Math 402/502 Homework 1 – due Friday, January 24

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(1) Let variables range over the natural numbers $\mathbb{N}$, and let $P(n)$ be the relation that n is prime, $L(n, m)$ the relation that $n < m$, and $E(n)$ the relation that n is even.

(a) Express the following symbolic statement as an English sentence:

$$\forall n \exists m (L(n, m) \wedge P(m))$$

For all natural numbers n, there exists a natural number m such that n is less than m and m is prime. Alternatively, there always exists a prime natural number greater than any given natural number.

(b) Express the following English sentence as a symbolic statement. Do not use the $\exists!$ abbreviation.
"There is a unique natural number n which is both even and prime."

$$\exists n ((E(n) \wedge P(n)) \wedge (\forall m (E(m) \wedge P(m)) \implies m = n))$$

(2) Let $\oplus$ be a new connective representing exclusive or (XOR), so that $P \oplus Q$ is true precisely when exactly one of P and Q is true. Write a truth table for $\oplus$.

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

(3) Consider the following three statements, where P, Q, and R are three predicates:

- **Statement 1:** $(P \to Q) \to R$
- **Statement 2:** $P \to (Q \to R)$
- **Statement 3:** $(P \wedge Q) \to R$

(a) Show Statement 2 is logically equivalent to Statement 3.

P	Q	R	$P \to (Q \to R)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

P	Q	R	$(P \wedge Q) \to R$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

The above truth tables show that the Statement 2 & 3 are equivalent as for each permuation of P, Q, and R the two statements have the same truth value.

(b) Show Statement 1 is not logically equivalent to Statement 2.

P	Q	R	$(P \to Q) \to R$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

P	Q	R	$P \to (Q \to R)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

The above truth tables show that the Statement 1 & 2 are not equivalent as there are some permuations of P, Q, and R where the two statements have different truth values.

(4) The symmetric difference of two sets, $X \Delta Y$, is defined to consist of those sets which are elements of exactly one of the two sets.

Write a formula in the language of set theory which expresses the statement $Z = X \Delta Y$.

$$Z = (m \in X : m \notin Y) \wedge (m \in Y : m \notin X)$$

or

$$Z = X \cup Y - X \cap Y$$