Completely positive maps for higher-order unraveling of Lindblad master equations

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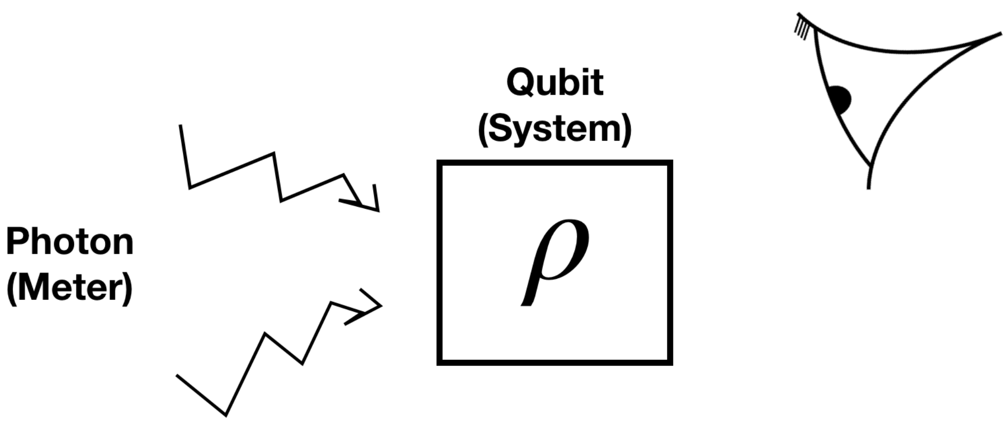
N. Wonglakhon[1*], H. M. Wiseman[2], A. Chantasri[1,2]

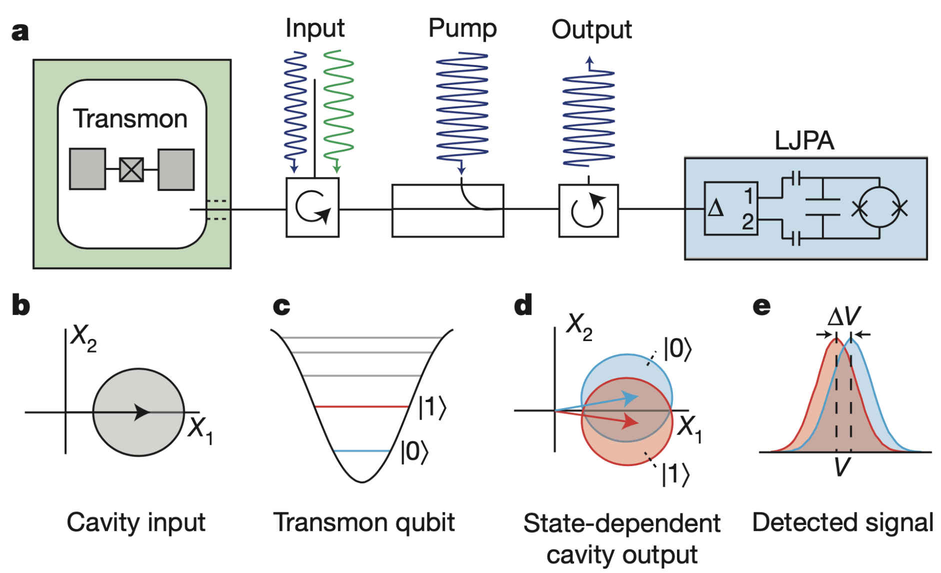
[1]Optical and Quantum Physics Laboratory, Department of Physics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand
[2]Centre for Quantum Dynamics, Griffith University, Nathan, Queensland 4111, Australia
[*]nat.wonglakhon@gmail.com

Introduction

- In quantum measurement, the system of interest is measured by interacting with a bath (meter). For example, one can measure the evolution of a qubit using photons via homodyne detection.



The quantum system is being measured.



The superconducting qubit experiment scheme.
[Source: S. J. Weber *et al*, Nature **511**, 570 (2014)]

- In this work, we focus on quantum continuous (weak) measurements.

Measurement Operation

- Coupling the system (ρ) with a bath $(\rho_{\mathrm{e}} = |\mathrm{e}_0\rangle\langle\mathrm{e}_0|)$ and calling them as a combined system, treated as a closed system under the Born-Markov assumption where if the bath is not measured, the system's dynamics can be described by the Lindblad master equation.

