Replicating John Philip 1972

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The first known expression for an effective slip length appeared in 1972, in a paper in ZAMP by John R. Philip entitled "Flows Satisfying Mixed No-Slip and No-Shear Conditions".

In the paper, John R. Philip says that the limit of

$$W_3 = \Im\left[\alpha^{-1}\cos^{-1}\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} - \Theta\right] \tag{1}$$

as $y \to \infty$ is

$$W_3 = \alpha^{-1}\ln\sec\alpha \tag{2}$$

Let us prove this forthwith.

$\Theta = x + iy$ is a complex number, α is real. Trig identities for *complex* cosine and *complex* exponential and log:

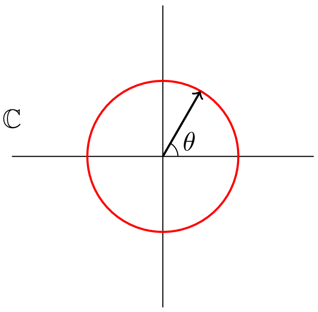
$$\arccos z = -i\ln(z + \sqrt{z^2-1}) \tag{3}$$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \tag{4}$$

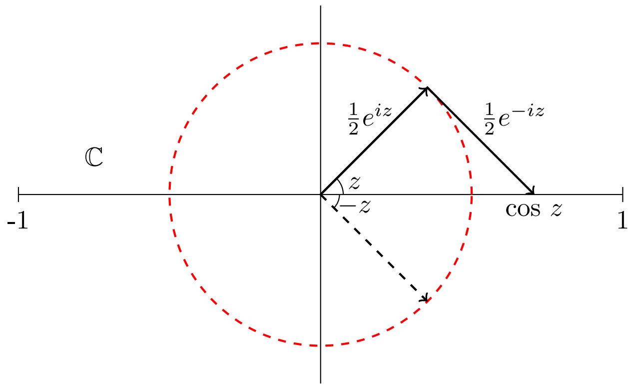
$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5}$$

Expand complex cosine term, dump negligible parts

In Euler's formula $e^{i\theta} = cis(\theta)$, if θ is *real*, then $e^{i\theta}$ traces out the unit circle in $\mathbb{C}$, with θ being the angle.



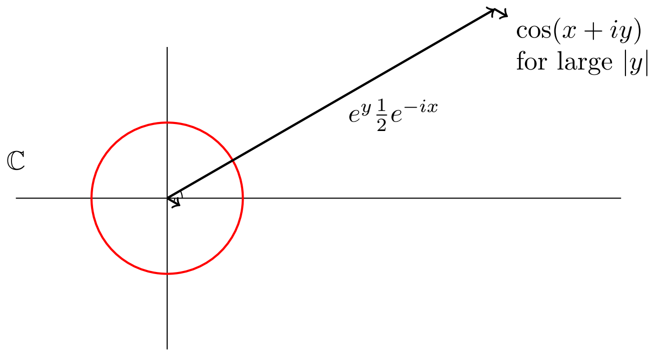
This gives insight into the $\cos z$ function. If z is real, then $\frac{1}{2}e^{iz}$ and $\frac{1}{2}e^{-iz}$ are two vectors of length $\frac{1}{2}$ that cycle in opposite directions, with z being the angle. Then $\cos z$ is the sum of the two vectors, which always ends on the real line between -1 and 1.



With this insight, it is useful to rewrite $\cos z$ as:

$$\cos(x+iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2} = e^y \frac{1}{2} e^{-ix} + e^{-y} \frac{1}{2} e^{ix} \quad (6)$$

Then it is clear that $\cos(x+iy)$ is the sum of two rotating vectors in $\mathbb{C}$ with amplitudes e^y and e^{-y}. A consequence is that for large y, e^y is *very* large, while e^{-y} is negligible, therefore $\cos(x+iy)$ is dominated by the vector $e^y \frac{1}{2} e^{-ix}$.



$$\text{Therefore} \quad \cos(x+iy) \to \frac{e^y e^{-ix}}{2} \quad \text{as} \quad y \to \infty \quad (7)$$

The complex cosine term in J. R. Philip's expression is:

$$\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} = \left\{\frac{\cos(\alpha(x+iy)}{\cos\alpha}\right\} = \left\{\frac{\cos(\alpha x + i\alpha y)}{\cos\alpha}\right\} \quad (8)$$

For any fixed real α, as $y \to \infty$,

$$\left\{\frac{\cos(\alpha x + i\alpha y)}{\cos\alpha}\right\} \to \left\{\frac{e^{\alpha y} e^{-i\alpha x}}{2\cos\alpha}\right\} \quad (9)$$

The magnitude of this complex number tends to infinity as $y \to \infty$.

