Dynamic Programming

Divide & Conquer + Lookup Table

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Overview

- The key idea of Divide & Conquer is to break a problem into smaller sub-problems and combine the result of those subproblems
- Some Problem can be divided into subproblems that is overlapping, i.e., same subproblem that happens more than once
    - If we use general D&C, each copies of the same subproblem will be solved repeatedly, wasting time
    - Dynamic Programming is a technique that use a look up table to store result of each sub-problem and immediately use it if any subproblem is required multiple times

Fibonacci Number

Fibonacci Number

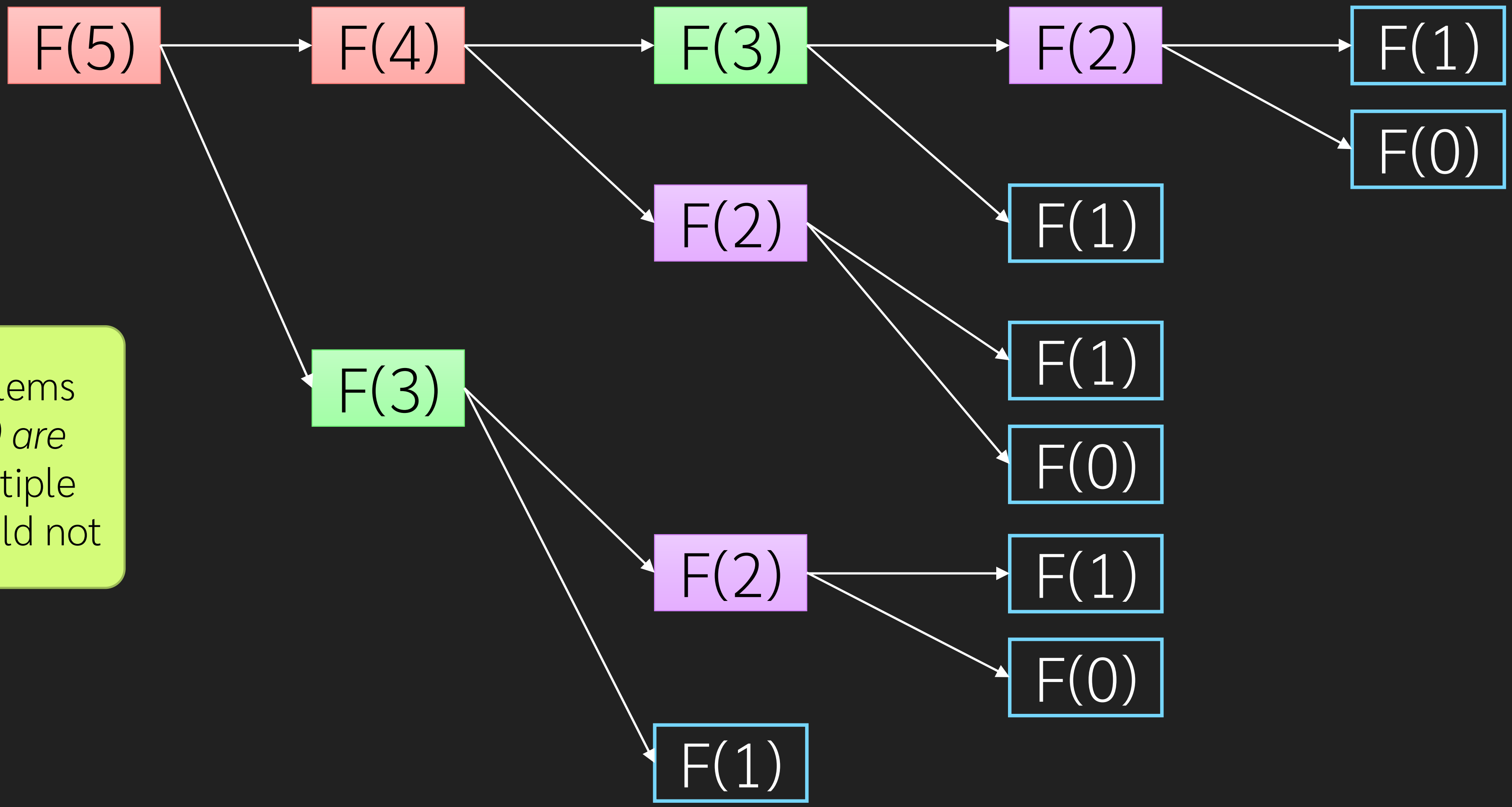
- Problem: compute F(N), the Fibonacci function of N
- Input:
  - An integer N >= 0
- Output:
  - F(N), according to

$$F(N) = \begin{cases} F(N-1) + F(N-2) & ; n > 1 \\ 1 & ; n = 1 \\ 0 & ; n = 0 \end{cases}$$

Can be solve directly using Divide & Conquer

Recursion Tree

```
int fibo(int n) {
  if (n == 1 || n == 0)
    return n;
  if (n >= 2)
    return fibo(n-1) + fibo(n-2);
}
```



Some subproblems *(F(3) and F(2)) are* computed multiple times, they should not

Memoization: Simplest form of Dynamic Programming

- Top-Down approach
- Remember what have been done, if the subproblem is needed again, use the remembered result

```
ResultType DC(Problem p) {
  if (p is trivial) {
    solve p directly
    return the result
  } else {


    divide p into p1,p2,...,pn
    for (i = 1 to n)
      ri = DC(pi)
    combine r1,r2,...,rn into r

    return r
  }
}
```

```
ResultType DP(Problem p) {
  if (p is trivial) {
    solve p directly
    return the result
  } else {
    if p is solved                      use
      return table.lookup(p);
    divide p into p1,p2,...,pn
    for (i = 1 to n)
      ri = DP(pi)
    combine r1,r2,...,rn into r
    table.save(p,r);                    remember
    return r
  }
}
```

Fibonacci: Top-Down DP

- table is an array[1..n] initialized by 0

```
int fibo_memo(int n) {
  if (n == 1 || n == 0)
    return n;

  if (n >= 2) {
    if (table[n] > 0) {
      return table[n];                              use
    }
    int value = fibo_memo(n-1) + fibo_memo(n-2);
    table[n] = value;                               remember
    return value;
  }
}
```

Exercise

- Draw recursion tree when we call fibo_memo(7)

```c
//table is a global variable
int fibo_memo(int n) {
  if (n == 1 || n == 0)
    return n;

  if (n >= 2) {
    if (table[n] > 0) {
      return table[n];
    }
    int value = fibo_memo(n-1) + fibo_memo(n-2);
    table[n] = value;
    return value;
  }
}
```

Bottom-up dynamic programming

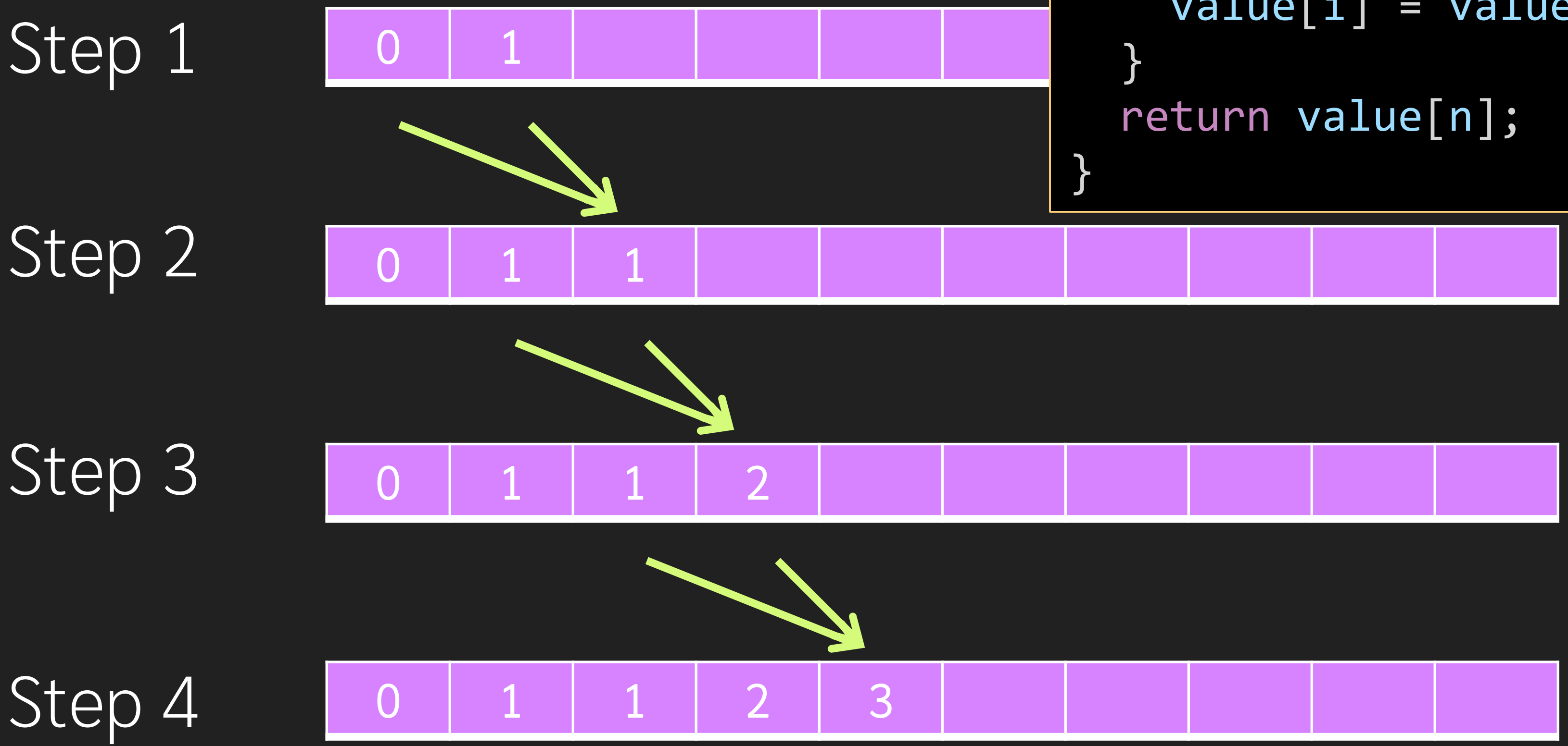
- Instead of relying on recursion to discover repetition of subproblems, we analyze the recursion directly and build table constructively from smaller subproblems
  - The initial subproblems are the ones from trivial case of Divide & Conquer recurrent relation
- Benefit: no-recursion, better runtime performance, (usually) easier to analyze
- Drawback: sometime, we build unnecessary sub-problem

