2\pi}{c}\int_0^\infty d\nu \int_0^{+1} \mu\, d\mu\, \kappa_\nu I_\nu \\ &= \frac{2\pi}{c}\int_0^\infty d\nu\, \kappa_\nu \int_0^{+1} I_\nu \mu d\mu \\ &= \frac{1}{c}\int_0^\infty \kappa_\nu F_\nu d\nu. \end{aligned} \tag{2.19}$$

From the bottom line of Eq. (2.19) we see that the K-integral is related to the flux of the radiation F_ν. Not every photon, however, interacts and collides with the matter, and hence the opacity correction κ_ν is also needed, representing effectively the fraction of radiation interacting with the matter. If we now integrate over all the frequencies we obtain

$$g_{\rm rad} = \frac{1}{c}\,\kappa F \tag{2.20}$$

Setting it equal to g we can solve for the Eddington flux as

$$F_{\rm Edd} = \frac{gc}{\kappa}\frac{1}{\sqrt{1-u}} = \frac{GMc}{R^2\kappa}\frac{1}{\sqrt{1-u}}. \tag{2.21}$$

*for a more in depth discussion, see e.g., [65] G. B. Rybicki and A. P. Lightman. 1979; [66] J. Frank, A. King, and D. J. Raine. 2002.

†Energy of a photon is $E = h\nu = mc^2$, from which we obtain the photon rest mass of $h\nu/c^2 = E/c^2$. Momentum, on the other hand, is just velocity times the mass, so for a photon it is $E/c^2 \times c = E/c$.

Usually in many astrophysical scenarios the opacity is dominated by the electron Thomson scattering opacity, $\sigma_\mathrm{T} = 6.65 \times 10^{-25}\,\mathrm{cm}^{-2}$. Mass for a hydrogen plasma is, on the other hand, mainly set by the protons, hence, opacity per mass is then

$$\kappa \approx \frac{\sigma_\mathrm{T}}{m_\mathrm{p}}. \tag{2.22}$$

Finally, be calculating the total radiation flux through the stellar surface, we can define a quantity called luminosity as

$$L = 4\pi R^2 F(R), \tag{2.23}$$

where $F(R)$ is the outcoming radiation flux at the surface. Using the aforementioned equation, we can then define the Eddington luminosity for a star as

$$L_\mathrm{Edd} = \frac{4\pi G M c m_\mathrm{p}}{\sigma_\mathrm{T}} \approx 1.3 \times 10^{38} \left(\frac{M}{M_\odot}\right) \mathrm{erg\,s^{-1}}. \tag{2.24}$$

For neutron stars close to $T \sim 10^7$ K this limit can be reached, after which the atmosphere is momentarily blown away.

2.2.4 Radiative heat transfer

Let us also shortly characterize the time-dependency of the energy transfer in the atmosphere.* Such an analysis will also be observationally important because in some cases we do see short bursts of energy coming from the neutron stars, called X-ray bursts.†

If we deposit an energy Q into somewhere deep below the atmosphere, this energy will be transferred to the surface, and then freely radiated to us. In general, this heat energy is

$$Q = m C_P T, \tag{2.25}$$

where m is the mass of the cooling layer, C_P is the specific heat capacity at a constant pressure, and T is the temperature of the layer. Assuming, as a first approximation, constant opacity for the atmosphere, the heat is transferred simply according to Stefan-Boltzmann law as

$$\frac{dQ}{dt} = -A\sigma_\mathrm{SB}\frac{T^4}{\tau} = -A\sigma_\mathrm{SB} T_\mathrm{eff}^4, \tag{2.26}$$

where t is now the time, A is the area that the heat radiation propagates through, T_eff is the effective temperature of the atmosphere, and τ is the optical depth, as given before. Temperature solved from this is simply

$$T_\mathrm{eff} = \left(\frac{3A\sigma_\mathrm{SB}}{mC_P} t\right)^{-1/3}. \tag{2.27}$$

Assuming all of the energy goes into radiation, we then obtain

$$L = -\frac{dQ}{dt} = A\sigma_\mathrm{SB} T_\mathrm{eff}^4 = (A\sigma_\mathrm{SB})^{-1/3} \left(\frac{3}{mC_P} t\right)^{-4/3}. \tag{2.28}$$

Hence, energy deposited below the atmosphere will radiate to us and decay as $\propto t^{-4/3}$. In general, the heat capacity is $C_P = C_P(T)$, but the main results hold, that observationally we should see a powerlaw decay of luminosity, if an excess energy is instantaneously injected or released into the deeper layers of the star.

*for a more in depth discussion, see [67] A. Cumming and J. Macbeth. *ApJ*. (2004); [68] J. J. M. in't Zand *et al*. *A&A*. (2014).

†for a review, see e.g., [69] W. H. G. Lewin, J. van Paradijs, and R. E. Taam. *Space Sci. Rev.* (1993); [70] D. K. Galloway *et al*. *ApJS*. (2008).

2.3 Crust

Below the gaseous atmosphere, a solidified layer of matter exists, called crust.* Between the atmosphere and crust, a liquid ocean of ions also exists, but the interface is not very strict and the matter is smoothly evolving from an one state to another. The solidified crust is also typically divided into an outer and inner layers, but the interface is again ambiguous. The pressure here is almost fully given by the degenerate electrons, and hence the matter is familiar already from white dwarfs. In the beginning, the electrons can be taken to be non-relativistic but after about $\rho \sim 10^6 \ \mathrm{g\,cm^{-3}}$ they turn into ultra-relativistic because of the increasing density.

By definition, the outer crust is a layer in the neutron star interiors where the plasma consists of electrons and nuclei, whereas the inner crust is characterized by an additional appearance of neutrons that start to drip out from the extremely neutron-rich nuclei. The density this occurs is called the neutron drip density and is of around $\rho_{\mathrm{ND}} \sim 4\times10^{11}\,\mathrm{g\,cm^{-3}}$. The outer crust, when defined to begin from the atmosphere at $\rho \sim 10^3\,\mathrm{g\,cm^{-3}}$ and continue to about ρ_{ND} is only about some hundred meters in thickness. The characteristics of the matter are strongly dependent on the Coulomb interactions of charged particles that form a solid Coulomb crystal.

