

Chapter 1: Introduction

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0.1 Background

How fast ~~is~~ the water at the bottom of the canal? Is it moving or at rest? The first person to answer this question (indeed, perhaps the first to ask it; entertainment was limited in those days) was probably Daniel Bernoulli. He claimed: the water is at rest. This pronouncement was made in 1738, but it was not until the twentieth century that this notion ~~was~~ accepted as universally valid. This notion is of course the ‘no slip’ boundary condition of fluid mechanics.

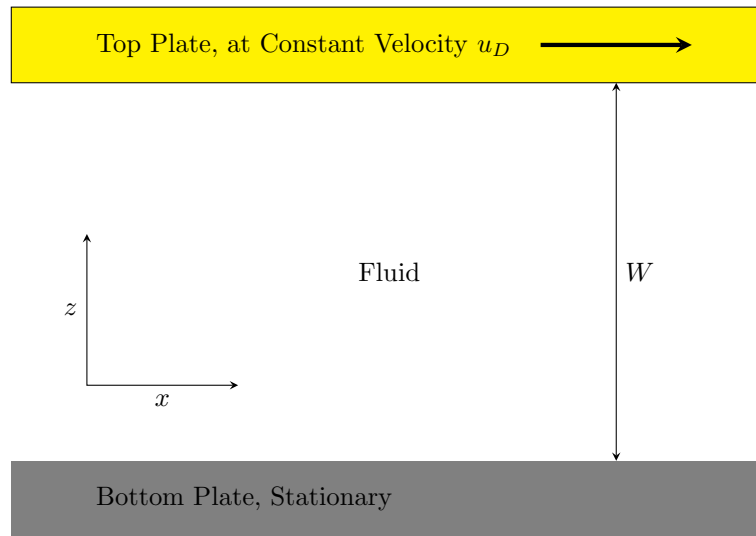
The foundations of fluid mechanics were laid by Navier and Stokes by about 1840. Their eponymous equation captured the fact that an element of fluid obeys Newton’s Second Law $F = ma$ just as surely as ‘solid’ matter. But the Navier-Stokes equations are not enough: While they describe behaviour in the *bulk* of the fluid, to generate a complete description of the fluid, we need information about what goes on at the *boundaries*.

Unlike the case of the bulk fluid obeying Newton’s universal laws of motion, there seemed to be no fundamental *a priori* principle at work on the boundary. However, there was a body of experimental work. Interestingly, no clear picture — let alone a clear principle — had emerged from the experiments. Bernoulli’s early result had later been contradicted by experiments showing *slip*. That is, the fluid in contact with the solid surface moved, or slipped, along the solid. By the 1840s, the issue had become sufficiently controversial that Stokes himself was commissioned to sort the matter out. At length, he determined that fluid did *not* slip along the solid surface. But the matter didn’t end there; for example, Helmholtz found evidence of slip in the 1860s. The issue wasn’t fully settled until 1927, when Tausz and Körosy published a book in which previous evidence of slip was dismissed as experimental artefact. An overview of this early history (and much more) is available in the excellent 2005 review article by Chiara Neto *et al* [1].

Thus, by the early twentieth century, the no slip boundary condition had become cemented in fluid mechanics textbooks. However, by the the twenty-first century, evidence was beginning to accumulate that the no slip condition sometimes does not hold. Before discussing this, it will help to introduce some mathematical machinery — specifically, the concept of *slip length*.

Slip Length

Let us consider what is surely the simplest case of fluid flow: Couette flow. In this canonical regime, some fluid sits between two parallel solid plates. One of the plates is moving at a constant velocity, with respect to the other plate. (For clarity we shall often assume that the bottom plate is stationary, and the top plate moves; another convention is that the two plates have equal and opposite velocities.) The plates are considered to be infinite planes; likewise, the interstitial fluid extends infinitely in all horizontal directions, so that there are no ‘edges’ to the system. We are interested in the flow velocity field of the fluid at ~~equilibrium~~.

