Euler-Milstein Numerical Scheme for High-precision Stochastic Process Simulation of Quantum Trajectories

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Motivation

In quantum measurement, the system of interest is measured by interacting with a meter (or an environment). For example, one can measure the evolution of a qubit using photons via homodyne detection.

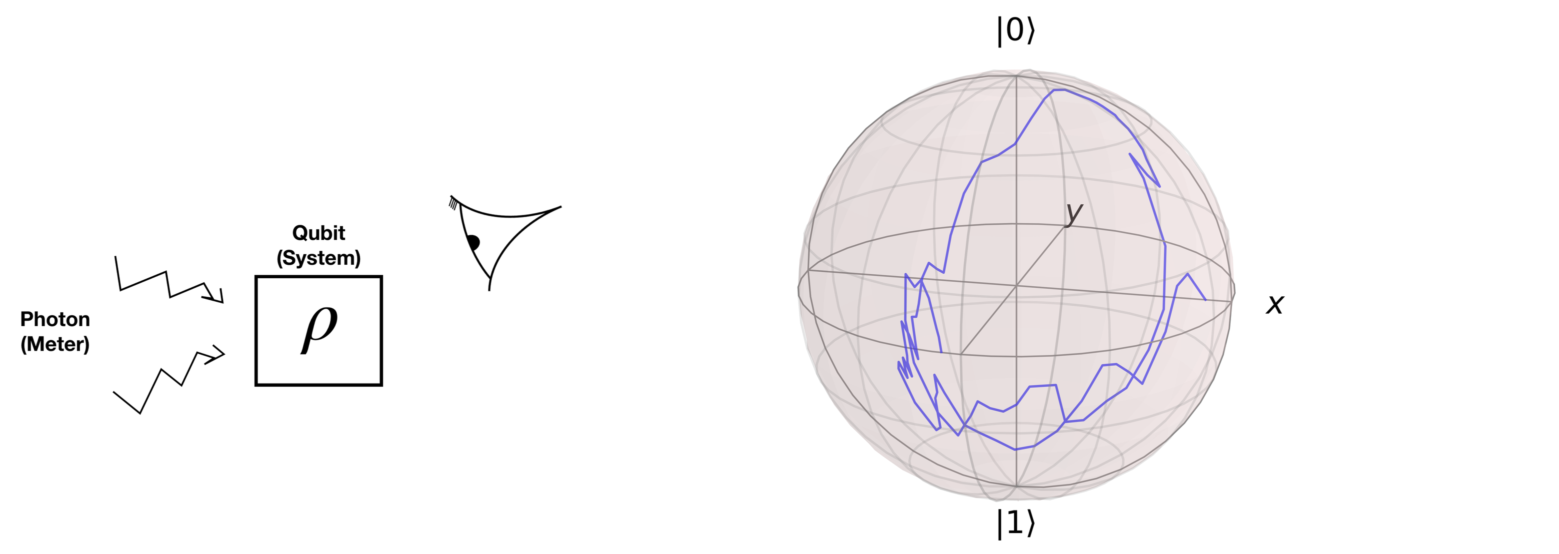

Figure 1: (Left) This figure displays the system and the meter. (Right) This figure displays the individual qubit trajectory for continuous measurement in Bloch sphere.

For continuous measurement, system's dynamics conditioned on a diffusive measurement records can be described by the Itô stochastic master equation, which works well in *small* dt. This is because the equation is of the first order of the Wiener increment dW. However, in real measurements, the time scale cannot be infinitesimal. We therefore use the Euler-Milstein approach [1] to calculate quantum trajectories and then apply the approach to real experimental data of superconducting transmon qubits. We show the comparison of individual trajectories between the Euler-Milstein approach and the original "quantum Bayesian" approach used in [2]. We also show averaged trajectories comparing among different approaches: Euler-Milstein, quantum Bayesian, and the analytical solution of the Lindblad master equation.

