Getting to Orbit

1.1 Introduction

Getting a spacecraft to orbit is difficult. Rocket science is one of those things that seems simple at first, and it is. It's not complicated, but it is very difficult. There are several things that have to happen, in a particular order, without failing and without human interaction in order to get a mass to orbit. Making sure that all happens correctly is a matter of good engineering and some trial and error. There is a lot we can learn from physics as to what is involved to get to orbit. This paper explores the physics in a ground up, step-by-step fashion to get to orbit.

1.2 Terms

Lets start be defining terms. There is a glossary at the end, but I prefer to get some of the more confusing and important terms defined first. LEO is low earth orbit. Mass Fraction is a byproduct of the rocket equation. It comes from the fact that the final velocity of a rocket is proportional to the fraction of the vehicle weight that is burned as fuel. I_{sp} is the specific impulse. Another term from the rocket equation that shows up a lot. It's the efficiency of a rocket motor.

2.1 Getting Started

A great place to start is to find an analytical solution for a rocket and solve for the height it will travel. Put everything in the 'up' axis and ignoring a ton of things and making some assumptions will result in an easy place to start. Then we can add in more complicated terms. At some point it will be impossible to make an analytical solution anymore (when you start getting into complex air resistance models) and a numerical approach will be required. At this point the analysis techniques will change a little but a lot will still apply.

2.2 The Simplest Case

The simplest case is a one dimensional ballistic trajectory, all in the z direction with constant gravity and no air resistance. Figure 2.1 shows the forces on the object. Finding an equation of motion is as simple as adding all the forces, which in this case is just gravity.

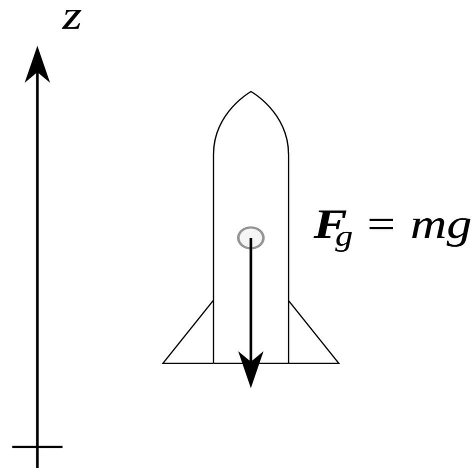

Figure 2.1 Force on a simple rocket

We apply Newtons 2nd law and we get:

$$\boldsymbol{F}=m\boldsymbol{a}=-mg \quad \textbf{(2.1)}$$

$$ma=-mg \quad \textbf{(2.2)}$$

$$a=-g \quad \textbf{(2.3)}$$

Note that we take g to be an acceleration in the negative direction. Integrating once we get a function for the velocity of the projectile.

$$v=-gt+v_0 \quad \textbf{(2.4)}$$

Integrating twice will get the height.

$$h=-\frac{1}{2}gt^2+v_0t+z_0 \quad \textbf{(2.5)}$$

We can solve for the time it takes because we know it will be at it's apogee when the velocity reaches 0.

$$0=-gt+v_0 \quad \textbf{(2.6)}$$

$$t=\frac{v_0}{g} \quad \textbf{(2.7)}$$

Substituting t into equation 2.5 we get

$$h=-\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 v_0\left(\frac{v_0}{g}\right)+z_0 \quad \textbf{(2.8)}$$

$$\boxed{h=\frac{1}{2}\frac{v_0^2}{g}+z_0} \quad \textbf{(2.9)}$$

2.3 Simplest Case with Energy

In this case because we are dealing with conservative forces and simple motion so we can work with energy. This should give us the same result as the equation 2.9 so this is a good test for our earlier work. In this case we have a projectile with some initial kinetic energy (KE) in a potential field. Conservation of energy will apply and the KE will be turned into potential energy (PE).

$$KE=PE \quad \textbf{(2.10)}$$

KE is the same as it always is and PE is a simple function of height.

$$\frac{1}{2}m v_0^2=m g \Delta h \quad \textbf{(2.11)}$$

Solving for Δh gives

$$\Delta h=\frac{1}{2}\frac{v_0^2}{g} \quad \textbf{(2.12)}$$

Height in potential energy is always relative so if we want to find the final height we start at some z_0 and add the change in height we get equation 2.9 again.

2.4 Adding Thrust

At this point we only have a projectile and not a rocket. Rockets move with thrust. Thrust can be thought of as the application of a more generic form or Newtons 2nd law

$$\boldsymbol{F} = \dot{\boldsymbol{p}} \tag{2.13}$$

$$F = \frac{dp}{dt} \tag{2.14}$$

$$F\, dt = dp \tag{2.15}$$

Thrust is the force resulting from a change in momentum. A rocket changes it's momentum by pushing a small amount of mass at some velocity out one end (see figure 2.2). So the new momentum of a rocket is it change in velocity times it's new mass plus the momentum of the exhaust. Here we introduce the speed of the exhaust gasses as u . We assume this is constant for the flight of the rocket. This turns out to be a pretty good assumption. The burn profiles of larger rockets tend to be relatively flat. That is to say that once the motor gets started it burns as a relatively constant rate.