The inner crust is taken to continue all the way down to the crust-core interface, where the matter turns liquid again. This layer is about one kilometer thick. Here, at the bottom of the crust, the density is already close to the nuclear density of $\rho_{\mathrm{n}} = 2.8 \times 10^{14}\,\mathrm{g\,cm^{-3}}$ but the exact location of the transition depends on the detailed microphysics of the core. The fraction of free neutrons grows with the increasing density. Because the normal nuclei nuclei here are immersed into free neutron gas, the nuclear interactions play a crucial role in defining the matter. Finally, the nuclei disappear totally when we enter the core. Before that, however, the nuclei form complicated structures that evolve together with the density. This region is also known as the nuclear pasta phase, as the different molecular structures are named after the pasta types that they resemble.†

2.3.1 Fermi gases

The matter in the crust is degenerate.‡ Let us discuss the physics behind degenerate matter and the related important concept of Fermi energy now. Elementary 6-dimensional phase space cell of any particle is bounded by the Heisenberg uncertainty principle as

$$\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3, \tag{2.29}$$

where $\Delta x \Delta y \Delta z$ is the volume in ordinary space, $\Delta p_x \Delta p_y \Delta p_z$ the volume in the momentum space, and h is the previously defined Planck constant. In accordance with the quantum mechanics, there is room for only one particle of any kind inside the elementary cell. In general the number density is given as

$$n_{\mathrm{av}} \ = \frac{g}{\exp[(E-\mu)/k_{\mathrm{B}}T] \pm 1}, \tag{2.30}$$

where + is used for fermions (such as electrons and protons), and − for bosons (such as photons). Additionally, the g is the number of different quantum states a particle may have inside the cell, E is the total particle energy, and μ is the chemical potential explained more carefully later on. Number of particles with momentum between p and $p + dp$ is

$$n(p)dp = n_{\mathrm{av}} \frac{4\pi p^2}{h^3} dp, \tag{2.31}$$

*for a review, see, e.g., [71] P. Haensel, A. Y. Potekhin, and D. G. Yakovlev. 2007; [72] N. Chamel and P. Haensel. *Living Reviews in Relativity*. (2008); [73] C. Bertulani and J. Piekarewicz. 2012.

†see e.g. [74] M. E. Caplan and C. J. Horowitz. *ArXiv e-prints*. (2016).

‡for a more in depth discussion of the topic, see e.g., [75] L. D. Landau and E. M. Lifshitz. 1980.

i.e., particles inside a spherical shell with a surface $4\pi p^2$ and thickness dp. Number density of particles with all momenta is then

$$n = \int_0^\infty n(p)dp \tag{2.32}$$

and the density is then simply

$$\rho = nm, \tag{2.33}$$

where m is the mass of the particle. The energy in equation (2.30) is the total energy $E \equiv E_{\mathrm{tot}} = E_0 + E_{\mathrm{k}}$, composed of the rest-mass energy $E_0 = mc^2$ and kinetic energy E_{k}. In special relativity, there exists a relation for the total energy as $E_{\mathrm{tot}}^2 = (mc^2)^2 + (pc)^2$, so

$$E = mc^2 \left[1 + \left(\frac{p}{mc}\right)^2\right]^{1/2}, \tag{2.34}$$

and the kinetic energy is then simply

$$E_{\mathrm{k}} = mc^2 \left[\left(1 + \left(\frac{p}{mc}\right)^2\right)^{1/2} - 1\right]. \tag{2.35}$$

Simple asymptotic limits can be obtained for the kinetic energy as

$$\begin{aligned} E &\approx \frac{p^2}{2m} \quad p \ll mc \text{ (non-relativistic case)} \\ E &\approx pc \quad p \gg mc \text{ (ultra-relativistic case).} \end{aligned} \tag{2.36}$$

Let us now focus on fermions only, and hence select the +-sign (Fermi-Dirac distribution) from the Eq. (2.30). Fermions are spin $\frac{1}{2}$ particles so we can also set $g = 2$. The chemical potential μ in Eq. (2.30) is expressed as

$$\mu = mc^2 + \epsilon_{\mathrm{F}} \tag{2.37}$$

where ϵ_{F} is now the so-called Fermi energy. On the other hand, it can also be defined as

$$E - \mu = E_{\mathrm{k}} - \epsilon_{\mathrm{F}} \tag{2.38}$$

i.e., difference between the kinetic and Fermi energy. Hence, the number density of fermions in general is

$$n_{\mathrm{f}} = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp(\frac{E_{\mathrm{k}} - \epsilon_{\mathrm{F}}}{k_{\mathrm{B}} T}) + 1}. \tag{2.39}$$

If Fermi energy $\epsilon_{\mathrm{F}} \ll 0$ the distribution will be Maxwellian even for small kinetic energies. On the other hand, if $\epsilon_{\mathrm{F}} \gg k_{\mathrm{B}} T$ we can divide the integrand of Eq. (2.39) into two distinct regions of

$$n_{\mathrm{f}} = \begin{cases} p^2 & \text{if } E_{\mathrm{k}} \ll \epsilon_{\mathrm{F}} \\ 0 & \text{if } E_{\mathrm{k}} \gg \epsilon_{\mathrm{F}} \end{cases} \tag{2.40}$$

where the transition occurs rather sharply at $E_{\mathrm{k}} = \epsilon_{\mathrm{F}}$. This allows us to define a characteristic momentum related to the transition, called Fermi momentum p_{F}, so that

$$n_{\mathrm{f}} \approx \frac{8\pi}{h^3} \int_0^{p_{\mathrm{F}}} p^2 dp = \frac{8\pi}{3} \left(\frac{p_{\mathrm{F}}}{h}\right)^3. \tag{2.41}$$

Physically this can be interpreted such that when the temperature decreases, the fermions start to occupy all the quantum states starting from the one with lowest energy all the way up to the Fermi energy. Because of the Pauli exclusion principle, no more than one fermion can exists in the same quantum state so the levels are filled in order, and all the higher states will remain empty. Hence, the highest momenta possible in the degenerate matter is the Fermi momentum

$$p_{\mathrm{F}} = \left(\frac{3n_{\mathrm{f}}}{8\pi}\right)^{1/3} h. \tag{2.42}$$

2.3.2 Why neutrons then?

Let us first consider an ideal gas of degenerate electron-proton-neutron plasma to understand the basic composition of the crust.* In a degenerate plasma all the quantum states are filled up all the way to the Fermi energy, as we just learned. Normal beta-decay mode for the neutrons, on the other hand, is

$$n \rightarrow p + e^- + \bar{\nu}_e, \tag{2.43}$$

that describes the possible path of how a neutron n will decay into a proton p, electron e^-, and electron neutrino $\bar{\nu}_e$. It is because of this decay, that we do not expect to see any free neutron flying around. Such a decay is, however, might be blocked because there is no room for an emission of an extra electron e^- or a proton p.