- If the bath is measured, we obtain a conditioned evolution. The measurement operator is given by

$$\hat{K}_{\mathrm{r}} = \langle\mathrm{e}_{\mathrm{r}}|\hat{U}(t+\Delta t, t)|\mathrm{e}_0\rangle, \tag{1}$$

where $|\mathrm{e}_0\rangle$ and $|\mathrm{e}_{\mathrm{r}}\rangle$ are the initial and measured eigenstates of the bath's observable, and $\hat{U}(t+\Delta t, t)$ describes the evolution of the combined system.

- The quantum trajectory **conditioned on measurement records** can be calculated via

$$\rho_{\mathrm{r}}(t+\Delta t) = \frac{\hat{K}_{\mathrm{r}}\rho(t)\hat{K}_{\mathrm{r}}^{\dagger}}{\mathrm{Tr}[\hat{K}_{\mathrm{r}}\rho(t)\hat{K}_{\mathrm{r}}^{\dagger}]}. \tag{2}$$

Measurement Operation

Properties of a measurement operator

- We use k to indicate the satisfying order in Δt for each property.

- CP: Complete positivity,

$$\boxed{\mathcal{C}_k = \sum_{\mathrm{r}} \hat{K}_{\mathrm{r}}^{\dagger} \hat{K}_{\mathrm{r}} = \hat{1} + \mathcal{O}(\Delta t^{k+1}). \qquad (3)}$$

- Unraveling Lindblad master equation, $\mathcal{U}_{\infty} = \sum_{\mathrm{r}} \hat{K}_{\mathrm{r}} \rho(t) \hat{K}_{\mathrm{r}}^{\dagger} = e^{\Delta t \mathcal{L} \bullet} \rho(t),$

  or the expansion

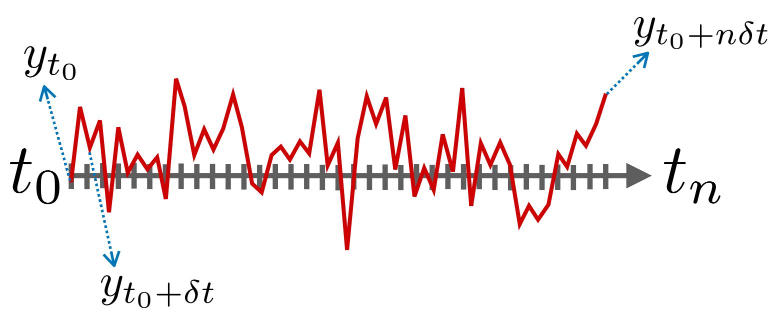
$$\boxed{\mathcal{U}_k = \left[\hat{1} + \sum_{k=1}^{\infty} \frac{(\Delta t \mathcal{L} \bullet)^k}{k!} \right] \rho(t), \qquad (4)}$$

  where the superoperator $\mathcal{L}$ defined as

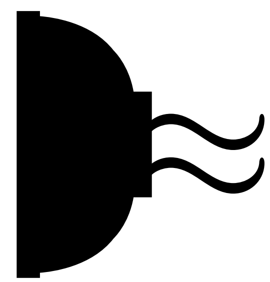
$$\mathcal{L} \bullet = -i[\hat{H}, \bullet] + \sum_j \mathcal{D}[\hat{c}_j] \bullet .$$

Time resolution

"True" (but unknown) records with close-to-infinitesimal δt

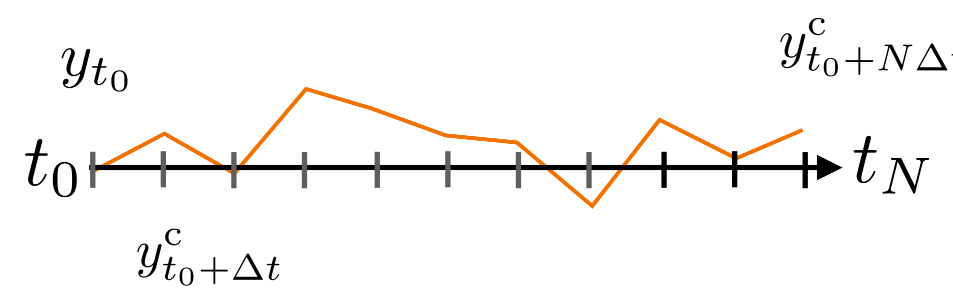


We do not have access to these records, but we can use the quantum theory to simulate.



Measurement with finite time resolution

What happens in experiments: Coarse-grained records with finite Δt



These records can be obtained from the actual measurement.