Quantum Trajectories

Consider an imperfect homodyne detection, the state update is given by (in the Itô's interpretation).

$$\frac{d\rho}{dt} = -i[\hat{H},\rho] + \sum_{\mu=1}^{N}\mathcal{D}[\hat{V}]\rho + \mathcal{D}[\hat{c}]\rho + \sqrt{\eta}\left(\mathcal{H}[\hat{c}]\rho\right)\frac{dW}{dt}, \tag{1}$$

where the superoperator $\mathcal{D}[\hat{A}]\rho = \hat{A}\rho\hat{A}^\dagger - \frac{1}{2}\left(\hat{A}^\dagger\hat{A}\rho + \rho\hat{A}^\dagger\hat{A}\right)$ describes the *decoherence channels*, while the superoperator $\mathcal{H}[\hat{A}]\rho = \hat{A}\rho + \rho\hat{A}^\dagger - \mathrm{Tr}\left[\hat{A}\rho + \rho\hat{A}^\dagger\right]\rho$ describes the *back-action*, and $\eta_r \in [0,1]$ is a detection efficiency. The Libdblad operator $\hat{c}$ and $\hat{V}$ represent the measured channel and unmeasured channel, respectively.

Euler-Milstein Scheme

Written in time-discrete form, we extend the stochastic equation to higher orders in dW of an Itô stochastic process $x_{n+1} = x_n + f(x_n)\Delta t + \sum_{r=1}^{m} g_r(x_n)\mathrm{d}W_{r,n}$ as

$$x_{n+1} = x_n + f(x_n)\Delta t + \sum_{r=1}^{m} g_r(x_n)\mathrm{d}W_{r,n} + \sum_{r,s=1}^{m}\frac{\partial g_s(x_n)}{\partial x}\cdot g_r(x_n)\left(\frac{\mathrm{d}W_{r,n}\mathrm{d}W_{s,n} - \delta_{r,s}\Delta t}{2}\right), \tag{2}$$

where $f(x_n)$ and $g_{r,s}(x_n)$ are arbitrary functions.

Rouchon et al. [1] transformed Eq. (1) using the Euler-Milstien Eq. (2) where x corresponds to ρ and $g_r(x)$ corresponds to $\sqrt{\eta_r}\mathcal{H}[\hat{c}_r]\rho$. Assuming that $[\hat{c}_r, \hat{c}_s] = 0$ and the following identity $\frac{\partial g_s}{\partial x}\cdot g_r = \frac{\partial g_r}{\partial x}\cdot g_s$, we apply Rouchon's technique to a quantum system. We then obtain a reduced form as (with three decoherence channels, where one is measured.)

$$\begin{aligned}\rho_{n+1} = \rho_n &+ \Big(-i[\hat{H},\rho_n] + \sum_{\mu=1}^{2}\mathcal{D}[\hat{V}_\mu]\rho_n + \mathcal{D}[\hat{c}]\rho_n\Big)\Delta t \\ &+ \sqrt{\eta}\mathcal{H}[\hat{c}]\rho_n\Delta W_n + \frac{\eta}{2}\Big(\hat{c}^2\rho_n + \rho_n(\hat{c}^\dagger)^2 + 2\hat{c}\rho_n\hat{c}^\dagger \\ &- \mathrm{Tr}\left[\hat{c}^2\rho_n + \rho_n(\hat{c}^\dagger)^2 + 2\hat{c}\rho_n\hat{c}^\dagger\right]\rho_n - 2\,\mathrm{Tr}\left[\hat{c}\rho_n + \rho_n\hat{c}^\dagger\right]\left(\hat{c}\rho_n + \rho_n\hat{c}^\dagger\right) \\ &+ 2\,\mathrm{Tr}\left[\hat{c}\rho_n + \rho_n\hat{c}^\dagger\right]\mathrm{Tr}\left[\hat{c}\rho_n + \rho_n\hat{c}^\dagger\right]\rho_n\Big)(\Delta W_n^2 - \Delta t).\end{aligned} \tag{3}$$

