Spectral and Modular Analysis of #P Problems

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Abstract

We present various analytic and number theoretic results concerning the #SAT problem as reflected when reduced into a #PART problem. As an application we propose a heuristic to probabilistically estimate the solution of #SAT problems.

1 Overview

#SAT is the problem of counting the number of satisfying assignments to a given 3CNF formula, while #PART is the problem of counting the number of zero partitions in a given set of integers. Precise definitions will be given later on. We present various results concerning #PART and analyze their connection with #SAT. On section 2 we skim some preliminaries. Section 3 presents the core of the analytic setting by analyzig the #PART problem as manipulations over product of cosines. Section 4 derives a modular-arithmetic formula for computing #PART, and section 5 presents implications to complexity theory. Section 6 deals with asymptotics, and section 7 deals with variances and correlations. Section 8 propose how multiple reductions may give probabilistic answer to #SAT as a consequence our analysis. Section 9 summarizes the highlights of the paper.

2 Preliminaries

Our setting is counting the number of solutions given an instance of the Partition problem. We sometimes use custom terminology as there is no unified one.

Definition 2.1. Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$, a *Partition* σ of $\mathbf{x}$ is some $\sigma \in \{-1,1\}^n$. The *size* of the partition σ is $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^{n} \sigma_k x_k$. A partition is called a *zero partition* if its size is zero. The problem #PART is to determine the number of zero partitions given $\mathbf{x}$. The problem PART is deciding whether a zero partition exists or not for $\mathbf{x}$. The *Weak* setting of the problem is when $\mathbf{x}$ is supplied in unary radix, and the *Strong* setting is when it is supplied in binary radix (or another format with same efficiency), therefore the input size is logarithmically smaller on the strong setting.

#PART is in #P complexity class. The setting of #PART after being reduced from the counting Boolean Satisfiability problem (#SAT) is n integers to partition each having up to $\mathcal{O}(n)$ binary digits (where n is linear in the size of the CNF formula), demonstrating why the rather strong setting is of interest. In fact, there exist polynomial time algorithms solving the weak setting of PART, notably Dynamic Programming algorithms, as well as the formula derived on Theorem 4.1 below. However, solving PART under the strong setting is not possible in polynomial time (as a function of the input's length), unless P=NP.

#SAT can be reduced to #SUBSET-SUM using an algorithm described in [1], while various slight variations appear on the literature. We summarize the reductions on the Appendix.

3 Analytic Setting

Theorem 3.1. *Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ then the (probability-theoretic) characteristic function of the random variable $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^{n} x_k \sigma_k$ over uniform $\sigma \in \{-1,1\}^n$ is $\prod_{k=1}^{n} \cos(x_k t)$.*

Proof. Consider the formula $2\cos a \cos b = \cos(a+b) + \cos(a-b)$ and the cosine being even function to see that:

$$\psi(t) \equiv \psi(x_1, \ldots, x_n, t) \equiv \prod_{k=1}^{n} \cos(x_k t) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \cos(t \langle \mathbf{x}, \sigma \rangle) = \mathbb{E}\left[e^{it\langle \mathbf{x}, \sigma \rangle}\right] \tag{1}$$

□

Corollary 3.2. *Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ then*

$$2^n \int_0^1 \prod_{k=1}^{n} \cos(2\pi x_k t)\, dt \tag{2}$$

is the number of zero partitions of $\mathbf{x}$.

Proof. Following (1) and integrating both sides. Stronger statements are possible (e.g. for $\mathbf{x} \in \mathbb{R}^n$ or $\mathbf{x} \in \mathbb{C}^n$) using characteristic function inversion theorems. □

Theorem 3.3. *Given $\{n, N\} \subset \mathbb{N}, j \in \mathbb{Z}, \mathbf{x} \in \mathbb{N}^n$ then*

$$\frac{2^n}{N} \sum_{m=1}^{N} e^{2\pi i j \frac{m}{N}} \prod_{k=1}^{n} \cos\left(2\pi x_k \frac{m}{N}\right) \tag{3}$$

is the number of partitions of $\mathbf{x}$ having size that is divisible by N with remainder j.

