ISLR Notes and Exercises

Nathaniel

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Chapter 3: Linear Regression

Prediction and Inference

From Cross Valided, Inference: Given a set of data you want to infer how the output is generated as a function of the data.

Prediction: Given a new measurement, you want to use an existing data set to build a model that reliably chooses the correct identifier from a set of outcomes."

I must say the definitions of inference, in statistics and econmetrics, confuse me. ETM (2004): "If we are to interpret any given set of OLS parameter estimates, we need to know, at least approximately, how $\hat{\beta}$ is actually distributed. For purposes of **inference**, the most important feature of the distribution of any vector of parameter estimates is the matrix of its central second moments.

It seems that inference accoding to ETM (2004), concerns about the distribution (1st and 2nd monments) of the estimator while, in ISLR (2013), inference, is about the change in response variable due to the change of an estimated parameters of a specified model.

Potenital Problems of Regression:

- Non-linearity of the response-predictor relationships.
- Correlation of error terms.
- Non-constant variance of error terms (heteroscedasticity).
- Outliers.
- High-leverage points.
- Collinearity.

Frisch-Waugh-Lovell Theorem

1. The OLS estimates of $\hat{\beta}$ from regressions

$$y = X_1\beta_1 + X_2\beta_2 + u$$

   and

$$M_1 y = M_1 X_2 \beta_2 + residuals$$

   are numerically identical, where $P_X = X(X^TX)^{-1}X^T$ and $M_x = I - P_X = I - X(X^TX)^{-1}X^T$
2. The residuals from the one-step and teo-step regressions are numerically identical (ETM P.69).

Think of the two-step model as regressing the residual of y on X_1 onto the residual of $X1$ on X_2. The Theorm would shed important insight on the problem of collinearity.

For concepts such as hypothesis testing (t-test and F-test), SSE, R^2, variance inflation factor (VIE), Mallow's C_p leverage point, influential point, AIC, BIC, forward selection, backward selection, mixed selection, see the ISLR book or the solution below for detail.

See ETM for the hat matrix and why "We say that observations for which h_t is large have high leverage or are leverage points. A leverage point is not necessarily influential, but it has the potential to be influential."

See also the `broom` package for implementation of models and graphics.

KNN regression vs KNN classification

Question 2

KNN regression averages the closest observations to estimate prediction, KNN classifier assigns classification group based on majority of closest observations.

KNN regression: given a value for K and a prediction point x_0, KNN regression first identifies the K training observations that are closest to x_0, represented by N_0. It then estimates $f(x_0)$ using the average of all the training responses in N_0. Then

$$\hat{f}(x_0) = \frac{1}{K} \sum_{x_i \in N_o} y_i$$

As usual, the greater the K, the more "smooth" the hyperplane, the lower the variance ($Var(\hat{f}(x_0))$, the more likely of the result of $Bias(\hat{f}(x_0))$. K can be determined by Cross-Validation.

KNN classifier: given a positive K-nearest integer K and a test observation x_0, the KNN classifier first identifies the neighbors K points in the training data that are closest to x_0, represented by N_0. It then estimates the conditional probability for class j as the fraction of points in N_0 whose response values equal j:

$$Pr(Y = j | X = x_0) = \frac{1}{K} \sum_{i \in N_0} I(y_i = j)$$

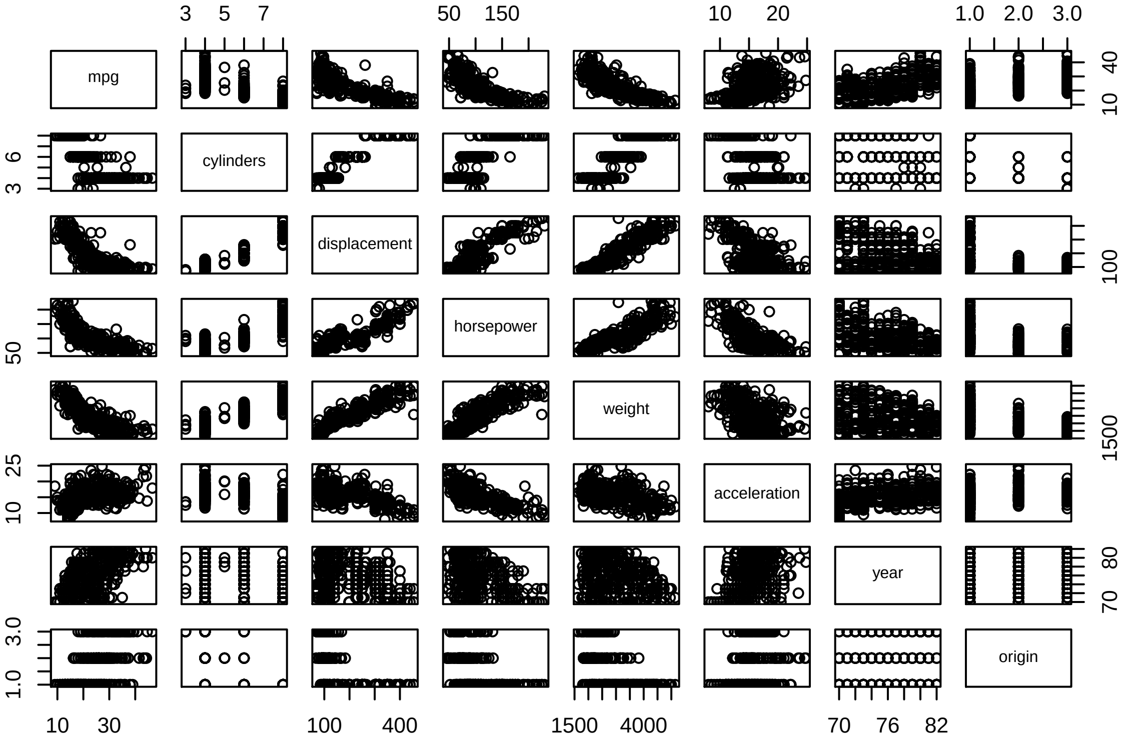
Finally, KNN applies Bayes rule and classifies the test observation x_0 to the class with the largest probability.

Applied Questions

Question 9

This question aims to teach standard regrassion prcedures.

```
Auto = read.csv("/Users/nathaniellai/Desktop/R_Notes/R_ISLR/ISLR_data/Auto.csv", header=T, na.strings="
Auto = na.omit(Auto)
pairs(Auto[,-9])
```



```
cor(subset(Auto, select=-name))
```

```
##                     mpg  cylinders displacement horsepower     weight
## mpg           1.0000000 -0.7776175   -0.8051269 -0.7784268 -0.8322442
## cylinders    -0.7776175  1.0000000    0.9508233  0.8429834  0.8975273
## displacement -0.8051269  0.9508233    1.0000000  0.8972570  0.9329944
## horsepower   -0.7784268  0.8429834    0.8972570  1.0000000  0.8645377
## weight       -0.8322442  0.8975273    0.9329944  0.8645377  1.0000000
## acceleration  0.4233285 -0.5046834   -0.5438005 -0.6891955 -0.4168392
## year          0.5805410 -0.3456474   -0.3698552 -0.4163615 -0.3091199
## origin        0.5652088 -0.5689316   -0.6145351 -0.4551715 -0.5850054
##              acceleration       year     origin
## mpg             0.4233285  0.5805410  0.5652088
## cylinders      -0.5046834 -0.3456474 -0.5689316
## displacement   -0.5438005 -0.3698552 -0.6145351
## horsepower     -0.6891955 -0.4163615 -0.4551715
## weight         -0.4168392 -0.3091199 -0.5850054
## acceleration    1.0000000  0.2903161  0.2127458
## year            0.2903161  1.0000000  0.1815277
## origin          0.2127458  0.1815277  1.0000000
```

