Statistical Properties of Trigonometric Functions with Applications

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Observe that

$$4\sum_{m=0}^{\infty}\cos(mx)\cos(my) = \sum_{n=1}^{\infty} e^{im(x+y)}+e^{im(x-y)} \tag{1}$$

$$= \frac{1}{1-e^{i(x+y)}}+\frac{1}{1-e^{i(x-y)}} \tag{2}$$

$$= \begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} \tag{3}$$

Recalling that $\frac{1}{1-t}+\frac{1}{1-\frac{1}{t}}=1$ for all nonzero complex t, then by the formulae of sum-of-angles and geometric progression we write:

$$\lim_{r\to 1^-} 4\sum_{m=0}^{\infty} r^m\cos(mx)\cos(my) = \lim_{r\to 1^-}\sum_{n=1}^{\infty} r^m e^{im(x+y)}+r^m e^{im(x-y)} \tag{4}$$

$$= \lim_{r\to 1^-}\frac{1}{1-re^{i(x+y)}}+\frac{1}{1-re^{i(x-y)}} \tag{5}$$

$$= \begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} \tag{6}$$

if x, y cannot ever meet 2π on some integer multiple.

The following theorems generalize this idea.

1 Main Results

Lemma 1.1. *For all complex x:*

$$\sum_{m=1}^{\infty}\frac{1}{m}e^{imx} = -\ln\left(1-e^{ix}\right) \tag{7}$$

provided the sum is convergent.

Proof. Follows immediately from the Taylor series of the logarithm $\sum_{k=1}^{\infty}\frac{t^k}{k}=-\ln(1-t)$: □

Lemma 1.2. *For all complex x:*

$$\sum_{m=1}^{\infty}\frac{\cos(mx)}{m} = \ln 2+\ln\sin x \tag{8}$$

provided the sum is convergent.

Proof. By Euler's formula and Lemma 1.1:

$$2\sum_{m=1}^{\infty}\frac{\cos(mx)}{m} = \sum_{m=1}^{\infty}\frac{e^{imx}+e^{-imx}}{m} \tag{9}$$

$$= -\ln\left[\left(1-e^{ix}\right)\left(1+e^{ix}\right)\right] \tag{10}$$

$$= -\ln(2-\cos 2x) \tag{11}$$

$$= -\ln\left(4\sin^2 x\right) \tag{12}$$

$$= -2\ln\frac{1}{2\sin x} \tag{13}$$

and the lemma follows. □

Lemma 1.3. *For all complex x:*

$$\sum_{m=1}^{\infty}\frac{\sin(mx)}{m} = -i\ln\left(\frac{1-e^{ix}}{2\sin x}\right) = -i\ln\left(\frac{1-\cos x - i\sin x}{2\sin x}\right) = -i\ln\left(\frac{\cos^2\frac{x}{2}}{\sin x}-\frac{i}{2}\right) \tag{14}$$

$$\sum_{m=1}^{\infty}\frac{\sin(mx)}{m} = -i\ln\left(1-e^{ix}\right)-\ln 2-\ln\sin x \tag{15}$$

provided the sum is convergent.

Proof. Follows immediately from the Taylor series of the logarithm $\sum_{k=1}^{\infty}\frac{t^k}{k}=-\ln(1-t)$: □

On this section we denote

$$f(m)\equiv\prod_{k=1}^{n}\cos(x_k m) \tag{16}$$

where $\mathbf{x}$ is a vector of n numbers to partition at the scope of the partition problem. Specific requirements from $\mathbf{x}$ depend on the discussed scope and will be well defined every time.

Theorem 1.4. *Fix $\mathbf{x}$ to be a vector of n nonzero algebraic numbers. Then for all nonzero algebraic number t the following identities hold:*

$$\sum_{\sigma\in\{-1,1\}^n}\ln\left|\sin\langle\mathbf{x},\sigma\rangle\right| = -\frac{2^n}{\sqrt{2}}\sum_{m=1}^{\infty}\frac{1}{m}f(2tm) \tag{17}$$

$$\sum_{\sigma\in\{-1,1\}^n}\ln\left|\cos\langle\mathbf{x},\sigma\rangle\right| = -\frac{2^n}{\sqrt{2}}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(2tm) \tag{18}$$

$$\sum_{\sigma\in\{-1,1\}^n}\ln\left|\tan\langle\mathbf{x},\sigma\rangle\right| = \sum_{m=1}^{\infty}\frac{2^n}{m+\frac{1}{2}}\prod_{k=1}^{n}f(4tm+t) \tag{19}$$