Proof. Following (1)

$$\frac{2^n}{N} \sum_{m=1}^{N} \prod_{k=1}^{n} \cos\left(2\pi x_k \frac{m}{N}\right) = \sum_{\sigma \in \{-1,1\}^n} \frac{1}{N} \sum_{m=1}^{N} e^{2\pi i \frac{m}{N} \langle \mathbf{x}, \sigma \rangle} \tag{4}$$

The sum of the roots of unity on the rhs of (4) is zero if N does not divide $\langle \mathbf{x}, \sigma \rangle$, and is one if N does divide it, therefore (4) is equal to

$$\sum_{u=-\infty}^{\infty} c_{uN} \tag{5}$$

where c_u denotes the number of partitions that sum to u:

$$c_u = |\{\sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = u\}| \tag{6}$$

As for the remainder, observe that

$$\sum_{\sigma \in \{-1,1\}^n} e^{2\pi i t(\langle \mathbf{x}, \sigma \rangle + j)} = e^{2\pi i t j} 2^n \prod_{k=1}^{n} \cos(2\pi x_k t) \tag{7}$$

□

4 Modular Arithmetic Formula

Theorem 4.1. *Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$, the number of $\mathbf{x}$'s zero partition out of all possible 2^n partitions is encoded as a binary number in the binary digits of*

$$\prod_{k=1}^{n} \left[1 + 4^{nx_k}\right] \tag{8}$$

from the s'th dight to the $s+n$ digit, where $s = n\langle \mathbf{x}, 1\rangle$.

Proof. We write down the following sum and perform substitution according to (1):

$$S = \frac{1}{n}\sum_{m=1}^{n} \psi\left(\frac{2\pi m}{n} + i\ln 2\right) = \sum_{\sigma\in\{-1,1\}^n} \frac{2^{-\langle \mathbf{x},\sigma\rangle}}{n} \sum_{m=1}^{n} e^{\frac{2\pi i m}{n}\langle \mathbf{x},\sigma\rangle} \tag{9}$$

multiplying all x_k by n (while preserving partitions, since we can always multiply all x_k by the same factor and keep the exact number of zero partitions) puts $e^{\frac{2\pi i m}{n}\langle n\mathbf{x},\sigma\rangle} = 1$ and we get:

$$S = \sum_{\sigma\in\{-1,1\}^n} 2^{-n\langle \mathbf{x},\sigma\rangle} = \sum_{u=-\infty}^{\infty} c_u 2^{-u} \tag{10}$$

where c_u is defined in. Recalling that $\sum_{u=-\infty}^{\infty} c_u = 2^n$ and c_u are all positive, while on (10) being multiplied by distinct powers $2^{\pm n}$, therefore the summands' binary digits never interfere with each other. Recalling that c_0 is our quantity of interest, we have shown that the number of zero partitions in $\mathbf{x}$ is encoded at

$$\left\lfloor \frac{2^n}{n}\sum_{m=1}^{n}\prod_{k=1}^{n}\cos\left[nx_k\left(\frac{2\pi m}{n} + i\ln 2\right)\right]\right\rfloor \mod 2^n \tag{11}$$

$$= \left\lfloor \prod_{k=1}^{n}\left[2^{nx_k} + 2^{-nx_k}\right]\right\rfloor \mod 2^n = \left\lfloor 2^{-n\sum_{k=1}^{n} x_k}\prod_{k=1}^{n}\left[1 + 2^{2nx_k}\right]\right\rfloor \mod 2^n \tag{12}$$

Set

$$M = \prod_{k=1}^{n}\left[1 + 2^{2nx_k}\right] = \sum_{\sigma\in\{0,1\}^n} 2^{2n\langle \mathbf{x},\sigma\rangle} \tag{13}$$

then (12) tells us that the number of zero partitions is encoded as a binary number in the binary digits of M, from the s'th dight to the $s+n$ digit. □

Note that the substitution in (9) could take a simpler form. Put $t = i\ln 2$ in (1):

$$\prod_{k=1}^{n}\cosh\left(x_k\ln 2\right) = \prod_{k=1}^{n}\left[2^{x_k} + 2^{-x_k}\right] = \mathbb{E}\left[2^{\langle \mathbf{x},\sigma\rangle}\right] \tag{14}$$