Let us then only focus on the decay of the most energetic neutrons with an energy equal to the Fermi energy $\epsilon_{\mathrm{F}}(n)$, where the related particle species is defined inside the parentheses. Co-existence of neutrons, protons, and electrons is then guaranteed (at zero temperature) if

$$\epsilon_{\mathrm{F}}(n) = \epsilon_{\mathrm{F}}(p) + \epsilon_{\mathrm{F}}(e^-). \tag{2.44}$$

Massive neutrons and protons are to a good approximation non-relativistic up to a densities of $10^{14}\,\mathrm{g\,cm^{-3}}$, and hence energy is simply a sum of their rest mass energy and kinetic energy

$$\epsilon_{\mathrm{F}}(n) \approx m_n c^2 + \frac{p_{\mathrm{F}}(n)^2}{2m_n}, \tag{2.45}$$

and

$$\epsilon_{\mathrm{F}}(p) \approx m_p c^2 + \frac{p_{\mathrm{F}}(p)^2}{2m_p}. \tag{2.46}$$

Electrons, on the other hand, are already ultra-relativistic, and so

$$\epsilon_{\mathrm{F}}(e^-) \approx p_{\mathrm{F}}(e^-) c^2. \tag{2.47}$$

Also note that $n_p = n_e$, as the star is electrically neutral.

From this we find relation of the $n_n/n_p \sim 1/200$ by taking into account the rest mass difference $m_p - m_n = 2.6\,\mathrm{MeV}\,c^2$, for example at $\rho \sim 10^{14}\,\mathrm{g\,cm^{-3}}$. Thus, we conclude that the matter inside is expected to be neutron rich, even though normally the neutrons would β-decay back to protons and electrons. Not only is the degeneracy then responsible for the pressure but it is also the source of the neutron enrichment.

*see e.g. [76] A. C. Phillips. 1994.

2.3.3 Degenerate electron gas

Let us next consider the equation of state for the crust. As we have seen, the pressure in the crust originates from the degenerate electron gas. Physics behind this are quite simple and we repeat the calculations here to introduce the reader to the topic. The result also bears some historical value as these are exactly the equations that were introduced by Dirac[*], Fowler[†], Frenkel[‡], Anderson[§], Stoner[¶], and Chandrasekhar[‖].More thorough discussion of the electron thermodynamics is given, for example, in Refs.[**]

The behavior of the matter in the crust is dominated mainly by the electrons. For this reason, it can be characterized by the electron number density n_e and temperature T_e, hereafter just T in this section. Instead of n_e, let us use the electron Fermi momentum p_F (2.42) as a measure of the number density. It is convenient to describe it in the units of electron rest mass, as

$$x_r \equiv \frac{p_F}{m_e c}, \tag{2.48}$$

also known as the relativity parameter.[††] We will also need the more general relativistic form of the Fermi energy

$$\epsilon_F = c^2 \sqrt{(m_e c)^2 + p_F^2}, \tag{2.49}$$

that for a strongly degenerate gas, has the meaning of the chemical potential μ. Finally, the electron Fermi temperature is

$$T_F = \frac{m_e c^2}{k_B} \left(\sqrt{1 + \left(\frac{p_F}{m_e c} \right)^2} - 1 \right) = T_r(\gamma_r - 1), \tag{2.50}$$

where a typical temperature is

$$T_r = \frac{m_e c^2}{k_B} \sim 6 \times 10^9 \text{ K}, \tag{2.51}$$

and a relativistic scaling factor is defined as

$$\gamma_r \equiv \sqrt{1 + x_r^2}. \tag{2.52}$$

Then, the temperature can also be expressed in units of T_r as

$$t_r \equiv \frac{T}{T_r}. \tag{2.53}$$

Using these definitions, it is easy to characterize how relativistic the electron gas is. We can divide it into three regions of

- non-relativistic, for which $t_r \ll 1$ and/or $x_r \ll 1$,
- mildly-relativistic, $t_r \sim 1$ and/or $x_r \sim 1$,

[*] [7] P. A. M. Dirac. *Proceedings of the Royal Society of London Series A*. (1925).
[†] [8] R. H. Fowler. *MNRAS*. (1926).
[‡] [13] J. Frenkel. *Zeitschrift für Physik*. (1928).
[§] [10] W. Anderson. *Zeitschrift für Physik*. (1929).
[¶] [9] E. C. Stoner. *Philos. Mag.* (1930).
[‖] [11] S. Chandrasekhar. *MNRAS*. (1931).
[**] [77] S. Chandrasekhar. 1939; [78] E. L. Schatzman. 1958; [79] E. E. Salpeter. *ApJ*. (1961); [80] R. F. Tooper. *ApJ*. (1969); [81] Y. B. Zeldovich and I. D. Novikov. 1971; [75] L. D. Landau and E. M. Lifshitz. 1980; [82] S. I. Blinnikov. *Soviet Astronomy Letters*. (1987); [83] D. G. Yakovlev and D. A. Shalybkov. *Astrophysics and Space Physics Reviews*. (1989); [71] P. Haensel, A. Y. Potekhin, and D. G. Yakovlev. 2007.
[††] [79] E. E. Salpeter. *ApJ*. (1961).

- ultra-relativistic, $t_r \gg 1$ and/or $x_r \gg 1$.

Similarly, we can scale the temperature with the Fermi temperature to get a so-called degeneracy parameter

$$\Theta_F \equiv \frac{T}{T_F}. \tag{2.54}$$

It can be then used to characterize the degeneracy of the plasma to regions of

- non-degenerate, for which $\Theta_F \gg 1$,
- mildly degenerate, $\Theta_F \sim 1$,
- and strongly degenerate $\Theta_F \ll 1$.

Moving on from an ideal gaseous plasma in the atmosphere, we can start by introducing corrections produced by the closely packed charged particles. In practice we use the so-called ion-sphere model to describe our Coulomb liquid of ions. We now assume that our ions are emerged into a sea of rigid electron background that takes care of the charge neutrality. In order to couple the number density of electrons and the mass of the plasma, let us first define the mean charge per mass ratio of the plasma as

$$\mu_{Z,A} = \frac{\langle Z \rangle}{\langle A \rangle}, \tag{2.55}$$

where $\langle Z \rangle$ is the mean charge number of the atomic nuclei (for an one-component plasma it is simply Z), and $\langle A \rangle$ is the average number of nucleons bound by one nucleus. For most plasmas, $\mu_{Z,A} \approx \frac{1}{2}$.

