

On Approximating Hard Integrals with the Double-Exponential Formula

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Abstract

Approximating $I_{\#PART} = \frac{1}{2} \int_{-1}^1 \prod_{k=1}^n \cos(x_k \pi t) dt$ to within an accuracy of 2^{-n} is equivalent to counting the number of equal-sum partitions of a set of positive integers $\{x_k\}_{k=1}^n$, and is thus a $\#P$ problem. Efficient numerical integration methods such as the double exponential formula, also known as tanh-sinh quadrature, have been around from the mid 70's. Taking note of the hardness of approximating $I_{\#PART}$ we argue that unless $P=NP$ the proven rates of convergence of such methods cannot possibly be correct. In addition we provide a novel and generalized theorem that implies the Double Exponential, having an elementary and constructive proof.

1 Overview

The Partition Counting Problem ($\#PART$) is the following: given n positive integers $\{x_k\}_{k=1}^n$, in how many ways is it possible to divide them into two equal-sum subsets. Analytic and number-theoretic approaches to this problem can be found in many works, many seem to go back to the classic monograph [1] by Kac. If the input $\{x_k\}$ is given in binary rather unary radix, then solving this problem in polynomial time wrt the input's length would prove $P=\#P$ and would also entail $P=NP$. Assuming the exponential time hypothesis, $\#PART$ cannot be solved in polynomial time.

The treatment in [1] and subsequently in many other places e.g. [2, 3, 4] express the number of equal-sum partitions by the integral

$$2^n I_{\#PART} = 2^{n-1} \int_{-1}^1 \prod_{k=1}^n \cos(x_k \pi t) dt$$

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An elementary proof of this result is provided in the ensuing.

The double-exponential (DE) tanh-sinh quadrature is a numerical integration technique whose convergence rate has been proven to be exponential with the number of evaluation points [4, 5, 6, 7]. It is currently considered as the fastest high-precision quadrature technique. Noting the hardness of approximating $I_{\#PART}$ we argue here that, unless $P=NP$, the DE convergence rate as stated in [4, 5, 6, 7] cannot be correct.

Comparing to currently known results, we compute a finer runtime complexity for a wider families of quadratures, and classify the ones that result with exponential convergence as tanh-sinh.

2 The partition problem

Given $n \in \mathbb{N}$ and $\{x_k\}_{k=1}^n \subset \mathbb{Z}$, we seek $\sigma \in \{-1, 1\}^n$ such that $\langle \sigma, \mathbf{x} \rangle = 0$, where $\langle \sigma, \mathbf{x} \rangle = \sum_{k=1}^n \sigma_k x_k$ denotes the inner product. Deciding whether such σ exists is a NP Complete problem, while counting how many such σ 's exists, is in $\#P$. We assume that the inputs $\{x_k\}$ are given in binary radix and denote by d_k the number of binary digits of x_k . The partition problem is known to be Weak-NP since it has a polynomial-time algorithm if the input is supplied in unary radix. To get a feeling about typical dimensions of hard problems, the reduction of n -clause and k -variables 3SAT into the partition problem ends up with $\mathcal{O}(n+k)$ integers to partition, each having $\mathcal{O}(n+k)$ digits [10]. The exponential time hypothesis therefore implies that it is impossible to solve the partition problem in runtime complexity of $\mathcal{O}(\text{poly}(\sum_{k=1}^n d_k))$.

The counting version of the partition problem is equivalent to the following definite integral:

Lemma 1. *Let $\{x_k\}_{k=1}^n \subset \mathbb{Z}$ be integers given in binary radix. Let also $\psi(t) = \prod_{k=1}^n \cos(\pi x_k t)$. Then evaluating $I_{\#PART} = \frac{1}{2} \int_{-1}^1 \psi(t) dt$ up to accuracy of n binary digits is in $\#P$.*

Proof. This lemma can be proved in many interesting ways, all seem to go back to the classical monograph by Kac [1]. Slightly different proofs of this lemma may be found in [2, 4]. Our derivation is based on the formula

$$\prod_{k=1}^n \cos(z_k) = 2^{-n} \sum_{\sigma \in \{-1, 1\}^n} \cos \langle \sigma, \mathbf{z} \rangle \quad (1)$$

for every $\mathbf{z} \in \mathbb{C}^n$, which follows from a repeated application of the identity

$$4 \cos(z_1) \cos(z_2) = \cos(z_1 + z_2) + \cos(z_1 - z_2) + \cos(-z_1 + z_2) + \cos(-z_1 - z_2) \quad (2)$$

Using this the integral reads

$$\begin{aligned}
I_{\# \text{PART}} &= 2^{-n-1} \sum_{\sigma \in \{-1,1\}^n} \int_{-1}^1 \cos(\pi t \langle \sigma, \mathbf{z} \rangle) dt = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{\sin \pi \langle \sigma, \mathbf{z} \rangle}{\pi \langle \sigma, \mathbf{z} \rangle} \\
&= 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \begin{cases} 1 & \text{if } \langle \sigma, \mathbf{z} \rangle = 0 \\ 0 & \text{if } \langle \sigma, \mathbf{z} \rangle \neq 0 \end{cases} \quad (3)
\end{aligned}$$

Thus, $I_{\# \text{PART}}$ is precisely the fraction of zero partitions for $\{x_k\}_{k=1}^n$ divided by 2^n . This also explains why an accuracy of at least 2^{-n} is required. \square

3 Quadrature Formulas

The DE formula approximates an integral using a weighted sum of $2N + 1$ terms. The convergence rate of this method to the actual integral is exponential in N for well-behaved integrands [4, 5, 6, 7].

