

APPENDIX II-C-1. UNITING AHS WITH ENGINEERING: MODULE LINKING MUSIC AND ENGINEERING DEVELOPED FOR A NEW COURSE—“ENGINEERING THE ACOUSTICAL WORLD”, HARVARD SCHOOL OF ENGINEERING AND APPLIED SCIENCES (SEAS)

I prepared the following course materials (two 1.5 hour classes) for “Engineering the Acoustical World” on the topic “What’s so great about the overtone series?” Intentional spaces in the handouts allowed students to take notes or work out the answers to questions. Though answers didn’t appear in the handouts, they were provided to the students for later reference.

1. The overtone series pops out of the wave equation
2. The overtone series, standing waves, harmonics
3. A quick recap of our prior class
4. Pythagoras: scales based on proportional string lengths
5. From Pythagoras to Ptolemy to tempered tuning
6. Modal scales
7. Taking advantage of the overtone series for orchestration

PDFs of these materials start on the next page.

2018-02-13 The 1-dimensional wave Equation: what does it have to do with the harmonic (or overtone) series?

The 1-d wave equation is given by $\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y(x,t)}{\partial t^2}$.

What is $y(x,t)$?

ANS: The vertical displacement of the string from its equilibrium position at any time t . The displacement of the string at any position x along the string and for any time t . For any point in time and at any position x along the string x , the wave eqn tells you the displacement of the string. Plug in any time at any position, and $y(x,t)$ will tell you the value of the displacement of the wave. $y(x,t)$?

Because $y(x,t)$ is a function of 2 variables x and t , it will tell us the displacement of the string (x) at any point in time (t).

What is c ?

ANS: the velocity with which a small disturbance (a wave) moves along the string.

Goal: to see how the harmonic series pops out of the solution to the 1-d wave equation.

Solving the 1-d wave equation means finding the vertical displacement $y(x,t)$ of the string from its equilibrium position.

What is the equilibrium position of the string?

To get the solution $y(x,t)$, we'll need two boundary conditions and two initial conditions because the wave equation is a 2nd order partial differential equation in x and t .

The boundary conditions are easy. The displacement of the string must be 0 at the nut ($x = 0$) and must also be 0 at the bridge ($x = L$). For our work today, we're most interested in the boundary conditions.

But first, note the LHS of the equation has only space derivatives of the 1-d variable x associated with it. The RHS has only time derivatives associated with respect to time t .

This suggests we try a product solution for the wave equation:

$y(x,t) = U(x)T(t)$ where $U(x)$ contains only x -dependent terms and $T(t)$ contains only t -dependent terms.

Substitute the product solution $y(x,t) = U(x)T(t)$ into the 1-d wave equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y(x,t)}{\partial t^2}.$$

ANS. $\frac{\partial^2 y(x,t)}{\partial x^2} = T(t) \frac{d^2 U(x)}{dx^2} = \text{LHS of the 1-d wave equation}$

$\frac{1}{c^2} \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{1}{c^2} U(x) \frac{d^2 T(t)}{dt^2} = \text{the RHS of the 1-d wave equation}$

Therefore, $T(t) \frac{d^2 U(x)}{dx^2} = \frac{1}{c^2} U(x) \frac{d^2 T(t)}{dt^2}$

Arrange the equation (algebra) so that the LHS is expressed solely in terms of $U(x)$ and the RHS expressed only i.t.o. $T(t)$:

The above equation $\frac{1}{U(x)} \frac{d^2 U(x)}{dx^2} = \frac{1}{c^2} \frac{1}{T(t)} \frac{d^2 T(t)}{dt^2}$

must be satisfied for any and all possible values of x and t . It results from separating the variables so that the LHS is in terms of x and the RHS is expressed in terms of t . Thus, we've seen a nice use of the "separation of variables" technique.

But how can a function of x (the LHS) = a function of t (the RHS)?

ANS The only way this can happen is if both sides are equal to a constant which we deliberately choose to set equal to $-k^2$

We call the constant $-k^2$ that arises from use of the separation of variables technique the "separation constant".

Using the separation of variables technique we end up with 2 ordinary differential equations:

$\frac{1}{U(x)} \frac{d^2 U(x)}{dx^2} = -k^2$ and $\frac{1}{c^2} \frac{1}{T(t)} \frac{d^2 T(t)}{dt^2} = -k^2$ which we can re-write as:

$\frac{d^2 U(x)}{dx^2} + k^2 U(x) = 0$ [spatial wave eqn] and $\frac{d^2 T(t)}{dt^2} + k^2 c^2 T(t) = 0$ [temporal wave eqn]

To solve the space domain wave eqn for $U(x)$, we use the boundary conditions.

To solve the time domain wave eqn for $T(t)$, we use the initial conditions.

Then the total solution will be $y(x,t) = U(x)T(t)$.

