DRAFT: Spectral and Modular Analysis of #P Problems

Ohad Asor

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Abstract

We present various analytic and number theoretic results concerning the #SAT problem as reflected when reduced into a #PART problem. As an application we propose a heuristic to probabilistically estimate the solution of #SAT problems.

1 Overview

#SAT is the problem of counting the number of satisfying assignments to a given 3CNF formula, while #PART is the problem of counting the number of zero partitions in a given set of integers. Precise definitions will be given later on. We present various results concerning #PART and analyze their connection with #SAT. On section 2 we skim some preliminaries. Section 3 presents the core of the analytic setting by analyzig the #PART problem as manipulations over product of cosines. Section 4 derives a modular-arithmetic formula for computing #PART, and section 5 presents implications to complexity theory. Section 6 deals shortly with asymptotic normality, and on section 7 we present miscellaneus results, where section 8 propose how multiple reductions may give probabilistic answer to #SAT as a consequence our analysis. Section 9 summarizes the highlights of the paper.

2 Preliminaries

Our setting is counting the number of solutions given an instance of the Partition problem. We sometimes use custom terminology as there is no unified one.

Definition 2.1. Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$, a *Partition* σ of $\mathbf{x}$ is some $\sigma \in \{-1,1\}^n$. The *size* of the partition σ is $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^{n} \sigma_k x_k$. A partition is called a *zero partition* if its size is zero. The problem #PART is to determine the number of zero partitions given $\mathbf{x}$. The problem PART is deciding whether a zero partition exists or not for $\mathbf{x}$. The *Weak* setting of the problem is when $\mathbf{x}$ is supplied in unary radix, and the *Strong* setting is when it is supplied in binary radix (or another format with same efficiency), therefore the input size is logarithmically smaller on the strong setting.

#PART is in #P complexity class. The setting of #PART after being reduced from the counting Boolean Satisfiability problem (#SAT) is n integers to partition each having up to $\mathcal{O}(n)$ binary digits (where n is linear in the size of the CNF formula), demonstrating why the rather strong setting is of interest. In fact, there exist polynomial time algorithms solving the weak setting of PART, notably Dynamic Programming algorithms, as well as the formula derived on Theorem 4.1 below. However, solving PART under the strong setting is not possible in polynomial time (as a function of the input's length), unless P=NP.

#SAT can be reduced to #SUBSET-SUM using an algorithm described in [1], while various slight variations appear on the literature. We summarize the reductions on the Appendix.

3 Analytic Setting

Theorem 3.1. *Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ then the (probability-theoretic) characteristic function of the random variable $\langle \mathbf{x}, \sigma \rangle = \sum_{k=1}^{n} x_k \sigma_k$ over uniform $\sigma \in \{-1, 1\}^n$ is $\prod_{k=1}^{n} \cos(x_k t)$.*

Proof. Consider the formula $2 \cos a \cos b = \cos(a+b) + \cos(a-b)$ and the cosine being even function to see that:

$$\psi(t) \equiv \psi(x_1, \ldots, x_n, t) \equiv \prod_{k=1}^{n} \cos(x_k t) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \cos(t \langle \mathbf{x}, \sigma \rangle) = \mathbb{E}\left[e^{it\langle \mathbf{x}, \sigma \rangle}\right] \tag{1}$$

□

Corollary 3.2. *Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$ then*

$$2^n \int_0^1 \prod_{k=1}^{n} \cos(2\pi x_k t)\, dt \tag{2}$$

is the number of zero partitions of $\mathbf{x}$.

Proof. Following (1) and integrating both sides. Stronger statements are possible (e.g. for $\mathbf{x} \in \mathbb{R}^n$ or $\mathbf{x} \in \mathbb{C}^n$) using characteristic function inversion theorems. □

Theorem 3.3. *Given $\{n, N\} \subset \mathbb{N}, j \in \mathbb{Z}, \mathbf{x} \in \mathbb{N}^n$ then*

$$\frac{2^n}{N} \sum_{m=1}^{N} e^{2\pi i j \frac{m}{N}} \prod_{k=1}^{n} \cos\left(2\pi x_k \frac{m}{N}\right) \tag{3}$$

is the number of partitions of $\mathbf{x}$ having size that is divisible by N with remainder j.

