On Approximating Hard Integrals with the Double-Exponential Formula

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25 Dec 2015

Abstract

Approximating $I_{\#\mathrm{PART}} = \frac{1}{2}\int_{-1}^{1}\prod_{k=1}^{n}\cos\left(x_k\pi t\right)dt$ to within an accuracy of 2^{-n} is equivalent to counting the number of equal-sum partitions of a set of positive integers $\{x_k\}_{k=1}^{n}$, and is thus a #P problem. Efficient numerical integration methods such as the double exponential formula, also known as tanh-sinh quadrature, have been around from the mid 70's. Taking note of the hardness of approximating $I_{\#\mathrm{PART}}$ we argue that unless P=NP the proven rates of convergence of such methods cannot possibly be correct.

1 Overview

The Partition Counting Problem (#PART) is the following: given n positive integers $\{x_k\}_{k=1}^{n}$, in how many ways is it possible to divide them into two equal-sum subsets. Analytic and number-theoretic approaches to this problem can be found in many works, many seem to go back to the classic monograph [1] by Kac. If the input $\{x_k\}$ is given in binary rather unary radix, then solving this problem in polynomial time wrt the input's length would prove P=#P and would also entail P=NP. Assuming the exponential time hypothesis, #PART cannot be solved in polynomial time.

The treatment in [1] and subsequently in many other places e.g. [2, 3, 4] express the number of equal-sum partitions by the integral

$$2^n I_{\#\mathrm{PART}} = 2^{n-1}\int_{-1}^{1}\prod_{k=1}^{n}\cos\left(x_k\pi t\right)dt$$

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An elementary proof of this result is provided in the ensuing.

The double-exponential (DE) tanh-sinh quadrature is a numerical integration technique whose convergence rate has been proven to be exponential with the number of evaluation points [4, 5, 6, 7]. It is currently considered as the fastest high-precision quadrature technique. Noting the hardness of approximating $I_{\#\mathrm{PART}}$ we argue here that, unless P=NP, the DE convergence rate as stated in [4, 5, 6, 7] cannot be correct.

2 The partition problem

Given $n \in \mathbb{N}$ and $\{x_k\}_{k=1}^n \subset \mathbb{Z}$, we seek $\sigma \in \{-1,1\}^n$ such that $\langle \sigma, \mathbf{x} \rangle = 0$, where $\langle \sigma, \mathbf{x} \rangle = \sum_{k=1}^n \sigma_k x_k$ denotes the inner product. Deciding whether such σ exists is a NP Complete problem, while counting how many such σ's exists, is in #P. We assume that the inputs $\{x_k\}$ are given in binary radix and denote by d_k the number of binary digits of x_k. The partition problem is known to be Weak-NP since it has a polynomial-time algorithm if the input is supplied in unary radix. To get a feeling about typical dimensions of hard problems, the reduction of n-clause and k-variables 3SAT into the partition problem ends up with $\mathcal{O}(n+k)$ integers to partition, each having $\mathcal{O}(n+k)$ digits [10]. The exponential time hypothesis therefore implies that it is impossible to solve the partition problem in runtime complexity of $\mathcal{O}\left(\mathrm{poly}\left(\sum_{k=1}^n d_k\right)\right)$.

The counting version of the partition problem is equivalent to the following definite integral:

Lemma 1. *Let $\{x_k\}_{k=1}^n \subset \mathbb{Z}$ be integers given in binary radix. Let also $\psi(t) = \prod_{k=1}^n \cos(\pi x_k t)$. Then evaluating $I_{\#PART} = \frac{1}{2}\int_{-1}^{1} \psi(t)\, dt$ up to accuracy of n binary digits is in #P.*

Proof. This lemma can be proved in many interesting ways, all seem to go back to the classical monograph by Kac [1]. Slightly different proofs of this lemma may be found in [2, 4]. Our derivation is based on the formula

$$\prod_{k=1}^{n} \cos(z_k) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \cos\langle \sigma, \mathbf{z} \rangle \tag{1}$$

for every $\mathbf{z} \in \mathbb{C}^n$, which follows from a repeated application of the identity

$$4\cos(z_1)\cos(z_2) = \cos(z_1+z_2) + \cos(z_1-z_2) + \cos(-z_1+z_2) + \cos(-z_1-z_2) \tag{2}$$