Express Arccos as Logarithm

Using the identity $\arccos z = -i \ln(z + \sqrt{z^2 - 1})$, W_3 becomes:

$$W_3 = \Im\left[-i\alpha^{-1} \ln\left(\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} + \sqrt{\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2 - 1}\right) - \Theta\right] \quad (10)$$

As the magnitude of a complex number z tends to infinity, subtracting 1 from it has a negligible effect. Therefore, as $y \to \infty$, $\sqrt{z^2 - 1} \to z$, and furthermore,

$$\text{For} \quad z = x + iy, \quad \ln(z + \sqrt{z^2 - 1}) \to \ln(2z) \quad \text{as} \quad y \to \infty \quad (11)$$

Hence,

$$W_3 \to \Im\left[-i\alpha^{-1} \ln\left(2\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}\right) - \Theta\right] \quad \text{as} \quad y \to \infty \quad (12)$$

Substituting for the cos expression in the limit, we have

$$\lim_{y\to\infty} W_3 = \Im\left[-i\alpha^{-1}\ln\left(\frac{e^{\alpha y}e^{-i\alpha x}}{\cos\alpha}\right) - \Theta\right] \tag{13}$$

$$= \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha(y-ix)}\frac{1}{\cos\alpha}\right) - \Theta\right] \tag{14}$$

Apply the logarithm law $\log xy = \log x + \log y$, giving:

$$\lim_{y\to\infty} W_3 = \Im\left[-i\alpha^{-1}\left(\ln e^{\alpha(y-ix)} + \ln\left\{\frac{1}{\cos\alpha}\right\}\right) - \Theta\right] \tag{15}$$

Apply the log definition: $\ln e^z = z$:

$$\lim_{y\to\infty} W_3 = \Im\left[-i\alpha^{-1}\left(\alpha(y-ix) + \ln\left\{\frac{1}{\cos\alpha}\right\}\right) - \Theta\right] \tag{16}$$

$$= \Im\left[-i\alpha^{-1}\alpha(y-ix) - i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - \Theta\right] \tag{17}$$

$$= \Im\left[(i^2x - iy) - i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - \Theta\right] \tag{18}$$

$$= \Im\left[-(x+iy) - i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - \Theta\right] \tag{19}$$

$$= \Im\left[-\Theta - i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - \Theta\right] \tag{20}$$

$$= \Im\left[-2\Theta - i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\}\right] \tag{21}$$

$$= \Im\left[-2x - 2iy - i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\}\right] \tag{22}$$

$$= -2y - \alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} \tag{23}$$

The problem is the minus sign in $\arccos z = -i\ln(z + \sqrt{z^2-1})$. Without it, all is well.

UPDATE: The minus sign appears to be just a convention. Adopting the other convention fixes the problem.

Alternative Square Root Sign.

We have

$$\text{For} \quad z = x + iy, \quad \ln(z + \sqrt{z^2-1}) \to \ln(2z) \quad \text{as} \quad y \to \infty \tag{24}$$

but there is another possibility.

$$z + \sqrt{z^2 - 1} = z + \sqrt{z^2(1 - \frac{1}{z^2})} \tag{25}$$

$$\approx z + \sqrt{z^2(1 - \frac{1}{z^2} + \frac{1}{4z^4})} \tag{26}$$

$$= z + \sqrt{z^2(1 - \frac{1}{2z^2})^2} \tag{27}$$

$$= z \pm z(1 - \frac{1}{2z^2}) \tag{28}$$

$$\tag{29}$$

$$z + \sqrt{z^2 - 1} = z + z(1 - \frac{1}{2z^2}) = 2z - \frac{1}{2z^2} \tag{30}$$

$$\text{or} \tag{31}$$

$$z + \sqrt{z^2 - 1} = z - z(1 - \frac{1}{2z^2}) = z - z + \frac{1}{2z^2} = \frac{1}{2z^2} \tag{32}$$

Thus, we now try

$$\ln(\frac{1}{2z^2}) = \ln((\sqrt{2}z)^{-2}) = -2\ln(\sqrt{2}z) \tag{33}$$

Hence,

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} + \sqrt{\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2 - 1}\right) - \Theta\right] \tag{34}$$

is equal to:

$$W_3 = \Im\left[2i\alpha^{-1}\ln\left(\sqrt{2}\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}\right) - \Theta\right] \tag{35}$$