Let us now begin by defining a so-called electron sphere radius as

$$r_e = \left(\frac{4\pi n_e}{3} \right)^{-1/3}. \tag{2.56}$$

We can also parameterize the strength between Coulomb (charge) interactions by considering a ratio of potential energy to the kinetic energy with

$$\Gamma_e = \frac{e^2}{r_e k_B T} \approx 22.75 \left(\frac{10^6\,\mathrm{K}}{T} \right) \left(\frac{\rho}{10^6\,\mathrm{g\,cm^{-3}}} \right)^{1/3} \mu_{Z,A}^{1/3}, \tag{2.57}$$

where $e = 4.80 \times 10^{-10}$ esu is the electron charge. Similarly, for ion with a charge number of Z_i, we can define the ion-sphere radius

$$r_i = r_e Z_i^{1/3}, \tag{2.58}$$

that encapsulates enough area to be charge neutral, when considering a static electron-induced background from n_e. Ion Coulomb coupling factor is similarly

$$\Gamma_i = \Gamma_e Z_i^{5/3} = \frac{(Z_i e)^2}{r_i k_B T}. \tag{2.59}$$

At high temperatures, the electrons form a classical Boltzmann gas. When the temperature is decreased, the plasma will then, without a phase-transition, become a strongly coupled Coulomb liquid corresponding to the neutron star ocean. If the temperature is decreased further, the plasma will then transform into a

Coulomb crystal with a phase transition. The gaseous regime can be constrained to be at $\Gamma_\mathrm{i} \ll 1$, or $T \gg T_\mathrm{B}$, where the T_B is given as

$$T_\mathrm{B} = \frac{Z^2 e^2}{a_i k_\mathrm{B}} \approx 2.28 \times 10^7 \left(\frac{\rho}{10^6\,\mathrm{g\,cm^{-3}}}\right)^{1/3} \mu_{Z,A}^{1/3}\,\mathrm{K}. \tag{2.60}$$

The pressure for such a system can be obtained via the standard thermodynamical relation

$$P = \left(\frac{\partial F}{\partial V}\right)_T, \tag{2.61}$$

where F is the Helmholtz free energy and V is the volume of the system.* It is useful to divide the free energy into an ideal part F_id corresponding to non-interacting particles, and to the excess part F_ex, leading to a splitting of

$$F(V, T) = F_\mathrm{id} + F_\mathrm{ex}. \tag{2.62}$$

Similarly, we could decompose the ideal effects from ions to $F_\mathrm{id}^{(i)}$, and from electrons to $F_\mathrm{id}^{(e)}$, giving

$$F(V, T) = F_\mathrm{id}^{(i)} + F_\mathrm{id}^{(e)} + F_\mathrm{ex}. \tag{2.63}$$

Most important non-ideal deviation for the plasma is from the Coulomb interaction between ions and electrons, and between electrons in the rigid background $F_\mathrm{ex} \approx F_\mathrm{ii}$. This free energy decomposition then induces similar splitting for the pressure

$$P \approx P_\mathrm{id}^{(i)} + P_\mathrm{id}^{(e)} + P_\mathrm{ii}. \tag{2.64}$$

A simple order-of-magnitude estimate for the relative strength of the different terms yields that $P_\mathrm{id}^{(e)}$ is the main contributor for the pressure. Other terms, given as $|P_\mathrm{part}|/P_\mathrm{id}^{(e)}$, result in leading order terms of†

$$\frac{P_\mathrm{id}^{(i)}}{P_\mathrm{id}^{(e)}} \approx \frac{\Theta_\mathrm{r}^{(i)}}{Z} \tag{2.65}$$

$$\frac{P_\mathrm{ii}}{P_\mathrm{id}^{(e)}} \approx \alpha_\mathrm{f} \frac{\gamma_\mathrm{r}}{x_\mathrm{r}}, \tag{2.66}$$

where $\Theta_\mathrm{i}^{(i)}$ is the temperature in units of nucleon Fermi temperature (see Eq. 2.54), $\alpha_\mathrm{f} = e^2/\hbar c \approx 1/137$ is the fine-structure constant, and $\hbar = h/2\pi$. From here it is easy to see that only the Coulomb interaction term P_ii, should give a measurable corrections to the pressure at low temperatures. This suggests that for the pressure P we can use the approximation

$$P \approx P_\mathrm{id}^{(e)} + P_\mathrm{ii}. \tag{2.67}$$

In the weak-coupling limit ($\Gamma_\mathrm{i} \ll 1$) the Debye-Hückel result for the excess free energy is a sufficiently good approximation, expressed as‡§

$$\frac{F_\mathrm{ex}}{V} = \frac{1}{\sqrt{3}} n_\mathrm{i} k T \Gamma_\mathrm{i}^{3/2}, \tag{2.68}$$

* [75] L. D. Landau and E. M. Lifshitz. 1980.

† [83] D. G. Yakovlev and D. A. Shalybkov. *Astrophysics and Space Physics Reviews*. (1989).

‡ [84] P. Debye and E. Hückel. *Physikalische Zeitschrift*. (1923); [75] L. D. Landau and E. M. Lifshitz. 1980; [85] S. L. Shapiro and S. A. Teukolsky. 1983; [86] D. R. Dewitt *et al.* *Phys. Rev. A*. (1996).

§ Debye-Hückel approximation relays on the assumption that the charge density surrounding an ion is described by electrostatics (Poisson's equation) and the distribution of the charge around the ion itself by thermal motions of electrons (Boltzmann's equation).

where $n_{\rm i}$ is the ion number density. Hence, the pressure correction due to the Coulomb interactions can be simply presented as

$$P_{\rm ii} \approx -0.3 n_{\rm i} \frac{Z^2 e^2}{r_{\rm i}}. \tag{2.69}$$

by using Eq. (2.61). Note also that $\Gamma_{\rm i} = \Gamma_{\rm i}(V)$.

