

Let  $n \in \mathbb{N}$ ,  $\mathbf{x} \in \mathbb{N}^n$  and consider the formula  $2 \cos a \cos b = \cos(a+b) + \cos(a-b)$  and the cosine being even function to see that:

$$\psi(t) = 2^n \prod_{k=1}^n \cos(x_k t) = \sum_{\sigma \in \{-1,1\}^n} \cos t \langle \mathbf{x}, \sigma \rangle = 2 \sum_{\sigma \in \{-1,1\}^n} e^{it \langle \mathbf{x}, \sigma \rangle} \quad (1)$$

where  $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^n \sigma_k x_k$  and counting the number of  $\sigma \in \{-1,1\}^n$  satisfying  $\langle \mathbf{x}, \sigma \rangle = 0$  is a #P problem. We write down the following sum just for fun and substitute (1) in it:

$$S = \frac{1}{2n+2} \sum_{m=1}^{n+1} \psi \left( \frac{2\pi m}{n+1} + i \ln 2 \right) = \sum_{\sigma \in \{-1,1\}^n} 2^{-\langle \mathbf{x}, \sigma \rangle} \sum_{m=1}^{n+1} \frac{1}{n+1} e^{2\pi i \frac{m}{n+1} \langle \mathbf{x}, \sigma \rangle} \quad (2)$$

the summation of roots of unity equals zero iff  $n+1$  does not divide  $\langle \mathbf{x}, \sigma \rangle$ , and if it does divide then it sums to  $n+1$ . Using this fact and denoting the number of partitions that sum to  $u$  by  $c_u = |\{\sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = u\}|$ , we get

$$S = \sum_{u=0}^{\infty} c_{u(n+1)} 2^{-(n+1)u} \quad (3)$$

recalling that  $\sum_{u=0}^{\infty} c_u = 2^n$  and  $c_u$  are all positive, while in (3) being multiplied by distinct powers  $2^{-n-1}$ , therefore the summands' binary digits never interfere with each other and can never grow as large as 1, except when  $u = 0$ . Recalling that  $c_0$  is our quantity of interest, we have proved that the number of zero partitions in  $\mathbf{x}$  is

$$\left\lfloor \frac{2^{n-1}}{n+1} \sum_{m=1}^{n+1} \prod_{k=1}^n \cos \left[ x_k \left( \frac{2\pi m}{n+1} + i \ln 2 \right) \right] \right\rfloor \quad (4)$$