# Analytical Tools for Second Order Systems

A. RLC circuit example

Recall that the Laplace transform of the output $V_o(s)$ in terms of the circuit's system function $H(s)$ and the Laplace transform of the input $V_i(s)$ is given by:

$$V_o(s) = H(s)V_i(s) = \frac{2}{s^2+3s+2}\left(\frac{1}{s}\right)$$

Then we did a partial fraction expansion (pfe) to express $V_o(s)$ as a sum of terms, to each of which we can apply the inverse Laplace transform, using the Table Lookup method:

$$V_o(s) = \frac{2}{(s+2)(s+1)}\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} .^1$$
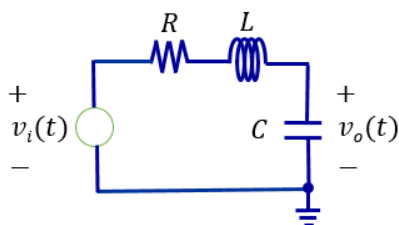
Solving for the constants gave: $V_o(s) = \frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}$, where $A=1$, $B=1$, $C=2$. For the RLC circuit with component values $R=3\Omega$, $L=1H$, $C=0.5F$, we found the circuit was damped by solving the equation $2\zeta\omega_n = \frac{R}{L}$ for ζ, with $\omega_n = \frac{1}{\sqrt{LC}}$:

$$\zeta = \frac{3}{2\frac{1}{\sqrt{LC}}} = \frac{3}{2\frac{1}{\sqrt{0.5}}} 1.06 > 1.$$

What other information can we deduce from this particular circuit?

Turns out, quite a bit!

Recall the system function $H(s) = \frac{V_o(s)}{V_i(s)} = \frac{2}{(s+2)(s+1)}$.   Eqn (*)

We want to find the "poles" of the system function $H(s) = \frac{num(s)}{denom(s)} = \frac{N(s)}{D(s)}$. Why would we want to find the poles? They actually characterize the behavior of the system, e.g., whether the system is stable or whether "ringing" will occur in the output signal. Eventually we'll use them to deduce a 'back-of-the-envelope' picture of the circuit's frequency response.

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[1] N.B. A partial fraction expansion of a function $F(s)$ is only valid if the degree of the denominator of $F(s)$ is greater than the degree of the numerator of $F(s)$. Here, the degree of the denominator is 3 and the degree of the numerator is 0.

Here's how we find the poles:

1. Setup the "characteristic equation" of $H(s)$. The characteristic equation is given by $D(s) = 0$. We'll see that the characteristic equation determines the behavior of the RLC circuit. In fact, it characterizes the behavior of any linear, time-invariant system,[2] of which the RLC circuit is but one example!

But why do we set the denominator equal to 0? If we do so, what happens to $H(s)$?

The poles of $H(s)$ are the locations in the complex s-plane where the magnitude of the system function $|H(s)|$ has an infinite value.

2. So let's find the poles of $H(s)$ for the above RLC circuit.
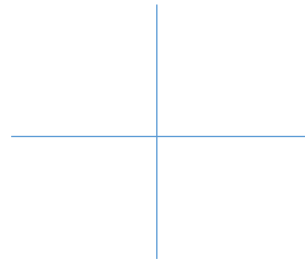
3. Graph the poles of $H(s)$ in the s-plane.

Can you always expect $H(s)$ to be a ratio of polynomials in s, i.e., a rational function? Hint: first check your differential equation, then see footnote at bottom of p. 5.

A rational system function $H(s)$ can always be written in the form

$$H(s) = K \frac{(s - s_{z1})(s - s_{z2})...(s - s_{zL})}{(s - s_{p1})(s - s_{p2})...(s - s_{pM})}, \text{ where L represents the number of roots of the numerator}$$

polynomial, M denotes the number of roots of the denominator polynomial, and K signifies a constant.

4. Calculate the "zeros" of $H(s)$ for the RLC circuit. The zeros of $H(s)$ are the locations in the complex s-plane where the magnitude of the system function $|H(s)|$ equals 0.

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[2] A system is linear (L) if and only if the properties of superposition and scaling hold. A system is time-invariant (TI) if a time delay or time advance of the input signal leads to an identical time-shift in the output signal. In other words, except for a time-shift in the output, a TI system responds exactly the same way no matter when the input signal is applied. We'll be covering these concepts more formally in a later handout.