For reasons that will soon become obvious, we want to focus today on solving the spatial wave equation

$$\frac{d^2U(x)}{dx^2} + k^2U(x) = 0 \text{ for } U(x) \text{ and applying the two boundary conditions: } U(0) = 0 \text{ and } U(L) = 0.$$

Remember the boundary condition $U(0) = 0$ results from the guitar string being fixed at the nut (no vertical displacement of the string is possible). Similarly, the boundary condition $U(L) = 0$ occurs because the guitar string is also fixed at the bridge so the string can't move at the bridge.

These boundary conditions suggest the solution to the *spatial* wave equation will somehow involve standing waves!

What might be a good test solution(s) for $U(x)$?

Let's try $U(x) = \sin kx$

Test this now by substituting the function $\sin kx$ for $U(x)$ in the *spatial* wave equation:

$$\frac{d^2U(x)}{dx^2} + k^2U(x) = 0$$

ANS $-k^2 \sin kx + k^2 \sin kx = 0$. It checks! So now check if $\sin kx$ satisfies the boundary conditions:

$U(0) = 0$ and $U(L) = 0$.

ANS $U(x) = \sin kx$. Therefore, when $x = 0$, $U(0) = \sin k(0) = 0$ for every value of k . So the test solution $U(x) = \sin kx$ satisfies the first boundary condition.

Now check if the test solution $U(x) = \sin kx$ satisfies the 2nd boundary condition: $U(L) = \sin kL = 0$:

ANS $U(L) = 0$ as long $kL = n\pi$, for $n = 1, 2, 3, \dots$

What value (or values) of k will satisfy this 2nd boundary condition?

ANS Solving for k : $k = \frac{n\pi}{L}$, $n = 1, 2, 3, \dots$

So what is the solution $U(x)$ to the spatial ODE wave equation $\frac{d^2U(x)}{dx^2} + k^2U(x) = 0$?

Thus, $U(x) = \sin \frac{n\pi}{L}x$ gives infinitely many solutions to the spatial ODE wave equation! All of these solutions comprise $U(x)$.

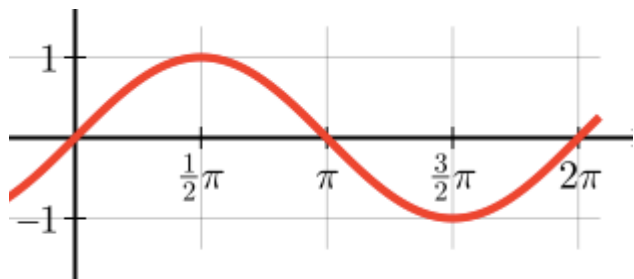
Now we're ready to find and draw the first three of these solutions for $U(x)$, i.e., where $n = 1, 2, 3, \dots$

1. Set $n = 1$ and you'll find $U(x) = \sin \frac{(1)\pi}{L}x$ for values of x ranging from 0 to L .

For starters, try $x = 0$, $x = L$, and $x = L/2$: substitute these into $U(x) = \sin \frac{(1)\pi}{L}x$ and use the values to draw the resulting standing wave:



(For those who are rusty with the sin function, here it is. $\frac{n\pi}{L}x$ maps to the horizontal axis.)



2. Set $n = 2$ and you'll find $U(x) = \sin \frac{(2)\pi}{L} x$ for values of x ranging from 0 to L .

Try $x = 0$, $x = L$, $x = L/4$, and $x = 3L/4$: substitute these into $U(x) = \sin \frac{(2)\pi}{L} x$ and use the values to draw the resulting waveform:

3. Set $n = 3$ and draw the standing wave that results:



First mark the nodes!

Set $n = 3$ and you'll find $U(x) = \sin \frac{(3)\pi}{L} x$ for values of x ranging from 0 to L .

Try $x = 0$, $x = L$, $x = L/3$, $x = 2L/3$, $x = L/6$, $x = L/2$, $x = 5L/6$.

These first 4 values for x will give you the nodes of the standing wave, i.e., the string does not move at these nodes. The next 3 values will give you the antinodes of the string, i.e., the string achieves maximum displacement.

Find the frequencies associated with each of the above standing waves.

ANS

For $n = 1$: $F_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$ which is exactly what we found earlier, where our old velocity value v is

equivalent to the velocity c used in the wave equation, i.e., $F_1 = \frac{c}{2L}$.

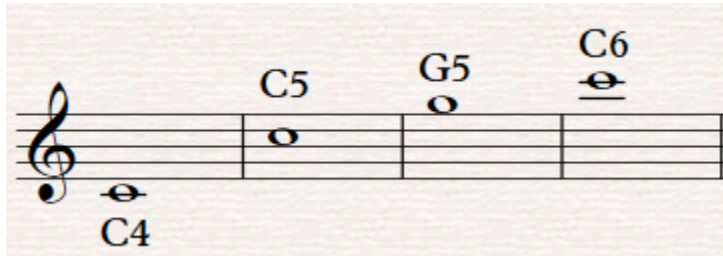
For $n = 2$: $F_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2F_1 = \frac{c}{L}$.

