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Acknowledgements:

3.3.2 **For each of the following, prove that there is at least one $x \in \mathbb{R}$ which satisfies the given equation.**
a. $e^x = x^3$

Proof.
Let $f(x) = e^x$ and $g(x) = x^3$. Notice that f and g are both continuous on $\mathbb{R}$. Now let $h(x) = f(x) - g(x)$.
h must be continuous on $\mathbb{R}$ since f, g are both continuous on $\mathbb{R}$.
Now, consider $h(x) > 0 \implies f(x)$ lies on top of $g(x)$ and $f(x) < 0$ $\implies g(x)$ lies on top of $f(x)$, when graphed.
If $h(x) = 0$, then $e^x = x^3$ is satisfied.
It suffices to find an $x \in \mathbb{R} \ni h(x) > 0$ and an $x \ni h(x) < 0$.
So let $x = 1$, $h(1) \approx 1.72$ and let $x = 2$, $h(2) \approx -0.61$.
Since, h is continuous on $\mathbb{R}$, h is closed on $[1, 2]$, and $h(1) \leq 0 \leq h(2)$, $\exists$ at at least one $x \in \mathbb{R} \ni f(x) = 0$ by the Intermediate Value Therem.

$\square$

3.3.4 **If $f : [a, b] \to [a, b]$ is continuous, then f has a fixed point; that is, there is a $c \in [a, b]$ such that $f(c) = c$.**

Proof.
Let $f : [a, b] \to [a, b]$.
If $f(a) = a$ or $f(b) = b$, then we have shown $f(c) = c$, otherwise, we may assume that $a < f(a) < f(b) < b$.
Consider $g(c) = f(c) - c$. Then $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. [AP]
Since $g(a) > 0$ and $g(b) < 0, \exists c \in [a, b] \ni 0 = g(c) = f(c) - c$.
$\therefore$ if $f : [a, b] \to [a, b]$ is continuous, then f has a fixed point; that is, there is a $c \in [a, b]$ such that $f(c) = c$. $\square$