Proof. Write

$$\lim_{r\to1^-}2^n\sum_{m=1}^{\infty}r^m f(m) = \lim_{r\to1^-}\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}r^m e^{im\langle\mathbf{x},\sigma\rangle} \tag{20}$$

$$= \lim_{r\to1^-}\sum_{\sigma\in\{-1,1\}^n}\frac{1}{1-re^{i\langle\mathbf{x},\sigma\rangle}} \tag{21}$$

$$= \begin{cases}-\infty & \exists\sigma\in\{-1,1\}^n \mid \langle\mathbf{x},\sigma\rangle=0\\ 0 & \forall\sigma\in\{-1,1\}^n,\langle\mathbf{x},\sigma\rangle\neq0\end{cases} \tag{22}$$

since $\sum_{k=1}^{\infty} \frac{t^k}{k} = \ln \frac{1}{1-t}$ we can write

$$\lim_{r\to 1^-} -2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) = \lim_{r\to 1^-} \sum_{\sigma\in\{-1,1\}^n} \sum_{m=1}^{\infty} \frac{r^m}{m} e^{im\langle \mathbf{x},\sigma\rangle} \tag{23}$$

$$= \lim_{r\to 1^-} \sum_{\sigma\in\{-1,1\}^n} \ln \frac{1}{1-re^{i\langle \mathbf{x},\sigma\rangle}} \tag{24}$$

implying:

$$\lim_{r\to 1^-} \exp\left(-2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m)\right) = \lim_{r\to 1^-} \prod_{\sigma\in\{-1,1\}^n} \left(1-re^{i\langle \mathbf{x},\sigma\rangle}\right) \tag{25}$$

for $r \to 1$ we have $1-re^{i\langle \mathbf{x},\sigma\rangle} = 0$ iff $\langle \mathbf{x},\sigma\rangle$ is a zero partition. Taking the limit:

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{f(m)}{m}\right) = \prod_{\sigma\in\{-1,1\}^n} \left(1-e^{i\langle \mathbf{x},\sigma\rangle}\right) \tag{26}$$

$$= \prod_{\sigma\in\{-1,1\}^n} \sqrt{2\sin^2 \frac{\langle \mathbf{x},\sigma\rangle}{2}} \tag{27}$$

where the latter equalit is due to every σ there exists a matching $-\sigma$ and

$$\left(1-e^{i\langle \mathbf{x},\sigma\rangle}\right)\left(1-e^{-i\langle \mathbf{x},\sigma\rangle}\right) = 2-2\cos\langle \mathbf{x},\sigma\rangle \tag{28}$$

$$= 2\sin^2 \frac{\langle \mathbf{x},\sigma\rangle}{2} \tag{29}$$

concluding

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{f(2m)}{m}\right) = 2^{2^{n-1}} \prod_{\sigma\in\{-1,1\}^n} |\sin\langle \mathbf{x},\sigma\rangle| \tag{30}$$

$$\Longrightarrow \frac{1}{\sqrt{2}} \exp\left(-\sum_{m=1}^{\infty} \frac{f(2m)}{m}\right) = \sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n} |\sin\langle \mathbf{x},\sigma\rangle|} \tag{31}$$

Equivalently,

$$-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m) = -\sum_{\sigma\in\{-1,1\}^n} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{im\langle \mathbf{x},\sigma\rangle} \tag{32}$$

$$= \sum_{\sigma\in\{-1,1\}^n} \ln\left(1+e^{i\langle \mathbf{x},\sigma\rangle}\right) \tag{33}$$

or

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m)\right) = \prod_{\sigma\in\{-1,1\}^n} \left(1+e^{i\langle \mathbf{x},\sigma\rangle}\right) \tag{34}$$

$$= \prod_{\sigma\in\{-1,1\}^n} \sqrt{2\cos^2 \frac{\langle \mathbf{x},\sigma\rangle}{2}} \tag{35}$$

$$\Longrightarrow \frac{1}{\sqrt{2}} \exp\left(-\sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m)\right) = \sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n} |\cos\langle \mathbf{x},\sigma\rangle|} \tag{36}$$

dividing (23) by (28) we conclude:

$$\exp\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(2m)-\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right) \tag{37}$$

$$= \quad \exp\left(\sum_{m=1}^{\infty}\frac{2f(4m+2)}{2m+1}\right) \tag{38}$$

$$= \quad \sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\tan\langle\mathbf{x},\sigma\rangle\right|} \tag{39}$$

