Statistical Properties of Trigonometric Functions with Applications to NP-Complete Problems

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Observe that

$$4\sum_{m=0}^{\infty}\cos(mx)\cos(my) = \sum_{n=1}^{\infty} e^{im(x+y)}+e^{im(x-y)} \tag{1}$$

$$= \frac{1}{1-e^{i(x+y)}}+\frac{1}{1-e^{i(x-y)}} \tag{2}$$

$$= \frac{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}}{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}} \tag{3}$$

$$= \begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} \tag{4}$$

Recalling that $\frac{1}{1-t}+\frac{1}{1-\frac{1}{t}}=1$ for all nonzero complex t, then by the formulae of sum-of-angles and geometric progression we write:

$$\lim_{r\to 1^-} 4\sum_{m=0}^{\infty} r^m\cos(mx)\cos(my) = \lim_{r\to 1^-}\sum_{n=1}^{\infty} r^m e^{im(x+y)}+r^m e^{im(x-y)} \tag{5}$$

$$= \lim_{r\to 1^-}\frac{1}{1-re^{i(x+y)}}+\frac{1}{1-re^{i(x-y)}} \tag{6}$$

$$= \begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} \tag{7}$$

if x, y cannot ever meet 2π on some integer multiple. Similarly, for $\mathbf{x}\in\mathbb{R}^n$ such that $\mathbf{x}$'s elements are lineraly independent of π over the rationals:

$$\lim_{r\to 1^-} 1-2^n\sum_{m=1}^{\infty} r^m\prod_{k=1}^{n}\cos(x_k m) = \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\frac{1}{1-re^{i\langle \mathbf{x},\sigma\rangle}} \tag{8}$$

$$= \begin{cases}-\infty & \exists\sigma\in\{-1,1\}^n \mid \langle\mathbf{x},\sigma\rangle=0\\ 0 & \forall\sigma\in\{-1,1\}^n, \langle\mathbf{x},\sigma\rangle\neq 0\end{cases} \tag{9}$$

since $\sum_{k=1}^{\infty}\frac{t^k}{k}=-\ln(1-t)$ we can write

$$\lim_{r\to 1^-} 1-2^n\sum_{m=1}^{\infty}\frac{r^m}{m}\prod_{k=1}^{n}\cos(x_k m) = \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{r^m}{m}e^{im\langle\mathbf{x},\sigma\rangle} \tag{10}$$

$$= \lim_{r\to 1^-}\sum_{\sigma\in\{-1,1\}^n}\ln\frac{1}{1-re^{i\langle\mathbf{x},\sigma\rangle}} \tag{11}$$

implying:

$$\lim_{r\to 1^-}\exp\left(-1+2^n\sum_{m=1}^{\infty}\frac{r^m}{m}\prod_{k=1}^{n}\cos\left(x_k m\right)\right)=\lim_{r\to 1^-}\prod_{\sigma\in\{-1,1\}^n}\left(1-re^{i\langle\mathbf{x},\sigma\rangle}\right) \tag{12}$$

for $r\to 1$ we have $1-re^{i\langle\mathbf{x},\sigma\rangle}=0$ iff $\langle\mathbf{x},\sigma\rangle$ is a zero partition. Taking the limit:

$$\begin{aligned}\exp\left(-1+2^n\sum_{m=1}^{\infty}\frac{1}{m}\prod_{k=1}^{n}\cos\left(x_k m\right)\right) &= \prod_{\sigma\in\{-1,1\}^n}\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right) &(13)\\ &= \prod_{\sigma\in\{-1,1\}^n}\sqrt{2\sin^2\frac{\langle\mathbf{x},\sigma\rangle}{2}} &(14)\end{aligned}$$

since for every σ there exists a matching $-\sigma$ and

$$\begin{aligned}\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right)\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right) &= 2-2\cos\langle\mathbf{x},\sigma\rangle &(15)\\ &= 2\sin^2\frac{\langle\mathbf{x},\sigma\rangle}{2} &(16)\\ & &(17)\end{aligned}$$

we may write

$$0\le\exp\left(-2+2^n\left(-\ln 2+2\sum_{m=1}^{\infty}\frac{1}{m}\prod_{k=1}^{n}\cos\left(2x_k m\right)\right)\right)=\prod_{\sigma\in\{-1,1\}^n}\sin^2\langle\mathbf{x},\sigma\rangle\le 1 \tag{18}$$

equivalently,

$$\begin{aligned}-2^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\prod_{k=1}^{n}\cos\left(x_k m\right) &= -\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{im\langle\mathbf{x},\sigma\rangle} &(19)\\ &= \sum_{\sigma\in\{-1,1\}^n}\ln\left(1+e^{i\langle\mathbf{x},\sigma\rangle}\right) &(20)\end{aligned}$$

or

$$\begin{aligned}\exp\left(-2^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\prod_{k=1}^{n}\cos\left(x_k m\right)\right) &= \prod_{\sigma\in\{-1,1\}^n}\left(1+e^{i\langle\mathbf{x},\sigma\rangle}\right) &(21)\\ &= \prod_{\sigma\in\{-1,1\}^n}\sqrt{2\cos^2\frac{\langle\mathbf{x},\sigma\rangle}{2}} &(22)\\ \implies \exp\left(-2^n\left(\ln 2+2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\prod_{k=1}^{n}\cos\left(2x_k m\right)\right)\right) &= \prod_{\sigma\in\{-1,1\}^n}\cos^2\langle\mathbf{x},\sigma\rangle &(23)\end{aligned}$$

dividing (10) and (13)

$$\exp\left(-2^n\left(\ln 2+2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\prod_{k=1}^{n}\cos\left(2x_k m\right)\right)+2-2^n\left(-\ln 2+2\sum_{m=1}^{\infty}\frac{1}{m}\prod_{k=1}^{n}\cos\left(2x_k m\right)\right)\right) \tag{24}$$

$$=\prod_{\sigma\in\{-1,1\}^n}\cot^2\langle\mathbf{x},\sigma\rangle \tag{25}$$

$$\implies \exp\left(1-2^n\sum_{m=1}^{\infty}\frac{\prod_{k=1}^{n}\cos\left(2x_k m\right)}{m+\frac{1}{2}}\right)=\prod_{\sigma\in\{-1,1\}^n}\left|\cot\langle\mathbf{x},\sigma\rangle\right| \tag{26}$$