On Approximating Hard Integrals with the Double-Exponential Formula

Ohad Asor* and Avishy Carmi†

25 Dec 2015

Abstract

Approximating $I_{\#\mathrm{PART}} = \frac{1}{2}\int_{-1}^{1}\prod_{k=1}^{n}\cos(x_k\pi t)\,dt$ to within an accuracy of 2^{-n} is equivalent to counting the number of equal-sum partitions of a set of positive integers $\{x_k\}_{k=1}^{n}$, and is thus a #P problem. Efficient numerical integration methods such as the double exponential formula, also known as tanh-sinh quadrature, have been around from the mid 70's. Taking note of the hardness of approximating $I_{\#\mathrm{PART}}$ we argue that unless P=NP the proven rates of convergence of such methods cannot possibly be correct.

1 Overview

The Partition Counting Problem (#PART) is the following: given n positive integers $\{x_k\}_{k=1}^{n}$, in how many ways is it possible to divide them into two equal-sum subsets. Analytic and number-theoretic approaches to this problem can be found in many works, many seem to go back to the classic monograph [1] by Kac. If the input $\{x_k\}$ is given in binary rather unary radix, then solving this problem in polynomial time wrt the input's length would prove P=#P and would also entail P=NP. Assuming the exponential time hypothesis, #PART cannot be solved in polynomial time.

The treatment in [1] and subsequently in many other places e.g. [2, 3, 4] express the number of equal-sum partitions by the integral

$$2^n I_{\#\mathrm{PART}} = 2^{n-1}\int_{-1}^{1}\prod_{k=1}^{n}\cos(x_k\pi t)\,dt$$

*ohadasor@gmail.com

†Center for Quantum Information Science and Technology & Faculty of Engineering Sciences, Ben-Gurion University of the Negev, Beer-Sheva 8410501, Israel. avishycarmi@gmail.com

An elementary proof of this result is provided in the ensuing.

The double-exponential (DE) tanh-sinh quadrature is a numerical integration technique whose convergence rate has been proven to be exponential with the number of evaluation points [4, 5, 6, 7]. It is currently considered as the fastest high-precision quadrature technique. Noting the hardness of approximating $I_{\#\mathrm{PART}}$ we argue here that, unless P=NP, the DE convergence rate as stated in [4, 5, 6, 7] cannot be correct.

2 The partition problem

Given $n \in \mathbb{N}$ and $\{x_k\}_{k=1}^n \subset \mathbb{Z}$, we seek $\sigma \in \{-1,1\}^n$ such that $\langle \sigma, \mathbf{x} \rangle = 0$, where $\langle \sigma, \mathbf{x} \rangle = \sum_{k=1}^n \sigma_k x_k$ denotes the inner product. Deciding whether such σ exists is a NP Complete problem, while counting how many such σ's exists, is in #P. We assume that the inputs $\{x_k\}$ are given in binary radix and denote by d_k the number of binary digits of x_k. The partition problem is known to be Weak-NP since it has a polynomial-time algorithm if the input is supplied in unary radix. To get a feeling about typical dimensions of hard problems, the reduction of n-clause and k-variables 3SAT into the partition problem ends up with $\mathcal{O}(n+k)$ integers to partition, each having $\mathcal{O}(n+k)$ digits [10]. The exponential time hypothesis therefore implies that it is impossible to solve the partition problem in runtime complexity of $\mathcal{O}\left(\mathrm{poly}\left(\sum_{k=1}^n d_k\right)\right)$.

The counting version of the partition problem is equivalent to the following definite integral:

Lemma 1. *Let $\{x_k\}_{k=1}^n \subset \mathbb{Z}$ be integers given in binary radix. Let also $\psi(t) = \prod_{k=1}^n \cos(\pi x_k t)$. Then evaluating $I_{\#PART} = \frac{1}{2}\int_{-1}^{1} \psi(t)\, dt$ up to accuracy of n binary digits is in #P.*

Proof. This lemma can be proved in many interesting ways, all seem to go back to the classical monograph by Kac [1]. Slightly different proofs of this lemma may be found in [2, 4]. Our derivation is based on the formula

$$\prod_{k=1}^{n} \cos(z_k) = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \cos\langle \sigma, \mathbf{z} \rangle \tag{1}$$

for every $\mathbf{z} \in \mathbb{C}^n$, which follows from a repeated application of the identity

$$4\cos(z_1)\cos(z_2) = \cos(z_1+z_2) + \cos(z_1-z_2) + \cos(-z_1+z_2) + \cos(-z_1-z_2) \tag{2}$$

