

# Statistical Properties of Trigonometric Functions with Applications

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Observe that

$$4 \sum_{m=0}^{\infty} \cos(mx) \cos(my) = \sum_{n=1}^{\infty} e^{im(x+y)} + e^{im(x-y)} \quad (1)$$

$$= \frac{1}{1 - e^{i(x+y)}} + \frac{1}{1 - e^{i(x-y)}} \quad (2)$$

$$= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (3)$$

Recalling that  $\frac{1}{1-t} + \frac{1}{1-\bar{t}} = 1$  for all nonzero complex  $t$ , then by the formulae of sum-of-angles and geometric progression we write:

$$\lim_{r \rightarrow 1^-} 4 \sum_{m=0}^{\infty} r^m \cos(mx) \cos(my) = \lim_{r \rightarrow 1^-} \sum_{n=1}^{\infty} r^m e^{im(x+y)} + r^m e^{im(x-y)} \quad (4)$$

$$= \lim_{r \rightarrow 1^-} \frac{1}{1 - r e^{i(x+y)}} + \frac{1}{1 - r e^{i(x-y)}} \quad (5)$$

$$= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (6)$$

if  $x, y$  cannot ever meet  $2\pi$  on some integer multiple.

The following theorems generalize this idea.

## 1 Main Results

**Lemma 1.1.** *For all complex  $x$ :*

$$\sum_{m=1}^{\infty} \frac{1}{m} e^{imx} = -\ln(1 - e^{ix}) \quad (7)$$

*provided the sum is convergent.*

*Proof.* Follows immediately from the Taylor series of the logarithm  $\sum_{k=1}^{\infty} \frac{t^k}{k} = -\ln(1-t)$ : □

**Lemma 1.2.** *For all complex  $x$ :*

$$\sum_{m=1}^{\infty} \frac{\cos(mx)}{m} = \ln 2 + \ln \sin x \quad (8)$$

*provided the sum is convergent.*

*Proof.* By Euler's formula and Lemma 1.1:

$$2 \sum_{m=1}^{\infty} \frac{\cos(mx)}{m} = \sum_{m=1}^{\infty} \frac{e^{imx} + e^{-imx}}{m} \quad (9)$$

$$= -\ln[(1 - e^{ix})(1 + e^{ix})] \quad (10)$$

$$= -\ln(2 - \cos 2x) \quad (11)$$

$$= -\ln(4 \sin^2 x) \quad (12)$$

$$= -2 \ln \frac{1}{2 \sin x} \quad (13)$$

and the lemma follows.  $\square$

On this section we denote

$$f(m) \equiv \prod_{k=1}^n \cos(x_k m) \quad (14)$$

where  $\mathbf{x}$  is a vector of  $n$  numbers to partition at the scope of the partition problem. Specific requirements from  $\mathbf{x}$  depend on the discussed scope and will be well defined every time.

**Theorem 1.3.** *Fix  $\mathbf{x}$  to be a vector of  $n$  nonzero algebraic numbers. Then for all nonzero algebraic number  $t$  the following identities hold:*

$$\sum_{\sigma \in \{-1,1\}^n} \ln |\sin \langle \mathbf{x}, \sigma \rangle| = -\frac{2^n}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{1}{m} f(2tm) \quad (15)$$

$$\sum_{\sigma \in \{-1,1\}^n} \ln |\cos \langle \mathbf{x}, \sigma \rangle| = -\frac{2^n}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2tm) \quad (16)$$

$$\sum_{\sigma \in \{-1,1\}^n} \ln |\tan \langle \mathbf{x}, \sigma \rangle| = \sum_{m=1}^{\infty} \frac{2^n}{m + \frac{1}{2}} \prod_{k=1}^n f(4tm + t) \quad (17)$$

*Proof.* Write

$$\lim_{r \rightarrow 1^-} 2^n \sum_{m=1}^{\infty} r^m f(m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} r^m e^{im \langle \mathbf{x}, \sigma \rangle} \quad (18)$$

$$= \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - r e^{i \langle \mathbf{x}, \sigma \rangle}} \quad (19)$$

$$= \begin{cases} -\infty & \exists \sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = 0 \\ 0 & \forall \sigma \in \{-1,1\}^n, \langle \mathbf{x}, \sigma \rangle \neq 0 \end{cases} \quad (20)$$

since  $\sum_{k=1}^{\infty} \frac{t^k}{k} = \ln \frac{1}{1-t}$  we can write

$$\lim_{r \rightarrow 1^-} -2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{r^m}{m} e^{im \langle \mathbf{x}, \sigma \rangle} \quad (21)$$