Fibonacci: Bottom-Up DP

- From the definition of F(N), we know that
  - F(n) needs to know F(n-1) and F(n-2)
  - In other words, if we know F(n-1) and F(n-2), then we can construct F(N)
- Initial Condition:
  - F(0) = 0, F(1) = 1
  - i.e., `table[0] = 0; table[1] = 1;`
- From the recurrent
  - `table[3] = table[2] + table[1]`
  - `table[4] = table[3] + table[2]`
  - …

Fibonacci: Bottom Up

```
int fibo_buttom_up(int n) {
  value[0] = 0;
  value[1] = 1;
  for (int i = 2;i <= n;++i) {
    value[i] = value[i-1] + value[i-2];
  }
  return value[n];
}
```



Optimized version of Bottom-Up Fibo

- From bottom up approach, we know that we only need two prior Fibonacci numbers (F(n-1) and F(n-2)) to compute the current Fibonacci number (F(n))
  - There is no need to lookup for F(n-3), F(n-4), … if we know F(n-1), and F(n-2)
  - Hence, no need to use entire table
    - Just remember two previous Fibonacci number

```
def fibo(n)
  if (n == 0 || n == 1)
    return n
  f2 = 0
  f1 = 1
  for i from 2 to n
    #calculate current
    f = f2 + f1
    #prepare f1 and f2 for next round
    f2 = f1
    f1 = f
  end
  return f
end
```

Binomial Coefficient

choose r things from n things

Example 2: Binomial Coefficient

- $C_{n,r}$ =how to choose r things from n things
  - We have a closed form solution
    - $C_{n,r} = n!/(\ r!*(n-r)!\ )$
- We also have recurrence relation of $C_{n,r}$
  - $C_{n,r} = C_{n-1,r} + C_{n-1,r-1}$

    $= 1 \quad ; r = 0$

    $= 1 \quad ; r = n$
- What is the subproblem?
- Do we have overlapping subproblem?

- Input:
  - Two integer r and n ($0 <= r <= n$)
- Output:
  - $C_{n,r}$

Binomial Coefficient

- Each subproblem is represented by 2 numbers, r and n
  - Hence, the table should be 2D

```
int bino_naive(int n,int r) {
  if (r == n) return 1;
  if (r == 0) return 1;

  int result = bino_naive(n-1,r) + bino_naive(n-1,r-1);
  return result;
}
```

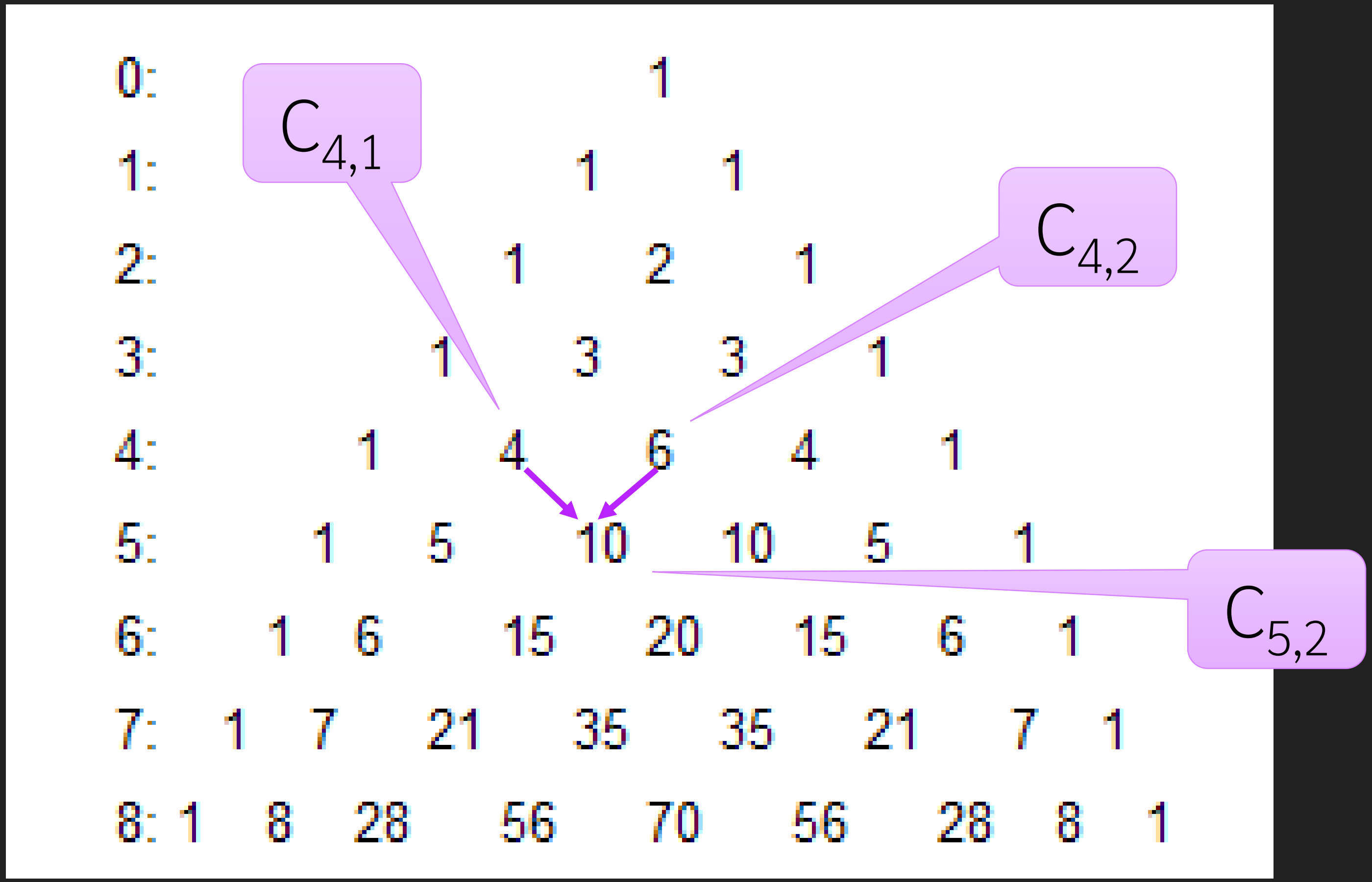
Binomial Coefficient: Top-Down (Memoization)

- table[0..n][0..n] is initialized by -1

```
int bino_memoize(int n,int r) {
  if (r == n) return 1;
  if (r == 0) return 1;

  if (table[n][r] != -1)
    return table[n][r];

  int result = bino_memoize(n-1,r) + bino_memoize(n-1,r-1);
  table[n][r] = result;

  return result;
}
```

Binomial Coefficient: Bottom Up

- Pascal triangle is a by-hand bottom-up DP of Binomial Coeff.



Binomial Coefficient: Bottom Up

	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1		1				
3	1			1			
4	1				1		
5	1					1	
6	1						1

```
int bino_DP(int n,int r) {
  for (int i = 0;i <= n;i++) {
    table[i][0] = 1;
    table[i][i] = 1;
  }
  for (int i = 1;i <= n;i++) {
    for (int j = 1;j < i;j++) {
      table[i][j] = table[i-1][j] + table[i-1][j-1];
    }
  }
  return table[n][r];
}
```

Question

- Is it possible to fill the table in different order?
- Does previous code solve subproblem that we does not need?
  - If yes, how to avoid?