Using this the integral reads

$$I_{\#\text{PART}} = 2^{-n-1} \sum_{\sigma \in \{-1,1\}^n} \int_{-1}^{1} \cos\left(\pi t \left\langle \sigma, \mathbf{x} \right\rangle\right) dt = 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \frac{\sin \pi \left\langle \sigma, \mathbf{z} \right\rangle}{\pi \left\langle \sigma, \mathbf{z} \right\rangle}$$

$$= 2^{-n} \sum_{\sigma \in \{-1,1\}^n} \begin{cases} 1 & \text{if } \left\langle \sigma, \mathbf{z} \right\rangle = 0 \\ 0 & \text{if } \left\langle \sigma, \mathbf{z} \right\rangle \neq 0 \end{cases} \tag{3}$$

Thus, $I_{\#\text{PART}}$ is precisely the fraction of zero partitions for $\{x_k\}_{k=1}^{n}$ divided by 2^n. This also explains why an accuracy of at least 2^{-n} is required. □

3 Double-Exponential formula

The DE formula approximates an integral using a weighted sum of $2N+1$ terms. The convergence rate of this method to the actual integral is exponential in N for well-behaved integrands [4, 5, 6, 7].

Recall that the Hardy space H^2 is the space of all functions f satisfying

$$\sup_{r \in [0,1)} \left[\frac{1}{2\pi} \int_{0}^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta \right]^2 < \infty$$

and recall that our integrand $\psi(t) = \prod_{k=1}^{n} \cos(\pi x_k t)$ is holomorphic and is bounded over any finite-measure complex region, so $\psi(t) \in H^2$.

The main result in [5] is its Theorem 5.1. Restating it using a simplified notation:

Theorem 2. *Let $f \in H^2$, $N \in \mathbb{N}$, $h > 0$. Let also $w(t) = \tanh\left(\frac{\pi}{2}\sinh t\right)$. Approximating the integral*

$$I_f = \int_{-1}^{1} f(t)\, dt = \int_{-\infty}^{\infty} f(w(u))\, w'(u)\, du$$

using the sum

$$\hat{I}_f = h \sum_{m=-N}^{N} f(w(mh))\, w'(mh)$$

has an approximation error of

$$\left| I_f - \hat{I}_f \right|^2 \leq \mathcal{O}\left(e^{-cN}\right) + \left(1 + \frac{4}{\pi h}\right) \mathcal{O}\left(e^{-\frac{1}{\pi}e^{N-1}}\right)$$

for some constant $c > 0$ independent of f, h and N.

A proof for an error bound $\mathcal{O}\left(e^{\frac{-cN}{\log N}}\right)$ can be found in [6]. See also [7].

Corollary 3. *Let $f \in H^2$ satisfy $|f(t)| \leq M$ and $|f'(t)| \leq L$ for all $t \in [-1,1]$, and set $g(t) = f(w(t))\, w'(t)$ where $w(t) = \tanh\left(\frac{\pi}{2}\sinh t\right)$. Then $\int_{-1}^{1} f(t)\, dt$ can be calculated up to n digits, within:*

- $\mathcal{O}(n)$ evaluations of g, at
- $\mathcal{O}(n + \log_2(M + 2L))$ digits of precision of g's input, and
- $\mathcal{O}(n)$ digits of precision of g's output.

Proof. From Theorem 2 we can see that as the number of evaluations N doubles, so does the number of preicsion digits, i.e. $\mathcal{O}\left(e^{-c2N}\right) = \mathcal{O}\left(\left[e^{-cN}\right]^2\right)$ so we proved the desired number of evaluations. To have n digit approximation of I_f we set $N \approx n/2$. next show that each summand in $\hat{I}_f$ should be evaluated with a precision of n digits if I_f is to be approximated to within the desired accuracy.

Note that for all real t, $|w''(t)| \leq 2$ and $|w'(t)| \leq 2$. These together with the triangle inequality allows bounding the numerical error of evaluating $g(\cdot)$:

$$\begin{aligned} |g(t+\epsilon) - g(t)| &\approx \left|\epsilon g'(t) + \mathcal{O}\left(\epsilon^2\right)\right| \leq |\epsilon g'(t)| + \mathcal{O}\left(\epsilon^2\right) \\ &\leq |\epsilon| \left| f(w(t))\, w''(t) + f'(w(t))\, [w'(t)]^2 \right| + \mathcal{O}\left(\epsilon^2\right) \\ &\leq 2|\epsilon|\, |M + 2L| + \mathcal{O}\left(\epsilon^2\right) \end{aligned} \tag{4}$$

Suppose $\epsilon = 2^{-p}$ where p is the number of digits of precision required for each evaluation. Employing Kahan summation algorithm [8, 9] while summing the terms of $\hat{I}_f$ relaxes the need for extra bits of accuracy which are normally taken to compensate for errors. We require $|g(t + 2^{-p}) - g(t)| \leq 2^{-n}$ so it is sufficient to have $2^{1-p}(M + 2L) + \mathcal{O}(2^{-2p}) \leq 2^{-n}$ from which we recognize that the relation between p and n is at most linear, or more accurately $p \approx n + 1 + \log_2(M + 2L)$. $\square$

Impossibility result

#SAT is the problem of counting the number of satisfying assignments of a CNF formula. It is the counting problem associated with a Strong-NP problem, the Boolean Satisfiability problem. The preceding analysis suggests that unless Theorem 2 and possibly other proven convergence rates of the DE formula turn up wrong in the case of $I_{\#\mathrm{PART}}$, #SAT may be solved in polynomial time.

