We now prove a more generalized result independently of previous derivations, and show how the tanh-sinh quadrature convergence rates follow from it. First we take a superset of the holomorphic functions, what we call here functions with the first Taylor property. We simply ask for the remainder of the first order Taylor approximation to hold and to bound the error from the actual function's value, as in Taylor's theorem.

Definition 1. We say that a function f has the *First Taylor Property* (FTP) over some interval, if for all x, y in that interval there exists ξ in-between them such that we have

$$f(x+y) = f(x) + (y-x) f'(x) + \frac{1}{2}(y-x)^2 f''(\xi) \tag{1}$$

Corollary 2. *Every function that Taylor theorem applies to over some interval, has the FTP over that interval.*

Theorem 3. *Let $n \in \mathbb{N}$, given f such that for all $t \in [-1,1]$:*

$$|f(t)| \leq 1, |f'(t)| \leq A, |f''(t)| \leq B \tag{2}$$

and given odd $w(t) = -w(-t)$ such that for all $t \in \mathbb{R}$:

$$|w(t)| \leq 1, |w'(t)| \leq C, |w''(t)| \leq D, |w'''(t)| \leq E, w(\pm\infty) = \pm 1 \tag{3}$$

and $w'(t) \neq 0$ for all $t \neq 0$. We allow the derivatives of w to be nonexists at $t = 0$, but we still require them to be bounded. Assume further that $g(t) = f(w(t))\, w'(t)$ has the FTP. Then

$$\left| \int_{-1}^{1} f(t)\, dt - 2^{1-N} \sum_{m=-N}^{N} g(hm)\, w'(hm) \right| \leq 2^{-n} \tag{4}$$

where

$$h = 2^{-\frac{1}{2}n}\sqrt{2BC^3 + 6ACD + 2},\ \ N = 2hw^{-1}\left(2^{-n-3} - 1\right) \tag{5}$$

Proof. Put $g(t) = f(w(t))\, w'(t)$ as the function to be evaluated. If using Kahan summation, then we need to calculate $g(t)$ only up to the desired integral accuracy, namely n digits. So we wish to find $\epsilon > 0$ such that $|g(t+\epsilon) - g(t)| \leq 2^{-n}$ for all $t \in [-1,1]$. By that, we should never sample g in granularity higher than ϵ since the contribution would be insignificant according to the given accuracy requirements and assuming Kahan summation. We note that

$$|g'(t)| = \left| f'(w(t))\, [w'(t)]^2 + f(w(t))\, w''(t) \right| \leq AC^2 + D \tag{6}$$

and

$$|g''(t)| = \left| f''(w(t))\left[w'(t)\right]^3 + 3f'(w(t))\, w'(t)\, w''(t) + f(w(t))\, w'''(t) \right|$$

$$\leq BC^3 + 3ACD + E \tag{7}$$

using g's FTP, there exists ξ such that:

$$|g(t+\epsilon) - g(t)| = \left| \epsilon g'(t) + \frac{1}{2} g''(\xi)\, \epsilon^2 \right| \leq |\epsilon g'(t)| + \left| \frac{1}{2} g''(\xi)\, \epsilon^2 \right|$$

$$\leq \epsilon \left(AC^2 + D\right) + \frac{BC^3 + 3ACD + E}{2} \epsilon^2 \tag{8}$$

we would like to have this less than 2^{-n}. We have an increasing quadratic:

$$\epsilon \left(AC^2 + D\right) + \frac{BC^3 + 3ACD + E}{2} \epsilon^2 \leq 2^{-n} \tag{9}$$

$$a = BC^3 + 3ACD + E, b = AC^2 + D$$

setting

$$\alpha = \frac{-\sqrt{b^2 + a2^{1-n}} - b}{a}$$

$$\beta = \frac{\sqrt{b^2 + a2^{1-n}} - b}{a} \tag{10}$$

we get

$$\alpha < \epsilon < \beta \tag{11}$$

picking the positive root, we set $\epsilon = \beta$.

In order to get such evaluation and know that our integral indeed converges, we have only 2^{1+p} possible different values for $t \in [-1, 1]$. Indeed, we need not sample the interval $[-1, 1]$ only, and now we turn to calculate the desired sampling interval.