Substitute in the cosine expression:

$$W_3 = \Im\left[2i\alpha^{-1}\ln\left(\sqrt{2}\frac{e^{\alpha y}e^{-i\alpha x}}{2\cos\alpha}\right) - \Theta\right] \tag{36}$$

$$= \Im\left[2i\alpha^{-1}\ln\left(\sqrt{2}^{-1}e^{\alpha(y-ix)}\frac{1}{\cos\alpha}\right) - \Theta\right] \tag{37}$$

Apply logarithm law $\log xy = \log x + \log y$

$$W_3 = \Im\left[2i\alpha^{-1}\left(\ln(\sqrt{2}^{-1}) + \ln e^{\alpha(y-ix)} + \ln\left\{\frac{1}{\cos\alpha}\right\}\right) - \Theta\right] \tag{38}$$

Apply log definition:

$$W_3 = \Im\left[2i\alpha^{-1}\left(\ln(\sqrt{2}^{-1}) + \alpha(y - ix) + \ln\left\{\frac{1}{\cos\alpha}\right\}\right) - \Theta\right] \tag{39}$$

$$= \Im\left[2i\alpha^{-1}\ln(\sqrt{2}^{-1}) + 2i\alpha^{-1}\alpha(y - ix) + 2i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - \Theta\right] \tag{40}$$

$$= \Im\left[2i\alpha^{-1}\ln(\sqrt{2}^{-1}) + 2(x + iy) + 2i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - \Theta\right] \tag{41}$$

$$= \Im\left[2i\alpha^{-1}\ln(\sqrt{2}^{-1}) + 2\Theta + 2i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - \Theta\right] \tag{42}$$

$$= \Im\left[2i\alpha^{-1}\ln(\sqrt{2}^{-1}) + \Theta + 2i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\}\right] \tag{43}$$

$$= 2\alpha^{-1}\ln(\sqrt{2}^{-1}) + y + 2\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} \tag{44}$$

which is still wrong.

Complicated Old Version

Complex cos function can be expressed in terms of *real* cos and exponential functions.

$$\cos(x + iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}, \quad e^{a+ib} = e^a(\cos b + i\sin b) \tag{45}$$

!!!! Can simplify the following dramatically.... Add diagram...

$$\cos(x + iy) = \frac{e^{(ix-y)} + e^{(-ix+y)}}{2} = \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos -x + i\sin -x)}{2}$$

Converting the real cos and sin term back to complex $e^{i\theta}$:

$$\cos(x + iy) = \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2}$$

Now e^{ix} and e^{-ix} are unit vectors cycling around the unit circle (in opposite directions). The $e^{\pm y}$ coefficients give the length of the vectors. The cos function just adds them together, and halves the magnitude.

Therefore, at large y, a single huge vector $e^{y}e^{-ix}/2$ sweeps around the complex plane, with a negligible modifying vector $e^{-y}e^{ix}/2$ added to it.

$$\text{At large } y: \quad \cos(x + iy) \simeq \frac{e^{y}e^{-ix}}{2}$$

Similarly,

$$\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} = \left\{\frac{e^{\alpha y}e^{-i\alpha x} + e^{-\alpha y}e^{i\alpha x}}{2\cos\alpha}\right\} = e^{\alpha y}\left\{\frac{e^{-i\alpha x}}{2\cos\alpha}\right\} + e^{-\alpha y}\left\{\frac{e^{i\alpha x}}{2\cos\alpha}\right\}$$

Now, $\alpha = \frac{1}{2}\pi a/b$ and $0 < \alpha < \frac{1}{2}\pi$. The unit vectors $e^{\pm i\alpha x}$ just rotate faster or slower, depending on α — their magnitude remains unity.

Furthermore, as $\alpha \to 0$, $\cos\alpha \to 1$. But, as $\alpha \to \pi/2$, $\cos\alpha \to 0$, hence

$$\left\{\frac{e^{\pm i\alpha x}}{2\cos\alpha}\right\} \text{diverges as } \alpha \to \pi/2$$

However, on physical grounds a/b is limited to about 0.99 – probably well less. (i.e. at least 1% solid surface.) Thus, $\alpha \leq 0.99\pi/2$, so that $1/\cos\alpha < 100$.

$$\text{Therefore, magnitude of vectors } \left\{\frac{e^{\pm i\alpha x}}{2\cos\alpha}\right\} \text{is in range 1 to 100.}$$

