On Approximating Hard Integrals with the Double-Exponential Formula

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Abstract

Approximating $I_{\#\mathrm{PART}} = \frac{1}{2}\int_{-1}^{1}\prod_{k=1}^{n}\cos(x_k\pi t)\,dt$ to within an accuracy of 2^{-n} is equivalent to counting the number of equal-sum partitions of a set of positive integers $\{x_k\}_{k=1}^{n}$, and is thus a #P problem. Efficient numerical integration methods such as the double exponential formula, also known as tanh-sinh quadrature, have been around from the mid 70's. Taking note of the hardness of approximating $I_{\#\mathrm{PART}}$ we argue that unless P=NP the proven rates of convergence of such methods cannot possibly be correct. In addition we provide a novel and generalized theorem that implies the Double Exponential, having an elementary and constructive proof.

1 Overview

The Partition Counting Problem (#PART) is the following: given n positive integers $\{x_k\}_{k=1}^{n}$, in how many ways is it possible to divide them into two equal-sum subsets. Analytic and number-theoretic approaches to this problem can be found in many works, many seem to go back to the classic monograph [1] by Kac. If the input $\{x_k\}$ is given in binary rather unary radix, then solving this problem in polynomial time wrt the input's length would prove P=#P and would also entail P=NP. Assuming the exponential time hypothesis, #PART cannot be solved in polynomial time.

The treatment in [1] and subsequently in many other places e.g. [2, 3, 4] express the number of equal-sum partitions by the integral

$$2^n I_{\#\mathrm{PART}} = 2^{n-1}\int_{-1}^{1}\prod_{k=1}^{n}\cos(x_k\pi t)\,dt$$

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An elementary proof of this result is provided in the ensuing.

The double-exponential (DE) tanh-sinh quadrature is a numerical integration technique whose convergence rate has been proven to be exponential with the number of evaluation points [4, 5, 6, 7]. It is currently considered as the fastest high-precision quadrature technique. Noting the hardness of approximating $I_{\#\text{PART}}$ we argue here that, unless P=NP, the DE convergence rate as stated in [4, 5, 6, 7] cannot be correct.

Comparing to currently known results, we compute a finer runtime complexity for a wider families of quadratures, and classify the ones that result with exponential convergence as tanh-sinh.

2 The partition problem

Given $n \in \mathbb{N}$ and $\{x_k\}_{k=1}^n \subset \mathbb{Z}$, we seek $\sigma \in \{-1,1\}^n$ such that $\langle \sigma, \mathbf{x} \rangle = 0$, where $\langle \sigma, \mathbf{x} \rangle = \sum_{k=1}^n \sigma_k x_k$ denotes the inner product. Deciding whether such σ exists is a NP Complete problem, while counting how many such σ's exists, is in #P. We assume that the inputs $\{x_k\}$ are given in binary radix and denote by d_k the number of binary digits of x_k. The partition problem is known to be Weak-NP since it has a polynomial-time algorithm if the input is supplied in unary radix. To get a feeling about typical dimensions of hard problems, the reduction of n-clause and k-variables 3SAT into the partition problem ends up with $\mathcal{O}(n+k)$ integers to partition, each having $\mathcal{O}(n+k)$ digits [10]. The exponential time hypothesis therefore implies that it is impossible to solve the partition problem in runtime complexity of $\mathcal{O}\left(\text{poly}\left(\sum_{k=1}^n d_k\right)\right)$.

The counting version of the partition problem is equivalent to the following definite integral:

Lemma 1. *Let $\{x_k\}_{k=1}^n \subset \mathbb{Z}$ be integers given in binary radix. Let also $\psi(t) = \prod_{k=1}^n \cos(\pi x_k t)$. Then evaluating $I_{\#PART} = \frac{1}{2}\int_{-1}^{1} \psi(t)\, dt$ up to accuracy of n binary digits is in #P.*