Let us, as a final task, look into the main degenerate electron pressure term $P_{\rm id}^{(e)}$. Pressure is defined as a flux of momentum through a surface, hence for one particular wall it is

$$P = \int_0^\infty \langle v_x p_x \rangle n(p) dp, \tag{2.70}$$

where the number density $n(p)$ is desribed by the Fermi-Dirac distribution, given by Eq. (2.30). Velocity can be obtained from the standard expression

$$v = \frac{dE}{dp} = \frac{p}{m_{\rm e}} \left[1 + \left(\frac{p}{m_{\rm e} c} \right)^2 \right]^{-1/2}, \tag{2.71}$$

using the energy defined by Eq. (2.34). Here we have already inserted the electron mass $m = m_{\rm e} = 9.11 \times 10^{-28}$ g into the equations. Hence, in the three-dimensional description the pressure is given as one third of what we would get by applying Eq. (2.31),

$$P = \frac{1}{3} \int_0^\infty v(p)\, p\, n(p)\, dp = \frac{8\pi}{3 m_{\rm e} h^3} \int_0^\infty \frac{p}{\left[1 + \left(\frac{p}{m_{\rm e} c} \right)^2 \right]^{1/2}} \, p \, \frac{p^2}{\exp[(E-\mu)/k_{\rm B} T] + 1} \, dp \tag{2.72}$$

Free energy of the system is then equal to what is left from the chemical potential after substracting the electron rest-mass energy and the pressure contribution,

$$F = (\mu - m_{\rm e} c^2) n_{\rm e} - \frac{8\pi}{3 m_{\rm e} h^3} \int_0^\infty \frac{p^4 dp}{\left[1 + \left(\frac{p}{m_{\rm e} c} \right)^2 \right]^{1/2}} \frac{1}{\exp[(E-\mu)/k_{\rm B} T] + 1}. \tag{2.73}$$

Sommerfield expanding the free energy expression in powers of temperature, i.e., $t_{\rm r}$, we finally obtain*

$$\frac{F}{V} = \frac{m_{\rm e} c^2}{\bar{\lambda}_{\rm C}^3} \frac{1}{8\pi^2} \left(x_{\rm r} (1 + 2 x_{\rm r}^2) \gamma_{\rm r} - \ln(x_{\rm r} + \gamma_{\rm r}) + \frac{4\pi^2}{3} t_{\rm r}^2 x_{\rm r} \gamma_{\rm r} \right) + O(t_{\rm r}^4) \tag{2.74}$$

from which it is easy to obtain an expression for the pressure by applying Eq. (2.61),

$$P_{\rm id}^{(\rm e)} \approx \frac{P_{\rm r}}{8\pi^2} \left((\frac{2}{3} x_{\rm r}^2 - 1) \gamma_{\rm r} + \ln(x_{\rm r} + \gamma_{\rm r}) + \frac{4\pi^2}{9} t_{\rm r}^2 x_{\rm r} (\gamma_{\rm r} + \gamma_{\rm r}^{-1}) \right) \tag{2.75}$$

where again a typical pressure is given as

$$P_{\rm r} = \frac{m_{\rm e} c^2}{\bar{\lambda}_{\rm C}} \approx 1.4 \times 10^{25} \,{\rm dyn\, cm^{-2}}, \tag{2.76}$$

*see e.g., [83] D. G. Yakovlev and D. A. Shalybkov. *Astrophysics and Space Physics Reviews.* (1989).

where $\bar{\lambda}_C = \hbar/m_e c = 3.86 \times 10^{-11}$ cm is the reduced electron Compton wavelength.* Omitting the (small) temperature related term ($\propto t_r^2$) leads to the well-known Chandrasekhar equation of state of for a perfect, completely degenerate electron gas.†

In the non-relativistic and ultra-relativistic regimes we can apply the asymptotic limits of Eq. (2.36). Then, the pressure given in Eq. (2.75) takes a simple polytropic form of

$$P_{\mathrm{id}}^{(\mathrm{e})} \approx \frac{P_r}{9\pi^2 \gamma_{\mathrm{AD}}} x_r^{3\gamma_{\mathrm{AD}}}, \tag{2.77}$$

where the polytropic index is given as $\gamma_{\mathrm{AD}} = \frac{5}{3}$ or $\gamma_{\mathrm{AD}} = \frac{4}{3}$ for the non-relativistic ($x_r \ll 1$) and the ultra-relativistic ($x_r \gg 1$) cases, respectively. Recall also that $x_r \propto n_e^{1/3} \propto \rho^{1/3}$. The transition occures at $x_r \approx 1$, corresponding to about $\rho \sim 10^6\,\mathrm{g\,cm^{-3}}$.

Finally, as suggested by our analysis given in Eq.(2.65), the ideal degenerate electron gas pressure accompanied with the ion Coulomb correction will give us a rather good approximation for the equation of state as

$$P(x_r) \approx P_{\mathrm{id}}^{(\mathrm{e})} + P_{\mathrm{ii}} = \frac{P_r}{8\pi^2}\left((\frac{2}{3}x_r^2 - 1)\gamma_r + \ln(x_r + \gamma_r) \right) - 0.3 n_i \frac{Z^2 e^2}{r_i}, \tag{2.78}$$

remaining valid in a large density range of $10^4 < \rho < 10^{10}\,\mathrm{g\,cm^{-3}}$. Below $\rho < 10^4\,\mathrm{g\,cm^{-3}}$ the plasma is gaseous and the ideal gas law description (2.16) is better suited in modeling the equation of state. Beyond $\rho > 10^{10}\,\mathrm{g\,cm^{-3}}$ the densities become so high that nuclear degeneracy pressure and more importantly, the mutual nuclear intearactions start to play an important role.

