

Modular Arithmetic Problem in $\#P$

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Abstract

Given n integers x_1, \dots, x_n in binary (or higher) radix, calculating the n LSB bits of the integer part of $\prod_{k=1}^n [2^{n x_k} + 2^{-n x_k}]$ is a $\#P$ problem. The calculation clearly has a pseudo-polynomial time complexity, as it is polynomial if the input would be supplied in unary format.

Let $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ and consider the formula $2 \cos a \cos b = \cos(a+b) + \cos(a-b)$ and the cosine being even function to see that:

$$\psi(t) = 2^n \prod_{k=1}^n \cos(x_k t) = \sum_{\sigma \in \{-1, 1\}^n} \cos t \langle \mathbf{x}, \sigma \rangle = \sum_{\sigma \in \{-1, 1\}^n} e^{it \langle \mathbf{x}, \sigma \rangle} \quad (1)$$

where $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^n \sigma_k x_k$ and counting the number of $\sigma \in \{-1, 1\}^n$ satisfying $\langle \mathbf{x}, \sigma \rangle = 0$ is a $\#P$ problem. We write down the following sum just for fun and substitute (1) in it:

$$S = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{2\pi m}{n} + i \ln 2\right) = \sum_{\sigma \in \{-1, 1\}^n} \frac{2^{-\langle \mathbf{x}, \sigma \rangle}}{n} \sum_{m=1}^n e^{\frac{2\pi i m}{n} \langle \mathbf{x}, \sigma \rangle} \quad (2)$$

multiplying all x_k by n (while preserving partitions), $e^{\frac{2\pi i m}{n} \langle n\mathbf{x}, \sigma \rangle} = 1$ so we get:

$$S = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{2\pi m}{n} + i \ln 2\right) = \sum_{\sigma \in \{-1, 1\}^n} 2^{-\langle \mathbf{x}, \sigma \rangle} \quad (3)$$

Denoting the number of partitions that sum to u by

$$c_u = |\{\sigma \in \{-1, 1\}^n \mid \langle \mathbf{x}, \sigma \rangle = u\}| \quad (4)$$

then

$$S = \sum_{u=-\infty}^{\infty} c_{nu} 2^{-nu} \quad (5)$$

Recalling that $\sum_{u=-\infty}^{\infty} c_u = 2^n$ and c_u are all positive, while in (3) being multiplied by distinct powers $2^{\pm n}$, therefore the summands' binary digits never interfere with each other and can never grow as large as 1, except when $u = 0$.

Recalling that c_0 is our quantity of interest, we have proved that the number of zero partitions in \mathbf{x}

$$\left\lfloor \frac{2^n}{n} \sum_{m=1}^n \prod_{k=1}^n \cos \left[nx_k \left(\frac{2\pi m}{n} + i \ln 2 \right) \right] \right\rfloor \pmod{2^n} \quad (6)$$

$$= \left\lfloor \prod_{k=1}^n [2^{nx_k} + 2^{-nx_k}] \right\rfloor \pmod{2^n} \quad (7)$$

$$\left(= \left\lfloor 2^{-n \sum_{k=1}^n x_k} \prod_{k=1}^n [1 + 4^{nx_k}] \right\rfloor \pmod{2^n} \right) \quad (8)$$

$$\left(= \left\lfloor 2^{-n \sum_{k=1}^n x_k} \mathbb{E}_{\sigma \in \{0,1\}^n} [4^{n \langle \mathbf{x}, \sigma \rangle}] \right\rfloor \pmod{2^n} \right) \quad (9)$$