Proof. This lemma can be proved in many interesting ways, all seem to go back to the classical monograph by Kac [1]. Slightly different proofs of this lemma may be found in [2, 4]. Our derivation is based on the formula

$$\prod_{k=1}^{n} \cos(z_k) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \cos\langle \sigma, \mathbf{z} \rangle \tag{1}$$

for every $\mathbf{z} \in \mathbb{C}^n$, which follows from a repeated application of the identity

$$4\cos(z_1)\cos(z_2) = \cos(z_1+z_2) + \cos(z_1-z_2) + \cos(-z_1+z_2) + \cos(-z_1-z_2) \tag{2}$$

Using this the integral reads

$$I_{\#\mathrm{PART}} = 2^{-n-1} \sum_{\sigma \in \{-1,1\}^n} \int_{-1}^{1} \cos\left(\pi t \left\langle \sigma, \mathbf{x} \right\rangle\right) dt = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{\sin \pi \left\langle \sigma, \mathbf{z} \right\rangle}{\pi \left\langle \sigma, \mathbf{z} \right\rangle}$$

$$= 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \begin{cases} 1 & \text{if } \left\langle \sigma, \mathbf{z} \right\rangle = 0 \\ 0 & \text{if } \left\langle \sigma, \mathbf{z} \right\rangle \neq 0 \end{cases} \tag{3}$$

Thus, $I_{\#\mathrm{PART}}$ is precisely the fraction of zero partitions for $\{x_k\}_{k=1}^{n}$ divided by 2^n. This also explains why an accuracy of at least 2^{-n} is required. □

3 Quadrature Formulas

The DE formula approximates an integral using a weighted sum of $2N+1$ terms. The convergence rate of this method to the actual integral is exponential in N for well-behaved integrands [4, 5, 6, 7].

Recall that the Hardy space H^2 is the space of all functions f satisfying

$$\sup_{r \in [0,1)} \left[\frac{1}{2\pi} \int_{0}^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta \right]^2 < \infty$$

and recall that our integrand $\psi(t) = \prod_{k=1}^{n} \cos(\pi x_k t)$ is holomorphic and is bounded over any finite-measure complex region, so $\psi(t) \in H^2$.

The main result in [5] is its Theorem 5.1. Restating it using a simplified notation:

Theorem 2. *Let $f \in H^2$, $N \in \mathbb{N}$, $h > 0$. Let also $w(t) = \tanh\left(\frac{\pi}{2}\sinh t\right)$. Approximating the integral*

$$I_f = \int_{-1}^{1} f(t)\, dt = \int_{-\infty}^{\infty} f(w(u))\, w'(u)\, du$$

using the sum

$$\hat{I}_f = h \sum_{m=-N}^{N} f(w(mh))\, w'(mh)$$

has an approximation error of

$$\left| I_f - \hat{I}_f \right|^2 \leq \mathcal{O}\left(e^{-cN}\right) + \left(1 + \frac{4}{\pi h}\right) \mathcal{O}\left(e^{-\frac{1}{\pi} e^{N-1}}\right)$$

for some constant $c > 0$ independent of f, h and N.

A proof for an error bound $\mathcal{O}\left(e^{\frac{-cN}{\log N}}\right)$ can be found in [6]. See also [7].

We now prove a more generalized result independently of previous derivations.

Theorem 3. *Let $n \in \mathbb{N}$, given analytic f such that for all $t \in [-1, 1]$:*

$$|f(t)| \leq A, |f'(t)| \leq B, |f''(t)| \leq C$$

and given w such that for all $t \in \mathbb{R}$:

$$w(t) = w(-t), |w(t)| \leq D, |w'(t)| \leq E, |w''(t)| \leq F,$$
$$|w'''(t)| \leq G, w(\infty) = 1, w(-\infty) = -1$$

and $w'(t) \neq 0$ for all $t \neq 0$. Then

$$\left| \int_{-1}^{1} f(t)\, dt - 2^{1-N} \sum_{m=-N}^{N} f(w(m))\, w'(m) \right| \leq 2^{-n}$$

where

$$p = \left\lceil -\log_2 \frac{\sqrt{\left(BE^2 + AF\right)^2 - 2^{1-n}\left(CE^3 + 3BEF + AG\right)} - BE^2 - AF}{CE^3 + 3BEF + AG} \right\rceil$$

