Statistical Properties of Trigonometric Functions with Applications

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1 Background

Observe that

$$\begin{aligned}
4\sum_{m=0}^{\infty}\cos(mx)\cos(my) &= \sum_{n=1}^{\infty}e^{im(x+y)}+e^{im(x-y)} && (1)\\
&= \frac{1}{1-e^{i(x+y)}}+\frac{1}{1-e^{i(x-y)}} && (2)\\
&= \frac{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}}{1-e^{i(x-y)}-e^{i(x+y)}+e^{2xi}} && (3)\\
&= \begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} && (4)
\end{aligned}$$

Recalling that $\frac{1}{1-t}+\frac{1}{1-\frac{1}{t}}=1$ for all nonzero complex t, then by the formulae of sum-of-angles and geometric progression we write:

$$\begin{aligned}
\lim_{r\to 1^-}4\sum_{m=0}^{\infty}r^m\cos(mx)\cos(my) &= \lim_{r\to 1^-}\sum_{n=1}^{\infty}r^m e^{im(x+y)}+r^m e^{im(x-y)} && (5)\\
&= \lim_{r\to 1^-}\frac{1}{1-re^{i(x+y)}}+\frac{1}{1-re^{i(x-y)}} && (6)\\
&= \begin{cases}\infty & (x-y)(x+y)=0\\ 1 & (x-y)(x+y)\neq 0\end{cases} && (7)
\end{aligned}$$

if x, y cannot ever meet 2π on some integer multiple.

The following theorem generalizes this result. Note that determining whether the products on the theorem's statements equal zero or not (equivalently, whether the infinite sums go to minus inifnity) is an NP-Hard problem. A structure of Mercer kernels also emerges from our identites. We will look on such applications later on this note.

2 Main Result

Theorem 2.1. *Fix* $\mathbf{x} \in \mathbb{Q}^n$ *then for all nonzero* $t \in \mathbb{Q}$ *the following identities hold:*

$$\sum_{\sigma \in \{-1,1\}^n} \ln |\sin \langle \mathbf{x}, \sigma \rangle| = -\frac{2^n}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{1}{m} \prod_{k=1}^{n} \cos (2x_k tm) \tag{8}$$

$$\sum_{\sigma \in \{-1,1\}^n} \ln |\cos \langle \mathbf{x}, \sigma \rangle| = -\frac{2^n}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \prod_{k=1}^{n} \cos (2x_k tm) \tag{9}$$

$$\sum_{\sigma \in \{-1,1\}^n} \ln |\tan \langle \mathbf{x}, \sigma \rangle| = \sum_{m=1}^{\infty} \frac{2^n}{m + \frac{1}{2}} \prod_{k=1}^{n} \cos (4x_k tm + 2x_k t) \tag{10}$$

Proof. Denote

$$f(m) \equiv \prod_{k=1}^{n} \cos (x_k m) \tag{11}$$

and write

$$\begin{aligned} \lim_{r \to 1^-} 2^n \sum_{m=1}^{\infty} r^m f(m) &= \lim_{r \to 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} r^m e^{im\langle \mathbf{x}, \sigma \rangle} && (12) \\ &= \lim_{r \to 1^-} \sum_{\sigma \in \{-1,1\}^n} \frac{1}{1 - re^{i\langle \mathbf{x}, \sigma \rangle}} && (13) \\ &= \begin{cases} -\infty & \exists \sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = 0 \\ 0 & \forall \sigma \in \{-1,1\}^n, \langle \mathbf{x}, \sigma \rangle \neq 0 \end{cases} && (14) \end{aligned}$$

since $\sum_{k=1}^{\infty} \frac{t^k}{k} = \ln \frac{1}{1-t}$ we can write

$$\begin{aligned} \lim_{r \to 1^-} -2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) &= \lim_{r \to 1^-} \sum_{\sigma \in \{-1,1\}^n} \sum_{m=1}^{\infty} \frac{r^m}{m} e^{im\langle \mathbf{x}, \sigma \rangle} && (15) \\ &= \lim_{r \to 1^-} \sum_{\sigma \in \{-1,1\}^n} \ln \frac{1}{1 - re^{i\langle \mathbf{x}, \sigma \rangle}} && (16) \end{aligned}$$