5 Hardness of Integration

Corollary 5.1. $\mathbf{x} \in \mathbb{Q}^n$ *has a zero partition if and only if*

$$\int_0^\infty \prod_{k=1}^n \cos(x_k t)\, dt = \infty \tag{15}$$

and does not have a zero partition if and only if

$$\int_0^\infty \prod_{k=1}^n \cos(x_k t)\, dt = 0 \tag{16}$$

Proof. Follows from Theorem 4.1, the integrand being periodic, change of variable to support rationals, and the integral over a single period being nonnegative for all inputs. □

Corollary 5.2. *There is no algorithm that takes any function that can be evaluated in polynomial time, and decides in polynomial time whether its integral over the real line is zero (conversley, infinity) unless P=NP.*

Theorem 5.3. *[Theorem 2.2 on [3]] If u is an analytic function satisfying $|u(z)| \le M$ in $\frac{1}{r} \le |z| \le r$ ($z \in \mathbb{C}$) for some $r > 1$, then for any $N \ge 1$ the trapezoid rule with N points will be far from the exact integral by no more than $\frac{4\pi M}{r^N - 1}$.*

Corollary 5.4. *If for every #PART instance it is possible to efficiently find a function w such that given ψ as in (1) that corresponds the problem's instance, and*

$$u(z) = \psi(w(z))\, w'(z) \tag{17}$$

is computable in polynomial time (wrt the input length and the desired output accuracy) and satisfies the conditions of Theorem 4.4 with $r = 2$ and $M = \mathcal{O}(poly(\sum_{k=1}^n x_k))$, then P=NP.

Proof. Observe that ψ behaves like $e^{x_k t}$ for imaginary input. It therefore satisfies $M = e^{r\sum_{k=1}^n x_k}$ at the setting of Theorem 4.4. For exponential convergence wrt PART's input length we need $\frac{4\pi M}{r^N - 1}$ diminish exponentially. Therefore if we can change the variable of integration in (17) using some w and result with $M = \mathcal{O}(\text{poly}(\sum_{k=1}^n x_k))$, we could estimate the integral in (17) to our desired accuracy (2^{-n}) in subexponential time. □

Remark 5.5. The desired accuracy mentioned in Corollary 5.4 is the same accuracy desired from the integral (typically n binary digits for our integrand, as (1) suggests). This is due to Kahan summation algorithm ([4]). We can compute the integrand only up to that accuracy when we use the trapezoid rule, as long as we perform the summation according to Kahan's algorithm (in constant multiplicative cost). It is evident that ψ as for itself can be computed in polynomial time to the desired accuracy for all real $t \in (0, 1)$, even under the strong setting of #PART:

Denote by $M(n)$ the complexity of multiplying two n-digit numbers up to accuracy of $2n$. Then multiplying three numbers can be done by multiplying the first two in no more than $M(n)$ and taking only the first n digits of the result. Afterwards we're left again with two n-digit numbers to multiply, ending with total complexity of no more than $2M(n)$. Continuing this way, the complexity of multiplying n numbers up to precision of n digits takes no more than $\mathcal{O}((n-1)M(n))$. Note that the multiplicands need not be more accurate than n digits, since higher digits won't impact lower digits in the result as long as we multiply numbers in $(0, 1)$, as in cosine. As for computing every single cosine, observe that $\cos 2^{-n} = \sum_{k=0}^{\infty} \frac{1}{(2k)!} 4^{-nk}$ prescribes the digits of the result nicely right away up to a single division and with linearly growing precision. It can also be achieved directly from the input's digits, by writing $x = \sum_k d_k 2^{-k} \implies \prod_k \left(e^{2^{-k}}\right)^{d_k}$ where in binary we have $d_k \in \{0,1\}$, suggesting $\cos 2^{\pm n}$ to be precomputed. We also note that the formulas for $\cos(a+b), \sin(a+b)$ can be applied to calculate the trigonometric functions of n-digit binary number in linear amount of arithmetic operations, by simply following its 1 digits and taking $b = 2^{-k}$ for all k up to n. Therefore computing the cosine in concern is $\mathcal{O}(nM(n))$ per one input, so we end up with complexity of maximum $\mathcal{O}\left(n^2 M^2(n)\right) \approx \mathcal{O}\left(n^5\right)$ per computing the integrand once up to the desired accuracy. Recalling that #SAT grows quadratically when reduced to #PART, we reach $\mathcal{O}\left(\ell^{10}\right)$ per single integrand evaluation where ℓ is the length of the CNF formula.