However, in practice, there are restrictions on the time resolution:

- The Markov assumption will be broken if the measurement is done with "too small" time resolution Δt (smaller than system-bath correlation time).

- On the other hand, estimation errors could be accumulated hugely if the measurement is done with "too large" time resolution Δt.

Goal

- Several existing approaches have been proposed to estimate the quantum trajectory.
- However, those existing approaches are not enough and can reach large errors in the measurement with finite Δt (not satisfy the properties to higher order in Δt).
- We then propose a new map, the high-order completely positive map, which satisfies the following properties to higher order in Δt,
  - the completeness relation $(\mathcal{C}_k)$,
  - unraveling the Lindblad master equation $(\mathcal{U}_k)$,

    where k is the order in Δt of satisfying the two properties.

Quantum Trajectory Approaches

In this work, we consider 4 different approaches, namely

- Ito map (1993) $\hat{M}_{\rm I}$,
- Rouchon-Ralph (2015) map $\hat{M}_{\rm R}$,
- Guevara-Wiseman (2020) map $\hat{M}_{\rm G}$,
- high-order completely positive map (WWC) $\hat{M}_{\rm W}$.

For simplicity, we review the following approaches with

- single measured channel $\hat{c}$,
- considering in no unitary evolution case $(\hat{H} = 0)$,
- assuming that the measurement efficiency (η) is unity (no extra dephasing).

Quantum Trajectory Approaches

Ito map

- This map is the simplest approach, derived from Ito stochastic master equation, giving

$$\hat{M}_{\mathrm{I}}(\hat{c}) = \hat{1} - \tfrac{1}{2}\hat{c}^\dagger\hat{c}\Delta t + \hat{c}y_t\Delta t, \tag{5}$$

where y_t is the measurement record.

- We show that the properties of this map yield $\mathcal{C}_1^{\mathrm{I}}$ and $\mathcal{U}_1^{\mathrm{I}}$.

Rouchon-Ralph map

- This map is derived from Euler-Milstein scheme, where the map is given by

$$\hat{M}_{\mathrm{R}}(\hat{c}) = \hat{M}_{\mathrm{I}} + \tfrac{1}{2}\hat{c}^2(y^2\Delta t^2 - \Delta t). \tag{6}$$

- We show that the properties of this map yield $\mathcal{C}_1^{\mathrm{R}}$ and $\mathcal{U}_1^{\mathrm{R}}$.

Quantum Trajectory Approaches

Guevara-Wiseman map

- This map is proposed to remove the non-zero terms of CP to high-order in Δt, giving

$$\hat{M}_{\mathrm{G}}(\hat{c}) = \hat{M}_{\mathrm{I}} - \frac{1}{8}(\hat{c}^\dagger \hat{c})^2 \Delta t^2. \tag{7}$$

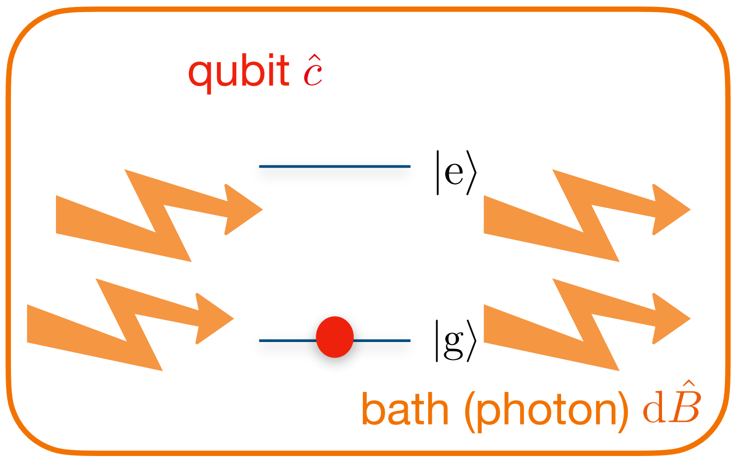
- We show that the properties of this map yield $\mathcal{C}_2^{\mathrm{G}}$ and $\mathcal{U}_1^{\mathrm{G}}$.

