Chapter 1

Periodic Functions Weakly Converge To Their Mean

In this Appendix we prove that periodic functions weakly converge to their mean, a fact we use in Chapter 6. To keep the Appendix self-contained, we start by defining weak convergence, using the definitions duplicated in Chapter 6.

1.1 Weak Convergence

Consider a sequence of functions defined by:

$$f_n = \sin(nx) \tag{1.1}$$

the first three functions of which appear in Figure (1.1).

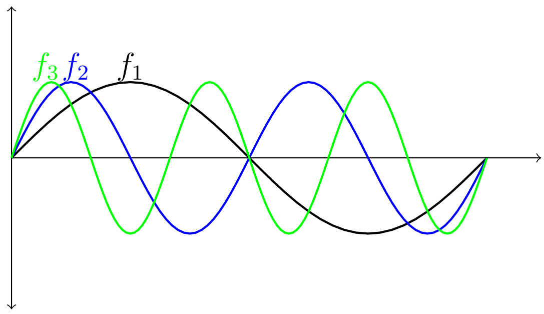

Figure 1.1: The first three functions in the sequence $\sin(nx)$.

As n increases, the period of the sine wave gets smaller and smaller, but the amplitude is unchanged. In the limit as $n \to \infty$, the waveform gets infinitely 'spiky'. What does the sequence converge to? There is no intuitive sense of the sinewave sequence getting 'closer to' some limit function. In fact, the sequence does not strongly converge.

However, there is a sense in which the function sequence converges.

We multiply each function in the sequence by an arbitrary test function g, and integrate, thus creating a sequence of integrals:

$$\int g f_n \, dx \tag{1.2}$$

If the sequence of integrals (strongly) converges to a limit integral:

$$\int g f_n \, dx \to \int g f \, dx \tag{1.3}$$

then we say that f_n **weakly converges** to f, and the 'limit function' f appearing in the limit integral is known as the **weak limit.** This is also written:

$$f_n \rightharpoonup f \tag{1.4}$$

1.1.1 Periodic Functions Weakly Converge to their Mean

It is a 'standard result' that periodic functions weakly converge to their mean. In a 2002 paper [1], Lukkassen and Wall state: "We have not found proofs of [this] fact in the literature. The aim of this paper is to present such proofs." Their paper provides a rigorous proof (and generalization) of this proof. Here, however, we present a simple intuitive proof, suitable for this thesis.

Consider our example of a sine wave sequence, together with an arbitrary test function g, integrated over the domain 0 to 2π. Each integral in the sequence is of the form:

$$\int_0^{2\pi} g(x)\sin(nx)\;dx \tag{1.5}$$

Over the domain 0 to 2π, the function $\sin(nx)$ has exactly n periods, each of width $2\pi/n$. We chop up the integral into n separate integrals, each with a subdomain of width $2\pi/n$.

$$\sum_{k=1}^{n}\int_{(k-1)\frac{2\pi}{n}}^{k\frac{2\pi}{n}} g(x)\sin(nx)\;dx \tag{1.6}$$

This is shown in Figure (1.2).

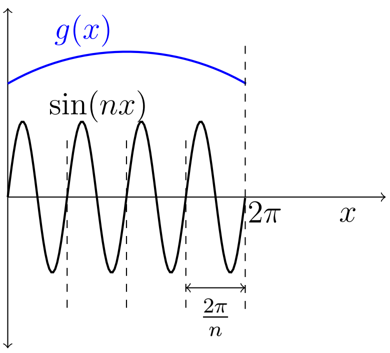

Figure 1.2: The periodic function $\sin(nx)$ and test function $g(x)$.

By a change of variable, we 'stretch' the domain so that in terms of the new variable, the period of the sine wave is again 2π. The domain is dilated by factor n and now has width $2\pi n$. With change of variable $x = t/n$, we have $dx = 1/n\, dt$. Pulling the Jacobian $1/n$ out of the sum, we have:

$$\frac{1}{n}\sum_{k=1}^{n}\int_{(k-1)2\pi}^{k2\pi} g\left(\frac{t}{n}\right)\sin(t)\, dt, \qquad x = \frac{t}{n}, \quad dx = \frac{1}{n}dt \tag{1.7}$$

Put another way, we move the n dependence from the sine function to the test function g. See Figure (1.3).

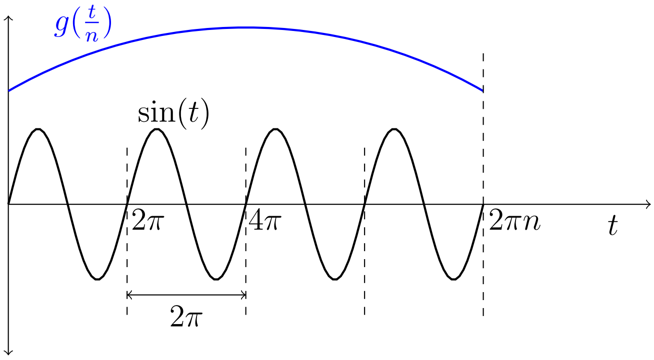

Figure 1.3: Change of variable dilates the domain.

