We now prove a more generalized result independently of previous derivations.

Theorem 1. *Let $n \in \mathbb{N}$, given analytic f such that for all $t \in [-1, 1]$:*

$$|f(t)| \leq 1, |f'(t)| \leq A, |f''(t)| \leq B$$

and given even $w(t) = w(-t)$ such that for all $t \in \mathbb{R}$:

$$|w(t)| \leq 1, |w'(t)| \leq C, |w''(t)| \leq D, |w'''(t)| \leq E, w(\pm\infty) = \pm 1$$

and $w'(t) \neq 0$ for all $t \neq 0$. Then

$$\left| \int_{-1}^{1} f(t)\, dt - 2^{1-N} \sum_{m=-N}^{N} f(w(m))\, w'(m) \right| \leq 2^{-n}$$

where

$$N = 2^{n-1} w^{-1} \left(2^{-n-3} - 1\right) \left[BC^3 + 3ACD + E + 2AC^2 + 2D\right]$$

Proof. Put $g(t) = f(w(t))\, w'(t)$ as the function to be evaluated. If using Kahan summation, then we need to calculate $g(t)$ only up to the desired integral accuracy, namely n digits. So we wish to find ϵ such that

$$|g(t+\epsilon) - g(t)| \leq 2^{-n}$$

and then we should never sample g in granlarity higher than ϵ, since the contribution would be insignificant according to the given accuracy requirements. We note that

$$|g'(t)| = \left| f'(w(t)) \left[w'(t)\right]^2 + f(w(t))\, w''(t) \right| \leq AC^2 + D$$

and

$$|g''(t)| = \left| f''(w(t)) \left[w'(t)\right]^3 + 3f'(w(t))\, w'(t)\, w''(t) + f(w(t))\, w'''(t) \right|$$

$$\leq BC^3 + 3ACD + E$$

using Taylor's theorem, there exists ξ such that:

$$|g(t+\epsilon) - g(t)| = \left| \epsilon g'(t) + \frac{1}{2} g''(\xi)\, \epsilon^2 \right| \leq |\epsilon g'(t)| + \left| \frac{1}{2} g''(\xi)\, \epsilon^2 \right|$$

$$\leq |\epsilon| \left(AC^2 + D\right) + \frac{BC^3 + 3ACD + E}{2} \epsilon^2$$

we would like to have this less than 2^{-n}, so for positive ϵ we have an increasing quadratic:

$$\epsilon\left(AC^2+D\right)+\frac{BC^3+3ACD+E}{2}\epsilon^2\leq 2^{-n}$$

$$\Longrightarrow\ \frac{-\sqrt{\left(AC^2+D\right)^2-2^{1-n}\left(BC^3+3ACD+E\right)}-AC^2-D}{BC^3+3ACD+E}<\epsilon<\frac{\sqrt{(AC^2+D)^2-2^{1-n}(BC^3+}}{BC^3+3AC}$$

picking the smaller in absolute value

$$\epsilon=\frac{\sqrt{\left(AC^2+D\right)^2-2^{1-n}\left(BC^3+3ACD+E\right)}-AC^2-D}{BC^3+3ACD+E}$$

In order to get such evaluation and know that our integral indeed converges, we have only 2^{1+p} possible different values for $t\in[-1,1]$. Indeed, we need not sample the interval $[-1,1]$ only, and now we turn to calculate the desired sampling interval.

Recalling w is even, we seek $z>0$ such that

$$\int\limits_{-\infty}^{-z}g\left(t\right)dt\leq 2^{-n-3}$$

since we'd like to have $\int_{-\infty}^{-z}g\left(t\right)dt+\int_{-z}^{z}g\left(t\right)dt+\int_{z}^{\infty}g\left(t\right)dt$ up to accuracy of 2^{-n}, therefore we ask for which z the tails are negligible. Iindeed it is sufficient to have

$$\int\limits_{-\infty}^{z}g\left(t\right)dt\leq\int\limits_{-\infty}^{z}w'\left(t\right)dt\leq 1+w\left(z\right)\leq 2^{-n-3}\ \Longrightarrow\ z\leq w^{-1}\left(2^{-n-3}-1\right)$$

Our function is indeed invertible over the negavite half line since its derivative never vanish. So our interval is

$$|t|\leq w^{-1}\left(2^{-n-3}-1\right)$$

with granularity of 2^p above, ending up with total of

$$w^{-1}\left(2^{-n-3}-1\right)\frac{BC^3+3ACD+E}{\sqrt{\left(AC^2+D\right)^2-2^{1-n}\left(BC^3+3ACD+E\right)}-AC^2-D}$$

function evaluations, since these are all possible inputs on this interval by the desired and implied accuracy. We can observe that the asymptotic behavior of (1) wrt n is decreasing exponentially as long as w^{-1} decreases faster than square root. $\square$

Finding a fast-diminising w^{-1} is apparently easy: in fact anything faster than quadrator polynomial would imply exponential convergence wrt n.

For DE we have:

$$w^{-1}\left(2^{-n-3}-1\right)\frac{BC^3+3ACD+E}{\sqrt{\left(AC^2+D\right)^2-2^{1-n}\left(BC^3+3ACD+E\right)}-AC^2-D}$$

$$=\sinh^{-1}\tanh^{-1}\left(2^{-n-3}-1\right)\frac{BC^3+3ACD+E}{\sqrt{\left(AC^2+D\right)^2-2^{1-n}\left(BC^3+3ACD+E\right)}-AC^2-D}$$

and this thing goes down not only quadratically but to the fourth power. try on maxima diff(asinh(atanh(2^(-x-3)-1)),x)/diff((sqrt(c^2-2^(1-x)*b)-c),x);