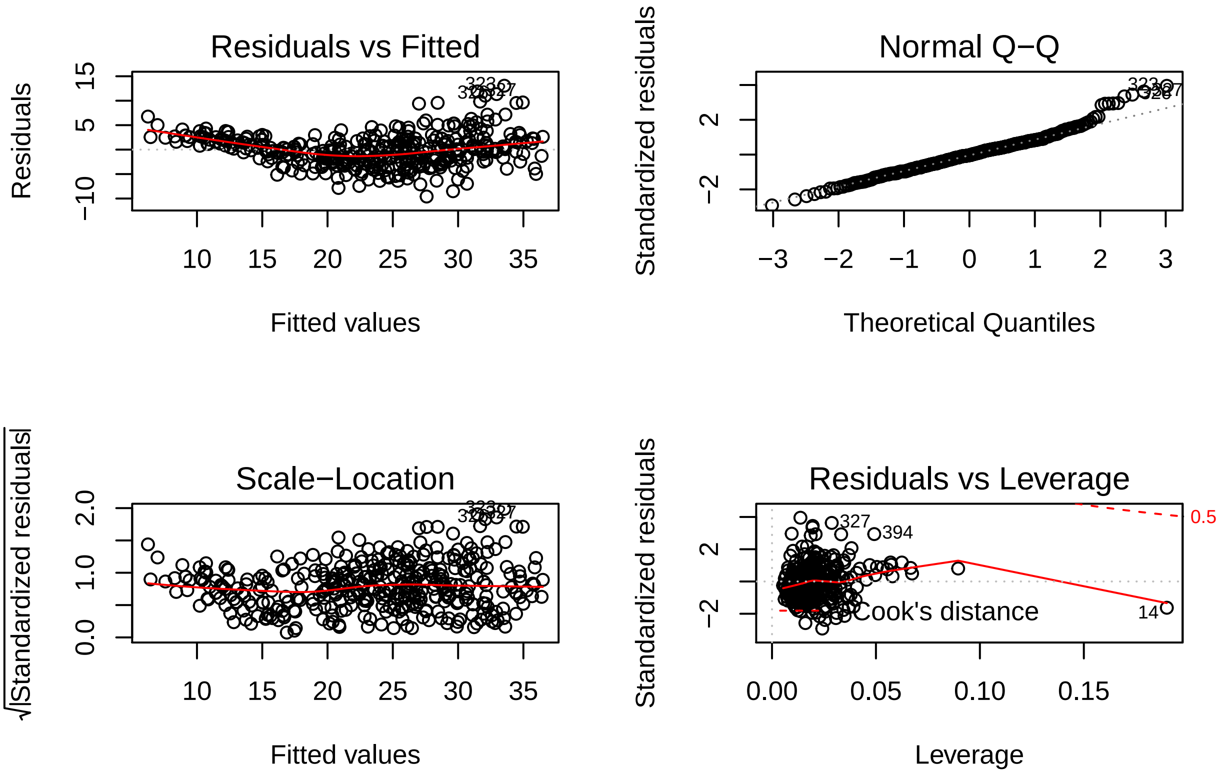
```
lm.fit0 <- lm(mpg ~ . -name, data=Auto)
summary(lm.fit0)
```

```
##
## Call:
## lm(formula = mpg ~ . - name, data = Auto)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -9.5903 -2.1565 -0.1169  1.8690 13.0604
```

```
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)    
## (Intercept)  -17.218435   4.644294  -3.707  0.00024 ***
## cylinders     -0.493376   0.323282  -1.526  0.12780    
## displacement   0.019896   0.007515   2.647  0.00844 ** 
## horsepower    -0.016951   0.013787  -1.230  0.21963    
## weight        -0.006474   0.000652  -9.929  < 2e-16 ***
## acceleration   0.080576   0.098845   0.815  0.41548    
## year           0.750773   0.050973  14.729  < 2e-16 ***
## origin         1.426141   0.278136   5.127 4.67e-07 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.328 on 384 degrees of freedom
## Multiple R-squared:  0.8215, Adjusted R-squared:  0.8182 
## F-statistic: 252.4 on 7 and 384 DF,  p-value: < 2.2e-16
```

Overall, the model supports a relationship bwteen predictors and the response, as suggested by the low p-value of the F test. Of the seven variables (excluding the incept), displacement, wight, year and oruginhave statistically significant effects onmpgwhilecylinders,horsepower, andaccelerationdo not. The variable,year, indicates that, for every one year,mpgincreases by the0.7507727‘. In other words, cars become more fuel efficient every year.

```
par(mfrow=c(2,2))
plot(lm.fit0)
```



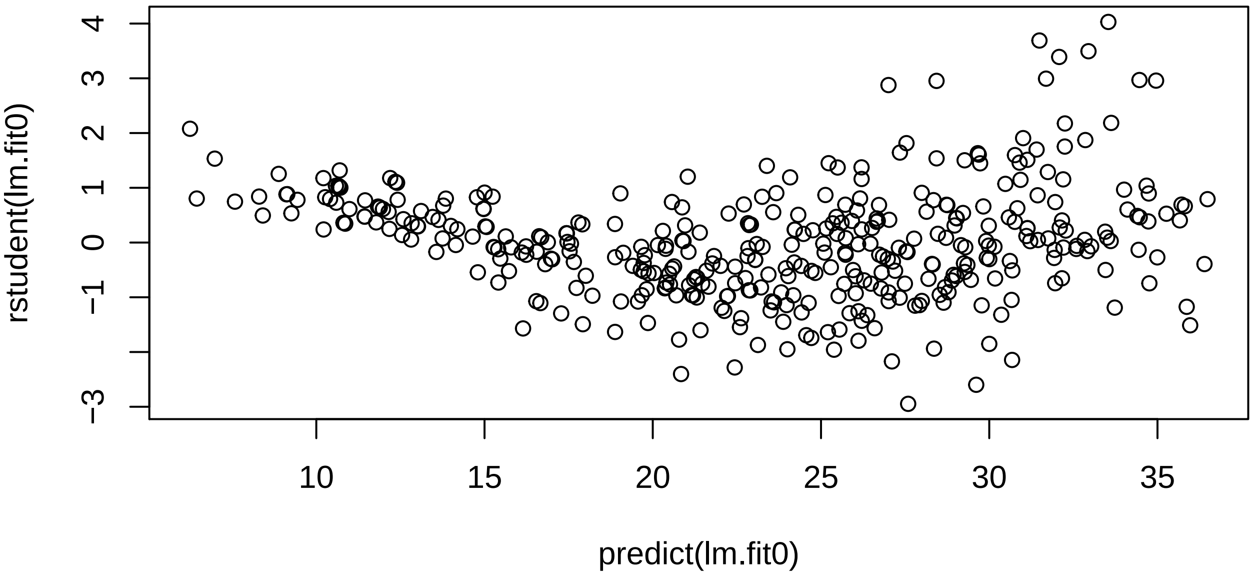
From the Residual vs Fitted plot, there seems to be a quadratic relationship between the residuals and the fitted values, suggesting that non-linearity between the predictors and response. Polynomial regression or non-linear transformation such as interaction of the variables may be needed.