We note that a period of $\sin(t)$ is the same for all k, so we use *only* the integral from 0 to 2π, and 'transport' the appropriate bit of $g(t)$ back to the interval 0 to 2π. This is accomplished by adding $(k-1)(2\pi/n)$ to the argument of $g(t/n)$. For clarity, we shall change variables again, $t \to \xi$, to highlight the fact that while the domain of t is the interval 0 to $2\pi n$, the domain of ξ is only the interval 0 to 2π.

$$\frac{1}{n}\sum_{k=1}^{n}\int_{0}^{2\pi} g\left(\frac{\xi}{n} + (k-1)\frac{2\pi}{n}\right)\sin(\xi)\, d\xi, \qquad x = \frac{\xi}{n} \tag{1.8}$$

The reduced domain is shown in Figure (1.4).

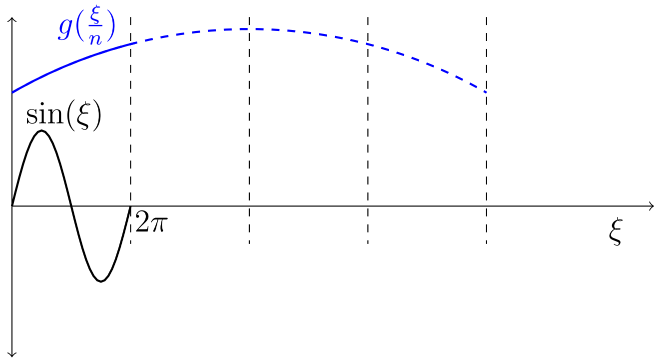

Figure 1.4: Reduced domain with g parameterised by k.

So we have:

$$\frac{1}{n}\int_0^{2\pi} g\left(\frac{\xi}{n}\right)\sin(\xi)\, d\xi + \frac{1}{n}\int_0^{2\pi} g\left(\frac{\xi+2\pi}{n}\right)\sin(\xi)\, d\xi + \cdots \tag{1.9}$$

The sum can go under a single integral sign, and the $\sin(\xi)$ common factor can be pulled out of the sum:

$$\frac{1}{n}\int_0^{2\pi} \sin(\xi)\sum_{k=1}^{n} g\left(\frac{\xi+(k-1)2\pi}{n}\right)\, d\xi \tag{1.10}$$

For later convenience, introduce a 2π and shift the $1/n$ factor:

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(\xi)\, \sum_{k=1}^{n} g\left(\frac{\xi+(k-1)2\pi}{n}\right)\frac{2\pi}{n}\, d\xi \tag{1.11}$$

What happens as $n \to \infty$? The summation term can be written:

$$\sum_{k=1}^{n} g\left((k-1)\frac{2\pi}{n}+\frac{\xi}{n}\right)\frac{2\pi}{n} \tag{1.12}$$

Since ξ is between 0 and 2π, the ξ/n term is between 0 and $2\pi/n$. As k ranges from 1 to n, the $g(k)$ term provides n ‘samples’ of the function at discrete points a distance $2\pi/n$ apart, with the starting point offset from 0 by the amount ξ/n. Each sample $g(k)$ is multiplied by the width of the inter-sample distance, giving n rectangles to sum up. In the limit $n \to \infty$, this is one definition of the Riemann integral of g over the interval 2π.

$$\lim_{n\to\infty}\sum_{k=1}^{n} g\left((k-1)\frac{2\pi}{n}+\frac{\xi}{n}\right)\frac{2\pi}{n}=\int_0^{2\pi} g(x)\;dx \tag{1.13}$$

The geometry of this Riemann integral is depicted in Figure (1.5).

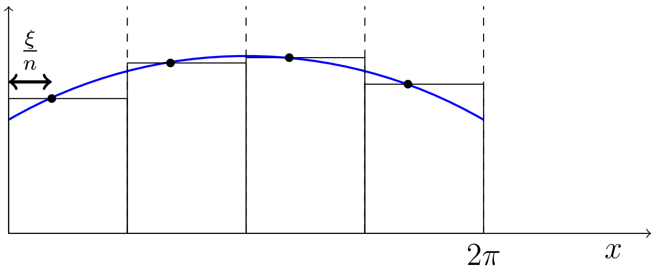

Figure 1.5: The geometry of the Riemann integral of g.

Thus we have:

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(\xi)\int_0^{2\pi} g(x)\;dx\;d\xi=\left(\frac{1}{2\pi}\int_0^{2\pi}\sin(\xi)\;d\xi\right)\left(\int_0^{2\pi} g(x)\;dx\right) \tag{1.14}$$

Now the integral with the sine function defines the **mean** of a function:

$$\langle\sin(x)\rangle=\frac{1}{2\pi}\int_0^{2\pi}\sin(\xi)\;d\xi \tag{1.15}$$

Therefore, we have shown that:

$$\lim_{n\to\infty}\int_0^{2\pi} g(x)\sin(nx)\;dx=\langle\sin(x)\rangle\int_0^{2\pi} g(x)\;dx \tag{1.16}$$

Or:

$$\int_0^{2\pi} g(x)\sin(nx)\;dx\to\int_0^{2\pi} g(x)\,\langle\sin(x)\rangle\;\;dx \tag{1.17}$$

The mean of $\sin(x)$ happens to be zero, so the limit vanishes. But the foregoing argument holds for *any* periodic function. Therefore we have shown that **periodic functions weakly converge to their mean:**

$$\int g f_n\;dx\to\int g\,\langle f\rangle\;\;dx \tag{1.18}$$

Bibliography

[1] Dag Lukkassen and Peter Wall. On weak convergence of locally periodic functions. *Journal of Nonlinear Mathematical Physics*, 9:45–57, 2002.