6 Asymptotics

Conjecture 6.1. *For all even n, for all $\mathbf{x} \in \mathbb{N}^n$ the number of $\mathbf{x}$'s zero partitions is no more than the number of zero partitions of vector of size n with all its elements equal 1. Namely, never more than $\binom{n}{\frac{1}{2}n}$ zero partitions.*

Furthermore, for all odd n, for all $\mathbf{x} \in \mathbb{N}^n$ the number of $\mathbf{x}$'s zero partitions is no more than the number of zero partitions of vector of size n with all its elements equal 1 except one element that equals 2.

Observe that the conjecture says that for all $\mathbf{x} \in \mathbb{N}^n$ we have

$$\int_0^1 \prod_{k=1}^{n} \cos\left(2\pi x_k t\right) dt \leq \int_0^1 \cos^n\left(2\pi t\right) dt \tag{18}$$

note that $\int_0^1 \cos^n(t)\, dt$ approaches to a gaussian as n tends to infinity:

$$\lim_{n\to\infty} \int_0^{\sqrt{n}} \cos^n \frac{2\pi t}{\sqrt{n}} dt = \lim_{n\to\infty} \int_0^{\sqrt{n}} \left[1 - \frac{4\pi^2 t^2}{2n} + \mathcal{O}\left(n^{-\frac{3}{2}}\right)\right]^n dt \tag{19}$$

$$= \int_0^{\infty} e^{-2\pi^2 t^2} dt = \frac{1}{\sqrt{8\pi}} \approx 0.1994 \tag{20}$$

as the standard Fourier transform derivation of the Central Limit Theorem suggests. Similarly, observe that due to (1):

$$\left[\prod_{k=1}^{n} \cos\left(\frac{t}{\sqrt{N}} x_k\right)\right]^N = \left[\sum_{\sigma\in\{\pm 1\}^n} 2^{-n} \cos\left(\frac{t}{\sqrt{N}} \langle \mathbf{x}, \sigma\rangle\right)\right]^N \tag{21}$$

$$= \left[\sum_{\sigma\in\{\pm 1\}^n} 2^{-n} \left(1 - \frac{t^2}{N} \langle \mathbf{x}, \sigma\rangle^2 + \mathcal{O}\left(N^{-\frac{3}{2}}\right)\right)\right]^N = \left[1 - 2^{-n} \frac{t^2}{N} \sum_{\sigma\in\{\pm 1\}^n} \langle \mathbf{x}, \sigma\rangle^2\right]^N \tag{22}$$

$$= \left[1 - 2^{-n} \frac{t^2}{N} \sum_{\sigma\in\{\pm 1\}^n} \langle \mathbf{x}, \sigma\rangle^2\right]^N \tag{23}$$

this doesn't tell us much as $\langle \mathbf{x}, \sigma \rangle$ depends on the size of $\mathbf{x}$. We therefore normalize $\langle \mathbf{x}, \sigma \rangle \to \frac{\langle \mathbf{x}, \sigma \rangle}{\sqrt{\sum_{k=1}^{n} x_k^2}}$, then we use Theorem 7.1 that says $\sum_{k=1}^{n} x_k^2 = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \langle \mathbf{x}, \sigma \rangle^2$ and get:

$$\lim_{N\to\infty} \left[\prod_{k=1}^{n} \cos\left(\frac{t}{\sqrt{N}} x_k \right) \right]^N = \lim_{N\to\infty} \left[1 - \frac{t^2 2^{-n}}{N \sum_{k=1}^{n} x_k^2} \sum_{\sigma\in\{\pm 1\}^n} \langle \mathbf{x}, \sigma \rangle^2 \right]^N = \lim_{N\to\infty} \left[1 - \frac{t^2}{N} \right]^N = e^{-\frac{1}{2}t^2} \tag{24}$$

we see that under our normalization we indeed get a normalized distribution, even for $n > 1$.