$$= \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \ln \frac{1}{1 - r e^{i \langle \mathbf{x}, \sigma \rangle}} \quad (22)$$

implying:

$$\lim_{r \rightarrow 1^-} \exp \left( -2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) \right) = \lim_{r \rightarrow 1^-} \prod_{\sigma \in \{-1,1\}^n} (1 - r e^{i \langle \mathbf{x}, \sigma \rangle}) \quad (23)$$

for  $r \rightarrow 1$  we have  $1 - re^{i\langle \mathbf{x}, \sigma \rangle} = 0$  iff  $\langle \mathbf{x}, \sigma \rangle$  is a zero partition. Taking the limit:

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{f(m)}{m}\right) = \prod_{\sigma \in \{-1,1\}^n} (1 - e^{i\langle \mathbf{x}, \sigma \rangle}) \quad (24)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (25)$$

where the latter equalit is due to every  $\sigma$  there exists a matching  $-\sigma$  and

$$(1 - e^{i\langle \mathbf{x}, \sigma \rangle}) (1 - e^{-i\langle \mathbf{x}, \sigma \rangle}) = 2 - 2 \cos \langle \mathbf{x}, \sigma \rangle \quad (26)$$

$$= 2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2} \quad (27)$$

concluding

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{f(2m)}{m}\right) = 2^{2^{n-1}} \prod_{\sigma \in \{-1,1\}^n} |\sin \langle \mathbf{x}, \sigma \rangle| \quad (28)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \exp\left(-\sum_{m=1}^{\infty} \frac{f(2m)}{m}\right) = \sqrt[2^n]{\prod_{\sigma \in \{-1,1\}^n} |\sin \langle \mathbf{x}, \sigma \rangle|} \quad (29)$$

Equivalently,

$$-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m) = - \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} \quad (30)$$

$$= \sum_{\sigma \in \{-1,1\}^n} \ln(1 + e^{i\langle \mathbf{x}, \sigma \rangle}) \quad (31)$$

or

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m)\right) = \prod_{\sigma \in \{-1,1\}^n} (1 + e^{i\langle \mathbf{x}, \sigma \rangle}) \quad (32)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \cos^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (33)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \exp\left(-\sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m)\right) = \sqrt[2^n]{\prod_{\sigma \in \{-1,1\}^n} |\cos \langle \mathbf{x}, \sigma \rangle|} \quad (34)$$

dividing (23) by (28) we conclude:

$$\exp\left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m) - \sum_{m=1}^{\infty} \frac{f(2m)}{m}\right) \quad (35)$$

$$= \exp\left(\sum_{m=1}^{\infty} \frac{2f(4m+2)}{2m+1}\right) \quad (36)$$

$$= \sqrt[2^n]{\prod_{\sigma \in \{-1,1\}^n} |\tan \langle \mathbf{x}, \sigma \rangle|} \quad (37)$$

□

*Remark 1.4.* By (24)

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m)\right) = \prod_{\sigma \in \{-1,1\}^n} (1 - re^{i\langle \mathbf{x}, \sigma \rangle}) \quad (38)$$

setting  $r = \frac{1}{2}$ :

$$\exp \left( -2^n \sum_{m=1}^{\infty} \frac{f(m)}{m2^m} \right) = \prod_{\sigma \in \{-1,1\}^n} \left( 1 - \frac{1}{2} e^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (39)$$

doing calculation similarly to (26):

$$\left( 1 - \frac{1}{2} e^{i\langle \mathbf{x}, \sigma \rangle} \right) \left( 1 - \frac{1}{2} e^{-i\langle \mathbf{x}, \sigma \rangle} \right) = \frac{5}{4} + \cos \langle \mathbf{x}, \sigma \rangle \quad (40)$$

implying

$$\exp \left( - \sum_{m=1}^{\infty} \frac{4^n f(m)}{2^m m} \right) = \prod_{\sigma \in \{-1,1\}^n} \sqrt[2^{n+1}]{\frac{5}{4} + \cos \langle \mathbf{x}, \sigma \rangle} \quad (41)$$

which can give a great deal in estimating the distribution of the cosines of the partitions, while calculating it converges exponentially fast due to the  $2^{-m}$  coefficient. Similar and tighter bounds are possible and even converge faster e.g. for  $r = \frac{1}{3}$ . Similarly for  $r = -\frac{1}{2}$ :

$$\exp \left( - \sum_{m=1}^{\infty} \frac{4^n f(m)}{(-2)^m m} \right) = \prod_{\sigma \in \{-1,1\}^n} \sqrt[2^{n+1}]{\frac{5}{4} - \cos \langle \mathbf{x}, \sigma \rangle} \quad (42)$$

