Trigonometric Correlations and Equidistributions

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Abstract

The trigonometric system sin/cos has maybe the most fundumental property of being orthogonal. That is, their L2 inner product vanish for distinct pairs. They therefore admit zero correlations. We show that this is not the case under a class of spaces of measure zero that have significant interest on various areas e.g. NP Complete problems, Normal numbers, Equidisributed sequences, and Additive combinatorics.

Observe that

$$4\sum_{m=0}^{\infty}\cos(mx)\cos(my)=\sum_{n=1}^{\infty}e^{im(x+y)}+e^{im(x-y)}=\frac{1}{1-e^{i(x+y)}}+\frac{1}{1-e^{i(x-y)}} \tag{1}$$

$$=\frac{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}}{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}}=\begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} \tag{2}$$

if x, y cannot ever meet π on some integer multiple. Similarly, for $\mathbf{x}\in\mathbb{R}^n$ such that $\mathbf{x}$'s elements are lineraly independent of π over the rationals:

$$\sum_{m=1}^{\infty}\prod_{k=1}^{n}\cos(x_k m)=\begin{cases}\infty & \exists\sigma\in\{-1,1\}^n \mid \langle\mathbf{x},\sigma\rangle=0\\ 0 & \forall\sigma\in\{-1,1\}^n,\langle\mathbf{x},\sigma\rangle\neq 0\end{cases} \tag{3}$$

we see that when $\mathbf{x}$ is *partitionable*, i.e. it has a *zero partition* which is a $\sigma\in\{-1,1\}^n$ satisfying $\langle\mathbf{x},\sigma\rangle=0$, then the sum of the product of cosines behave in a positively correlated way, while in case the set is *unpartitionable* it admits zero expection.

This property can be seen immediately by using the same sum-of-angle trick as in the transition from (1) to (2):

$$2^n\prod_{k=1}^{n}\cos(x_k m)=\sum_{\sigma\in\{-1,1\}^n}\cos(m\langle\mathbf{x},\sigma\rangle)=\sum_{\sigma\in\{-1,1\}^n}e^{im\langle\mathbf{x},\sigma\rangle} \tag{4}$$

and can easily be proved by induction. In case that $\mathbf{x}$ is partitionable, we always have a positive constant term on the rhs of (4), which is exactly the number of zero partitions it has. Note that this is due to the partition count being invariant under multiplication by natural m. While summing for various m the positive constant adds up to infinity, while other cosines are quite equidistributed (c.f. the equidistribution theorem) therefore summing to zero. This gives an intuition why (3) indeed works.

Recall that

$$\cot x = \sum_{m=0}^{\infty} \frac{(-1)^m \, 4^m B_n x^{2m-1}}{(2m!)} \tag{5}$$

where the Bernoulli numbers are asymptotically

$$B_{2m} \sim 4\sqrt{\pi m} \left[\frac{m}{\pi e}\right]^{2m} \tag{6}$$

therefore by Stirling's approximation

$$\frac{4^m B_{2m}}{(2m!)} \sim 2\left(\pi^4 e^8\right)^{-m} \tag{7}$$

the Taylor coefficients of coth decrease exponentially fast as $2\left(3 \times 10^{-6}\right)^m$. Also, cot tends to infinity at real integer multiples of π while finite everywhere else. We can therefore write

$$2^n \sum_{m=1}^{\infty} \frac{(-4)^m \, \pi^{2m-1} B_{2m}}{(2m!)} \prod_{k=1}^{n} \cos\left((2m-1)\, x_k\right) = \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{(-4)^m \, \pi^{2m-1} B_n}{(2m!)} \left(e^{i\langle \mathbf{x}, \sigma \rangle}\right)^{2m-1} \tag{8}$$

$$= \sum_{\sigma \in \{-1,1\}^n} \cot\left(\pi e^{i\langle \mathbf{x}, \sigma \rangle}\right) \begin{cases} = \infty & \exists \sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = 0 \\ \neq \infty & \forall \sigma \in \{-1,1\}^n, \langle \mathbf{x}, \sigma \rangle \neq 0 \end{cases} \tag{9}$$

but since the terms on our series decrease even more than exponentially fast we need to sum only a few in order to get convergence, unless the product of cosines flips the signs and accurately weights our series to make it always diverge.