PSN PG1

8/28/24 A: $U = \{0,1,2,3,4,5,6,7,8,9\}$, $A = \{0,1,2\}$, $B = \{0,2,4,6\}$

$|A| = 3$

$A \cup B = \{0,1,2,4,6\}$

$A \cap B = \{0,2\}$

$A - B = \{1\}$

$A \oplus B = A \cup B - A \cap B = \{1,4,6\}$

$A \times B = \{(0,0),(0,2),(0,4),(0,6),(1,0),(1,2),(1,4),(1,6),(2,0),(2,2),(2,4),(2,6)$

$|A \times B| = 3 \times 4 = 12$

$\overline{A} = \{3,4,5,6,7,8,9\}$

8/28/24 B: What is meant by a paradox? And why does this lead to a paradox?

If R contains all of the sets that do not contain themselves, then if R incudes itself then it cannot contain itself, and if it does not include itself, it has to.

8/30/24 A: Give Membership Tables for operations $\oplus$ & $\overline{B}$

A	B	$A \oplus B$	$\overline{A}$	$\overline{B}$
0	0	0	1	1
0	1	1	1	0
1	0	1	0	1
1	1	0	0	0

8/30/24 B: Prove $(A \cup B) - C = (A - C) \cup (B - C)$ using Membership tables

A	B	C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A-C) \cup (B-C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	1	0	1	1	0	1	1
0	1	1	1	0	0	0	0
1	0	0	1	1	1	0	1
1	0	1	1	0	0	0	0
1	1	0	1	1	1	1	1
1	1	1	1	0	0	0	0

PSN PG2

8/30/24 C: $U=\{0,1,2,3,4,5,6,7,8,9\}$, $A=\{0,1,2\}$, $B=\{0,2,4,6\}$

$P(B)=\{\{\},\{0\},\{2\},\{4\},\{6\},\{0,2\},\{0,4\},\{0,6\},\{2,4\},\{2,6\},\{4,6\},$
$\{0,2,4\},\{0,2,6\},\{0,4,6\},\{2,4,6\},\{0,2,4,6\}\}$

$|P(A\cup B)| = 2^n = 2^5 = 32$

$|P(A\times B)| \quad 2^{|A|\times|B|} = 2^{3\times4} = 2^{12} = 4096$

$|P(A)\times P(B)| \quad |P(A)|\times|P(B)| = 2^3\times2^4 = 2^7 = 128$

9/4/24 A: Let p & q be prime... $n=pq$ ... Apply Principle Inclusion-Exclusion...

Show the number of positive nums $< n$ are rel. prime to n

$U=\{1,\ldots,n\}$ $A=\{p,2p,\ldots,qp\}$ $B=\{q,2q,\ldots,pq\}$

Either p or q: $|A\cup B| = |A|+|B|-|A\cap B| = q+p-1$

Since both are prime, U is rel. prime if it is divisible by neither p or q

This means $|\overline{A\cup B}| = n-|A\cup B| = n-(q+p-1) = pq-q-p+1 = (p-1)(q-1)$

9/4/24 B: Using set complement and Incl-Excl w/ 3 sets, obtain a formula for the # of numbers between 1-500, inclusive that are rel. prime to 60.

Not rel. prime: 60 f. $\rightarrow$ 2, 3, 5 so:

$U=\{1,\ldots,500\}$ $A=\{2,4,\ldots,500\}$ $B=\{3,6,\ldots,498\}$

$C=\{5,10,\ldots,500\}$

$|\overline{A\cup B\cup C}| = |U|-|A\cup B\cup C| = |U|-(|A|+|B|+|C|-|A\cap B|-|A\cap C|$
$-|B\cap C|+|A\cap B\cap C|) = 500-(\frac{500}{2}+\frac{500}{3}+\frac{500}{5}-\frac{500}{6}-\frac{500}{10}-\frac{500}{15}+\frac{500}{30})$
$= 500-(250+166+100-83-50-33+16) = 500-366 = 134$ numbers

9/6/24 A: Greedy Algorithm w/o Nickels; is it correct still?

