Chapter 1

Replicating John Philip 1972

The first known expression for an effective slip length appeared in 1972, in a paper in ZAMP by John R. Philip entitled "Flows Satisfying Mixed No-Slip and No-Shear Conditions" [1].

In the paper, John R. Philip says that the limit of

$$W_3 = \Im\left[\alpha^{-1}\cos^{-1}\left\{\frac{\cos(\alpha\Theta)}{\cos\alpha}\right\} - \Theta\right] \tag{1.1}$$

as $y \to \infty$ is

$$W_3 = \alpha^{-1}\ln\sec\alpha \tag{1.2}$$

Let us prove this forthwith.

$\Theta = x + iy$ is a complex number, α is real. Trig identities for *complex* cosine and exponential:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{1.3}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.4}$$

1.1 Expand cosine term, dump negligible parts

In Euler's formula $e^{i\theta} = cis(\theta)$, if θ is *real*, then $e^{i\theta}$ traces out the unit circle in $\mathbb{C}$, with θ being the angle.

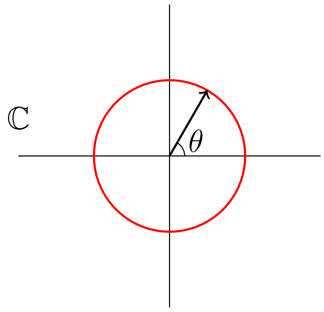

Figure 1.1: Euler's formula $e^{i\theta}$ for real θ .

This gives insight into the $\cos z$ function. If z is real, then $\frac{1}{2}e^{iz}$ and $\frac{1}{2}e^{-iz}$ are two vectors of length $\frac{1}{2}$ that cycle in opposite directions, with z being the angle. Then $\cos z$ is the sum of the two vectors, which always ends on the real line between -1 and 1, as shown in Figure (1.2).

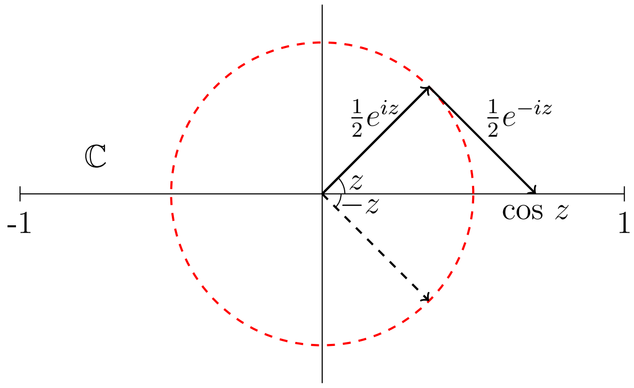

Figure 1.2: The complex cosine.

With this insight, it is useful to rewrite $\cos z$ as:

$$\cos(x+iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = e^{y}\frac{1}{2}e^{-ix} + e^{-y}\frac{1}{2}e^{ix} \tag{1.5}$$

Then it is clear that $\cos(x+iy)$ is the sum of two rotating vectors in $\mathbb{C}$ with amplitudes e^{y} and e^{-y}. A consequence is that for large y, e^{y} is *very* large, while e^{-y} is negligible, therefore $\cos(x+iy)$ is dominated by the vector $e^y\frac{1}{2}e^{-ix}$. See Figure (1.3).

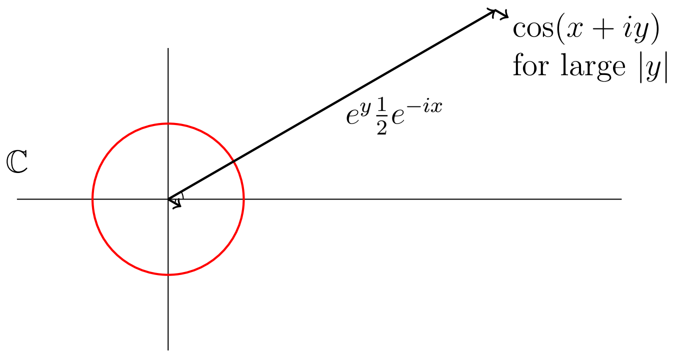

Figure 1.3: Complex cosine at large $|y|$.

$$\text{Therefore} \quad \cos(x+iy) \to \frac{e^y e^{-ix}}{2} \quad \text{as} \quad y \to \infty \tag{1.6}$$

$$\cos z \to \frac{1}{2}e^{-iz} \quad \text{as} \quad y \to \infty \tag{1.7}$$

1.2 Inverse Cosine at Large y

As $y \to \infty$:

$$w = \cos z \to \frac{1}{2}e^{-iz} \tag{1.8}$$

Solve $w = \cos z$ for z to get:

$$\arccos w = z$$

Likewise solve $w = \frac{1}{2}e^{-iz}$ for z:

$$w = \frac{1}{2}e^{-iz}$$
$$2w = e^{-iz}$$
$$\ln(2w) = -iz$$
$$i\ln(2w) = -i^2 z$$
$$i\ln(2w) = z$$

Equate the two expressions to obtain the inverse cosine in terms of a logarithm:

$$\arccos z = i \ln(2z) \tag{1.9}$$

1.3 Put into J. R. Philip's Expression

$$W_3 = \Im \left[\alpha^{-1} \cos^{-1} \left\{ \frac{\cos(\alpha\Theta)}{\cos\alpha} \right\} - \Theta \right] \tag{1.10}$$

As $y \to \infty$, the cosine expression may be substituted:

$$W_3 = \Im \left[\alpha^{-1} \cos^{-1} \left\{ \frac{\frac{1}{2} e^{-i\alpha\Theta}}{\cos\alpha} \right\} - \Theta \right] \tag{1.11}$$

And the inverse cosine expression may also be substituted:

$$W_3 = \Im \left[i\alpha^{-1} \ln \left\{ 2 \frac{\frac{1}{2} e^{-i\alpha\Theta}}{\cos\alpha} \right\} - \Theta \right] \tag{1.12}$$

$$W_3 = \Im \left[i\alpha^{-1} \ln \left\{ e^{-i\alpha\Theta} \frac{1}{\cos\alpha} \right\} - \Theta \right] \tag{1.13}$$

Recall that $\ln ab = \ln a + \ln b$.

$$W_3 = \Im \left[i\alpha^{-1} \ln \left\{ e^{-i\alpha\Theta} \right\} + i\alpha^{-1} \ln \left\{ \frac{1}{\cos\alpha} \right\} - \Theta \right] \tag{1.14}$$

Invoke definition of logarithm: $\ln e^z = z$.

$$W_3 = \Im \left[i\alpha^{-1} \left\{ -i\alpha\Theta \right\} + i\alpha^{-1} \ln \left\{ \frac{1}{\cos\alpha} \right\} - \Theta \right] \tag{1.15}$$

$$W_3 = \Im \left[\Theta + i\alpha^{-1} \ln \left\{ \frac{1}{\cos\alpha} \right\} - \Theta \right] \tag{1.16}$$

$$W_3 = \Im \left[i\alpha^{-1} \ln \left\{ \frac{1}{\cos\alpha} \right\} \right] \tag{1.17}$$

$$W_3 = \alpha^{-1} \ln \sec\alpha \tag{1.18}$$

We have demonstrated that which we set out to prove.

Bibliography

[1] John R. Philip. Flows satisfying mixed no-slip and no-shear conditions. *Journal of Applied Mathematics and Physics (ZAMP)*, 23:353, 1972.