Ergodicity of Unsatisfiable Boolean Formulas

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Abstract

We present some trigonometric identities that also express the solution to NP-Complete problems in forms of infinite series.

We say that a vector $\mathbf{x} \in \mathbb{C}^n$ is *partitionable* if there is an assignment of $\pm$ such that it sums to zero, or more precisely: if there exists $\sigma \in \{-1,1\}^n$ such that $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^{n} x_k \sigma_k = 0$. We call a *partition* to the quentity $\langle \mathbf{x}, \sigma \rangle$ given some $\sigma \in \{-1,1\}^n$. If $\langle \mathbf{x}, \sigma \rangle = 0$ then we call it a *zero partition*. Determining whether a zero partition exists given $\mathbf{x} \in \mathbb{N}^n$ in binary (or higher) radix is a Strong NP-Complete problem. Counting the number of zero partitions is therefore a #P problem. We will therefore be interested in counting or detecting zero partitions, preferrably on a hard instances e.g. reduced from SAT problems as we will emphasize later on.

1 Main Result

Theorem 1.1. *The following identities hold for* $\mathbf{x} \in \mathbb{R}_+^n$*:*

$$\prod_{m=0}^{\infty} \exp \frac{2^{n+1} \prod_{k=1}^{n} \cos\left(\frac{2\pi x_k m}{\sum_{c=1}^{n} x_c}\right)}{2m+1} = \prod_{\sigma \in \{-1,1\}^n} \left[\frac{i \sin \langle \mathbf{x}, \sigma \rangle}{\cos \langle \mathbf{x}, \sigma \rangle - 1}\right]^{e^{-i\langle \mathbf{x}, \sigma \rangle}} \tag{1}$$

$$\prod_{m=0}^{\infty} \exp \frac{-2^{n} \prod_{k=1}^{n} \cos\left(x_k m\right)}{m+1} = \prod_{\sigma \in \{-1,1\}^n} \left(1 - e^{-i\langle \mathbf{x}, \sigma \rangle}\right)^{e^{i\langle \mathbf{x}, \sigma \rangle}} \tag{2}$$

Moreover, both (1) and (2) equal zero if and only if $\mathbf{x}$ *is partitionable.*

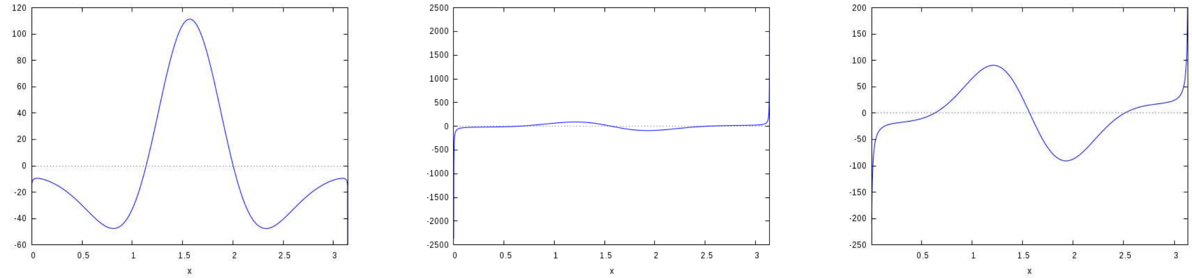
Proof. Denote

$$f(m) = \prod_{k=1}^{n} \cos(x_k m) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} e^{i\langle \mathbf{x}, \sigma \rangle} \tag{3}$$

as can be easily verified due to angle addition formulae. Define

$$\psi_0(z) = \sum_{m=0}^{\infty} z^m f(m) = \sum_{\sigma \in \{-1,1\}^n} \sum_{m=0}^{\infty} z^m e^{im\langle \mathbf{x}, \sigma \rangle} = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - z e^{i\langle \mathbf{x}, \sigma \rangle}} \tag{4}$$

Figure 1: Plots of $\left[\frac{i\sin x}{\cos x-1}\right]^{e^{-ix}}$



The leftmost is the realpart, while the midle is the imaginary part, both over $[0, \pi]$. The rightmost is the imaginary part over $\left[\frac{1}{100}, \pi - \frac{1}{100}\right]$

for complex $|z| < 1$ and due to summation of geometric progression, and write

$$\psi_0(z) + \psi_0(-z) = \sum_{m=0}^{\infty} 2z^{2m} f(2m) = 2^{-n} \sum_{\sigma\in\{-1,1\}^n} \frac{2}{1 - z^2 e^{2i\langle \mathbf{x},\sigma\rangle}} \tag{5}$$

integrating wrt z:

$$\sum_{m=0}^{\infty} \frac{2z^{2m+1}}{2m+1} f(2m) = 2^{-n} \sum_{\sigma\in\{-1,1\}^n} e^{-i\langle \mathbf{x},\sigma\rangle} \ln \frac{ze^{i\langle \mathbf{x},\sigma\rangle} + 1}{ze^{i\langle \mathbf{x},\sigma\rangle} - 1} \tag{6}$$

for $z = 1$ we get

$$\sum_{m=0}^{\infty} \frac{2^{n+1} f(2m)}{2m+1} = \sum_{\sigma\in\{-1,1\}^n} e^{-i\langle \mathbf{x},\sigma\rangle} \ln \frac{e^{i\langle \mathbf{x},\sigma\rangle} + 1}{e^{i\langle \mathbf{x},\sigma\rangle} - 1} \tag{7}$$