For $n = 3$: $F_3 = \frac{v}{\lambda_3} = \frac{v}{\frac{2L}{3}} = 3F_1 = \frac{c}{\frac{2L}{3}} = \frac{3c}{2L}$.

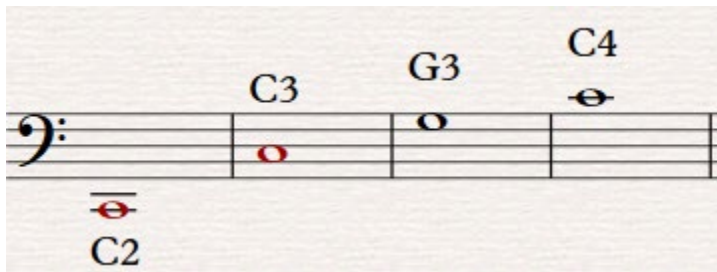
2018-02-13 What's so great about the OT series (also known as the harmonic series)?

First, let's use music notation to define the overtone series based on fundamental C4 (equivalent terminology: harmonic series based on harmonic 1 = C4)

Here are the first 4 pitches of the overtone series based on fundamental C4 = middle C:

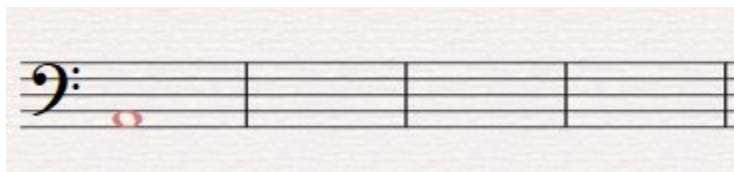


Here are the first 4 pitches of the overtone series based on fundamental C2 = 2 octaves below middle C:



What are the intervals separating the above pitches? If you're rusty with musical notation, just count lines and spaces. For ex., to find C3, the octave above C2, count 8 lines and spaces, including the starting line or space.

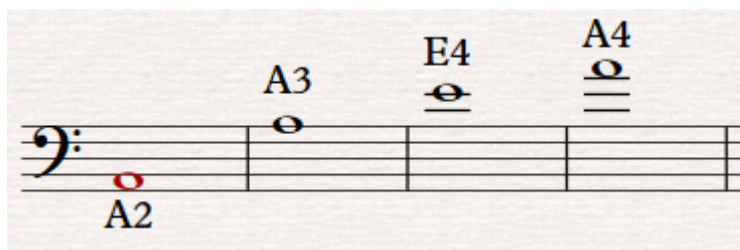
Write the first 4 pitches of the overtone series based on fundamental A2 for bass clef (or A3 for treble clef).



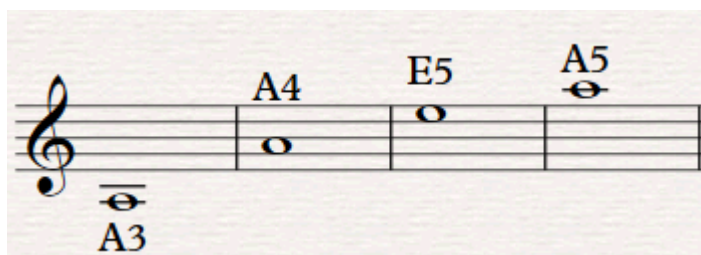
Note: the above 4 pitches correspond to frequencies 110, 220, 330, 440



Note: the above 4 pitches correspond to frequencies 220, 440, 660, 880



ANS



ANS

Where does the harmonic series come from?

<https://www.khanacademy.org/science/ap-physics-1/ap-mechanical-waves-and-sound/modal/v/standing-waves-on-strings>

Let's consider a guitar string, i.e., a string fastened at 2 ends (e.g., secured at the nut and bridge of a guitar)

The nut of a guitar (the horizontal bar just below the head stock):



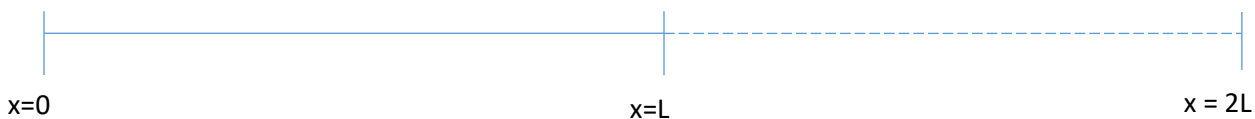
The nut of a guitar



The bridge of a guitar

Here's the guitar string, fixed at both ends, at $x=0$ and $x=L$.

Draw the standing wave associated with the fundamental (harmonic 1):



Draw the standing wave associated with harmonic 2:



What would the standing wave for the 3rd harmonic look like?