Maximum Subarray Sum

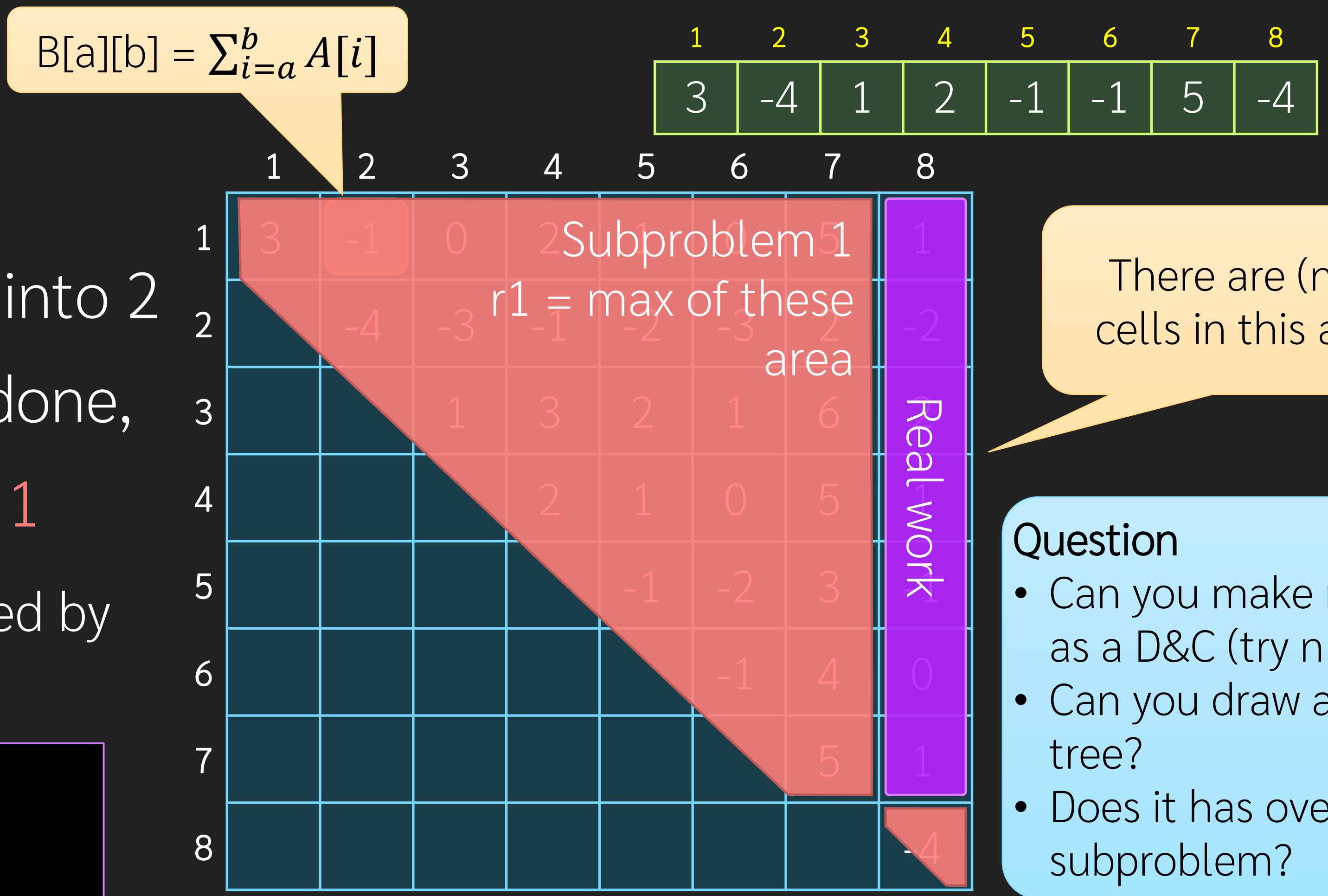
Revisiting

The problem

- Given array A[1..n] of numbers, may contain negative number
  - Find a non-empty subarray A[p..q] such that the summation of the values in the subarray is maximum
- Input:
  - A[1..n]
- Output:
  - p and q, where 1 <= p <= q <= n and summation of A[p..q] is maximum
- Example:
  - A = [1, 4, 2, 3] output: 1 and 4
  - A = [-2, -1, -3, -5] output: 2 and 2
  - A = [2, 3, -6, 4, -2, 3, -5, -4, 3] output: 4 and 6

D&C by n-1

- Instead of divining n into 2 of n/2 as previously done, we divide by n-1 and 1
  - The real work is solved by another D&C



There are (n-1) cells in this area

Question
- Can you make max_suffix as a D&C (try n -> n-1)
- Can you draw a recursion tree?
- Does it has overlapping subproblem?

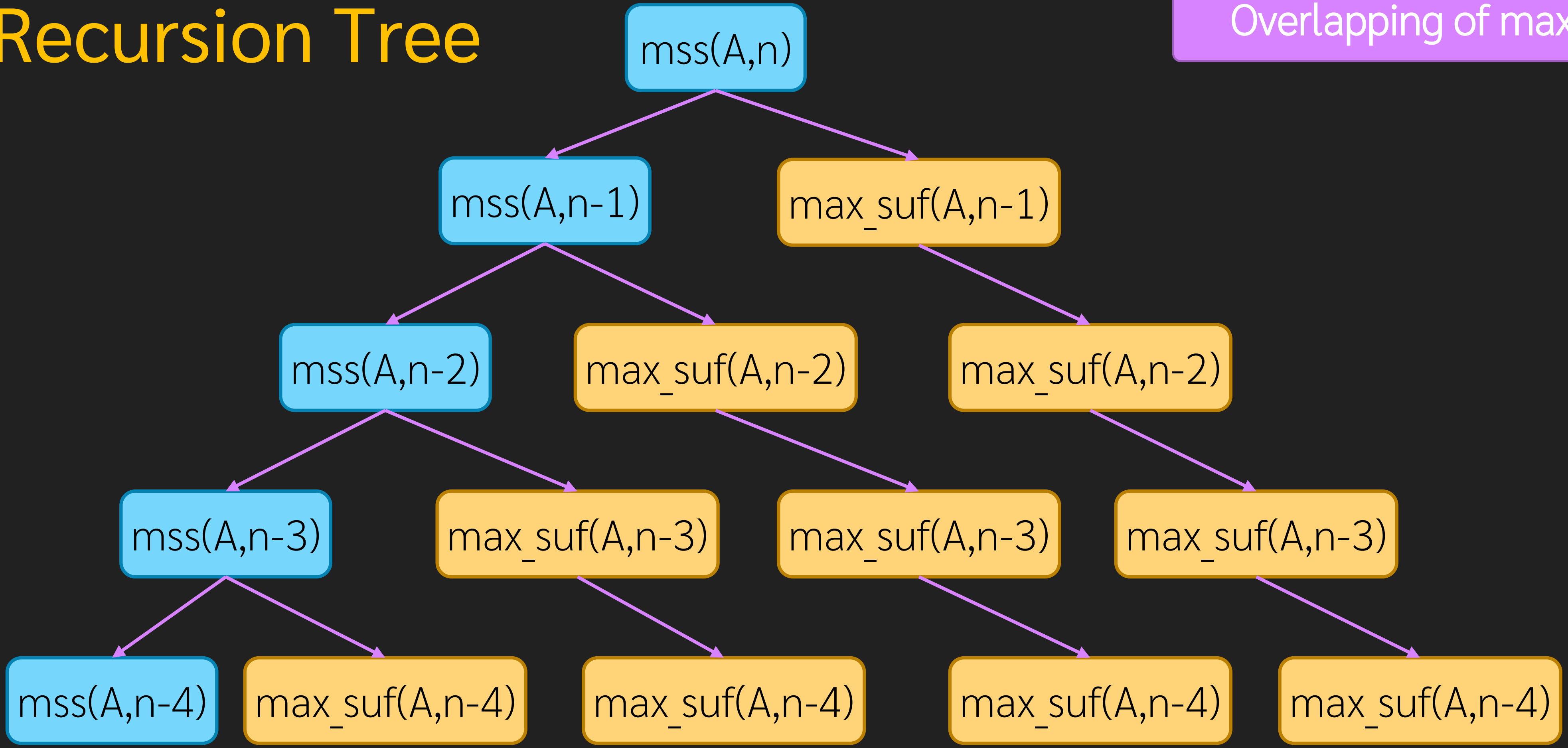
```
def mss(A,stop)
  if (stop == 1)
    return A[1]
  r1 = mss(A,stop-1)
  r2 = A[stop]
  r3 = max_suffix(A,stop-1)+A[stop]
  return max(r1,r2,r3)
end
```

```
def max_suffix(A,stop)
  if stop == 1
    return A[1]
  return max(A[stop],
             A[stop]+max_suffix(A,stop-1))
end
```

max_suffix(A,m) = $\max_{1 \le k \le m} \sum_{i=k}^{m} A[i]$

Recursion Tree

Overlapping of max_suf



MSS with Dynamic Programming

```
def mss(A,stop)
  if (stop == 1)
    return A[1]
  r1 = mss(A,stop-1)
  r2 = A[stop]
  r3 = max_suffix(A,stop-1)+A[stop]
  return max(r1,r2,r3)
end
```

```
def max_suffix(A[1..n],stop,table[1..n],done[1..n])
  if stop == 1
    return A[1]
  if (done[stop])
    return table[stop]

  table[stop] =
    max(A[stop],
        A[stop]+max_suffix(A,stop-1))
  done[stop] = true
  return table[stop]
end
```

- Memoization (top-down) approach
- Since the value of max_suffix can be negative, we need another table to determine whether this subproblem is already solved
  - done[1..n] is initialized as false

Bottom-Up approach

- Direct version
  - Build max_sur first
  - Calculate mss from 1 to n
- Optimized version (Kadane's Algorithm)
  - See that we need only one max_suf

```
def mss_bottom_up(A[1..n])
  max_suf is array [1..n]
  max_suf[1] = A[1]
  for i from 2 to n
    max_suf[i] = max(max_suf[i-1]+A[i],A[i])

  mss = A[1]
  for i from 2 to n
    mss = max(mss,
              max(A[i],
                  max_suf[i-1]))
  return mss
end
```

