Statistical Properties of Trigonometric Functions with Applications to NP-Complete Problems

January 15, 2016

1 Main Result

Theorem 1.1. *For all* $\mathbf{x} \in \mathbb{N}^n$

$$\exp\left(-2^{-n} + \sum_{m=1}^{\infty} \frac{\prod_{k=1}^{n} \cos(2x_k m)}{m + \frac{1}{2}}\right) = \sqrt[2^n]{\prod_{\sigma \in \{-1,1\}^n} |\tan\langle \mathbf{x}, \sigma \rangle|} \tag{1}$$

and

$$\exp\left(-2^{-n} - \ln\sqrt{2} + \sum_{m=1}^{\infty} \frac{\prod_{k=1}^{n} \cos(2x_k m)}{m}\right) = \sqrt[2^n]{\prod_{\sigma \in \{-1,1\}^n} |\sin\langle \mathbf{x}, \sigma \rangle|} \tag{2}$$

Proof. Observe that

$$\begin{aligned}
4\sum_{m=0}^{\infty} \cos(mx)\cos(my) &= \sum_{n=1}^{\infty} e^{im(x+y)} + e^{im(x-y)} && (3)\\
&= \frac{1}{1-e^{i(x+y)}} + \frac{1}{1-e^{i(x-y)}} && (4)\\
&= \frac{1-e^{i(x-y)} - e^{i(x+y)} + e^{2xi}}{1-e^{i(x-y)} - e^{i(x+y)} + e^{2xi}} && (5)\\
&= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} && (6)
\end{aligned}$$

Recalling that $\frac{1}{1-t} + \frac{1}{1-\frac{1}{t}} = 1$ for all nonzero complex t, then by the formulae of sum-of-angles and geometric progression we write:

$$\begin{aligned}
\lim_{r\to 1^-} 4\sum_{m=0}^{\infty} r^m \cos(mx)\cos(my) &= \lim_{r\to 1^-} \sum_{n=1}^{\infty} r^m e^{im(x+y)} + r^m e^{im(x-y)} && (7)\\
&= \lim_{r\to 1^-} \frac{1}{1-re^{i(x+y)}} + \frac{1}{1-re^{i(x-y)}} && (8)\\
&= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} && (9)
\end{aligned}$$

if x, y cannot ever meet 2π on some integer multiple. Denote

$$f(m) \equiv \prod_{k=1}^{n} \cos(x_k m) \tag{10}$$

and write

$$\lim_{r\to1^-}2^n\sum_{m=1}^{\infty}r^m f(m) = \lim_{r\to1^-}\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}r^m e^{im\langle\mathbf{x},\sigma\rangle} \tag{11}$$

$$= \lim_{r\to1^-}\sum_{\sigma\in\{-1,1\}^n}\frac{1}{1-re^{i\langle\mathbf{x},\sigma\rangle}} \tag{12}$$

$$= \begin{cases}-\infty & \exists\sigma\in\{-1,1\}^n \mid \langle\mathbf{x},\sigma\rangle=0\\ 0 & \forall\sigma\in\{-1,1\}^n, \langle\mathbf{x},\sigma\rangle\neq0\end{cases} \tag{13}$$

since $\sum_{k=1}^{\infty}\frac{t^k}{k}=\ln\frac{1}{1-t}$ we can write

$$\lim_{r\to1^-}-2^n\sum_{m=1}^{\infty}\frac{r^m}{m}f(m) = \lim_{r\to1^-}\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{r^m}{m}e^{im\langle\mathbf{x},\sigma\rangle} \tag{14}$$

$$= \lim_{r\to1^-}\sum_{\sigma\in\{-1,1\}^n}\ln\frac{1}{1-re^{i\langle\mathbf{x},\sigma\rangle}} \tag{15}$$

implying:

$$\lim_{r\to1^-}\exp\left(-2^n\sum_{m=1}^{\infty}\frac{r^m}{m}f(m)\right)=\lim_{r\to1^-}\prod_{\sigma\in\{-1,1\}^n}\left(1-re^{i\langle\mathbf{x},\sigma\rangle}\right) \tag{16}$$

for $r\to1$ we have $1-re^{i\langle\mathbf{x},\sigma\rangle}=0$ iff $\langle\mathbf{x},\sigma\rangle$ is a zero partition. Taking the limit:

$$\exp\left(-2^n\sum_{m=1}^{\infty}\frac{f(m)}{m}\right) = \prod_{\sigma\in\{-1,1\}^n}\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right) \tag{17}$$

$$= \prod_{\sigma\in\{-1,1\}^n}\sqrt{2\sin^2\frac{\langle\mathbf{x},\sigma\rangle}{2}} \tag{18}$$

where the latter equalit is due to every σ there exists a matching $-\sigma$ and

$$\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right)\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right) = 2-2\cos\langle\mathbf{x},\sigma\rangle \tag{19}$$

$$= 2\sin^2\frac{\langle\mathbf{x},\sigma\rangle}{2} \tag{20}$$

concluding

$$\exp\left(-2^n\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right) = 2^{2^{n-1}}\prod_{\sigma\in\{-1,1\}^n}\left|\sin\langle\mathbf{x},\sigma\rangle\right| \tag{21}$$

$$\Longrightarrow \exp\left(-2^n\left(\ln\sqrt{2}+\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right)\right) = \prod_{\sigma\in\{-1,1\}^n}\left|\sin\langle\mathbf{x},\sigma\rangle\right| \tag{22}$$

equivalently,

$$-2^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(m) = -\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{im\langle\mathbf{x},\sigma\rangle} \tag{23}$$

$$= \sum_{\sigma\in\{-1,1\}^n}\ln\left(1+e^{i\langle\mathbf{x},\sigma\rangle}\right) \tag{24}$$

or

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m)\right) = \prod_{\sigma\in\{-1,1\}^n} \left(1+e^{i\langle \mathbf{x},\sigma\rangle}\right) \tag{25}$$

$$= \prod_{\sigma\in\{-1,1\}^n} \sqrt{2\cos^2 \frac{\langle \mathbf{x},\sigma\rangle}{2}} \tag{26}$$

$$\Longrightarrow \exp\left(-2^n \left(\ln\sqrt{2} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m)\right)\right) = \prod_{\sigma\in\{-1,1\}^n} \left|\cos\langle \mathbf{x},\sigma\rangle\right| \tag{27}$$

dividing (22) by (27) and taking the root we conclude:

$$e^{2^n\left(\ln\sqrt{2}+\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(2m)\right)} e^{-2^n\left(\ln\sqrt{2}+\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right)} \tag{28}$$

$$= \exp\left(2^n\left(\textstyle\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(2m) - \frac{1}{m}f(2m)\right)\right) \tag{29}$$

$$= \exp\left(\textstyle\sum_{m=1}^{\infty}\frac{2}{2m+1}f(4m+2)\right) \tag{30}$$

$$= \sqrt[2^n]{\textstyle\prod_{\sigma\in\{-1,1\}^n}\left|\tan\langle \mathbf{x},\sigma\rangle\right|} \tag{31}$$

□