Using this the integral reads

$$I_{\#\mathrm{PART}} = 2^{-n-1} \sum_{\sigma\in\{-1,1\}^n} \int_{-1}^{1} \cos\left(\pi t \left\langle \sigma, \mathbf{x} \right\rangle\right) dt = 2^{-n} \sum_{\sigma\in\{-1,1\}^n} \frac{\sin \pi \left\langle \sigma, \mathbf{z} \right\rangle}{\pi \left\langle \sigma, \mathbf{z} \right\rangle}$$

$$= 2^{-n} \sum_{\sigma\in\{-1,1\}^n} \begin{cases} 1 & \text{if } \left\langle \sigma, \mathbf{z} \right\rangle = 0 \\ 0 & \text{if } \left\langle \sigma, \mathbf{z} \right\rangle \neq 0 \end{cases} \quad (3)$$

Thus, $I_{\#\mathrm{PART}}$ is precisely the fraction of zero partitions for $\{x_k\}_{k=1}^n$ divided by 2^n. This also explains why an accuracy of at least 2^{-n} is required. □

3 Double-Exponential formula

The DE formula approximates an integral using a weighted sum of $2N+1$ terms. The convergence rate of this method to the actual integral is exponential in N for well-behaved integrands [4, 5, 6, 7].

Recall that the Hardy space H^2 is the space of all functions f satisfying

$$\sup_{r\in[0,1)} \left[\frac{1}{2\pi} \int_{0}^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta \right]^2 < \infty$$

and recall that our integrand $\psi\left(t\right) = \prod_{k=1}^{n} \cos\left(\pi x_k t\right)$ is holomorphic and is bounded over any finite-measure complex region, so $\psi\left(t\right) \in H^2$.

The main result in [5] is its Theorem 5.1. Restating it using a simplified notation:

Theorem 2. *Let $f \in H^2$, $N \in \mathbb{N}$, $h > 0$. Let also $w\left(t\right) = \tanh\left(\frac{\pi}{2}\sinh t\right)$. Approximating the integral*

$$I_f = \int_{-1}^{1} f\left(t\right) dt = \int_{-\infty}^{\infty} f\left(w\left(u\right)\right) w'\left(u\right) du$$

using the sum

$$\hat{I}_f = h \sum_{m=-N}^{N} f\left(w\left(mh\right)\right) w'\left(mh\right)$$

has an approximation error of

$$\left| I_f - \hat{I}_f \right|^2 \leq \mathcal{O}\left(e^{-cN}\right) + \left(1 + \frac{4}{\pi h}\right) \mathcal{O}\left(e^{-\frac{1}{\pi}e^{N-1}}\right)$$

for some constant $c > 0$ independent of f, h and N.

A proof for an error bound $\mathcal{O}\left(e^{\frac{-cN}{\log N}}\right)$ can be found in [6]. See also [7].

Corollary 3. *Let $f \in H^2$ be holomorphic and satisfy $|f(t)| \leq M, |f'(t)| \leq L$ and $|f''(t)| \leq H$ for all $t \in [-1,1]$, and set $g(t) = f(w(t))w'(t)$ where $w(t) = \tanh\left(\frac{\pi}{2}\sinh t\right)$. Then $\int_{-1}^{1} f(t)\,dt$ can be calculated up to n digits, within:*

- $\mathcal{O}(n)$ evaluations of g, at
- $\frac{1}{3}n + \frac{1}{3}\log_2\left(\left[\frac{\pi\left(2-\pi^2\right)}{4}\left[2M^2 + 4LM\right] + 24LM + 48L^2 + 16HM + 32HL\right]\right)$ digits of precision of g's input, and
- n digits of precision of g's output.

Proof. From Theorem 2 we can see that as the number of evaluations N doubles, so does the number of preicsion digits, i.e. $\mathcal{O}\left(e^{-c2N}\right) = \mathcal{O}\left(\left[e^{-cN}\right]^2\right)$ so we proved the desired number of evaluations. To have n digit approximation of I_f we set $N \approx n/2$. next show that each summand in $\hat{I}_f$ should be evaluated with a precision of n digits if I_f is to be approximated to within the desired accuracy.