2.4 Core

After the relatively well-known crust, a physical no-man's land begins when we enter the core. At about half of the nuclear density, $\rho_n \approx 2.8 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, the nuclei become so neutron-rich that more neutrons will leak out then what is left bounded. Hence, normal atoms can no longer exists, and the matter becomes an uniform plasma of neutrons, protons, and electrons. In the end, we will have a liquid sea of free neutrons with a several per cent admixture of protons and electrons. The neutrons and protons, interacting via the nuclear forces, constitute a strongly interacting Fermi liquid, whereas electrons form an almost ideal Fermi gas. Instead of purely degenerate non-interacting matter, like the electrons in the crust, the nucleons become close enough to interact with each other. When in close contact, the repulsive short-range neutron-neutron interaction introduces a considerable stiffening of the equation of state. Hence, a many-body theory is needed in order to describe the matter in the core. If we take the effective range of strong nuclear forces to be of around $r_{\mathrm{sn}} \sim 10^{-13}$ cm, we can easily estimate the order-of-magnitude of the density when they become important by equating the mean distance of nucleons with r_{sn}. In this case we obtain $\rho_{\mathrm{sn}} \sim m_n/V = 3 \times 1.66 \times 10^{-24}\,\mathrm{g}/4\pi r_{\mathrm{sn}}^3 \sim 4 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, a density range surprisingly close to more sophisticated calculations giving $\rho \sim (1.5 - 2) \times 10^{14}\,\mathrm{g\,cm^{-3}}$.‡

A density range between $0.5\,\rho_n < \rho < 2\,\rho_n$ $(1.4 - 5.6) \times 10^{14}\,\mathrm{g\,cm^{-3}})$ is named the *outer core*. For neutron stars it is already several kilometers thick and constitutes a substantial part of the total mass of the star. The matter consists of so-called $npe\mu$ composition, referring to neutrons n, protons p, electrons e, and in some models to muons μ. The basic physics of this matter is determined by charge neutrality, β-equilibrium, and many-body nuclear interactions. Beyond $\rho > 2\,\rho_n \approx 5.6 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, we have the *inner core*. Its composition is even more unknown and the results here become heavily model-dependent. This is

*Normal Compton wavelength is simply $\lambda_e = 2\pi \times \bar{\lambda}_C = h/m_e c = 2.43 \times 10^{-10}$ cm.

† [9] E. C. Stoner. *Philos. Mag.* (1930); [87] S. Chandrasekhar. *MNRAS.* (1935); [77] S. Chandrasekhar. 1939.

‡such estimates were first presented by Hund [88] F. Hund. *Zeitschrift fur Physik.* (1936).

mainly because, in addition to the many-body forces, the exact particle composition is unclear. In addition to the $npe\mu$ matter, we might have hyperonization (Σ^-, Λ, or more generally sometimes labeled as H), pion condensation (π), kaon condensation (K), or even a phase-transition to a (pure) quark matter (q). Inner core of a neutron star can be many kilometers thick and the central densities of the most massive stars can go up to $\rho \approx (10-15)\times \approx (2.8-4.2)\times 10^{15}\,\mathrm{g\,cm^{-3}}$.

2.4.1 Polytropes

Instead of trying to obtain a very uncertain nuclear physics description of the equation of state, let us take another route. As the strongly degenerate equation of state relevant for neutron stars is mainly just the relation between pressure and the density, we can try and phenomenologically parameterize this relation in contrast to describing the accurate but complicated microphysics. One such a parameterization is a polytropic presentation of the $P(\rho)$ function defined as

$$P(\rho) = K\rho^{\gamma}, \tag{2.79}$$

where γ is a polytropic index and is a measure of the *stiffness* of the matter, and K is just a normalization factor. Instead of settling for only one polytrope (actually called a monotrope then), we can link them together to form a piecewise polytropic description of the equation of state. Such a phenomenological description was pioneer by Read et al.* In practice, we can then generalize Eq. (2.79) such that

$$P(\rho) = K_i\,\rho^{\gamma_i}, \quad \text{for } \rho_{i-1} \le \rho \le \rho_i, \tag{2.80}$$

for transition densities $\rho_0 < \rho_1 < \rho_2 < \ldots < \rho_{i-1} < \rho_i < \rho_{i+1}$, and so on.

Note that ρ here is the rest-mass density. The full energy density can be obtained from the first law of thermodynamics as

$$d\frac{\epsilon}{\rho} = -\frac{P}{c^2}d\frac{1}{\rho}, \tag{2.81}$$

that has an immediate integral of

$$\frac{\epsilon}{\rho} = (1+a) + \frac{1}{c^2}\frac{1}{\gamma-1}K\rho^{\gamma-1}, \tag{2.82}$$

where a is a continuity constant. Now

$$\lim_{\rho\to 0}\frac{\epsilon}{\rho} = \frac{1}{c^2}, \tag{2.83}$$

implying $a = 0$ for $\rho = 0$. For the piecewise description we then have (for $\gamma_i \neq 0$)

$$\epsilon(\rho) = (1+a_i)\rho + \frac{1}{c^2}\frac{K_i}{\gamma-1}\rho_i^{\gamma} \tag{2.84}$$

where

$$a_i = \frac{\epsilon(\rho_{i-1})}{\rho_{i-1}} - 1 - \frac{1}{c^2}\frac{K_i}{\gamma_i - 1}\rho_{i-1}^{\gamma_i-1}, \tag{2.85}$$

defines the energy density continuous.

* [89] J. S. Read *et al*. *Phys. Rev. D.* (2009).

2.4.2 Library for the equation of state of the core

As noticed by Read et al.*, in most cases, just three piecewise polytropes are enough to give a faithful description of the accurately calculated equations of states, up to a rms error of about 2% in pressure. Here we consider few (relatively) modern descriptions for the dense matter equation of state and parameterize them with polytropes. We follow the naming convention presented in the previous section and divide the compositions into

- normal nuclear matter ($npe\mu$),
- normal nuclear matter spiced up with hyperons ($npe\mu + H$),
- normal nuclear matter together with more exotic particles like pion and kaon condensates ($npe\mu + \pi + K$), and
- matter consisting of (or normal matter spiced up with) quarks (q or $npe\mu + q$).

The different equations of states we describe here are mainly presented and used in Refs.† and fitted with polytropes by Ref.‡ Additionally, we require that the maximum mass of the equation of state fulfills the $2\,M_\odot$ mass limit as set by astrophysical measurements of double pulsars.§ For $npe\mu$ composition we include models computed with

- potential method using SLy effective nuclear interaction that is of Skyrme-type¶,
- four variational method EoSs, APR3/4‖ and WFF1/2**,
- two relativistic Brueckner-Hartree-Fock calculations, ENG†† and MPA1‡‡, and
- two relativistic mean-field theory models, MS1 and MS1b (same as MS1 but with lower symmetry energy)§§.

For hyperon models ($npe\mu + H$) we include

- one variant of relativistic mean-field theory model, H4¶¶.

Here the introduction of hyperons into the matter considerably softens the equation of state and the stars constructed with these models do not end up reaching the $2\,M_\odot$ limit. Similarly, models where mesons, like pion and kaon condensates ($npe\mu + \pi$ and/or $npe\mu + K$) are taken into account, end up not being stiff enough. Finally, for the hybrid nuclear matter and quark matter compositions ($npe\mu + q$) we include

- mixed APR nuclear matter and color-flavor-locked quark matter EoS ALF2.***

Again, for pure quark matter composition, the equation of state ends up being too soft. Even in the case of the ALF2, the transition density from the nuclear matter to quark matter is defined at a relatively late phase corresponding to $\rho = 3\,\rho_{\rm n}$.