Recall that the Hardy space H^2 is the space of all functions f satisfying

$$\sup_{r \in [0,1)} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right] < \infty$$

and recall that our integrand $\psi(t) = \prod_{k=1}^n \cos(\pi x_k t)$ is holomorphic and is bounded over any finite-measure complex region, so $\psi(t) \in H^2$.

The main result in [5] is its Theorem 5.1. Restating it using a simplified notation:

Theorem 2. *Let $f \in H^2$, $N \in \mathbb{N}$, $h > 0$. Let also $w(t) = \tanh(\frac{\pi}{2} \sinh t)$. Approximating the integral*

$$I_f = \int_{-1}^1 f(t) dt = \int_{-\infty}^{\infty} f(w(u)) w'(u) du$$

using the sum

$$\hat{I}_f = h \sum_{m=-N}^N f(w(mh)) w'(mh)$$

has an approximation error of

$$\left| I_f - \hat{I}_f \right|^2 \leq \mathcal{O}(e^{-cN}) + \left(1 + \frac{4}{\pi h} \right) \mathcal{O}\left(e^{-\frac{1}{\pi} e^{N-1}} \right)$$

for some constant $c > 0$ independent of f , h and N .

A proof for an error bound $\mathcal{O}\left(e^{\frac{-cN}{\log N}}\right)$ can be found in [6]. See also [7].

We now prove a more generalized result independently of previous derivations.

Theorem 3. *Let $n \in \mathbb{N}$, given analytic f such that for all $t \in [-1, 1]$:*

$$|f(t)| \leq 1, |f'(t)| \leq A, |f''(t)| \leq B$$

and given w such that for all $t \in \mathbb{R}$:

$$w(t) = w(-t), |w(t)| \leq 1, |w'(t)| \leq C, |w''(t)| \leq D,$$

$$|w'''(t)| \leq E, w(\infty) = 1, w(-\infty) = -1$$

and $w'(t) \neq 0$ for all $t \neq 0$. Then

$$\left| \int_{-1}^1 f(t) dt - 2^{1-N} \sum_{m=-N}^N f(w(m)) w'(m) \right| \leq 2^{-n}$$

where

$$p = \left\lceil -\log_2 \frac{\sqrt{(AC^2 + D)^2 - 2^{1-n}(BC^3 + 3ACD + E)} - AC^2 - D}{BC^3 + 3ACD + E} \right\rceil$$

(interestingly, this implies that “better” w has lower E, F and higher G) and

$$N = w^{-1} \left(\frac{2^{-n-3}}{A} - 1 \right) 2^p$$

If, in addition, $\lim_{x \rightarrow \infty} \sqrt{x} w^{-1}(x) = 0$ at least polynomially fast, then N increases only polynomially wrt n since then $N = \mathcal{O}\left(2^{-\frac{1}{2}n} w^{-1}(2^{-n})\right)$, as in the tanh-sinh case.

Proof. Put $g(t) = f(w(t)) w'(t)$ as the function to be evaluated. If using Kahan summation, then we need to calculate $g(t)$ only up to the desired integral accuracy, namely n digits. So we wish to find ϵ such that

$$|g(t + \epsilon) - g(t)| \leq 2^{-n}$$

using Taylor theorem, there exists ξ such that:

$$|g(t + \epsilon) - g(t)| = \left| \epsilon g'(t) + \frac{1}{2} g''(\xi) \epsilon^2 \right|$$

we note that

$$|g'(t)| = \left| f'(w(t)) [w'(t)]^2 + f(w(t)) w''(t) \right| \leq AC^2 + D$$

and

$$\begin{aligned} |g''(t)| &= \left| f''(w(t)) [w'(t)]^3 + 3f'(w(t)) w'(t) w''(t) + f(w(t)) w'''(t) \right| \\ &\leq BC^3 + 3ACD + E \end{aligned}$$

but

$$\begin{aligned} \left| \epsilon g'(t) + \frac{1}{2} g''(\xi) \epsilon^2 \right| &\leq |\epsilon g'(t)| + \left| \frac{1}{2} g''(\xi) \epsilon^2 \right| \\ &\leq |\epsilon| (AC^2 + D) + \frac{BC^3 + 3ACD + E}{2} \epsilon^2 \end{aligned}$$

the last term is clearly increasing with positive ϵ . Equating it to 2^{-n} we get one positive root:

$$\epsilon = \frac{\sqrt{(AC^2 + D)^2 - 2^{1-n} (BC^3 + 3ACD + E)} - AC^2 - D}{BC^3 + 3ACD + E}$$

So far we have shown that we need to calculate $g(t)$ up to accuracy of n digits, and this will give us, using Kahan summation, the integral up to the desired accuracy. Since we have just shown that we need no more than

$$p = \left\lceil -\log_2 \frac{\sqrt{(AC^2 + D)^2 - 2^{1-n} (BC^3 + 3ACD + E)} - AC^2 - D}{BC^3 + 3ACD + E} \right\rceil$$

for t in order to get such evaluation and know that our integral indeed converges, we have only 2^p possible different values for $t \in [-1, 1]$. Indeed, we need not sample the interval $[-1, 1]$ only, and now we turn to calculate the desired sampling interval.

