

# DRAFT: Spectral and Modular Analysis of #P Problems

Ohad Asor

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## Abstract

We present various analytic and number theoretic results concerning the #SAT problem as reflected when reduced into a #PART problem.

## 1 Overview

#SAT is the problem of counting the number of satisfying assignment to a given 3CNF formula, while #PART is the problem of counting the number of zero partitions in a given set of integers. Precise definitions will be given later on. Those problems lie on the complexity class #P, as whether merely deciding if the count is zero or not is an NP-Complete problem. We present various results concerning #PART and analyze their connection with #SAT. On section 2 we skim some preliminaries. Section 3 deals with number theoretic aspects. Section 4 presents the solution as a definite integral. Section 6 presents probabilistic viewpoint of the derivations. On section 7 we present miscellaneous results, and section 8 is devoted to analyze how multiple reductions can give probabilistic answer to #SAT as a consequence our analysis.

## 2 Preliminaries

Our setting is counting the number of solutions given an instance of the Partition problem:

**Definition 1.** Given  $n \in \mathbb{N}$ ,  $\mathbf{x} \in \mathbb{N}^n$ , a *Partition*  $\sigma$  of  $\mathbf{x}$  is some  $\sigma \in \{-1, 1\}^n$ . The *size* of the partition  $\sigma$   $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^n \sigma_k x_k$ . A partition is called a *zero partition* if its size is zero. The problem #PART is the following to determine the number of zero partitions given  $\mathbf{x}$ . The problem PART is deciding whether a zero partition exists or not for  $\mathbf{x}$ . The *Weak* setting of the problem is when  $\mathbf{x}$  is supplied in unary radix, and the *Strong* setting is when it is supplied in binary radix (or another format with same efficiency), therefore the input size is logarithmically smaller on the strong setting.

#PART is in #P complexity class. The setting of the #PART after being reduced from the counting Boolean Satisfiability problem (SAT) is  $n$  integers to partition each having up to  $\mathcal{O}(n)$  binary digits, demonstrating why the rather strong setting is of interest. In fact,

there exists polynomial time algorithms given the weak setting of PART, notably Dynamic Programming algorithms, as well as the formula derived on Theorem 2 below. However, solving PART on the strong setting is not possible in polynomial time (as a function of the input length), unless  $P=NP$ .

$\#SAT$  can be reduced to  $\#SUBSET-SUM$  using an algorithm described in [1], while various slight variations appear on the literature. We summarize here this reduction:

**Reduction of  $\#SAT$  to  $\#SUBSET-SUM$**  Given variables  $x_1, \dots, x_l$  and clauses  $c_1, \dots, c_k$  and let natural  $b \geq 6$ . we construct a set  $S$  and a target  $t$  such that the resulted subset-sum problem requires finding a subset of  $S$  that sums to  $t$ . The number  $t$  is  $l$  ones followed by  $k$  3s (i.e. of the form 1111...3333).  $S$  contains four groups of numbers  $y_1, \dots, y_l, z_1, \dots, z_l, g_1, \dots, g_k, h_1, \dots, h_k$  where  $g_i = h_i = b^{k-i}$ , and  $y_i, z_i$  are  $b^{k+l-i}$  plus  $b^m$  for  $y_i$  if variable  $i$  appears positively in clause  $m$ , or for  $z_i$  if variable  $i$  appears negated in clause  $m$ . Then, every subset that sum to  $t$  matches to a satisfying assignment in the input CNF formula and vice versa, as proved in [1].

**Reduction of  $\#SUBSET-SUM$  to  $\#PART$**  Given  $S, t$  as before and denote by  $s = \sum_{x \in S} x$  the sum of  $S$  members, the matching PART problem is  $S \cup \{2s - t, s + t\}$ . Here too all solutions to both problems are preserved by the reduction and can be translated in both directions.

### 3 $\#PART$ as Modular Arithmetic Problem

**Theorem 2.** Given  $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ , denote  $M = \prod_{k=1}^n [1 + 4^{nx_k}]$  and  $s = n \langle \mathbf{x}, 1 \rangle$ . Denote the binary digits of  $M$  by  $m_i$  such that  $M = \sum_i m_i 2^i$ . Then  $\sum_{j=0}^n m_{s+j} 2^j$  is the number of  $\mathbf{x}$ 's zero partition out of all possible  $2^n$  partitions.