The Scale-Location plot, also known as the Spread-Location plot, shows if residuals are spread equally along the ranges of predictors. Homoscedasticity seems not to hold in the model as indicated by the non-horizontal line with equally (randomly) spread points.

The QQ plot displays a steeper slope on the right tail, implying a positive skewness of the residuals.

The Residual vs Leverage plot suggests that `buick estate wagon (sw)` (obversation 14) has high leverage, despite not a high magnitude residual.

```
plot(predict(lm.fit0), rstudent(lm.fit0))
```



There are possible outliers as seen in the plot of studentized residuals because there are data with a value greater than 3.

```
# Interaction Terms
lm.fit0 <- lm(mpg ~ . -name, data=Auto)
lm.fit1 <- lm(mpg~cylinders+weight*cylinders+year+origin, data=Auto)
lm.fit2 <- lm(mpg~acceleration+weight*acceleration+year+origin, data=Auto)
lm.fit3 <- lm(mpg~horsepower+weight*horsepower+year+origin, data=Auto)
summary(lm.fit0)
```

```
##
## Call:
## lm(formula = mpg ~ . - name, data = Auto)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -9.5903 -2.1565 -0.1169  1.8690 13.0604
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -17.218435   4.644294  -3.707  0.00024 ***
## cylinders     -0.493376   0.323282  -1.526  0.12780
## displacement   0.019896   0.007515   2.647  0.00844 **
## horsepower    -0.016951   0.013787  -1.230  0.21963
## weight        -0.006474   0.000652  -9.929  < 2e-16 ***
## acceleration   0.080576   0.098845   0.815  0.41548
## year           0.750773   0.050973  14.729  < 2e-16 ***
## origin         1.426141   0.278136   5.127 4.67e-07 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.328 on 384 degrees of freedom
## Multiple R-squared:  0.8215, Adjusted R-squared:  0.8182
## F-statistic: 252.4 on 7 and 384 DF,  p-value: < 2.2e-16
```

```r
summary(lm.fit1)
```

```
##
## Call:
## lm(formula = mpg ~ cylinders + weight * cylinders + year + origin,
##     data = Auto)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -11.5277  -1.7587  -0.2015   1.5147  12.7885
##
## Coefficients:
##                     Estimate Std. Error t value Pr(>|t|)
## (Intercept)       3.8845201  4.5378720   0.856   0.3925
## cylinders        -4.4875789  0.5639369  -7.958 1.97e-14 ***
## weight           -0.0144268  0.0010804 -13.353  < 2e-16 ***
## year              0.8115738  0.0455113  17.832  < 2e-16 ***
## origin            0.5345940  0.2506425   2.133   0.0336 *
## cylinders:weight  0.0013967  0.0001637   8.533 3.30e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.078 on 386 degrees of freedom
## Multiple R-squared:  0.8464, Adjusted R-squared:  0.8444
## F-statistic: 425.5 on 5 and 386 DF,  p-value: < 2.2e-16
```

```r
summary(lm.fit2)
```

```
##
## Call:
## lm(formula = mpg ~ acceleration + weight * acceleration + year +
##     origin, data = Auto)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -8.1780 -2.0869  0.1691  1.7532 12.2679
##
## Coefficients:
##                        Estimate Std. Error t value Pr(>|t|)
## (Intercept)         -4.639e+01  5.612e+00  -8.266 2.26e-15 ***
## acceleration         1.602e+00  2.352e-01   6.813 3.67e-11 ***
## weight               1.481e-03  1.124e-03   1.318 0.188315
## year                 8.134e-01  4.762e-02  17.081  < 2e-16 ***
## origin               9.390e-01  2.478e-01   3.789 0.000175 ***
## acceleration:weight -5.024e-04  7.457e-05  -6.738 5.85e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.169 on 386 degrees of freedom
## Multiple R-squared:  0.8372, Adjusted R-squared:  0.8351
## F-statistic: 397.1 on 5 and 386 DF,  p-value: < 2.2e-16
```

```r
summary(lm.fit3)
```

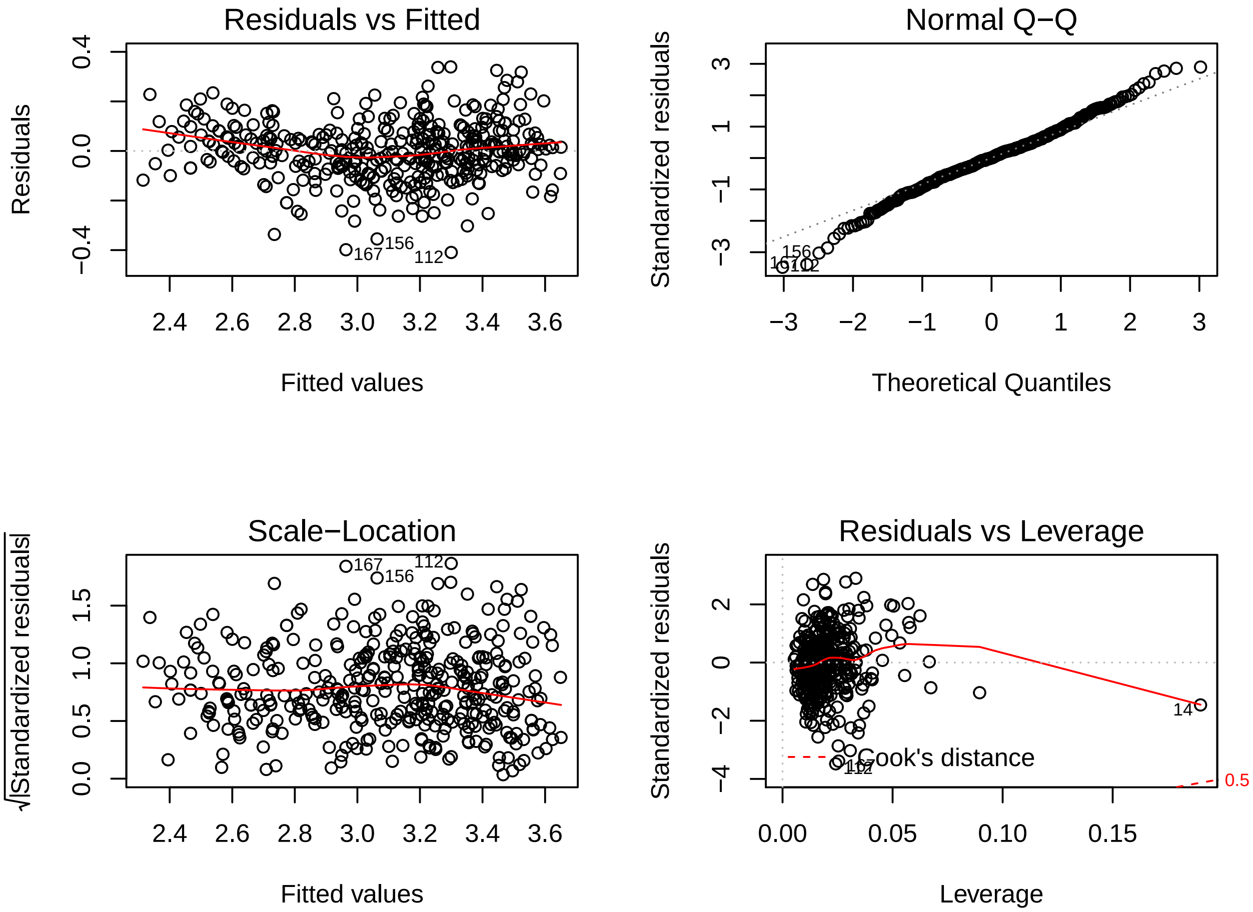
```
##
## Call:
## lm(formula = mpg ~ horsepower + weight * horsepower + year +
##     origin, data = Auto)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -8.6051 -1.7722 -0.1304  1.5205 12.0369
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## (Intercept)         8.145e-01  3.969e+00   0.205  0.83753
## horsepower         -2.160e-01  2.055e-02 -10.514  < 2e-16 ***
## weight             -1.106e-02  6.343e-04 -17.435  < 2e-16 ***
## year                7.677e-01  4.464e-02  17.195  < 2e-16 ***
## origin              7.224e-01  2.328e-01   3.103  0.00206 **
## horsepower:weight   5.501e-05  5.051e-06  10.891  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.931 on 386 degrees of freedom
## Multiple R-squared:  0.8608, Adjusted R-squared:  0.859
## F-statistic: 477.5 on 5 and 386 DF,  p-value: < 2.2e-16
```