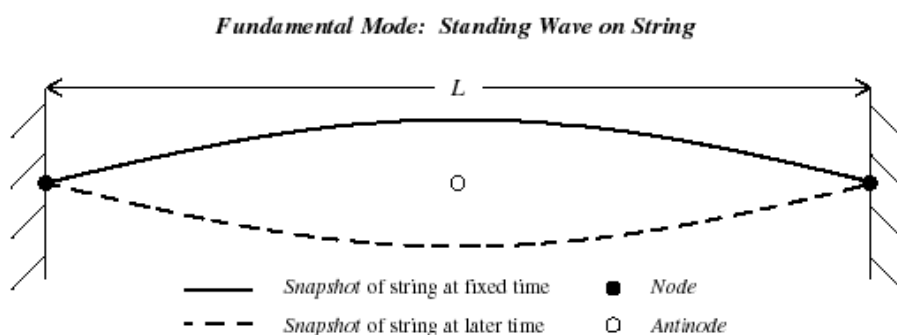
Therefore, $\lambda_n = \frac{2L}{n}$ gives the wavelength λ of the n th harmonic. Ex: λ of the 38th harmonic = $2L/38$.

λ of the 23rd harmonic = $2L/23$.

This formula assumes you have string fixed at both ends, so have nodes at both ends. Holds for stringed instruments since on all instruments with a string, both ends are fixed.

Finding the harmonic frequencies associated with the above harmonic wavelengths: relating wavelength and frequency

Recall the wavelength λ_1 of the fundamental standing wave: $\lambda_1 = \frac{2L}{1}$. Thus, the wavelength of the fundamental standing wave is twice the length of L.



What is the frequency F_1 associated with wavelength λ_1 of the fundamental standing wave, also known as harmonic 1?

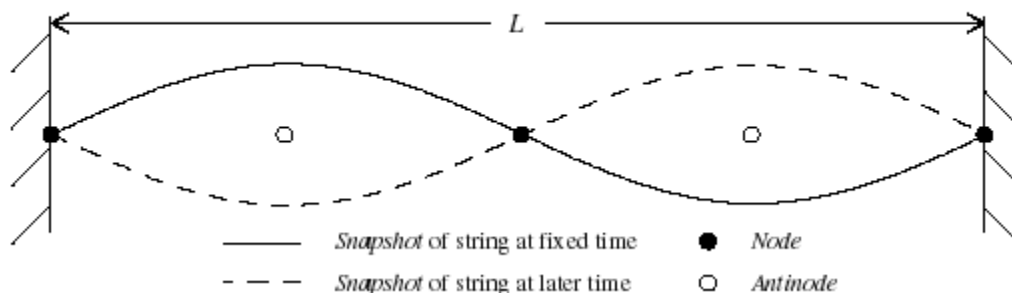
We use the expression $F = \frac{v}{\lambda}$, where F denotes frequency, v is the velocity with which a small disturbance (wave) moves along the string (the wave speed), and λ denotes the wavelength of the sin wave associated with the frequency F .

Just as we found the harmonic series in terms of special λ 's, we want to find the harmonic series in terms of special frequencies.

$F_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$. So pick some reference fundamental frequency. Let's choose A440: $F_1 = 440$.

Find the frequency F_2 associated with the wavelength λ_2 of the standing wave whose period is L ?

Second Harmonic: Standing Wave on String

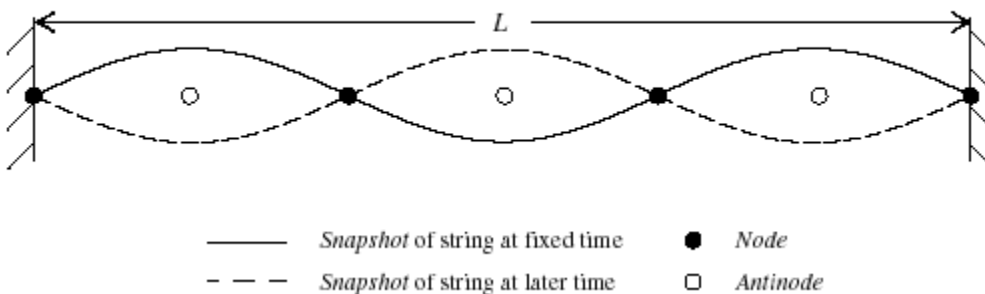


1. First use $\lambda_n = \frac{2L}{n}$ to find $\lambda_2 = \frac{2L}{2} = L$.
2. Now use $F = \frac{v}{\lambda}$ to calculate the frequency associated with a wave with wavelength

$$\lambda_2 = \frac{2L}{2} = L : \quad F_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2F_1. \text{ Btw, what is } F_1?$$

Find the frequency F_3 associated with the standing wave that sounds the 3rd harmonic.

Third Harmonic: Standing Wave on String



1. First find wavelength λ_3 of the standing wave shown in the above picture:

2. Now use $F = \frac{v}{\lambda}$ to calculate the frequency associated with $\lambda_3 = \frac{2L}{3}$:

Remember, there are an infinite number of these standing waves simultaneously occurring in a string's vibration! So we can keep calculating and calculating ...

