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Wade 1.3.8

Suppose that $E, A, B \subset \mathbb{R}$ and $E = A \cup B$. Prove that if E has a supremum and both A and B are nonempty, then $\sup A$ and $\sup B$ both exist, and $\sup E$ is one of the numbers $\sup A$ or $\sup B$.

Proof.
Let $a \in A$, then $a \in A \cup B$.
Now, let $b \in B$, then $b \in A \cup B$.
if $\sup E$ exists, $\sup E \leq M$ $\forall$ upper bounds $M \in E$.
so $\forall a, b \in A, B$ respectively, $a, b \leq M$.
Since $A, B \neq \varnothing$ and bounded above, $\exists \sup A, \sup B$. [Completeness Axiom]
Let $S_1, S_2 = \sup A, \sup B$ respectively.
$S_1 \geq S_2 \implies \forall x \in E, x \leq S_1 \implies \sup E = \sup A$.
$S_2 \geq S_1 \implies \forall x \in E, x \leq S_2 \implies \sup E = \sup B$.
$\therefore \sup E$ and $A, B \neq \varnothing \implies \exists \sup A, \sup B$ and $\sup E = \sup A$ or $\sup B$. $\square$