Statistical Properties of Trigonometric Functions with Applications to NP-Complete Problems

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1 Background

Observe that

$$\begin{aligned} 4\sum_{m=0}^{\infty}\cos(mx)\cos(my) &= \sum_{n=1}^{\infty} e^{im(x+y)}+e^{im(x-y)} && (1)\\ &= \frac{1}{1-e^{i(x+y)}}+\frac{1}{1-e^{i(x-y)}} && (2)\\ &= \frac{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}}{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}} && (3)\\ &= \begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} && (4)\end{aligned}$$

Recalling that $\frac{1}{1-t}+\frac{1}{1-\frac{1}{t}}=1$ for all nonzero complex t, then by the formulae of sum-of-angles and geometric progression we write:

$$\begin{aligned} \lim_{r\to 1^-} 4\sum_{m=0}^{\infty} r^m\cos(mx)\cos(my) &= \lim_{r\to 1^-}\sum_{n=1}^{\infty} r^m e^{im(x+y)}+r^m e^{im(x-y)} && (5)\\ &= \lim_{r\to 1^-}\frac{1}{1-re^{i(x+y)}}+\frac{1}{1-re^{i(x-y)}} && (6)\\ &= \begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} && (7)\end{aligned}$$

if x, y cannot ever meet 2π on some integer multiple.

The following theorem generalizes this result. Note that determining whether the products on the theorem's statements equal zero or not is an NP-Hard problem.

2 Main Result

Theorem 2.1. *For all* $\mathbf{x}\in\mathbb{N}^n$ *the following identities hold:*

$$\sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\sin\langle\mathbf{x},\sigma\rangle\right|} = \frac{1}{\sqrt{2}}\prod_{m=1}^{\infty}\exp\left(-\frac{1}{m}\prod_{k=1}^{n}\cos\left(2x_km\right)\right) \tag{8}$$

$$\sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\cos\langle\mathbf{x},\sigma\rangle\right|} = \frac{1}{\sqrt{2}}\prod_{m=1}^{\infty}\exp\left(-\frac{(-1)^m}{m}\prod_{k=1}^{n}\cos\left(2x_km\right)\right) \tag{9}$$

$$\sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\tan\langle\mathbf{x},\sigma\rangle\right|} = \prod_{m=1}^{\infty}\exp\left(\frac{1}{m+\frac{1}{2}}\prod_{k=1}^{n}\cos\left(8x_km+2x_k\right)\right) \tag{10}$$

Proof. Denote

$$f(m)\equiv\prod_{k=1}^{n}\cos\left(x_km\right) \tag{11}$$

and write

$$\lim_{r\to 1^-}2^n\sum_{m=1}^{\infty}r^mf(m) = \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}r^me^{im\langle\mathbf{x},\sigma\rangle} \tag{12}$$

$$= \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\frac{1}{1-re^{i\langle\mathbf{x},\sigma\rangle}} \tag{13}$$

$$= \begin{cases}-\infty & \exists\sigma\in\{-1,1\}^n\mid\langle\mathbf{x},\sigma\rangle=0\\ 0 & \forall\sigma\in\{-1,1\}^n,\langle\mathbf{x},\sigma\rangle\neq 0\end{cases} \tag{14}$$

since $\sum_{k=1}^{\infty}\frac{t^k}{k}=\ln\frac{1}{1-t}$ we can write

$$\lim_{r\to 1^-}-2^n\sum_{m=1}^{\infty}\frac{r^m}{m}f(m) = \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{r^m}{m}e^{im\langle\mathbf{x},\sigma\rangle} \tag{15}$$

$$= \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\ln\frac{1}{1-re^{i\langle\mathbf{x},\sigma\rangle}} \tag{16}$$

implying:

$$\lim_{r\to 1^-}\exp\left(-2^n\sum_{m=1}^{\infty}\frac{r^m}{m}f(m)\right)=\lim_{r\to 1^-}\prod_{\sigma\in\{-1,1\}^n}\left(1-re^{i\langle\mathbf{x},\sigma\rangle}\right) \tag{17}$$

for $r\to 1$ we have $1-re^{i\langle\mathbf{x},\sigma\rangle}=0$ iff $\langle\mathbf{x},\sigma\rangle$ is a zero partition. Taking the limit:

$$\exp\left(-2^n\sum_{m=1}^{\infty}\frac{f(m)}{m}\right) = \prod_{\sigma\in\{-1,1\}^n}\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right) \tag{18}$$

$$= \prod_{\sigma\in\{-1,1\}^n}\sqrt{2\sin^2\frac{\langle\mathbf{x},\sigma\rangle}{2}} \tag{19}$$

where the latter equalit is due to every σ there exists a matching $-\sigma$ and

$$\begin{aligned}\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right)\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right) &= 2-2\cos\langle\mathbf{x},\sigma\rangle && (20)\\ &= 2\sin^2\frac{\langle\mathbf{x},\sigma\rangle}{2} && (21)\end{aligned}$$

concluding

$$\begin{aligned}\exp\left(-2^n\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right) &= 2^{2^{n-1}}\prod_{\sigma\in\{-1,1\}^n}\left|\sin\langle\mathbf{x},\sigma\rangle\right| && (22)\\ \implies \frac{1}{\sqrt{2}}\exp\left(-\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right) &= \sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\sin\langle\mathbf{x},\sigma\rangle\right|} && (23)\end{aligned}$$

equivalently,

$$\begin{aligned}-2^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(m) &= -\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{im\langle\mathbf{x},\sigma\rangle} && (24)\\ &= \sum_{\sigma\in\{-1,1\}^n}\ln\left(1+e^{i\langle\mathbf{x},\sigma\rangle}\right) && (25)\end{aligned}$$

or

$$\begin{aligned}\exp\left(-2^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(m)\right) &= \prod_{\sigma\in\{-1,1\}^n}\left(1+e^{i\langle\mathbf{x},\sigma\rangle}\right) && (26)\\ &= \prod_{\sigma\in\{-1,1\}^n}\sqrt{2\cos^2\frac{\langle\mathbf{x},\sigma\rangle}{2}} && (27)\\ \implies \frac{1}{\sqrt{2}}\exp\left(-\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(2m)\right) &= \sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\cos\langle\mathbf{x},\sigma\rangle\right|} && (28)\end{aligned}$$

dividing (23) by (28) we conclude:

$$\begin{aligned}&\exp\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(2m)-\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right) && (29)\\ &\qquad= \exp\left(\textstyle\sum_{m=1}^{\infty}\frac{2f(4m+2)}{2m+1}\right) && (30)\\ &\qquad= \sqrt[2^n]{\textstyle\prod_{\sigma\in\{-1,1\}^n}\left|\tan\langle\mathbf{x},\sigma\rangle\right|} && (31)\end{aligned}$$

$\square$