Recalling w is even, we seek $z > 0$ such that $\int_{-\infty}^{-z} g(t)\, dt \leq 2^{-n-3}$ since we'd like to have $\int_{-\infty}^{-z} g(t)\, dt + \int_{-z}^{z} g(t)\, dt + \int_{z}^{\infty} g(t)\, dt$ up to accuracy of 2^{-n}, therefore we ask for which z the tails are negligible. Iindeed it is sufficient to have

$$\int_{-\infty}^{z} g(t)\, dt \leq \int_{-\infty}^{z} w'(t)\, dt \leq 1 + w(z) \leq 2^{-n-3} \implies z \leq w^{-1}\left(2^{-n-3} - 1\right) \tag{12}$$

Our function is indeed invertible over the negavite half line since its derivative never vanish. So our interval is $|t| \leq w^{-1}\left(2^{-n-3}-1\right)$ with granularity of 2^p above, ending up with total of

$$\left(\sqrt{b^2+a2^{1-n}}-b\right)w^{-1}\left(2^{-n-3}-1\right) \leq \sqrt{2a}2^{-\frac{1}{2}n}w^{-1}\left(2^{-n-3}-1\right) \tag{13}$$

since $\sqrt{b^2+ax}-b \leq \sqrt{ax}$ for all positive x, a, b. To see this, square both sides, expand and get $b^2+ax-2b\sqrt{b^2+ax}+b^2 \leq ax$ that simplifies into $b \leq \sqrt{b^2+ax}$.

function evaluations, since these are all possible inputs on this interval by the desired and implied accuracy. □

We observe in (13) that N satisfy

$$N = \mathcal{O}\left[2^{\frac{1}{2}n}w^{-1}\left(2^{-n}-1\right)\right] \tag{14}$$

as n grows large. Of course, the number of sampled points must go up when we want more accuracy. But can it go up only polynomialy? We write

$$w^{-1}\left(2^{-n}-1\right) = 2^{-\frac{1}{2}n}\text{poly}\left(n\right) \tag{15}$$

implying

$$w^{-1}\left(t\right) = \sqrt{1+t}\,\text{poly}\left(\log_2\left(1+t\right)\right) \tag{16}$$

for $t \in (-1, 1)$. It should be noted that for quadrature uses, we don't need to calculate $w\left(t\right)$ but we can simply evaluate the inverse and get unordered values of w. Note that for $t \in [0, 1]$, $\sqrt{1+t}\log_2\left(1+t\right)$ behave asyomptotically like $-\tan\frac{\pi t}{2}$ as their ratio is always finite, even though they're both unbounded. Indeed, $-2^{\frac{1}{2}n}\tan\frac{\pi}{2}\left(2^{-n}-1\right) \sim -e^n\tan\frac{\pi}{2}\left(e^{-2n}-1\right)$ while it can be verified that rather $\lim_{n\to\infty} -e^{2n}\tan\frac{\pi}{2}\left(e^{-2n}-1\right) = \frac{2}{\pi}$ therefore $\tan\frac{\pi}{2}\left(e^{-2n}-1\right)$ goes to infinity as fast as e^{2n} rather our desired e^n. We conclude by that, that the quadrature $w\left(t\right) = \frac{2}{\pi}\tan^{-1}t$ would imply polynomial growts in N wrt n, by our theorem. We observe that

$$C = \left|\frac{\partial}{\partial t}\frac{2}{\pi}\tan^{-1}t\right| \leq \frac{2}{\pi}, D = \left|\frac{\partial^2}{\partial t^2}\frac{2}{\pi}\tan^{-1}t\right| \leq \frac{\sqrt{27}}{4\pi}, E = \left|\frac{\partial^3}{\partial t^3}\frac{2}{\pi}\tan^{-1}t\right| \leq \frac{4}{\pi} \tag{17}$$

so (5) turns into

$$N = \frac{1}{\pi}\tan^{-1}\left(2^{-n-3}-1\right)2^{2-\frac{1}{2}n}\sqrt{2+\frac{\sqrt{81}}{\pi^2}A+\frac{16}{\pi^3}B} \tag{18}$$