ANS

Use $\lambda_n = \frac{2L}{n}$ to find $\lambda_3 = \frac{2L}{3}$, i.e., the wavelength of the standing wave producing the 3rd harmonic is

2/3 of the length L of the string. Therefore, $F_3 = \frac{v}{\lambda_3} = \frac{v}{\frac{2L}{3}} = 3F_1$

Another approach to the harmonic series: The Wave Equation We'll see that the harmonic series falls right out of the solution to the one-dimensional wave equation!

S 2018-02-14 Recapping Tuesday's class

We approached the OT series (aka harmonic series) 3 different ways:

1. Musically

2. By finding special wavelengths that 'fit' the boundaries on the guitar string: $\lambda_n = \frac{2L}{n}$, $n = 1, 2, 3, \dots$

We saw that each of these special wavelengths produced a particular standing wave such that the standing wave must be 0 at the endpoints of the string. Why?

Then we used an expression relating wavelength λ to frequency F to find a special frequency associated with each of the special wavelengths:

$$F_n = \frac{v}{\lambda_n}, \quad n = 1, 2, 3, \dots$$

We saw that the special frequencies associated with the special wavelengths are the same frequencies found in the OT series, i.e., they are found by multiplying the fundamental (or harmonic 1) by successive integer values, $n = 1, 2, 3, 4$, and so on. Thus we get frequencies that replicate the harmonic series: $1F, 2F, 3F, 4F, 5F, \dots$

3. The harmonic series pops out of the 1-dimensional wave equation $\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y(x,t)}{\partial t^2}$

What is $y(x,t)$? On Tuesday, we defined $y(x,t)$ as the vertical displacement of the string from its equilibrium position, without consideration of time. Though we didn't focus on the temporal component of the wave eqn, for completeness we now define $y(x,t)$ as the displacement of the string at any position x along the string and for any time t .

$$y(x,t) = U(x) T(t)$$

$$\frac{d^2 U(x)}{dx^2} + k^2 U(x) = 0 \quad \text{the spatial wave equation as an ordinary differential equation}$$

$U(x) = \sin \frac{n\pi}{L} x$, where $n = 1, 2, 3, \dots$, satisfied the spatial wave eqn as well as the boundary conditions

imposed by the guitar string being fixed at both ends. By plotting $\sin \frac{n\pi}{L} x$ against x , we got the same standing wave shapes seen on the video. If you haven't done so already, you really want to try your

hand at plotting $\sin \frac{n\pi}{L} x$ vs. x on the guitar strings provided in Tuesday's 2nd handout. You will see the exact standing wave shapes take place before your very eyes for every $n = 1, 2, 3, \dots$!



$U(x) = \sin \frac{n\pi}{L} x$, $n = 1, 2, 3, \dots$, gives infinitely many solutions to the spatial ODE wave equation! All of these solutions comprise $U(x)$. Why is this the case? Why an infinite number of solutions?

We saw that standing waves are predicted by the wave equation and that they have the same shapes as those seen in approach #2. This is because the spatial solution to the wave equation $U(x) = \sin \frac{n\pi}{L} x$ is a sine function that satisfies the boundary conditions of a guitar string.

By the way, what are the boundary conditions?

What is the significance of $U(0) = 0$ and $U(L) = 0$?

Why is $\lambda_n = \frac{2L}{n}$, where $n = 1, 2, 3, \dots$, useful?

Why is $F_n = \frac{v}{\lambda_n}$, where $n = 1, 2, 3, \dots$, useful?

What is the significance of $F_1 = \frac{v}{\lambda_1}$, $F_2 = \frac{v}{\lambda_2} = 2F_1$, $F_3 = \frac{v}{\lambda_3} = 3F_1$, and so on?

What kind of series is this: $F_1, 2F_1, 3F_1, 4F_1, 5F_1, \dots$

Give another name for this series:

S 2018-02-14 Pythagoras builds pentatonic and diatonic scales based on proportional lengths of strings

Pentatonic scale example: Ravel Mother Goose Suite (Laideronnette, Empress of the Pagodas)

<https://www.youtube.com/watch?v=Zk4EKnRfmGw>

Pythagoras made 3 important observations:

1. Two strings with the same length, tension, and thickness will sound the same when plucked
2. Two strings with different lengths (assuming same length, tension, and thickness) will have different pitches that generally sound dissonant
3. But for certain lengths, two strings—despite their differing lengths—will sound different pitches that actually sound good together

Specifically, if the two string lengths are such that string2 is half as long as string1, then string2 will sound a pitch an octave higher than string1.

Or if string2 is $\frac{2}{3}$ as long as string1, then string2 will sound a pitch a perfect fifth higher than string1.

We can thus define intervals in terms of ratios of string lengths!

Making a pentatonic scale using Pythagoras' string ratios

Assume a string that when plucked gives a certain pitch, e.g., D4 (frequency is 293.7 Hz which we'll round to 294 Hz). For convenience, we'll say the length of the string is L.

