Consider the function

$$\psi_0(z) = \sum_{m=0}^{\infty} z^m f(m) \tag{1}$$

for complex $|z| < 1$ where

$$f(m) = 2^n \prod_{k=1}^{n} \cos(x_k m) = \sum_{\sigma\in\{-1,1\}^n} e^{i\langle \mathbf{x},\sigma\rangle} \tag{2}$$

as we've already seen due to angle addition formulae. This implies

$$\psi_0(z) = \sum_{m=0}^{\infty} z^m f(m) = \sum_{\sigma\in\{-1,1\}^n} \sum_{m=0}^{\infty} z^m e^{im\langle \mathbf{x},\sigma\rangle} = \sum_{\sigma\in\{-1,1\}^n} \frac{1}{1 - ze^{i\langle \mathbf{x},\sigma\rangle}} \tag{3}$$

where the last equality is due to summation of geometric progression.

Integrating and dividing by $-z$, we define:

$$\psi_1(z) \equiv -\frac{1}{z}\int^{z} \psi_0(z')\,dz' = \sum_{m=0}^{\infty} \frac{-z^m}{m+1} f(m) = -\sum_{\sigma\in\{-1,1\}^n} \frac{1}{ze^{i\langle \mathbf{x},\sigma\rangle}} \ln \frac{1}{1 - ze^{i\langle \mathbf{x},\sigma\rangle}} \tag{4}$$

observe that the rhs is the Dirichlet space representation prescribed explicitly by its reproducing kernel, while (3) is on the Bergman space and kernel. Moreover:

$$\sum_{m=0}^{\infty} \frac{-z^m}{m+1} f(m) = \ln \prod_{\sigma\in\{-1,1\}^n} \left(1 - ze^{i\langle \mathbf{x},\sigma\rangle}\right)^{z^{-1}e^{-i\langle \mathbf{x},\sigma\rangle}} \tag{5}$$

$$\implies \psi(z) \equiv e^{2^n z\psi_1(z)} = \prod_{m=0}^{\infty} \exp \frac{-2^n z^{m+1} f(m)}{m+1} = \prod_{\sigma\in\{-1,1\}^n} \left(1 - ze^{i\langle \mathbf{x},\sigma\rangle}\right)^{\left(e^{-i\langle \mathbf{x},\sigma\rangle}\right)} \tag{6}$$

is a Blaschke product, where the zeros are detemined by all possible partitions. This funtion is holomorphic over the whole complex plane, and we wish to decide whether it vanishes at $z = 1$:

$$\psi(1) = \prod_{m=0}^{\infty} \exp \frac{-2^n f(m)}{m+1} = \prod_{\sigma\in\{-1,1\}^n} \left(1 - e^{-i\langle \mathbf{x},\sigma\rangle}\right)^{e^{i\langle \mathbf{x},\sigma\rangle}} \leq 4^{-2^n} \tag{7}$$

$$\prod_{m=0}^{\infty} \exp \frac{f(m)}{m+1} \leq 4 \tag{8}$$

since indeed

$$\left| \left(1 - e^{i\langle \mathbf{x},\sigma\rangle}\right)^{e^{-i\langle \mathbf{x},\sigma\rangle}} \right| \leq \frac{1}{4} \tag{9}$$

or even a stronger bound as follows, since in fact for $|z| = 1$:

$$\frac{-1}{100} \leq \left| \left(1 - \frac{1}{z}\right)^{z} \right| - z(1-z) \leq \frac{1}{40} \tag{10}$$

so

$$\prod_{m=0}^{\infty} \exp \frac{f(m)}{m+1} \le \max_{\sigma} \left| \left(1 - e^{-i\langle \mathbf{x},\sigma\rangle}\right)^{e^{i\langle \mathbf{x},\sigma\rangle}} \right| \le \frac{1}{40} + \max_{\sigma} \left| e^{i\langle \mathbf{x},\sigma\rangle} \left(1 - e^{i\langle \mathbf{x},\sigma\rangle}\right) \right| = \frac{1}{40} + 2 \quad (11)$$

we observe the closeness to $\prod_k e^{\frac{(-1)^k}{k}} = 2$, so $f(m)$ must be very close to the alternating harmonic sequence, implying almost maximal entropy if the set is unpartitionable.