Replicating John Philip 1972

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John R. Philip says that the limit of

$$W_3 = \Im\left[\alpha^{-1}\cos^{-1}\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} - \Theta\right]$$

as $y \to \infty$ is

$$W_3 = \alpha^{-1}\ln\sec\alpha$$

Let's replicate.

$\Theta = x + iy$ is a complex number, α is real. Trig identities for *complex* cosine and *complex* exponential and log:

$$\arccos z = -i\ln(z + \sqrt{z^2-1})$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Strategy: Expand complex cosine term, dump negligible parts.

Complex cos function can be expressed in terms of *real* cos and exponential functions.

$$\cos(x+iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}, \quad e^{a+ib} = e^a(\cos b + i\sin b)$$

$$\cos(x+iy) = \frac{e^{(ix-y)} + e^{(-ix+y)}}{2} = \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos -x + i\sin -x)}{2}$$

Converting the real cos and sin term back to complex $e^{i\theta}$:

$$\cos(x+iy) = \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2}$$

Now e^{ix} and e^{-ix} are unit vectors cycling around the unit circle (in opposite directions). The $e^{\pm y}$ coefficients give the length of the vectors. The cos function just adds them together, and halves the magnitude.

Therefore, at large y, a single huge vector $e^{y}e^{-ix}/2$ sweeps around the complex plane, with a negligible modifying vector $e^{-y}e^{ix}/2$ added to it.

$$\text{At large } y: \quad \cos(x+iy) \simeq \frac{e^y e^{-ix}}{2}$$

Similarly,

$$\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} = \left\{\frac{e^{\alpha y}e^{-i\alpha x} + e^{-\alpha y}e^{i\alpha x}}{2\cos\alpha}\right\} = e^{\alpha y}\left\{\frac{e^{-i\alpha x}}{2\cos\alpha}\right\} + e^{-\alpha y}\left\{\frac{e^{i\alpha x}}{2\cos\alpha}\right\}$$

Now, $\alpha = \frac{1}{2}\pi a/b$ and $0 < \alpha < \frac{1}{2}\pi$. The unit vectors $e^{\pm i\alpha x}$ just rotate faster or slower, depending on α — their magnitude remains unity.

Furthermore, as $\alpha \to 0$, $\cos\alpha \to 1$. But, as $\alpha \to \pi/2$, $\cos\alpha \to 0$, hence

$$\left\{\frac{e^{\pm i\alpha x}}{2\cos\alpha}\right\} \text{diverges as } \alpha \to \pi/2$$

However, on physical grounds a/b is limited to about 0.99 – probably well less. (i.e. at least 1% solid surface.) Thus, $\alpha \leq 0.99\pi/2$, so that $1/\cos\alpha < 100$.

$$\text{Therefore, magnitude of vectors } \left\{\frac{e^{\pm i\alpha x}}{2\cos\alpha}\right\} \text{is in range 1 to 100.}$$

Obviously, at very large y, $e^{\alpha y}$ is arbitrarily large, while $e^{-\alpha y}$ is negligible. Hence

$$\text{At large } y: \quad \left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} \simeq e^{\alpha y}\left\{\frac{e^{-i\alpha x}}{2\cos\alpha}\right\} = e^{\alpha y}\{\beta\}$$

Strategy: Convert complex arccos to log form

Using the identity $\arccos z = -i\ln(z + \sqrt{z^2-1})$, W_3 becomes:

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} + \sqrt{\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2 - 1}\right) - \Theta\right]$$

Which is:

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}\{\beta\} + \sqrt{\left[e^{\alpha y}\{\beta\}\right]^2 - 1}\right) - \Theta\right]$$

Now, at large y, $\left[e^{\alpha y}\{\beta\}\right]^2$ is arbitrarily large, so $\sqrt{\left[e^{\alpha y}\{\beta\}\right]^2 - 1}$ is arbitrarily close to $e^{\alpha y}\{\beta\}$. Hence

$$W_3 \cong \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}\{\beta\} + e^{\alpha y}\{\beta\}\right) - \Theta\right]$$

$$= \Im\left[-i\alpha^{-1}\ln\left(2e^{\alpha y}\left\{\frac{e^{-i\alpha x}}{2\cos\alpha}\right\}\right) - \Theta\right]$$

$$= \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha(y-ix)}\left\{\frac{1}{\cos\alpha}\right\}\right)-\Theta\right]$$

