

Let $n \in \mathbb{N}$, $\mathbf{x} \in \mathbb{N}^n$ and consider the formula $2 \cos a \cos b = \cos(a+b) + \cos(a-b)$ and the cosine being even function to see that:

$$\psi(t) = 2^n \prod_{k=1}^n \cos(x_k t) = \sum_{\sigma \in \{-1,1\}^n} \cos t \langle \mathbf{x}, \sigma \rangle = 2 \sum_{\sigma \in \{-1,1\}^n} e^{it \langle \mathbf{x}, \sigma \rangle} \quad (1)$$

Denote the number of partitions that sum to k by

$$c_k = |\{\sigma \in \{-1,1\}^n \mid \langle \mathbf{x}, \sigma \rangle = k\}| \quad (2)$$

we write down the following sum just for fun and substitute (1) and (2) in it:

$$S = \frac{1}{2n} \sum_{m=1}^n \psi\left(2\pi \frac{m}{n} + i \ln 2\right) = \sum_{\sigma \in \{-1,1\}^n} 2^{-\langle \mathbf{x}, \sigma \rangle} \sum_{m=1}^n \frac{1}{n} e^{2\pi i \frac{m}{n} \langle \mathbf{x}, \sigma \rangle} \quad (3)$$

the summation of roots of unity equals zero iff n does not divide $\langle \mathbf{x}, \sigma \rangle$, and if it does divide then it sums to n . So

$$S = \sum_{k=-\infty}^{\infty} c_{kn} 2^{-nk} \leq c_0 + 2^{1-n} \quad (4)$$

We have proved that \mathbf{x} has a zero partition iff

$$\sum_{m=1}^n \prod_{k=1}^n \cos\left(2\pi \frac{m}{n} + i \ln 2\right) \geq \frac{2n}{2^n} \quad (5)$$