Quantum Trajectory Approaches

High-order completely positive map

- We derived a new map from the original definition of the measurement operator in Eq. (1) for the qubit (bosonic) bath coupling.

$$\hat{K}_{\mathrm{r}} = \langle \mathrm{e_r} | \hat{U}(t+\Delta t, t) | \mathrm{e_0} \rangle \qquad (1)$$



$$\hat{U}(t+\Delta t, t) = \exp\left[\hat{c}\,\mathrm{d}\hat{B}^{\dagger} - \hat{c}^{\dagger}\mathrm{d}\hat{B}\right]$$

Where $\hat{U}(t+\Delta t, t)$ is the unitary evolution of the system and bath.

- After projecting to the homodyne eigenstate, giving

$$\hat{M}_{\mathrm{W}}(\hat{c}) = \hat{M}_{\mathrm{R}} + \tfrac{1}{8}(\hat{c}^{\dagger}\hat{c})^2\Delta t^2 - \tfrac{1}{4}y_t\Delta t^2(\hat{c}^{\dagger}\hat{c}^2 + \hat{c}\hat{c}^{\dagger}\hat{c}). \qquad (8)$$

- We show that the properties of this map yield $\mathcal{C}_2^{\mathrm{W}}$ and $\mathcal{U}_2^{\mathrm{W}}$.

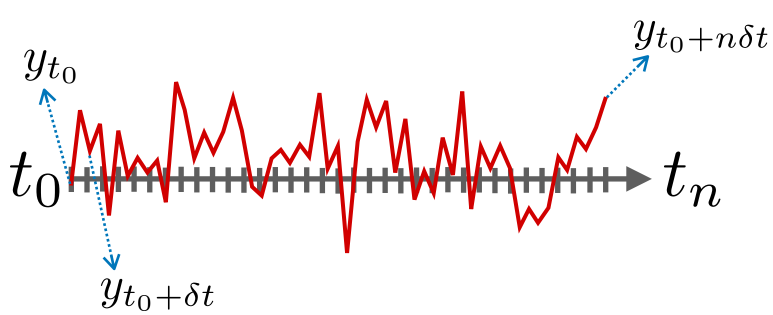
Quantum Trajectory Approaches

Comparison among existing maps

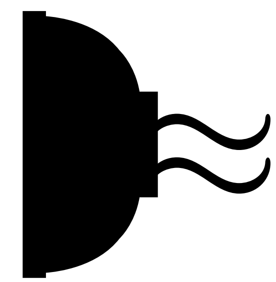
<table>
<tr><th>Properties/ maps</th><th>Ito</th><th>Rouchon-Ralph</th><th>Guevara-Wiseman</th><th>High-order completely positive map</th></tr>
<tr><td>$\mathcal{C}_k$</td><td colspan="2">$k = 1;$
$\hat{1} + \mathcal{O}(\Delta t^2)$</td><td colspan="2">$k = 2;$
$\hat{1} + \mathcal{O}(\Delta t^3)$</td></tr>
<tr><td>$\mathcal{U}_k$</td><td colspan="3">$k = 1;$
$\rho + \mathcal{L}\rho\Delta t + \mathcal{O}(\Delta t^2)$</td><td>$k = 2;$
$\rho + \mathcal{L}\rho\Delta t + \frac{1}{2}\mathcal{L}^2\rho\Delta t^2 + \mathcal{O}(\Delta t^3)$</td></tr>
</table>

Time resolution in Simulation

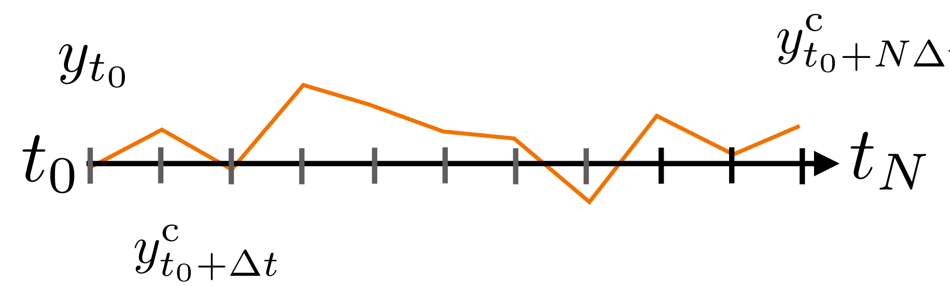
Simulated “true” records with close-to-infinitesimal δt



We do not have access to these records, but we can use the quantum theory to simulate.



What happens in experiments: Coarse-grained records with Δt



These records can be obtained from the actual measurement.

