Hearthstone Specialist format: Nash equlibrium strategy

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Specialist format is used in official Hearthstone (Blizzard's hit collectible card game) competitions. Here's how it works:

- Both players create a set of three decks of 30 cards: primary, secondary and tertiary
  - Secondary and tertiary decks have to share no less than 25 cards with the primary deck
- In the first game of the match both players have to use their primary deck
- Starting from game two players can secretly pick any deck, no matter the result of prior games
- Matches are best-of-3 or best-of-5, although most matches are best-of-3

Since players may randomize their deck choice (random.org memes have always been a staple of competitive Hearthstone community), one is tempted to find a Nash equilibrium in mixed strategies based on a matrix of winrates (how Player One's (P1's) decks match up against Player Two's (P2's) decks). Let's establish some terms useful for such analysis.

Let w_{ij} be P1's i-ry deck's expected winrate against P2's j-ry deck, and W be a matrix representation of these winrates:

$$W = \begin{pmatrix} w_{1,1} & w_{1,2} & w_{1,3} \\ w_{2,1} & w_{2,2} & w_{2,3} \\ w_{3,1} & w_{3,2} & w_{3,3} \end{pmatrix}.$$

Let $m_{kl}(S)$ be expected *match* winrate of P1, where k and l are P1's and P2's match scores respectively and S is a strategy profile with respect to deck choices. Let $M(S)$ be a matrix representation of these winrates. Then for a best-of-3 (S references are omitted for clarity)

$$M = \begin{pmatrix} m_{0,0} & m_{0,1} \\ m_{1,0} & m_{1,1} \end{pmatrix},$$

and for best of n

$$M = \begin{pmatrix} m_{0,0} & m_{0,1} & m_{0,2} & \cdots & m_{0,n-1} \\ m_{1,0} & m_{1,1} & m_{1,2} & \cdots & m_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1,0} & m_{n-1,} & m_{n-1,2} & \cdots & m_{n-1,n-1} \end{pmatrix}.$$

Note that every game "moves" the match down or to the right on the M matrix. Now, let's compute $M(S^*)$, where S^* is Nash equilibrium in mixed strategies for the best-of-N case. Note that every game after the first one is generated by the same choice of decks, and the first one is played with primary decks. This recurring subgame is a zero-sum game represented by payoff matrix W. It is obvious that

$$m_{n-1,n-1} = V(W),$$

where V is value of the game. It is also easy to show that the match (disregarding the game one for now) has a binomial nature - if the match score is $i - j$, P1 has to succeed in $n - i$ trials of $2n - 1 - i - j$ remaining to win. Also note that

$$m_{0,0} = w_{11} m_{1,0} + (1 - w_{11}) m_{0,1}$$

It then follows that

$$m_{i,j} = \begin{cases} w_{11} m_{1,0} + (1 - w_{11}) m_{0,1}, & \text{if } i = j = 0 \\ \sum_{k=n-i}^{2n-1-i-j} \binom{2n-1-i-j}{k} V^k(W)(1 - V(W))^{2n-1-i-j-k}, & \text{otherwise} \end{cases}$$

Thus, P1's winrate for the match itself is

$$m_{0,0} = w_{11} m_{1,0} + (1 - w_{11}) m_{0,1} =$$

$$= w_{11} \sum_{k=n-1}^{2n-2} \binom{2n-2}{k} V^k(W)(1 - V(W))^{2n-2-k} + (1 - w_{11}) \sum_{k=n}^{2n-2} \binom{2n-2}{k} V^k(W)(1 - V(W))^{2n-2-k}$$

It is easy to show that $V(W)$ is the solution of a simple linear programming problem that also produces Nash equilibrium strategy profiles (see any game theory textbook, e.g. Owen, 1968) -

$$V(W) = \max_{S} \lambda$$

, where

$$S = \begin{cases} w_{11}x_1 + w_{21}x_2 + w_{31}x_3 \geq V(W) \\ w_{12}x_1 + w_{22}x_2 + w_{32}x_3 \geq V(W) \\ w_{13}x_1 + w_{23}x_2 + w_{33}x_3 \geq V(W) \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

So, expected match winrate given deck winrates is a lot easier to compute for Specialist format than for formats with variable sets of available decks (e.g. Last Hero Standing, Conquest) or complex ban rules (e.g. Dima *RDU* Radu's Strike format). An interactive Shiny app for calculating match winrates is available at *bruh.shinyapps.com*.