Approaches for high-precision stochastic simulation of quantum trajectories

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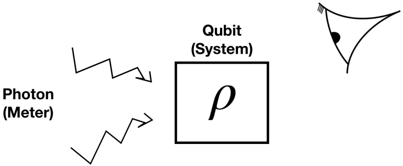
Motivation

In quantum measurement, the system of interest is measured by interacting with a meter (or an environment). For example, one can measure the evolution of a qubit using photons via homodyne detection.



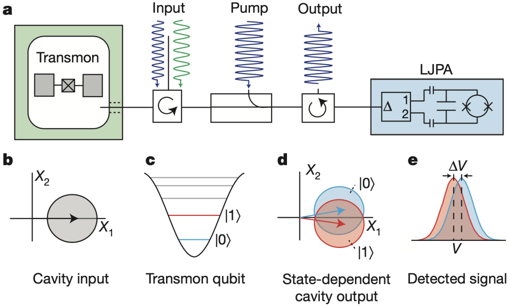


Figure: **(Left)** This figure displays the system and the meter. **(Right)** This figure displays the superconducting qubit experiment scheme† where we can construct the continuous records from the measurement signals.

† S. J. Weber *et al*, Nature **511**, 570 (2014).

Motivation (2)

In this work, we focus on quantum continuous (weak) measurements, which is a new concept different from the "projective" collapsing style we learn in textbooks.

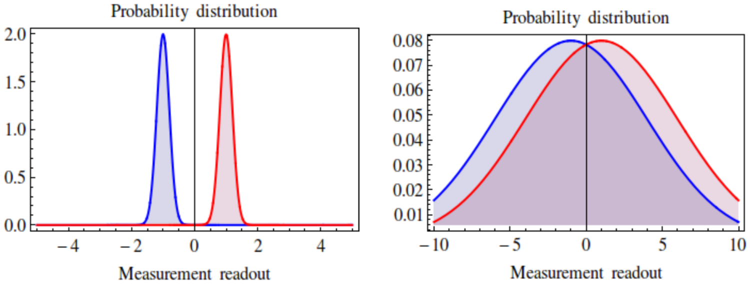


Figure: This figure[†] displays the probability distribution of the measurement readout of a qubit experiment (blue lines for $|0\rangle$ and red lines for $|1\rangle$). **(Left)** Strong (projective) measurement **(Right)** Weak measurement.

[†] A. Chantasri (*Stochastic Path Integral Formalism For Continuous Quantum Measurement* (2016))

Motivation (3)

The diffusive measurement records are used to simulate the “quantum trajectories” describing the conditioned system’s dynamics,

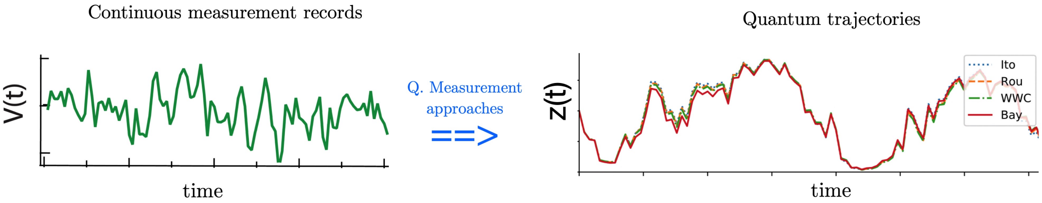


Figure: This figure† displays the idea of the quantum trajectory.

which can be described by the Itô stochastic master equation. Such equation works well in *small* dt. That is because the equation is of the first order of the Wiener increment dW.

†The figure is manipulated from, S. J. Weber *et al*, Nature **511**, 570 (2014).

Motivation (4)

- However, in real measurements, we cannot do measurement with infinitesimal time resolution $\mathrm{d}t$.

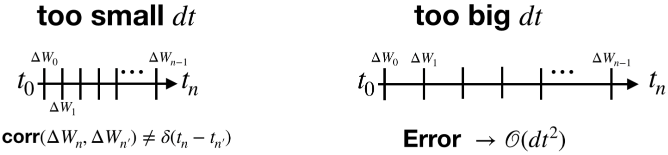


Figure: This figure displays the problem of time scale of experiment. Left: If we do measurement with a too small $\mathrm{d}t$, each step of the measurement record is not independent of each other (non-Markovian). Right: If we do measurement with a too big $\mathrm{d}t$, we get an error in $\mathcal{O}(\mathrm{d}t^2)$.

- There are several approaches proposed to reduce the error in quantum trajectories, which are (1) the Euler-Milstein approach, (2) the high-order completely positive map, and (3) the quantum Bayesian approach.
- In this work, we aim to make a comparison among the four approaches (including the usual Itô map).