*Proof.* Consider the formula  $2 \cos a \cos b = \cos(a + b) + \cos(a - b)$  and the cosine being even function to see that:

$$\psi(t) = 2^n \prod_{k=1}^n \cos(x_k t) = \sum_{\sigma \in \{-1,1\}^n} \cos(t \langle \mathbf{x}, \sigma \rangle) = \sum_{\sigma \in \{-1,1\}^n} e^{it \langle \mathbf{x}, \sigma \rangle} \quad (1)$$

We write down the following sum and perform substitution according to (1):

$$S = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{2\pi m}{n} + i \ln 2\right) = \sum_{\sigma \in \{-1,1\}^n} \frac{2^{-\langle \mathbf{x}, \sigma \rangle}}{n} \sum_{m=1}^n e^{\frac{2\pi i m}{n} \langle \mathbf{x}, \sigma \rangle} \quad (2)$$

multiplying all  $x_k$  by  $n^{-1}$  puts  $e^{\frac{2\pi i m}{n} \langle n\mathbf{x}, \sigma \rangle} = 1$  and we get:

$$S = \sum_{\sigma \in \{-1,1\}^n} 2^{-n \langle \mathbf{x}, \sigma \rangle} \quad (3)$$

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<sup>1</sup>While preserving partitions, since we can always multiply all  $x_k$  by the same factor and keep the exact number of zero partitions.

Denoting the number of partitions that sum to  $u$  by

$$c_u = |\{\sigma \in \{-1, 1\}^n \mid \langle n\mathbf{x}, \sigma \rangle = u\}| \quad (4)$$

then

$$S = \sum_{u=-\infty}^{\infty} c_u 2^{-u} \quad (5)$$

Recalling that  $\sum_{u=-\infty}^{\infty} c_u = 2^n$  and  $c_u$  are all positive, while in (3) being multiplied by distinct powers  $2^{\pm n}$ , therefore the summands' binary digits never interfere with each other and can never grow as large as 1, except when  $u = 0$ . Recalling that  $c_0$  is our quantity of interest, we have proved that the number of zero partitions in  $\mathbf{x}$

$$\left\lfloor \frac{2^n}{n} \sum_{m=1}^n \prod_{k=1}^n \cos \left[ nx_k \left( \frac{2\pi m}{n} + i \ln 2 \right) \right] \right\rfloor \mod 2^n \quad (6)$$

$$= \left\lfloor \prod_{k=1}^n [2^{nx_k} + 2^{-nx_k}] \right\rfloor \mod 2^n \quad (7)$$

$$= \left\lfloor 2^{-n \sum_{k=1}^n x_k} \prod_{k=1}^n [1 + 2^{2nx_k}] \right\rfloor \mod 2^n \quad (8)$$

Set

$$M = \prod_{k=1}^n [1 + 2^{2nx_k}] = \sum_{\sigma \in \{0,1\}^n} 2^{2n\langle \mathbf{x}, \sigma \rangle} \quad (9)$$

then (8) tells us that the number of zero partitions is encoded as a binary number in the binary digits of  $M$ , from the  $s$ 'th digit to the  $s + n$  digit.  $\square$

**Theorem 3.** *Given  $n \in \mathbb{N}, N \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$  then*

$$\frac{1}{N} \sum_{m=1}^N \prod_{k=1}^n \cos \left( 2\pi x_k \frac{m}{N} \right) \quad (10)$$

*is the number of partitions of  $\mathbf{x}$  having size that is divisible by  $N$  without remainder.*

*Proof.* Following (1):

$$\frac{1}{N} \sum_{m=1}^N \prod_{k=1}^n \cos \left( 2\pi x_k \frac{m}{N} \right) = \sum_{\sigma \in \{-1,1\}^n} \frac{1}{N} \sum_{m=1}^N e^{2\pi i \frac{m}{N} \langle \mathbf{x}, \sigma \rangle} = \sum_{u=-\infty}^{\infty} c_{uN} \quad (11)$$

where  $c$  is defined in (4), and the sum of the roots of unity on the rhs is zero if  $N$  does not divide  $\langle \mathbf{x}, \sigma \rangle$ , and is one if  $N$  does divide it.  $\square$