Insignificant variables's effect to `mpg` maybe captured by *synergy* or interaction terms. Demonstarated aboved, interaction bwteen `weight` and `cylinders` (`lm.fit1`), bwteen `weight` and `acceleration` (`lm.fit2`), bwteen `weight` and and `horsepower` (`lm.fit3`) are all statistically significant.

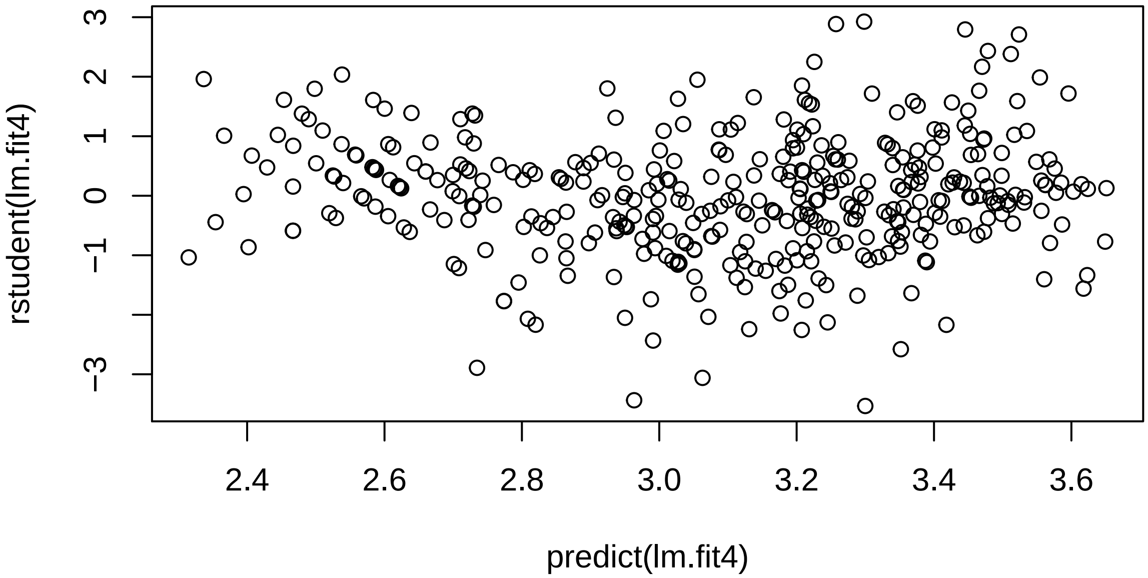
```r
# Non-linear Transformations of the Predictors
lm.fit4<-lm(log(mpg)~cylinders+displacement+horsepower+weight+acceleration+year+origin,data=Auto)
summary(lm.fit2)
```

```
##
## Call:
## lm(formula = mpg ~ acceleration + weight * acceleration + year +
##     origin, data = Auto)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -8.1780 -2.0869  0.1691  1.7532 12.2679
##
## Coefficients:
##                        Estimate Std. Error t value Pr(>|t|)
## (Intercept)          -4.639e+01  5.612e+00  -8.266 2.26e-15 ***
## acceleration          1.602e+00  2.352e-01   6.813 3.67e-11 ***
## weight                1.481e-03  1.124e-03   1.318 0.188315
## year                  8.134e-01  4.762e-02  17.081  < 2e-16 ***
## origin                9.390e-01  2.478e-01   3.789 0.000175 ***
## acceleration:weight -5.024e-04  7.457e-05  -6.738 5.85e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.169 on 386 degrees of freedom
## Multiple R-squared:  0.8372, Adjusted R-squared:  0.8351
## F-statistic: 397.1 on 5 and 386 DF,  p-value: < 2.2e-16
```

```
par(mfrow=c(2,2))
plot(lm.fit4)
```



```
par(mfrow=c(1,1))
plot(predict(lm.fit4),rstudent(lm.fit4))
```

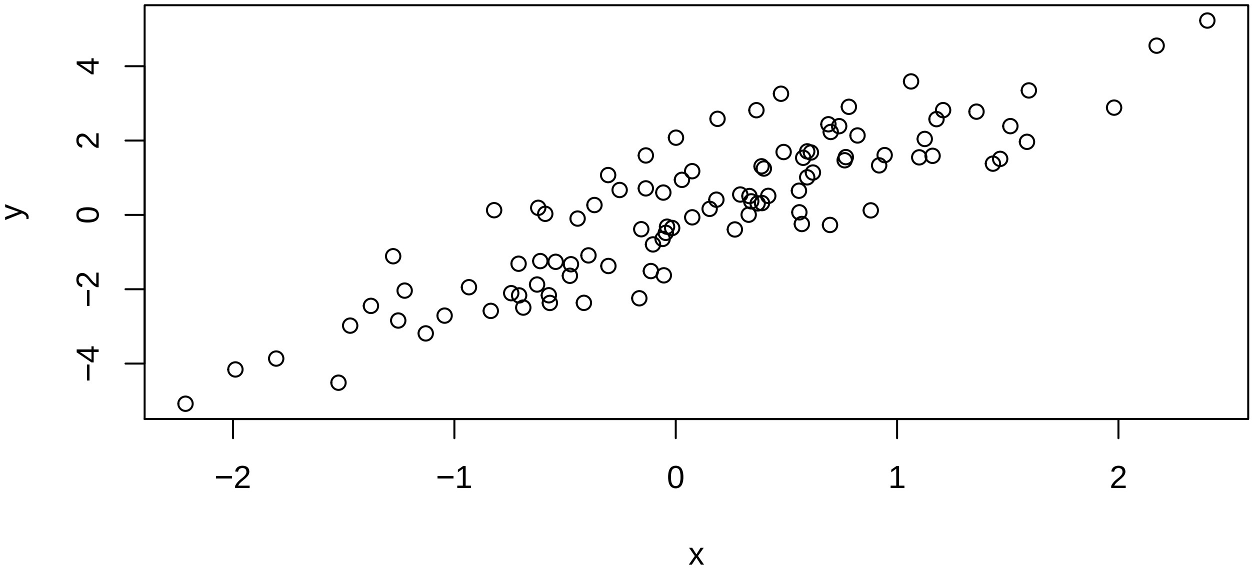


As indicated by the Residual vs Fitted plot, QQ plot and the Scale-Location plot, heteroskedasticity appears to be a feature in the previous model with linear predictors. Also in the scatter matrix, `displacement`, `horsepower` and `weight` show a similar nonlinear pattern against response `mpg`. Non-linear transformations of the predictors may be appropriate. Using `log(mpg)` as the response variable, the outputs show that log transform of `mpg` yield a higher R^2 and residuals more normally distributed.