Measurement Operation

- Coupling the system (ρ) with a meter ($\rho_{\mathrm{e}} = |e_0\rangle\langle e_0|$) and calling both as a combined system.
- Assuming that the combined system is a closed system.

For a conditioned evolution, the measurement operator is given by

$$\hat{M}_k = \langle e_k|\hat{U}(t+\mathrm{d}t, t)|e_0\rangle, \tag{1}$$

where $|e_0\rangle$ and $|e_k\rangle$ are the initial and "measured" states of the meter, and $\hat{U}(t+\mathrm{d}t, t)$ describes the evolution of the combined system.

For an unconditioned evolution, summing over all the meter's states, then the system's evolution is described by a Lindblad superoperator $\mathcal{L}$

$$e^{\mathcal{L}(t+\mathrm{d}t,t)}\rho(t) \propto \sum_k \hat{M}_k \rho(t) \hat{M}_k^\dagger. \tag{2}$$

[M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).]

Quantum Trajectory Approaches

In this work, we consider 4 different approaches, namely

- Itô map $\hat{M}_{\mathrm{I}}$,
- Euler-Milstein approach (Rou) $\hat{M}_{\mathrm{R}}$,
- high-order completely positive map (WWC) $\hat{M}_{\mathrm{W}}$,
- quantum Bayesian approach $\hat{M}_{\mathrm{B}}$.

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For simplicity, we review the following approaches with

- a single measured channel $\hat{L}$,
- considering in no unitary evolution case ($\hat{H} = 0$),
- assuming that the measurement efficiency (η) is perfect (no extra dephasing, i.e., $\eta = 1$).

Quantum Trajectory Approaches (2)

- **Itô map** : This map is derived from Itô stochastic master equation[1], giving

$$\hat{M}_{\mathrm{I}} = \hat{1} + \left(\hat{L}y(t) - \frac{1}{2}\hat{L}^{\dagger}\hat{L}\right)\mathrm{d}t, \tag{3}$$

where $y(t)\mathrm{d}t = \mathrm{Tr}\left(\hat{L}\rho(t) + \rho(t)\hat{L}^{\dagger}\right)\mathrm{d}t + \mathrm{d}W$ is the measurement readout and $\mathrm{d}W$ is the Wiener process.

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- **Euler-Milstein** : This approach proposed by Rouchon and Ralph[2]. The measurement operator is given by

$$\hat{M}_{\mathrm{R}} = \hat{M}_{\mathrm{I}} + \frac{1}{2}\hat{L}^{2}(y(t)^{2}\mathrm{d}t^{2} - \mathrm{d}t). \tag{4}$$

[1] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2010),

[2] P. Rouchon and J. F. Ralph, Phys. Rev. A **91**, 012118 (2015).

Quantum Trajectory Approaches (3)

- **High-order completely positive map** : This approach proposed by myself and co-authors[1]. The measurement operator is given by

$$\hat{M}_{\mathrm{W}} = \hat{M}_{\mathrm{R}} + \frac{1}{8}\left(\hat{L}^\dagger \hat{L}\right)^2 \mathrm{d}t^2 - \frac{1}{4}\, y(t)\left[\hat{L}^\dagger \hat{L}^2 + \hat{L}\hat{L}^\dagger \hat{L}\right]\mathrm{d}t^2. \tag{5}$$

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- **Quantum Bayesian approach** : This particular approach was introduced for the qubit z-measurement[2]. The measurement operator is given by

$$\hat{M}_{\mathrm{B}} \propto \exp\Bigg\{\Bigg[-\frac{\mathrm{d}t}{4\tau}\big(\sqrt{\tau}y(t) - \hat{\sigma}_z\big)^2\Bigg]\Bigg\}, \tag{6}$$

where we have assumed $\hat{L} = \sqrt{1/(4\tau)}\hat{\sigma}_z$ and τ is the characteristic measurement time.

[1] N. Wonglakhon, H. M. Wiseman, A. Chantasri, "Completely positive maps for higher-order unraveling of Lindblad master equations," In Preparation (2020), [2] A. N. Korotkov, Phys. Rev. B **60**, 5737(1999).

How Can We Use the Map to Construct Quantum Trajectories?

The state update of the system after doing measurement is given by

$$\rho(t+\mathrm{d}t) \quad = \quad \frac{\hat{M}_k \rho(t) \hat{M}_k^\dagger}{\mathrm{Tr}\left[\hat{M}_k \rho(t) \hat{M}_k^\dagger\right]}, \tag{7}$$

where $\mathrm{Tr}\left[\hat{M}_k \rho(t) \hat{M}_k^\dagger\right]$ is the probability of getting the measured outcome k, and $\hat{M}_k$ represents any of the operators $\hat{M}_{\mathrm{I}}$, $\hat{M}_{\mathrm{R}}$, $\hat{M}_{\mathrm{W}}$ or $\hat{M}_{\mathrm{B}}$.

