

Statistical Properties of Trigonometric Functions with Applications to NP-Complete Problems

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1 Main Result

Theorem 1.1. *For all $\mathbf{x} \in \mathbb{N}^n$*

$$\prod_{m=1}^{\infty} \exp \left(\frac{1}{m + \frac{1}{2}} \prod_{k=1}^n \cos(8x_k m + 2x_k) \right) = \sqrt[n]{\prod_{\sigma \in \{-1,1\}^n} |\tan \langle \mathbf{x}, \sigma \rangle|} \quad (1)$$

and

$$\frac{1}{\sqrt{2}} \prod_{m=1}^{\infty} \exp \left(-\frac{1}{m} \prod_{k=1}^n \cos(2x_k m) \right) = \sqrt[n]{\prod_{\sigma \in \{-1,1\}^n} |\sin \langle \mathbf{x}, \sigma \rangle|} \quad (2)$$

Proof. Observe that

$$4 \sum_{m=0}^{\infty} \cos(mx) \cos(my) = \sum_{n=1}^{\infty} e^{im(x+y)} + e^{im(x-y)} \quad (3)$$

$$= \frac{1}{1 - e^{i(x+y)}} + \frac{1}{1 - e^{i(x-y)}} \quad (4)$$

$$= \frac{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}}{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}} \quad (5)$$

$$= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (6)$$

Recalling that $\frac{1}{1-t} + \frac{1}{1-\bar{t}} = 1$ for all nonzero complex t , then by the formulae of sum-of-angles and geometric progression we write:

$$\lim_{r \rightarrow 1^-} 4 \sum_{m=0}^{\infty} r^m \cos(mx) \cos(my) = \lim_{r \rightarrow 1^-} \sum_{n=1}^{\infty} r^m e^{im(x+y)} + r^m e^{im(x-y)} \quad (7)$$

$$= \lim_{r \rightarrow 1^-} \frac{1}{1 - r e^{i(x+y)}} + \frac{1}{1 - r e^{i(x-y)}} \quad (8)$$

$$= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (9)$$

if x, y cannot ever meet 2π on some integer multiple. Denote

$$f(m) \equiv \prod_{k=1}^n \cos(x_k m) \quad (10)$$

and write

$$\lim_{r \rightarrow 1^-} 2^n \sum_{m=1}^{\infty} r^m f(m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} r^m e^{im\langle \mathbf{x}, \sigma \rangle} \quad (11)$$

$$= \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - re^{i\langle \mathbf{x}, \sigma \rangle}} \quad (12)$$

$$= \begin{cases} -\infty & \exists \sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = 0 \\ 0 & \forall \sigma \in \{-1,1\}^n, \langle \mathbf{x}, \sigma \rangle \neq 0 \end{cases} \quad (13)$$

since $\sum_{k=1}^{\infty} \frac{t^k}{k} = \ln \frac{1}{1-t}$ we can write

$$\lim_{r \rightarrow 1^-} -2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{r^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} \quad (14)$$

$$= \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \ln \frac{1}{1 - re^{i\langle \mathbf{x}, \sigma \rangle}} \quad (15)$$

implying:

$$\lim_{r \rightarrow 1^-} \exp \left(-2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) \right) = \lim_{r \rightarrow 1^-} \prod_{\sigma \in \{-1,1\}^n} \left(1 - re^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (16)$$

for $r \rightarrow 1$ we have $1 - re^{i\langle \mathbf{x}, \sigma \rangle} = 0$ iff $\langle \mathbf{x}, \sigma \rangle$ is a zero partition. Taking the limit:

$$\exp \left(-2^n \sum_{m=1}^{\infty} \frac{f(m)}{m} \right) = \prod_{\sigma \in \{-1,1\}^n} \left(1 - e^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (17)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (18)$$

where the latter equalit is due to every σ there exists a matching $-\sigma$ and

$$\left(1 - e^{i\langle \mathbf{x}, \sigma \rangle} \right) \left(1 - e^{-i\langle \mathbf{x}, \sigma \rangle} \right) = 2 - 2 \cos \langle \mathbf{x}, \sigma \rangle \quad (19)$$

$$= 2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2} \quad (20)$$

concluding

$$\exp \left(-2^n \sum_{m=1}^{\infty} \frac{f(2m)}{m} \right) = 2^{2^{n-1}} \prod_{\sigma \in \{-1,1\}^n} |\sin \langle \mathbf{x}, \sigma \rangle| \quad (21)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \exp \left(- \sum_{m=1}^{\infty} \frac{f(2m)}{m} \right) = \sqrt[2^n]{\prod_{\sigma \in \{-1,1\}^n} |\sin \langle \mathbf{x}, \sigma \rangle|} \quad (22)$$

equivalently,

$$-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m) = - \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} \quad (23)$$

$$= \sum_{\sigma \in \{-1,1\}^n} \ln \left(1 + e^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (24)$$

or

$$\exp \left(-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m) \right) = \prod_{\sigma \in \{-1,1\}^n} \left(1 + e^{i \langle \mathbf{x}, \sigma \rangle} \right) \quad (25)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \cos^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (26)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \exp \left(- \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m) \right) = \sqrt[n]{\prod_{\sigma \in \{-1,1\}^n} |\cos \langle \mathbf{x}, \sigma \rangle|} \quad (27)$$

dividing (22) by (27) we conclude:

$$\exp \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m) - \sum_{m=1}^{\infty} \frac{f(2m)}{m} \right) \quad (28)$$

$$= \exp \left(\sum_{m=1}^{\infty} \frac{2f(4m+2)}{2m+1} \right) \quad (29)$$

$$= \sqrt[n]{\prod_{\sigma \in \{-1,1\}^n} |\tan \langle \mathbf{x}, \sigma \rangle|} \quad (30)$$

□