Proof. Following (1)

$$\frac{2^n}{N} \sum_{m=1}^{N} \prod_{k=1}^{n} \cos\left(2\pi x_k \frac{m}{N}\right) = \sum_{\sigma \in \{-1,1\}^n} \frac{1}{N} \sum_{m=1}^{N} e^{2\pi i \frac{m}{N} \langle \mathbf{x}, \sigma \rangle} \tag{4}$$

The sum of the roots of unity on the rhs of (4) is zero if N does not divide $\langle \mathbf{x}, \sigma \rangle$, and is one if N does divide it, therefore (4) is equal to

$$\sum_{u=-\infty}^{\infty} c_{uN} \tag{5}$$

where c_u denotes the number of partitions that sum to u:

$$c_u = \left| \{\sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = u\} \right| \tag{6}$$

As for the remainder, observe that

$$\sum_{\sigma \in \{-1,1\}^n} e^{2\pi i t (\langle \mathbf{x}, \sigma \rangle + j)} = e^{2\pi i t j} 2^n \prod_{k=1}^{n} \cos(2\pi x_k t) \tag{7}$$

□

Conjecture 3.4. *For all even n, for all $\mathbf{x} \in \mathbb{N}^n$ the number of $\mathbf{x}$'s zero partitions is no more than the number of zero partitions of vector of size n with all its elements equal 1. Namely, never more than $\binom{n}{\frac{1}{2}n}$ zero partitions.*

Furthermore, for all odd n, for all $\mathbf{x} \in \mathbb{N}^n$ the number of $\mathbf{x}$'s zero partitions is no more than the number of zero partitions of vector of size n with all its elements equal 1 except one element that equals 2.

4 Modular Arithmetic Formula

Theorem 4.1. *Given $n \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$, the number of $\mathbf{x}$'s zero partition out of all possible 2^n partitions is encoded as a binary number in the binary digits of*

$$\prod_{k=1}^{n} \left[1 + 4^{nx_k}\right] \tag{8}$$

from the s'th dight to the $s+n$ digit, where $s = n\,\langle \mathbf{x}, 1 \rangle$.

Proof. We write down the following sum and perform substitution according to (1):

$$S = \frac{1}{n} \sum_{m=1}^{n} \psi\left(\frac{2\pi m}{n} + i \ln 2\right) = \sum_{\sigma \in \{-1,1\}^n} \frac{2^{-\langle \mathbf{x}, \sigma \rangle}}{n} \sum_{m=1}^{n} e^{\frac{2\pi i m}{n} \langle \mathbf{x}, \sigma \rangle} \tag{9}$$

multiplying all x_k by n (while preserving partitions, since we can always multiply all x_k by the same factor and keep the exact number of zero partitions) puts $e^{\frac{2\pi i m}{n} \langle n\mathbf{x}, \sigma \rangle} = 1$ and we get:

$$S = \sum_{\sigma \in \{-1,1\}^n} 2^{-n\langle \mathbf{x}, \sigma \rangle} = \sum_{u=-\infty}^{\infty} c_u 2^{-u} \tag{10}$$

where c_u is defined in. Recalling that $\sum_{u=-\infty}^{\infty} c_u = 2^n$ and c_u are all positive, while on (10) being multiplied by distinct powers $2^{\pm n}$, therefore the summands' binary digits never interfere with each other. Recalling that c_0 is our quantity of interest, we have shown that the number of zero partitions in $\mathbf{x}$ is encoded at

$$\left\lfloor \frac{2^n}{n} \sum_{m=1}^{n} \prod_{k=1}^{n} \cos\left[nx_k \left(\frac{2\pi m}{n} + i \ln 2 \right) \right] \right\rfloor \mod 2^n \tag{11}$$

$$= \left\lfloor \prod_{k=1}^{n} \left[2^{nx_k} + 2^{-nx_k} \right] \right\rfloor \mod 2^n \tag{12}$$

$$= \left\lfloor 2^{-n\sum_{k=1}^{n} x_k} \prod_{k=1}^{n} \left[1 + 2^{2nx_k} \right] \right\rfloor \mod 2^n \tag{13}$$