Question 11

This and the next questions (as well as question 5) ask about simple linear regression without an intercept.

```r
set.seed (1)
x=rnorm (100)
y=2*x+rnorm (100)
plot(x,y)
```



```r
# Regress y on x. Result is highly significant
lm.fit0 <- lm(y~x+0)
summary(lm.fit0)
```

```
##
## Call:
## lm(formula = y ~ x + 0)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.9154 -0.6472 -0.1771  0.5056  2.3109
##
## Coefficients:
##   Estimate Std. Error t value Pr(>|t|)
## x   1.9939     0.1065   18.73   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9586 on 99 degrees of freedom
## Multiple R-squared:  0.7798, Adjusted R-squared:  0.7776
## F-statistic: 350.7 on 1 and 99 DF,  p-value: < 2.2e-16
```

```r
# Regress x on y. Result is highly significant
lm.fit1 <- lm(x~y+0)
summary(lm.fit1)
```

```
##
## Call:
## lm(formula = x ~ y + 0)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -0.8699 -0.2368  0.1030  0.2858  0.8938
##
## Coefficients:
##   Estimate Std. Error t value Pr(>|t|)
## y  0.39111    0.02089   18.73   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4246 on 99 degrees of freedom
## Multiple R-squared:  0.7798, Adjusted R-squared:  0.7776
## F-statistic: 350.7 on 1 and 99 DF,  p-value: < 2.2e-16
```

First, the multiple R^2, adjusted R^2, t-statistics, and F-statistics are the same in the two models. Second, since $\hat{x} = \hat{\beta}_x y$ versus $\hat{y} = \hat{\beta}_y x$, so the betas should be inverse of each other ($\hat{\beta}_x = \frac{1}{\hat{\beta}_y}$) but they are somewhat off here ($\frac{1}{0.39111} = 2.557 \neq 1.994$).

```r
lm.fit = lm(y~x)
lm.fit2 = lm(x~y)
summary(lm.fit)
```

```
##
## Call:
## lm(formula = y ~ x)
```

```
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.8768 -0.6138 -0.1395  0.5394  2.3462
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.03769    0.09699  -0.389    0.698
## x            1.99894    0.10773  18.556   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9628 on 98 degrees of freedom
## Multiple R-squared:  0.7784, Adjusted R-squared:  0.7762
## F-statistic: 344.3 on 1 and 98 DF,  p-value: < 2.2e-16
```

```
summary(lm.fit2)
```

```
##
## Call:
## lm(formula = x ~ y)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.90848 -0.28101  0.06274  0.24570  0.85736
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.03880    0.04266    0.91    0.365
## y            0.38942    0.02099   18.56   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4249 on 98 degrees of freedom
## Multiple R-squared:  0.7784, Adjusted R-squared:  0.7762
## F-statistic: 344.3 on 1 and 98 DF,  p-value: < 2.2e-16
```

The t-statistics are the same.

Question 12

Generate an example in R with n = 100 observations in which the coefficient estimate for the regression of X onto Y is different from the coefficient estimate for the regression of Y onto X.

```
# Question 11a is the example in point.
```

Generate an example in R with n = 100 observations in which the coefficient estimate for the regression of X onto Y is the same as the coefficient estimate for the regression of Y onto X.

Focus on the denominator in equation 3.38. If $\sum(x_{i'}^2) = \sum(y_{i'}^2)$, $\hat{\beta}$ of regressing y on x will be equal to that of regressing x on y. To illusrate, see

```
set.seed(1)
x <- rnorm(100)
# Generate random sample (i.e. y ) from x without replacement
y <- -sample(x, 100)
```

```
# suh that:
sum(x^2)==sum(y^2)
```

```
## [1] TRUE
```

```
lm.fit_x <- lm(y~x+0)
lm.fit_y <- lm(x~y+0)
summary(lm.fit_x)
```

```
##
## Call:
## lm(formula = y ~ x + 0)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.3926 -0.6877 -0.1027  0.5124  2.2315
##
## Coefficients:
##   Estimate Std. Error t value Pr(>|t|)
## x -0.02148    0.10048  -0.214    0.831
##
## Residual standard error: 0.9046 on 99 degrees of freedom
## Multiple R-squared:  0.0004614,  Adjusted R-squared:  -0.009635
## F-statistic: 0.0457 on 1 and 99 DF,  p-value: 0.8312
```

```
summary(lm.fit_y)
```

```
##
## Call:
## lm(formula = x ~ y + 0)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.2400 -0.5154  0.1213  0.6788  2.3959
##
## Coefficients:
##   Estimate Std. Error t value Pr(>|t|)
## y -0.02148    0.10048  -0.214    0.831
##
## Residual standard error: 0.9046 on 99 degrees of freedom
## Multiple R-squared:  0.0004614,  Adjusted R-squared:  -0.009635
## F-statistic: 0.0457 on 1 and 99 DF,  p-value: 0.8312
```

Question 14

This problem focuses on the collinearity problem.

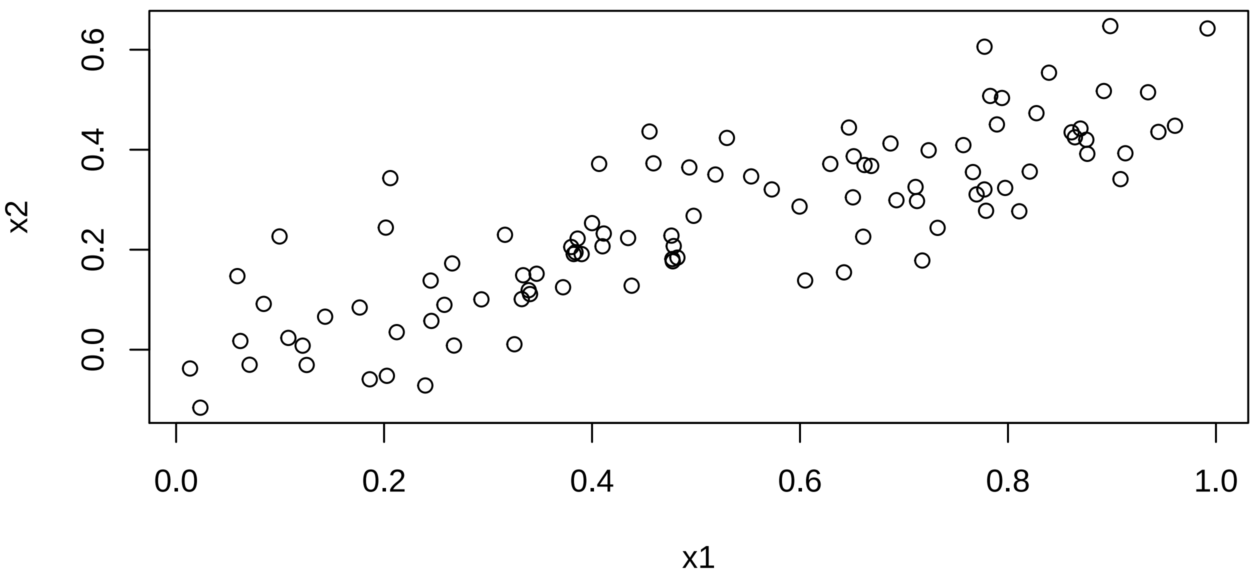
```
set.seed (1)
x1=runif (100)
x2 =0.5*x1+rnorm (100) /10
y=2+2*x1 +0.3*x2+rnorm (100)
```

