Binomial Quadrature Method with Applications to #P Problems

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Assume we are interested in numerically evaluating the definite integral of a function f. Assume further that f admits either the following sine or cosine series representation:

$$f(x) = \sum_{m=0}^{\infty} a_m \cos(2\pi m x) \tag{1}$$

or

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \sin(2\pi m x) \tag{2}$$

and we are interested in estimating a_0. Last, f may admit a Fourier series representation with real coefficients:

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m e^{2\pi i m x} \tag{3}$$

while again a_0 is nothing but the integral of the function which is the Fourier transform at zero:

$$\int_0^{2\pi} f(t)\, e^{it0} dt \tag{4}$$

where we took the interval 2π arbitrarity.

The Trapezoid rule of order N has been shown to "eliminate" all coefficients that aren't divisible by N hence admits very fast convergence. Nevertheless, on some cases, the structure of the integrand might require an orthogonal treatment, of the form of reducing all coefficients at once rather eliminating some of them.

Our derivation relies on the identities

$$\cos^n x = \begin{cases} 2^{1-n} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \cos((n-2k)x) & n \text{ odd} \\ 2^{-n} \binom{n}{\frac{n}{2}} + 2^{1-n} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \cos((n-2k)x) & n \text{ even} \end{cases} \tag{5}$$

$$\sin^n x = \begin{cases} 2^{1-n} \sum_{k=0}^{\frac{n-1}{2}} (-1)^{\frac{n-2k-1}{2}} \binom{n}{k} \sin((n-2k)x) & n \text{ odd} \\ 2^{-n} \binom{n}{\frac{n}{2}} + 2^{1-n} \sum_{k=0}^{\frac{n-1}{2}} (-1)^{\frac{n-2k}{2}} \binom{n}{k} \cos((n-2k)x) & n \text{ even} \end{cases} \tag{6}$$

which implies, for example, under the case of (1):

$$\frac{3}{4}f\left(\sqrt{2}\right)+\frac{1}{4}f\left(3\sqrt{2}\right)=\sum_{m=0}^{\infty}a_m\left[\frac{3}{4}\cos\left(2\pi m\sqrt{2}\right)+\frac{1}{4}\cos\left(6\pi m\sqrt{2}\right)\right]=\sum_{m=0}^{\infty}a_m\cos^3\left(2\pi m\sqrt{2}\right) \tag{7}$$

the choice of $\sqrt{2}$ was arbitrarily made to make $2\pi mx$ not to meed an integer multiple of π, but any other trick would do it. We therefore recuded the cosines from a number in $(-1,1)$ to its third power, so it must shrink in absolute value. Continuing this way, using n samples we can raise all cosines to the n'th power. This will of course leave us with a_0 with some kind of exponential convergence. Similar treatment is possible for the cases of (2) and (3) using (5) and (6).

Observe that the formulae for even n aren't very helpful in case we don't know the quantity $\sum_{m=0}^{\infty}a_m$ or whether this quantity is even finite, therefore we're unable to use the free constant additive term on (5) and (6) for even n.

This rationale can further be applied to problems beyond integration, and we show another application here namely approximating the counting version of the partition problem.

Consider the identity

$$\prod_{k=1}^{n}\cos x_k=\mathbb{E}\cos\langle\mathbf{x},\sigma\rangle \tag{8}$$

where the expectation is taken over all 2^n *partitions* $\sigma\in\{-1,1\}^n$ and $\langle\mathbf{x},\sigma\rangle=\sum_{k=1}^{n}x_k\sigma_k$ for some given $\mathbf{x}\in\mathbb{Q}^n$ (can also be assumed to be algebraic numbers). We're interested in estimating the quantity

$$Z=\left|\{\sigma\in\{-1,1\}^n \mid \langle\mathbf{x},\sigma\rangle=0\}\right| \tag{9}$$

to be called the number of $\mathbf{x}$'s *zero partitions*. Solving this problem is a #P problem, where merely deciding whether $Z=0$ or $Z\neq 0$ is an NP-Complete problem.

The rhs of (8) contains therefore either a constant quantity that equals $2^{-n}Z$ and arises from the cosine of zero partitions. Other quantities have absolute value strictly lower than 1 since $\langle\mathbf{x},\sigma\rangle$ cannot ever meet an integer multiple of π. Taking N samples

$$s_{m\in[N]}=\prod_{k=1}^{n}\cos\left(x_k m\right) \tag{10}$$

for some odd N we can apply (5) directly on s_m while they actually operate linearly on the rhs of (8), which will turn into

$$\mathbb{E}\cos^N\langle\mathbf{x},\sigma\rangle \tag{11}$$

therefore as N grows we get exponentially convergent approximation to the number of zero partitions from the formula

$$\sum_{m=0}^{\frac{N-1}{2}}\binom{N}{m}\prod_{k=1}^{n}\cos\left(\left(N-2m\right)x_k\right) \tag{12}$$