□

Corollary 1.5. *(31) interestingly implies that if $2^{-n}\langle\mathbf{x},\mathbf{1}\rangle\le 1$ for positive algebraic x's:*

$$\frac{1}{\sqrt{2}}\exp\left(-\sum_{m=1}^{\infty}\frac{f\left(2^{1-n}m\right)}{m}\right)=\sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\sin 2^{-n}\langle\mathbf{x},\sigma\rangle\right|}\approx\sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\langle\mathbf{x},\sigma\rangle\right|} \tag{40}$$

Theorem 1.6. *The following identities hold for $\mathbf{x}\in\mathbb{R}_+^n$ TBD*

Proof. Define

$$\psi_0(z)=\sum_{m=0}^{\infty}z^m f(m)=\sum_{\sigma\in\{-1,1\}^n}\sum_{m=0}^{\infty}z^m e^{im\langle\mathbf{x},\sigma\rangle}=2^{-n}\sum_{\sigma\in\{-1,1\}^n}\frac{1}{1-ze^{i\langle\mathbf{x},\sigma\rangle}} \tag{41}$$

for complex $|z|<1$ and due to summation of geometric progression, and write

$$\psi_0(z)+\psi_0(-z)=\sum_{m=0}^{\infty}2z^{2m}f(2m)=2^{-n}\sum_{\sigma\in\{-1,1\}^n}\frac{2}{1-z^2e^{2i\langle\mathbf{x},\sigma\rangle}} \tag{42}$$

integrating wrt z:

$$\sum_{m=0}^{\infty}\frac{2z^{2m+1}}{2m+1}f(2m)=2^{-n}\sum_{\sigma\in\{-1,1\}^n}e^{-i\langle\mathbf{x},\sigma\rangle}\ln\frac{ze^{i\langle\mathbf{x},\sigma\rangle}+1}{ze^{i\langle\mathbf{x},\sigma\rangle}-1} \tag{43}$$

for $z=1$ we get

$$\sum_{m=0}^{\infty}\frac{2^{n+1}f(2m)}{2m+1}=\sum_{\sigma\in\{-1,1\}^n}e^{-i\langle\mathbf{x},\sigma\rangle}\ln\frac{e^{i\langle\mathbf{x},\sigma\rangle}+1}{e^{i\langle\mathbf{x},\sigma\rangle}-1} \tag{44}$$

$$=\sum_{\sigma\in\{-1,1\}^n}e^{-i\langle\mathbf{x},\sigma\rangle}\ln\left[-i\cot\frac{\langle\mathbf{x},\sigma\rangle}{2}\right]=\sum_{\sigma\in\{-1,1\}^n}\ln\left[\frac{i\sin\langle\mathbf{x},\sigma\rangle}{\cos\langle\mathbf{x},\sigma\rangle-1}\right]^{e^{-i\langle\mathbf{x},\sigma\rangle}} \tag{45}$$

by exponentiation:

$$\prod_{m=0}^{\infty}\exp\frac{-2^{n+1}f(2m)}{2m+1}=\prod_{\sigma\in\{-1,1\}^n}\left[\frac{\cos\langle\mathbf{x},\sigma\rangle-1}{i\sin\langle\mathbf{x},\sigma\rangle}\right]^{e^{-i\langle\mathbf{x},\sigma\rangle}} \tag{46}$$

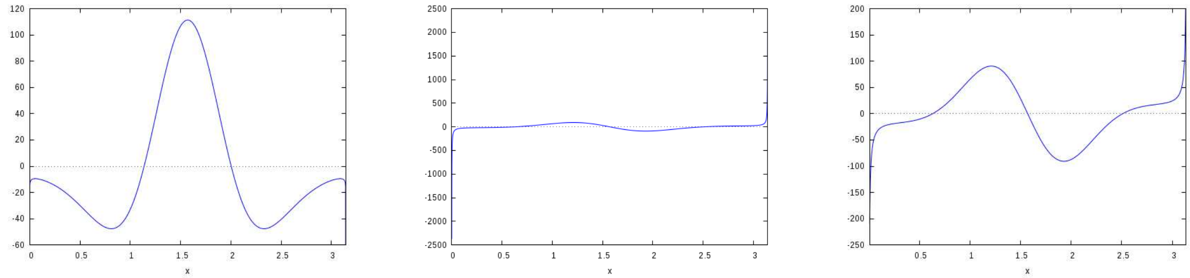
which equals zero iff $\mathbf{x}$ is partitionable. it is interesting to observe the plot of $\left[\frac{i\sin x}{\cos x-1}\right]^{e^{-ix}}$ as shown in Figure 1.