Kadane’s Algorithm

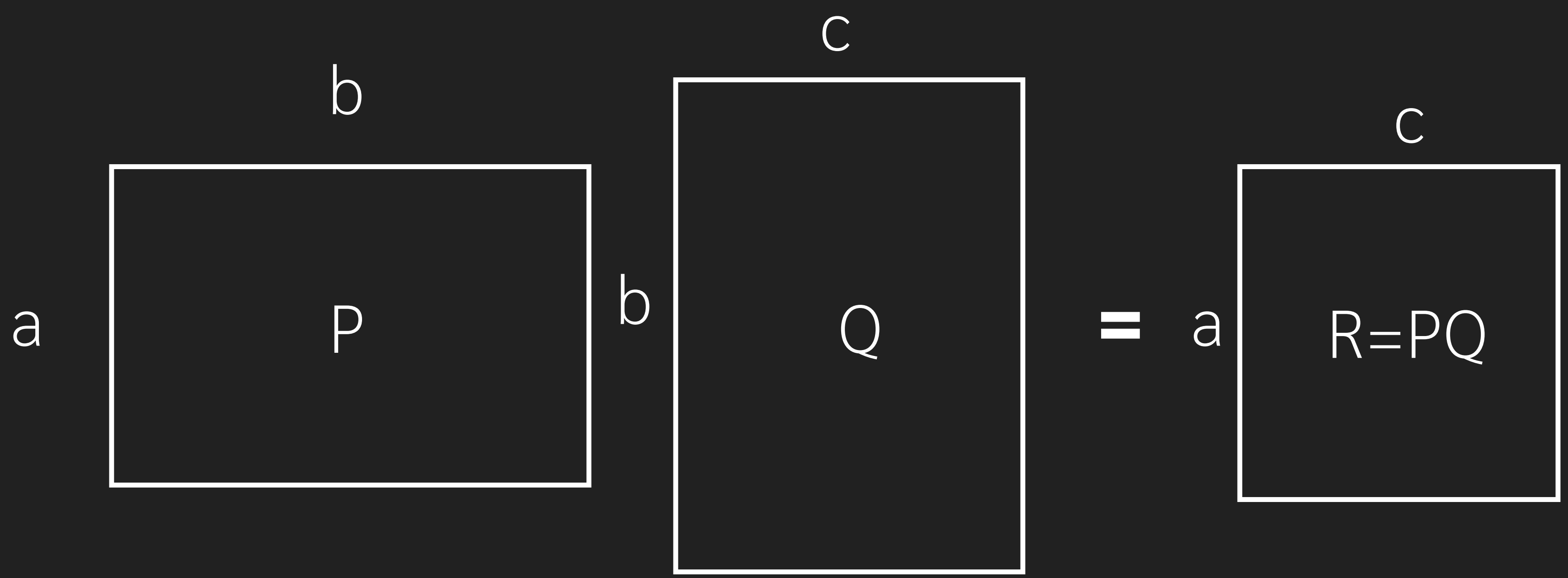
```
def kadane(A[1..n])
  suf = A[1]
  mss = A[1]
  for i from 2 to n
    suf = max(A[i],suf+A[i])
    mss = max(mss,suf)
  return mss
end
```

- Calculate both mss and suf on the fly
- Original problem was proposed by Ulf Grenander in 1977
  - Originally 2D problem, convert to 1D to gain insight
- O(n log n) D&C proposed by Michael Shamos
- Joseph Born Kadane heared the problem in a seminar and propose O(n)

Matrix Chain Multiplication

Non-trivial bottom-up

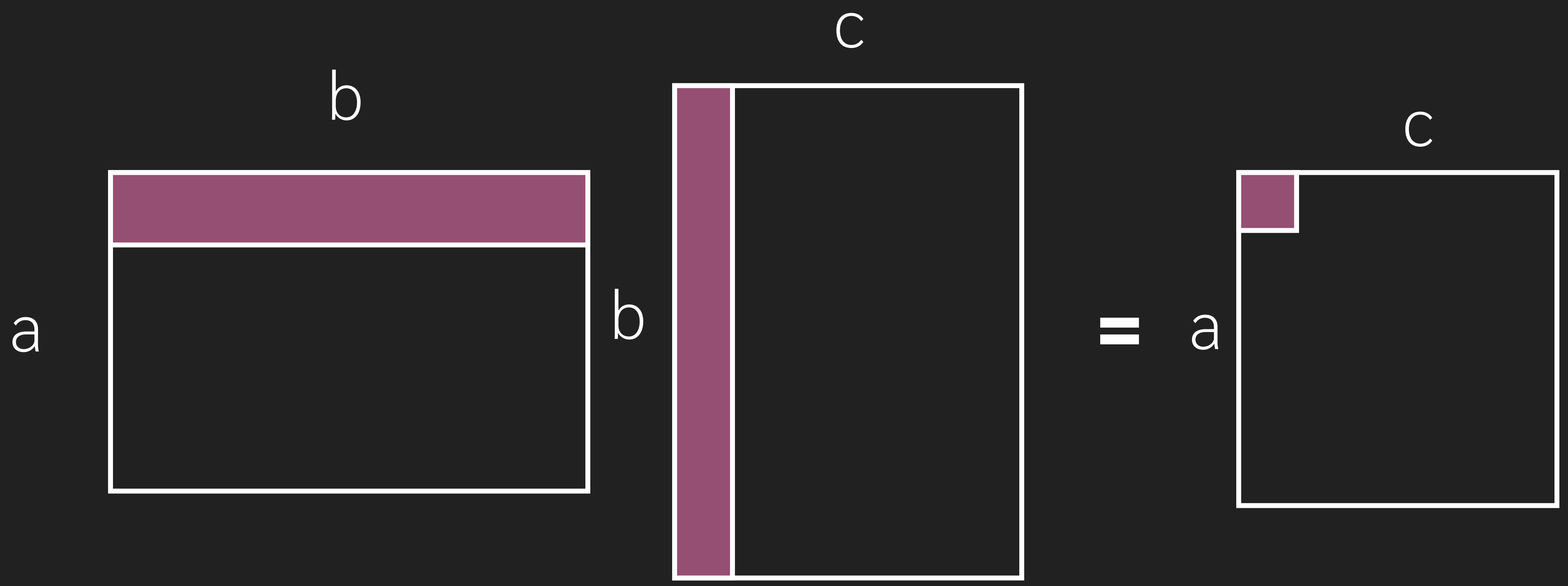
Matrix Multiplication



P = matrix with a rows and b columns

Q = matrix with b rows and c columns

Multiplying the Matrix



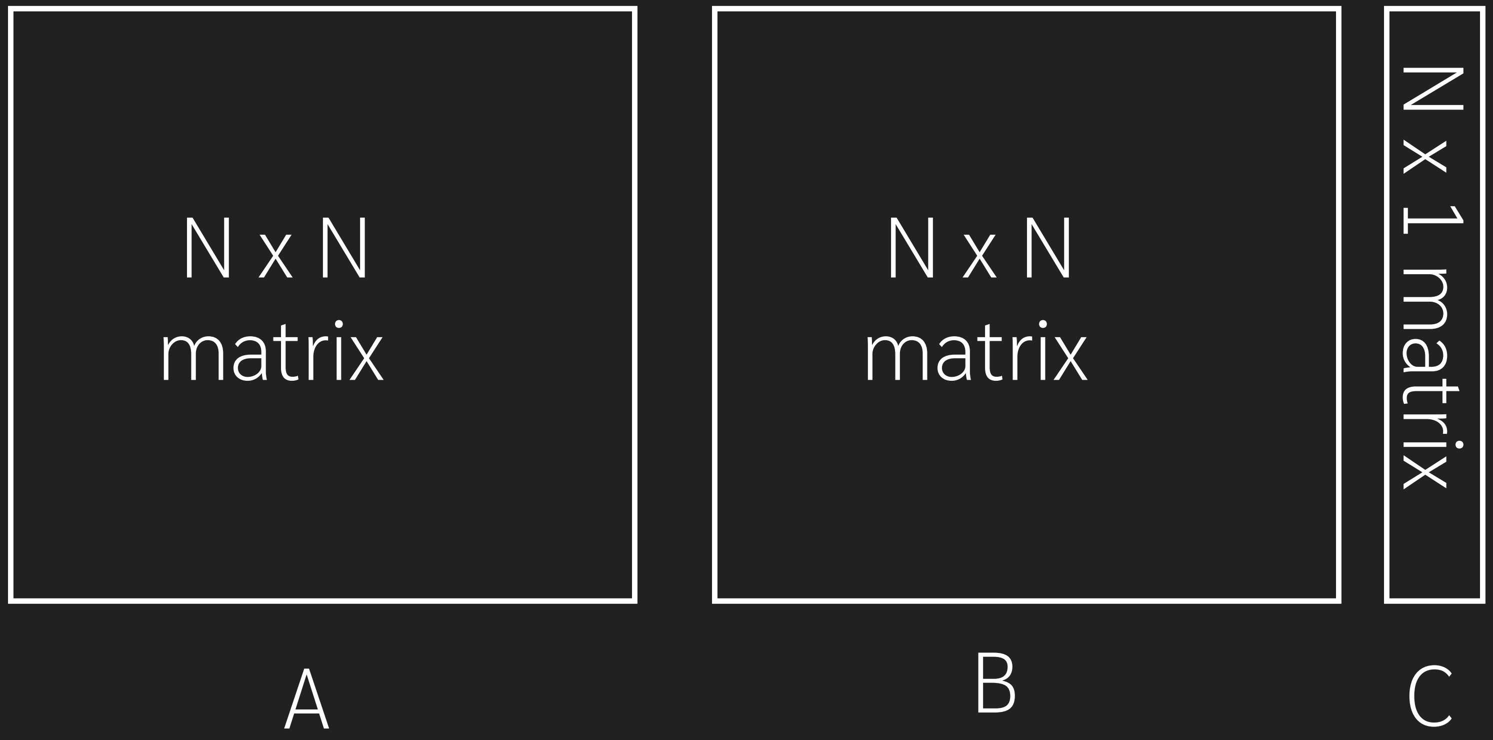
Time used = $\Theta(abc)$

Naïve Method

```
for (i = 1; i <= a;i++) {
  for (j = 1; i <= c;j++) {
     sum = 0;
     for (k = 1;k <= b;k++) {
       sum += P[i][k] * Q[k][j];
     }
     R[i][j] = sum;
  }
}
```

