Dimensional Analysis of Perturbation Height

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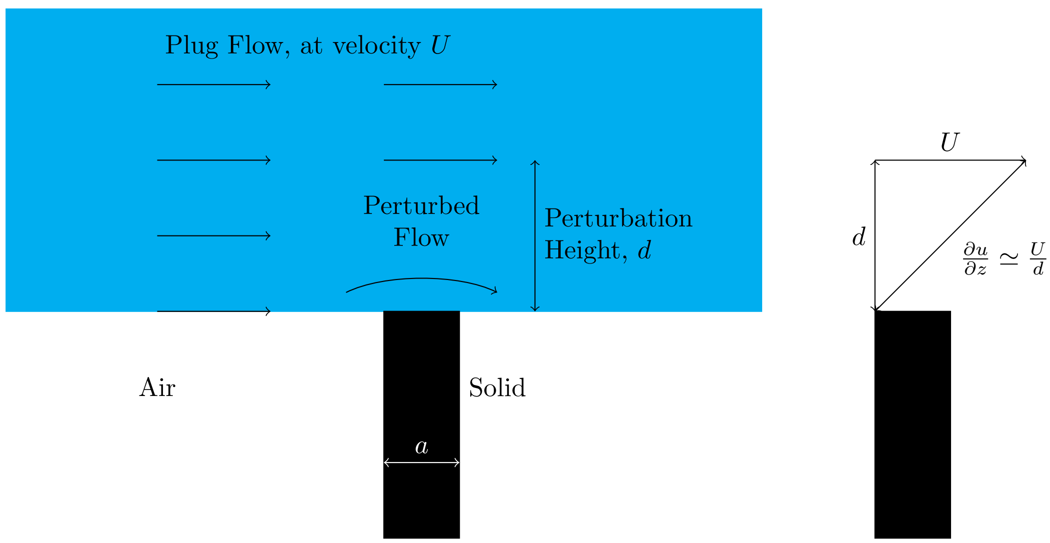
Observe:



What is perturbation height d?
It will be a function of some fundamental physical parameters:

- Ridge width a
- Velocity U
- Viscosity η
- Density ρ
- Pressure p

In other words $d = g(a, U, \eta, \rho, p)$. Or equivalently, $f(d, a, U, \eta, \rho, p) = 0$.
Dimensionally,

$$d = [m], \quad a = [m], \quad U = [ms^{-1}], \quad \eta = [kgs^{-1}m^{-1}], \quad \rho = [kgm^{-3}], \quad p = [kgm^{-1}s^{-2}]$$

$$d = [m], \quad a = [m], \quad U = \left[\frac{m}{s}\right], \quad \eta = \left[\frac{kg}{ms}\right], \quad \rho = \left[\frac{kg}{m^3}\right], \quad p = \left[\frac{kg}{ms^2}\right]$$

Units are arbitrary, so we want the height d as a *ratio* of the width a.

$$\frac{d}{a} = Dimensionless\ f(U, \eta, \rho, p)$$

Furthermore, the physical variables will appear as *powers*, multiplied by some constant, or as the argument of functions such as log and cosine. Those functions are dimensionless, so their arguments must be dimensionless. Wrapping those functions into the dimensionless variable C, we have something like:

$$\frac{d}{a} = C\ U^w \eta^x \rho^y p^z$$

Now, the physical variables on the right hand side must be in powers such that the units cancel out to be a dimensionless constant.

$$\begin{aligned} C &= \left[\frac{m}{s}\right]^w \left[\frac{kg}{ms}\right]^x \left[\frac{kg}{m^3}\right]^y \left[\frac{kg}{ms^2}\right]^z \\ &= \left[\frac{m^w kg^x kg^y kg^z}{s^w m^x s^x m^{3y} m^z s^{2z}}\right] \\ &= \left[\frac{m^w kg^{x+y+z}}{m^{x+3y+z} s^{w+x+2z}}\right] = \left[kg^{x+y+z} m^{w-x-3y-z} s^{-w-x-2z}\right] \end{aligned}$$

