Let $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ and consider the formula $2\cos a \cos b = \cos(a+b) + \cos(a-b)$ and the cosine being even function to see that:

$$\psi(t) = 2^n \prod_{k=1}^{n} \cos(x_k t) = \sum_{\sigma \in \{-1,1\}^n} \cos t \langle \mathbf{x}, \sigma \rangle = 2 \sum_{\sigma \in \{-1,1\}^n} e^{it\langle \mathbf{x}, \sigma \rangle} \tag{1}$$

where $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^{n} \sigma_k x_k$ and counting the number of $\sigma \in \{-1,1\}^n$ satisfying $\langle \mathbf{x}, \sigma \rangle = 0$ is a #P problem. We write down the following sum just for fun and substitute (1) in it:

$$S = \frac{1}{2n+2} \sum_{m=1}^{n+1} \psi\left(\frac{2\pi m}{n+1} + i \ln 2\right) = \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{n+1} \frac{2^{-\langle \mathbf{x}, \sigma \rangle}}{n+1} e^{\frac{2\pi i m}{n+1} \langle \mathbf{x}, \sigma \rangle} \tag{2}$$

the summation of roots of unity equals zero iff $n+1$ does not divide $\langle \mathbf{x}, \sigma \rangle$, and if it does divide then it sums to $n+1$. Using this fact and denoting the number of partitions that sum to u by $c_u = |\{\sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = u\}|$, we get

$$S = \sum_{u=0}^{\infty} c_{u(n+1)} 2^{-(n+1)u} \tag{3}$$

recalling that $\sum_{u=0}^{\infty} c_u = 2^n$ and c_u are all positive, while in (3) being multiplied by distinct powers 2^{-n-1}, therefore the summands' binary digits never interfere with each other and can never grow as large as 1, except when $u = 0$. Recalling that c_0 is our quantity of interest, we have proved that the number of zero partitions in $\mathbf{x}$ is

$$\left\lfloor \frac{2^{n-1}}{n+1} \sum_{m=1}^{n+1} \prod_{k=1}^{n} \cos\left[x_k \left(\frac{2\pi m}{n+1} + i \ln 2\right)\right] \right\rfloor \tag{4}$$

$$= \left\lfloor \frac{2^{2n-1}}{2n+2} \sum_{m=1}^{n+1} \prod_{k=1}^{n} \left[e^{x_k\left(\frac{2\pi i m}{n+1} - \ln 2\right)} + e^{x_k\left(-\frac{2\pi i m}{n+1} + \ln 2\right)}\right] \right\rfloor \tag{5}$$

$$= \left\lfloor \frac{4^{n-1}}{n+1} \sum_{m=1}^{n+1} \prod_{k=1}^{n} \left[2^{-x_k} e^{\frac{2\pi i m x_k}{n+1}} + 2^{x_k} e^{-\frac{2\pi i m x_k}{n+1}}\right] \right\rfloor \tag{6}$$

assuming $\frac{x_k}{n+1}$ is integer (otherwise multiply $\mathbf{x}$ while preserving partitions):

$$= \left\lfloor \frac{4^{n-1}}{n+1} \prod_{k=1}^{n} \left[2^{-x_k} + 2^{x_k}\right] \right\rfloor \tag{7}$$