- True trajectories using $\delta t = 6.3 \times 10^{-4}\ T_\gamma.$

- Coarse-grain record to $\Delta t = 0.025\ T_\gamma.$

Simulations

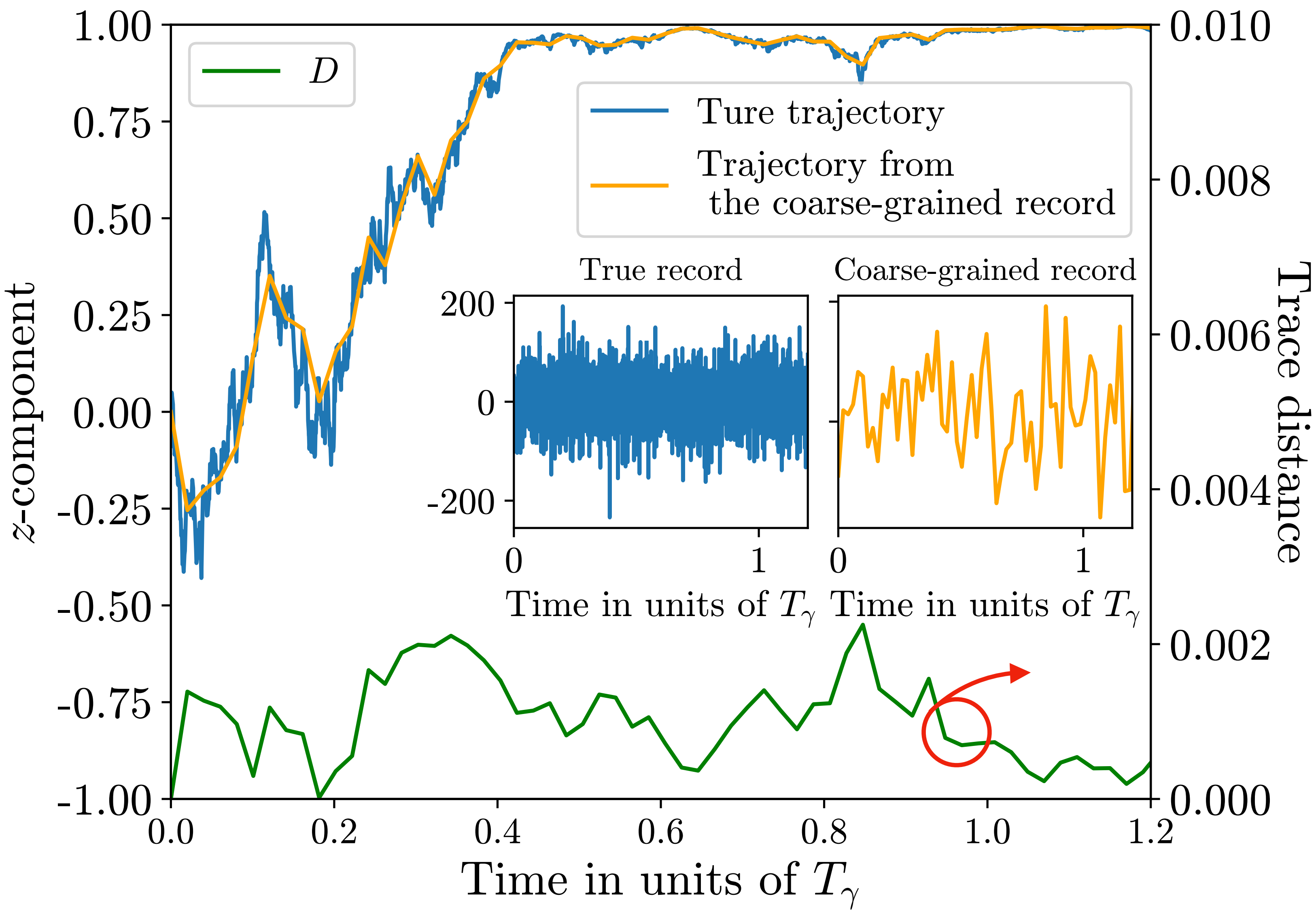
- We will use the coarse-grained records to estimate the quantum trajectory via the maps.

- The comparison among the maps will be calculated by the Trace Distance D defined as

$$D(\rho_j, \rho_k) = \frac{1}{2}\mathrm{Tr}|\rho_j - \rho_k|.$$

Simulation

Generating measurement records



- True trajectories using $\delta t = 6.3 \times 10^{-4}\ T_\gamma$.
- Coarse-grain record to $\Delta t = 0.025\ T_\gamma$.