The type of elements used in the circuit, as well as the topology of the network, constrain where the poles and zeros may lie in the s-plane. For our RLC circuit, a second order circuit, the fact that the element values are real numbers forces the pole and zero locations to lie either along the $\text{Re}(s) = \sigma$ axis or to occur in complex conjugate pairs.

Moreover, ==if positive R's, L's, and C's comprise the circuit, then the poles of the system function must lie in the LHP (left half of the s-plane).==

Finally, system functions whose poles lie in the LHP are stable. If any pole exists in the RHP (right half of the s-plane), the system is unstable.

6. Is the RLC circuit stable?

B. LC circuit: an ideal oscillator

If there were no resistor in the circuit, then we'd have an LC circuit. Short circuit the resistor in the circuit so that R = 0. $L=1H, C=0.5F$.
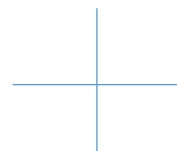
1. Find the differential equation for the LC circuit. Assume
   the circuit starts from rest, i.e., all initial conditions = 0.
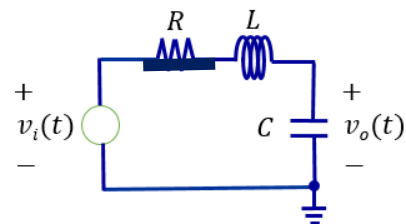
2. Find $H(s) = \dfrac{V_c(s)}{V_{in}(s)} = \dfrac{V_{out}(s)}{V_{in}(s)}$.

3. What is the natural frequency ω_n of the LC circuit? Just curious, what's the damping ratio ζ.

4. Find the poles of the LC circuit and indicate them on the s-plane:

Idealized LC circuits are lossless and the poles must all lie on the Im(s) axis!

5. Suppose the input to the LC circuit is $v_{in}(t) = u(t)$. Determine $v_c(t) = v_{out}(t)$.

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6. Use MATLAB to check your answer for $v_{out}(t)$ due to a unit step input.

syms s t;

F = (1/s) - s/(s^2 + 2);

ilaplace(F)

7. Use MATLAB to plot $v_{out}(t)$, the response of the circuit due to the unit step, aka the "step response."

t = (0:0.1:20);

plot(t, 1 - cos(2^(1/2)*t))

8. Now suppose $v_{in}(t) = \sin \sqrt{2}\ t\ u(t)$. Find $V_{out}(s)$ and use MATLAB to calculate $v_c(t) = v_{out}(t)$

9. Use MATLAB to plot $v_c(t) = v_{out}(t)$.

10. What can you deduce from the plot? Justify your observations.

11. A big take-away is now evident! Once we know the system function $H(s) = \dfrac{V_{out}(s)}{V_{in}(s)}$ for an LTI (linear, time-invariant) system, we can find $v_{out}(t)$ of the LC circuit for **any** input, e.g., step input or sinusoidal input, by first finding $V_{out}(s) = H(s)V_{in}(s)$, then taking the inverse Laplace transform of $V_{out}(s)$, either by hand or with a numerical calculator such as MATLAB.

Remember what ESA is about: using tools to arrive at a desired design that you're confident will work, rather than using trial and error to hopefully find something that works.

Similarly, for our RLC circuit, we can use the system function $H(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{2}{(s+2)(s+1)}$ to find $v_{out}(t)$ of the RLC circuit for **a given input.**

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C. <u>Applying MATLAB functions</u> to verify what was calculated for the damped RLC circuit on p. 1

While we're on the topic of MATLAB, let's also verify the poles of $H(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{2}{(s+2)(s+1)}$.

Use the roots command and apply it to the denominator polynomial.

      roots([1 3 2])

MATLAB can also find the constants for a partial fraction expansion, such as that depicted on p. 1:

$$V_o(s) = \frac{2}{(s+2)(s+1)}\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}.$$

Use the syms function to create a symbolic object that is automatically assigned to a MATLAB variable with the same name:

\>\> syms s

\>\> Vout = 2/(s^3 + 3*s^2 + 2*s);

\>\> partfrac(Vout);

\>\> ans

 ans = 1/(s + 2) - 2/(s + 1) + 1/s

* Given a system modeled by a linear, constant coefficient, ordinary differential equation (LCC ODE), its system function H(s) will be a rational function (a ratio of polynomials in s).