## 2 Dirichlet Space Formulation

**Theorem 2.1.** *The following identities hold for  $\mathbf{x} \in \mathbb{R}_+^n$  TBD*

*Proof.* Define

$$\psi_0(z) = \sum_{m=0}^{\infty} z^m f(m) = \sum_{\sigma \in \{-1,1\}^n} \sum_{m=0}^{\infty} z^m e^{im\langle \mathbf{x}, \sigma \rangle} = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - ze^{i\langle \mathbf{x}, \sigma \rangle}} \quad (43)$$

for complex  $|z| < 1$  and due to summation of geometric progression, and write

$$\psi_0(z) + \psi_0(-z) = \sum_{m=0}^{\infty} 2z^{2m} f(2m) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{2}{1 - z^2 e^{2i\langle \mathbf{x}, \sigma \rangle}} \quad (44)$$

integrating wrt  $z$ :

$$\sum_{m=0}^{\infty} \frac{2z^{2m+1}}{2m+1} f(2m) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} e^{-i\langle \mathbf{x}, \sigma \rangle} \ln \frac{ze^{i\langle \mathbf{x}, \sigma \rangle} + 1}{ze^{i\langle \mathbf{x}, \sigma \rangle} - 1} \quad (45)$$

for  $z = 1$  we get

$$\sum_{m=0}^{\infty} \frac{2^{n+1} f(2m)}{2m+1} = \sum_{\sigma \in \{-1,1\}^n} e^{-i\langle \mathbf{x}, \sigma \rangle} \ln \frac{e^{i\langle \mathbf{x}, \sigma \rangle} + 1}{e^{i\langle \mathbf{x}, \sigma \rangle} - 1} \quad (46)$$

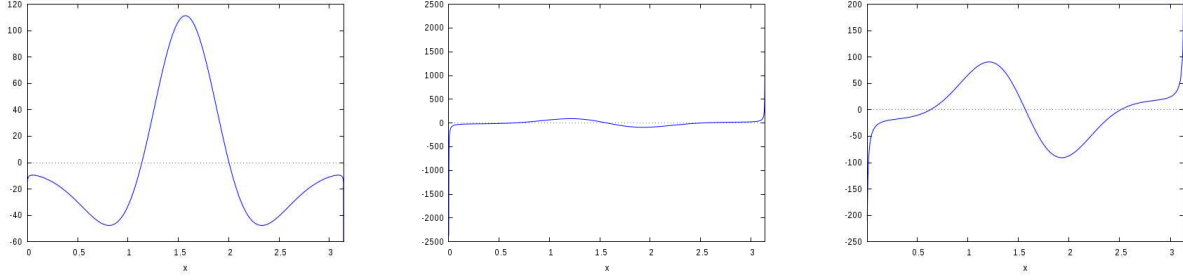
$$= \sum_{\sigma \in \{-1,1\}^n} e^{-i\langle \mathbf{x}, \sigma \rangle} \ln \left[ -i \cot \frac{\langle \mathbf{x}, \sigma \rangle}{2} \right] = \sum_{\sigma \in \{-1,1\}^n} \ln \left[ \frac{i \sin \langle \mathbf{x}, \sigma \rangle}{\cos \langle \mathbf{x}, \sigma \rangle - 1} \right] e^{-i\langle \mathbf{x}, \sigma \rangle} \quad (47)$$

by exponentiation:

$$\prod_{m=0}^{\infty} \exp \frac{-2^{n+1} f(2m)}{2m+1} = \prod_{\sigma \in \{-1,1\}^n} \left[ \frac{\cos \langle \mathbf{x}, \sigma \rangle - 1}{i \sin \langle \mathbf{x}, \sigma \rangle} \right] e^{-i\langle \mathbf{x}, \sigma \rangle} \quad (48)$$

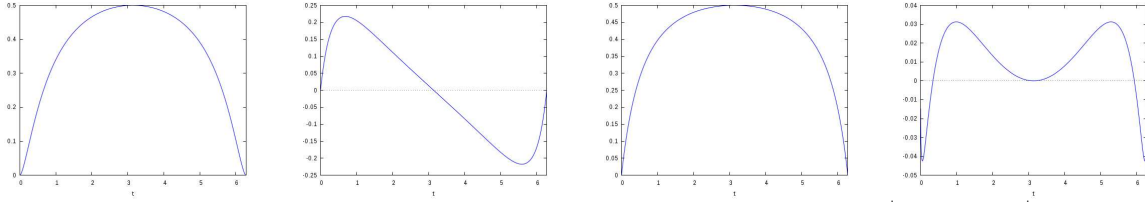
which equals zero iff  $\mathbf{x}$  is partitionable. it is interesting to observe the plot of  $\left[ \frac{i \sin x}{\cos x - 1} \right] e^{-ix}$  as shown in Figure 1.

