

# Statistical Properties of Trigonometric Functions with Applications

January 15, 2016

## 1 Background

Observe that

$$4 \sum_{m=0}^{\infty} \cos(mx) \cos(my) = \sum_{n=1}^{\infty} e^{im(x+y)} + e^{im(x-y)} \quad (1)$$

$$= \frac{1}{1 - e^{i(x+y)}} + \frac{1}{1 - e^{i(x-y)}} \quad (2)$$

$$= \frac{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}}{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}} \quad (3)$$

$$= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (4)$$

Recalling that  $\frac{1}{1-t} + \frac{1}{1-\frac{1}{t}} = 1$  for all nonzero complex  $t$ , then by the formulae of sum-of-angles and geometric progression we write:

$$\lim_{r \rightarrow 1^-} 4 \sum_{m=0}^{\infty} r^m \cos(mx) \cos(my) = \lim_{r \rightarrow 1^-} \sum_{n=1}^{\infty} r^m e^{im(x+y)} + r^m e^{im(x-y)} \quad (5)$$

$$= \lim_{r \rightarrow 1^-} \frac{1}{1 - r e^{i(x+y)}} + \frac{1}{1 - r e^{i(x-y)}} \quad (6)$$

$$= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (7)$$

if  $x, y$  cannot ever meet  $2\pi$  on some integer multiple.

The following theorem generalizes this result. Note that determining whether the products on the theorem's statements equal zero or not (equivalently, whether the infinite sums go to minus infinity) is an NP-Hard problem. A structure of Mercer kernels also emerges from our identities. We will look on such applications later on this note.

## 2 Main Result

**Theorem 2.1.** Fix  $\mathbf{x}$  to be a vector of  $n$  nonzero algebraic numbers. Then for all nonzero algebraic number  $t$  the following identities hold:

$$\sum_{\sigma \in \{-1,1\}^n} \ln |\sin \langle \mathbf{x}, \sigma \rangle| = -\frac{2^n}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{1}{m} \prod_{k=1}^n \cos(2x_k t m) \quad (8)$$

$$\sum_{\sigma \in \{-1,1\}^n} \ln |\cos \langle \mathbf{x}, \sigma \rangle| = -\frac{2^n}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \prod_{k=1}^n \cos(2x_k t m) \quad (9)$$

$$\sum_{\sigma \in \{-1,1\}^n} \ln |\tan \langle \mathbf{x}, \sigma \rangle| = \sum_{m=1}^{\infty} \frac{2^n}{m + \frac{1}{2}} \prod_{k=1}^n \cos(4x_k t m + 2x_k t) \quad (10)$$

*Proof.* Denote

$$f(m) \equiv \prod_{k=1}^n \cos(x_k m) \quad (11)$$

and write

$$\lim_{r \rightarrow 1^-} 2^n \sum_{m=1}^{\infty} r^m f(m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} r^m e^{im \langle \mathbf{x}, \sigma \rangle} \quad (12)$$

$$= \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - r e^{i \langle \mathbf{x}, \sigma \rangle}} \quad (13)$$

$$= \begin{cases} -\infty & \exists \sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = 0 \\ 0 & \forall \sigma \in \{-1,1\}^n, \langle \mathbf{x}, \sigma \rangle \neq 0 \end{cases} \quad (14)$$

since  $\sum_{k=1}^{\infty} \frac{t^k}{k} = \ln \frac{1}{1-t}$  we can write

$$\lim_{r \rightarrow 1^-} -2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{r^m}{m} e^{im \langle \mathbf{x}, \sigma \rangle} \quad (15)$$

$$= \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \ln \frac{1}{1 - r e^{i \langle \mathbf{x}, \sigma \rangle}} \quad (16)$$

implying:

$$\lim_{r \rightarrow 1^-} \exp \left( -2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) \right) = \lim_{r \rightarrow 1^-} \prod_{\sigma \in \{-1,1\}^n} (1 - r e^{i \langle \mathbf{x}, \sigma \rangle}) \quad (17)$$

for  $r \rightarrow 1$  we have  $1 - r e^{i \langle \mathbf{x}, \sigma \rangle} = 0$  iff  $\langle \mathbf{x}, \sigma \rangle$  is a zero partition. Taking the limit:

$$\exp \left( -2^n \sum_{m=1}^{\infty} \frac{f(m)}{m} \right) = \prod_{\sigma \in \{-1,1\}^n} (1 - e^{i \langle \mathbf{x}, \sigma \rangle}) \quad (18)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (19)$$

where the latter equalit is due to every  $\sigma$  there exists a matching  $-\sigma$  and

$$(1 - e^{i\langle \mathbf{x}, \sigma \rangle}) (1 - e^{-i\langle \mathbf{x}, \sigma \rangle}) = 2 - 2 \cos \langle \mathbf{x}, \sigma \rangle \quad (20)$$

$$= 2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2} \quad (21)$$

concluding

$$\exp \left( -2^n \sum_{m=1}^{\infty} \frac{f(2m)}{m} \right) = 2^{2^{n-1}} \prod_{\sigma \in \{-1,1\}^n} |\sin \langle \mathbf{x}, \sigma \rangle| \quad (22)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \exp \left( - \sum_{m=1}^{\infty} \frac{f(2m)}{m} \right) = 2^n \sqrt{\prod_{\sigma \in \{-1,1\}^n} |\sin \langle \mathbf{x}, \sigma \rangle|} \quad (23)$$

equivalently,

$$-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m) = - \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} \quad (24)$$

$$= \sum_{\sigma \in \{-1,1\}^n} \ln (1 + e^{i\langle \mathbf{x}, \sigma \rangle}) \quad (25)$$

or

$$\exp \left( -2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m) \right) = \prod_{\sigma \in \{-1,1\}^n} (1 + e^{i\langle \mathbf{x}, \sigma \rangle}) \quad (26)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \cos^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (27)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \exp \left( - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m) \right) = 2^n \sqrt{\prod_{\sigma \in \{-1,1\}^n} |\cos \langle \mathbf{x}, \sigma \rangle|} \quad (28)$$

dividing (23) by (28) we conclude:

$$\exp \left( \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m) - \sum_{m=1}^{\infty} \frac{f(2m)}{m} \right) \quad (29)$$

$$= \exp \left( \sum_{m=1}^{\infty} \frac{2f(4m+2)}{2m+1} \right) \quad (30)$$

$$= 2^n \sqrt{\prod_{\sigma \in \{-1,1\}^n} |\tan \langle \mathbf{x}, \sigma \rangle|} \quad (31)$$

□