Set

$$M = \prod_{k=1}^{n} \left[1 + 2^{2nx_k} \right] = \sum_{\sigma \in \{0,1\}^n} 2^{2n\langle \mathbf{x}, \sigma \rangle} \tag{14}$$

then (12) tells us that the number of zero partitions is encoded as a binary number in the binary digits of M, from the s'th dight to the $s+n$ digit. ∎

Note that the substitution in (9) could take a simpler form. Put $t = i \ln 2$ in (1):

$$\prod_{k=1}^{n} \cosh\left(x_k \ln 2\right) = \prod_{k=1}^{n} \left[2^{x_k} + 2^{-x_k}\right] = \mathbb{E}\left[2^{\langle \mathbf{x}, \sigma \rangle}\right] \tag{15}$$

5 Hardness of Integration

Corollary 5.1. $\mathbf{x} \in \mathbb{Q}^n$ *has a zero partition if and only if*

$$\int_0^\infty \prod_{k=1}^{n} \cos\left(x_k t\right) dt = \infty \tag{16}$$

and does not have a zero partition if and only if

$$\int_0^\infty \prod_{k=1}^{n} \cos\left(x_k t\right) dt = 0 \tag{17}$$

Proof. Follows from Theorem 4.1, the integrand being periodic, change of variable to support rationals, and the integral over a single period being nonnegative for all inputs. □

Corollary 5.2. *There is no algorithm that takes any function that can be evaluated in polynomial time, and decides in polynomial time whether its integral over the real line is zero (conversley, infinity) unless P=NP.*

Theorem 5.3. *[Theorem 2.2 on [3]] If u is an analytic function satisfying $|u(z)| \leq M$ in $\frac{1}{r} \leq |z| \leq r$ for some $r > 1$, then for any $N \geq 1$ the trapezoid rule with N points will be far from the exact integral by no more than $\frac{4\pi M}{r^N - 1}$.*

Corollary 5.4. *If for every* #PART *instance it is possible to efficiently find a function w such that given ψ as in (1) that corresponds the problem's instance, and*

$$u(z) = \psi\left(w(z)\right) w'(z) \tag{18}$$

is computable in polynomial time (wrt the input length and the desired output accuracy) and satisfies the conditions of Theorem 4.4 with $r = 2$ and $M = \mathcal{O}\left(poly\left(\sum_{k=1}^{n} x_k\right)\right)$, then P=NP.

Proof. Observe that ψ behaves like $e^{x_k t}$ for imaginary input. It therefore satisfies $M = e^{r \sum_{k=1}^{n} x_k}$ at the setting of Theorem 4.4. For exponential convergence wrt PART's input length we need $\frac{4\pi M}{r^N - 1}$ diminish exponentially. Therefore if we can change the variable of integration in (17) using some w and result with $M = \mathcal{O}\left(\text{poly}\left(\sum_{k=1}^{n} x_k\right)\right)$, we could estimate the integral in (17) to our desired accuracy (2^{-n}) in subexponential time. □

Remark 5.5. The desired accuracy mentioned in Corollary 5.4 is the same accuracy desired from the integral (typically n binary digits for our integrand, as (1) suggests). This is due to Kahan summation algorithm ([4]). We can compute the integrand only up to that accuracy when we use the trapezoid rule, as long as we perform the summation according to Kahan's algorithm (in constant multiplicative cost).