Analysis

We implement the Euler-Milstein's approach to calculate qubit trajectories from superconducting qubit experiments. The initial state is at $x_0 = 0.88$ and $z_0 = 0$ of the Bloch coordinates. The system Hamiltonian is $\hat{H} = \frac{\Omega}{2}\hat{\sigma}_y$. Using the Bloch coordinates $z = \mathrm{Tr}[\rho\hat{\sigma}_z]$ and $x = \mathrm{Tr}[\rho\hat{\sigma}_x]$. In this project, the Lindblad operators that we have used are $\hat{L} = 1/\sqrt{4\eta\tau}\hat{\sigma}_z$ since we consider a z-measurement, $\hat{V}_1 = 1/\sqrt{T_1}\hat{\sigma}_-$ from T_1 dephasing and $\hat{V}_2 = \sqrt{\gamma/2}\hat{\sigma}_z$ from T_2 dephasing. We have

$$z_{n+1} = z_n - \Omega x_n + \frac{z_n+1}{T_1}\Delta t + \frac{(1-z_n^2)}{\sqrt{\tau}}\Delta W_n + \frac{z_n(z_n^2-1)}{\tau}\Delta W_n^2 - \Delta t, \tag{4}$$

$$x_{n+1} = x_n + (\Omega z_n - \Gamma x_n)\Delta t - \frac{x_n z_n}{\sqrt{\tau}}\Delta W_n + \frac{x_n}{2\tau}(2z_n^2 - 1)(\Delta W_n^2 - \Delta t), \tag{5}$$

where $\Gamma = \frac{1}{2T_1} + \gamma + \frac{1}{2\eta\tau}$ and τ is a measurement time scale. Here, we assume that there is no spontaneous emission of a photon (or $T_1 \to \infty$).

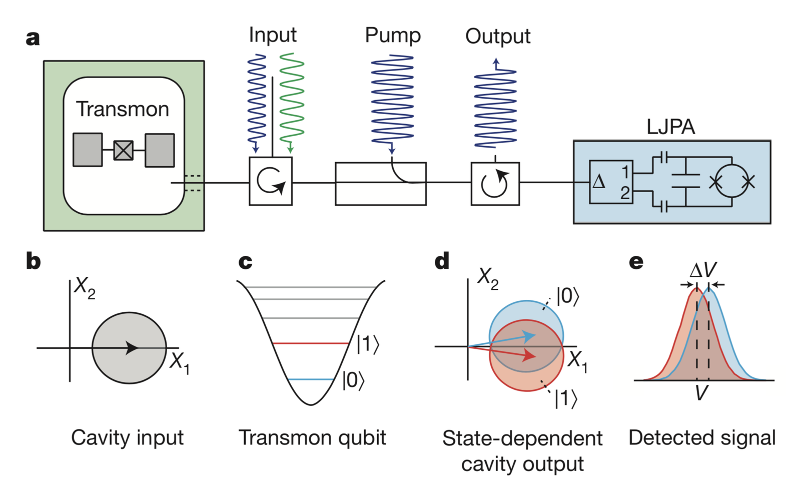

Figure 2: This figure displays the superconducting qubit experiment scheme [2].

Results from Numerical Simulations

Individual Trajectories

We generate an individual qubit trajectory from experimental records $r_n = z_n + \sqrt{\tau}\frac{\mathrm{d}W_n}{\mathrm{d}t}$ by using Rouchon's approach and the quantum Bayesian's approach [2] as in Figure 3. (Left).