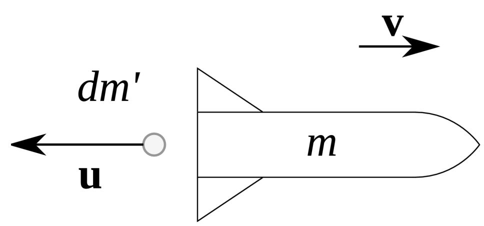

Figure 2.2 Thrust and momentum

$$dp = p(t+dt) - p(t) \tag{2.16}$$

$$dp = (m - d\acute{m})(v+dv) + d\acute{m}(v-u) - mv \tag{2.17}$$

Note $(m - d\acute{m})$ is the new mass of the rocket. $(v+dv)$ is the new velocity. mv is the new momentum, $p(t)$ and $d\acute{m}(v-u)$ is the exhaust gas momentum. Simplifying give us

$$dp = mdv - u\, d\acute{m} \tag{2.18}$$

We can relate the exhaust mass d acute m to the change in rocket mass as the same thing but with an opposite sign (*i.e.*, the rocket looses mass).

$$d\acute{m} = -dm \tag{2.19}$$

$$dp = mdv + udm \tag{2.20}$$

In free space a rocket there are no external forces so $dp = 0$. Solving for dv gives

$$dv = -u \frac{dm}{m} \tag{2.21}$$

Integrating over a change in mass gives us the famous rocket equation

$$\int_{v_0}^{v} dv = -u \int_{m_0}^{m_f} \frac{dm}{m} \tag{2.22}$$

$$v - v_0 = u \ln\left(\frac{m_0}{m_f}\right) \tag{2.23}$$

$$v = u \ln\left(\frac{m_0}{m_f}\right) + v_0 \tag{2.24}$$

The ratio of m_0 to m_f is called the mass fraction. U is related to I_{sp} by

$$u = I_{sp} \cdot g_0 \tag{2.25}$$

So you can write equation 2.24 in terms of the slightly more useful I_{sp} instead of u

$$v = g_0 I_{sp} \ln\left(\frac{m_0}{m_f}\right) + v_0 \tag{2.26}$$

Equation 2.26 is valid for rockets in free space. But we have gravity to deal with.

$$F = -mg = \frac{dp}{dt} \tag{2.27}$$

$$-mg\,dt = dp \tag{2.28}$$

$$-mg\,dt = mdv + udm \tag{2.29}$$

$$-mg = m \frac{dv}{dt} + u \frac{dm}{dt} \tag{2.30}$$

$$-mg = m\dot{v} + u\dot{m} \tag{2.31}$$

Here it becomes useful to consider the burn rate constant. So we introduce α in place of $\dot{m}$.

$$\dot{m} \equiv -\alpha \tag{2.32}$$

$$-mg = m\dot{v} - u\alpha \tag{2.33}$$

Now we can rearrange and solve for the acceleration.

$$-g = \frac{dv}{dt} - \frac{u\alpha}{m} \tag{2.34}$$

$$dv = \left(-g + \frac{u\alpha}{m}\right) dt \tag{2.35}$$

Since we know α is just $\dot{m}$ we can end up substituting dt for dm, since we don't care how long it takes to burn the fuel, only how much is burnt.

$$\frac{dm}{dt} = -\alpha \tag{2.36}$$

$$dt = \frac{-dm}{\alpha} \tag{2.37}$$

$$dv = \left(-g + \frac{u\alpha}{m}\right) - \frac{dm}{\alpha} \tag{2.38}$$

$$dv = \left(\frac{g}{\alpha} - \frac{u}{m}\right) dm \tag{2.39}$$

Integrating once gives us the velocity at burnout of the rocket

$$\int_0^{v_{bo}} dv = \frac{g}{\alpha} \int_{m_0}^{m_f} dm - u \int_{m_0}^{m_f} \frac{dm}{m} \tag{2.40}$$

$$\boxed{v_{bo} = \frac{-g}{\alpha}(m_0 - m_f) + u \ln\left(\frac{m_0}{m_f}\right)} \tag{2.41}$$

Once again we see mass fraction enter the equation. But again we see u and now α which are less convenient units. It would be nicer to see it in terms of fuel mass, burn time and I_{sp}.

$$m_{fuel} \equiv m_0 - m_f \tag{2.42}$$

$$m_{rocket} \equiv m_f \tag{2.43}$$

$$\alpha = \frac{m_{fuel}}{t_{bo}} \tag{2.44}$$

$$\boxed{v_{bo} = -g\,t_{bo} + g_0 I_{sp} \ln\left(1 + \frac{m_{fuel}}{m_{rocket}}\right)} \tag{2.45}$$

To find the height at burnout we have to integrate again.