**Theorem 4.** *Given  $\{n, j, N\} \subset \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$  then*

$$\frac{1}{N} \sum_{m=1}^N e^{2\pi i \frac{mnj}{N}} \prod_{k=1}^n \cos \left( 2\pi x_k \frac{m}{N} \right)$$

*is the number of partitions of  $\mathbf{x}$  such that after adding the partition's size the number  $j$ , the result is divisible by  $N$  without remainder.*

*Proof.* Following (1) and recalling that  $\langle \sigma, \sigma \rangle = n$  for all  $\sigma \in \{-1, 1\}^n$ :

$$\sum_{\sigma \in \{-1, 1\}^n} e^{2\pi i t \langle \mathbf{x} + j\sigma, \sigma \rangle} = \sum_{\sigma \in \{-1, 1\}^n} e^{2\pi i t (\langle \mathbf{x}, \sigma \rangle + nj)} = 2^n e^{2\pi i t n j} \prod_{k=1}^n \cos(2\pi x_k t)$$

and the result follows by derivation as in the proof of Theorem 3.  $\square$

## 4 Hardness of Integration

Note that the expression in (10) is nothing but the trapezoid rule of order  $N$  applied to the following integral:

**Theorem 5.** *Given  $n \in \mathbb{N}$ ,  $\mathbf{x} \in \mathbb{N}^n$  then*

$$\int_0^1 \prod_{k=1}^n \cos(2\pi x_k t) dt \tag{12}$$

*is the number of zero partitions of  $\mathbf{x}$ .*

*Proof.* Following (1):

$$2^n \prod_{k=1}^n \cos(x_k t) = \sum_{\sigma \in \{-1, 1\}^n} e^{it \langle \mathbf{x}, \sigma \rangle} \tag{13}$$

and integrating both sides.  $\square$

**Corollary 6.**  *$\mathbf{x} \in \mathbb{Q}^n$  has a zero partition if and only if*

$$\int_0^\infty \prod_{k=1}^n \cos(2\pi x_k t) dt = \infty \tag{14}$$

*and does not have a zero partition if and only if*

$$\int_0^\infty \prod_{k=1}^n \cos(2\pi x_k t) dt = 0 \tag{15}$$

*Proof.* Follows from Theorem 5, the integrand being periodic, and change of variable to support rationals.  $\square$

**Corollary 7.** *There is no algorithm that takes any function that can be evaluated in polynomial time, and decides in polynomial time whether its integral over the real line is zero (conversley, infinity) unless  $P=NP$ .*

*Proof.* Follows from Corollary 6.  $\square$

**Theorem 8.** Given  $\mathbf{x} \in \mathbb{C}^n, c > 1$  and  $w$  satisfying

$$|w(t + iy)| \leq 1 \quad w(0) = 0 \quad w(1) = 1 \quad |g(t + iy)| \leq 1 \quad (16)$$

for all real  $t$  and  $|y| \leq c$  and holomorphic there, and  $g$  is defined as

$$g(t) = w'(t) \prod_{k=1}^n \cos(x_k w(t)) \quad (17)$$

Then

$$\left| \int_0^1 \prod_{k=1}^n \cos(2\pi x_k t) dt - \frac{1}{N} \sum_{m=1}^N g\left(e^{2\pi i \frac{m}{N}}\right) \right| \leq \frac{2c^{-N}}{c-1} \quad (18)$$