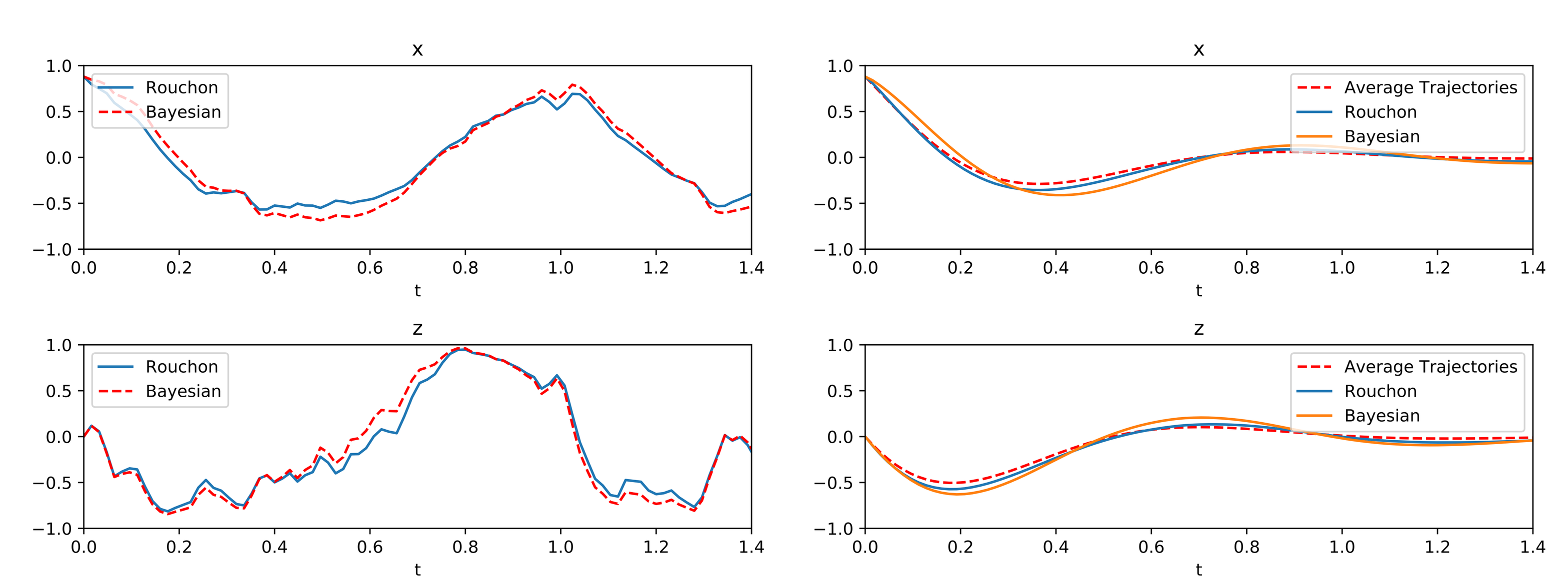

Figure 3: (Left) The comparison of an individual trajectories computed from Rouchon's approach and quantum Bayesian's approach using measurement records from the superconducting qubit experiment. Each record has 90 time steps with $\Delta t = 0.016\mu$s. (Right) This figure displays average trajectories which are calculated in 3 different approaches. By using $\Omega/2\pi = 1.08$ MHz, $\tau = 0.315271\mu$s, $\eta = 0.411932$, $\gamma = 2.26406$ MHz.

Averaged Trajectories

We cannot make a comparison using individual trajectories, therefore we show results for three averaged trajectories from the experimental data (150k trajectories) showing results for three different approaches: (a) Rouchon's approach, (b) quantum Bayesian, and (c) the analytical solution of the Lindblad master equation (Eqs. (4-5)) as in Figure 3. (Right).

Discussion

From the average trajectories in Figure 3, we can see that Rouchon's method gives an average closer to the analytic solution than the quantum Bayesian approach. However we also note that, in the experiments, there might be errors in the parameters, e.g. the qubit frequency, or T_1 and T_2 dephasing. Such errors can incorrectly give qubit trajectories. Therefore, in order to make a fair comparison between different approaches, we will need to do numerical simulations and analytical calculations, which will be in our future work [3].

References

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