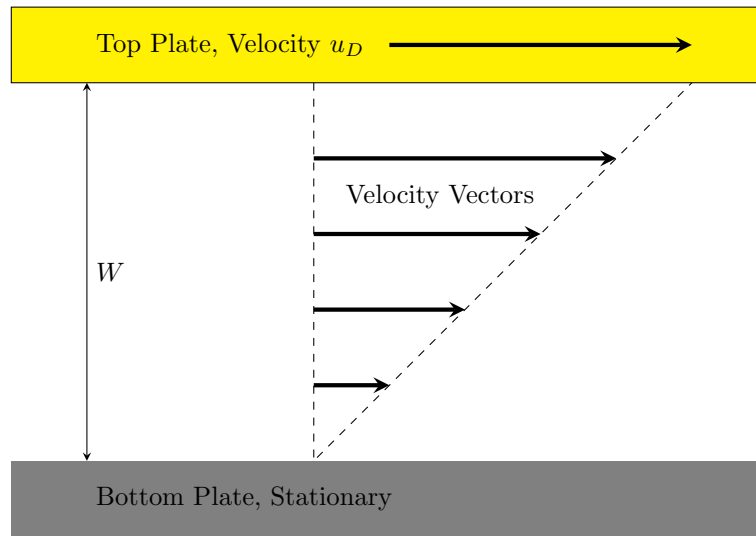


This setup is amenable to experimental measurement; the point of theoretical physics is to generate a prediction of the measurements using mathematical reasoning. To that end, we need a mathematical model of the physical situation. We work in \mathbb{R}^3 , with z being the vertical direction, and x the left-to-right horizontal direction. The fluid is modeled as a *continuum* velocity field $\vec{u}(x, y, z) = (u, v, w)$ obeying the Navier-Stokes equations at all points. The boundaries of the fluid – the two plates – are modeled as perfect x, y planes, separated by a width W . The bottom boundary is the plane $z = 0$, the top the plane $z = W$. The bottom boundary is stationary, while the top boundary has a constant velocity in the x direction of u_D .

To solve, a boundary condition is required. Let's start with the simple classical case of no slip. ~~The fluid sticks to the boundaries.~~ Thus, the fluid at the bottom boundary has zero velocity, while fluid at the top boundary has velocity ~~u_D~~ . The fluid is driven by the driving plate, and viscosity causes shear stress, accelerating fluid in the bulk. The solution at equilibrium is that the fluid velocity linearly ~~interpolates~~ between 0 at the bottom and u_D at the top:

$$u(z) = \frac{z}{W} u_D \quad (1)$$

Plotting the fluid velocity vectors shows the velocity gradient.



Now let's relax the no slip condition and consider some kind of *slip* boundary condition. The simplest such condition is that there is a finite slip velocity at the surface, and that velocity is proportional to the *shear rate* in the fluid. This condition was first proposed by Navier:

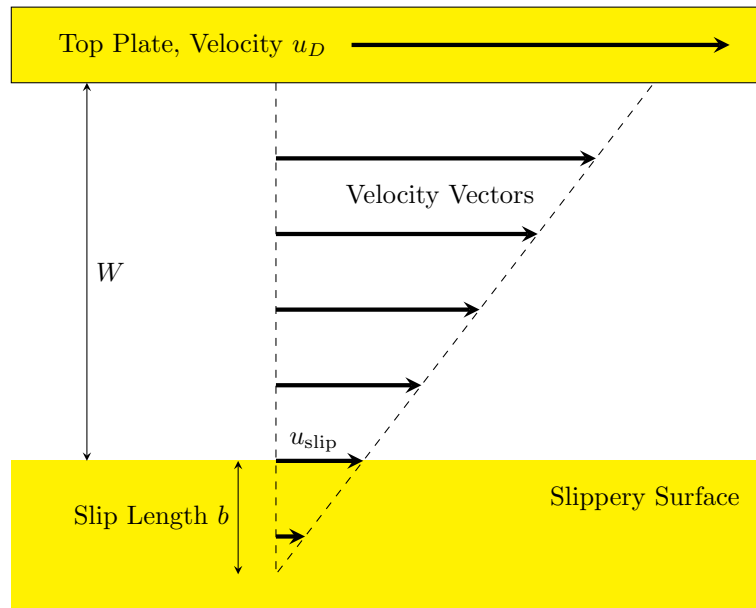
$$u_{\text{slip}} = b \frac{\partial u}{\partial z} \quad (2)$$

The constant of proportionality b , has units of length, and accordingly is known as the *slip length*.