No, since dimes could be better than quarter & pennies

Take 30¢: Greedy: 1 quarter, 5 pennies

Best: 3 dimes

9/6/24 B: Obtain upper bounds p_{10}, p_5, p_1

* With 1 nickel you can only have 1 dime

p_{10}: since quarters are 25, nickels are 5 $\rightarrow$ 30 can be obtained w/ 2 rather than 3, so UB is 2

p_5: since dimes are 10, UB is 1 since 2 can be replaced by 1 nickel

p_1: since this is the lowest and nickel is next, UB is 4, since 5 can be replaced by 1 nickel

PSN Pg 3

9/9/24 A: Show that $1+2+\dots+n = n(n+1)/2$

Basis Step: $1 = 1(1-0)/2$

Induction Step: for $n=k \rightarrow 1+2+\dots+k = k(k+1)/2$

for $n=k+1 \rightarrow (1+2+\dots+k)+k+1 = k(k+1)/2+k+1$

$= k^2/2 + k/2 + k + 1 = (k+1)(k+2)/2$

9/9/24 B: Prove $Fib(n) > 1.5^n$ for all $n \geq 11$

(Combining prev. results: $1.5^n < Fib(n) < 2^n$, for all $n \geq 11$)

Proposition: $Fib(n) > 1.5^n$ for all $n \geq 11$

Basis: $n=11$, $Fib(11) = 89 > 1.5^{11} = 86.497$

$n=12$, $Fib(12) = 144 > 1.5^{12} = 129.746$

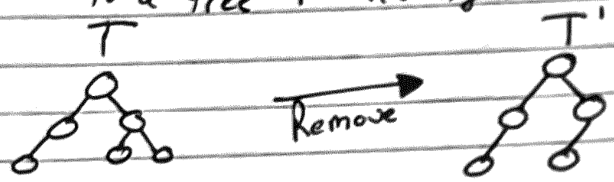
Strong Induction: Assume proposition is true from 11 to k

$Fib(n) > 1.5^n$, $n = 11, 12, \dots, k$

case $n=k+1$: $Fib(k+1) = Fib(k) + Fib(k-1) > 1.5^k + 1.5^{k-1}$

$= 1.5^{k+1} \rightarrow$ Completes induction step & proof

9/11/24 A: To apply induction hyp... perform operation reducing T to a tree T' having k nodes, how to do?



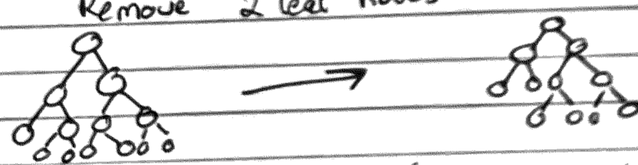
Hypothesis: $m(T') = n(T') - 1$

$\hookrightarrow m(T) = m(T') + 1 = (n(T') - 1) + 1$

$\hookrightarrow n(T') = n(T) - 1$

9/11/24 B: To apply ind. hyp. ... reduce T to T' with k leaf

Remove 2 leaf nodes



$\boxed{\begin{array}{l} L(T') = I(T') + 1 \\ L(T) = L(T') + 1 = (I(T') + 1) + 1 \\ = I(T) + 1 \end{array}}$

Assuming Y is not a leaf, it has child C. Since X and Y are at the same level, X is not the deepest layer. This contradicts assumption so X & Y are leaf nodes

PSN PG4

9/13/24 A: Which is equal to $p \to q$: $q \to p$, $\neg p \to \neg q$, $\neg q \to \neg p$?

p	q	$\neg q$	$\neg p$	$p \to q$	$\neg q \to \neg p$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	F	F	F
T	T	F	F	T	T

9/13/24 B: $p \leftrightarrow q$ is the same as $p \to q$ and $q \to p$

AND $p \leftrightarrow q$ is the same as $\neg(p \oplus q)$

Forgot oopsie

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$	$\neg(p \oplus q)$	$p \oplus q$
F	F	T	T	T	T	T	F
F	T	T	F	F	F	F	T
T	F	F	T	F	F	F	T
T	T	T	T	T	T	T	F