Obviously, at very large y, $e^{\alpha y}$ is arbitrarily large, while $e^{-\alpha y}$ is negligible. Hence

$$\text{At large } y: \quad \left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} \simeq e^{\alpha y}\left\{\frac{e^{-i\alpha x}}{2\cos\alpha}\right\} = e^{\alpha y}\{\beta\}$$

Strategy: Convert complex arccos to log form

Using the identity $\arccos z = -i\ln(z + \sqrt{z^2-1})$, W_3 becomes:

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} + \sqrt{\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2 - 1}\right) - \Theta\right]$$

Which is:

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}\{\beta\} + \sqrt{\left[e^{\alpha y}\{\beta\}\right]^2 - 1}\right) - \Theta\right]$$

Now, at large y, $\left[e^{\alpha y}\{\beta\}\right]^2$ is arbitrarily large, so $\sqrt{\left[e^{\alpha y}\{\beta\}\right]^2 - 1}$ is arbitrarily close to $e^{\alpha y}\{\beta\}$. Hence

$$W_3 \cong \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}\{\beta\} + e^{\alpha y}\{\beta\}\right) - \Theta\right]$$

$$= \Im\left[-i\alpha^{-1}\ln\left(2e^{\alpha y}\left\{\frac{e^{-i\alpha x}}{2\cos\alpha}\right\}\right) - \Theta\right]$$

$$= \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha(y-ix)}\left\{\frac{1}{\cos\alpha}\right\}\right) - \Theta\right]$$

Recall the logarithm product law: $\ln xy = \ln x + \ln y$. Thus,

$$W_3 \cong \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha(y-ix)}\right) - i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - \Theta\right]$$

Now recall the log definition: $\ln e^x = x$. Thus,

$$-i\alpha^{-1}\ln\left(e^{\alpha(y-ix)}\right) = -i\alpha^{-1}\alpha(y-ix) = -iy + i^2x = x + iy = \Theta$$

Then

$$W_3 \cong \Im\left[\Theta - i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - \Theta\right] = \Im\left[-i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\}\right]$$

Finally,

$$\lim_{y\to\infty} W_3 = -\alpha^{-1}\ln\sec\alpha$$

But what about that flippin' minus sign?

Transverse Slots

J. Philip also studied flow over *transverse* no-shear slots. He was only able to use Stokes flow, not Navier-Stokes flow. He uses the Stokes stream function.

The Couette flow solution is:

$$\psi_1 = \frac{1}{2}\tau_\infty y^2/\mu$$

The solution for 'Shear Stokes Flow over a Plate with a Regular Array of Transverse No-Shear Slots' will be of the form:

$$\psi_3 = \psi_1 + a\tau_\infty \Psi_3/\mu$$

The solution of Ψ_3 is:

$$\Psi_3 = \frac{1}{2}YW_3$$

So:

$$\psi_3 = \frac{1}{2}\tau_\infty y^2/\mu + a\tau_\infty \frac{1}{2}YW_3/\mu$$

Now, Y is the nondimensionalized y/a. So:

$$\psi_3 = \frac{\tau_\infty}{\mu}\left(\frac{1}{2}y^2 + \frac{1}{2}yW_3\right)$$

It is **very tempting** to say that x-velocity u is given by

$$u = \frac{\partial \psi}{\partial y}$$

Hence:

$$u = \frac{\tau_\infty}{\mu}\left(y + \frac{1}{2}W_3\right)$$

So that the slip length is:

$$b_{\text{eff}} = \frac{1}{2}W_3$$

which in the limit $y \to \infty$ is

$$b_{\text{eff}} = \frac{1}{\pi}\frac{b}{a}\ln\sec\frac{\pi}{2}\frac{a}{b}$$

But **unfortunately**, W_3 is a function of y, so must be differentiated *before* taking the limit.