On a different route, recall that by Wallis' integral formula for even n

$$q_n = \int_0^1 \cos^{2n}(\pi x t)\, dt = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi} n \Gamma(n)} \approx \frac{1}{n} \tag{25}$$

independently of x (as long as x is an integer), then by Cauchy-Scwarz inequality:

$$\int_0^1 \cos(\pi x_1 t) \prod_{k=2}^{n} \cos(\pi x_k t)\, dt \leq \sqrt{\left[\int_0^1 \cos^2(\pi x_1 t)\, dt \right] \left[\int_0^1 \prod_{k=2}^{n} \cos^2(\pi x_k t)\, dt \right]} \tag{26}$$

$$= \sqrt{\frac{1}{2} \int_0^1 \cos^2(\pi x_2 t) \prod_{k=3}^{n} \cos^2(\pi x_k t)\, dt} \leq \sqrt{\frac{1}{2} \sqrt{\frac{3}{8} \int_0^1 \prod_{k=3}^{n} \cos^4(\pi x_k t)\, dt}} \tag{27}$$

$$\cdots \leq \sqrt{q_{2^1} \sqrt{q_{2^2} \sqrt{q_{2^3} \sqrt{q_{2^4} \ldots}}}} \tag{28}$$

For the case $\log_2 n$ is an integer and recalling that the infinite sum $\sum_{m=1}^{\infty} m 2^{-m} = 2$ rapidly, write

$$\log_2 \sqrt{q_{2^1} \sqrt{q_{2^2} \sqrt{q_{2^3} \sqrt{q_{2^4} \ldots}}}} = \frac{1}{2} \sum_{k=1}^{\log_2 n} 2^{-k} \log_2 q_{2^k} \approx -\frac{1}{2} \sum_{k=1}^{\log_2 n} k 2^{-k} \approx -1 \tag{29}$$

$$\implies \sqrt{q_{2^1} \sqrt{q_{2^2} \sqrt{q_{2^3} \sqrt{q_{2^4} \ldots}}}} \approx \frac{1}{e} \approx 0.368 \tag{30}$$

as indeed empirical calculation suggests. We can see that neither this bound and the bound on (23) are as tight as the bound on (21) implied by the conjecture.

7 Second Order Statistics

Theorem 7.1. *Given $n \in \mathbb{N}, N \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$, the variance of the sizes of all partitions is the sum of the squares of the input. Formally:*

$$\sum_{k=1}^{n} x_k^2 = 2^{-n} \sum_{\sigma\in\{-1,1\}^n} \langle \mathbf{x}, \sigma \rangle^2 \tag{31}$$

while

$$\frac{2^n}{N^3} \sum_{m=1}^{N} \frac{\partial^2}{\partial t^2} \prod_{k=1}^{n} \cos(2\pi x_k t) \Bigg|_{t=\frac{m}{N}} \tag{32}$$

is the variance of the sizes of all partitions that their size is divisible by N without remainder.

Proof. Following (1) and differentiating:

$$\prod_{k=1}^{n} \cos\left(\pi x_k t\right) = 2^{-n} \sum_{\sigma\in\{-1,1\}^n} \cos\left(\pi t \left\langle \mathbf{x}, \sigma\right\rangle\right) \tag{33}$$

$$\implies \sum_{\ell=1}^{n} x_\ell \sin\left(\pi x_\ell t\right) \prod_{k\neq\ell}^{n} \cos\left(\pi x_k t\right) = 2^{-n} \sum_{\sigma\in\{-1,1\}^n} \left\langle \mathbf{x}, \sigma\right\rangle \sin\left(\pi t \left\langle \mathbf{x}, \sigma\right\rangle\right) \tag{34}$$

$$\implies \sum_{\ell=1}^{n}\sum_{\ell\prime=1}^{n} -x_\ell \sin\left(\pi x_\ell t\right) x_{\ell\prime} \sin\left(\pi x_{\ell\prime} t\right) \prod_{k\neq\ell,\ell\prime}^{n} \cos\left(\pi x_k t\right) + x_\ell^2 \prod_{k=1}^{n} \cos\left(\pi x_k t\right) \tag{35}$$

$$= 2^{-n} \sum_{\sigma\in\{-1,1\}^n} \left\langle \mathbf{x}, \sigma\right\rangle^2 \cos\left(\pi t \left\langle \mathbf{x}, \sigma\right\rangle\right) \tag{36}$$