Note that for all real t, $|w''(t)| \leq 2$ and $|w'(t)| \leq 2$. These together with Taylor's theorem and the triangle inequality allows bounding the numerical error of evaluating $g(\cdot)$:

$$\begin{aligned}
|g(t+\epsilon) - g(t)| &\approx \left|\epsilon g'(t) + \mathcal{O}\left(\frac{1}{2}g''(\xi)\epsilon^2\right)\right| \leq |\epsilon g'(t)| + \mathcal{O}\left(\frac{1}{2}g''(\xi)\epsilon^2\right) \\
&\leq |\epsilon|\left|f(w(t))w''(t) + f'(w(t))\left[w'(t)\right]^2\right| + \mathcal{O}\left(\frac{1}{2}g''(\xi)\epsilon^2\right) \\
&\leq |\epsilon|\,|2M + 4L| + \mathcal{O}\left(\frac{1}{2}g''(\xi)\epsilon^2\right) \\
&\leq |\epsilon|\,|2M + 4L| + \mathcal{O}\left(\frac{1}{2}g''(\xi)\epsilon^2\right) \quad (4)
\end{aligned}$$

Now to bound $\mathcal{O}\left(\frac{1}{2}g''(\xi)\epsilon^2\right)$ and recall that $|w'''(t)| \leq \frac{1}{4}\pi\left(2-\pi^2\right)$:

$$\begin{aligned}
|g''(t)| &= \left|f'(w(t))w''(t)w'(t) + f(w(t))w'''(t) + 2f'(w(t))w'(t)w''(t) + f''(w(t))\left[w'(t)\right]^3\right| \\
&= \left|3f'(w(t))w''(t)w'(t) + f''(w(t))\left[w'(t)\right]^3 + f(w(t))w'''(t)\right| \\
&\leq 12L + 8H + \frac{1}{4}\pi\left(2-\pi^2\right)M
\end{aligned}$$

combining with (4), we result with

$$\left|g\left(t+\epsilon\right)-g\left(t\right)\right|\leq\epsilon\left[2M+4L\right]+\epsilon^{2}\left[\frac{1}{4}\pi\left(2-\pi^{2}\right)M+12L+8H\right]$$

Suppose $\epsilon=2^{-p}$ where p is the number of digits of precision required for each evaluation. Employing Kahan summation algorithm [8, 9] while summing the terms of $\hat{I}_f$ relaxes the need for extra bits of accuracy which are normally taken to compensate for errors. We require $\left|g\left(t+2^{-p}\right)-g\left(t\right)\right|\leq 2^{-n}$ so it is sufficient to have $2^{-p}\left[2M+4L\right]+2^{-2p}\left[\frac{1}{4}\pi\left(2-\pi^{2}\right)M+12L+8H\right]\leq 2^{-n}$. Taking the logarithm and using Jensen inequality, it is sufficient to require

$$\begin{aligned}
&\log_{2}\left(2^{-p}\left[2M+4L\right]+2^{-2p}\left[\frac{1}{4}\pi\left(2-\pi^{2}\right)M+12L+8H\right]\right)\\
&\leq\log_{2}\left(2^{-p}\left[2M+4L\right]\right)+\log_{2}\left(2^{-2p}\left[\frac{1}{4}\pi\left(2-\pi^{2}\right)M+12L+8H\right]\right)\\
&=-p+\log_{2}\left[2M+4L\right]-2p+\log_{2}\left[\frac{1}{4}\pi\left(2-\pi^{2}\right)M+12L+8H\right]\\
&=-3p+\log_{2}\left(\left[2M+4L\right]\left[\frac{1}{4}\pi\left(2-\pi^{2}\right)M+12L+8H\right]\right)\\
&=-3p+\log_{2}\left(\left[\frac{\left[2M+4L\right]}{4}\pi\left(2-\pi^{2}\right)M+12L\left[2M+4L\right]+8H\left[2M+4L\right]\right]\right)\\
&=-3p+\log_{2}\left(\left[\frac{\pi\left(2-\pi^{2}\right)}{4}\left[2M^{2}+4LM\right]+24LM+48L^{2}+16HM+32HL\right]\right)\leq -n\\
&\implies p\geq\frac{1}{3}n+\frac{1}{3}\log_{2}\left(\left[\frac{\pi\left(2-\pi^{2}\right)}{4}\left[2M^{2}+4LM\right]+24LM+48L^{2}+16HM+32HL\right]\right)
\end{aligned}$$

□

Impossibility result

#SAT is the problem of counting the number of satisfying assignments of a CNF formula. It is the counting problem associated with a Strong-NP problem, the Boolean Satisfiability problem. The preceding analysis suggests that unless Theorem 2 and possibly other proven convergence rates of the DE formula turn up wrong in the case of $I_{\#\mathrm{PART}}$, #SAT may be solved in polynomial time.