In the model in use here, the slip length has a simple geometric interpretation: The velocity gradient at the wall may be linearly extrapolated *into* the wall. The distance into the wall at which the velocity *would* be zero, is the slip length.



4-1



At this point we stop to emphasize that the slip length is a feature of the *mathematical model*. We have assumed two things: First, the boundary between fluid and solid is a perfect Euclidean plane. Second, the fluid is a continuum, allowing for a well-defined velocity gradient at the boundary. Reconciling this Platonic ideal with a real physical experiment is non-trivial. We shall go into some depth with this issue shortly.

The slip length is a property of the fluid/solid interface, and can in principle vary in space. This leads us to the concept of *effective slip length*, the calculation of which is the subject of this thesis.

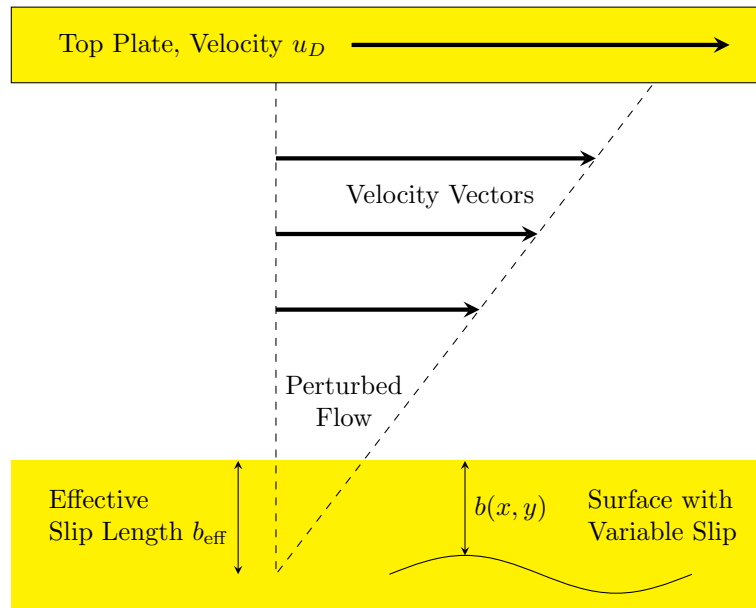
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Effective Slip Length



Working again with the same model, let us assume the slip length b is a parameter that varies over the solid surface. Again, we seek a velocity flow field description of the fluid. In this case, a simple solution is not readily apparent — the problem has essentially gained an extra dimension. In fact, there are no known analytical solutions to this general problem.

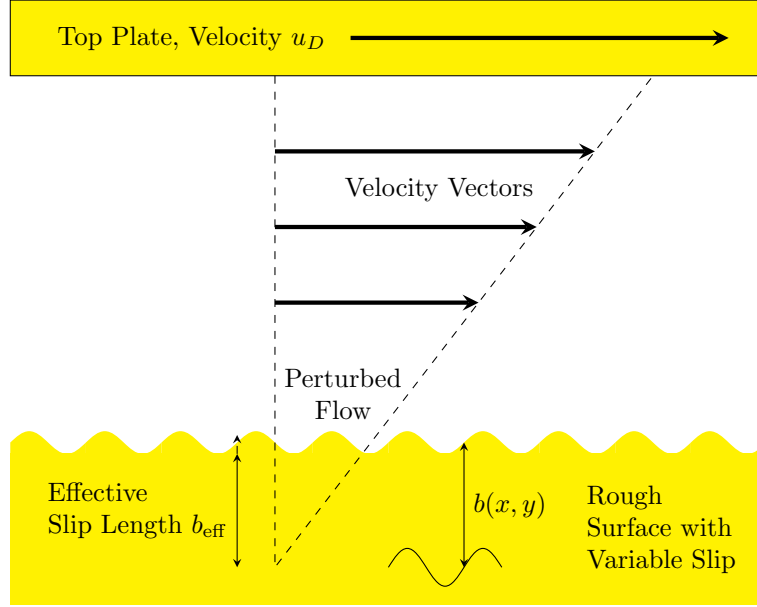
The flow field near the surface will no longer be a straightforward laminar flow, but will be perturbed by the spatial variations in slip length. However, we would expect those perturbations to decay with increasing height above the surface. At sufficient height, the perturbations will be essentially zero, and the flow will be uniform laminar flow. This uniform flow has a shear rate that does not vary over space. Therefore, the velocity gradient can be extrapolated down to the solid surface, and *into* the surface, defining an *effective* slip length.



