

# Statistical Properties of Trigonometric Functions with Applications to NP-Complete Problems

January 15, 2016

## 1 Main Result

**Theorem 1.1.** *For all  $\mathbf{x} \in \mathbb{N}^n$*

$$\exp \left( -2^{-n} + \sum_{m=1}^{\infty} \frac{\prod_{k=1}^n \cos(2x_k m)}{m + \frac{1}{2}} \right) = \sqrt[n]{\prod_{\sigma \in \{-1,1\}^n} |\tan \langle \mathbf{x}, \sigma \rangle|} \quad (1)$$

and

$$\exp \left( -2^{-n} - \ln \sqrt{2} + \sum_{m=1}^{\infty} \frac{\prod_{k=1}^n \cos(2x_k m)}{m} \right) = \sqrt[n]{\prod_{\sigma \in \{-1,1\}^n} |\sin \langle \mathbf{x}, \sigma \rangle|} \quad (2)$$

*Proof.* Observe that

$$4 \sum_{m=0}^{\infty} \cos(mx) \cos(my) = \sum_{n=1}^{\infty} e^{im(x+y)} + e^{im(x-y)} \quad (3)$$

$$= \frac{1}{1 - e^{i(x+y)}} + \frac{1}{1 - e^{i(x-y)}} \quad (4)$$

$$= \frac{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}}{1 - e^{i(x-y)} - e^{i(x+y)} + e^{2xi}} \quad (5)$$

$$= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (6)$$

Recalling that  $\frac{1}{1-t} + \frac{1}{1-\bar{t}} = 1$  for all nonzero complex  $t$ , then by the formulae of sum-of-angles and geometric progression we write:

$$\lim_{r \rightarrow 1^-} 4 \sum_{m=0}^{\infty} r^m \cos(mx) \cos(my) = \lim_{r \rightarrow 1^-} \sum_{n=1}^{\infty} r^m e^{im(x+y)} + r^m e^{im(x-y)} \quad (7)$$

$$= \lim_{r \rightarrow 1^-} \frac{1}{1 - r e^{i(x+y)}} + \frac{1}{1 - r e^{i(x-y)}} \quad (8)$$

$$= \begin{cases} \infty & (x-y)(x+y) = 0 \\ 1 & (x-y)(x+y) \neq 0 \end{cases} \quad (9)$$

if  $x, y$  cannot ever meet  $2\pi$  on some integer multiple. Denote

$$f(m) \equiv \prod_{k=1}^n \cos(x_k m) \quad (10)$$

and write

$$\lim_{r \rightarrow 1^-} 1 - 2^n \sum_{m=1}^{\infty} r^m f(m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - r e^{i\langle \mathbf{x}, \sigma \rangle}} \quad (11)$$

$$= \begin{cases} -\infty & \exists \sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = 0 \\ 0 & \forall \sigma \in \{-1,1\}^n, \langle \mathbf{x}, \sigma \rangle \neq 0 \end{cases} \quad (12)$$

since  $\sum_{k=1}^{\infty} \frac{t^k}{k} = -\ln(1-t)$  we can write

$$\lim_{r \rightarrow 1^-} 1 - 2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) = \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{r^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} \quad (13)$$

$$= \lim_{r \rightarrow 1^-} \sum_{\sigma \in \{-1,1\}^n} \ln \frac{1}{1 - r e^{i\langle \mathbf{x}, \sigma \rangle}} \quad (14)$$

implying:

$$\lim_{r \rightarrow 1^-} \exp \left( -1 + 2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) \right) = \lim_{r \rightarrow 1^-} \prod_{\sigma \in \{-1,1\}^n} \left( 1 - r e^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (15)$$

for  $r \rightarrow 1$  we have  $1 - r e^{i\langle \mathbf{x}, \sigma \rangle} = 0$  iff  $\langle \mathbf{x}, \sigma \rangle$  is a zero partition. Taking the limit:

$$\exp \left( -1 + 2^n \sum_{m=1}^{\infty} \frac{1}{m} f(m) \right) = \prod_{\sigma \in \{-1,1\}^n} \left( 1 - e^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (16)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (17)$$

since for every  $\sigma$  there exists a matching  $-\sigma$  and

$$\left( 1 - e^{i\langle \mathbf{x}, \sigma \rangle} \right) \left( 1 - e^{-i\langle \mathbf{x}, \sigma \rangle} \right) = 2 - 2 \cos \langle \mathbf{x}, \sigma \rangle \quad (18)$$

$$= 2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2} \quad (19)$$

we may write

$$\exp \left( -2 + 2^n \left( -\ln 2 + 2 \sum_{m=1}^{\infty} \frac{1}{m} f(2m) \right) \right) = \prod_{\sigma \in \{-1,1\}^n} \sin^2 \langle \mathbf{x}, \sigma \rangle \quad (20)$$

equivalently,

$$-2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m) = - \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} \quad (21)$$

$$= \sum_{\sigma \in \{-1,1\}^n} \ln \left( 1 + e^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (22)$$

or

$$\exp \left( -2^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(m) \right) = \prod_{\sigma \in \{-1,1\}^n} \left( 1 + e^{i\langle \mathbf{x}, \sigma \rangle} \right) \quad (23)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \cos^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} \quad (24)$$

$$\implies \exp \left( -2^n \left( \ln 2 + 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m) \right) \right) = \prod_{\sigma \in \{-1,1\}^n} \cos^2 \langle \mathbf{x}, \sigma \rangle \quad (25)$$

dividing (20) by (25):

$$\exp \left( 2^n \left( \ln 2 + 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m} f(2m) \right) - 2 + 2^n \left( -\ln 2 + 2 \sum_{m=1}^{\infty} \frac{1}{m} f(2m) \right) \right) \quad (26)$$

$$= \prod_{\sigma \in \{-1,1\}^n} \tan^2 \langle \mathbf{x}, \sigma \rangle \quad (27)$$

$$\Rightarrow \exp \left( -2^{-n} + \sum_{m=1}^{\infty} \frac{\prod_{k=1}^n f(2m)}{m + \frac{1}{2}} \right) = \sqrt[n]{\prod_{\sigma \in \{-1,1\}^n} |\tan \langle \mathbf{x}, \sigma \rangle|} \quad (28)$$

□