Statistical Properties of Trigonometric Functions with Applications to NP-Complete Problems

January 15, 2016

Observe that

$$4\sum_{m=0}^{\infty}\cos(mx)\cos(my)=\sum_{n=1}^{\infty}e^{im(x+y)}+e^{im(x-y)}=\frac{1}{1-e^{i(x+y)}}+\frac{1}{1-e^{i(x-y)}} \tag{1}$$

$$=\frac{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}}{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}}=\begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} \tag{2}$$

Recalling that $\frac{1}{1-t}+\frac{1}{1-\frac{1}{t}}=1$ for all nonzero complex t, then by the formulae of sum-of-angles and geometric progression we write:

$$\lim_{r\to 1^-}4\sum_{m=0}^{\infty}r^m\cos(mx)\cos(my)=\lim_{r\to 1^-}\sum_{n=1}^{\infty}r^me^{im(x+y)}+r^me^{im(x-y)} \tag{3}$$

$$=\lim_{r\to 1^-}\frac{1}{1-re^{i(x+y)}}+\frac{1}{1-re^{i(x-y)}}=\begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} \tag{4}$$

if x,y cannot ever meet 2π on some integer multiple. Similarly, for $\mathbf{x}\in\mathbb{R}^n$ such that $\mathbf{x}$'s elements are lineraly independent of π over the rationals:

$$\lim_{r\to 1^-}1-2^n\sum_{m=1}^{\infty}r^m\prod_{k=1}^{n}\cos(x_km)=\lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\frac{1}{1-re^{i\langle\mathbf{x},\sigma\rangle}}=\begin{cases}-\infty & \exists\sigma\in\{-1,1\}^n \mid \langle\mathbf{x},\sigma\rangle=0\\ 0 & \forall\sigma\in\{-1,1\}^n,\langle\mathbf{x},\sigma\rangle\neq 0\end{cases} \tag{5}$$

since $\sum_{k=1}^{\infty}\frac{t^k}{k}=-\ln(1-t)$ we can write

$$\lim_{r\to 1^-}1-2^n\sum_{m=1}^{\infty}\frac{r^m}{m}\prod_{k=1}^{n}\cos(x_km)=\lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{r^m}{m}e^{im\langle\mathbf{x},\sigma\rangle}=\lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\ln\frac{1}{1-re^{i\langle\mathbf{x},\sigma\rangle}} \tag{6}$$

implying:

$$\lim_{r\to 1^-}\exp\left(-1+2^n\sum_{m=1}^{\infty}\frac{r^m}{m}\prod_{k=1}^{n}\cos(x_km)\right)=\lim_{r\to 1^-}\prod_{\sigma\in\{-1,1\}^n}\left(1-re^{i\langle\mathbf{x},\sigma\rangle}\right) \tag{7}$$

for $r\to 1$ we have $1-re^{i\langle\mathbf{x},\sigma\rangle}=0$ iff $\langle\mathbf{x},\sigma\rangle$ is a zero partition. Taking the limit:

$$\exp\left(-1+2^n\sum_{m=1}^{\infty}\frac{1}{m}\prod_{k=1}^{n}\cos(x_km)\right)=\prod_{\sigma\in\{-1,1\}^n}\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right) \tag{8}$$

since for every σ there exists a matching $-\sigma$ and $\sqrt{\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right)\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right)}=\sqrt{2-2\cos\langle\mathbf{x},\sigma\rangle}=\sqrt{2}\left|\sin\frac{\langle\mathbf{x},\sigma\rangle}{2}\right|$ we may write

$$\exp\left(-1+2^n\sum_{m=1}^{\infty}\frac{1}{m}\prod_{k=1}^{n}\cos\left(x_k m\right)\right)=\prod_{\sigma\in\{-1,1\}^n}\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right) \tag{9}$$

$$0\leq\exp\left(-1+2^n\left(\ln\frac{1}{2}+\sum_{m=1}^{\infty}\frac{1}{m}\prod_{k=1}^{n}\cos\left(2x_k m\right)\right)\right)=2^{2^{n-1}}\left|\prod_{\sigma\in\{-1,1\}^n}\sin\langle\mathbf{x},\sigma\rangle\right|\leq 1 \tag{10}$$

$$0\leq\exp\left(-1+2^n\left(\ln\frac{1}{2}+\sum_{m=1}^{\infty}\frac{1}{m}\prod_{k=1}^{n}\cos\left(2x_k m\right)\right)\right)=2^{2^{n-1}}\left|\prod_{\sigma\in\{-1,1\}^n}\sin\langle\mathbf{x},\sigma\rangle\right|\leq 1 \tag{11}$$

or

$$0\leq\exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\prod_{k=1}^{n}\cos\left(2x_k m\right)\right)=\frac{1}{2}\sqrt[2^n]{\left|\prod_{\sigma\in\{-1,1\}^n}\sin\langle\mathbf{x},\sigma\rangle\right|}\leq\frac{1}{2}e^{2^{-n}}\approx\frac{1}{2} \tag{12}$$

$$0\leq\exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\prod_{k=1}^{n}\cos\left(2x_k m\right)\right)=\frac{1}{2}\sqrt[2^n]{\left|\prod_{\sigma\in\{-1,1\}^n}\sin\langle\mathbf{x},\sigma\rangle\right|}\leq\frac{1}{2}e^{2^{-n}}\approx\frac{1}{2} \tag{13}$$

where equality to zero takes place iff $\mathbf{x}$ is partitionable. The similarity to the alternating Harmonic series is remarkable. Indeed, intuitively, partitions are about alternating signs.

We now wish to estimate a probabilistic lower bound to $\sin\left(m\langle\mathbf{x},\sigma\rangle\right)$ over natural m, in order to know how many summands on (8) we have to sum in order to get a probabilistic confidence interval to the null hypothesis that $\mathbf{x}$ is partitionable. We can do various apparently-independent trials, due to the equidistribution theorem, and the partitions agnostic to multiplication of the inputs by a constant, and due to the reduction from SAT to PART allowing many various PART problems while all are simultanously partitionable or not.