Observe that

$$\prod_{m=0}^{\infty}\left(1+\frac{f(2m)}{2m+1}\right)\ge\prod_{m=0}^{\infty}\exp\frac{-f(2m)}{2m+1}\ge\prod_{m=0}^{\infty}\left(1-\frac{f(2m)}{2m+1}\right) \tag{47}$$

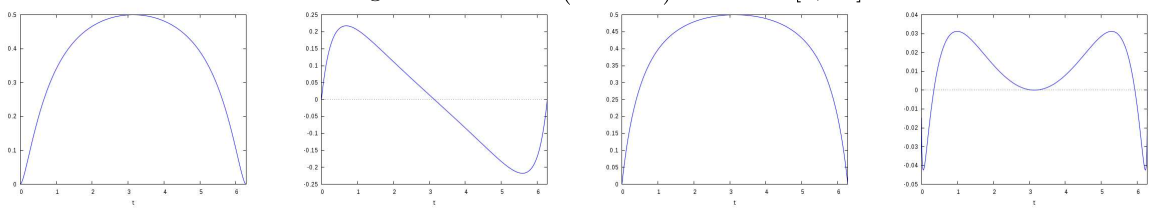
which is a nontrivial result if recalling the structure of f. The rhs of (28) goes to zero if the set is partitionable, and the lhs goes to infinity if the set is unpartitionable. TBD: explain this by rate-of-interest example and show it rigorously by calculating the drift of this geometric motion.

Figure 1: Plots of $\left[\frac{i\sin x}{\cos x-1}\right]^{e^{-ix}}$



The leftmost is the realpart, while the midle is the imaginary part, both over $[0,\pi]$. The rightmost is the imaginary part over $\left[\frac{1}{100},\pi-\frac{1}{100}\right]$

Figure 2: Plots of $\left(1-e^{-it}\right)^{e^{it}}$ for $t\in[0,2\pi]$



Leftmost is real part, middle is imaginary part, afterwards - the norm, and the rightmost is $\left|\left(1-e^{-it}\right)^{e^{it}}\right|-\sqrt{\frac{t}{2\pi}\left(1-\frac{t}{2\pi}\right)}$.

We therefore require $\mathbf{x}$ to satisfy $\langle\mathbf{x},\mathbf{1}\rangle<\pi$. This way we can see that (46) converges to zero iff $\mathbf{x}$ is partitionable.

On a different route, beginning from (**??**), integrating (we may do so due to Vitali's convergence theorem) and dividing by $-z$, we define:

$$\psi_1(z)\equiv-\frac{1}{z}\int^{z}\psi_0(z')\,dz'=\sum_{m=0}^{\infty}\frac{-z^m}{m+1}f(m)=-2^{-n}\sum_{\sigma\in\{-1,1\}^n}\frac{1}{ze^{i\langle\mathbf{x},\sigma\rangle}}\ln\frac{1}{1-ze^{i\langle\mathbf{x},\sigma\rangle}}\tag{48}$$

observe that the rhs is the Dirichlet space representation prescribed explicitly by its reproducing kernel, while **??** is on Bergman space. We also write

$$\sum_{m=0}^{\infty}\frac{-2^nz^m}{m+1}f(m)=\ln\prod_{\sigma\in\{-1,1\}^n}\left(1-ze^{i\langle\mathbf{x},\sigma\rangle}\right)^{z^{-1}e^{-i\langle\mathbf{x},\sigma\rangle}}\tag{49}$$

$$\implies\psi(z)\equiv e^{2^nz\psi_1(z)}=\prod_{m=0}^{\infty}\exp\frac{-2^nz^{m+1}f(m)}{m+1}=\prod_{\sigma\in\{-1,1\}^n}\left(1-ze^{i\langle\mathbf{x},\sigma\rangle}\right)^{\left(e^{-i\langle\mathbf{x},\sigma\rangle}\right)}\tag{50}$$

and see that the zeros are detemined by all possible partitions. This funtion is holomorphic over the whole complex plane and we wish to evaluate it at $z=1$:

$$\psi(1)=\prod_{m=0}^{\infty}\exp\frac{-2^nf(m)}{m+1}=\prod_{\sigma\in\{-1,1\}^n}\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right)^{e^{i\langle\mathbf{x},\sigma\rangle}}\leq2^{-2^n}\tag{51}$$

since $\left|\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right)^{e^{i\langle\mathbf{x},\sigma\rangle}}\right|\leq\frac{1}{2}$ (see Figure 2), implying:

$$\sum_{m=0}^{\infty}\frac{1}{m+1}\prod_{k=1}^{n}\cos(x_km)\leq-\ln2\tag{52}$$

$\square$

References

[1] Sipser, "Introduction to the Theory of Computation". International Thomson Publishing (1996).

[2] Kac, "Statistical Independence in Probability, Analysis and Number Theory". Carus Mathematical Monographs, No. 12, Wiley, New York (1959)