(interestingly, this implies that "better" w has lower E, F and higher G) and

$$N = w^{-1}\left(\frac{2^{-n-3}}{A} - 1\right) 2^p$$

If, in addition, $\lim_{x\to\infty} \sqrt{x} w^{-1}(x) = 0$ at least polynomialy fast, then N increases only polynomialy wrt n since then $N = \mathcal{O}\left(2^{-\frac{1}{2}n} w^{-1}\left(2^{-n}\right)\right)$. For the special case $A = 1, w(t) = t$:

$$N = \frac{C\left(1 - 2^{-n-3}\right)}{\sqrt{B^2 - 2^{1-n}C} - B}$$

Proof. Put $g(t) = f(w(t))\, w'(t)$ as the function to be evaluated. If using Kahan summation, then we need to calculate $g(t)$ only up to the desired integral accuracy, namely n digits. So we wish to find ϵ such that

$$|g(t + \epsilon) - g(t)| \leq 2^{-n}$$

using Taylor theorem, there exists ξ such that:

$$|g(t + \epsilon) - g(t)| = \left| \epsilon g'(t) + \frac{1}{2} g''(\xi)\, \epsilon^2 \right|$$

we note that

$$|g'(t)| = \left|f'(w(t))\left[w'(t)\right]^2 + f(w(t))\,w''(t)\right| \le BE^2 + AF$$

and

$$|g''(t)| = \left|f''(w(t))\left[w'(t)\right]^3 + 3f'(w(t))\,w'(t)\,w''(t) + f(w(t))\,w'''(t)\right|$$

$$\le CE^3 + 3BEF + AG$$

but

$$\left|\epsilon g'(t) + \frac{1}{2}g''(\xi)\,\epsilon^2\right| \le |\epsilon g'(t)| + \left|\frac{1}{2}g''(\xi)\,\epsilon^2\right|$$

$$\le |\epsilon|\left(BE^2 + AF\right) + \frac{CE^3 + 3BEF + AG}{2}\epsilon^2$$

the last term is clearly increasing with positive ϵ. Equating it to 2^{-n} we get one positive root:

$$\epsilon = \frac{\sqrt{\left(BE^2 + AF\right)^2 - 2^{1-n}\left(CE^3 + 3BEF + AG\right)} - BE^2 - AF}{CE^3 + 3BEF + AG}$$

So far we have shown that we need to calculate $g(t)$ up to accuracy of n digits, and this will give us, using Kahan summation, the integral up to the desired accuracy. Since we have just shown that we need no more than

$$p = \left\lceil -\log_2 \frac{\sqrt{\left(BE^2 + AF\right)^2 - 2^{1-n}\left(CE^3 + 3BEF + AG\right)} - BE^2 - AF}{CE^3 + 3BEF + AG}\right\rceil$$

for t in order to get such evaluation and know that our integral indeed converges, we have only 2^p possible different values for $t \in [-1, 1]$. Indeed, we need not sample the interval $[-1, 1]$ only, and now we turn to calculate the desired sampling interval.

Recalling w is even, we seek $z < 0$ such that

$$\int_{-\infty}^{z} g(t)\,dt \le 2^{-n-3}$$

indeed it is sufficient to have

$$\int_{-\infty}^{z} g(t)\,dt \le A\int_{-\infty}^{z} w'(t)\,dt \le A\left(1 + w(z)\right) \le 2^{-n-3}$$

$$\implies z \leq w^{-1}\left(\frac{2^{-n-3}}{A}-1\right)$$

our function is indeed invertible over the negavite half line since its derivative never vanish. So our interval is

$$|t| \leq w^{-1}\left(\frac{2^{-n-3}}{A}-1\right)$$

with granularity of p above, ending up with total of

$$2w^{-1}\left(\frac{2^{-n-3}}{A}-1\right)\frac{\sqrt{\left(BE^2+AF\right)^2-2^{1-n}\left(CE^3+3BEF+AG\right)}-BE^2-AF}{CE^3+3BEF+AG}$$

function evaluations, since these are all possible inputs on this interval by the desired and implied accuracy. We can observe that the asymptotic behavior of (1) wrt n is decreasing exponentially as long as w^{-1} decreases faster than square root. $\square$