and (19) follows by substituting $t = 0$. (19) can be proved using Parseval identity as well. Turning to (20):

$$\frac{2^n}{N} \sum_{m=1}^{N} \frac{\partial^2}{\partial t^2} \prod_{k=1}^{n} \cos\left(2\pi x_k t\right)\bigg|_{t=\frac{m}{N}} = \sum_{\sigma\in\{-1,1\}^n} \left\langle \mathbf{x}, \sigma\right\rangle^2 \cos\left(2\pi \frac{m}{N} \left\langle \mathbf{x}, \sigma\right\rangle\right) = \sum_{u=-\infty}^{\infty} u^2 N^2 c_{Nu} \tag{37}$$

due to aliasing of roots of unity, and c_{Nu} the number of partitions whose size is divisible by Nu as in (4). □

Remark 7.2. It is easy to derive all moments and cumulants of our random variable since we're given its characteristic function.

Theorem 7.3. *Given $n \in \mathbb{N}, N \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$, then among all $\mathbf{x}$'s partitions that divide by $N > 0$ with remainder j, the correlation of the sign of x_1, x_2 (wlog) on those partition is given by:*

$$-\int_0^{\pi} t^2 \sin\left(x_1 t\right) \sin\left(x_2 t\right) \prod_{k=3}^{n} \cos\left(x_k t\right) dt \tag{38}$$

Proof. Obtained immediately by differentiating (1) and differentiating wrt x_1, x_2 and integrating wrt t. □

8 Estimating #SAT

The numbers produced by the reduction from #SAT to #PART have digits that does not exceed 4, and if using radix 6, they never even carry. Therefore the very same digits produced by the reduction can be interpreted in any radix larger than 5, being reduced to a different #PART problem. Still, it is guaranteed that the number of solution to those #PART problems are independent of the radix, as they're all reduced from the same #SAT problem. This property might be used to approximate #SAT using results as Theorem 3.3. We can obtain the number of partitions that their size divides a given number N in polynomial time wrt n (the number of numbers to partition) and the number of digits of x_k. Nevertheless, it takes exponential time in the number of digits of N.

The probability that there exists a partition with nonzero size that is divisible by a given prime p is roughly

$$\mathcal{P}\left[p | \left\langle \mathbf{x}, \sigma\right\rangle\right] \approx 1 - \left(1 - \frac{1}{p}\right)^{2^n} \tag{39}$$

taking K reductions of a single #SAT problem instance and a set P of primes, the probability that on reductions there exists a partition with size divisible by a given prime p that is not a zero partition is roughly

$$\prod_{p\in P}\prod_{k=1}^{K}\mathcal{P}\left[p|\left\langle \mathbf{x}_k,\sigma\right\rangle\right]\approx\prod_{p\in P}\left[1-\left(1-\frac{1}{p}\right)^{2^n}\right]^K\leq\exp\left(-K\sum_{p\in P}\left(1-\frac{1}{p}\right)^{2^n}\right) \tag{40}$$

recalling that for $x\in[0,1]$ we have $e^{-x}\geq 1-x$. This doesn't seem to be helpful since it seem to require exponentially many or exponentially large primes or reductions.

9 Discussion

On Theorem 3.3 we have seen that we can efficiently query for the number of partitions that divide by N with remainder j. It is interesting to see that positively solving PART (resp. SAT) by guesses is straight-forward: we just try partitions (resp. substitutions) and if we're lucky to find a zero partition (resp. SAT) then we solved the problem. On the other hand, how can we do one trial and possibly decide that the set is unpartitionable (resp. UNSAT)? Our analysis suggest such a method. If we query for the number partitions that are divisible by N with $j=0$ and get zero, then we know that the set is unpartitionable. Similarly, if we do the same for $j\neq 0$ and happen to get 2^n, we know that $\mathbf{x}$ does not have a zero partition. Those trials are argubaly independent due to the pseudo-randomness of the mod operation.

On Section 4 #PART (equivalently, #SAT) is reduced into a problem of computing the k'th digit of the result of the multiplications of numbers of the form $100\dots001$, i.e. two ones only and zeros between them. In fact, this result is independent on the radix chosen (given it is not too small). On this setting, the number of zeros has polynomial amount of digits wrt the problem's input size, while the number of multiplicands is also polynomial.

