Chapter 1

Tensor Identities

In this Appendix, we introduce the double dot product of two tensors, and work with the velocity gradient tensor to ultimately derive the tensor identity

$$\nabla^2\vec{u}\cdot\vec{g} = \nabla\cdot((\nabla\vec{u}+\nabla\vec{u}^T)\cdot\vec{g}) - 2\mathbf{E}(\vec{u}):\mathbf{E}(\vec{g})$$

for use in Chapter 6.

1.0.1 Tensor Double Dot Product

The double dot product of two tensors, also known as the Frobenius inner product, is a generalization of the vector inner product:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad Z = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \quad A : Z = ax + by + cz + dw \tag{1.1}$$

As expected, addition distributes over this form of multiplication:

$$(A+B):(Z+W) = A:Z + A:W + B:Z + B:W \tag{1.2}$$

1.0.2 Tensor Vector Divergence Identity

For a tensor T and vector $\vec{g}$:

$$T : \nabla\vec{g} = T_{11}\partial_x g_x + T_{12}\partial_x g_y + T_{21}\partial_y g_x + T_{22}\partial_y g_y \tag{1.3}$$

and

$$\nabla \cdot T = \begin{bmatrix} \partial_x & , & \partial_y \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \partial_x T_{11} + \partial_y T_{21} & , & \partial_x T_{12} + \partial_y T_{22} \end{bmatrix} \tag{1.4}$$

so that

$$(\nabla \cdot T) \cdot \vec{g} = g_x \partial_x T_{11} + g_x \partial_y T_{21} + g_y \partial_x T_{12} + g_y \partial_y T_{22} \tag{1.5}$$

Furthermore

$$T \cdot \vec{g} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} T_{11} g_x + T_{12} g_y \\ T_{21} g_x + T_{22} g_y \end{bmatrix} \tag{1.6}$$

Therefore

$$\begin{aligned}
\nabla \cdot (T \cdot \vec{g}) &= \partial_x (T_{11} g_x) + \partial_x (T_{12} g_y) + \partial_y (T_{21} g_x) + \partial_y (T_{22} g_y) && (1.7) \\
&= g_x \partial_x T_{11} + T_{11} \partial_x g_x + g_y \partial_x T_{12} + T_{12} \partial_x g_y && (1.8) \\
&\quad + g_x \partial_y T_{21} + T_{21} \partial_y g_x + g_y \partial_y T_{22} + T_{22} \partial_y g_y && (1.9) \\
&= [T_{11} \partial_x g_x + T_{12} \partial_x g_y + T_{21} \partial_y g_x + T_{22} \partial_y g_y] && (1.10) \\
&\quad + [g_x \partial_x T_{11} + g_x \partial_y T_{21} + g_y \partial_x T_{12} + g_y \partial_y T_{22}] && (1.11) \\
&= T : \nabla\vec{g} + (\nabla \cdot T) \cdot \vec{g} && (1.12)
\end{aligned}$$

We have shown:

$$\nabla \cdot (T \cdot \vec{g}) = T : \nabla\vec{g} + (\nabla \cdot T) \cdot \vec{g} \tag{1.13}$$

1.0.3 Application to Velocity Gradient Tensor

Substituting $T = \nabla\vec{u}$ in the identity gives:

$$\nabla \cdot (\nabla\vec{u} \cdot \vec{g}) = \nabla\vec{u} : \nabla\vec{g} + (\nabla \cdot \nabla\vec{u}) \cdot \vec{g} \tag{1.14}$$

and the 'vector Laplacian' is defined:

$$\nabla \cdot \nabla\vec{u} = \Big[\partial_x\partial_x u + \partial_y\partial_y u \quad , \quad \partial_x\partial_x v + \partial_y\partial_y v\Big] = \Big[\nabla^2 u \quad , \quad \nabla^2 v\Big] = \nabla^2\vec{u} \tag{1.15}$$

So

$$\nabla \cdot (\nabla\vec{u} \cdot \vec{g}) = \nabla\vec{u} : \nabla\vec{g} + \nabla^2\vec{u} \cdot \vec{g} \tag{1.16}$$