Recall the logarithm product law: $\ln xy = \ln x + \ln y$. Thus,

$$W_3 \cong \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha(y-ix)}\right)-i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\}-\Theta\right]$$

Now recall the log definition: $\ln e^x = x$. Thus,

$$-i\alpha^{-1}\ln\left(e^{\alpha(y-ix)}\right) = -i\alpha^{-1}\alpha(y-ix) = -iy + i^2x = x + iy = \Theta$$

Then

$$W_3 \cong \Im\left[\Theta - i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\}-\Theta\right] = \Im\left[-i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\}\right]$$

Finally,

$$\lim_{y\to\infty} W_3 = -\alpha^{-1}\ln\sec\alpha$$

But what about that flippin' minus sign?

Transverse Slots

J. Philip also studied flow over *transverse* no-shear slots. He was only able to use Stokes flow, not Navier-Stokes flow. He uses the Stokes stream function.

The Couette flow solution is:

$$\psi_1 = \frac{1}{2}\tau_\infty y^2/\mu$$

The solution for 'Shear Stokes Flow over a Plate with a Regular Array of Transverse No-Shear Slots' will be of the form:

$$\psi_3 = \psi_1 + a\tau_\infty \Psi_3/\mu$$

The solution of Ψ_3 is:

$$\Psi_3 = \frac{1}{2}YW_3$$

So:

$$\psi_3 = \frac{1}{2}\tau_\infty y^2/\mu + a\tau_\infty \frac{1}{2}YW_3/\mu$$

Now, Y is the nondimensionalized y/a. So:

$$\psi_3 = \frac{\tau_\infty}{\mu}\left(\frac{1}{2}y^2 + \frac{1}{2}yW_3\right)$$

It is **very tempting** to say that x-velocity u is given by

$$u = \frac{\partial\psi}{\partial y}$$

Hence:

$$u = \frac{\tau_\infty}{\mu}\left(y + \frac{1}{2}W_3\right)$$

So that the slip length is:

$$b_{\text{eff}} = \frac{1}{2}W_3$$

which in the limit $y \to \infty$ is

$$b_{\text{eff}} = \frac{1}{\pi}\frac{b}{a}\ln\sec\frac{\pi}{2}\frac{a}{b}$$

But **unfortunately**, W_3 is a function of y, so must be differentiated *before* taking the limit.

So in reality:

$$u = \frac{\tau_\infty}{\mu}\left(y + \frac{1}{2}W_3 + \frac{1}{2}y\frac{\partial W_3}{\partial y}\right)$$

Alrighty,

$$W_3 = \Im\left[\alpha^{-1}\cos^{-1}\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} - \Theta\right]$$

$$\frac{\partial}{\partial y} W_3 = \Im \left[\alpha^{-1} \frac{\partial}{\partial y} \cos^{-1} \left\{ \frac{\cos(\alpha\Theta)}{\cos\alpha} \right\} - \frac{\partial}{\partial y} \Theta \right]$$

$$\frac{\partial}{\partial y} \Theta = \frac{\partial}{\partial y} (x + \imath y)/a = \imath/a$$

$$\frac{\partial}{\partial y} \cos^{-1} \left\{ \frac{\cos(\alpha\Theta)}{\cos\alpha} \right\} = \frac{\partial}{\partial u} \cos^{-1}(u) \frac{\partial}{\partial y} \frac{\cos(\alpha\Theta)}{\cos\alpha}$$

$$= \frac{-1}{\sqrt{1-u^2}} \frac{\partial_t \cos(t) \alpha \partial_y \Theta}{\cos\alpha}$$

$$\frac{-1}{\sqrt{1-u^2}} \frac{-\sin(t) \imath\alpha/a}{\cos\alpha}$$

$$= \frac{-1}{\sqrt{1 - \left\{ \frac{\cos(\alpha\Theta)}{\cos\alpha} \right\}^2}} \frac{-\imath\alpha \sin(\alpha\Theta)}{a\cos\alpha}$$

Whew! Finally:

$$\frac{\partial}{\partial y} W_3 = \Im \left[\frac{\imath \sin(\alpha\Theta)}{a \cos\alpha \sqrt{1 - \left\{ \frac{\cos(\alpha\Theta)}{\cos\alpha} \right\}^2}} - \frac{\imath}{a} \right]$$

What can you do with that?

