Statistical Properties of Trigonometric Functions with Applications to NP-Complete Problems

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1 Background

Observe that

$$
4 \sum_{m=0}^{\infty} \cos(mx) \cos(my) \quad = \quad \sum_{n=1}^{\infty} e^{im(x+y)} + e^{im(x-y)} \tag{1}
$$

$$
= \quad \frac{1}{1 - e^{i(x+y)}} + \frac{1}{1 - e^{i(x-y)}} \tag{2}
$$

$$
= \quad \frac{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}}{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}} \tag{3}
$$

$$
= \quad \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \tag{4}
$$

Recalling that $\frac{1}{1-t} + \frac{1}{1-\frac{1}{t}} = 1$ for all nonzero complex t, then by the formulae of sum-of-angles and geometric progression we write:

$$
\lim_{r \to 1^-} 4 \sum_{m=0}^{\infty} r^m \cos(mx) \cos(my) \quad = \quad \lim_{r \to 1^-} \sum_{n=1}^{\infty} r^m e^{im(x+y)} + r^m e^{im(x-y)} \tag{5}
$$

$$
= \quad \lim_{r \to 1^-} \frac{1}{1 - r e^{i(x+y)}} + \frac{1}{1 - r e^{i(x-y)}} \tag{6}
$$

$$
= \quad \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \tag{7}
$$

if x, y cannot ever meet 2π on some integer multiple.

The following theorem generalizes this result. Note that determining whether the products on the theorem's statements equal zero or not (equivalently, whether the infinite sums go to minus inifnity) is an NP-Hard problem.

2 Main Result

Theorem 2.1. *For all* $\mathbf{x} \in \mathbb{N}^n$ *the following identities hold:*

$$\sum_{\sigma \in \{-1,1\}^n} \ln|\sin \langle \mathbf{x}, \sigma \rangle| \;=\; -\frac{2^n}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{1}{m} \prod_{k=1}^{n} \cos(2x_k m) \tag{8}$$

$$\sum_{\sigma \in \{-1,1\}^n} \ln|\cos \langle \mathbf{x}, \sigma \rangle| \;=\; -\frac{2^n}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \prod_{k=1}^{n} \cos(2x_k m) \tag{9}$$

$$\sum_{\sigma \in \{-1,1\}^n} \ln|\tan \langle \mathbf{x}, \sigma \rangle| \;=\; \sum_{m=1}^{\infty} \frac{2^n}{m + \frac{1}{2}} \prod_{k=1}^{n} \cos(4x_k m + 2x_k) \tag{10}$$

Proof. Denote

$$f(m) \equiv \prod_{k=1}^{n} \cos(x_k m) \tag{11}$$

and write

$$\lim_{r \to 1^-} 2^n \sum_{m=1}^{\infty} r^m f(m) \;=\; \lim_{r \to 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} r^m e^{im\langle \mathbf{x}, \sigma \rangle} \tag{12}$$

$$=\; \lim_{r \to 1^-} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - r e^{i\langle \mathbf{x}, \sigma \rangle}} \tag{13}$$

$$=\; \begin{cases} -\infty & \exists \sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = 0 \\ 0 & \forall \sigma \in \{-1,1\}^n,\, \langle \mathbf{x}, \sigma \rangle \neq 0 \end{cases} \tag{14}$$

since $\sum_{k=1}^{\infty} \frac{t^k}{k} = \ln \frac{1}{1-t}$ we can write

$$\lim_{r \to 1^-} -2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) \;=\; \lim_{r \to 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{r^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} \tag{15}$$

$$=\; \lim_{r \to 1^-} \sum_{\sigma \in \{-1,1\}^n} \ln \frac{1}{1 - r e^{i\langle \mathbf{x}, \sigma \rangle}} \tag{16}$$

implying:

$$\lim_{r \to 1^-} \exp\left(-2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) \right) = \lim_{r \to 1^-} \prod_{\sigma \in \{-1,1\}^n} \left(1 - r e^{i\langle \mathbf{x}, \sigma \rangle} \right) \tag{17}$$

for $r \to 1$ we have $1 - r e^{i\langle \mathbf{x}, \sigma \rangle} = 0$ iff $\langle \mathbf{x}, \sigma \rangle$ is a zero partition. Taking the limit:

