9/21

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Wade 1.3.4

Prove that a lower bound of a set need not be unique but the infimum of a given set E is unique.

Proof.
We must show that:

1 Lower bound need not be unique.

2 inf are unique.

1 Consider a set $E \subseteq \mathbb{R} \ni E \neq \varnothing$ and E is bounded below,
so, $\exists m \in \mathbb{R} \ni x \geq m \forall x \in E$.
Now, consider $n \leq m$, thus $\forall x \in E, x \geq n$.
It has been shown that two lower bounds of a set E may exist,
$\therefore$ a lower bound for a set need not be unique.

2 Assume there exists two inf of a set $E \subseteq \mathbb{R}$, i_1, i_2 respectively whereby both are lower bounds of E. Then, $i_1 \leq x$ and $i_2 \leq x \forall x \in E$.
By definition of infimum, $i_1 \leq i_2 \leq x$ and $i_2 \leq i_1 \leq x$ must be true.
The only case where this holds is when $i_1 = i_2$.
$\therefore$ the infimum of a given set E is unique.

□

Wade 1.3.7

a. Prove that if x is an upper bound of a set $E \subseteq \mathbb{R}$ and $x \in E$, then x is the supremum of E.

   Proof.
   Consider a nonempty set $E \subseteq \mathbb{R}$ that is bounded above and let $x \in E$.
   Let $x \in m \forall$ upper bounds m of E.
   $x \in m \implies \forall y \in E, y \leq x$.
   Since $x \in E$, $x \leq m$ By definition of supremum, $y \leq x \leq m$
   $\implies \sup E = x$.
   $\therefore$ if x is an upper bound of a set $E \subseteq \mathbb{R}$ and $x \in E$, then x is the supremum of E. □

b. Make and prove an analogous statement for the infimum of E.

Proof.
Consider a nonempty set $E \subseteq \mathbb{R}$ that is bounded above and let $x \in E$.
Let $x \in n \forall$ upper bounds n of E.
$x \in n \implies \forall y \in E, y \geq x$.
Since $x \in E$, $x \geq n$ By definition of infimum, $x \geq y \geq n \implies \inf E = x$. $\square$

c. Show by example that the converse of each of these statements is false.

Take for example, the set $E = (0, 1)$. Neither 0 nor 1 are in the set, yet the infinum and supremum of E is $\inf E = 0, \sup E = 1$
$\therefore$ since $\inf E = 0, \sup E = 1$ and $0, 1 \notin E$, the converse of these statement must be false.