Consider the function

$$\psi_0(z) = \sum_{m=0}^{\infty} z^m f(m) \tag{1}$$

for complex $|z| < 1$ where

$$f(m) = 2^n \prod_{k=1}^{n} \cos(x_k m) = \sum_{\sigma \in \{-1,1\}^n} e^{i\langle \mathbf{x}, \sigma \rangle} \tag{2}$$

as we've already seen due to angle addition formulae. This implies

$$\psi_0(z) = \sum_{m=0}^{\infty} z^m f(m) = \sum_{\sigma \in \{-1,1\}^n} \sum_{m=0}^{\infty} z^m e^{im\langle \mathbf{x}, \sigma \rangle} = \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - ze^{i\langle \mathbf{x}, \sigma \rangle}} \tag{3}$$

where the last equality is due to summation of geometric progression.

Integrating and dividing by $-z$, we define:

$$\psi_1(z) \equiv -\frac{1}{z} \int^{z} \psi_0(z')\, dz' = \sum_{m=0}^{\infty} \frac{-z^m}{m+1} f(m) = -\sum_{\sigma \in \{-1,1\}^n} \frac{1}{ze^{i\langle \mathbf{x}, \sigma \rangle}} \ln \frac{1}{1 - ze^{i\langle \mathbf{x}, \sigma \rangle}} \tag{4}$$

observe that the rhs is the Dirichlet space representation prescribed explicitly by its reproducing kernel, while (3) is on the Bergman space and kernel. Moreover:

$$\sum_{m=0}^{\infty} \frac{-z^m}{m+1} f(m) = \ln \prod_{\sigma \in \{-1,1\}^n} \left(1 - ze^{i\langle \mathbf{x}, \sigma \rangle}\right)^{z^{-1} e^{-i\langle \mathbf{x}, \sigma \rangle}} \tag{5}$$

$$\implies \psi(z) \equiv e^{2^{-n} z \psi_1(z)} = \prod_{m=0}^{\infty} \exp \frac{-2^{-n} z^{m+1} f(m)}{m+1} = \prod_{\sigma \in \{-1,1\}^n} \left(1 - ze^{i\langle \mathbf{x}, \sigma \rangle}\right)^{\left(e^{-i\langle \mathbf{x}, \sigma \rangle}\right)} \tag{6}$$

is a Blaschke product, where the zeros are detemined by all possible partitions. This funtion is holomorphic over the whole complex plane, and we wish to decide whether it vanishes at $z = 1$:

$$\psi(1) = \prod_{m=0}^{\infty} \exp \frac{-2^{-n} f(m)}{m+1} = \prod_{\sigma \in \{-1,1\}^n} \left(1 - e^{-i\langle \mathbf{x}, \sigma \rangle}\right)^{e^{i\langle \mathbf{x}, \sigma \rangle}} \le 2^{-2^n} \tag{7}$$

as will be justified right away, yet equivalent to:

$$\implies \prod_{m=0}^{\infty} \exp \frac{f(m)}{m+1} \le 2 \implies \prod_{m=0}^{\infty} \exp \frac{\prod_{k=1}^{n} \cos(x_k m)}{m+1} \le 2^{2^{-n}} \tag{8}$$

$$\implies \sum_{m=0}^{\infty} \frac{2^n}{m+1} \prod_{k=1}^{n} \cos(x_k m) \le \ln 2 \tag{9}$$

indeed

$$\left|\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right)^{e^{-i\langle\mathbf{x},\sigma\rangle}}\right| \leq \frac{1}{2} \tag{10}$$

or even a stronger bound as follows, since in fact for $|z|=1$:

$$\left|\left|\left(1-\frac{1}{z}\right)^{z}\right|-\sqrt{z\left(1-z\right)}\right| \leq \frac{1}{20} \tag{11}$$

so

$$\prod_{m=0}^{\infty}\exp\frac{f\left(m\right)}{m+1} \leq \max_{\sigma}\left|\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right)^{e^{i\langle\mathbf{x},\sigma\rangle}}\right| \leq \frac{1}{20}+\max_{\sigma}\left|e^{i\langle\mathbf{x},\sigma\rangle}\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right)\right| = \frac{1}{20}+2 \tag{12}$$

we observe the closeness to $\prod_{k} e^{\frac{(-1)^{k}}{k}} = 2$, so $f\left(m\right)$ must be very close to the alternating harmonic sequence, implying almost maximal entropy as the set contains more partitions close to the average of the x's, as can be seen by plotting $\left|\left(1-e^{-it}\right)^{e^{it}}\right|$ for $t \in [0, 2\pi]$: