Trigonometric Identities, Ergodicity of Sums of Subsets, and NP Complete Problems

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Abstract

We present an analysis of the Partition problem and derive some apparently-novel trigonometric identities that also express the solution to NP-Complete problems in forms of infinite series and powers of trigonometric functions.

We say that a vector $\mathbf{x} \in \mathbb{C}^n$ is *partitionable* if there is an assignment of $\pm$ such that it sums to zero, or more precisely: if there exists $\sigma \in \{-1, 1\}^n$ such that $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^n x_k \sigma_k = 0$. We call a *partition* to the quentity $\langle \mathbf{x}, \sigma \rangle$ given some $\sigma \in \{-1, 1\}^n$. If $\langle \mathbf{x}, \sigma \rangle = 0$ then we call it a *zero partition*. Determining whether a zero partition exists given $\mathbf{x} \in \mathbb{N}^n$ in binary (or higher) radix is a Strong NP-Complete problem. Counting the number of zero partitions is therefore a #P problem. We will therefore be interested in counting or detecting zero partitions, preferrably on a hard instances e.g. reduced from SAT problems as shown on the Appendix.

We present several interesting results concerning the Partition problem. Afterwards we present our main result being the following identities for all $\mathbf{x} \in \mathbb{R}_+^n$:

$$\prod_{m=0}^{\infty} \exp \frac{-2^n \prod_{k=1}^n \cos(x_k m)}{m+1} = \prod_{\sigma \in \{-1,1\}^n} \left(1 - e^{-i\langle \mathbf{x}, \sigma \rangle}\right)^{e^{i\langle \mathbf{x}, \sigma \rangle}}$$

$$\prod_{m=0}^{\infty} \exp \frac{2^{n+1} \prod_{k=1}^n \cos\left(\frac{2\pi x_k m}{\sum_{c=1}^n x_c}\right)}{2m+1} = \prod_{\sigma \in \{-1,1\}^n} \left[\frac{i \sin \langle \mathbf{x}, \sigma \rangle}{\cos \langle \mathbf{x}, \sigma \rangle - 1}\right]^{e^{-i\langle \mathbf{x}, \sigma \rangle}}$$

1 Background

We present some basic probabilistic and number-theoretic setting and results regarding the Partition problem.

Theorem 1.1. *Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ then the (probability-theoretic) characteristic function of the random variable $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^n x_k \sigma_k$ over uniform $\sigma \in \{-1, 1\}^n$ is $\prod_{k=1}^n \cos(x_k t)$.*

Proof. Consider the formula $2\cos a \cos b = \cos(a+b) + \cos(a-b)$ and the cosine being even function to see that:

$$\prod_{k=1}^{n} \cos(x_k t) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \cos(t \langle \mathbf{x}, \sigma \rangle) = \mathbb{E}\left[e^{it\langle \mathbf{x}, \sigma \rangle}\right] \tag{1}$$

□

Corollary 1.2. *Given* $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ *then*

$$2^n \int_0^1 \prod_{k=1}^{n} \cos(2\pi x_k t)\, dt \tag{2}$$

is the number of zero partitions of $\mathbf{x}$.

Proof. Following (1) and integrating both sides. Stronger statements are possible (e.g. for $\mathbf{x} \in \mathbb{R}^n$ or $\mathbf{x} \in \mathbb{C}^n$) using characteristic function inversion theorems. □

Theorem 1.3. *Given* $\{n, N\} \subset \mathbb{N}, j \in \mathbb{Z}, \mathbf{x} \in \mathbb{N}^n$ *then*

$$\frac{2^n}{N} \sum_{m=1}^{N} e^{2\pi i j \frac{m}{N}} \prod_{k=1}^{n} \cos\left(2\pi x_k \frac{m}{N}\right) \tag{3}$$

is the number of partitions of $\mathbf{x}$ *having size that is divisible by* N *with remainder* j.

Proof. Following (1)

$$\frac{2^n}{N} \sum_{m=1}^{N} \prod_{k=1}^{n} \cos\left(2\pi x_k \frac{m}{N}\right) = \sum_{\sigma \in \{-1,1\}^n} \frac{1}{N} \sum_{m=1}^{N} e^{2\pi i \frac{m}{N} \langle \mathbf{x}, \sigma \rangle} \tag{4}$$

The sum of the roots of unity on the rhs of (4) is zero if N does not divide $\langle \mathbf{x}, \sigma \rangle$, and is one if N does divide it, therefore (4) is equal to

$$\sum_{u=-\infty}^{\infty} c_{uN} \tag{5}$$

where c_u denotes the number of partitions that sum to u:

$$c_u = \left|\{\sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = u\}\right| \tag{6}$$

As for the remainder, observe that

$$\sum_{\sigma \in \{-1,1\}^n} e^{2\pi i t(\langle \mathbf{x}, \sigma \rangle + j)} = e^{2\pi i t j} 2^n \prod_{k=1}^{n} \cos(2\pi x_k t) \tag{7}$$

□

The following conjecture might have already stated and proven, though the author is not aware of such a result:

Conjecture 1.4. *For all even n, for all $\mathbf{x} \in \mathbb{N}^n$ the number of $\mathbf{x}$'s zero partitions is no more than the number of zero partitions of vector of size n with all its elements equal 1. Namely, never more than $\binom{n}{\frac{1}{2}n}$ zero partitions.*

Furthermore, for all odd n, for all $\mathbf{x} \in \mathbb{N}^n$ the number of $\mathbf{x}$'s zero partitions is no more than the number of zero partitions of vector of size n with all its elements equal 1 except one element that equals 2.

Theorem 1.5. *Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$, the number of $\mathbf{x}$'s zero partition out of all possible 2^n partitions is encoded as a binary number in the binary digits of*

$$\prod_{k=1}^{n} [1 + 4^{nx_k}] \tag{8}$$

from the s'th dight to the $s+n$ digit, where $s = n\,\langle \mathbf{x},1 \rangle$.

Proof. We write down the following sum and perform substitution according to (1):

$$S = \frac{1}{n}\sum_{m=1}^{n} \psi\left(\frac{2\pi m}{n} + i\ln 2\right) = \sum_{\sigma\in\{-1,1\}^n} \frac{2^{-\langle \mathbf{x},\sigma\rangle}}{n} \sum_{m=1}^{n} e^{\frac{2\pi i m}{n}\langle \mathbf{x},\sigma\rangle} \tag{9}$$

multiplying all x_k by n (while preserving partitions, since we can always multiply all x_k by the same factor and keep the exact number of zero partitions) puts $e^{\frac{2\pi i m}{n}\langle n\mathbf{x},\sigma\rangle} = 1$ and we get:

$$S = \sum_{\sigma\in\{-1,1\}^n} 2^{-n\langle \mathbf{x},\sigma\rangle} = \sum_{u=-\infty}^{\infty} c_u 2^{-u} \tag{10}$$

where c_u is defined in. Recalling that $\sum_{u=-\infty}^{\infty} c_u = 2^n$ and c_u are all positive, while on (10) being multiplied by distinct powers $2^{\pm n}$, therefore the summands' binary digits never interfere with each other. Recalling that c_0 is our quantity of interest, we have shown that the number of zero partitions in $\mathbf{x}$ is encoded at

$$\left\lfloor \frac{2^n}{n} \sum_{m=1}^{n} \prod_{k=1}^{n} \cos\left[n x_k \left(\frac{2\pi m}{n} + i\ln 2\right)\right]\right\rfloor \mod 2^n \tag{11}$$

$$= \left\lfloor \prod_{k=1}^{n} \left[2^{nx_k} + 2^{-nx_k}\right]\right\rfloor \mod 2^n = \left\lfloor 2^{-n\sum_{k=1}^{n} x_k} \prod_{k=1}^{n} \left[1 + 2^{2nx_k}\right]\right\rfloor \mod 2^n \tag{12}$$

Set

$$M = \prod_{k=1}^{n} \left[1 + 2^{2nx_k}\right] = \sum_{\sigma\in\{0,1\}^n} 2^{2n\langle \mathbf{x},\sigma\rangle} \tag{13}$$

then (12) tells us that the number of zero partitions is encoded as a binary number in the binary digits of M, from the s'th dight to the $s+n$ digit. ∎

Note that the substitution in (9) could take a simpler form. Put $t = i\ln 2$ in (1):

$$\prod_{k=1}^{n}\cosh\left(x_k\ln 2\right) = \prod_{k=1}^{n}\left[2^{x_k}+2^{-x_k}\right] = \mathbb{E}\left[2^{\langle \mathbf{x},\sigma\rangle}\right] \tag{14}$$