O(abc)

Matrix Chain Multiplication



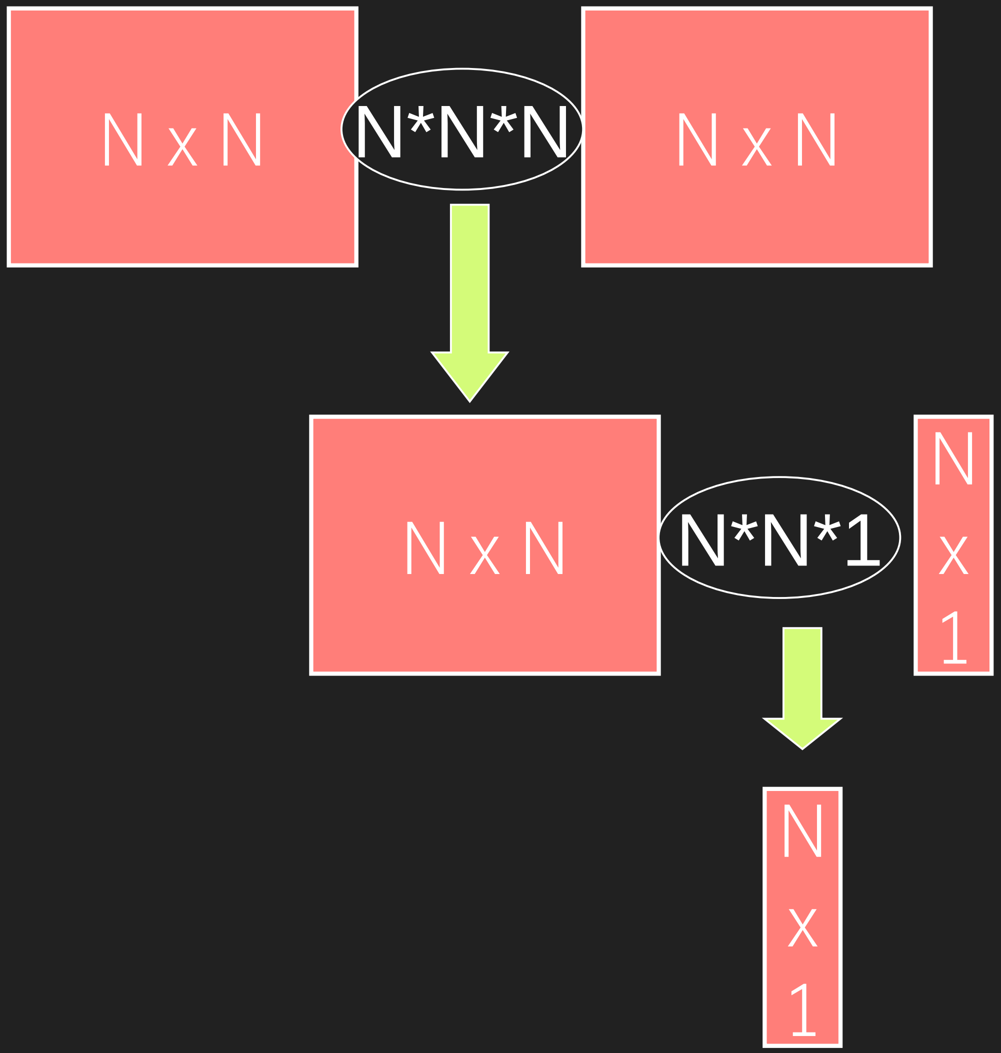
How to compute ABC ?

Matrix Multiplication

- $ABC = (AB)C = A(BC)$
- $(AB)C$ differs from $A(BC)$?
  - Same result, different efficiency

- What is the cost of $(AB)C$?
- What is the cost of $A(BC)$?

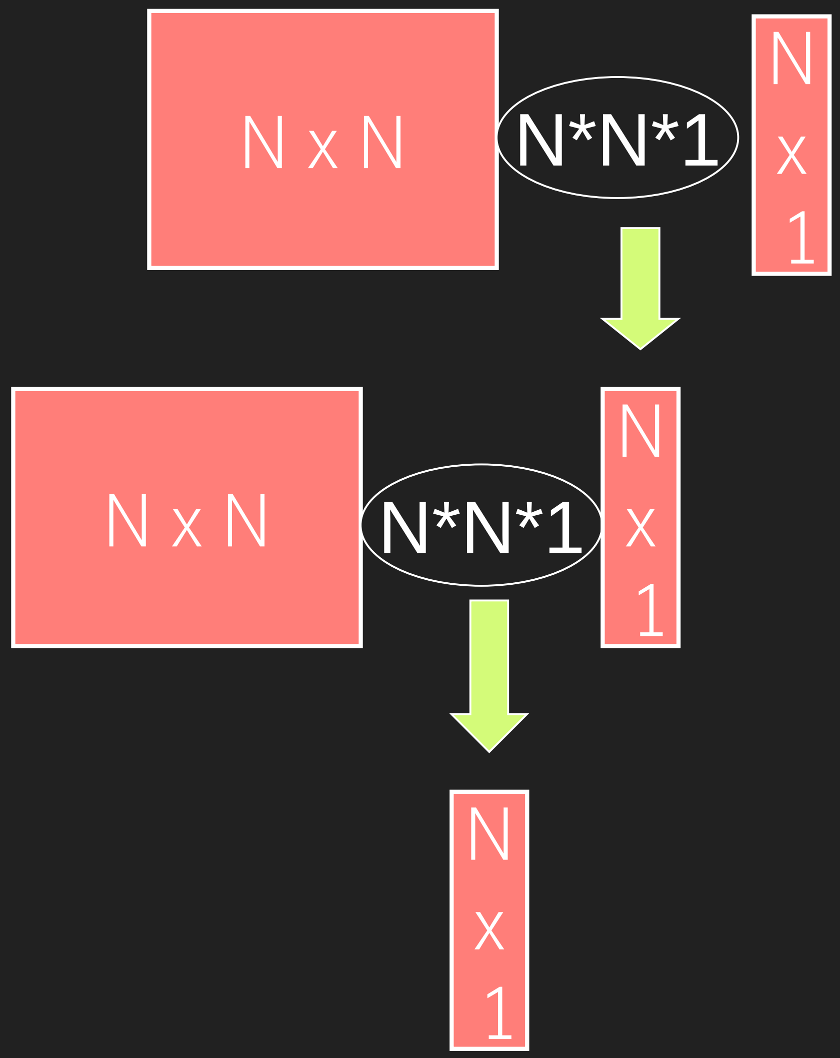
(AB)C



- Total = N^3+N^2

A(BC)

- Total = $2N^2$



The Problem

- Input:
  - $a_1, a_2, a_3, \ldots, a_n$
- Output:
  - The order of multiplication
    - How to parenthesize the chain
  - How many multiplication is needed
- Example Instance:
  - Input: 10 10 10 1 Output: $(B_1(B_2B_3))$ 200

These represents the size of the n-1 matrices B_1 .. B_{n-1}

$a_1 \times a_2$	B_1
$a_2 \times a_3$	B_2
$a_3 \times a_4$	B_3
....	
$a_{n-1} \times a_n$	B_{n-1}

More Example

INPUT

- $a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$
- $10 \times 5 \times 1 \times 5 \times 10 \times 2$

$B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5$

Possible Output

$((B_1B_2)(B_3B_4))B_5$

$(B_1B_2)((B_3B_4)B_5)$

$(B_1((B_2B_3)B_4))B_5$

And much more...

Consider the Output

What do

$(B_1B_2)((B_3B_4)B_5)$

$(B_1B_2)(B_3(B_4B_5))$

have in common?

What do

$((B_1B_2)(B_3B_4))B_5$

$(((B_1B_2)B_3)B_4))B_5$

have in common?