6 Asymptotic Normality

Observe that the Conjecture 3.4 says that for all $\mathbf{x} \in \mathbb{N}^n$ we have

$$\int_0^1 \prod_{k=1}^{n} \cos\left(2\pi x_k t\right) dt \leq \int_0^1 \cos^n\left(2\pi t\right) dt \tag{19}$$

note that $\int_0^1 \cos^n\left(t\right) dt$ approaches to a gaussian as n tends to infinity:

$$\lim_{n\to\infty} \int_0^{\sqrt{n}} \cos^n \frac{2\pi t}{\sqrt{N}} dt = \lim_{n\to\infty} \int_0^{\sqrt{n}} \left[1 - \frac{4\pi^2 t^2}{2n} + \mathcal{O}\left(\frac{1}{n}\right)\right]^n dt \tag{20}$$

$$= \int_0^{\infty} e^{-2\pi^2 t^2} dt = \frac{1}{\sqrt{8\pi}} \approx 0.1994 \tag{21}$$

as the standard Fourier transform derivation of the Central Limit Theorem suggests. Similarly, if we take a vector $\mathbf{x}$ and equally add more copies of its elements up to infinity (e.g. transforming $\{1,2,3\}$ into $\{1,1,1,2,2,2,3,3,3\}$), we get asymptotic amount of zero partition written as:

$$\lim_{N\to\infty} \int_0^{\sqrt{N}} \prod_{k=1}^{n} \cos^N \left(\frac{2\pi x_k t}{\sqrt{N}}\right) dt \tag{22}$$

note that now the limit is wrt N since we still have base n numbers, just copied N times. Continuing:

$$= \lim_{N\to\infty} \int_0^{\sqrt{N}} \prod_{k=1}^{n} \left[1 - \frac{4\pi^2 x_k^2 t^2}{2N} + \mathcal{O}\left(\frac{1}{N}\right)\right]^N dt \tag{23}$$

$$= \int_0^{\infty} e^{-2\pi^2 t^2 \sum_{k=1}^{n} x_k^2} dt = \frac{1}{\sqrt{8\pi \sum_{k=1}^{n} x_k^2}} \tag{24}$$

It is interesting to see that the resulted gaussian is diagonalized, i.e. no correlations between the x_k's at the asymptote on this special case of having infinitely many copies. This means that the fact that numbers are being copied will always govern any other property of the numbers, except the single quantity $\sqrt{\sum_{k=1}^{n} x_k^2}$.

7 Additional Results

Theorem 7.1. *Given $n \in \mathbb{N}, N \in \mathbb{N}, \mathbf{x} \in \mathbb{N}^n$, the variance of the sizes of all partitions is the sum of the squares of the input. Formally:*

$$\sum_{k=1}^{n} x_k^2 = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \langle \mathbf{x}, \sigma \rangle^2 \tag{25}$$

while

$$\frac{2^n}{N^3} \sum_{m=1}^{N} \frac{\partial^2}{\partial t^2} \prod_{k=1}^{n} \cos\left(2\pi x_k t\right) \Bigg|_{t=\frac{m}{N}} \tag{26}$$

is the variance of the sizes of all partitions that their size is divisible by N without remainder.

Proof. Following (1) and differentiating:

$$\prod_{k=1}^{n}\cos\left(\pi x_k t\right)=2^{-n}\sum_{\sigma\in\{-1,1\}^n}\cos\left(\pi t\left\langle\mathbf{x},\sigma\right\rangle\right) \tag{27}$$

$$\implies\sum_{\ell=1}^{n}x_\ell\sin\left(\pi x_\ell t\right)\prod_{k\neq\ell}^{n}\cos\left(\pi x_k t\right)=2^{-n}\sum_{\sigma\in\{-1,1\}^n}\left\langle\mathbf{x},\sigma\right\rangle\sin\left(\pi t\left\langle\mathbf{x},\sigma\right\rangle\right) \tag{28}$$

$$\implies\sum_{\ell=1}^{n}\sum_{\ell\prime=1}^{n}-x_\ell\sin\left(\pi x_\ell t\right)x_{\ell\prime}\sin\left(\pi x_{\ell\prime}t\right)\prod_{k\neq\ell,\ell\prime}^{n}\cos\left(\pi x_k t\right)+x_\ell^2\prod_{k=1}^{n}\cos\left(\pi x_k t\right) \tag{29}$$

$$=2^{-n}\sum_{\sigma\in\{-1,1\}^n}\left\langle\mathbf{x},\sigma\right\rangle^2\cos\left(\pi t\left\langle\mathbf{x},\sigma\right\rangle\right) \tag{30}$$

and (19) follows by substituting $t=0$. (19) can be proved using Parseval identity as well. Turning to (20):

