

Statistical Properties of Trigonometric Functions with Applications to NP-Complete Problems

January 15, 2016

Observe that

$$4 \sum_{m=0}^{\infty} \cos(mx) \cos(my) = \sum_{n=1}^{\infty} e^{im(x+y)} + e^{im(x-y)} = \frac{1}{1 - e^{i(x+y)}} + \frac{1}{1 - e^{i(x-y)}} \quad (1)$$

$$= \frac{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}}{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}} = \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (2)$$

Recalling that $\frac{1}{1-t} + \frac{1}{1-\bar{t}} = 1$ for all nonzero complex t , then by the formulae of sum-of-angles and geometric progression we write:

$$\lim_{r \rightarrow 1^-} 4 \sum_{m=0}^{\infty} r^m \cos(mx) \cos(my) = \lim_{r \rightarrow 1^-} \sum_{n=1}^{\infty} r^m e^{im(x+y)} + r^m e^{im(x-y)} \quad (3)$$

$$= \lim_{r \rightarrow 1^-} \frac{1}{1 - re^{i(x+y)}} + \frac{1}{1 - re^{i(x-y)}} = \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (4)$$

if x, y cannot ever meet 2π on some integer multiple. Similarly, for $\mathbf{x} \in \mathbb{R}^n$ such that \mathbf{x} 's elements are linearly independent of π over the rationals:

$$\lim_{r \rightarrow 1^-} 1 - 2^n \sum_{m=1}^{\infty} r^m \prod_{k=1}^n \cos(x_k m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - re^{i\langle \mathbf{x}, \sigma \rangle}} = \begin{cases} -\infty & \exists \sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = 0 \\ 0 & \forall \sigma \in \{-1,1\}^n, \langle \mathbf{x}, \sigma \rangle \neq 0 \end{cases} \quad (5)$$

since $\sum_{k=1}^{\infty} \frac{t^k}{k} = -\ln(1-t)$ we can write

$$\lim_{r \rightarrow 1^-} 1 - 2^n \sum_{m=1}^{\infty} \frac{r^m}{m} \prod_{k=1}^n \cos(x_k m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{r^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \ln \frac{1}{1 - re^{i\langle \mathbf{x}, \sigma \rangle}} \quad (6)$$

implying:

$$\lim_{r \rightarrow 1^-} \exp \left(-1 + 2^n \sum_{m=1}^{\infty} \frac{r^m}{m} \prod_{k=1}^n \cos(x_k m) \right) = \lim_{r \rightarrow 1^-} \prod_{\sigma \in \{-1,1\}^n} \left(1 - re^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (7)$$

for $r \rightarrow 1$ we have $1 - re^{i\langle \mathbf{x}, \sigma \rangle} = 0$ iff $\langle \mathbf{x}, \sigma \rangle$ is a zero partition. Taking the limit:

$$\exp \left(-1 + 2^n \sum_{m=1}^{\infty} \frac{1}{m} \prod_{k=1}^n \cos(x_k m) \right) = \prod_{\sigma \in \{-1,1\}^n} \left(1 - e^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (8)$$

since for every σ there exists a matching $-\sigma$ and $(1 - e^{i\langle \mathbf{x}, \sigma \rangle})(1 - e^{-i\langle \mathbf{x}, \sigma \rangle}) = 2 - 2\cos \langle \mathbf{x}, \sigma \rangle = 2\sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}$ we may write

$$\exp \left(-1 + 2^n \sum_{m=1}^{\infty} \frac{1}{m} \prod_{k=1}^n \cos(x_k m) \right) = \prod_{\sigma \in \{-1,1\}^n} \sqrt{2\sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (9)$$

or

$$0 \leq \exp \left(-2 + 2^n \left(-\ln 2 + 2 \sum_{m=1}^{\infty} \frac{1}{m} \prod_{k=1}^n \cos(2x_k m) \right) \right) = \prod_{\sigma \in \{-1,1\}^n} \sin^2 \langle \mathbf{x}, \sigma \rangle \leq 1 \quad (10)$$

equivalently,

$$-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \prod_{k=1}^n \cos(x_k m) = - \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} = \sum_{\sigma \in \{-1,1\}^n} \ln(1 + e^{i\langle \mathbf{x}, \sigma \rangle}) \quad (11)$$

or

$$\exp \left(-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \prod_{k=1}^n \cos(x_k m) \right) = \prod_{\sigma \in \{-1,1\}^n} (1 + e^{i\langle \mathbf{x}, \sigma \rangle}) = \prod_{\sigma \in \{-1,1\}^n} \sqrt{2\cos^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (12)$$

$$\Rightarrow \exp \left(-2^n \left(\ln 2 + 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \prod_{k=1}^n \cos(2x_k m) \right) \right) = \prod_{\sigma \in \{-1,1\}^n} \cos^2 \langle \mathbf{x}, \sigma \rangle \quad (13)$$

dividing (10) and (13)

$$\exp \left(-2^n \left(\ln 2 + 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \prod_{k=1}^n \cos(2x_k m) \right) + 2 - 2^n \left(-\ln 2 + 2 \sum_{m=1}^{\infty} \frac{1}{m} \prod_{k=1}^n \cos(2x_k m) \right) \right) \quad (14)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \cot^2 \langle \mathbf{x}, \sigma \rangle \quad (15)$$

$$\Rightarrow \exp \left(1 - 2^n \sum_{m=1}^{\infty} \frac{\prod_{k=1}^n \cos(2x_k m)}{m + \frac{1}{2}} \right) = \prod_{\sigma \in \{-1,1\}^n} |\cot \langle \mathbf{x}, \sigma \rangle| \quad (16)$$