

Let  $n \in \mathbb{N}$ ,  $\mathbf{x} \in \mathbb{N}^n$  and

$$\psi(t) = 2^n \prod_{k=1}^n \cos(x_k t) = \sum_{\sigma \in \{-1,1\}^n} \cos t \langle \mathbf{x}, \sigma \rangle = 2 \sum_{\sigma \in \{-1,1\}^n} e^{it \langle \mathbf{x}, \sigma \rangle} \quad (1)$$

recalling the cosine is even. Put

$$c_k = |\{\sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = k\}| \quad (2)$$

is the number of partitions that sum to  $k$ , then

$$\frac{1}{2n} \sum_{m=1}^n \psi\left(2\pi \frac{m}{n} + i \ln 2\right) = \sum_{\sigma \in \{-1,1\}^n} 2^{-\langle \mathbf{x}, \sigma \rangle} \sum_{m=1}^n \frac{1}{n} e^{2\pi i \frac{m}{n} \langle \mathbf{x}, \sigma \rangle} \quad (3)$$

$$= \sum_{k=-\infty}^{\infty} c_{kn} 2^{-nk} \leq c_0 + 2^{1-n} \quad (4)$$

where the transition from (4) to (5) is due to summation of roots of unity, that sum to zero on (4) iff  $n$  does not divide  $\langle \mathbf{x}, \sigma \rangle$ , and if it does divide then they sum to  $n$ . We have proved that  $\mathbf{x}$  has a zero partition iff

$$\sum_{m=1}^n \prod_{k=1}^n \cos\left(2\pi \frac{m}{n} + i \ln 2\right) \geq \frac{2n}{2^n} \quad (5)$$