We now prove a more generalized result independently of previous derivations.

Theorem 1. *Let $n \in \mathbb{N}$, given analytic f such that for all $t \in [-1,1]$:*

$$|f(t)| \le 1, |f'(t)| \le A, |f''(t)| \le B$$

and given even $w(t) = w(-t)$ such that for all $t \in \mathbb{R}$:

$$|w(t)| \le 1, |w'(t)| \le C, |w''(t)| \le D, |w'''(t)| \le E, w(\infty) = L$$

and $w'(t) \neq 0$ for all $t \neq 0$. Then

$$\left| \int_{-1}^{1} f(t)\, dt - 2^{1-N} \sum_{m=-N}^{N} f(w(m))\, w'(m) \right| \le 2^{-n}$$

where

$$N = w^{-1}\left(2^{-n-3} - 1\right) 2^{p}$$

and

$$p = \left\lceil -\log_2 \frac{\sqrt{\left(AC^2 + D\right)^2 - 2^{1-n}\left(BC^3 + 3ACD + E\right)} - AC^2 - D}{BC^3 + 3ACD + E} \right\rceil$$

If, in addition, $\lim_{x\to\infty} \sqrt{x} w^{-1}(x) = 0$ at least polynomialy fast, then N increases only polynomialy wrt n since then $N = \mathcal{O}\left(2^{-\frac{1}{2}n} w^{-1}\left(2^{-n}\right)\right)$, as in the tanh-sinh case.

Proof. Put $g(t) = f(w(t))\, w'(t)$ as the function to be evaluated. If using Kahan summation, then we need to calculate $g(t)$ only up to the desired integral accuracy, namely n digits. So we wish to find ϵ such that

$$|g(t+\epsilon) - g(t)| \le 2^{-n}$$

and then we should never sample g in granlarity higher than ϵ, since the contribution would be insignificant according to the given accuracy requirements. We note that

$$|g'(t)| = \left| f'(w(t)) \left[w'(t)\right]^2 + f(w(t))\, w''(t) \right| \le AC^2 + D$$

and

$$|g''(t)| = \left| f''(w(t)) \left[w'(t)\right]^3 + 3f'(w(t))\, w'(t)\, w''(t) + f(w(t))\, w'''(t) \right|$$

$$\leq BC^3 + 3ACD + E$$

using Taylor's theorem, there exists ξ such that:

$$|g(t+\epsilon) - g(t)| = \left|\epsilon g'(t) + \frac{1}{2}g''(\xi)\epsilon^2\right| \leq |\epsilon g'(t)| + \left|\frac{1}{2}g''(\xi)\epsilon^2\right|$$

$$\leq |\epsilon|\left(AC^2 + D\right) + \frac{BC^3 + 3ACD + E}{2}\epsilon^2$$

so we seek to have $|\epsilon|(AC^2 + D) + \frac{BC^3+3ACD+E}{2}\epsilon^2 \leq 2^{-n}$. Put $\epsilon = 2^{-p}$, and require

$$\left(BC^3 + 3ACD + E\right)2^{-p-1} \pm 2^{-p}\left(AC^2 + D\right) \leq 2^{-n}$$

$$\implies \frac{1}{2}\left(BC^3 + 3ACD + E\right) \pm \left(AC^2 + D\right) \leq 2^{p-n}$$

$$\implies n + \log_2\left[\frac{1}{2}\left(BC^3 + 3ACD + E\right) \pm \left(AC^2 + D\right)\right] \leq p$$

In order to get such evaluation and know that our integral indeed converges, we have only 2^{1+p} possible different values for $t \in [-1, 1]$. Indeed, we need not sample the interval $[-1, 1]$ only, and now we turn to calculate the desired sampling interval.

Recalling w is even, we seek $z > 0$ such that

$$\int_{-\infty}^{-z} g(t)\,dt \leq 2^{-n-3}$$

since we'd like to have $\int_{-\infty}^{-z} g(t)\,dt + \int_{-z}^{z} g(t)\,dt + \int_{z}^{\infty} g(t)\,dt$ up to accuracy of 2^{-n}, therefore we ask for which z the tails are negligible. Iindeed it is sufficient to have

$$\int_{-\infty}^{z} g(t)\,dt \leq \int_{-\infty}^{z} w'(t)\,dt \leq L + w(z) \leq 2^{-n-3} \implies z \leq w^{-1}\left(2^{-n-3} - L\right)$$

Our function is indeed invertible over the negavite half line since its derivative never vanish. So our interval is

$$|t| \leq w^{-1}\left(2^{-n-3} - L\right)$$

with granularity of 2^p above, ending up with total of

$$2^n w^{-1}\left(2^{-n-3} - L\right)\left[BC^3 + 3ACD + E \pm 2\left(AC^2 + D\right)\right]$$

function evaluations, since these are all possible inputs on this interval by the desired and implied accuracy. We can observe that the asymptotic behavior of (1) wrt n is decreasing exponentially as long as w^{-1} decreases faster than square root. $\square$

Finding a fast-diminising w^{-1} is apparently easy: in fact anything faster than quadrator polynomial would imply exponential convergence wrt n. But the immediate functions that come to mind like t^2 would fail to have reasonable (or finite) C, D, E. The double-exponential formula has this rare property. Take

$$w_{DE}(t) = \tanh \sinh t,\ w_{DE}^{-1}(t) = \sinh^{-1} \tanh^{-1} t$$

omitting the $\frac{\pi}{2}$ constant for simplicity. Then

$$|w'_{DE}(t)| \leq 1, |w''_{DE}(t)| \leq 1, |w'''_{DE}(t)| \leq 2, w_{DE}(\infty) = 1$$

so the number of points is:

$$2^n w^{-1}\left(2^{-n-3} - L\right)\left[BC^3 + 3ACD + E \pm 2\left(AC^2 + D\right)\right]$$

$$= 2^n \sinh^{-1} \tanh^{-1}\left(2^{-n-3} - 1\right)\left[B + (3 \pm 2) A + 2 \pm 2\right]$$

to observe the grows wrt n, we analyze the asymptote of

$$\frac{\sinh^{-1} \tanh^{-1} t}{t}$$

by l'Hopital rule, it tends to

$$\partial_t \left[\sinh^{-1} \tanh^{-1} t\right] = \frac{1}{(1-t)^2 \sqrt{1 + \left[\tanh^{-1} t\right]^2}} = \frac{1}{(1-t)^2 \sqrt{1 + \frac{1}{4} \log^2 \frac{1+t}{1-t}}}$$

so we observe at least quadratic conevrgence to zero. Therefore $2^n w^{-1}\left(2^{-n-3} - L\right)$ will decrease exponentially fast wrt n.