9/16/24 A: Give alternate proof than ~~by~~ using truth tables that

$$(p \wedge \neg q) \to (p \oplus r) \Longleftrightarrow \neg p \vee q \vee \neg r$$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \wedge \neg q$	$p \oplus r$	$(p \wedge \neg q) \to (p \oplus r)$	$\neg p \vee q$	$\neg p \vee q \vee \neg r$
F	F	F	T	T	T	F	F	T	T	T
F	F	T	T	T	F	F	T	T	T	T
F	T	F	T	F	T	F	F	T	T	T
F	T	T	T	F	F	F	T	T	T	T
T	F	F	F	T	T	T	T	T	F	T
T	F	T	F	T	F	T	F	F	F	F
T	T	F	F	F	T	F	T	T	T	T
T	T	T	F	F	F	F	F	T	T	T

PSNPGS

9/16/24 B: Prove that this is a tautology:

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee$$
$$(p \wedge \neg q \wedge \neg r) \vee (\neg q \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

...using dist law $= (p \vee \neg p) \wedge (q \vee \neg q) \wedge (r \vee \neg r) = T \wedge T \wedge T = T$

...this also every outcome in a truth table, meaning it could be asking that it is an outcome which is always true

9/18/24 A: Put formula $\mathcal{F}$ in DNF: $(p \wedge \neg q) \leftrightarrow (p \oplus r)$

p q r	$\neg p$	$\neg q$	$\neg r$	$p \wedge \neg q$	$p \oplus r$	$\mathcal{F}$	Clause Conjunction
F F F	T	T	T	F	F	T	$\rightarrow \neg p \wedge \neg q \wedge \neg r$
F F T	T	T	F	F	T	F	... $\neg p \wedge \neg q \wedge r$
F T F	T	F	T	F	F	T	$\rightarrow \neg p \wedge q \wedge \neg r$
F T T	T	F	F	F	T	F	... $\neg p \wedge q \wedge r$
T F F	F	T	T	T	T	T	$\rightarrow p \wedge \neg q \wedge \neg r$
T F T	F	T	F	T	F	F	... $p \wedge \neg q \wedge r$
T T F	F	F	T	F	T	F	... $p \wedge q \wedge \neg r$
T T T	F	F	F	F	F	T	$\rightarrow p \wedge q \wedge r$

$\mathcal{F} \Leftrightarrow (\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$

Put formula $\mathcal{F}$ in CNF

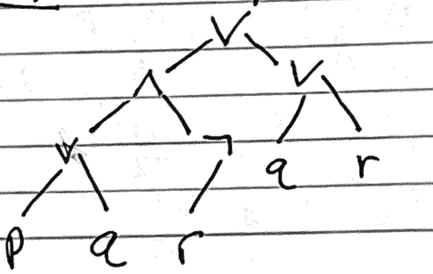
$\mathcal{F} \Leftrightarrow \neg(\neg \mathcal{F})$

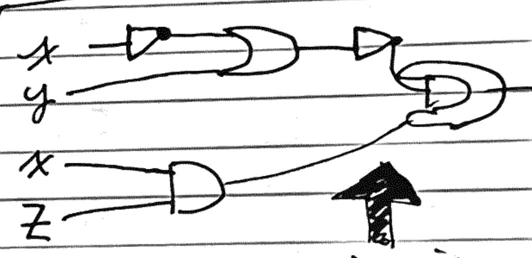
$\Leftrightarrow \neg (\neg p \wedge \neg q \wedge r) \wedge \neg(\neg p \wedge q \wedge r) \wedge \neg(p \wedge \neg q \wedge r) \wedge \neg(p \wedge q \wedge \neg r)$

$\Leftrightarrow (\neg\neg p \vee \neg\neg q \vee \neg r) \wedge (\neg\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg\neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee \neg\neg r)$

$\mathcal{F} \Leftrightarrow (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \rightarrow$ CNF

9/18/24 B: Give the expression tree for $([(p \vee q) \wedge (\neg r)] \vee (q \vee r))$





9/18/24 C: Obtain Formula associated with the circuit

$\neg(\neg x \vee y) \vee (x \wedge z)$

PSN PG6

9/20/24 A: for clause size of 2, find conj. of size 3 that is equivalent.

