We now prove a more generalized result independently of previous derivations, and show how the tanh-sinh quadrature convergence rates follow from it. First we take a superset of the holomorphic functions, what we call here functions with the first Taylor property. We simply ask for the remainder of the first order Taylor approximation to hold and to bound the error from the actual function's value, as in Taylor's theorem.

Definition 1. We say that a function f has the *First Taylor Property* (FTP) over some interval, if for all x, y in that interval there exists ξ in-between them such that we have

$$f\left(x+y\right)=f\left(x\right)+\left(y-x\right)f'\left(x\right)+\frac{1}{2}\left(y-x\right)^{2}f''\left(\xi\right) \tag{1}$$

Corollary 2. *Every function that Taylor theorem applies to over some interval, has the FTP over that interval.*

Theorem 3. *Let $n\in\mathbb{N}$, given f such that for all $t\in[-1,1]$:*

$$\left|f\left(t\right)\right|\le 1,\left|f'\left(t\right)\right|\le A,\left|f''\left(t\right)\right|\le B \tag{2}$$

and given even $w\left(t\right)=w\left(-t\right)$ such that for all $t\in\mathbb{R}$:

$$\left|w\left(t\right)\right|\le 1,\left|w'\left(t\right)\right|\le C,\left|w''\left(t\right)\right|\le D,\left|w'''\left(t\right)\right|\le E,w\left(\pm\infty\right)=\pm 1 \tag{3}$$

and $w'\left(t\right)\neq 0$ for all $t\neq 0$. Assume further that $g\left(t\right)=f\left(w\left(t\right)\right)w'\left(t\right)$ has the FTP. Then

$$\left|\int_{-1}^{1}f\left(t\right)dt-2^{1-N}\sum_{m=-N}^{N}g\left(hm\right)w'\left(hm\right)\right|\le 2^{-n} \tag{4}$$

where

$$N=2w^{-1}\left(2^{-n-3}-1\right)\frac{BC^{3}+3ACD+E}{\sqrt{\left(AC^{2}+D\right)^{2}-2^{1-n}\left(BC^{3}+3ACD+E\right)}-AC^{2}-D} \tag{5}$$

Proof. Put $g\left(t\right)=f\left(w\left(t\right)\right)w'\left(t\right)$ as the function to be evaluated. If using Kahan summation, then we need to calculate $g\left(t\right)$ only up to the desired integral accuracy, namely n digits. So we wish to find $\epsilon>0$ such that $\left|g\left(t+\epsilon\right)-g\left(t\right)\right|\le 2^{-n}$ for all $t\in[-1,1]$. By that, we should never sample g in granularity higher than ϵ since the contribution would be insignificant according to the given accuracy requirements and assuming Kahan summation. We note that

$$\left|g'\left(t\right)\right|=\left|f'\left(w\left(t\right)\right)\left[w'\left(t\right)\right]^{2}+f\left(w\left(t\right)\right)w''\left(t\right)\right|\le AC^{2}+D \tag{6}$$

and

$$|g''(t)| = \left|f''(w(t))[w'(t)]^3 + 3f'(w(t))w'(t)w''(t) + f(w(t))w'''(t)\right|$$
$$\leq BC^3 + 3ACD + E \tag{7}$$

using g's FTP, there exists ξ such that:

$$|g(t+\epsilon) - g(t)| = \left|\epsilon g'(t) + \frac{1}{2}g''(\xi)\epsilon^2\right| \leq |\epsilon g'(t)| + \left|\frac{1}{2}g''(\xi)\epsilon^2\right|$$
$$\leq \epsilon\left(AC^2 + D\right) + \frac{BC^3 + 3ACD + E}{2}\epsilon^2 \tag{8}$$

we would like to have this less than 2^{-n}. We have an increasing quadratic:

$$\epsilon\left(AC^2 + D\right) + \frac{BC^3 + 3ACD + E}{2}\epsilon^2 \leq 2^{-n} \tag{9}$$

setting

$$\alpha = \frac{-\sqrt{(AC^2 + D)^2 + 2^{1-n}(BC^3 + 3ACD + E)} - AC^2 - D}{BC^3 + 3ACD + E}$$
$$\beta = \frac{\sqrt{(AC^2 + D)^2 + 2^{1-n}(BC^3 + 3ACD + E)} - AC^2 - D}{BC^3 + 3ACD + E} \tag{10}$$

we get

$$\alpha < \epsilon < \beta \tag{11}$$

picking the positive root, we set $\epsilon = \beta$.

In order to get such evaluation and know that our integral indeed converges, we have only 2^{1+p} possible different values for $t \in [-1, 1]$. Indeed, we need not sample the interval $[-1, 1]$ only, and now we turn to calculate the desired sampling interval.

Recalling w is even, we seek $z > 0$ such that

$$\int_{-\infty}^{-z} g(t)\,dt \leq 2^{-n-3} \tag{12}$$

since we'd like to have $\int_{-\infty}^{-z} g(t)\,dt + \int_{-z}^{z} g(t)\,dt + \int_{z}^{\infty} g(t)\,dt$ up to accuracy of 2^{-n}, therefore we ask for which z the tails are negligible. Iindeed it is sufficient to have

$$\int_{-\infty}^{z} g(t)\,dt \leq \int_{-\infty}^{z} w'(t)\,dt \leq 1 + w(z) \leq 2^{-n-3} \implies z \leq w^{-1}\left(2^{-n-3} - 1\right) \tag{13}$$

Our function is indeed invertible over the negavite half line since its derivative never vanish. So our interval is $|t| \leq w^{-1}\left(2^{-n-3} - 1\right)$ with granularity of 2^p above, ending up with total of

$$\frac{2}{\beta} w^{-1}\left(2^{-n-3} - 1\right) \tag{14}$$

function evaluations, since these are all possible inputs on this interval by the desired and implied accuracy. $\square$

Let

$$w(t) = \frac{\sqrt{4t^2 + 1} - 1}{2t} \tag{15}$$

then

$$w^{-1}(t) = \frac{1}{\frac{1}{t} - t} \left(= t \sum_{\ell=0}^{\infty} t^{2\ell} \right) \tag{16}$$

and $C = 1, D = \frac{3}{2}, E = 6$ so

$$N = \frac{2^{n+7} - 2^{n+4}}{2^{n+6} - 2^{n+3} - 2^{n+3} + 2} \frac{B + \frac{9}{2}A + 6}{\sqrt{\left(A + \frac{3}{2}\right)^2 - 2^{1-n}\left(\frac{9}{2}A + B + 6\right)} - A - \frac{3}{2}} \tag{17}$$

as we can see, N decreases exponentially wrt n in a rate of $2^{\frac{1}{2}n}$.