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Acknowledgements:

Wade Exercise 3.1.1 Using Definition 3.1, prove that each of the following limits exists.

c. $\lim_{x\to 1} x^3 + 2x + 1 = 4.$

Proof.
Let $f(x) = x^3 + 2x + 1$ and Let $\epsilon > 0$.
Set $\delta = \min\left\{1, \frac{\epsilon}{9}\right\}$.
Assume $0 < |x - 1| < \delta$.
Then,

$$\begin{aligned} |x-1| < 1 &\implies -1 < x - 1 < 1 && \text{[FTAV]} \\ &\implies 0 < x < 2 \\ &\implies x \in (0, 2) \\ &\text{Also note that } |x-1| < \frac{\epsilon}{9} \end{aligned}$$

Then,

$$\begin{aligned} |f - 4| &= |x^3 + 2x + 1 - 4| && \text{[Substitution/Algebra]} \\ &= |x^3 + 2x - 3| \\ &= |x-1||x^2 + x + 3| \\ &< |x-1|(|x|^2 + |x| + 3) \\ &= \frac{\epsilon}{9}((2)^2 + (2) + 3) \\ &= \frac{\epsilon}{9}9 \\ &< \epsilon. \end{aligned}$$

$\therefore f \to 4$ as $x \to 1$. $\square$

d. $\lim_{x\to 0} x^3 \sin(e^{x^2}) = 0$

Proof.
Let $f(x) = x^3 \sin(e^{x^2})$ and Let $\epsilon > 0$.
Set $\delta = \min\left\{1, \frac{\epsilon}{\sin(e)}\right\}$.
Assume $0 < |x - 0| < \delta$.
Notice $|x| < 1$.
Then,

$$\begin{aligned} |x| < 1 &\implies -1 < x < 1 && \text{[FTAV]} \\ &\implies x \in (-1, 1) \\ &\text{Also note that } |x-1| < \frac{\epsilon}{\sin(e)} \end{aligned}$$

Then,

$$\begin{aligned}
|f-0| &= |x^3 \sin(e^{x^2}) - 0| && \text{[Substitution/Algebra]} \\
&= |x^3 \sin(e^{x^2})| \\
&= |x||x^2 \sin(e^{x^2})| \\
&< |x|(|x|^2 \sin(e^{|x|^2})) \\
&= \frac{\epsilon}{\sin(e)}((1)^2 \sin(e^{(1)^2})) \\
&= \frac{\epsilon}{\sin(e)} \sin(e) \\
&< \epsilon.
\end{aligned}$$

$\therefore f \to 0$ as $x \to 0$. $\square$