Natural Numbers Must Be Preferred

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It is well known that uniform distribution over the natural numbers cannot exist according to the axioms of probability. This means several things: First, the sentence “pick a natural number randomally without preferring one number on another” doesn’t make sense. Well, it is kind of obvious that this is quite a problematic sentence, but under the axioms of probability, the sentence “pick a real number between 0 and 1 randomally without preferring one number on another” makes perfect sense. It is the continuous uniform distribution. Probability theory knows to do the magic of taking infinite set and give every element a zero-but-nonzero probability. But this magic doesn’t work for the natural numbers. We could never assign equal probability p to every number, such that the sum of those infinite p’s would equal one. If p is positive, then the sum would turn infinite. And if $p = 0$, then the sum is zero as well.

Secondly, this means that if someone somehow picks a natural number by drawing values from some known or unknown distribution, then there must exists the smallest most-probable natural number. Since the probabilities of all naturals cannot be equal as no uniform distribution over them exists, there must be a set of numbers that have the maximal probability, and this set must have a least element.

This might sound even more paradoxical. Since if I ask you to pick a natural number, indeed I have no idea whatsoever which number you picked. So from my point of view of lack of information, I cannot tell which number is more likely. So doesn’t the distribution from my point of view look completely uniform?

But in fact, it can be shown that indeed I can tell which number you most likely picked!

We know that $\frac{2}{3}$ of the naturals doesn’t divide in 3. Therefore, most chances that the number you picked does not divide by 3. Same for $\frac{4}{5}$, the fraction of numbers divisible by the next prime, 5. Continuing this way, there’s a chance of more than 50% you didn’t select a number that is divisible by 3 or 5 or any other prime, where it is >50% for every prime - let alone the probability of altogether. On the other hand, all naturals are divisible by one, and half of them are divisible by 2.

That’s the main idea. But observe that we actually get an Euler product here. The probability for a number not to be divisible by p is $1 - \frac{1}{p}$. Therefore, the probability that a number isn’t divisible by any prime is indeed zero:

$$\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)=\frac{1}{\zeta\left(1\right)}=0 \tag{1}$$

where the ζ is Riemman's. The probability that a number isn't divisible by any <u>even</u> power of a prime is

$$\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^2}\right)=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}\approx \%60.79$$

so I'm not going to put my money on those numbers.