Solving $B_1 B_2 B_3 B_4 \ldots B_{n-1}$

Min cost of

(1) B_1 Subproblem

(2) Subproblem Subproblem

(3) Subproblem Subproblem

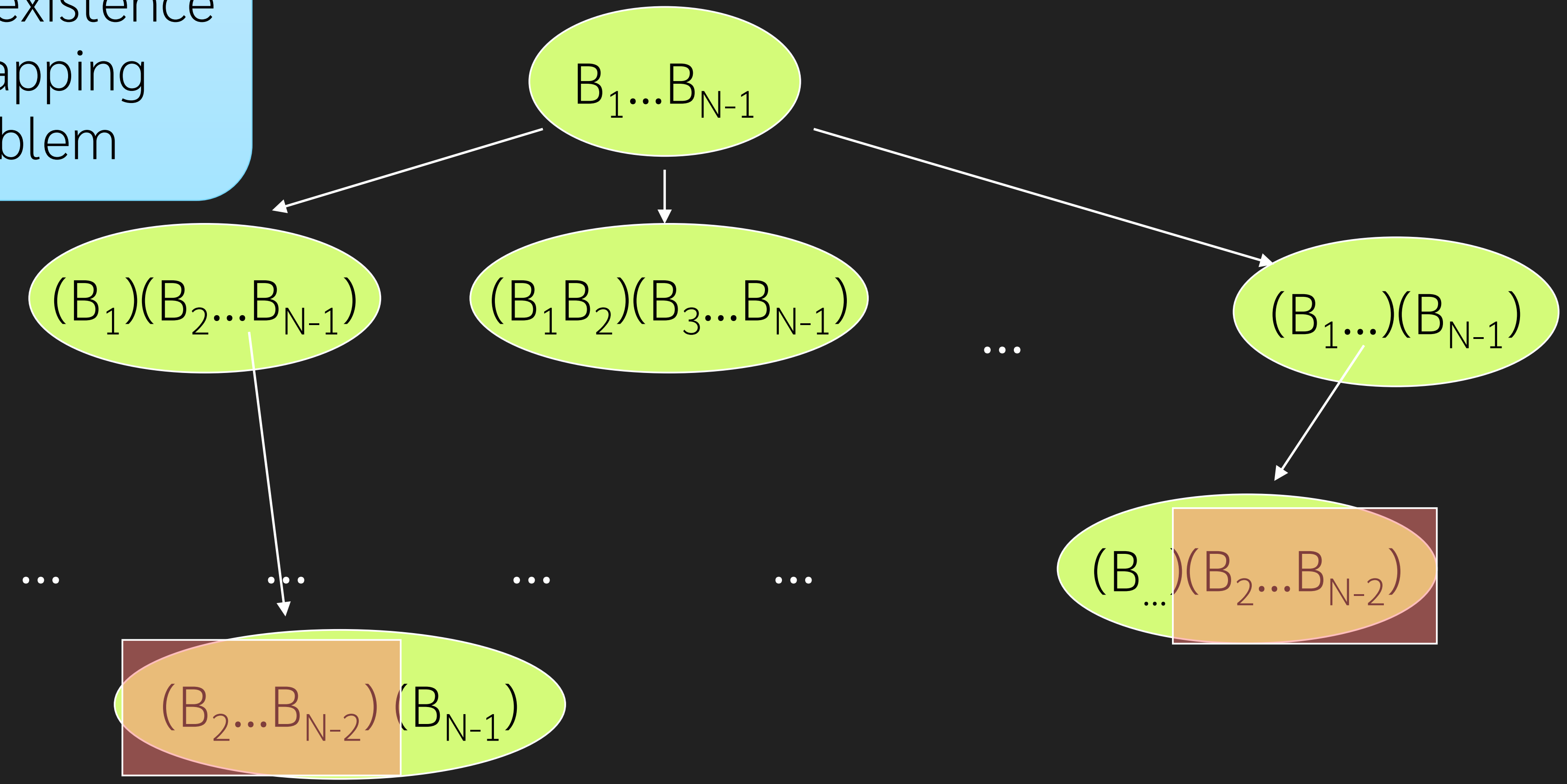
…

(n-2) Subproblem B_{n-1}

- Each options ((1)..(n-2)) has 1 or 2 subproblems
- Sub problem is described by indices of left and right matrix
  - Needs 2 integers to describe a subproblem
- No overlapping subproblem (yet)

Overlapping Subproblem

Have to dig deeper to identify existence of overlapping subproblem



Deriving the Recurrence Relation for D&C

- mcm(l,r)
  - The least cost to multiply $B_l \dots B_r$
- The solution is mcm(1,n-1)
- Initial Case, when (r – l) <= 1 (one or two matrices)
  - mcm(x,x) = 0
  - mcm(x,x+1) = a[x] * a[x+1] * a[x+2]

The Recurrence Relation

- Recursion Case

Subproblems | Final multiplication

$$\mathrm{mcm}(l,r) = \min \text{ of } \begin{cases} \text{min cost of } \; B_l \; \boxed{\mathrm{mcm}(l+1,r)} & + \, a_l \bullet a_{l+1} \bullet a_{r+1} \\ \text{min cost of } \; \boxed{\mathrm{mcm}(l, l+1)} \; \boxed{\mathrm{mcm}(l+2,r)} & + \, a_l \bullet a_{l+2} \bullet a_{r+1} \\ \text{min cost of } \; \boxed{\mathrm{mcm}(l, l+2)} \; \boxed{\mathrm{mcm}(l+3,r)} & + \, a_l \bullet a_{l+3} \bullet a_{r+1} \\ \ldots & \\ \text{min cost of } \; \boxed{\mathrm{mcm}(l, r-1)} \; B_r & + \, a_l \bullet a_r \bullet a_{r+1} \end{cases}$$

Divide & Conquer

```
int mcm(int l,int r) {
  if (l < r) {
    minCost = MAX_INT;
    for (int i = l;i < r;i++) {
      my_cost = mcm(l,i) + mcm(i+1,r) + (a[l] * a[i+1] * a[r+1]);
      minCost = min(my_cost,minCost);
    }
    return minCost;
  } else {
    return 0;
  }
}
```

Using bottom-up DP

- Design the table
  - M[i][j] = the best solution (min cost) for multiplying $B_i \ldots B_j$
    - M[i][j] stores mcm(i,j)
  - The solution is at M[1][n-1]
- Trivial Case
  - What is M[x][x] ?
    - No multiplication, M[x][x] = 0
- Simple case
  - What is M[x][x+1] ?
  - $B_x B_{x+1}$
  - Only one solution = $a_x * a_{x+1} * a_{x+2}$

What is M[i,j]?

- General case
  - What is M[x][x+k] ?
  - $B_x B_{x+1} B_{x+2} \ldots B_{x+k}$

$$\text{min of} \begin{cases} M[x][x] + M[x+1][x+k] & + \, a_x \bullet a_{x+1} \bullet a_{x+k+1} \\ M[x][x+1] + M[x+2][x+k] & + \, a_x \bullet a_{x+2} \bullet a_{x-k+1} \\ M[x][x+2] + M[x+3][x+k] & + \, a_x \bullet a_{x+3} \bullet a_{x+k+1} \\ \ldots & \\ M[x][x+k-1] + M[x+k][x+k] & + \, a_x \bullet a_{x+k} \bullet a_{x+k+1} \end{cases}$$

Filling the Table

M[1,1]

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

M[1,6]
(our
solution)

Filling the Table

Trivial case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Plus $a_1 \cdot a_2 \cdot a_6$

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Plus $a_1 \bullet a_3 \bullet a_6$

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Plus $a_1 \cdot a_4 \cdot a_6$

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Plus $a_1 \bullet a_5 \bullet a_6$

Filling the Table

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Filling the Table

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Example

- $a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$
- $10 \times 5 \times 1 \times 5 \times 10 \times 2$

  $B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5$

Example

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array}$$
$$10 \times 5 \times 1 \times 5 \times 10 \times 2$$

	1	2	3	4	5
1	0				
2		0			
3			0		
4				0	
5					0

Example

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array}$$

$$\boxed{10 \times 5 \times 1} \times 5 \times 10 \times 2$$

	1	2	3	4	5
1	0	50			
2		0			
3			0		
4				0	
5					0

Example

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 10 & \times\ 5 & \times\ 1 & \times\ 5 & \times\ 10 & \times\ 2 \end{array}$$

$$10 \boxed{\times 5 \times 1 \times 5} \times 10 \times 2$$

	1	2	3	4	5
1	0	50			
2		0	25		
3			0		
4				0	
5					0

Example

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

$10 \times 5 \times \boxed{1 \times 5 \times 10} \times 2$

	1	2	3	4	5
1	0	50			
2		0	25		
3			0	50	
4				0	
5					0

Example

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

$10 \times 5 \times 1 \times \boxed{5 \times 10 \times 2}$

	1	2	3	4	5
1	0	50			
2		0	25		
3			0	50	
4				0	100
5					0

Example

$$\begin{matrix} a_1 & & a_2 & & a_3 & & a_4 & & a_5 & & a_6 \\ 10 & \times & 5 & \times & 1 & \times & 5 & \times & 10 & \times & 2 \end{matrix}$$

	1	2	3	4	5
1	0	50			
2		0	25		
3			0	50	
4				0	100
5					0

Example

Option 1 = 0 + 25 + 10 x 5 x 5 = 275

Option 2 = 50 + 0 + 10 x 1 x 5 = 100

a_1 a_2 a_3 a_4 a_5 a_6

$\boxed{10} \times \boxed{5} \times 1 \times \boxed{5} \times 10 \times 2$

	1	2	3	4	5
1	0	50			
2		0	25		
3			0	50	
4				0	100
5					0

Example

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 10 \times & 5 \times & 1 \times & 5 \times & 10 \times & 2 \end{array}$$