The form of the linear model is $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$ where $\beta_0 = 2$, $\beta_1 = 2$ and $\beta_2 = 0.3$.

```
cor(x1,x2)
```

```
## [1] 0.8351212
```

```
plot(x1,x2)
```



```
lm.fit <- lm(y~x1+x2)
summary(lm.fit)
```

```
##
## Call:
## lm(formula = y ~ x1 + x2)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.8311 -0.7273 -0.0537  0.6338  2.3359
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   2.1305     0.2319   9.188 7.61e-15 ***
## x1            1.4396     0.7212   1.996   0.0487 *
## x2            1.0097     1.1337   0.891   0.3754
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.056 on 97 degrees of freedom
## Multiple R-squared:  0.2088, Adjusted R-squared:  0.1925
## F-statistic:  12.8 on 2 and 97 DF,  p-value: 1.164e-05
```

Estimated beta coefficients are $\hat{\beta}_0 = 2.13$, $\hat{\beta}_1 = 1.44$ and $\hat{\beta}_2 = 1.01$. Coefficient for x1 is statistically significant but the coefficient for x2 is not. Null hypothesis for x_1, $H_0 : \beta_1 = 0$, is rejected at 0.01 significant level while that of x_2, $H_0 : \beta_2 = 0$, is retained.

```
par(mfrow=c(2,1), mar=c(2, 3, 2, 1), mgp=c(2, 0.8, 0))
lm.fit1 <- lm(y~x1)
summary(lm.fit1)
```

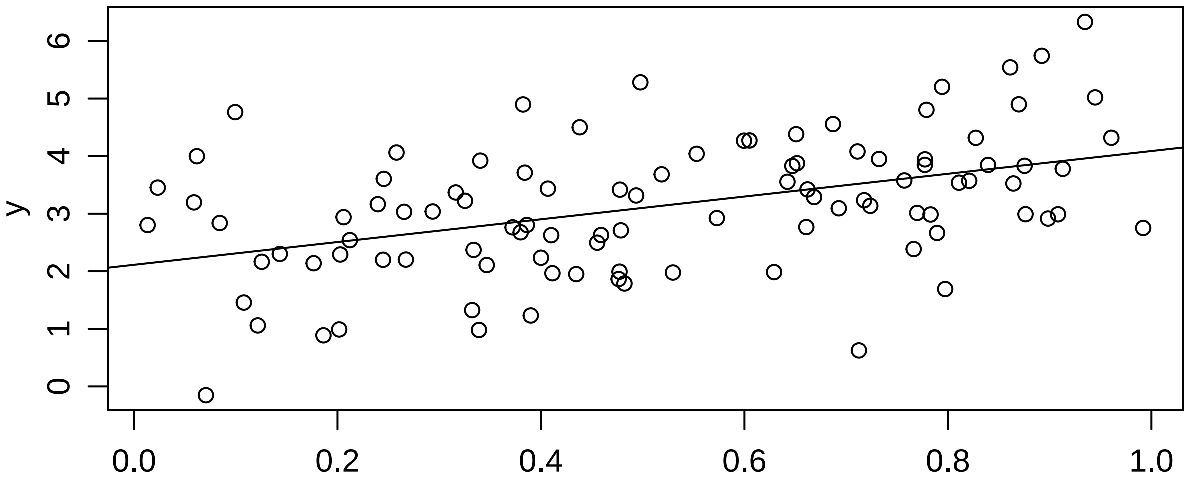
```
##
## Call:
## lm(formula = y ~ x1)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.89495 -0.66874 -0.07785  0.59221  2.45560
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   2.1124     0.2307   9.155 8.27e-15 ***
## x1            1.9759     0.3963   4.986 2.66e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.055 on 98 degrees of freedom
## Multiple R-squared:  0.2024, Adjusted R-squared:  0.1942
## F-statistic: 24.86 on 1 and 98 DF,  p-value: 2.661e-06
```

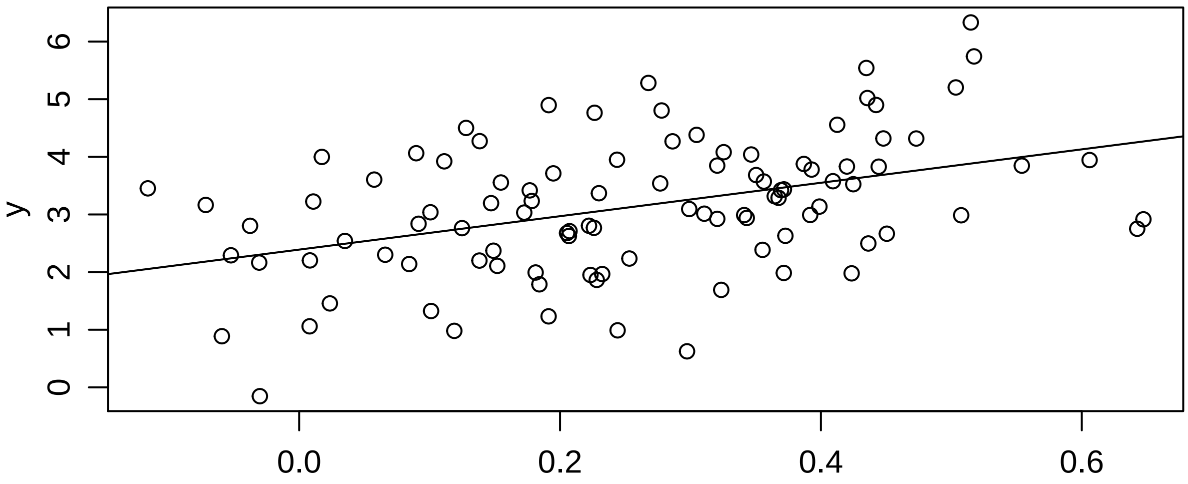
```
plot(x1,y)
abline(lm.fit1)

lm.fit2 <- lm(y~x2)
summary(lm.fit2)
```

```
##
## Call:
## lm(formula = y ~ x2)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.62687 -0.75156 -0.03598  0.72383  2.44890
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   2.3899     0.1949   12.26  < 2e-16 ***
## x2            2.8996     0.6330    4.58 1.37e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.072 on 98 degrees of freedom
## Multiple R-squared:  0.1763, Adjusted R-squared:  0.1679
## F-statistic: 20.98 on 1 and 98 DF,  p-value: 1.366e-05
```

```
plot(x2,y)
abline(lm.fit2)
```





Individually, both x_1 and x_2 enter the simple regression model with highly significant statistical levels.