$x \vee (y \wedge \neg y) \vee (z \wedge \neg z) = x \vee F \vee F = x$ ... Dist Law

$(x \vee y \vee z) \wedge (x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee \neg y \vee \neg z)$

9/20/24 B: for clause size of 2, find conj. of size 3 that is equivalent

$(x \vee y) \vee (z \wedge \neg z) = x \vee y \vee F = x \vee y$ ... Dist Law

$(x \vee y) = (x \vee y) \vee (z \wedge \neg z) = (x \vee y \vee z) \wedge (x \vee y \vee \neg z)$

9/23/24 A:

$$A(x_1, x_2) = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} a_{00} x_0 & a_{01} x_1 \\ a_{10} x_0 & a_{11} x_1 \end{bmatrix}$$

$$B(x_1, x_2) = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} b_{00} x_0 & b_{01} x_1 \\ b_{10} x_0 & b_{11} x_1 \end{bmatrix}$$

Compute $A \circ B$:

$$\begin{bmatrix} a_{00} b_{00} + a_{01} b_{10} & a_{00} b_{01} + a_{01} b_{11} \\ a_{10} b_{00} + a_{11} b_{10} & a_{10} b_{01} + a_{11} b_{11} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

9/23/24 B: Let f be invertible from Y to Z and g be invertible from X to Y

Show that the inverse: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$(f \circ g)[(g^{-1} \circ f^{-1})(x)] = f[g(g^{-1}[f^{-1}(x)])]$

$= f(f^{-1}(x)) = x$

9/25/24 A: Find the inverse relation of $\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

$\{(1,1),(2,1),(3,1),(4,1),(2,2),(3,2),(4,2),(3,3),(4,3),(4,4)\}$

9/25/24 B: $R = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$ $S = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$

Construct $R \circ S$ as for each (s_1, s_2) in S, for each (r_1, r_2) in R if $s_2 = r_1$ then (s_1, r_2) in $R \circ S$

$R \circ S = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$

9/25/24 C: Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$ Find R^n where $n = 1,2,3,4,5$

$R^1 = R = \{(1,1),(2,1),(3,2),(4,3)\}$

$R^2 = R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$

$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

$R^4 = R^3 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

$R^5 = R^4 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

PSN PG7

9/27/24 A: Define the equivalence relation R corresponding to P and prove equiv. relation

$P = \{A_i \mid i \in I\} \rightarrow$ Define R that "belongs to the same set"

$(x,y) \in R$ iff x & y belong to same set $P \rightarrow i \in I, x, y \in A_i$

To show R is an equivalence relation we must show Symmetric, Reflexsive, Transitive

<u>Symmetric:</u> $(x,x) \in R \rightarrow$ every element x is in the same set as itself

<u>Reflexive:</u> $(x,y) \in R \Rightarrow (y,x) \in R \rightarrow$ if they are in the same set, they both are

<u>Transitive:</u> $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$

if x & y are in the same set and y & z are in the same set then x & z are too

9/30/24 A: Show that relation R given by xRy whenever $x \equiv y \pmod{n}$

i.e. whenever n divides $x-y$ is an equiv. rel. on set of $\mathbb{Z}$ of integers

$a|b$ denotes $\frac{a}{b}$

<u>Reflexive:</u> $n|(x-x) \Rightarrow x \equiv x \pmod{n}$

<u>Symmetric:</u> $x \equiv y \pmod{n} \Rightarrow n|(x-y) \Rightarrow n|(y-x) \Rightarrow y \equiv x \pmod{n}$

<u>Transitive:</u> $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n} \Rightarrow n|(x-y)$ and $n|(y-z)$

$\Rightarrow n|(x-y)+(y-z) \Rightarrow n|(x-z) \Rightarrow x \equiv z \pmod{n}$

9/30/24 B: Obtain addition and multiplication tables for $n=7$

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

10/2/24 A: Convert 250 to binary

$250 = 128+64+32+16+8+2$: 11111010

10/2/24 B: Show num decimal digits d of n is approx $\log_{10} n$

What about binary? octal? hexadecimal?

