Modular Arithmetic Problem in #P

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Abstract

Given n integers $x_1, ..., x_n$ in binary (or higher) radix, calculating the n LSB bits of the integer part of $\prod_{k=1}^{n}\left[2^{nx_k}+2^{-nx_k}\right]$ is a #P problem. The calculation clearly has a pseudo-polynomial time complexity, as it is polynomial if the input would be supplied in unary format.

Let $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ and consider the formula $2\cos a \cos b = \cos(a+b) + \cos(a-b)$ and the cosine being even function to see that:

$$\psi(t) = 2^n \prod_{k=1}^{n} \cos(x_k t) = \sum_{\sigma\in\{-1,1\}^n} \cos t\langle \mathbf{x},\sigma\rangle = \sum_{\sigma\in\{-1,1\}^n} e^{it\langle \mathbf{x},\sigma\rangle} \tag{1}$$

where $\langle \mathbf{x},\sigma\rangle = \sum_{k=1}^{n}\sigma_k x_k$ and counting the number of $\sigma \in \{-1,1\}^n$ satisfying $\langle \mathbf{x},\sigma\rangle = 0$ is a #P problem. We write down the following sum just for fun and substitute (1) in it:

$$S = \frac{1}{n}\sum_{m=1}^{n}\psi\left(\frac{2\pi m}{n} + i\ln 2\right) = \sum_{\sigma\in\{-1,1\}^n} \frac{2^{-\langle \mathbf{x},\sigma\rangle}}{n}\sum_{m=1}^{n} e^{\frac{2\pi i m}{n}\langle \mathbf{x},\sigma\rangle} \tag{2}$$

multiplying all x_k by n (while preserving partitions), $e^{\frac{2\pi i m}{n}\langle n\mathbf{x},\sigma\rangle} = 1$ so we get:

$$S = \frac{1}{n}\sum_{m=1}^{n}\psi\left(\frac{2\pi m}{n} + i\ln 2\right) = \sum_{\sigma\in\{-1,1\}^n} 2^{-\langle \mathbf{x},\sigma\rangle} \tag{3}$$

Denoting the number of partitions that sum to u by

$$c_u = \left|\{\sigma \in \{-1,1\}^n \mid \langle \mathbf{x},\sigma\rangle = u\}\right| \tag{4}$$

then

$$S = \sum_{u=-\infty}^{\infty} c_{nu} 2^{-nu} \tag{5}$$

Recalling that $\sum_{u=-\infty}^{\infty} c_u = 2^n$ and c_u are all positive, while in (3) being multiplied by distinct powers $2^{\pm n}$, therefore the summands' binary digits never interfere with each other and can never grow as large as 1, except when $u = 0$.

Recalling that c_0 is our quantity of interest, we have proved that the number of zero partitions in $\mathbf{x}$

$$\left\lfloor \frac{2^n}{n} \sum_{m=1}^{n} \prod_{k=1}^{n} \cos\left[nx_k \left(\frac{2\pi m}{n} + i \ln 2 \right) \right] \right\rfloor \mod 2^n \tag{6}$$

$$= \left\lfloor \prod_{k=1}^{n} \left[2^{nx_k} + 2^{-nx_k} \right] \right\rfloor \mod 2^n \tag{7}$$

$$= \left\lfloor 2^{-n \sum_{k=1}^{n} x_k} \prod_{k=1}^{n} \left[1 + 2^{2nx_k} \right] \right\rfloor \mod 2^n \tag{8}$$

Denote $s = n \sum_{k=1}^{n} x_k$ and

$$M = \prod_{k=1}^{n} \left[1 + 2^{2nx_k} \right] = \sum_{\sigma \in \{0,1\}^n} 4^{n \langle \mathbf{x}, \sigma \rangle} \tag{9}$$

then (8) tells us that the number of zero partitions is encoded as a binary number in the binary digits of M, from the s'th dight to the $s+n$ digit.