Corollary 4. *Theorem 2 is incorrect for otherwise #SAT may be solved in polynomial time.*

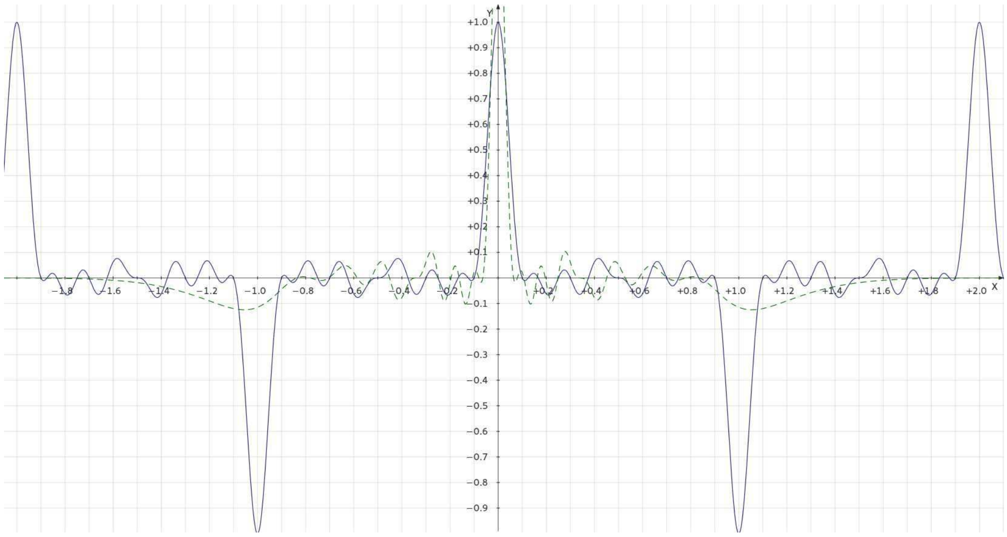

Figure 1: The function $f(t) = \cos(\pi t)\cos(2\pi t)\cos(3\pi t)\cos(4\pi t)\cos(5\pi t)\cos(6\pi t)$ (continuous line) and $f(w(t))w'(t)$ (dashed).

Proof. Reducing #SAT with n clauses and k variables into #PART ends up with $\mathcal{O}(n+k)$ numbers to partition each having $\mathcal{O}(n+k)$ digits [10]. By Lemma 1 this problem is equivalent to approximating $n+k$ digits of the integral $I_{\#\mathrm{PART}}$. Our integrand clearly fulfills the conditions of Corollary 3 and so the number of evaluations needed to compute the $(n+k)$-digit approximation $\hat{I}_{\#\mathrm{PART}}$ is linear in $n+k$. Because evaluating the integrand once costs polynomial time the corollary follows. $\square$

4 Concluding remark

The DE convergence rates should be reexamined for they currently suggest the existence of a polynomial time solution to a #P problem, though obviously we are unable to rule out the possibility that P=NP.

5 Acknowledgments

The authors thank HunterMinerCrafter for many valuable discussions.

References

[1] Kac, M. *Statistical Independence in Probability, Analysis and Number Theory.* Carus Mathematical Monographs, No. 12, Wiley, New York (1959).

[2] Johnson, D. S., Garey, M. *Computers and Intractability.* W. H. Freeman and Company (1979).

[3] Krantz, S.G. *Handbook of Logic and Proof Techniques for Computer Science.* Springer-Verlag New York, NY, USA. (2002).

[4] Moore, C., Mertens, S. *The Nature of Computation.* Oxford University Press, Inc., New York, NY, USA. (2010).

[5] Borwein, J. M., Lingyun, Y. Quadratic Convergence of the Tanh-sinh Quadrature Rule. *Mathematics of Computation.* (2006).

[6] Hidetosi, T., Masatake, M. Double Exponential Formulas for Numerical Integration. *Publications of the Research Institute for Mathematical Sciences*, Vol. 9 (3), pp. 721 – 741. (1974).

[7] Trefethen, L.N., Weideman, J.A.C. The Exponentially Convergent Trapezoidal Rule. *SIAM Review*, Vol. 56 (3) pp. 385–458. (2014)

[8] Kahan, W. Further remarks on reducing truncation errors. *Communications of the ACM*, 8 (1): 40. (1965).

[9] Higham, N. J. The accuracy of floating point summation. *SIAM Journal on Scientific Computing*, 14 (4): 783–799. (1993).

[10] Sipser, M. *Introduction to the Theory of Computation.* International Thomson Publishing (1996).