Let $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ and consider the formula $2\cos a \cos b = \cos(a+b) + \cos(a-b)$ and the cosine being even function to see that:

$$\psi(t) = 2^n \prod_{k=1}^{n} \cos(x_k t) = \sum_{\sigma \in \{-1,1\}^n} \cos t \langle \mathbf{x}, \sigma \rangle = 2 \sum_{\sigma \in \{-1,1\}^n} e^{it\langle \mathbf{x}, \sigma \rangle} \tag{1}$$

where $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^{n} \sigma_k x_k$ and counting the number of $\sigma \in \{-1,1\}^n$ satisfying $\langle \mathbf{x}, \sigma \rangle = 0$ is a #P problem. We write down the following sum just for fun and substitute (1) in it:

$$S = \frac{1}{2n+2} \sum_{m=1}^{n+1} \psi\left(2\pi \frac{m}{n+1} + i \ln 2\right) = \sum_{\sigma \in \{-1,1\}^n} 2^{-\langle \mathbf{x}, \sigma \rangle} \sum_{m=1}^{n+1} \frac{1}{n+1} e^{2\pi i \frac{m}{n+1} \langle \mathbf{x}, \sigma \rangle} \tag{2}$$

the summation of roots of unity equals zero iff $n+1$ does not divide $\langle \mathbf{x}, \sigma \rangle$, and if it does divide then it sums to $n+1$. Using this fact and denoting the number of partitions that sum to k by $c_k = |\{\sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = k\}|$, we get

$$S = \sum_{k=-\infty}^{\infty} c_{k(n+1)} 2^{-(n+1)k} \tag{3}$$

recalling that $\sum_{k=-\infty}^{\infty} c_k = 2^n$ and are all positive, while in (3) being multiplied by distinct powers 2^{-n-1}, therefore the summands' binary digits never interfere with each other and can never grow as large as 1, except where $k = 0$. Recalling that c_0 is our quantity of interest, we have proved that the number of zero partitions in $\mathbf{x}$ is

$$\left\lfloor \frac{2^{n-1}}{n+1} \sum_{m=1}^{n+1} \prod_{k=1}^{n} \cos\left(2\pi \frac{m}{n+1} + i \ln 2\right) \right\rfloor \tag{4}$$