Effective Slip Length of Rough Surface

We now extend the model, allowing the solid ~~slip~~ surface to be ~~non flat~~. ~~The surface is now rough, perhaps corrugated.~~ We allow the local slip length to vary over the surface, as before.

The definition of effective slip is the same as before, but with an important extension: The position of the solid/liquid interface is now a *range*. Therefore, the effective slip length can be defined with respect to any position in that range. We let the surface be a function $h(x, y)$, (nominally centered at $z = 0$), bounded by h_{\min} and h_{\max} . Define $\Delta h = h_{\max} - h_{\min}$. Then plausible definitions of b_{eff} lie in the range $[b_0, b_0 + \Delta h]$, where b_0 is the minimum *effective* slip length — not the minimum *local* slip length.



Purpose of Thesis

The purpose of this thesis is to present a theory with which to predict the effective slip length, given ~~the varying~~ local slip length:

$$b_{\text{eff}} = f(b(x, y)) \quad (3)$$

Furthermore, the theory can deal with a rough surface:

$$b_{\text{eff}} = f(b(x, y), h(x, y)) \quad (4)$$

We present a formula for b_{eff} , in the idealized case where surface roughness is periodic, and the local slip length varies with the same period, in the limiting case where the local slip length is always large compared to the period of surface roughness.

The answer turns out to be that b_{eff} is the area-weighted *harmonic* average of the local slip length, where the area in question is the area of contact between solid and liquid.

$$b_{\text{eff}} = \left\langle \frac{\sqrt{1 + h'^2}}{b} \right\rangle^{-1} \quad (5)$$

The core of this thesis is a rigorous derivation of this result.

We also confirm, using a completely different mathematical technique, the special case of a flat surface:

$$b_{\text{eff}} = \left\langle \frac{1}{b} \right\rangle^{-1} \quad (6)$$

8-1

And show that for a flat surface, in the opposite limiting case where the period is large compared to any local slip length, that b_{eff} is the area-weighted average of local slip lengths:

$$b_{\text{eff}} = \langle b \rangle \quad (7)$$

Application of Thesis

The question of whether or not fluid slips at the solid boundary is fundamental to fluid mechanics in the following sense: To solve the Navier Stokes equation, a boundary condition is needed, and the no-slip condition was accepted by consensus only after lengthy controversy. However, in almost all practical applications, plumbing, drainage, ship building etc., ~~any possible slip length is too small to matter~~. At the macro scale, the no-slip boundary condition is observed.

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8-4

Only recently has the possibility of fluid slip become important, in the new fields of microfluidics and nanofluidics. In these fields, researchers and engineers fabricate devices in which fluid flows through pipes with a width measured in microns, or even smaller. In such channels, the high surface area to volume ratio works strongly against fluid flow. If the no-slip condition could be relaxed, then flow rates could be increased significantly.

Hydrophobic channel walls give the possibility of a non-zero intrinsic slip length. However, larger slip effects are attempted by creating surfaces that contain regions of *air*, perhaps trapped in small pockets on the surface. Then the fluid flows partly over an air-water interface, which is expected to have a large intrinsic slip length.