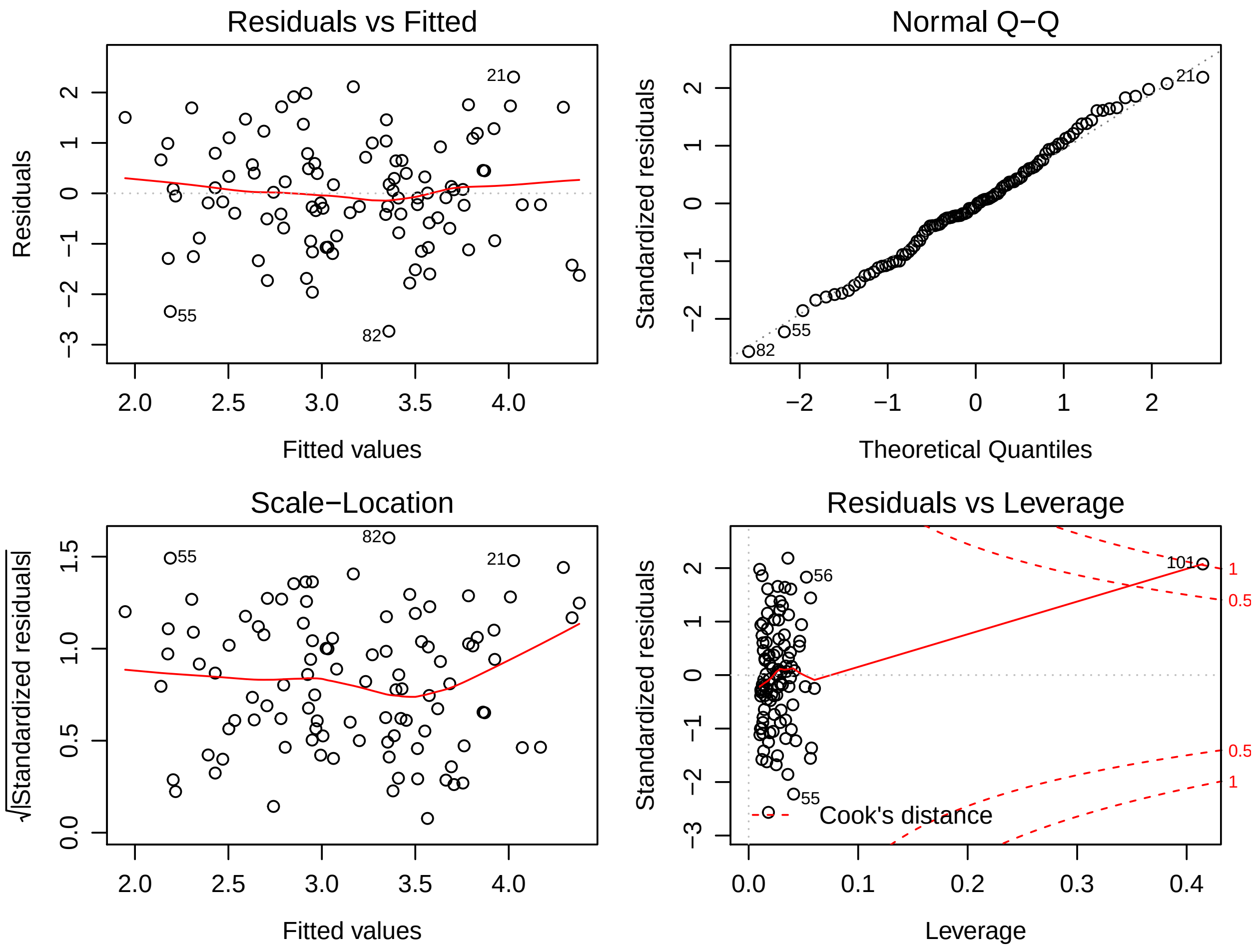
There is no contradiction. The problem lies in collinearity. It is hard to distinguish their individual effects from the combined effects when regressed upon together.

```
x1=c(x1, 0.1)
x2=c(x2, 0.8)
y=c(y,6)
par(mfrow=c(2,2), mar=c(3.5, 3.5, 2, 1), mgp=c(2.4, 0.8, 0))
# regression with both x1 and x2
fit.lm <- lm(y~x1+x2)
summary(fit.lm)
```

```
##
## Call:
## lm(formula = y ~ x1 + x2)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.73348 -0.69318 -0.05263  0.66385  2.30619
##
## Coefficients:
```

```
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)   2.2267     0.2314   9.624 7.91e-16 ***
## x1            0.5394     0.5922   0.911  0.36458
## x2            2.5146     0.8977   2.801  0.00614 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.075 on 98 degrees of freedom
## Multiple R-squared:  0.2188, Adjusted R-squared:  0.2029
## F-statistic: 13.72 on 2 and 98 DF,  p-value: 5.564e-06
```

```r
plot(fit.lm)
```

