Solution

We notice that the probability that there exists at least one such path in an infinite binary tree can be expressed as $1 - \text{P(Complementary Event)} = 1 - \text{P(Every path in the tree sums up to at least 2)}$.
To simplify later expressions, we now define the following function:

$P : \mathbb{N} \to [0, 1]$,
$P(n) :=$ Probability that every path of an infinite complete binary tree sums up to at least n.

We then notice that when considering a complete infinite binary tree, we can make the following case distinction when trying to express the probability that every path in the tree sums up to at least 2:

Case 1: The root node has value 1

In this case it suffices to know the probability that every path in both subtrees of the root sum up to at least 1, since that would mean that every path of the entire tree would sum up to at least 2. We express this probability as

$$\begin{aligned} &(1-p) \cdot \text{P(Every path of both subtrees sums up to at least1)} \\ &= (1-p) \cdot P(1) \cdot P(1) \\ &= (1-p) \cdot P(1)^2. \end{aligned} \tag{1}$$

Note that a subtree of a complete infinite binary tree is a complete and infinite binary tree. Which is why we can use P in the context of the two subtrees.

Case 2: The root node has value 0

In this case we need to know the probability that every path in each of the subtrees of root sums up to at least 2, since the root node did not contribute anything to this sum. Notice that we are then asking the same question as in the beginning, but with regard to the two subtrees of root, rather than the entire tree. The corresponding probability can be expressed as

$$\begin{aligned} &p \cdot \text{P(Every path of both subtrees sums up to at least 2)} \\ &= p \cdot P(2) \cdot P(2) \\ &= p \cdot P(2)^2. \end{aligned} \tag{2}$$

Having considered these two cases, we now have a recursive expression for the probability that every path in a given complete infinite binary tree sums up to at least 2, namely the sum of (1) and (2):

$$P(2) = (1-p) \cdot P(1)^2 + p \cdot P(2)^2 \tag{3}$$

To make use of this expression, we also need to find a formula for $P(1)$. Using a recursive approach we get

$$\begin{aligned} P(1) &= (1-p) + pP(1)^2. \\ \implies 0 &= pP(1)^2 - P(1) + (1-p). \\ \implies P(1) &= \frac{1 \pm |1-2p|}{p}. \\ \implies P(1) &= \begin{cases} 1, & \text{if } p \le \frac{1}{2} \\ \frac{1-p}{p}, & \text{otherwise} \end{cases} \end{aligned}$$

As assuming $p \le \frac{1}{2}$ leads to a contradiction[1] we get

$$P(1) = \frac{1-p}{p} \tag{4}$$

[1] Assume $p \le \frac{1}{2}$. We get $P(1) = 1$. Substituting this into (3) and fixing $P(2) = \frac{1}{2}$ we get $p = \frac{2}{3}$, which is a contradiction to $p \le \frac{1}{2}$.

Substituting (4) into (3) and fixing $P(2) = \frac{1}{2}$ we get

$$\frac{1}{2} = (1-p) \cdot \left(\frac{1-p}{p}\right)^2 + \frac{p}{4}. \tag{5}$$

Copy this into WolframAlpha and we get $p \approx 0.5306035754$, which is the solution to this puzzle.