*Proof.* Since  $g$  is analytic, it has a uniformly and absolutely convergent Fourier series for  $0 \leq r \leq c$ :

$$g(re^{i\theta}) = \sum_{d=0}^{\infty} b_d r^d e^{id\theta} \quad (19)$$

$$|g(re^{i\theta})| \leq e^{-\pi r \sum_k x_k} \prod_{k=1}^n \cos(\pi i r x_k) \leq e^{-\pi r \sum_k x_k} \left( e^{\sum_{k=1}^n \pi r x_k} \right) = 1 \quad (20)$$

for  $z \leq 1$ , we see that the  $d$ 'th derivative of (20) evaluated on  $r = 0$  simply yields  $d!b_d$ . By Cauchy's integral formula

$$|g^{(d)}(0)| = \left| \frac{d!}{2\pi i} \int_{|z|=c} \frac{g(z)}{z^{d+1}} dz \right| \leq \frac{d!}{c^d} \quad (21)$$

implying

$$b_d \leq c^{-d} \quad (22)$$

rewriting the sum (19) using the Fourier series:

$$\frac{1}{N} \sum_{m=1}^N \sum_{d=0}^{\infty} b_d e^{2\pi i d \frac{m}{N}} = \sum_{d=0}^{\infty} b_d \equiv S_N \quad (23)$$

where the first equality is by eliminating the roots of unity that sum to one or zero. We observe that

$$|S_N - S_{N-1}| \leq 2 \sum_{d=N+1}^{\infty} |b_d| \leq \sum_{d=N+1}^{\infty} c^{-d} = \frac{c^{-N^2}}{c^N - 1} \quad (24)$$

bounding the sum of all increments up to infinity:

$$\sum_{m=N}^{\infty} |S_m - S_{m-1}| \leq \sum_{m=N}^{\infty} \frac{c^{-m^2}}{c^m - 1} \leq 2 \sum_{m=N}^{\infty} c^{-m} = \frac{2c^{-N}}{c-1} \quad (25)$$

Note that the last inequality in (26) has a lot of room to be tightened and the asymptotic decrease is rather  $\mathcal{O}(c^{-3N})$ . The proof is complete recalling that  $\lim_{N \rightarrow \infty} S_N$  indeed equals to the integral as the trapezoid rule is nothing but a Riemann partition, and the changing the integration variable into  $w$ .  $\square$

## 5 Probabilistic Setting

**Corollary 9.** *Given  $\mathbf{x} \in \mathbb{C}^n$ , the function  $\prod_{k=1}^n \cos(x_k t)$  is the characteristic function of the random variable that takes uniformly distributed  $\sigma \in \{-1, 1\}^n$  and returns  $\langle \mathbf{x}, \sigma \rangle$ .*

*Proof.* The characteristic function is generally defined as  $\mathbb{E}[e^{itX}]$  and on our case follows from reading (1) as

$$\prod_{k=1}^n \cos(x_k t) = \mathbb{E}_{\sigma \in \{-1, 1\}^n} [\cos t \langle \mathbf{x}, \sigma \rangle] = \mathbb{E}_{\sigma \in \{-1, 1\}^n} [e^{it \langle \mathbf{x}, \sigma \rangle}] \quad (26)$$

□

**Theorem 10.** *Given  $n \in \mathbb{N}, N \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ , the variance of the sizes of all partitions is the sum of the squares of the input. Formally:*

$$\sum_{k=1}^n x_k^2 = 2^{-n} \sum_{\sigma \in \{-1, 1\}^n} \langle \mathbf{x}, \sigma \rangle^2 \quad (27)$$

while

$$\frac{2^n}{N^3} \sum_{m=1}^N \frac{\partial^2}{\partial t^2} \prod_{k=1}^n \cos(2\pi x_k t) \Big|_{t=\frac{m}{N}} \quad (28)$$

is the variance of the sizes of all partitions that their size is divisible by  $N$  without remainder.

*Proof.* Following (1) and differentiating:

$$\prod_{k=1}^n \cos(\pi x_k t) = 2^{-n} \sum_{\sigma \in \{-1, 1\}^n} \cos(\pi t \langle \mathbf{x}, \sigma \rangle) \quad (29)$$

$$\Rightarrow \sum_{\ell=1}^n x_\ell \sin(\pi x_\ell t) \prod_{k \neq \ell}^n \cos(\pi x_k t) = 2^{-n} \sum_{\sigma \in \{-1, 1\}^n} \langle \mathbf{x}, \sigma \rangle \sin(\pi t \langle \mathbf{x}, \sigma \rangle) \quad (30)$$