Figure 1: Plots of  $\left[ \frac{i \sin x}{\cos x - 1} \right] e^{-ix}$



The leftmost is the real part, while the middle is the imaginary part, both over  $[0, \pi]$ . The rightmost is the imaginary part over  $[\frac{1}{100}, \pi - \frac{1}{100}]$

Figure 2: Plots of  $(1 - e^{-it}) e^{it}$  for  $t \in [0, 2\pi]$



Leftmost is real part, middle is imaginary part, afterwards - the norm, and the rightmost is  $\left| (1 - e^{-it}) e^{it} \right| - \sqrt{\frac{t}{2\pi} (1 - \frac{t}{2\pi})}$ .

Observe that

$$\prod_{m=0}^{\infty} \left( 1 + \frac{f(2m)}{2m+1} \right) \geq \prod_{m=0}^{\infty} \exp \frac{-f(2m)}{2m+1} \geq \prod_{m=0}^{\infty} \left( 1 - \frac{f(2m)}{2m+1} \right) \quad (49)$$

which is a nontrivial result if recalling the structure of  $f$ . The rhs of (28) goes to zero if the set is partitionable, and the lhs goes to infinity if the set is unpartitionable. TBD: explain this by rate-of-interest example and show it rigorously by calculating the drift of this geometric motion.

We therefore require  $\mathbf{x}$  to satisfy  $\langle \mathbf{x}, \mathbf{1} \rangle < \pi$ . This way we can see that (48) converges to zero iff  $\mathbf{x}$  is partitionable.

On a different route, beginning from (??), integrating (we may do so due to Vitali's convergence theorem) and dividing by  $-z$ , we define:

$$\psi_1(z) \equiv -\frac{1}{z} \int^z \psi_0(z') dz' = \sum_{m=0}^{\infty} \frac{-z^m}{m+1} f(m) = -2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{ze^{i\langle \mathbf{x}, \sigma \rangle}} \ln \frac{1}{1 - ze^{i\langle \mathbf{x}, \sigma \rangle}} \quad (50)$$

observe that the rhs is the Dirichlet space representation prescribed explicitly by its reproducing kernel, while ?? is on Bergman space. We also write

$$\sum_{m=0}^{\infty} \frac{-2^n z^m}{m+1} f(m) = \ln \prod_{\sigma \in \{-1,1\}^n} \left( 1 - ze^{i\langle \mathbf{x}, \sigma \rangle} \right)^{z^{-1} e^{-i\langle \mathbf{x}, \sigma \rangle}} \quad (51)$$

$$\implies \psi(z) \equiv e^{2^n z \psi_1(z)} = \prod_{m=0}^{\infty} \exp \frac{-2^n z^{m+1} f(m)}{m+1} = \prod_{\sigma \in \{-1,1\}^n} \left( 1 - ze^{i\langle \mathbf{x}, \sigma \rangle} \right)^{e^{-i\langle \mathbf{x}, \sigma \rangle}} \quad (52)$$

and see that the zeros are determined by all possible partitions. This function is holomorphic over the whole complex plane and we wish to evaluate it at  $z = 1$ :

$$\psi(1) = \prod_{m=0}^{\infty} \exp \frac{-2^n f(m)}{m+1} = \prod_{\sigma \in \{-1,1\}^n} \left(1 - e^{-i\langle \mathbf{x}, \sigma \rangle}\right)^{e^{i\langle \mathbf{x}, \sigma \rangle}} \leq 2^{-2^n} \quad (53)$$

since  $\left| \left(1 - e^{-i\langle \mathbf{x}, \sigma \rangle}\right)^{e^{i\langle \mathbf{x}, \sigma \rangle}} \right| \leq \frac{1}{2}$  (see Figure 2), implying:

$$\sum_{m=0}^{\infty} \frac{1}{m+1} \prod_{k=1}^n \cos(x_k m) \leq -\ln 2 \quad (54)$$

□

## References

- [1] Sipser, “Introduction to the Theory of Computation”. International Thomson Publishing (1996).
- [2] Kac, “Statistical Independence in Probability, Analysis and Number Theory”. Carus Mathematical Monographs, No. 12, Wiley, New York (1959)