Corollary 4. *Theorem 2 is incorrect for otherwise #SAT may be solved in polynomial time.*

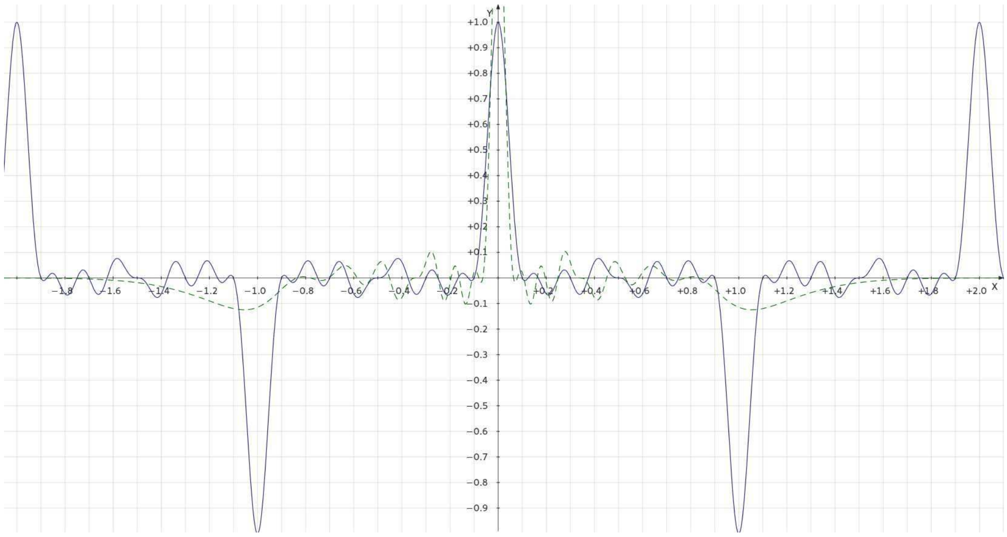

Figure 1: The function $f(t) = \cos(\pi t)\cos(2\pi t)\cos(3\pi t)\cos(4\pi t)\cos(5\pi t)\cos(6\pi t)$ (continuous line) and $f(w(t))w'(t)$ (dashed).

Proof. Reducing #SAT with n clauses and k variables into #PART ends up with $\mathcal{O}(n+k)$ numbers to partition each having $\mathcal{O}(n+k)$ digits [10]. By Lemma 1 this problem is equivalent to approximating $n+k$ digits of the integral $I_{\#\text{PART}}$. Our integrand clearly fulfills the conditions of Corollary 3 and so the number of evaluations needed to compute the $(n+k)$-digit approximation $\hat{I}_{\#\text{PART}}$ is linear in $n+k$. Because evaluating the integrand once costs polynomial time the corollary follows. $\square$

Remark 5. The actual precisions needed for the partition problem according to Corollary 3 is as follows. $|\psi(t)| \leq 1, |\psi'(t)| \leq nx_m, |\psi''(t)| \leq n^2x_m^2$ where x_m is the largest input number, d_m is its number of digits, and n is the number of numbers to partition, the required precision of the input to the composed integrand is

$$\frac{1}{3}n + \frac{1}{3}\log_2\left(\left[\frac{\pi\left(2-\pi^2\right)}{4}\left[2+4nx_m\right] + 24nx_m + 48n^2x_m^2 + 16n^2x_m^2 + 32n^3x_m^3\right]\right)$$
$$\leq \frac{1}{3}n + \frac{1}{3}\log_2\left[128n^3x_m^3\right] \leq \frac{n+7}{3} + d_m + \log_2(n)$$

4 Concluding remark

The DE convergence rates should be reexamined for they currently suggest the existence of a polynomial time solution to a #P problem, though obviously we are unable to rule out the possibility that P=NP.

5 Acknowledgments

The authors thank HunterMinerCrafter for many valuable discussions.

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