For n (3 dig in decimal) $100 \leq n < 999$ or $10^2 \leq n \leq 10^3 - 1$

$\log_{10}(n+1) \leq d \leq 1 + \log 10\, n$; $d \approx \log_{10} n$

Binary: $d \approx \log_2 n$, Oct.: $d \approx \log_8 n$, Hex: $d \approx \log_{16} n$

PSNPG8

10/4/24 A: Using Euclid's algorithm compute gcd(585, 1035)

$585 = 0 \cdot 1035 + 585$ $\quad 1035 = 1 \cdot 585 + 450$

$585 = 1 \cdot 450 + 135$ $\quad 450 = 3 \cdot 135 + 45$ $\quad 135 = 3 \cdot 45 + 0$

$\gcd(585, 1035) = \gcd(1035, 585) = \gcd(585, 450) =$
$\gcd(450, 135) = \gcd(135, 45) = \gcd(45, 0) = 45$

10/4/24 B: Solve $g = \gcd(a, b) = sa + tb$ for $a = 1035$, $b = 585$

$1035 = 1 \cdot 585 + 450 \Rightarrow 450 = 1035 - 1 \cdot 585$

$585 = 1 \cdot 450 + 135 \Rightarrow 135 = 585 - 1 \cdot 450$

$450 = 3 \cdot 135 + 45 \Rightarrow 45 = 450 - 3 \cdot 135$

$135 = 3 \cdot 45 + 0$

$45 = 450 - 3 \cdot 135$

$45 = 450 - 3 \cdot (585 - 1 \cdot 450)$

$45 = 4 \cdot 450 - 3 \cdot 585$

$45 = 4 \cdot (1035 - 1 \cdot 585) - 3 \cdot 585$

$45 = 4 \cdot 1035 - 7 \cdot 585$

This means for $g = \gcd(a, b) = sa\ tb$ for $a = 7700$, $b = 3000$

obtaining $g = 45$, $s = 4$, $t = -7$

PSN PG9

10/14/24A: Describe algorithm for computing private key $d=e^{-1}(\bmod\ \phi(n))$

Show that private key $d=s$, where $se+t\varphi(n)=1$, then compute s

$g=\gcd(e,\varphi(n))=1$, compute s & t : $se+t\varphi(n)=g=1$

mod $\varphi(n)$ both sides : $se\equiv 1(\bmod\ \varphi(n))$

or equivalently : $s=e^{-1}(\bmod\ \varphi(n))$

10/16/24A: Obtain a formula for average degree α of a vertex in terms of order & size (#edges, #vertices swapped :))

$$\alpha=\frac{\sum_{v\in V} d(v)}{n}=\frac{2m}{n} \text{ since } \sum_{v\in V} d(v)=2m$$

Since $\delta\le\alpha\le\Delta \longrightarrow \delta\le\frac{2m}{n}\le\Delta$: δ: min degree, Δ: max degree

10/16/24B: How many edges does the complete graph K_n have?

Complete graph is r-regular w/ $r=n-1$, so: $m=\frac{nr}{2}=\frac{n(n-1)}{2}=\frac{n^2}{2}-\frac{n}{2}$

10/18/24A: Show relation R where uRv iff u is connected to v is an equivalence relation on V

Reflexive: There is a path from $u\to u$: $(u,u)\in R$

Symmetric: If there is a path $u\to v$, there is the reverse path $v\to u$

Transitive: If there is a path $u\to v$ & path $v\to w$, putting together yields $u\to w$

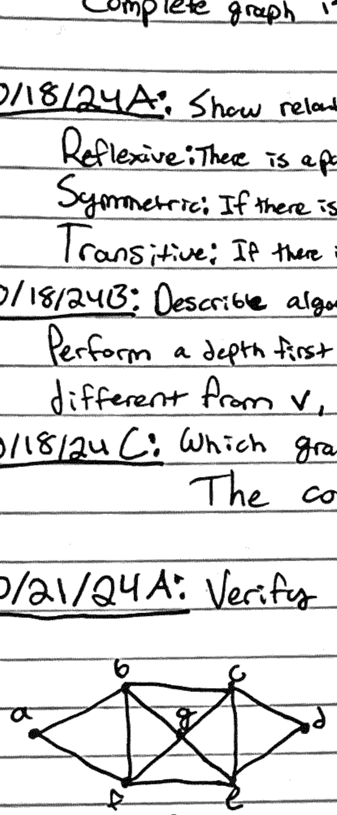
10/18/24B: Describe algorithm for proper 2-coloring → equal to determining if graph bipartite