$$\Rightarrow \sum_{\ell=1}^n \sum_{\ell'=1}^n -x_\ell \sin(\pi x_\ell t) x_{\ell'} \sin(\pi x_{\ell'} t) \prod_{k \neq \ell, \ell'}^n \cos(\pi x_k t) + x_\ell^2 \prod_{k=1}^n \cos(\pi x_k t) \quad (31)$$

$$= 2^{-n} \sum_{\sigma \in \{-1, 1\}^n} \langle \mathbf{x}, \sigma \rangle^2 \cos(\pi t \langle \mathbf{x}, \sigma \rangle) \quad (32)$$

and (27) follows by substituting  $t = 0$ . (27) can be proved using Parseval identity as well. Turning to (28):

$$\frac{2^n}{N} \sum_{m=1}^N \frac{\partial^2}{\partial t^2} \prod_{k=1}^n \cos(2\pi x_k t) \Big|_{t=\frac{m}{N}} = \sum_{\sigma \in \{-1, 1\}^n} \langle \mathbf{x}, \sigma \rangle^2 \cos\left(2\pi \frac{m}{N} \langle \mathbf{x}, \sigma \rangle\right) \quad (33)$$

$$= \sum_{u=-\infty}^{\infty} u^2 N^2 c_{Nu} \quad (34)$$

due to aliasing of roots of unity, and  $c_{Nu}$  the number of partitions whose size is divisible by  $Nu$  as in (4). □

*Remark 11.* It is easy to derive all moments and cumulants of our random variable since we're given its characteristic function. Note that it involves Bernoulli numbers.

## 6 Additional Results and Conjectures

**Theorem 12.** *Let  $Z^{\mathbf{x}}$  be the number of zero partitions of a vector of naturals  $X$ . Let  $D_x^{\mathbf{x}}$  be the number of zero partitions of  $X$  after multiplying one of its elements by two, where this element is denoted by  $x$ . Let  $A_x^{\mathbf{x}}$  be the number of zero partitions of  $X$  after appending it  $x$  (so now  $x$  appears at least twice). Then*

$$Z^{\mathbf{x}} = D_x^{\mathbf{x}} + A_x^{\mathbf{x}} \quad (35)$$

*Proof.* Denote

$$\psi(x_1, \dots, x_n) = 2^n \int_0^\pi \prod_{k=1}^n \cos(x_k t) dt \quad (36)$$

then, using the identity  $\cos 2x = 2 \cos^2 x - 1$ :

$$\psi(x_1, \dots, 2x_m, \dots, x_n) = 2^n \int_0^\pi \cos(2x_m t) \prod_{k \neq m}^n \cos(x_k t) dt \quad (37)$$

$$\begin{aligned} &= 2^n \int_0^\pi [2 \cos^2(x_m t) - 1] \prod_{k \neq m}^n \cos(x_k t) dt \\ \implies &\psi(x_1, \dots, x_m, \dots, x_n) - \psi(x_1, \dots, 2x_m, \dots, x_n) = \\ &2^{n+1} \int_0^\pi \cos^2(x_m t) \prod_{k \neq m}^n \cos(x_k t) dt = \psi(x_1, \dots, x_m, \dots, x_n, x_m) \end{aligned} \quad (38)$$

and the result follows by Theorem 3.  $\square$

**Conjecture 13.** *For all even  $n$ , for all  $\mathbf{x} \in \mathbb{N}^n$  the number of  $\mathbf{x}$ 's zero partitions is no more than the number of zero partitions of vector of size  $n$  with all its elements equal 1. Namely, never more than  $\binom{n}{\frac{1}{2}n}$  zero partitions.*

*Furthermore, for all odd  $n$ , for all  $\mathbf{x} \in \mathbb{N}^n$  the number of  $\mathbf{x}$ 's zero partitions is no more than the number of zero partitions of vector of size  $n$  with all its elements equal 1 except one element that equals 2.*

## 7 #SAT Reduction Revisited

The important aspect we focus here is that the numbers produced by the reduction have digits that does not exceed 4, and if using radix 6, they never even carry. Therefore the very same digits produced by the reduction can be interpreted in any radix larger than 5, being reduced to a different #PART problem, yet we're still guaranteed that the number of solution

to those #PART problems are independent of the radix, as they're all reduced from the same #SAT problem.