So in reality:

$$u = \frac{\tau_\infty}{\mu}\left(y + \frac{1}{2}W_3 + \frac{1}{2}y\frac{\partial W_3}{\partial y}\right)$$

Alrighty,

$$W_3 = \Im\left[\alpha^{-1}\cos^{-1}\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} - \Theta\right]$$

$$\frac{\partial}{\partial y}W_3 = \Im\left[\alpha^{-1}\frac{\partial}{\partial y}\cos^{-1}\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} - \frac{\partial}{\partial y}\Theta\right]$$

$$\frac{\partial}{\partial y}\Theta = \frac{\partial}{\partial y}(x + \imath y)/a = \imath/a$$

$$\frac{\partial}{\partial y}\cos^{-1}\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} = \frac{\partial}{\partial u}\cos^{-1}(u)\frac{\partial}{\partial y}\frac{\cos(\alpha\Theta)}{\cos\alpha}$$

$$= \frac{-1}{\sqrt{1-u^2}}\frac{\partial_t \cos(t)\alpha\partial_y\Theta}{\cos\alpha}$$

$$\frac{-1}{\sqrt{1-u^2}}\frac{-\sin(t)\imath\alpha/a}{\cos\alpha}$$

$$= \frac{-1}{\sqrt{1-\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2}}\frac{-\imath\alpha\sin(\alpha\Theta)}{a\cos\alpha}$$

Whew! Finally:

$$\frac{\partial}{\partial y}W_3 = \Im\left[\frac{\imath\sin(\alpha\Theta)}{a\cos\alpha\sqrt{1-\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2}} - \frac{\imath}{a}\right]$$

What can you do with that?

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Old rough working

So we can reason as follows:

$$\text{At large } y: \quad \left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} \simeq \left\{\frac{e^{\alpha y}e^{-i\alpha x}}{2\cos\alpha}\right\}$$

$$\text{Obviously, } \lim_{y\to\infty} \cos(x+iy) = \infty$$

in the sense of infinity in any direction.

But, we can say that as y gets very large, $\cos(x+iy)$ behaves like:

$$\frac{e^y(\cos -x + i\sin -x)}{2} = \frac{e^y(\cos x - i\sin x)}{2}$$

$$\text{So} \quad \left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} \text{ behaves like } \left\{\frac{e^{\alpha y}(\cos\alpha x - i\sin\alpha x)}{2\cos\alpha}\right\}$$

Therefore, for arbitrarily large y, define:

$$e^{\alpha y}\left\{\frac{\cos\alpha x - i\sin\alpha x}{2\cos\alpha}\right\} = e^{\alpha y}\{\beta\}$$

Where β is a complex number with magnitude on the order of 1.

Then $\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2$ behaves like $e^{2\alpha y}\{\beta\}^2$

Sooooo....

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} + \sqrt{\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\}^2 - 1}\right) - \Theta\right]$$

at large y behaves like:

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}\{\beta\} + \sqrt{(e^{\alpha y})^2\{\beta\}^2 - 1}\right) - \Theta\right]$$

Now, $e^{\alpha y}$ is humungous, so $\sqrt{(e^{\alpha y})^2\{\beta\}^2 - 1}$ is very, very close to $\sqrt{(e^{\alpha y})^2\{\beta\}^2}$ which is $e^{\alpha y}\{\beta\}$. Hence

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}\{\beta\} + e^{\alpha y}\{\beta\}\right) - \Theta\right]$$

$$= \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}2\{\beta\}\right) - \Theta\right]$$

$$= \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}2\left\{\frac{\cos\alpha x - i\sin\alpha x}{2\cos\alpha}\right\}\right) - \Theta\right]$$

$$= \Im\left[-i\alpha^{-1}\ln\left(e^{\alpha y}(\cos\alpha x - i\sin\alpha x)\left\{\frac{1}{\cos\alpha}\right\}\right) - \Theta\right]$$

Now $e^{\alpha y}(\cos\alpha x - i\sin\alpha x) = e^{\alpha y}(\cos(-\alpha x) + i\sin(-\alpha x))$
which is $e^{\alpha y - i\alpha x} = e^{\alpha(y-ix)}$
Thus

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left(e^{-\alpha(y-ix)}\left\{\frac{1}{\cos\alpha}\right\}\right) - \Theta\right]$$

All good. Now recall the log law: $\ln xy = \ln x + \ln y$. Thus

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} - i\alpha^{-1}\ln\left(e^{-\alpha(y-ix)}\right) - \Theta\right]$$

Further recall that $\ln e^x = x$. So that

$$\ln\left(e^{-\alpha(y-ix)}\right) = -\alpha(y-ix)$$

And so the third term simplifies to Θ:

$$-i\alpha^{-1}\ln\left(e^{-\alpha(y-ix)}\right) = -i\alpha^{-1}(-\alpha(y-ix))$$

$$= iy - i^2x = x + iy = \Theta$$

Hence

$$W_3 = \Im\left[-i\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\} + \Theta - \Theta\right]$$

$$W_3 = -\alpha^{-1}\ln\left\{\frac{1}{\cos\alpha}\right\}$$