Perform a depth first search, search when v brings in u, color u different from v, if monochromatic edge → not bipartite → cannot be 2-colored

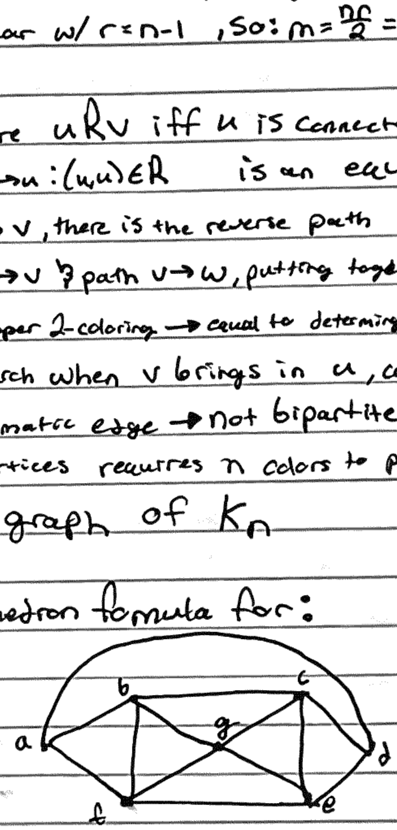
10/18/24C: Which graph on n vertices requires n colors to properly color

The complete graph of K_n

10/21/24A: Verify Euler's Polyhedron formula for:



G



H

G: $n=7$, $m=12$, $f=7 \to n-m+f=2 \to 7-12+7=2$ ✓

H: $n=7$, $m=13$, $f=8 \to n-m+f=2 \to 7-13+8=2$ ✓

PSN PG 10

10/21/24B: Using average degree apply induction to show every planar graph can be properly 6-colored

Proposition: every planar graph can be properly vertex 6-colored

Basis: Clearly true for 1 vertex, color the vertex any 6 colors

Induction: Assume true for all planar graphs w/ k vertices

Consider a graph w/ $k+1$ vertices: pick vertex v of degree at most 5,

Let G' be graph from G by deleting vertex v and all incident edges

then G' is planar and has k vertices $\rightarrow G'$ can be properly vertex 6-colored

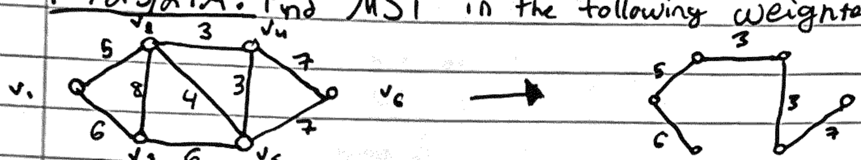
Color all vertices different from v the same as G' using

a different color from any of the vertices in the neighborhood (adjacent to v)

which can be done since there are at most 5 vertices in the neighborhood of v, so this is a construction of proper 6-coloring

of G, completing the induction step. QED

10/23/24A: Find MST in the following weighted graph:



Using Kruskal's for lowest weigh first $= 24$

10/23/24B: Prove corollary Eulerian Trail Theorem.

Suppose G is connected w/ exactly two vertices x & y of odd degree.

Construct G' from G by adding xy edge, By prev theorem,

G' contain Eulerian Circuit. Removing edge xy yields Eulerian Trail.

The converse of these steps are true aswell

10/25/24A: Obtain formula for #edges & diameter of H_k

$m = \frac{rn}{2}$, H_k: $r=k$, $n=2^k$ $\longrightarrow$ $m = \frac{k2^k}{2} = k2^{k-1}$ & $k = \log_2 n$ since they differ by one bit

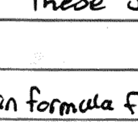
10/25/24B: Show that Dirac's ~~T~~ theorem is as strong as possible,

give counterexample of graph w/ every vertex has degree atleast $\frac{n}{2}-1$

a) Counter example where graph is not connected for $n=10$

b) find counter example for any n, where n is even

Take 2 K_5



And they will be disconnected

For n even, take 2 ~~K~~ $K_{n/2}$ & every vertex has degree atleast $\frac{n}{2}-1$

PSN PG11

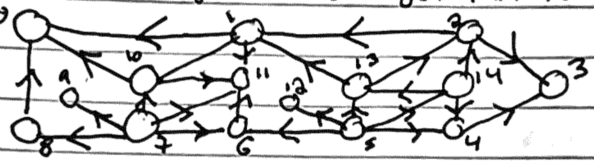
10/28/24A: Is a graph or digraph more general?