$$= \sum_{\sigma\in\{-1,1\}^n} e^{-i\langle \mathbf{x},\sigma\rangle} \ln \left[-i \cot \frac{\langle \mathbf{x},\sigma\rangle}{2}\right] = \sum_{\sigma\in\{-1,1\}^n} \ln \left[\frac{i\sin\langle \mathbf{x},\sigma\rangle}{\cos\langle \mathbf{x},\sigma\rangle - 1}\right]^{e^{-i\langle \mathbf{x},\sigma\rangle}} \tag{8}$$

by exponentiation:

$$\prod_{m=0}^{\infty} \exp \frac{2^{n+1} f(2m)}{2m+1} = \prod_{\sigma\in\{-1,1\}^n} \left[\frac{i\sin\langle \mathbf{x},\sigma\rangle}{\cos\langle \mathbf{x},\sigma\rangle - 1}\right]^{e^{-i\langle \mathbf{x},\sigma\rangle}} \tag{9}$$

it is interesting to observe the plot of $\left[\frac{i\sin x}{\cos x-1}\right]^{e^{-ix}}$ as shown in Figure 1.

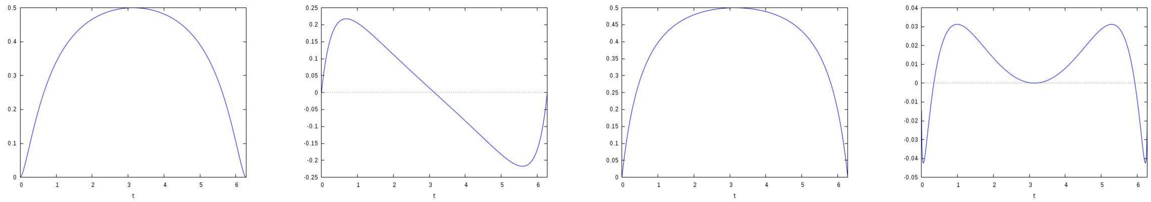
We therefore require $\mathbf{x}$ to satisfy $\langle \mathbf{x}, \mathbf{1}\rangle < \pi$. This way we can see that (9) converges to zero iff $\mathbf{x}$ is partitionable.

On a different route, beginning from (3), integrating and dividing by $-z$, we define:

$$\psi_1(z) \equiv -\frac{1}{z} \int^{z} \psi_0(z')\, dz' = \sum_{m=0}^{\infty} \frac{-z^m}{m+1} f(m) = -2^{-n} \sum_{\sigma\in\{-1,1\}^n} \frac{1}{ze^{i\langle \mathbf{x},\sigma\rangle}} \ln \frac{1}{1 - ze^{i\langle \mathbf{x},\sigma\rangle}} \tag{10}$$

observe that the rhs is the Dirichlet space representation prescribed explicitly by its reproducing kernel, while 3 is on Bergman space. We also write

Figure 2: Plots of $\left(1-e^{-it}\right)^{e^{it}}$ for $t\in[0,2\pi]$



Leftmost is real part, middle is imaginary part, afterwards - the norm, and the rightmost is $\left|\left(1-e^{-it}\right)^{e^{it}}\right|-\sqrt{\frac{t}{2\pi}\left(1-\frac{t}{2\pi}\right)}$.

$$\sum_{m=0}^{\infty}\frac{-2^{n}z^{m}}{m+1}f\left(m\right)=\ln\prod_{\sigma\in\{-1,1\}^{n}}\left(1-ze^{i\langle\mathbf{x},\sigma\rangle}\right)^{z^{-1}e^{-i\langle\mathbf{x},\sigma\rangle}}\tag{11}$$

$$\implies\psi\left(z\right)\equiv e^{2^{n}z\psi_{1}(z)}=\prod_{m=0}^{\infty}\exp\frac{-2^{n}z^{m+1}f\left(m\right)}{m+1}=\prod_{\sigma\in\{-1,1\}^{n}}\left(1-ze^{i\langle\mathbf{x},\sigma\rangle}\right)^{\left(e^{-i\langle\mathbf{x},\sigma\rangle}\right)}\tag{12}$$

and see that the zeros are detemined by all possible partitions. This funtion is holomorphic over the whole complex plane and we wish to evaluate it at $z=1$. Since

$$\psi\left(1\right)=\prod_{m=0}^{\infty}\exp\frac{-2^{n}f\left(m\right)}{m+1}=\prod_{\sigma\in\{-1,1\}^{n}}\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right)^{e^{i\langle\mathbf{x},\sigma\rangle}}\leq 2^{-2^{n}}\tag{13}$$

since$\left|\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right)^{e^{i\langle\mathbf{x},\sigma\rangle}}\right|\leq\frac{1}{2}$ (see Figure 2), implying:

$$\sum_{m=0}^{\infty}\frac{1}{m+1}\prod_{k=1}^{n}\cos\left(x_{k}m\right)\leq-\ln 2\tag{14}$$

we observe the closeness to $\prod_{k}e^{\frac{(-1)^{k}}{k}}=2$, so $f\left(m\right)$ might sometimes be very close to the alternating harmonic sequence, implying almost maximal entropy when the set is unpartitionable and contains more partitions close to the average of the x's.

$\square$