Corollary 1.6. *$\mathbf{x}\in\mathbb{Q}^n$ has a zero partition if and only if*

$$\int_0^\infty \prod_{k=1}^{n}\cos\left(x_k t\right)dt = \infty \tag{15}$$

and does not have a zero partition if and only if

$$\int_0^\infty \prod_{k=1}^{n}\cos\left(x_k t\right)dt = 0 \tag{16}$$

Proof. Follows from Theorem 4.1, the integrand being periodic, change of variable to support rationals, and the integral over a single period being nonnegative for all inputs. □

Corollary 1.7. *There is no algorithm that takes any function that can be evaluated in polynomial time, and decides in polynomial time whether its integral over the real line is zero (conversley, infinity) unless P=NP.*

Theorem 1.8. *Given $n\in\mathbb{N},\mathbf{x}\in\mathbb{R}^n$ then*

$$\sum_{m=0}^{\infty}\prod_{k=1}^{n}\cos\left(x_k m\right) = \begin{cases}\infty & \exists\sigma\in\{-1,1\}^n \mid \langle \mathbf{x},\sigma\rangle = 0\\ \frac{1}{2} & \forall\sigma\in\{-1,1\}^n, \langle \mathbf{x},\sigma\rangle \neq 0\end{cases} \tag{17}$$

Proof. Consider the function

$$g\left(r\right) = \sum_{m=0}^{\infty} r^m \prod_{k=1}^{n}\cos\left(x_k m\right) = \mathbb{E}\left[\sum_{m=0}^{\infty} r^m e^{im\langle \mathbf{x},\sigma\rangle}\right] = \mathbb{E}\left[\frac{1}{1-re^{i\langle \mathbf{x},\sigma\rangle}}\right] \tag{18}$$

according to (1) and by summation of geometric series. Since for complex $t\neq 1$ we have $\frac{1}{1-t}+\frac{1}{1-\frac{1}{t}} = 1$ therefore when limiting r to 1 the nonzero partitions $\langle \mathbf{x},\sigma\rangle \neq 0$ on (18)'s $\frac{1}{1-re^{i\langle \mathbf{x},\sigma\rangle}}+\frac{1}{1-re^{-i\langle \mathbf{x},\sigma\rangle}}$ simply equals 1 under taking the expectation and recalling that for any partition $\langle \mathbf{x},\sigma\rangle \neq 0$ there exists another partition $-\langle \mathbf{x},\sigma\rangle \neq 0$. On the other hand, since $r\to 1$ then $\langle \mathbf{x},\sigma\rangle = 0$ takes $\frac{1}{1-re^{i\langle \mathbf{x},\sigma\rangle}}$ to infinity, and (17) is proved. □

2 Main Result

Theorem 2.1. *The following identities hold for $\mathbf{x}\in\mathbb{R}_+^n$:*

$$\prod_{m=0}^{\infty}\exp\frac{2^{n+1}\prod_{k=1}^{n}\cos\left(\frac{2\pi x_k m}{\sum_{c=1}^{n}x_c}\right)}{2m+1} = \prod_{\sigma\in\{-1,1\}^n}\left[\frac{i\sin\langle \mathbf{x},\sigma\rangle}{\cos\langle \mathbf{x},\sigma\rangle - 1}\right]^{e^{-i\langle \mathbf{x},\sigma\rangle}} \tag{19}$$

$$\prod_{m=0}^{\infty} \exp \frac{-2^n \prod_{k=1}^{n} \cos\left(x_k m\right)}{m+1} = \prod_{\sigma \in \{-1,1\}^n} \left(1 - e^{-i\langle \mathbf{x},\sigma\rangle}\right)^{e^{i\langle \mathbf{x},\sigma\rangle}} \tag{20}$$

Moreover, both (19) and (20) equal zero if and only if $\mathbf{x}$ *is partitionable.*

Proof. Denote

$$f\left(m\right) = \prod_{k=1}^{n} \cos\left(x_k m\right) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} e^{i\langle \mathbf{x},\sigma\rangle} \tag{21}$$

as can be easily verified due to angle addition formulae. Define

$$\psi_0\left(z\right) = \sum_{m=0}^{\infty} z^m f\left(m\right) = \sum_{\sigma \in \{-1,1\}^n} \sum_{m=0}^{\infty} z^m e^{im\langle \mathbf{x},\sigma\rangle} = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - ze^{i\langle \mathbf{x},\sigma\rangle}} \tag{22}$$