Since digraphs only differ in a directional component, it would lose a piece of information if transformed into a graph. A graph does not do this in reverse, making it more general

10/30/24A: Show a finite DAG must contain a Source & Sink.

Since digraph is acyclic, we cannot return to the same vertex, so we must reach a sink, reversing orientation means we meet Source

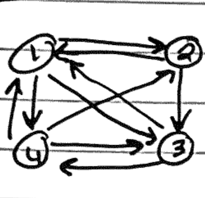
10/30/24B: Use straightforward algorithm to find topological sort for:



1. find v so all vertices in in-neighborhood have been visited
2. Insert v to end of list
3. Mark v as visited

5 7 8 9 12 14 13 2 1 4 3 10 0 6 11

11/1/24A: Give page rank equations, matrix equation, transpose & verify it is the matrix random walk



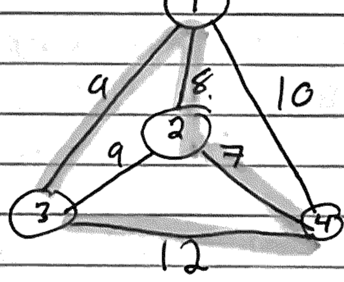
$$R_1 = \tfrac{1}{2}R_2 + \tfrac{1}{2}R_3 + \tfrac{1}{3}R_4$$
$$R_2 = \tfrac{1}{3}R_1 + \tfrac{1}{3}R_4$$
$$R_3 = \tfrac{1}{3}R_1 + \tfrac{1}{2}R_2 + \tfrac{1}{3}R_4$$
$$R_4 = \tfrac{1}{3}R_1 + \tfrac{1}{2}R_3$$

$$R = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 \end{pmatrix} R \quad \text{where } R = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$$

$$R = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}^T R$$

All rows involve fractions that sum to 1 so it is a random walk matrix on the web digraph.

11/4/24A: Solve The TSP for:



$$1 \xrightarrow{} 2 \xrightarrow{} 4 \xrightarrow{} 3 \xrightarrow{} 1$$
$$8 \quad 7 \quad 12 \quad 9$$

$$8 + 7 + 12 + 9 = 15 + 21 = 36$$

36 is the solution to the TSP

PSN PG12

11/4/24B: a) How many permutations of 7 elements: $7! = 5040$

b) How many seatings of 10 people at circular table : $9! = 362{,}880$

c) How many 5-permutations of an 8 element set: $P(8,5) = \frac{8!}{3!} = 6720$

d) How many subsets of size 5 of an 8 element set: $C(8,5) = \frac{P(8,5)}{5!} = \frac{6720}{5!} = 56$

11/6/24A: How many poker hands w/ two pairs (of dif. rank) are there?

$C[13,2] \times C[4,2] \times C[4,2] \times C[11,1] \times C[4,1]$

$78 \times 6 \times 6 \times 11 \times 4 = 123552$

11/6/24B: How many bridge hands have a void in two suits?

$C[26,13] \times C[4,2]$

11/6/24C: How many permutations of engineering are there?

e: 3, n: 3, g: 2, i: 2, r: 1 = 11 total

$P(11; 3,3,2,2,1) = \frac{11!}{3!3!2!2!1!}$

11/8/24A: Give comb. proof of Pascal's Identity: $C[n,k] = C[n-1,k] + C[n-1,k-1]$

Let A be set of all k subsets of $S = \{1, \dots, n\} \rightarrow |A| = C[n,k]$

X is subsets not containing 1 & Y contains 1 $\rightarrow |X| = C[n-1,k]$

$|Y| = C[n-1,k-1]$, $|A| = |X| + |Y| \longrightarrow C[n,k] = C[n-1,k] + C[n-1,k-1]$