	1	2	3	4	5
1	0	50	100 (2)		
2		0	25		
3			0	50	
4				0	100
5					0

(2) means that the minimal solution is by dividing at B_2

Example

Option 1 = 0+ 50 + 5x 1 x 10 = 100

Option 2 = 25+ 0 + 5x 5 x 10 = 275

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 10 \times & \boxed{5} \times & \boxed{1} \times & 5 \times & \boxed{10} \times & 2 \end{array}$$

	1	2	3	4	5
1	0	50	100 (2)		
2		0	25		
3			0	50	
4				0	100
5					0

Example

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 10 \times & 5 \times & 1 \times & 5 \times & 10 \times & 2 \end{array}$$

	1	2	3	4	5
1	0	50	100 (2)		
2		0	25	100 (2)	
3			0	50	70 (4)
4				0	100
5					0

Example

Option 1 = 0+ 100 + 10x 5 x 10 = 600

Option 2 = 50+ 50 + 10x 1 x 10 = 200

Option 2 = 100+ 0 + 10x 5 x 10 = 600

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

$\boxed{10} \times \boxed{5} \times 1 \times 5 \times \boxed{10} \times 2$

	1	2	3	4	5
1	0	50	100 (2)		
2		0	25	100 (2)	
3			0	50	70 (4)
4				0	100
5					0

Example

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 10 \times & 5 \times & 1 \times & 5 \times & 10 \times & 2 \end{array}$$

	1	2	3	4	5
1	0	50	100 (2)	200 (2)	
2		0	25	100 (2)	
3			0	50	70 (4)
4				0	100
5					0

Example

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array}$$
$$10 \times 5 \times 1 \times 5 \times 10 \times 2$$

	1	2	3	4	5
1	0	50	100 (2)	200 (2)	
2		0	25	100 (2)	80 (2)
3			0	50	70 (4)
4				0	100
5					0

Example

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 10 \times & 5 \times & 1 \times & 5 \times & 10 \times & 2 \end{array}$$

	1	2	3	4	5
1	0	50	100 (2)	200 (2)	140 (2)
2		0	25	100 (2)	80 (2)
3			0	50	70 (4)
4				0	100
5					0

Analysis

- There is $O(n^2)$ cell to be filled
  - Each cell has $O(n)$ options
- This totals to $O(n^3)$

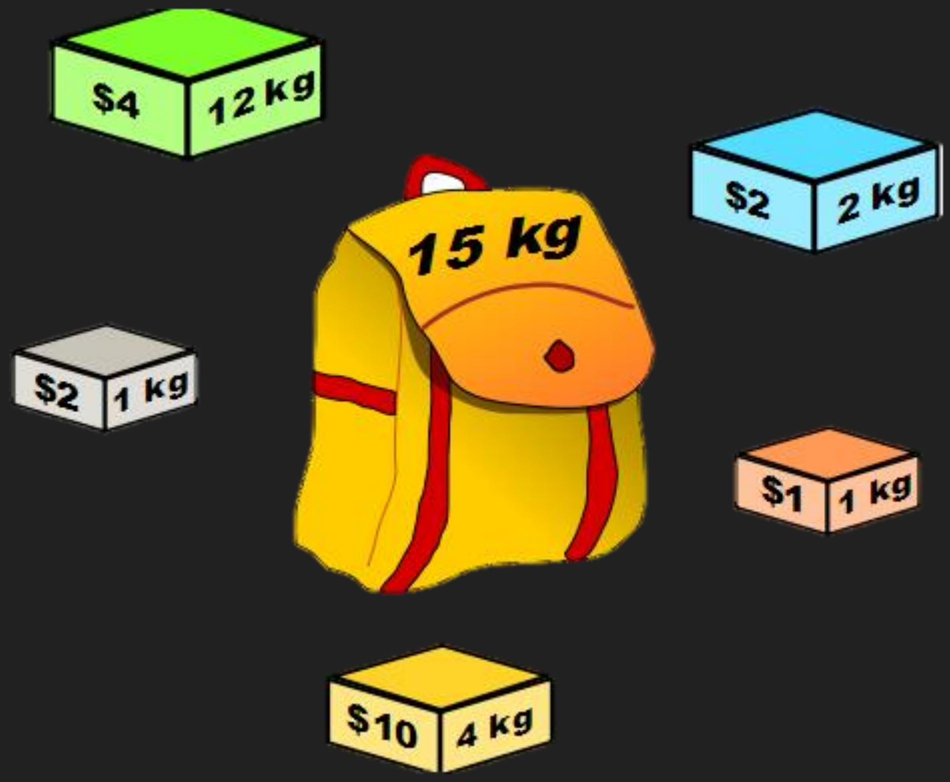
Can you write a code for Bottom-up DP of Matrix Chain Multiplication Problem?

Also can your code build the actual solution (the parenthesis of Bi, not just the minimum cost)

0-1 Knapsack Problem

Knapsack Problem

- Given a sack, able to hold W kg
- Given a list of objects
  - Each has a value and a weight
- Try to pack the object in the sack so that the total value is maximized



from http://en.wikipedia.org/wiki/File:Knapsack.svg

Variation

- Rational Knapsack
  - Object is like a gold bar, we can cut it into pieces, each has the same value/weight ratio
- 0-1 Knapsack
  - Object cannot be broken, we have to choose to take (1) or leave (0) the object
    - W = 50
    - Objects = (60, 10) (100, 20) (120, 30)
    - Best solution = second and third

The Problem

- Input:
  - A number W, the capacity of the sack
  - n values of weight and price
    - w_i = weight of the i^{th} items
    - p_i = price of the i^{th} item
- Output:
  - A subset S of {1,2,3,...,n} such that
    - $\sum_{i \in S} p_i$ is maximum
    - $\sum_{i \in S} w_i \leq W$

- Example Instance
  - W = 50
  - Pi = 60, 100, 120
  - wi = 10, 20, 30
  - Best solution = second and third

Naïve approach

```
def knapsack(W,w[1..n],p[1..n],idx,pick[1..n])
  if (idx == 0)
    sum_price = 0
    sum_weight = 0
    for i from 1 to n
      if pick[i]
        sum_price += p[i]
        sum_weight += w[i]
    if (sum_weight <= W && sum_price > max)
      max = sum_price

  pick[idx] = false
  knapsack(W,w,p,idx-1,pick)
  pick[idx] = true
  knapsack(W,w,p,idx-1,pick)
end
```

- Try every possible combination of {1,2,3,...n}
- Test whether a combination satisfies the weight constraint
  - If so, remember the best one
  - Start with knapsack(W,w,p,n,[1..n])
  - max is global var
  - $\theta(2^n * n)$

Another Naïve approach

- Keep track of remaining weight, sum the total price along the way
- What is the benefit of this approach?

```
def knapsack(W,w[1..n],p[1..n],idx,remain)
  if (idx == 0)
    return 0
  if (remain >= w[idx])
    #r1 is that we don't pick item #idx
    r1 = knapsack(W,w,p,idx-1,remain)
    #r2 is that we pick item #idx
    r2 = knapsack(W,w,p,idx-1,remain - w[idx]) + p[idx]
    return max(r1,r2)
  else
    return knapsack(W,w,p,idx-1,remain)
end
```

The Recurrence Relation

- $K(a,b)$ = the best total price when and only item number 1 to number a is considered and the knapsack is of size b

- $K(a,b) = 0$ when $a = 0$ or $b = 0$
- $K(a,b) = K(a-1,b)$ when $w_a > b$
- $K(a,b) = \max(K(a-1,b - w_a) + p_a ,$
  $K(a-1,b))$
- The solution is at $K(n,W)$

The Failed Attempt #1

- Let *K(a)* be the best total value when we consider only item number *1* to number *a* and the weight limit is *W*
  - The answer is at *K(n)*
  - By definition, *K(n)* and *K(n-1)* and *K(n-2)*... all consider the same weight limit
- Let's say that the answer contains item number n
  - Also by definition, its means that K(n) = K(n-1) + pn
  - However, K(n-1) will consider the problem thinking that the weight limit is the same (not reduced by weight of item number n)
  - It is wrong to say that $K(a) = \max(K(a-1) + p_a , K(a-1))$
  - It is not possible to have a recurrence relation that does not consider W