$$\frac{2^n}{N}\sum_{m=1}^{N}\frac{\partial^2}{\partial t^2}\prod_{k=1}^{n}\cos\left(2\pi x_k t\right)\bigg|_{t=\frac{m}{N}}=\sum_{\sigma\in\{-1,1\}^n}\left\langle\mathbf{x},\sigma\right\rangle^2\cos\left(2\pi\frac{m}{N}\left\langle\mathbf{x},\sigma\right\rangle\right)=\sum_{u=-\infty}^{\infty}u^2N^2c_{Nu} \tag{31}$$

due to aliasing of roots of unity, and c_{Nu} the number of partitions whose size is divisible by Nu as in (4). □

Remark 7.2. It is easy to derive all moments and cumulants of our random variable since we're given its characteristic function.

Theorem 7.3. *Let $Z^{\mathbf{x}}$ be the number of zero partitions of a vector of naturals X. Let $D_x^{\mathbf{x}}$ be the number of zero partitions of X after multiplying one if its elements by two, where this element is denoted by x. Let $A_x^{\mathbf{x}}$ be the number of zero partitions of X after appending it x (so now x appears at least twice). Then*

$$Z^{\mathbf{x}}=D_x^{\mathbf{x}}+A_x^{\mathbf{x}} \tag{32}$$

Proof. Denote

$$\psi\left(x_1,\ldots,x_n\right)=2^n\int_0^\pi\prod_{k=1}^{n}\cos\left(x_k t\right)dt \tag{33}$$

then, using the identity $\cos 2x=2\cos^2 x-1$:

$$\psi\left(x_1,\ldots,2x_m,\ldots,x_n\right)=2^n\int_0^\pi\cos\left(2x_m t\right)\prod_{k\neq m}^{n}\cos\left(x_k t\right)dt \tag{34}$$

$$=2^n\int_0^\pi\left[2\cos^2\left(x_m t\right)-1\right]\prod_{k\neq m}^{n}\cos\left(x_k t\right)dt$$

$$\implies\psi\left(x_1,\ldots,x_m,\ldots,x_n\right)-\psi\left(x_1,\ldots,2x_m,\ldots,x_n\right)= \tag{35}$$

$$2^{n+1}\int_0^\pi\cos^2\left(x_m t\right)\prod_{k\neq m}^{n}\cos\left(x_k t\right)dt=\psi\left(x_1,\ldots,x_m,\ldots,x_n,x_m\right)$$

□

8 Estimating #SAT

The numbers produced by the reduction from #SAT to #PART have digits that does not exceed 4, and if using radix 6, they never even carry. Therefore the very same digits produced by the reduction can be interpreted in any radix larger than 5, being reduced to a different #PART problem. Still, it is guaranteed that the number of solution to those #PART problems are independent of the radix, as they're all reduced from the same #SAT problem. This property might be used to approximate #SAT using results as Theorem 3.3. We can obtain the number of partitions that their size divides a given number N in polynomial time wrt n (the number of numbers to partition) and the number of digits of x_k. Nevertheless, it takes exponential time in the number of digits of N.

The probability that there exists a partition with nonzero size that is divisible by a given prime p is roughly

$$\mathcal{P}\left[p \middle| \langle \mathbf{x}, \sigma \rangle\right] \approx 1 - \left(1 - \frac{1}{p}\right)^{2^n} \tag{36}$$

taking K reductions of a single #SAT problem instance and a set P of primes, the probability that on reductions there exists a partition with size divisible by a given prime p that is not a zero partition is therefore roughly

$$\prod_{p \in P} \prod_{k=1}^{K} \mathcal{P}\left[p \middle| \langle \mathbf{x}_k, \sigma \rangle\right] \approx \prod_{p \in P} \left[1 - \left(1 - \frac{1}{p}\right)^{2^n}\right]^K \tag{37}$$

$$\leq \exp\left(-K \sum_{p \in P} \left(1 - \frac{1}{p}\right)^{2^n}\right)$$

recalling that for $x \in [0, 1]$ we have $e^{-x} \geq 1 - x$. This doesn't seem to be helpful since it seem to require exponentially many or exponentially large primes or reductions. However, if Conjecture 3.4 is true, then we can bound our hueristic approximation with rather

$$\exp\left(-K \sum_{p \in P} \left(1 - \frac{1}{p}\right)^{\binom{n}{\frac{1}{2}n}}\right) \tag{38}$$