Similarly for the transpose of the velocity gradient tensor:

$$\nabla \cdot (\nabla\vec{u}^T \cdot \vec{g}) = \nabla\vec{u}^T : \nabla\vec{g} + (\nabla \cdot \nabla\vec{u}^T) \cdot \vec{g} \tag{1.17}$$

Now, however, the last term vanishes:

$$\nabla \cdot \nabla\vec{u}^T = \Big[\partial_x\partial_x u + \partial_x\partial_y v \quad , \quad \partial_y\partial_x u + \partial_y\partial_y v\Big] \tag{1.18}$$

$$= \Big[\partial_x(\partial_x u + \partial_y v) \quad , \quad \partial_y(\partial_x u + \partial_y v)\Big] \tag{1.19}$$

$$= \Big[\partial_x(\nabla \cdot \vec{u}) \quad , \quad \partial_y(\nabla \cdot \vec{u})\Big] \tag{1.20}$$

$$= \Big[0 \quad , \quad 0\Big] \tag{1.21}$$

since we assume the fluid is **incompressible**, so $\nabla \cdot \vec{u} = 0$ everywhere.

Thus

$$\nabla \cdot (\nabla\vec{u}^T \cdot \vec{g}) = \nabla\vec{u}^T : \nabla\vec{g} \tag{1.22}$$

1.0.4 Deformation Rate Tensor Identity

Extending our notation slightly to include vector fields other than $\vec{u}$, recall that the deformation rate tensor is:

$$\mathbf{E}(\vec{u}) = \frac{\nabla\vec{u} + \nabla\vec{u}^T}{2} \tag{1.23}$$

so that:

$$2\mathbf{E}(\vec{g}) = \nabla\vec{g} + \nabla\vec{g}^T \tag{1.24}$$

Then the double dot product of two such tensors is:

$$2\mathbf{E}(\vec{u}) : 2\mathbf{E}(\vec{g}) = (\nabla\vec{u} + \nabla\vec{u}^T) : (\nabla\vec{g} + \nabla\vec{g}^T) \tag{1.25}$$

$$4\mathbf{E}(\vec{u}) : \mathbf{E}(\vec{g}) = \nabla\vec{u} : \nabla\vec{g} + \nabla\vec{u} : \nabla\vec{g}^T + \nabla\vec{u}^T : \nabla\vec{g} + \nabla\vec{u}^T : \nabla\vec{g}^T \tag{1.26}$$

Now, transposition affects the double dot product such that
$\nabla\vec{u} : \nabla\vec{g} = \nabla\vec{u}^T : \nabla\vec{g}^T$ and $\nabla\vec{u}^T : \nabla\vec{g} = \nabla\vec{u} : \nabla\vec{g}^T$, so

$$4\mathbf{E}(\vec{u}) : \mathbf{E}(\vec{g}) = 2\nabla\vec{u} : \nabla\vec{g} + 2\nabla\vec{u}^T : \nabla\vec{g} \tag{1.27}$$

$$2\mathbf{E}(\vec{u}) : \mathbf{E}(\vec{g}) = \nabla\vec{u} : \nabla\vec{g} + \nabla\vec{u}^T : \nabla\vec{g} \tag{1.28}$$

Finally,

$$\begin{aligned}
\nabla \cdot ((\nabla\vec{u} + \nabla\vec{u}^T) \cdot \vec{g}) &= \nabla \cdot (\nabla\vec{u} \cdot \vec{g} + \nabla\vec{u}^T \cdot \vec{g}) && (1.29)\\
&= \nabla \cdot (\nabla\vec{u} \cdot \vec{g}) + \nabla \cdot (\nabla\vec{u}^T \cdot \vec{g}) && (1.30)\\
&= \nabla^2\vec{u} \cdot \vec{g} + \nabla\vec{u} : \nabla\vec{g} + \nabla\vec{u}^T : \nabla\vec{g} && (1.31)\\
&= \nabla^2\vec{u} \cdot \vec{g} + 2\mathbf{E}(\vec{u}) : \mathbf{E}(\vec{g}) && (1.32)
\end{aligned}$$

Therefore:

$$\nabla^2\vec{u} \cdot \vec{g} = \nabla \cdot ((\nabla\vec{u} + \nabla\vec{u}^T) \cdot \vec{g}) - 2\mathbf{E}(\vec{u}) : \mathbf{E}(\vec{g}) \tag{1.33}$$