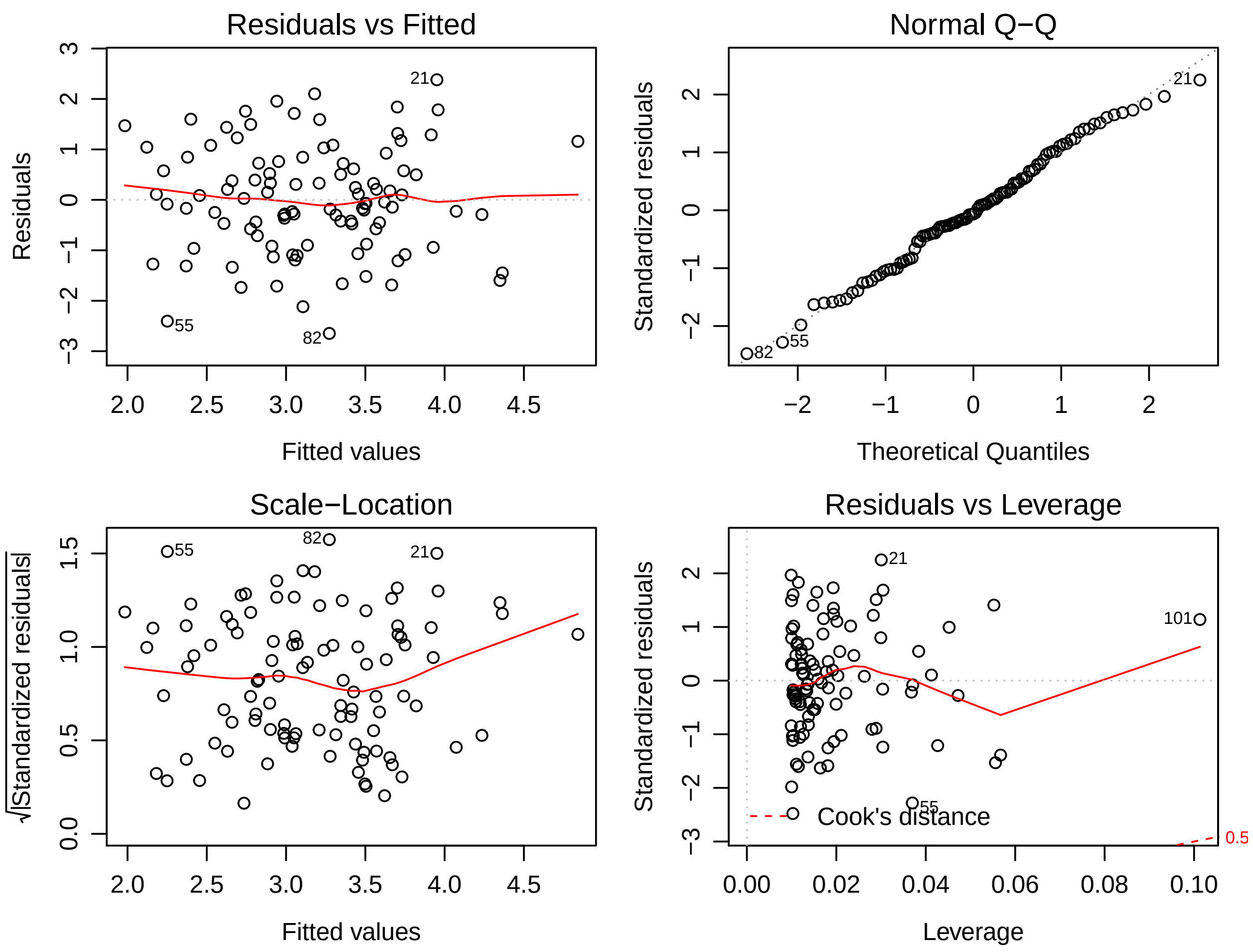


```r
# regression with x1 only
fit.lm1 <- lm(y~x2)
summary(fit.lm1)
```

```
##
## Call:
## lm(formula = y ~ x2)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.64729 -0.71021 -0.06899  0.72699  2.38074
```

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   2.3451     0.1912  12.264  < 2e-16 ***
## x2            3.1190     0.6040   5.164 1.25e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.074 on 99 degrees of freedom
## Multiple R-squared:  0.2122, Adjusted R-squared:  0.2042
## F-statistic: 26.66 on 1 and 99 DF,  p-value: 1.253e-06
```

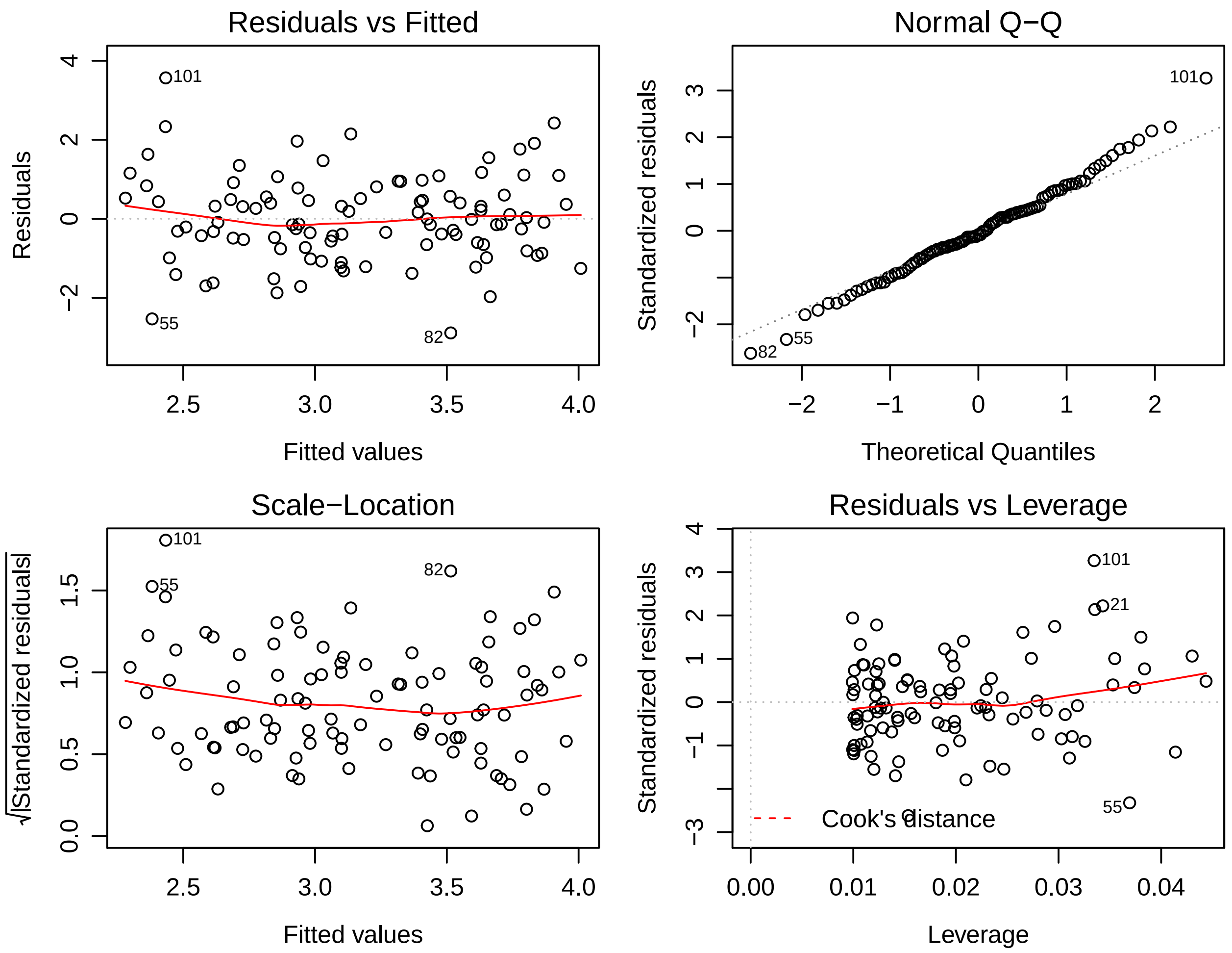
```
plot(fit.lm1)
```



```
# regression with x2 only
fit.lm2 <- lm(y~x1)
summary(fit.lm2)
```

```
##
## Call:
## lm(formula = y ~ x1)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.8897 -0.6556 -0.0909  0.5682  3.5665
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   2.2569     0.2390   9.445 1.78e-15 ***
## x1            1.7657     0.4124   4.282 4.29e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.111 on 99 degrees of freedom
## Multiple R-squared:  0.1562, Adjusted R-squared:  0.1477
## F-statistic: 18.33 on 1 and 99 DF,  p-value: 4.295e-05
```

```
plot(fit.lm2)
```



The new observation ($[y,\, x_1,\, x_2] = [6, 0.1, 0.8]$) is an outlier for x_2 and has high leverage for both x_1 and x_2. From the residuals vs leverage plot, observation 101 falls on the right hand side in all three models. In particular, it stands out as the red line is extensivelt tilted relative to the dotted black line indicating high leverage (Cook's Distance) for the model in which x_1 and x_2 are the predictors of y.