$\alpha\beta\gamma\delta$ Theorem

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Theorem. *To every natural m there exists reals $\alpha, \beta, \gamma, \delta$ such that for all real t:*

$$\cos(mt) = \alpha + (1-\alpha)\cos t + \beta \sin t$$

$$\sin(mt) = \gamma(1-\cos t) + \delta \sin t$$

Proof. Write $m = \sum_{i=0}^{|m|} m_i 2^i$ as m's binary expansion where $|m|$ is m's number of binary digits:

$$\cos\left(\sum_{i=0}^{|m|} m_i 2^i t\right)$$

take the first term out

$$\cos\left(m_0 t + \sum_{i=1}^{|m|} m_i 2^i t\right)$$

and use angle addition formula:

$$\cos(m_0 t)\cos\left(\sum_{i=1}^{|m|} m_i 2^i t\right) - \sin(m_0 t)\sin\left(\sum_{i=1}^{|m|} m_i 2^i t\right)$$

continuing this way we'll end up with sum of 2^d summands where d is the number of ones on m's binary representation, and every summand is a multiplication of d sines and cosines. All in all, since m_i are all either zero or one, the exponential sum reduce to a linear function in $\sin t, \cos t$ where the other terms are constants built from multiplications of $\cos 2^k, \sin 2^k$, i.e. we showed that

$$\cos(mt) = a + b\cos t + c\sin t$$

setting $t = 0$ we see that $b = 1 - a$. Equivalent derivation yields the result for sine with $b = -a$. $\square$