Recalling w is even, we seek $z < 0$ such that

$$\int_{-\infty}^z g(t) dt \leq 2^{-n-3}$$

indeed it is sufficient to have

$$\int_{-\infty}^z g(t) dt \leq A \int_{-\infty}^z w'(t) dt \leq 1 + w(z) \leq 2^{-n-3} \implies z \leq w^{-1}(2^{-n-3} - 1)$$

our function is indeed invertible over the negative half line since its derivative never vanish. So our interval is

$$|t| \leq w^{-1} (2^{-n-3} - 1)$$

with granularity of p above, ending up with total of

$$2w^{-1} (2^{-n-2} - 1) \frac{\sqrt{(AC^2 + D)^2 - 2^{1-n} (BC^3 + 3ACD + E) - AC^2 - D}}{BC^3 + 3ACD + E}$$

function evaluations, since these are all possible inputs on this interval by the desired and implied accuracy. We can observe that the asymptotic behavior of (1) wrt n is decreasing exponentially as long as w^{-1} decreases faster than square root. \square

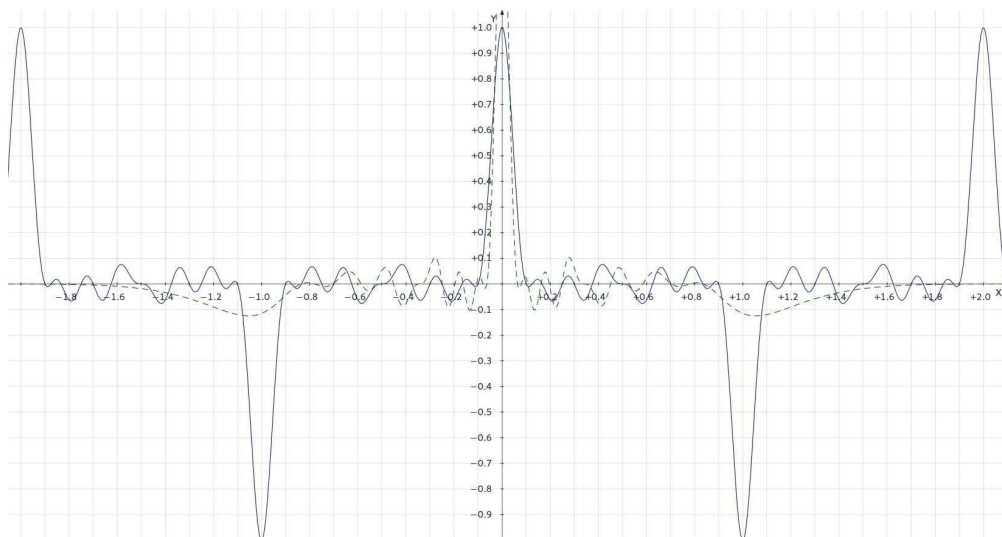


Figure 1: The function $f(t) = \cos(\pi t) \cos(2\pi t) \cos(3\pi t) \cos(4\pi t) \cos(5\pi t) \cos(6\pi t)$ (continuous line) and $f(w(t))w'(t)$ (dashed).

Impossibility result

$\#SAT$ is the problem of counting the number of satisfying assignments of a CNF formula. It is the counting problem associated with a Strong-NP problem, the Boolean Satisfiability problem. The preceding analysis suggests that unless Theorem 2 and possibly other proven convergence rates of the DE formula turn up wrong in the case of $I_{\#PART}$, $\#SAT$ may be solved in polynomial time.

Corollary 4. *Theorem 2 and 3 are incorrect for otherwise $\#SAT$ may be solved in polynomial time.*

Proof. Reducing #SAT with n clauses and k variables into #PART ends up with $\mathcal{O}(n+k)$ numbers to partition each having $\mathcal{O}(n+k)$ digits [10]. By Lemma 1 this problem is equivalent to approximating $n+k$ digits of the integral $I_{\text{\#PART}}$. Our integrand clearly fulfills the conditions of theorems 2 and 3 and so the number of evaluations needed to compute the $(n+k)$ -digit approximation $\hat{I}_{\text{\#PART}}$ is linear in $n+k$. Because evaluating the integrand once costs polynomial time, the corollary follows. \square

4 Concluding remark

The DE convergence rates should be reexamined for they currently suggest the existence of a polynomial time solution to a #P problem, though obviously we are unable to rule out the possibility that $P=NP$.

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