On Section 5 we showed that P!=NP implies a result of nonexistence of certain complex analytic functions. P!=NP also implies impossibility to decide in polynomial time whether a definite integral (with bounded, periodic, and polynomially evaluated integrand) equals either zero or infinity, and proved that a single evaluation of ψ can be done in polynomial time.

On Section 6 we computed some bounds that might be useful in further analysis. We saw that asymptotically, as more the x's are close to an integer multiplication of 2π, the more their partitions are dense as their variance is smaller.

On Section 7 we practically showed how it is possible to express correlations among different variables in a CNF formula, though we did not give a full development of this idea.

On Section 8 we have seen a heuristic estimation to #SAT, by taking advantage of the modular formulas we derived in Theorem 3.3. Interestingly, this method reveals relatively very little information about any single #PART problem, since we use reduce the #SAT problem instance into many #PART instances, taking advantage on the reduction promising us #PART problems with quite different modular properties, yet with exactly the same number of zero partitions. Nevertheless our bounds are quite weak and further research is needed to refine this technique of solving #SAT.

For additional further research, by Theorem 3.3 we can get successive estimates to the integral by selecting e.g. primes $N=2,3,5,\dots$. We could then accelerate this sequence using Shanks, Romberg, Pade or similar sequence-acceleration method.

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A Appendix

A.1 Reductions

Reduction of #SAT to #SUBSET-SUM Given variables $x_1, \dots, x_l$ and clauses $c_1, \dots, c_k$ and let natural $b \geq 6$. we construct a set S and a target t such that the resulted subset-sum problem requires finding a subset of S that sums to t. The number t is l ones followed by k 3s (i.e. of the form $1111 \dots 3333$). S contains four groups of numbers $y_1, \dots, y_l$, $z_1, \dots, z_l$, $g_1, \dots, g_k$, $h_1, \dots, h_k$ where $g_i = h_i = b^{k-i}$, and y_i, z_i are b^{k+l-i} plus b^m for y_i if variable i appears positively in clause m, or for z_i if variable i appears negated in clause m. Then, every subset that sum to t matches to a satisfying assignment in the input CNF formula and vice versa, as proved in [1].

Reduction of #SUBSET-SUM to #PART Given S, t as before and denote by $s = \sum_{x \in S} x$ the sum of S members, the matching PART problem is $S \cup \{2s - t, s + t\}$. Here too all solutions to both problems are preserved by the reduction and can be translated in both directions.

A.2 Miscellaneous

Theorem A.1. *Let $Z^{\mathbf{x}}$ be the number of zero partitions of a vector of naturals X. Let $D_x^{\mathbf{x}}$ be the number of zero partitions of X after multiplying one if its elements by two, where this element is denoted by x. Let $A_x^{\mathbf{x}}$ be the number of zero partitions of X after appending it x (so now x appears at least twice). Then*

$$Z^{\mathbf{x}} = D_x^{\mathbf{x}} + A_x^{\mathbf{x}} \tag{41}$$

Proof. Denote

$$\psi\left(x_1, \dots, x_n\right) = 2^n \int_0^\pi \prod_{k=1}^n \cos\left(x_k t\right) dt \tag{42}$$

then, using the identity $\cos 2x = 2\cos^2 x - 1$:

$$\psi\left(x_1, \dots, 2x_m, \dots, x_n\right) = 2^n \int_0^\pi \cos\left(2x_m t\right) \prod_{k \neq m}^n \cos\left(x_k t\right) dt \tag{43}$$

$$= 2^n \int_0^\pi \left[2\cos^2\left(x_m t\right) - 1\right] \prod_{k \neq m}^n \cos\left(x_k t\right) dt$$

$$\implies \psi\left(x_1,\ldots,x_m,\ldots,x_n\right)-\psi\left(x_1,\ldots,2x_m,\ldots,x_n\right)= \tag{44}$$

$$2^{n+1}\int\limits_0^{\pi}\cos^2\left(x_m t\right)\prod_{k\neq m}^{n}\cos\left(x_k t\right)dt=\psi\left(x_1,\ldots,x_m,\ldots,x_n,x_m\right)$$

$\square$