	Frequency	Keyboard	Note name	MIDI number
	4186.0			
	3951.1		C8	108
3729.3	3520.0		B7	107
3322.4	3136.0		A7	106
2960.0	2793.8		G7	104
	2637.0		F7	103
2489.0	2349.3		E7	102
2217.5	2093.0		D7	101
	1975.5		C7	99
1864.7	1760.0		B6	98
1661.2	1568.0		A6	97
1480.0	1396.9		G6	96
	1318.5		F6	94
1244.5	1174.7		E6	93
1108.7	1046.5		D6	92
	987.77		C6	91
932.33	880.00		B5	90
830.61	783.99		A5	88
739.99	698.46		G5	87
	659.26		F5	86
622.25	587.33		E5	85
554.37	523.25		D5	84
	493.88		C5	83
466.16	440.0		B4	82
415.30	392.00		A4	81
369.99	349.23		G4	80
	329.63		F4	79
311.13	293.67		E4	78
277.18	261.6		D4	77
	246.94		C4	76
233.08	220.00		B3	75
207.65	196.00		A3	74
185.00	174.61		G3	73
	164.81		F3	72
155.56	146.83		E3	71
138.59	130.81		D3	70
	123.47		C3	69
116.54	110.00		B2	68
103.83	97.999		A2	67
92.499	87.307		G2	66
	82.407		F2	65
77.782	73.416		E2	64
69.296	65.406		D2	63
	61.735		C2	62
58.270	55.000		B1	61
51.913	48.999		A1	60
46.249	43.654		G1	59
	41.203		F1	58
38.891	36.708		E1	57
34.648	32.703		D1	56
	30.868		C1	55
29.135	27.500		B0	54
			A0	53

The pentatonic scale based on D4 will have pitches D4, E4, G4, A4, B4, and D5. How can we get these pitches using Pythagorean ratios?

- $\frac{1}{2}$ of L sounds the octave above D4, i.e., D5 (check: $2 \times 294 \text{ Hz} = 588 \text{ Hz}$)
- $\frac{2}{3}$ of L sounds the perfect 5th above the D4, i.e., A4 (check: $\frac{3}{2} \times 294 \text{ Hz} = 441 \text{ Hz}$)
- To find E4, first find E5. Note that E5 is a perfect 5th up from A4. The string length for E5 will be $\frac{2}{3}$ of the string length needed to produce A4 (which was $\frac{2}{3}$ L). Therefore, the string length needed to give E5 is: $\frac{2}{3} \times \frac{2}{3} L = \frac{4}{9} L$. But we want the pitch E4, which is an octave lower than E5. So we multiply by 2: $2 \times \frac{4}{9} L = \frac{8}{9} L =$ length of the string that'll produce E4. (check: $\frac{9}{8} \times 294 \text{ Hz} = 330 \text{ Hz}$)

- Now find G4: G4 is a perfect 4th up from D4. Pythagoras found that a string whose length is $\frac{3}{4}$ of L will sound the perfect 4th above D4. Thus $\frac{3}{4}$ of L sounds G4. (check: $\frac{4}{3} \times 294 \text{ Hz} = 392 \text{ Hz}$)
- We already found A4.
- How to find B4? Note that B4 is a perfect 5th above E4. We already found the string length for E4 = $\frac{8}{9}$ of L = $\frac{8}{9}$ L. What should we do?

ANS Let's take the string of length $\frac{8}{9}$ L and take $\frac{2}{3}$ of it: $(\frac{8}{9} L) \times \frac{2}{3} = \frac{16}{27} L$ = length of the string needed to produce a pitch sounding a perfect 5th higher than E4. (check: $\frac{27}{16} \times 294 \text{ Hz} = 496 \text{ Hz}$)

You can just imagine Pythagoras' excitement in relating musical pitches to string lengths in ratios comprising the counting numbers!

Pythagoras then developed a diatonic scale

What is a diatonic scale?

(*) Pythagoras created the notes for part of his diatonic scale by progressing **up** by perfect 5ths from a given note. Suppose we want to start the scale with D4. A perfect 5th higher than D4 is A4, a perfect 5th higher than A4 is E5, and a perfect 5th higher than E5 is B__.

(**) To create more notes for the diatonic scale, he started at again at D4, but this time he found pitches that are perfect 5ths **down** from D4: G3, C3, F2.

But he wanted a diatonic scale bounded by an octave.

(***) So his goal was: D4, E4, F4, A4, B4, C5, D5.

So the E5 we found from (*) has to come down an octave to produce E4. What do you think he did?

The F2 we found in (**) has to come up two octaves, so just halve the string length that produces F2 to get F3. Then half the string length again to get F4.

The A4 we found in (*) is in the correct register, i.e., the 4th register according to the desired diatonic scale given in (***)).

The B5 we found in (*) has to come down an octave to give B4 so simply double its string length.

The C3 we found in (**) has to come up 2 octaves to give C5 so halve the string length that produces C3 to get C4. Then half the string length again to get C5.