for complex $|z| < 1$ and due to summation of geometric progression, and write

$$\psi_0\left(z\right) + \psi_0\left(-z\right) = \sum_{m=0}^{\infty} 2z^{2m} f\left(2m\right) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{2}{1 - z^2 e^{2i\langle \mathbf{x},\sigma\rangle}} \tag{23}$$

integrating wrt z:

$$\sum_{m=0}^{\infty} \frac{2z^{2m+1}}{2m+1} f\left(2m\right) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} e^{-i\langle \mathbf{x},\sigma\rangle} \ln \frac{ze^{i\langle \mathbf{x},\sigma\rangle} + 1}{ze^{i\langle \mathbf{x},\sigma\rangle} - 1} \tag{24}$$

for $z = 1$ we get

$$\sum_{m=0}^{\infty} \frac{2^{n+1} f\left(2m\right)}{2m+1} = \sum_{\sigma \in \{-1,1\}^n} e^{-i\langle \mathbf{x},\sigma\rangle} \ln \frac{e^{i\langle \mathbf{x},\sigma\rangle} + 1}{e^{i\langle \mathbf{x},\sigma\rangle} - 1} \tag{25}$$

$$= \sum_{\sigma \in \{-1,1\}^n} e^{-i\langle \mathbf{x},\sigma\rangle} \ln \left[-i \cot \frac{\langle \mathbf{x},\sigma\rangle}{2}\right] = \sum_{\sigma \in \{-1,1\}^n} \ln \left[\frac{i \sin\langle \mathbf{x},\sigma\rangle}{\cos\langle \mathbf{x},\sigma\rangle - 1}\right]^{e^{-i\langle \mathbf{x},\sigma\rangle}} \tag{26}$$

by exponentiation:

$$\prod_{m=0}^{\infty} \exp \frac{2^{n+1} f\left(2m\right)}{2m+1} = \prod_{\sigma \in \{-1,1\}^n} \left[\frac{i \sin\langle \mathbf{x},\sigma\rangle}{\cos\langle \mathbf{x},\sigma\rangle - 1}\right]^{e^{-i\langle \mathbf{x},\sigma\rangle}} \tag{27}$$

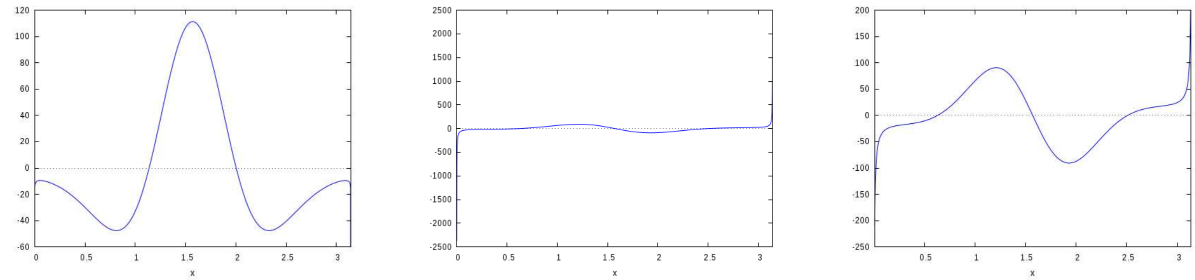
it is interesting to observe the plot of $\left[\frac{i\sin x}{\cos x - 1}\right]^{e^{-ix}}$ as shown in Figure 1.

We therefore require $\mathbf{x}$ to satisfy $\langle \mathbf{x}, \mathbf{1}\rangle < \pi$. This way we can see that (27) converges to zero iff $\mathbf{x}$ is partitionable.

On a different route, beginning from (21), integrating and dividing by $-z$, we define:

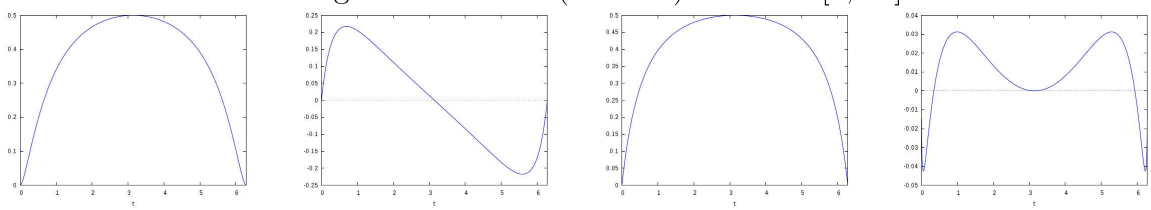
$$\psi_1\left(z\right) \equiv -\frac{1}{z} \int^{z} \psi_0\left(z'\right) dz' = \sum_{m=0}^{\infty} \frac{-z^m}{m+1} f\left(m\right) = -2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{ze^{i\langle \mathbf{x},\sigma\rangle}} \ln \frac{1}{1 - ze^{i\langle \mathbf{x},\sigma\rangle}} \tag{28}$$