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* [89] J. S. Read *et al*. *Phys. Rev. D*. (2009).

† [90] J. M. Lattimer and M. Prakash. *ApJ*. (2001); [91] M. Bejger, T. Bulik, and P. Haensel. *MNRAS*. (2005); [92] B. D. Lackey, M. Nayyar, and B. J. Owen. *Phys. Rev. D*. (2006).

‡ [89] J. S. Read *et al*. *Phys. Rev. D*. (2009).

§ [93] P. B. Demorest *et al*. *Nature*. (2010); [94] J. Antoniadis *et al*. *Science*. (2013); [95] E. Fonseca *et al*. *ApJ*. (2016).

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†† [99] L. Engvik *et al*. *ApJ*. (1996).

‡‡ [100] H. Müther, M. Prakash, and T. L. Ainsworth. *Physics Letters B*. (1987).

§§ [101] H. Müller and B. D. Serot. *Nuclear Physics A*. (1996).

¶¶ [92] B. D. Lackey, M. Nayyar, and B. J. Owen. *Phys. Rev. D*. (2006).

*** [102] M. Alford *et al*. *ApJ*. (2005).

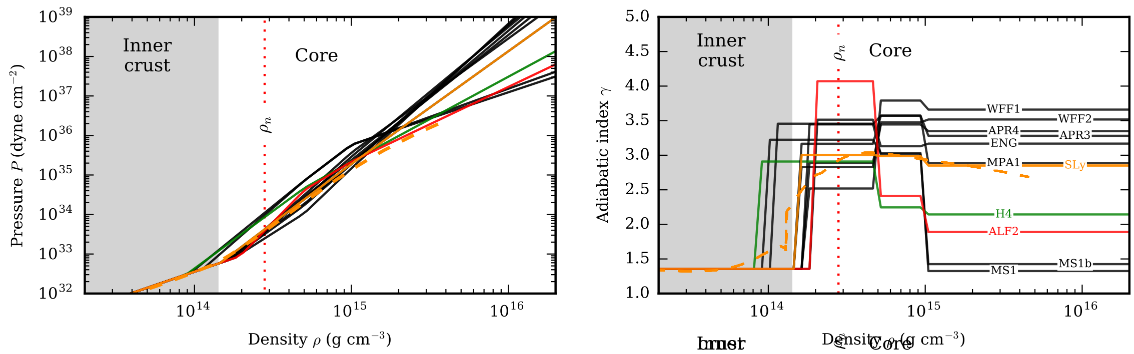

Figure 2.3: Equation of states for the core. *Left side:* Pressure versus density relation, $P(\rho)$. *Right side:* Corresponding adiabatic or the polytropic index evolution versus the density, $\gamma(\rho)$. Plain nuclear matter models are colored in black, hyperonic ($npe\mu + H$) with green, and quark (q) matter in red. Additionally, the exact SLy model (shown exceptionally in dark orange) is shown with dashed lines. Grey region highlights the crust densities and the red dotted vertical line shows the nuclear density of $\rho_{\mathrm{n}} = 2.8 \times 10^{14}\,\mathrm{g\,cm^{-3}}$.

These 11 equations of states end up giving quite good overview of different models present in the literature. We, however, stress that by no means is it a complete list. The corresponding $P(\rho)$ evolution of these models is shown in Fig. 2.3. Unlike the previously well-defined crust, here the theoretical uncertainties become obvious as there is quite a large scatter in between different models.

2.5 Tolman-Volkoff-Oppenheimer equations

Let us, as a final task, try and construct an actual model of a star from the equation of state. For this, we require the star to be in hydrostatic equilibrium so that the pressure gradient from opposes the gravity. In classical Newtonian form this can be expressed simply as

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}, \tag{2.86}$$

where r is the radial coordinate. In addition, we will need a connection for the mass m and radius r, that in spherical symmetry is

$$\frac{dm}{dr} = 4\pi r^2 \rho. \tag{2.87}$$

Taking into account the general relativistic corrections we get

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \times \frac{(1 + P/\rho c^2)(1 + 4\pi r^3 P/mc^2)}{1 - 2Gm/rc^2}. \tag{2.88}$$

This is the relativistic hydrostatic equilibrium equation as first derived by Tolman*, Oppenheimer and Volkoff†.‡ Difference in respect to the classical formulation originates from the source of gravity: in the Newtonian case it is the mass m, whereas in the general relativity it is the energy momentum tensor that

* [15] R. C. Tolman. *Physical Review*. (1939).

† [16] J. R. Oppenheimer and G. M. Volkoff. *Physical Review*. (1939).

‡ for an introduction, see, e.g., [103] R. M. Wald. 1984; [104] C. W. Misner, K. S. Thorne, and J. A. Wheeler. 1973.

depend both on the energy density and the pressure. As a result, both, energy and pressure, give rise to a gravitational field.

This has an important consequence for the stability of neutron stars: Successive increase in the pressure to counter the gravity is ultimately self-defeating. We can get an idea of this, by using an unphysical but analytically easier special case of constant density, i.e. incompressible equation of state, as our model. By solving the Eq. (2.86) for constant density $\rho(r) = \rho_0$, we obtain

$$P(r) = G\frac{2\pi}{3}\rho_0^2(R^2 - r^2) \tag{2.89}$$

whereas the general relativistic form described by Eq. (2.87) gives, after some tedious algebra,

$$P(r) = \rho_0^2 c^2 \frac{\sqrt{1 - u\left(\frac{r}{R}\right)^2} - \sqrt{1-u}}{3\sqrt{1-u} - \sqrt{1 - u\left(\frac{r}{R}\right)^2}}, \tag{2.90}$$

where again the compactness parameter $u = 2GM/Rc^2$, is used. Unlike the classical counter-part, the central pressure $P(0)$ here attains a maximum for $u_{\max} = \frac{8}{9}$ equal to

$$M_{\max} = \frac{4}{9}Rc^2/G \approx 3.01\left(\frac{R}{10\,\mathrm{km}}\right) M_\odot. \tag{2.91}$$