We now have the Pythagorean diatonic scale:

D4, E4, F4, G4, A4, B4, C5, D5 with the following ratios:

D4	E4	F4	G4	A4	B4	C5	D5
1	$\frac{9}{8}$	$\frac{32}{27}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{16}{9}$	2

Pythagoras went on to define a ratio that will produce a whole step: $9/8$ (e.g., check the whole step between G4 and A4: $9/8 \times 392 = 441 = A4$)

He also defined a ratio to produce a half step: $256/243$ (check the half step between E4 and F4: $256/243 \times 330 = 348 = F4$)

	Frequency	Keyboard	Note name	MIDI number
	4186.0		C8	108
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	3729.3		A7	106
	3322.4		G7	104
	2960.0		F7	102
	2637.0		E7	100
	2489.0		D7	99
	2217.5		C7	97
	1975.5		B6	95
	1864.7		A6	94
	1661.2		G6	92
	1480.0		F6	90
	1318.5		E6	88
	1244.5		D6	87
	1108.7		C6	85
	987.77		B5	83
	932.33		A5	82
	830.61		G5	80
	739.99		F5	78
	659.26		E5	76
	622.25		D5	75
	554.37		C5	73
	493.88		B4	71
	466.16		A4	70
	415.30		G4	68
	369.99		F4	66
	329.63		E4	64
	311.13		D4	63
	277.18		C4	61
	246.94		B3	59
	233.08		A3	58
	207.65		G3	56
	185.00		F3	54
	164.81		E3	52
	155.56		D3	51
	138.59		C3	49
	123.47		B2	47
	116.54		A2	46
	103.83		G2	44
	92.499		F2	42
	82.407		E2	40
	77.782		D2	39
	69.296		C2	37
	61.735		B1	35
	58.270		A1	34
	51.913		G1	32
	46.249		F1	30
	41.203		E1	29
	38.891		D1	27
	34.648		C1	25
	30.868		B0	23
	29.135		A0	22
	27.500			21

S 2018-02-14 Pythagorean tuning, Ptolemy's Just intonation, Equal Temperament

Pythagoras built the pentatonic scale, and by extension, a diatonic scale. But the M3 was considered dissonant (not 'harmonious') because it didn't align with the M3 heard in the harmonic series. The Greeks noticed that performers preferred integer ratios that aligned with the harmonic series to improve sonority.

During the 2nd c AD, Ptolemy proposed a Just intonation system to reflect what musicians actually played. He proposed a system based on the following ratios:

1 denotes the starting pitch, 2/1 is the octave above, 3/2 gives the 5th above the starting pitch, 4/3 gives the 4th above the starting pitch, 5/4 gives the M3 above the starting pitch, and 6/5 gives the m3 above the starting pitch

e.g., C2 = 65.2

$$3/2 \times 65.2 = 97.8 = G2$$

$$4/3 \times 97.8 = 130.4 \sim C3 (130.8)$$

$$5/4 \times 130.4 = 163 = \sim E3 (164.8)$$

$$6/5 \times 163 = 195.6 = \sim G3 (196)$$

Recall the OT series (harmonic series) --

	C3	G3	C4	E4	G4
C2 = harmonic1 = F ₁	C3 = harmonic2 = 2F ₁ Interval ratio = 2F ₁ / F ₁ = 2/1 -> P8	G3 = harmonic3 = 3F ₁ Interval ratio = 3F ₁ / 2F ₁ = 3/2 -> P5	C4 = harmonic4 = 4F ₁ Interval ratio = 4F ₁ / 3F ₁ = 4/3 -> P4	E4 = harmonic5 = 5F ₁ Interval ratio = 5F ₁ / 4F ₁ = 5/4 -> M3	G4 = harmonic6 = 6F ₁ Interval ratio = 6F ₁ / 5F ₁ = 6/5 -> m3

Though Just intonation enabled the M3 and m3 to be harmonious (because their ratios agree with the harmonic series), a chromatic scale built with Just intonation (to allow modulation and transposition) still posed problems. Not all 5ths were in tune.

Finally, theorists decided to temper (or adjust) the ratios so that some modulation would be possible. Mean-tone tempered scale allowed some transposition but the 5th was not 3:2.

But with equal temperament, all octaves match the OT series perfectly. And for harmonics 3 and 4 (P5 and P4), the equal temperament is pretty close to the Just intonation of the OT series. 5ths are a little flat and 4ths a little sharp, but not by much. When you realize that 100 cents constitute the m2 and an equal tempered 5th is less than 2 cents flatter than the 3/2 ratio of the OT series (and the 4th is less than 2 cents sharper than its Just intonation equivalent), the equal tempered scale looks pretty good.

The following table from Gareth Loy's *Musimathics* shows the error in cents between Just and equal-tempered chromatic intervals:

Table 3.6
Comparison of Natural and Equal-Tempered Chromatic Intervals

Degree	Name	Error	Degree	Name	Error
1	Unison	0.0	7	Tritone	-9.7763
2	Minor second	-11.731	8	Perfect fifth	-1.955
3	Major second	-3.910	9	Minor sixth	-13.686
4	Minor third	-15.641	10	Major sixth	15.641
5	Major third	13.686	11	Minor seventh	3.910
6	Perfect fourth	1.955	12	Major seventh	11.730

With equal temperament, those integer ratios valued by Pythagoras and Ptolemy are gone, except for the octave (still 2:1).