Figure 1: Plots of $\left[\frac{i\sin x}{\cos x-1}\right]^{e^{-ix}}$



The leftmost is the realpart, while the midle is the imaginary part, both over $[0,\pi]$. The rightmost is the imaginary part over $\left[\frac{1}{100},\pi-\frac{1}{100}\right]$

Figure 2: Plots of $\left(1-e^{-it}\right)^{e^{it}}$ for $t\in[0,2\pi]$



Leftmost is real part, middle is imaginary part, afterwards - the norm, and the rightmost is $\left|\left(1-e^{-it}\right)^{e^{it}}\right|-\sqrt{\frac{t}{2\pi}\left(1-\frac{t}{2\pi}\right)}$.

observe that the rhs is the Dirichlet space representation prescribed explicitly by its reproducing kernel, while 21 is on Bergman space. We also write

$$\sum_{m=0}^{\infty}\frac{-2^n z^m}{m+1}f(m)=\ln\prod_{\sigma\in\{-1,1\}^n}\left(1-ze^{i\langle\mathbf{x},\sigma\rangle}\right)^{z^{-1}e^{-i\langle\mathbf{x},\sigma\rangle}} \tag{29}$$

$$\implies \psi(z)\equiv e^{2^n z\psi_1(z)}=\prod_{m=0}^{\infty}\exp\frac{-2^n z^{m+1}f(m)}{m+1}=\prod_{\sigma\in\{-1,1\}^n}\left(1-ze^{i\langle\mathbf{x},\sigma\rangle}\right)^{\left(e^{-i\langle\mathbf{x},\sigma\rangle}\right)} \tag{30}$$

and see that the zeros are detemined by all possible partitions. This funtion is holomorphic over the whole complex plane and we wish to evaluate it at $z=1$. Since

$$\psi(1)=\prod_{m=0}^{\infty}\exp\frac{-2^n f(m)}{m+1}=\prod_{\sigma\in\{-1,1\}^n}\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right)^{e^{i\langle\mathbf{x},\sigma\rangle}}\le 2^{-2^n} \tag{31}$$

since$\left|\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right)^{e^{i\langle\mathbf{x},\sigma\rangle}}\right|\le\frac{1}{2}$ (see Figure 2), implying:

$$\sum_{m=0}^{\infty}\frac{1}{m+1}\prod_{k=1}^{n}\cos(x_k m)\le-\ln 2 \tag{32}$$

we observe the closeness to $\prod_k e^{\frac{(-1)^k}{k}}=2$, so $f(m)$ might sometimes be very close to the alternating harmonic sequence, implying almost maximal entropy when the set is unpartitionable and contains more partitions close to the average of the x's. $\square$

References

[1] Sipser, "Introduction to the Theory of Computation". International Thomson Publishing (1996).

[2] Kac, "Statistical Independence in Probability, Analysis and Number Theory". Carus Mathematical Monographs, No. 12, Wiley, New York (1959)

A Appendix

Reduction of #SAT to #SUBSET-SUM Given variables $x_1, \ldots, x_l$ and clauses $c_1, \ldots, c_k$ and let natural $b \geq 6$. we construct a set S and a target t such that the resulted subset-sum problem requires finding a subset of S that sums to t. The number t is l ones followed by k 3s (i.e. of the form $1111\ldots3333$). S contains four groups of numbers $y_1, \ldots, y_l$, $z_1, \ldots, z_l$, $g_1, \ldots, g_k$, $h_1, \ldots, h_k$ where $g_i = h_i = b^{k-i}$, and y_i, z_i are b^{k+l-i} plus b^m for y_i if variable i appears positively in clause m, or for z_i if variable i appears negated in clause m. Then, every subset that sum to t matches to a satisfying assignment in the input CNF formula and vice versa, as proved in [1].

Reduction of #SUBSET-SUM to #PART Given S, t as before and denote by $s = \sum_{x \in S} x$ the sum of S members, the matching PART problem is $S \cup \{2s - t, s + t\}$. Here too all solutions to both problems are preserved by the reduction and can be translated in both directions.