The question is then *how much* would flow rates increase with these exotic surfaces. This can be rephrased as 'what is the effective slip length of the surface?' This thesis offers a means of calculating the effective slip length for some cases.

Structure of Thesis

The core of this thesis is essentially the shortest possible physically and mathematically rigorous derivation of the harmonic mean formula for b_{eff} .

Chapter 2 ‘Does Slip Exist?’ reviews the literature on slip, focussing at reasonable depth on the (few) papers that credibly demonstrate intrinsic slip on a clean, atomically flat surface. It ultimately concludes that an intrinsic slip length of 10 - 20 nm has been seen on hydrophobic surfaces.

Chapter 3 ‘Mixed Slip Flow’ looks at flow over surfaces that are rough, or have a varying intrinsic slip length. The focus is on ‘superhydrophobic’ surfaces and nanobubbles, and the importance of carefully locating the $z = 0$ plane.

Chapter 4 ‘Effective Slip Length Expressions: Prior Work’ is a literature review of previous results for an effective slip length. Since the work in this thesis has been published as three papers, the work is easily placed in context with other peer-reviewed articles.

Chapter 5 ‘The Mathematical Model’ sets up the formal mathematical framework, including the definition and explanation of the velocity gradient tensor, and the simplification to Stokes flow.

Chapter 6 ‘The Homogenized Effective Slip Length’ is the core of the thesis: The technique for solving partial differential equations known as homogenization is explained in some detail, and then applied to the effective slip problem. An expression for b_{eff} is derived for the regime where the period is much smaller than any local slip length.

Chapter 7 ‘Perturbative Effective Slip Lengths’ uses the completely different technique of perturbation methods to derive b_{eff} for the special case of a flat surface. Also derived is another b_{eff} that applies in the contrasting regime where the period is much larger than slip lengths.

Chapter 8 ‘Numerics’ shows that numerical simulations using finite element modelling are in excellent agreement with our expression for b_{eff} .

Chapter 9 ‘Analogues of Effective Slip’ takes a brief look at other physical problems that happen to be described by the same mathematical model of Chapter 5, and therefore have the same solution. A problem in thermal insulation, and a problem in catalysis are given a brief treatment.

Chapter 10 ‘Conclusion’ concludes.

Bibliography

- [1] Chiara Neto, Drew R Evans, Elmar Bonaccorso, Hans-Jurgen Butt, and Vincent S. J. Craig. Boundary slip in newtonian liquids: a review of experimental studies. *Reports on Progress in Physics*, 68:2859, 2005.

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 \vec{u}_D
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solve what?
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technically the velocity is the direction and the speed, so this should say "constant velocity of magnitude u_D in the x direction"
- 3-4 6/11/2013 7:16 am, Shaun Hendy
This is such a nice introduction that a brief explanation of what viscosity is seems appropriate e.g. "causes shear stress as layers of fluid atoms slides over one another" or something like that.
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would normally refer to this as steady-state (i.e. $du/dt=0$) rather than equilibrium
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changes
- 3-7 6/11/2013 7:16 am, Shaun Hendy
Need to number and caption all figures, and the formal thesis style is to not use pictures in sentences. So need to say "gradient is shown in figure 0.1."
- 4-1 6/11/2013 7:16 am, Shaun Hendy
seems like this would be a good place to add a one sentence physical interpretation of this. That is the shear stress on the fluid at the boundary is proportional to the fluid velocity at the boundary?
- 5-1 6/11/2013 7:16 am, Shaun Hendy
A brief physical example here (e.g. superhydrophobic surfaces or teflon or something) would fit well here in your exposition of the problem.
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rough rather than just flat.

6-2 6/11/2013 7:16 am, Shaun Hendy

I don't like this. It presupposes that the slip length is defined by your geometric construction, which is not very physical. I would rather think about how you define an *effective* slip length when there is no planar surface to define it on. How about thinking of it in terms of an effective surface plane?

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Oh dear. This thesis obviously not as brief as it could be if you were to eliminate the tautologies.