The tempered 5th is $T_5 = 2^{\frac{7}{12}} = 1.498 \sim 1.5$. The tempered 4th is $T_4 = 2^{\frac{5}{12}} = 1.335 \sim 4/3$.

The take-away: equal temperament, with its 12 equally spaced half steps, allows transposition and modulation to any major or minor key. Its octaves are 2:1; its 5ths are reasonably close to 3/2; its 4ths close enough to 4/3. The vagaries of tuning are spread out over the 12 pitches comprising the octave. Basically, its advantages (modulation and transposition) won out over intonation.

Modal scales

Here is a handy guide to the 7 modal scales. Keep in mind that any of them can be transposed to another tonic, i.e., another starting note, by preserving the whole step/half step relationships given below. For ex., Ab mixolydian would read: Ab, Bb, C, Db, Eb, F, Gb, Ab. Note it maintains the same w w h w w h w relationships as the G mixolydian scale shown below.

Ionian c d e f g a b c w w h w w w h (w = whole step, h = half step)

Dorian d e f g a b c d w h w w w h w

Phrygian e f g a b c d e h w w w h w w

Lydian f g a b c d e f w w w h w w h

Mixolydian g a b c d e f g w w h w w h w

Aeolian a b c d e f g a w h w w h w w

Locrian b c d e f g a b h w w h w w w

S 2018-02-14 Taking advantage of the OT series for orchestration

Students in the Olin Conductorless Orchestra (OCO) select their own repertoire. In general, they favor big romantic pieces typically played by 70+ musicians (~50 strings and 24 wind/brass/percussion). But Olin has a small student body which has resulted in a non-traditional instrumentation since OCO's inception. This means every piece must be re-orchestrated to suit each semester's available instrumentation. For example, OCO winds and brass typically outnumber the strings, so balance has to be achieved through often unorthodox means.

The orchestra selected the fourth movement of Dvorak's Symphony No. 8 in G major, op. 88 (Allegro, ma non troppo). Dvorak scored his symphony for 2 flutes, 2 oboes, 2 clarinets, 2 bassoons, 4 horns, 2 trumpets, 2 trombones, bass trombone, tuba, timpani, and strings (32 violins, 12 violas, 12 cellos, and 8 double basses).

I then re-orchestrated the symphony for OCO's 18 players (**2 flutes, clarinet, bass clarinet, trumpet, baritone, 3 violins, 2 violas, 3 cellos, piano, and 3 percussionists**)

In the score below, Dvorak wrote a part for the first violin section that takes them ever higher into the stratosphere. But OCO only had two first violins, rather than 17. Playing the stratospheric passage as written would have resulted in a thin (and barely audible) climax.

So for the ending of the Dvorak, I brought our two first violins and single second violin each down an octave in their respective parts. Then I wrote a short part for the glockenspiel that would ping the desired climactic high notes. I counted on the octave overtones occurring in the lowered violin parts to contribute a string resonance to the glockenspiel part. (The glockenspiel is a high, bell-like instrument which sounds 2 octaves higher than written.)

Though Dvorak doesn't use a glockenspiel anywhere in his Eighth Symphony, I've often found it effective for adding "color" and sonority to OCO's instrumentation. Since we had a good player, I gave her extra parts throughout the work, including the momentous sprint to the end.

Here is the original Dvorak score showing the high first violin parts (top staff below)

The image displays a musical score for the first violin part of Dvorak's Symphony No. 8, specifically the fourth movement. The score is written for a single violin, with the key signature of G major (one sharp) and a 4/4 time signature. The tempo/mood is marked "Piu animato." and the dynamics are "ff" (fortissimo). The score consists of 12 measures. The first four measures feature a rapid, ascending scale-like pattern. The fifth measure is a whole rest, followed by a sixteenth rest, and then a sixteenth note G5. The final seven measures continue the high, rapid pattern, ending with a final G5. The notation includes many beamed sixteenth and thirty-second notes, indicating a very fast and high-pitched passage.

8.

ritard.

Più animato.



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Here is my re-orchestration which brings all three violins an octave lower than in the Dvorak score so that their octave overtones will coincide with an added glockenspiel part.

<https://www.youtube.com/watch?v=uGVKzEfd4qE>

443
Glock. *f*

443
Xyl. *f*

443
Pno. *sub. p cresc. ff*

443 **S** Più animato

443 SW *p cresc. ff*

443 AD *p cresc. ff*

443 JW *p cresc. ff*

451

Glock.

Xyl.

Mar.

Pno.

SW

AD

JW

461

Glock.

Xyl.

Mar.

Pno.

SW

AD

JW

in tempo

470

Glock. 

470 (S) |

Mar. 

470

Pno. 

470

SW 

470

AD 

470

JW 