Statistical Properties of Trigonometric Functions with Applications to NP-Complete Problems

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1 Main Result

Theorem 1.1. *For all* $\mathbf{x} \in \mathbb{N}^n$

$$\exp\left(2^{-n} - \sum_{m=1}^{\infty} \frac{\prod_{k=1}^{n} \cos(2x_k m)}{m+\frac{1}{2}}\right) = \sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n} \left|\cot\langle\mathbf{x},\sigma\rangle\right|} \tag{1}$$

and

$$\exp\left(-2^{-n} - \frac{1}{2}\ln 2 + \sum_{m=1}^{\infty} \frac{\prod_{k=1}^{n} \cos(2x_k m)}{m}\right) = \sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n} \left|\sin\langle\mathbf{x},\sigma\rangle\right|} \tag{2}$$

Proof. Observe that

$$\begin{aligned}
4\sum_{m=0}^{\infty} \cos(mx)\cos(my) &= \sum_{n=1}^{\infty} e^{im(x+y)} + e^{im(x-y)} && (3)\\
&= \frac{1}{1-e^{i(x+y)}} + \frac{1}{1-e^{i(x-y)}} && (4)\\
&= \frac{1-e^{i(x-y)} - e^{i(x+y)} + e^{2xi}}{1-e^{i(x-y)} - e^{i(x+y)} + e^{2xi}} && (5)\\
&= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} && (6)
\end{aligned}$$

Recalling that $\frac{1}{1-t} + \frac{1}{1-\frac{1}{t}} = 1$ for all nonzero complex t, then by the formulae of sum-of-angles and geometric progression we write:

$$\begin{aligned}
\lim_{r\to1^-} 4\sum_{m=0}^{\infty} r^m \cos(mx)\cos(my) &= \lim_{r\to1^-} \sum_{n=1}^{\infty} r^m e^{im(x+y)} + r^m e^{im(x-y)} && (7)\\
&= \lim_{r\to1^-} \frac{1}{1-re^{i(x+y)}} + \frac{1}{1-re^{i(x-y)}} && (8)\\
&= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} && (9)
\end{aligned}$$

if x, y cannot ever meet 2π on some integer multiple. Denote

$$f(m) \equiv \prod_{k=1}^{n} \cos(x_k m) \tag{10}$$

and write

$$\lim_{r\to 1^-} 1-2^n\sum_{m=1}^{\infty} r^m f(m) = \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\frac{1}{1-re^{i\langle \mathbf{x},\sigma\rangle}} \tag{11}$$

$$= \begin{cases} -\infty & \exists\sigma\in\{-1,1\}^n \,|\, \langle \mathbf{x},\sigma\rangle = 0 \\ 0 & \forall\sigma\in\{-1,1\}^n, \langle \mathbf{x},\sigma\rangle \neq 0 \end{cases} \tag{12}$$

since $\sum_{k=1}^{\infty}\frac{t^k}{k} = -\ln(1-t)$ we can write

$$\lim_{r\to 1^-} 1-2^n\sum_{m=1}^{\infty}\frac{r^m}{m} f(m) = \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{r^m}{m}e^{im\langle \mathbf{x},\sigma\rangle} \tag{13}$$

$$= \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\ln\frac{1}{1-re^{i\langle \mathbf{x},\sigma\rangle}} \tag{14}$$

implying:

$$\lim_{r\to 1^-}\exp\left(-1+2^n\sum_{m=1}^{\infty}\frac{r^m}{m}f(m)\right) = \lim_{r\to 1^-}\prod_{\sigma\in\{-1,1\}^n}\left(1-re^{i\langle \mathbf{x},\sigma\rangle}\right) \tag{15}$$

for $r\to 1$ we have $1-re^{i\langle \mathbf{x},\sigma\rangle}=0$ iff $\langle \mathbf{x},\sigma\rangle$ is a zero partition. Taking the limit:

$$\exp\left(-1+2^n\sum_{m=1}^{\infty}\frac{1}{m}f(m)\right) = \prod_{\sigma\in\{-1,1\}^n}\left(1-e^{i\langle \mathbf{x},\sigma\rangle}\right) \tag{16}$$

$$= \prod_{\sigma\in\{-1,1\}^n}\sqrt{2\sin^2\frac{\langle \mathbf{x},\sigma\rangle}{2}} \tag{17}$$

since for every σ there exists a matching $-\sigma$ and

$$\left(1-e^{i\langle \mathbf{x},\sigma\rangle}\right)\left(1-e^{-i\langle \mathbf{x},\sigma\rangle}\right) = 2-2\cos\langle \mathbf{x},\sigma\rangle \tag{18}$$

$$= 2\sin^2\frac{\langle \mathbf{x},\sigma\rangle}{2} \tag{19}$$

we may write

$$\exp\left(-2+2^n\left(-\ln 2+2\sum_{m=1}^{\infty}\frac{1}{m}f(2m)\right)\right) = \prod_{\sigma\in\{-1,1\}^n}\sin^2\langle \mathbf{x},\sigma\rangle \tag{20}$$

equivalently,

$$-2^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(m) = -\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{im\langle \mathbf{x},\sigma\rangle} \tag{21}$$

$$= \sum_{\sigma\in\{-1,1\}^n}\ln\left(1+e^{i\langle \mathbf{x},\sigma\rangle}\right) \tag{22}$$

or

$$\exp\left(-2^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(m)\right) = \prod_{\sigma\in\{-1,1\}^n}\left(1+e^{i\langle \mathbf{x},\sigma\rangle}\right) \tag{23}$$

$$= \prod_{\sigma\in\{-1,1\}^n}\sqrt{2\cos^2\frac{\langle \mathbf{x},\sigma\rangle}{2}} \tag{24}$$

$$\implies \exp\left(-2^n\left(\ln 2+2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(2m)\right)\right) = \prod_{\sigma\in\{-1,1\}^n}\cos^2\langle \mathbf{x},\sigma\rangle \tag{25}$$

dividing (25) by (20):

$$\exp\left(-2^n\left(\ln 2+2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f\left(2m\right)\right)+2-2^n\left(-\ln 2+2\sum_{m=1}^{\infty}\frac{1}{m}f\left(2m\right)\right)\right) \tag{26}$$

$$=\prod_{\sigma\in\{-1,1\}^n}\cot^2\left\langle\mathbf{x},\sigma\right\rangle \tag{27}$$

$$\implies\exp\left(2^{-n}-\sum_{m=1}^{\infty}\frac{\prod_{k=1}^{n}f\left(2m\right)}{m+\frac{1}{2}}\right)=\sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\cot\left\langle\mathbf{x},\sigma\right\rangle\right|} \tag{28}$$

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