This property might be used to approximate #SAT using the result of Theorem 3. We can obtain the number of partitions that their size divides a given number  $N$  in polynomial time wrt  $n$  (the number of numbers to partition) and the number of digits of  $x_k$ , and in exponential time in the number of digits of  $N$ .

The case of  $N = 2$  is uninteresting: it is easy to observe that all  $2^n$  possible partitions are either all even or all odd, and this can be decided in linear time. If they're all odd, then zero partition does not exist.

Picking a prime  $N > 2$ , we reduce a #sat problem into  $K$  different #part problems, each by using different radix. heuristically and intuitively, the number of partitions divisible by  $N$  among those  $K$  partition problems is independent between them. We could then use probabilistic reasoning to decide whether the CNF formula is satisfiable or not, without actually solving a single #part problem, but with multiplying probabilistic estimates of intuitively-independent  $K$  #part problems.

*Notation 14.* Denote by  $\mathcal{R}(\phi, b)$  a function that takes a 3CNF formula  $\phi$  and natural  $b > 1$ , and returns a vector  $\mathbf{x}_{\phi, b} \in \mathbb{N}^n$  for some  $n$  that depends on  $\phi$ , where all satisfying inputs of  $\phi$  are encoded as zero partitions of  $\mathbf{x}_{\phi, b}$ . Note that  $b \neq b' \implies \mathcal{R}(\phi, b) \neq \mathcal{R}(\phi, b')$ . We also denote the  $k$ 'th prime number by  $p_k$ . Given some vector of naturals  $\mathbf{x}'$  we denote by  $\mathcal{N}(\mathbf{x}', N)$  the number of  $\mathbf{x}'$ 's partitions that are divisible by  $N$ . Recall that due to (10):

$$\mathcal{N}(\mathbf{x}, N) = \frac{1}{N} \sum_{m=1}^N \prod_{k=1}^n \cos\left(2\pi x_k \frac{m}{N}\right) \quad (39)$$

Given  $N, n \in \mathbb{N}$  and a prime  $p$ , let  $X$  be a uniformly random vector of size  $N$  containing natural numbers smaller than  $2^n$ . Then the probability that neither of  $X$ 's elements are divisible by  $p$  is approximately

$$\left(1 - \frac{1}{p}\right)^N \quad (40)$$

for small  $p$  and large elements in  $X$ , while the probability that neither of  $X$ 's elements are divisible by a power  $k$  of all primes is approximately and asymptotically given by the Euler product

$$\prod_{p < 2^n \text{ prime}} \left(1 - \frac{1}{p^k}\right)^N \approx \zeta^{-N}(k) \quad (41)$$

$$\prod_{p \text{ prime}} \prod_{k=1}^{\left\lceil \frac{n}{\log_2 p} \right\rceil} \left(1 - \frac{1}{p^k}\right)^N \quad (42)$$

where the limit of the product wrt  $k$  is due to requiring  $p^k < 2^n$ . Since

$$2^{-x} > 1 - x \quad x \in (0, 1)$$

then (42) is no larger than

$$2^{-N \sum_{p \text{ prime}} \sum_{k=1}^{\left\lceil \frac{n}{\log_2 p} \right\rceil} \frac{1}{p^k}} = 2^{-N \sum_{p < 2^n \text{ prime}} \frac{1-p^{\left\lceil \frac{n}{\log_2 p} \right\rceil}}{1-p}} \leq 2^{-N 2^n} \quad (43)$$

where the inequality is achieved by taking  $p = 2^n$ .



## 8 Further Research

By using (11) we can get successive estimates to (12) by selecting e.g. primes  $N = 2, 3, 5, \dots$ . We then could accelerate this sequence using Shanks, Romberg, Pade or similar sequence-acceleration method.

It is also interesting to consider a paper-and-pencil algorithm for calculating a single digit of the result of numbers that are given in the following form: all numbers have the form  $1000\dots 0001$  so they're fully characterized by the number of zeros in the middle. The numbers are then given as naturals expressing the numbers of zeros, and we'd like to calculate the  $k$ 'th digit of the result of the multiplication of all those numbers, as (9) and Theorem 1 suggest.

## Acknowledgments

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## References

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