

# Modular Arithmetic Problem in #P

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## Abstract

Given  $n$  integers  $x_1, \dots, x_n$ , calculating  $[\prod_{k=1}^n [2^{n x_k} + 2^{-n x_k}]] \mod 2^n$  is a #P problem.

Let  $n \in \mathbb{N}$ ,  $\mathbf{x} \in \mathbb{N}^n$  and consider the formula  $2 \cos a \cos b = \cos(a+b) + \cos(a-b)$  and the cosine being even function to see that:

$$\psi(t) = 2^n \prod_{k=1}^n \cos(x_k t) = \sum_{\sigma \in \{-1, 1\}^n} \cos t \langle \mathbf{x}, \sigma \rangle = \sum_{\sigma \in \{-1, 1\}^n} e^{it \langle \mathbf{x}, \sigma \rangle} \quad (1)$$

where  $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^n \sigma_k x_k$  and counting the number of  $\sigma \in \{-1, 1\}^n$  satisfying  $\langle \mathbf{x}, \sigma \rangle = 0$  is a #P problem. We write down the following sum just for fun and substitute (1) in it:

$$S = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{2\pi m}{n} + i \ln 2\right) = \sum_{\sigma \in \{-1, 1\}^n} \frac{2^{-\langle \mathbf{x}, \sigma \rangle}}{n} \sum_{m=1}^n e^{\frac{2\pi i m}{n} \langle \mathbf{x}, \sigma \rangle} \quad (2)$$

the summation of roots of unity equals zero iff  $n$  does not divide  $\langle \mathbf{x}, \sigma \rangle$ , and if it does divide then it sums to  $n$ . Using this fact and denoting the number of partitions that sum to  $u$  by  $c_u = |\{\sigma \in \{-1, 1\}^n \mid \langle \mathbf{x}, \sigma \rangle = u\}|$ , we get

$$S = \sum_{u=-\infty}^{\infty} c_{nu} 2^{-nu} \quad (3)$$

recalling that  $\sum_{u=-\infty}^{\infty} c_u = 2^n$  and  $c_u$  are all positive, while in (3) being multiplied by distinct powers  $2^{\pm n}$ , therefore the summands' binary digits never interfere with each other and can never grow as large as 1, except when  $u = 0$ . Recalling that  $c_0$  is our quantity of interest, we have proved that the number of zero partitions in  $\mathbf{x}$

$$\left\lfloor \frac{2^n}{n} \sum_{m=1}^n \prod_{k=1}^n \cos \left[ x_k \left( \frac{2\pi m}{n} + i \ln 2 \right) \right] \right\rfloor \mod 2^n \quad (4)$$

$$= \left\lfloor \frac{1}{n} \sum_{m=1}^n \prod_{k=1}^n \left[ e^{x_k \left( \frac{2\pi i m}{n} - \ln 2 \right)} + e^{x_k \left( -\frac{2\pi i m}{n} + \ln 2 \right)} \right] \right\rfloor \mod 2^n \quad (5)$$

multiplying all  $x_k$  by  $n$  (while preserving partitions),  $e^{\frac{2\pi i m x_k n}{n}} = 1$  so we get:

$$= \left[ \prod_{k=1}^n [2^{nx_k} + 2^{-nx_k}] \right] \mod 2^n \quad (6)$$