Remark 4. For clear reasons, we'd be happy to have $w^{-1}\left(\frac{2^{-n}}{A}-1\right)=2^{-k}$ for some integer k. Assume $A=1$:

$$w\left(2^{-k}\right)=2^{-n}-1$$
$$w\left(t\right)=ct-1$$
$$\frac{y+1}{c}=t$$

suggesting

$$w\left(t\right)=1+\log_2\left(t\right)$$

Impossibility result

#SAT is the problem of counting the number of satisfying assignments of a CNF formula. It is the counting problem associated with a Strong-NP problem, the Boolean Satisfiability problem. The preceding analysis suggests that unless Theorem 2 and possibly other proven convergence rates of the DE formula turn up wrong in the case of $I_{\#\mathrm{PART}}$, #SAT may be solved in polynomial time.

Corollary 5. *Theorem 2 anr 3 are incorrect for otherwise #SAT may be solved in polynomial time.*

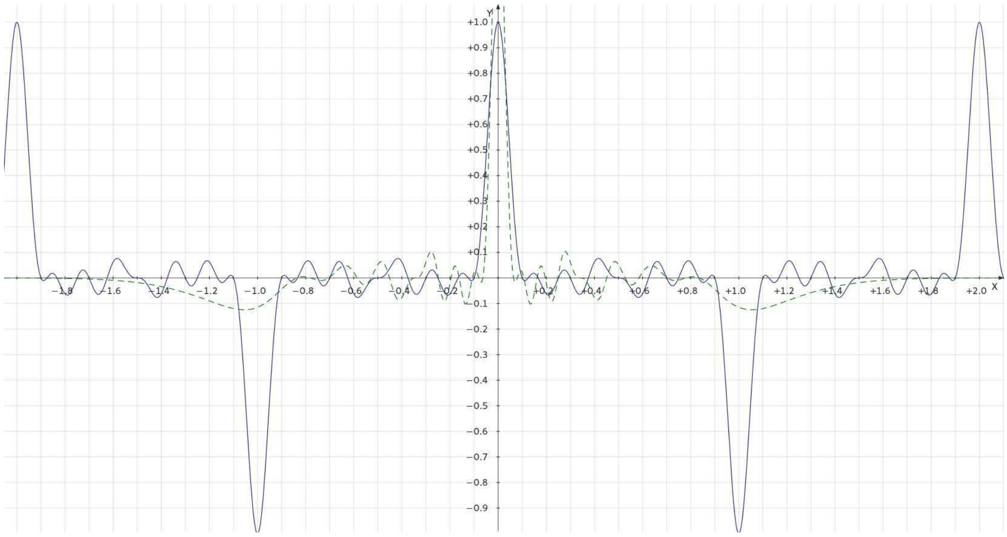

Figure 1: The function $f(t) = \cos(\pi t)\cos(2\pi t)\cos(3\pi t)\cos(4\pi t)\cos(5\pi t)\cos(6\pi t)$ (continuous line) and $f(w(t))w'(t)$ (dashed).

Proof. Reducing #SAT with n clauses and k variables into #PART ends up with $\mathcal{O}(n+k)$ numbers to partition each having $\mathcal{O}(n+k)$ digits [10]. By Lemma 1 this problem is equivalent to approximating $n+k$ digits of the integral $I_{\#\text{PART}}$. Our integrand clearly fulfills the conditions of theorems 2 and 3 and so the number of evaluations needed to compute the $(n+k)$-digit approximation $\hat{I}_{\#\text{PART}}$ is linear in $n+k$. Because evaluating the integrand once costs polynomial time, the corollary follows. $\square$

4 Concluding remark

The DE convergence rates should be reexamined for they currently suggest the existence of a polynomial time solution to a #P problem, though obviously we are unable to rule out the possibility that P=NP.

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