Even though the assumption of constant density is unphysical, the solution shows that the relativistic equations of hydrostatic balance have a maximum mass after which the star becomes unstable. This is an important feature of the equations as it can be used to, for example, rule out certain equations of states, given that we have measured some (large) mass to exists in Nature. It was also the source of confusion in the 1930s because the degenerate neutron gas equation of state gives $M_{\max} \approx 0.7\,M_\odot$, while the Chandrasekhar limit, i.e., the maximum mass for white dwarfs with degenerate electrons, give $M_{\mathrm{Ch}} \approx 1.44\,M_\odot$. From these arguments, it would seem that neutron stars can not exists because the maximum mass is less than that of the white dwarfs. Now we, of course, already know that it is the nuclear many-body interactions that have a huge impact on the equations of states and subsequently alter the maximum mass to be of around $M_{\max} \approx (1.5 - 3)\,M_\odot$. Similarly, the measurements of $M \approx 2.0\,M_\odot$ neutron stars, that we already used to rule out some equations of states in the previous section, put stringent constraints on the possible behavior of the matter.

Instead of looking at the different quantities, such as the pressure, as a function of the density, let us now finally solve the structure of the star given the derived equations of states. The results are obtained by numerically solving the Tolman-Oppenheimer-Volkoff equations using the previously presented crust model and the different equations of states for the core.* In Fig. 2.4 the dependency of the pressure and cumulative mass of the star against the radial coordinate (as measured starting from the core) is visualized for the SLy equation of state for a central density of $P_c = 8.9 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, corresponding to $M = 1.4\,M_\odot$. This visualizes the real dimensions of the neutron stars internal structure: Inner core spans about 6 km in height from the core while the outer core extends from 6 to 11 km. The full crust is only about ~ 1 km thick, and the atmosphere is too thin to be even visible. The core also consists of more than 99% of the total mass of the star.

Instead of fixed central pressure, we can let it span a large range, starting from some small value and ending to the pressure corresponding to the maximum mass, to obtain mass−radius, $M - R$ curves for the

*TOV-solver that is used is available from https:// github.com/natj/tov.

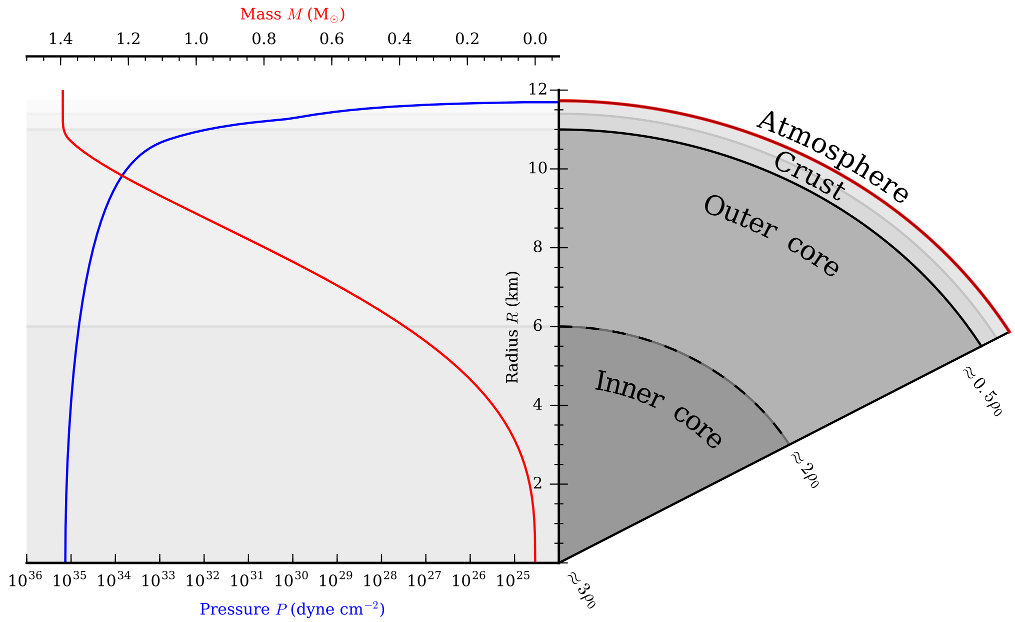

Figure 2.4: Overview of the neutron star structure for the SLy equation of state and core density of $P_c = 8.9 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, corresponding to $M = 1.4\,M_\odot$. Right side of the figure shows a schematic presentation of the star's interiors against the radial coordinate, whereas the left side shows the pressure (blue; bottom axis) and cumulative mass (red; top axis) evolution from the core to the surface.

equations of states. These are visualized in Fig. 2.5 along with the pressure−density relations that yield them. From here it is obvious how the large uncertainty in the nuclear physics of the core then translates into a large possible allowed radius range, from about 10 to 15 km. It also opens up a pathway of probing the nuclear physics with astrophysics because mass and radius measurements of real neutron stars can set constraints to the equation of state.

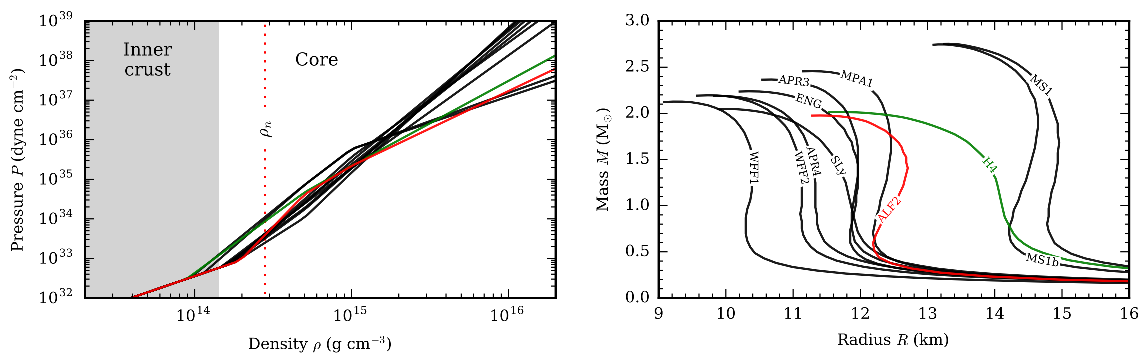

Figure 2.5: Equation of states and the resulting neutron star mass-radius curves for the different core models. *Left side:* Pressure versus density relation, $P(\rho)$. *Right side:* Corresponding $M - R$ relations. Symbols and colors are the same as in Fig. 2.3.

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