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{f(m)}{m} \right) \;=\; \prod_{\sigma \in \{-1,1\}^n} \left(1 - e^{i\langle \mathbf{x}, \sigma \rangle} \right) \tag{18}$$

$$=\; \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \tag{19}$$

where the latter equalit is due to every σ there exists a matching $-\sigma$ and

$$\left(1 - e^{i\langle \mathbf{x}, \sigma \rangle}\right)\left(1 - e^{-i\langle \mathbf{x}, \sigma \rangle}\right) \quad = \quad 2 - 2\cos\langle \mathbf{x}, \sigma \rangle \tag{20}$$

$$= \quad 2\sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2} \tag{21}$$

concluding

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{f(2m)}{m}\right) \quad = \quad 2^{2^{n-1}} \prod_{\sigma \in \{-1,1\}^n} |\sin\langle \mathbf{x}, \sigma \rangle| \tag{22}$$

$$\implies \frac{1}{\sqrt{2}} \exp\left(-\sum_{m=1}^{\infty} \frac{f(2m)}{m}\right) \quad = \quad \sqrt[2^n]{\prod_{\sigma \in \{-1,1\}^n} |\sin\langle \mathbf{x}, \sigma \rangle|} \tag{23}$$

equivalently,

$$-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m) \quad = \quad -\sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} \tag{24}$$

$$= \quad \sum_{\sigma \in \{-1,1\}^n} \ln\left(1 + e^{i\langle \mathbf{x}, \sigma \rangle}\right) \tag{25}$$

or

$$\exp\left(-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m)\right) \quad = \quad \prod_{\sigma \in \{-1,1\}^n} \left(1 + e^{i\langle \mathbf{x}, \sigma \rangle}\right) \tag{26}$$

$$= \quad \prod_{\sigma \in \{-1,1\}^n} \sqrt{2\cos^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \tag{27}$$

$$\implies \frac{1}{\sqrt{2}} \exp\left(-\sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m)\right) \quad = \quad \sqrt[2^n]{\prod_{\sigma \in \{-1,1\}^n} |\cos\langle \mathbf{x}, \sigma \rangle|} \tag{28}$$

dividing (23) by (28) we conclude:

$$\exp\left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m) - \sum_{m=1}^{\infty} \frac{f(2m)}{m}\right) \tag{29}$$

$$= \quad \exp\left(\sum_{m=1}^{\infty} \frac{2f(4m+2)}{2m+1}\right) \tag{30}$$

$$= \quad \sqrt[2^n]{\prod_{\sigma \in \{-1,1\}^n} |\tan\langle \mathbf{x}, \sigma \rangle|} \tag{31}$$

$$\square$$

$$2^{-n} \sum_{m=1}^{\infty} \frac{2f(4m+2)}{2m+1} = \sum_{\sigma \in \{-1,1\}^n} \ln|\tan\langle \mathbf{x}, \sigma \rangle| \tag{32}$$

$$2^{-n} \sum_{m=1}^{\infty} \frac{2f\left(4mt + 2t\right)}{2m + 1} = \sum_{\sigma \in \{-1,1\}^n} \ln|\tan \langle t\mathbf{x}, \sigma \rangle| \tag{33}$$

$$2^{-n} \sum_{m=1}^{\infty} \frac{2\left(4m + 2\right)^2}{2m + 1} f''\left(4mt + 2t\right) = \sum_{\sigma \in \{-1,1\}^n} \frac{-4 \langle \mathbf{x}, \sigma \rangle^2 \cos\left(2 \langle t\mathbf{x}, \sigma \rangle\right)}{\sin^2\left(2 \langle t\mathbf{x}, \sigma \rangle\right)} \tag{34}$$

$$2^{-n} \sum_{m=1}^{\infty} \frac{\left(4m + 2\right)^2}{2m + 1} f''\left(4mt + 2t\right) = \sum_{\sigma \in \{-1,1\}^n} \frac{-2 \langle \mathbf{x}, \sigma \rangle^2 \cos\left(2 \langle t\mathbf{x}, \sigma \rangle\right)}{\sin^2\left(2 \langle t\mathbf{x}, \sigma \rangle\right)} \tag{35}$$