9 Discussion

On Theorem 3.3 we have seen that we can efficiently query for the number of partitions that divide by N with remainder j. It is interesting to see that positively solving PART (resp. SAT) by guesses is straight-forward: we just try partitions (resp. substitutions) and if we're lucky to find a zero partition (resp. SAT) then we solved the problem. On the other hand, how can we do one trial and possibly decide that the set is unpartitionable (resp. UNSAT)? Our analysis suggest such a method. If we query for the number partitions that are divisible by N with $j = 0$ and get zero, then we know that the set is unpartitionable. Similarly, if we do the same for $j \neq 0$ and happen to get 2^n, we know that $\mathbf{x}$ does not have a zero partition. Those trials are argubaly independent due to the pseudo-randomness of the mod operation.

On Section 4 reduced #SAT into a problem of computing the k'th digit of the result of the multiplications of numbers of the form $100\ldots001$, i.e. two ones only and zeros between them. In fact, this result is independent on the radix chosen (given it is not too small). On #SAT setting, the number of zeros has polynomial amount of digits wrt the #SAT problem's input size, while

the number of multiplicands is also polynomial. Note that this means that the unary size of the multiplicands are exponential.

On Section 5 we showed that P!=NP implies a result of nonexistence of certain complex analytic functions. P!=NP also imlies impossibility to decide in polynomial time whether an integral (with bounded, periodoc, and polynomially computed integrand) converges to either zero or infinity.

On Section 8 we have seen that if Conjecture 3.4 is true, then we can give heuristic having approximate exponentially convergent probabilistic estimation to #SAT, by taking advantage of the modular formulas we derived in Theorem 3.3. Interestingly, this method reveals relatively very little information about any single #PART problem, since we use reduce the #SAT problem instance into many #PART instances, taking advantage on the reduction promising us #PART problems with quite different modular properties, yet with exactly the same number of zero partitions.

We also derived asymptotic normality and showed a method of computing the moments of the partitions as a random variable.

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A Appendix

A.1 Reductions

Reduction of #SAT to #SUBSET-SUM Given variables $x_1, \ldots, x_l$ and clauses $c_1, \ldots, c_k$ and let natural $b \geq 6$. we construct a set S and a target t such that the resulted subset-sum problem requires finding a subset of S that sums to t. The number t is l ones followed by k 3s (i.e. of the form $1111\ldots3333$). S contains four groups of numbers $y_1, \ldots, y_l$, $z_1, \ldots, z_l$, $g_1, \ldots, g_k$, $h_1, \ldots, h_k$ where $g_i = h_i = b^{k-i}$, and y_i, z_i are b^{k+l-i} plus b^m for y_i if variable i appears positively in clause m, or for z_i if variable i appears negated in clause m. Then, every subset that sum to t matches to a satisfying assignment in the input CNF formula and vice versa, as proved in [1].

Reduction of #SUBSET-SUM to #PART Given S, t as before and denote by $s = \sum_{x \in S} x$ the sum of S members, the matching PART problem is $S \cup \{2s - t, s + t\}$. Here too all solutions to both problems are preserved by the reduction and can be translated in both directions.

A.2 Further Research

1. By using (11) we can get successive estimates to (12) by selecting e.g. primes $N = 2, 3, 5, \ldots$. We then could accelerate this sequence using Shanks, Romberg, Pade or similar sequence-acceleration method.

2. Detecting whether x_1, x_2 appear with different (resp. equal) signs in some zero partition can be done by examining the correlations such as

$$-\sin(x_1 t)\sin(x_2 t)\prod_{k=3}^{n}\cos(x_k t) = 2^{-n}\sum_{\sigma\in\{-1,1\}^n}\sigma_1\sigma_2\cos(t\langle \mathbf{x},\sigma\rangle) \tag{39}$$