11/8/24B: Show $C[n,0] - C[n,1] + \dots + (-1)^n C[n,n] = 0$

$$0 = (1-1)^n = \sum_{i=0}^{n} C[n,i]\, 1^{n-i} (-1)^i = \sum_{i=0}^{n} (-1)^i C[n,i]$$

PSN PG13

11/18/24A: Compute all derangements for $n=2,3,4$ ⅋
Compute the fraction of permutations that are derangements for $n=2,3,4$

Fixed → $\boxed{12}$, 21 $\frac{1}{2}$ derangements

$\boxed{123, 132, 213}$, 231, 312, 321 $\frac{1}{2}$ derangements

1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314,
2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421,
4123, 4132, 4213, 4231, 4312, 4321 $9/24 = 3/8$ are derangements

11/21/24A: P(card drawn at random, 52 card deck will be a king or a heart)

$E = K \cup H$, $P(K \cup H) = P(K) + P(H) - P(K \cap H)$
$= 4/52 + 13/52 - 1/52 = 4/13$

swap order

11/21/24B: Uniform prob dens. func. on standard deck of cards. Determine P(Ace) ⅋ P(Jack ∪ Queen ∪ King) Face cards

$P(Ace) = 4 \cdot 1/52 = 4/52 = 1/13$

$P(\text{Face card}) = 4 \cdot 3 \cdot 1/52 = 12/52 = 3/13$

11/22/24A: Consider fair die, $S = \{1,2,3,4,5,6\}$, $A = \{1,2,3\}$, $B = \{3,4,5,6\}$ ⅋ $C = \{2,3,5\}$ ⅋ which are independent

$A \cap B = \{3,5\}$ $A \cap C = \{3,5\}$ $B \cap C = \{3,5\}$

$P(A) = 3/6 = 1/2$, $P(B) = 4/6 = 2/3$, $P(C) = 3/6 = 1/2$

A ⅋ B are independent, A ⅋ C are not, B ⅋ C are

11/22/24B: Compute conditional probability that the sum of dice is 2 or 11 given that the first is 1, 6

$E = \{(1,1), (5,6), (6,5)\} \rightarrow P(E) = 3/36 = 1/12$

$P(F) = 12/36 = 1/3$ $P(E \cap F) = 2/36 = 1/18$

$[1/18] / [1/3] = 1/6$

11/22/24C: distribution for random var R mapping sum of two dice roll of falling on

$i+j$	2	3	4	5	6	7	8	9	10	11	12
$P(R=i+j)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

PSN PG 14

11/25/24A: Verify $p(s) = p^k q^{n-k}$ is pdf for S where
$p = q = 1/2$ (show Probabilities in S sum to 1)
w/16: $p(s) = \frac{1}{2}^k \frac{1}{2}^{4-k} = \frac{1^4}{2} = \frac{1}{16}$ so $\sum_{s \in S} p(s) = 16(1/16) = 1$

11/25/24B: Use $E(X) = \sum_x x P(X=x)$ to compute expectation
of the sum of two dice & compare
$2(1/36) + 3(2/36) + 4(3/36) + 5(4/36) + 6(5/36) + 7(6/36)$
$+ 8(5/36) + 9(4/36) + 10(3/36) + 11(2/36) + 12(1/36) = 7$
previously you could use most common 7 to see expectation

(whole Paragraph)

11/27/24A: Suppose search elem X is twice... Assume n even...
$p_i = p_1, \; p_1/2 \quad i = n/2 \ldots n$

$$A(n) = \sum_{i=1}^{n} i p_i = \sum_{i=1}^{n/2} i p_1 + \sum_{i=n/2+1}^{n} i (p_1/2)$$

$$= \frac{p_1}{2}\left(\frac{n/2\,(n/2+1)}{2} + \frac{n(n+1)}{2}\right) = \frac{p_1}{16}(5n^2 + 6n)$$

since sum must equal 1:

$$\left(\frac{n}{2}\right)p_1 + \left(\frac{n}{2}\right)p_1/2 = 1 \Rightarrow p_1\left(\frac{3}{4}\right)n = 1 \Rightarrow p_1 = \frac{4}{3n}$$

$$A(n) = \frac{p_1}{16}(5n^2 + 6n) = \frac{4}{48n}(5n^2 + 6n) = \frac{5}{12}n + \frac{1}{2}$$