Thus, the units cancel iff the indices w, x, y, z simultaneously satisfy:

$$\begin{aligned} x + y + z &= 0 \\ w - x - 3y - z &= 0 \\ -w - x - 2z &= 0 \end{aligned}$$

We can express and solve these three simultaneous equations with matrix algebra.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & -3 & -1 \\ -1 & -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Gauss-Jordan reduction:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & -3 & -1 \\ -1 & -1 & 0 & -2 \end{bmatrix} \begin{matrix} \text{Swap} \\ \text{R1 \& R2} \\ \end{matrix} \begin{bmatrix} 1 & -1 & -3 & -1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & -2 \end{bmatrix} \begin{matrix} \\ \\ \text{R3 + R1} \end{matrix} \begin{bmatrix} 1 & -1 & -3 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -3 & -3 \end{bmatrix}$$

$$\begin{array}{r}\\ \\ \text{R3 + 2R2}\end{array}\begin{bmatrix}1 & -1 & -3 & -1\\ 0 & 1 & 1 & 1\\ 0 & 0 & -1 & -1\end{bmatrix}\begin{array}{l}\text{R1 + R2}\\ \\ \\ \end{array}\begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & -1 & -1\end{bmatrix}\text{R2 + R3}\begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & -1\end{bmatrix}$$

$$\begin{array}{r}\\ \\ \text{R3} \times \text{-1}\end{array}\begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\begin{array}{l}\text{R1 + 2R3}\\ \\ \\ \end{array}\begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

Our simultaneous equations have simplified to:

$$\begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}w\\ x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}\quad\Rightarrow\quad\begin{array}{c}w+2z=0\\ x=0\\ y+z=0\end{array}\quad\text{so}\quad\begin{array}{c}w=-2z\\ x=0\\ y=-z\end{array}$$

As a check, we substitute $w=-2z$, $x=0$ and $y=-z$ back into

$$\begin{array}{c}x+y+z=0\\ w-x-3y-z=0\\ -w-x-2z=0\end{array}\quad\text{getting}\quad\begin{array}{c}0+-z+z=0\\ -2z+0+3z-z=0\\ 2z-0-2z=0\end{array}\quad\text{which is}\quad\begin{array}{c}0=0\\ 0=0\\ 0=0\end{array}$$

Hence, our formula must be of the form:

$$\frac{d}{a}=C\,U^{-2z}\eta^{0}\rho^{-z}p^{z}$$

Interestingly, our formula is *not* a function of viscosity η. For the simplest case, $z=1$, we have:

$$\frac{d}{a}=C\frac{p}{\rho U^2}$$

The dimensionless quantity

$$\frac{\rho U^2}{p}=\frac{[kgm^{-3}][ms^{-1}]^2}{[kgm^{-1}s^{-2}]}=\frac{[kgm^{-1}s^{-2}]}{[kgm^{-1}s^{-2}]}$$

has the obscure name of the Ruark number.

Note that

$$\frac{d}{a}=C\left(\frac{\rho U^2}{p}\right)^7\quad\text{or}\quad\frac{d}{a}=\sqrt{\frac{\rho U^2}{p}}\cos\left(\frac{\rho U^2}{p}\right)$$

would be equally valid. How can we refine further?

Note: We have explicitly assumed that the ratio d/a is a function of U, η, ρ, p *only.* i.e. d and a do not appear on the right hand side. A more complete treatmeat would relax that assumption. Such a treatment is the Buckingham Pi Theorem.

Formal Treatment - Buckingham π Theorem

Formally, there are 6 dimensional variables, and 3 physical units.

