Let $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ and

$$\psi(t) = 2^n \prod_{k=1}^{n} \cos(x_k t) = \sum_{\sigma \in \{-1,1\}^n} \cos t \langle \mathbf{x}, \sigma \rangle \tag{1}$$

equivalently (recalling the cosine is even)

$$\psi(t) = \prod_{k=1}^{n} \left(e^{ix_k t} + e^{-ix_k t}\right) = 2 \sum_{\sigma \in \{-1,1\}^n} e^{it\langle \mathbf{x}, \sigma \rangle} \tag{2}$$

put

$$c_k = |\{\sigma \in \{-1,1\}^n \mid \{\langle \mathbf{x}, \sigma \rangle = k\}\}| \tag{3}$$

is the number of partitions that sum to k, then

$$\frac{1}{2n} \sum_{m=1}^{n} \psi\left(2\pi \frac{m}{n} + i \ln 2\right) = \sum_{\sigma \in \{-1,1\}^n} 2^{-\langle \mathbf{x}, \sigma \rangle} \sum_{m=1}^{n} \frac{1}{n} e^{2\pi i \frac{m}{n} \langle \mathbf{x}, \sigma \rangle} \tag{4}$$

$$= \sum_{k=-\infty}^{\infty} c_{kn} 2^{-nk} \leq c_0 + 2^{1-n} \tag{5}$$

where the transition from (4) to (5) is due to summation of roots of unity, that sum to zero on (4) iff n does not divide $\langle \mathbf{x}, \sigma \rangle$, and if it does divide then they sum to n. We have proved that $\mathbf{x}$ has a zero partition iff

$$\sum_{m=1}^{n} \prod_{k=1}^{n} \cos\left(2\pi \frac{m}{n} + i \ln 2\right) \geq \frac{2n}{2^n} \tag{6}$$