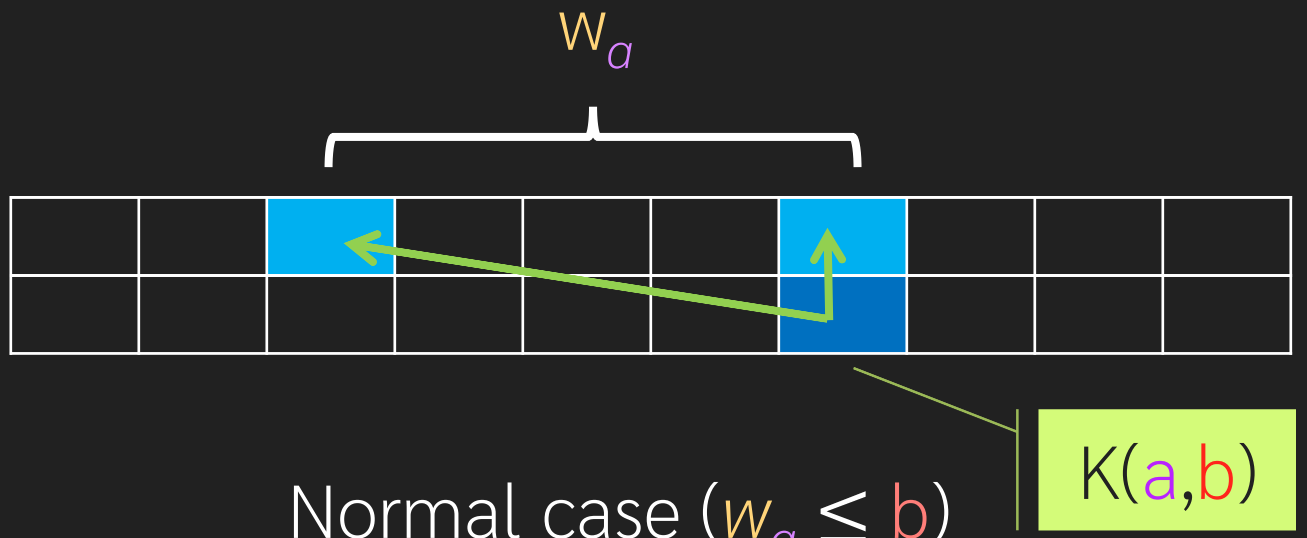
The Failed Attempt #2

- Let $K(b)$ be the best total value when the weight limit of the sack is b
  - The answer is at $K(W)$
- If the i^{th} item is in the best solution
  - $K(W) = K(W - w_i) + p_i$
- But, we don't really know that the i^{th} item is in the optimal solution
  - So, we try everything
  - $K(W) = max_{1 \leq i \leq n}(K(W - w_i) + p_i)$
- Is this our algorithm?
  - Yes, if and only if we allow each item to be selected multiple times (that is not true for this problem)

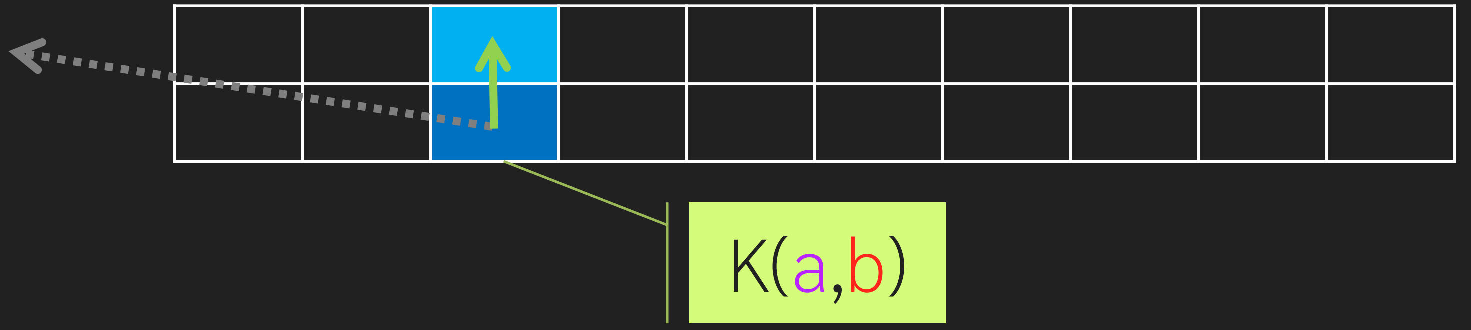
Exercise: Top-Down approach

- Write a top down dynamic programming approach using this recurrence relation
  - $K(a,b) = 0$ when $a = 0$ or $b = 0$
  - $K(a,b) = K(a-1,b)$ when $w_a > b$
  - $K(a,b) = \max(K(a-1,b - w_a) + p_a ,$
    $K(a-1,b))$
- Which data structure should we use to store result?
  - Should we use 2D array?
  - Should we use associative data structure such as std::map or std::unordered_map?

The Table for Bottom-Up



Normal case ($w_a \leq b$)



Too much weight ($w_a > b$)

- Row = item id (a)
- Col = weight (b)

Example

p = {4, 2, 2, 1, 10}
w ={12, 2, 1, 1, 4}   W = 15

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0																
1																
2																
3																
4																
5																

Example

p = {4, 2, 2, 1, 10}
w ={12, 2, 1, 1, 4}   W = 15

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0															
2	0															
3	0															
4	0															
5	0															

$K(a,b) = 0$ when $a = 0$ or $b = 0$

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 1 ($p_1=4$ $w_1=12$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0															
2	0															
3	0															
4	0															
5	0															

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 1 ($p_1=4$ $w_1=12$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0				
2	0															
3	0															
4	0															
5	0															

$K(a,b) = K(a-1,b)$ when $w_a > b$

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $\quad W = 15$

Fill row 1 $\quad (p_1=4 \quad w_1=12)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0															
3	0															
4	0															
5	0															

$$K(a,b) = \max(\ K(a-1, b - w_a) + p_a\ , \quad K(a-1,b)\ )$$

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 2 ($p_2=2$ $w_2=2$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0														
3	0															
4	0															
5	0															

$K(a,b) = K(a,b-1)$ when $w_b > a$

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 2 $(p_2=2 \quad w_2=2)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0														
3	0															
4	0															
5	0															

$K(a,b) = K(a-1,b)$ when $w_a > b$

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 2 ($p_2=2$ $w_2=2$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0	2	2	2	2	2	2	2	2	2	2	4	4	6	6
3	0															
4	0															
5	0															

$$K(a,b) = \max(\ K(a-1, b - w_a) + p_a\ ,\quad K(a-1,b)\ )$$

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 2 ($p_2=2$ $w_2=2$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0	2	2	2	2	2	2	2	2	2	2	4	4	6	6
3	0															
4	0															
5	0															

$$K(a,b) = \max(\, K(a-1, b - w_a) + p_a \,,\quad K(a-1,b)\,)$$

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 3 ($p_3 = 2$ $w_3 = 1$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0	2	2	2	2	2	2	2	2	2	2	4	4	6	6
3	0	2	2	4	4	4	4	4	4	4	4	4	4	6	6	8
4	0															
5	0															

$$K(a,b) = \max(\, K(a-1, b - w_a) + p_a \,,\quad K(a-1,b)\,)$$

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 3 ($p_3 = 2$ $w_3 = 1$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0	2	2	2	2	2	2	2	2	2	2	4	4	6	6
3	0	2	2	4	4	4	4	4	4	4	4	4	4	6	6	8
4	0															
5	0															

Example

$p = \{4, 2, 2, 1, 10\}$

$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 4 ($p_4=1$ $w_4=1$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0	2	2	2	2	2	2	2	2	2	2	4	4	6	6
3	0	2	2	4	4	4	4	4	4	4	4	4	4	6	6	8
4	0	2	3	4	5	5	5	5	5	5	5	5	5	6	7	8
5	0															

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 4 $(p_4 = 1 \quad w_4 = 1)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0	2	2	2	2	2	2	2	2	2	2	4	4	6	6
3	0	2	2	4	4	4	4	4	4	4	4	4	4	6	6	8
4	0	2	3	4	5	5	5	5	5	5	5	5	5	6	7	8
5	0															

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 5 ($p_5 = 10$ $w_5 = 4$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0	2	2	2	2	2	2	2	2	2	2	4	4	6	6
3	0	2	2	4	4	4	4	4	4	4	4	4	4	6	6	8
4	0	2	3	4	5	5	5	5	5	5	5	5	5	6	7	8
5	0	2	3	4	10	12	13	14	15	15	15	15	15	15	15	15

Example

$p = \{4, 2, 2, 1, 10\}$
$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Fill row 5 ($p_5 = 10$ $w_5 = 4$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0	2	2	2	2	2	2	2	2	2	2	4	4	6	6
3	0	2	2	4	4	4	4	4	4	4	4	4	4	6	6	8
4	0	2	3	4	5	5	5	5	5	5	5	5	5	6	7	8
5	0	2	3	4	10	12	13	14	15	15	15	15	15	15	15	15

Example

$p = \{4, 2, 2, 1, 10\}$

$w = \{12, 2, 1, 1, 4\}$ $W = 15$

Trace the solution backward to get the actual item number

We have item number 5,4,3,2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4
2	0	0	2	2	2	2	2	2	2	2	2	2	4	4	6	6
3	0	2	2	4	4	4	4	4	4	4	4	4	4	6	6	8
4	0	2	3	4	5	5	5	5	5	5	5	5	5	6	7	8
5	0	2	3	4	10	12	13	14	15	15	15	15	15	15	15	15

Bottom-Up Code

```
set all K[0][*] = 0 and all K[*][0] = 0
for a = 1 to n
  for b = 1 to W
    if (w[a] > b)
      K[a][b] = K[a - 1][b];
    else
      K[a][b] = max( K[a - 1][b – w[a]] + p[a] ,
                     K[a – 1][b] )
return K[n][W];
```

Can you write a code that generate the list of actual item that we take?

- Does this code generate too much subproblem?
- Does it generates one that we does not need?
- Is it better to use Top-Down approach?
  - Can you show some instance that Top-Down is better than Bottom-up (this code)

Analysis

- From Bottom-Up, it is clear that this is $O(Wn)$
- Original generate-all-solution method is $O(2^n)$
- Which one is better
  - In what case that $O(Wn)$ Dynamic Programming will benefit greatly (because there are several overlapping subproblems)