implying:

$$\lim_{r \to 1^-} \exp \left(-2^n \sum_{m=1}^{\infty} \frac{r^m}{m} f(m) \right) = \lim_{r \to 1^-} \prod_{\sigma \in \{-1,1\}^n} \left(1 - re^{i\langle \mathbf{x}, \sigma \rangle} \right) \tag{17}$$

for $r \to 1$ we have $1 - re^{i\langle \mathbf{x}, \sigma \rangle} = 0$ iff $\langle \mathbf{x}, \sigma \rangle$ is a zero partition. Taking the limit:

$$\begin{aligned} \exp \left(-2^n \sum_{m=1}^{\infty} \frac{f(m)}{m} \right) &= \prod_{\sigma \in \{-1,1\}^n} \left(1 - e^{i\langle \mathbf{x}, \sigma \rangle} \right) && (18) \\ &= \prod_{\sigma \in \{-1,1\}^n} \sqrt{2 \sin^2 \frac{\langle \mathbf{x}, \sigma \rangle}{2}} && (19) \end{aligned}$$

where the latter equalit is due to every σ there exists a matching $-\sigma$ and

$$\begin{aligned}\left(1-e^{i\langle\mathbf{x},\sigma\rangle}\right)\left(1-e^{-i\langle\mathbf{x},\sigma\rangle}\right) &= 2-2\cos\langle\mathbf{x},\sigma\rangle &(20)\\ &= 2\sin^2\frac{\langle\mathbf{x},\sigma\rangle}{2} &(21)\end{aligned}$$

concluding

$$\begin{aligned}\exp\left(-2^n\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right) &= 2^{2^{n-1}}\prod_{\sigma\in\{-1,1\}^n}\left|\sin\langle\mathbf{x},\sigma\rangle\right| &(22)\\ \Longrightarrow \frac{1}{\sqrt{2}}\exp\left(-\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right) &= \sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\sin\langle\mathbf{x},\sigma\rangle\right|} &(23)\end{aligned}$$

equivalently,

$$\begin{aligned}-2^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(m) &= -\sum_{\sigma\in\{-1,1\}^n}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{im\langle\mathbf{x},\sigma\rangle} &(24)\\ &= \sum_{\sigma\in\{-1,1\}^n}\ln\left(1+e^{i\langle\mathbf{x},\sigma\rangle}\right) &(25)\end{aligned}$$

or

$$\begin{aligned}\exp\left(-2^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(m)\right) &= \prod_{\sigma\in\{-1,1\}^n}\left(1+e^{i\langle\mathbf{x},\sigma\rangle}\right) &(26)\\ &= \prod_{\sigma\in\{-1,1\}^n}\sqrt{2\cos^2\frac{\langle\mathbf{x},\sigma\rangle}{2}} &(27)\\ \Longrightarrow \frac{1}{\sqrt{2}}\exp\left(-\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(2m)\right) &= \sqrt[2^n]{\prod_{\sigma\in\{-1,1\}^n}\left|\cos\langle\mathbf{x},\sigma\rangle\right|} &(28)\end{aligned}$$

dividing (23) by (28) we conclude:

$$\begin{aligned}&\exp\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m}f(2m)-\sum_{m=1}^{\infty}\frac{f(2m)}{m}\right) &(29)\\ &\quad= \exp\left(\textstyle\sum_{m=1}^{\infty}\frac{2f(4m+2)}{2m+1}\right) &(30)\\ &\quad= \sqrt[2^n]{\textstyle\prod_{\sigma\in\{-1,1\}^n}\left|\tan\langle\mathbf{x},\sigma\rangle\right|} &(31)\end{aligned}$$

□