# 10/14

*Nathan Keough*

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**Theorem 2.12 part iii**
$\lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}(x_n)\lim_{n\to\infty}(y_n)$

*Proof.*
Suppose $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
let $\epsilon > 0$. Recall that $\lim_{n\to\infty}(x_n) \implies |x_n - x| < \epsilon$ and
$\lim_{n\to\infty}(y_n) \implies |y_n - y| < \epsilon$.
Choose $N \in \mathbb{N} \ni n \leq N \implies |x_n - x| < \sqrt{\epsilon} < \epsilon$ and $|y_n - y| < \sqrt{\epsilon} < \epsilon$.
Thus $n \geq N \implies |(x_n y_n) - (xy)| \leq |x_n - x||y_n - y| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon$.
$\therefore \lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}(x_n)\lim_{n\to\infty}(y_n)$

$\square$

**Theorem 2.12 part iv**
Where $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$.
$\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{\lim_{n\to\infty}(x_n)}{\lim_{n\to\infty}(y_n)}$

*Proof.*
Let $\epsilon > 0$.
Since $y_n$ is a converging sequence, $|y_n|$ must be bounded.
Say $c \leq |y_n| \leq d$. Then $\frac{1}{|y_n|} \leq \frac{1}{c}$ where $c > 0$.
Since, $y_n \to y, \exists N \in \mathbb{N} \ni n \geq N \implies |y_n - y_0| < c|y_0|\epsilon$.
So, $n \geq N \implies$
$\left|\frac{1}{y_n} - \frac{1}{y_0}\right| = \left|\frac{y_0 - y_n}{y_n y_0}\right| = \left|(y_0 - y_n)\frac{1}{y_n}\frac{1}{y_0}\right| = |y_0 - y_n|\frac{1}{|y_n|}\frac{1}{|y_0|} < c|y_0|\epsilon\frac{1}{c}\frac{1}{|y_0|} = \epsilon$.
$\therefore \frac{1}{y_n} \to \frac{1}{y_0}$.
Notice $\frac{x_n}{y_n} = x_n\frac{1}{y_n} \implies x_0\frac{1}{y_0} = \frac{x_0}{y_0}$. [By Product Law]
$\therefore \frac{x_n}{y_n} \to \frac{x_0}{y_0}$ Thus it is shown that $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{\lim_{n\to\infty}(x_n)}{\lim_{n\to\infty}(y_n)}$ $\square$

**Wade 2.2.3ab using Theorem 2.12**
Find the limit (if it exists) of each of the following sequences.
A. $x_n = (2 + 3n - 4n^2)/(1 - 2n + 3n^2)$

*Proof.*
Let $x_n = (2 + 3n - 4n^2)/(1 - 2n + 3n^2)$ Then by 2.12, we can use limit laws to find the limit (if one exists).
$\lim_{n\to\infty} x_n$

$$= \lim_{n\to\infty}(2+3n-4n^2)/\lim_{n\to\infty}(1-2n+3n^2) \qquad [2.12.iv]$$
$$= \lim_{n\to\infty}(2)+\lim_{n\to\infty}(3n)+\lim_{n\to\infty}(-4n^2)/$$
$$\lim_{n\to\infty}(1)+\lim_{n\to\infty}(-2n)+\lim_{n\to\infty}(3n^2) \qquad [2.12.i]$$
$$= \lim_{n\to\infty}(2)+(3)\lim_{n\to\infty}(n)+(-4)\lim_{n\to\infty}(n^2)/$$
$$\lim_{n\to\infty}(1)+(-2)\lim_{n\to\infty}(n)+(3)\lim_{n\to\infty}(n^2) \qquad [2.12.ii]$$
$$= \lim_{n\to\infty}(2)+(3)\lim_{n\to\infty}(n)+(-4)\lim_{n\to\infty}(n)\lim_{n\to\infty}(n)/$$
$$\lim_{n\to\infty}(1)+(-2)\lim_{n\to\infty}(n)+(3)\lim_{n\to\infty}(n)\lim_{n\to\infty}(n) \qquad [2.12.iii]$$
$$= -\frac{4}{3} \qquad [\text{Product Law + Algebra}]$$

$\therefore \lim_{n\to\infty} x_n = -\frac{4}{3}$ □

B. $x_n = (n^3+n-2)/(2n^3+n-2)$

*Proof.*
Let $x_n = (n^3+n-2)/(2n^3+n-2)$ Then by 2.12, we can use limit laws to find the limit (if one exists).
$\lim_{n\to\infty} x_n$

$$= \lim_{n\to\infty}(n^3+n-2)/\lim_{n\to\infty}(2n^3+n-2) \qquad [2.12.iv]$$
$$= \lim_{n\to\infty}(n^3)+\lim_{n\to\infty}(n)+\lim_{n\to\infty}(-2)/$$
$$\lim_{n\to\infty}(2n^3)+\lim_{n\to\infty}(n)+\lim_{n\to\infty}(-2) \qquad [2.12.i]$$
$$= \lim_{n\to\infty}(n^3)+\lim_{n\to\infty}(n)+\lim_{n\to\infty}(-2)/$$
$$(2)\lim_{n\to\infty}(n^3)+\lim_{n\to\infty}(n)+\lim_{n\to\infty}(-2) \qquad [2.12.ii]$$
$$= \lim_{n\to\infty}(n)\lim_{n\to\infty}(n)\lim_{n\to\infty}(n)+$$
$$\lim_{n\to\infty}(n)+\lim_{n\to\infty}(-2)/$$
$$(2)\lim_{n\to\infty}(n)\lim_{n\to\infty}(n)\lim_{n\to\infty}(n)+$$
$$\lim_{n\to\infty}(n)+\lim_{n\to\infty}(-2) \qquad [2.12.iii]$$
$$= \frac{1}{2} \qquad [\text{Product Law + Algebra}]$$

$\therefore \lim_{n\to\infty} x_n = \frac{1}{2}$ □