So if f *expresses a valid physical law,* there will be 6 - 3 = 3 dimensionless variables, π_1, π_3 and π_2. Where we suspect

$$\pi_1 = \frac{d}{a} \quad \text{and} \quad \pi_2 = \frac{\rho U^2}{p}, \quad \pi_3 = \eta^0 = 1$$

Thus, $f(a, \rho, U, \eta, p, d) = 0$ is equivalent to $\Phi(\pi_1, \pi_2) = 0$. Which we have found to be

$$\pi_1 - C\pi_2 = 0$$

The dimensional matrix M is:

	a	ρ	U	η	p	d
m	1	-3	1	-1	-1	1
kg	0	1	0	1	1	0
s	0	0	-1	-1	-2	0

Basis of Nullspace from Column Echelon Form

The components of the dimensional matrix M are the exponents on the fundamental units. The components of the vector $\vec{x}$ are the exponents of the physical variables. The simultaneous equations that express the constraint that the physical formula be dimensionless are expressed as:

$$M\vec{x} = 0$$

We are solving for the appropriate exponents on the physical variables that satisfy the non-dimensionality requirement.

However, there will be a whole *family* of solutions. They form a space; the set of all $\vec{x}$ satisfying $M\vec{x} = 0$. The set is known as the *null space* or *kernel* of M. We would like a basis for the space. The basis vectors will represent dimensionless quantities like the Reynolds number.

To find a basis for the null space of M, we use the following technique: Glue the identity matrix I underneath M. Transpose the result, forming a new matrix A. Row reduce A until the part corresponding to M is in row echelon form. Then, the basis vectors are: any row of I with all zeros in the corresponding row of M.

Here goes:

$$
\left[\begin{array}{cccccc}
1 & -3 & 1 & -1 & -1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & -1 & -2 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]^T
=
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-3 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 1 & -2 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\begin{array}{r}
\\
\text{R3 + 3R1} \\
\text{R3 - R1} \\
\text{R4 + R1} \\
\text{R5 + R1} \\
\text{R6 - R1}
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\begin{array}{r}
\\
\\
\text{R3} \times \text{-1} \\
\text{R4 - R2} \\
\text{R5 - R2} \\
\\
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & -2 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & -2 & -2 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\begin{array}{r}
\\
\\
\\
\text{R4 + R3} \\
\text{R5 + 2R3} \\
\\
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

At this point we can stop; we have three basis vectors. They are the bottom three rows of what was the identity matrix:

$$
\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & -2 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}
$$

whose components are the exponents of our physical variables

$$
\begin{bmatrix} a & \rho & U & \eta & p & d \end{bmatrix}
$$

We are free to take the *negative* of any basis vectors. If we do this, then the corresponding dimensionless variables appear in the convenient form:

$$
\pi_1 = \frac{d}{a}, \quad \pi_2 = \frac{\rho U^2}{p}, \quad \pi_3 = \frac{\rho a U}{\eta}
$$

Then, the dimensionless number

$$\pi_3 = \frac{\rho a U}{\eta} = \frac{[kgm^{-3}][m][ms^{-1}]}{[kgm^{-1}s^{-1}]} = \frac{[kgm^{-1}s^{-1}]}{[kgm^{-1}s^{-1}]} = 1$$

is the familiar *Reynolds* number, while $\pi_2 = \frac{\rho U^2}{p}$ is the more obscure *Ruark* number.

The three vectors *are* linearly independent — none can be obtained by a linear combination of the other two, since each contains a unique physical variable (specifically d, p and η).

But π_3 is coupled to the other two. In some physical sense, the variables are not independent.

We have found $\Phi(\pi_1, \pi_2, \pi_3) = 0$, where

$$\pi_1 = \frac{d}{a}, \quad \pi_2 = \frac{\rho U^2}{p}, \quad \pi_3 = \frac{\rho a U}{\eta}$$

So whatever Φ is, d and a appear in it *to the same order*, with one being an inverse of the other. Therefore the ratio d/a can be pulled out, like so:

$$\frac{d}{a} = \Phi(Ru, Re)$$

$$\text{where} \quad Ru = \frac{\rho U^2}{p}, \quad Re = \frac{\rho a U}{\eta}$$

Conclusion

The ratio d/a is an unknown function of the Ruark number and the Reynolds number.

For a series of experiments with a given fluid at a given pressure:

For a given velocity, perturbation height d scales as anomaly width a.

It is not *generally true* that d scales as a.

Eg. for a fixed a, d may also scale as U, ρ or whatever.