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Here, the Bloch coordinates are defined as $x \equiv \mathrm{Tr}[\rho\hat{\sigma}_x]$, $z \equiv \mathrm{Tr}[\rho\hat{\sigma}_z]$.

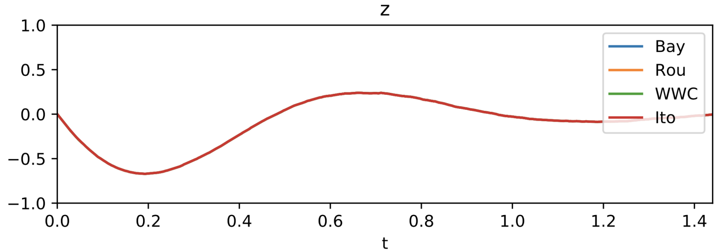
Numerical Simulation (for qubit z-measurement simulations)

The qubit rotates with the Hamiltonian $\hat{H} = \Omega/2\hat{\sigma}_y$ and the Lindblad operators are $\hat{L} = \sqrt{\Gamma}\hat{\sigma}_z$, $\hat{V} = \sqrt{\gamma}\hat{\sigma}_z$.

Initialise the qubit at $\rho(0) = 1/2(\hat{1} + \hat{\sigma}_x)$

Generate *true* trajectories with very small dt (we use d$t = 4 \times 10^{-4}$ μs) by using the Itô map. Then, collect the true measurement records.

Check the Lindblad average equation (get the same results)



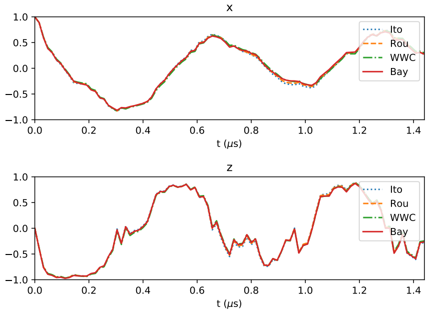
Numerical Simulation (for qubit z-measurement simulations)

In real measurements, too small time step is not possible, we then...

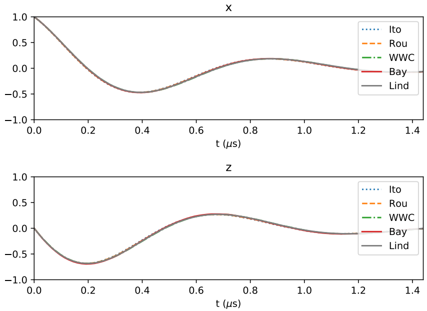
Rescale measurement records from dt to $\Delta t = 0.016\ \mu$s (by coarse graining method), obtaining coarse grained records.

↓

Implement the four approaches to simulate quantum trajectories.



→ average →



Trace distance calculations

In this work, we make the comparison among four approaches via the trace distance.

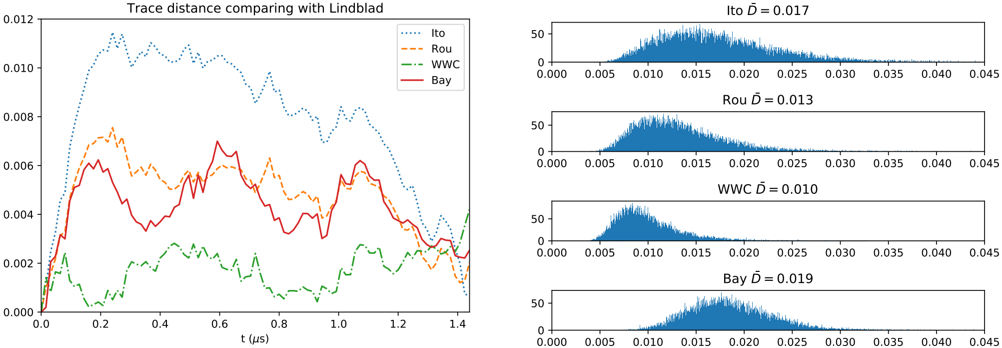


Figure: **(Left)** The changes in time of the trace distance between the averaged trajectory and the numerical Lindblad evolution. **(Right)** The trace distance of individual trajectories comparing true trajectories.

Conclusion

- Continuous measurements : Measurement records $\xrightarrow[\text{operators}]{\text{measurement}}$ Quantum trajectories (QTs)
- Generate true trajectories $\rightarrow$ Construct true measurement records
- Rescale true records $\rightarrow$ Implement four approaches to simulate QTs
- The trace distance calculation results show that the high-order completely positive map $\hat{M}_{\mathrm{W}}$ is the most accurate method.
- We can also confirm with some confidence that the time step $\Delta t = 0.016\ \mu s$ is a good time resolution (the least accurate approach, Itô map, still gives reasonably small errors).

References

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Q&A