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Old rough working

So we can reason as follows:

$$\text{At large } y: \quad \left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} \simeq \left\{\frac{e^{\alpha y}e^{-i\alpha x}}{2\cos\alpha}\right\}$$

$$\text{Obviously, } \lim_{y\to\infty}\cos(x+iy) = \infty$$

in the sense of infinity in any direction.

But, we can say that as y gets very large, $\cos(x+iy)$ behaves like:

$$\frac{e^{y}(\cos -x + i\sin -x)}{2} = \frac{e^{y}(\cos x - i\sin x)}{2}$$

$$\text{So} \quad \left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} \text{ behaves like } \left\{\frac{e^{\alpha y}(\cos\alpha x - i\sin\alpha x)}{2\cos\alpha}\right\}$$

Therefore, for arbitrarily large y, define:

$$e^{\alpha y}\left\{\frac{\cos\alpha x - i\sin\alpha x}{2\cos\alpha}\right\} = e^{\alpha y}\{\beta\}$$

Where β is a complex number with magnitude on the order of 1.

Then $\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2$ behaves like $e^{2\alpha y}\{\beta\}^2$

Sooooo....

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} + \sqrt{\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2 - 1}\right) - \Theta\right]$$

at large y behaves like:

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}\{\beta\} + \sqrt{(e^{\alpha y})^2\{\beta\}^2 - 1}\right) - \Theta\right]$$

Now, $e^{\alpha y}$ is humungous, so $\sqrt{(e^{\alpha y})^2\{\beta\}^2 - 1}$ is very, very close to $\sqrt{(e^{\alpha y})^2\{\beta\}^2}$ which is $e^{\alpha y}\{\beta\}$. Hence

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}\{\beta\} + e^{\alpha y}\{\beta\}\right) - \Theta\right]$$

$$= \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}2\{\beta\}\right) - \Theta\right]$$

$$= \Im \left[-i\alpha^{-1} \ln \left(e^{\alpha y} 2 \left\{ \frac{\cos \alpha x - i \sin \alpha x}{2 \cos \alpha} \right\} \right) - \Theta \right]$$

$$= \Im \left[-i\alpha^{-1} \ln \left(e^{\alpha y} (\cos \alpha x - i \sin \alpha x) \left\{ \frac{1}{\cos \alpha} \right\} \right) - \Theta \right]$$

Now $e^{\alpha y}(\cos \alpha x - i \sin \alpha x) = e^{\alpha y}(\cos(-\alpha x) + i \sin(-\alpha x))$
which is $e^{\alpha y - i\alpha x} = e^{\alpha(y-ix)}$
Thus

$$W_3 = \Im \left[-i\alpha^{-1} \ln \left(e^{-\alpha(y-ix)} \left\{ \frac{1}{\cos \alpha} \right\} \right) - \Theta \right]$$

All good. Now recall the log law: $\ln xy = \ln x + \ln y$. Thus

$$W_3 = \Im \left[-i\alpha^{-1} \ln \left\{ \frac{1}{\cos \alpha} \right\} - i\alpha^{-1} \ln \left(e^{-\alpha(y-ix)} \right) - \Theta \right]$$

Further recall that $\ln e^x = x$. So that

$$\ln \left(e^{-\alpha(y-ix)} \right) = -\alpha(y - ix)$$

And so the third term simplifies to Θ:

$$-i\alpha^{-1} \ln \left(e^{-\alpha(y-ix)} \right) = -i\alpha^{-1}(-\alpha(y - ix))$$

$$= iy - i^2 x = x + iy = \Theta$$

Hence

$$W_3 = \Im \left[-i\alpha^{-1} \ln \left\{ \frac{1}{\cos \alpha} \right\} + \Theta - \Theta \right]$$

$$W_3 = -\alpha^{-1} \ln \left\{ \frac{1}{\cos \alpha} \right\}$$