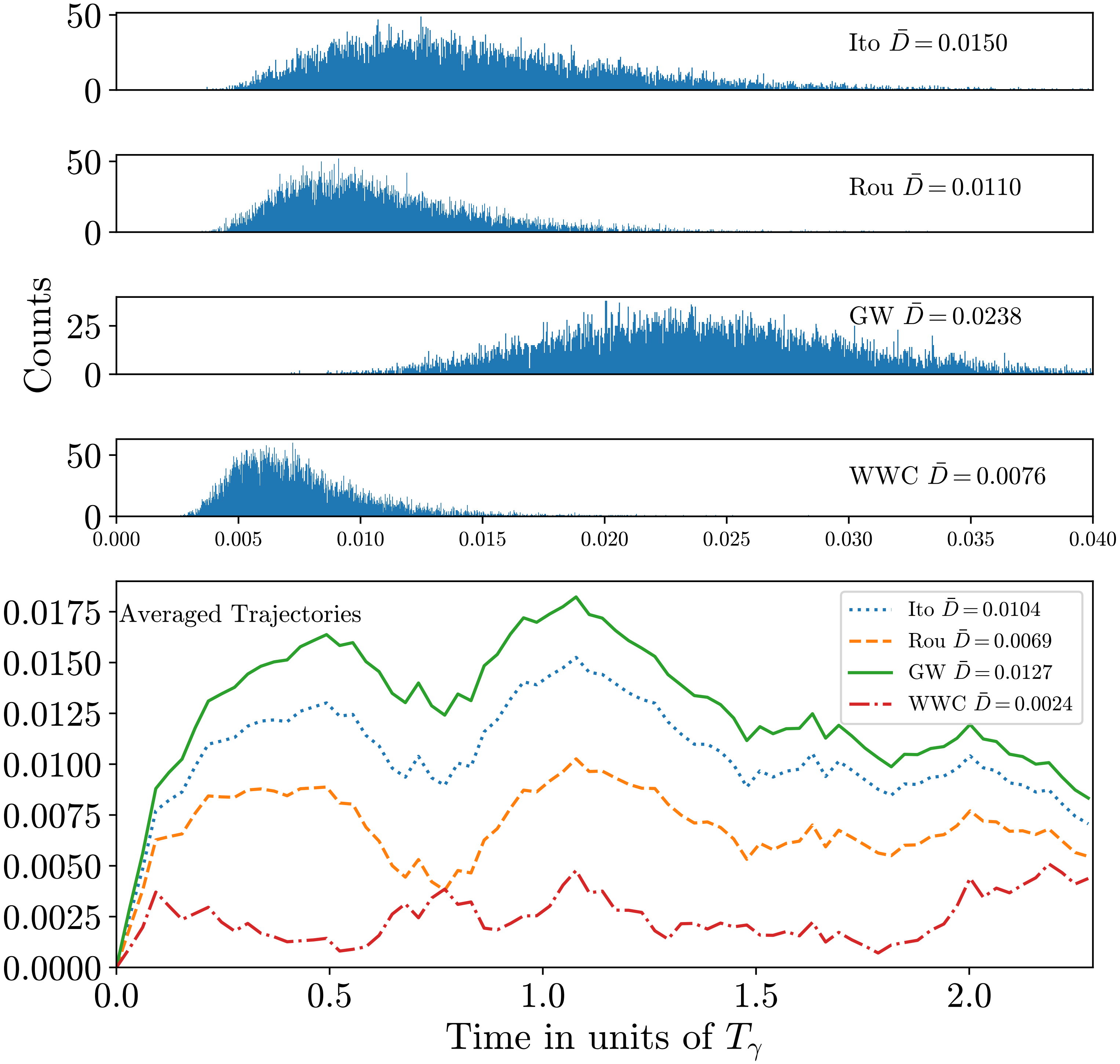
Setup

- The simulations are done for a practical case, adding the system's Hamiltonian and extra dephasing.

- $\hat{H} = (\Omega/2)\hat{\sigma}_y,$
- $\hat{c} = \sqrt{\gamma/2}\hat{\sigma}_z,$ where Ω is the Rabi frequency, γ is the coupling rate, and Γ is the dephasing rate.
- $\hat{V} = \sqrt{\Gamma/2}\hat{\sigma}_z,$

Simulation

Qubit z-measurement simulation



- The histograms show the trace distance between the true trajectories and the ones from each map.
- The bottom figure shows the trace distance of the averaged trajectories compared to the Lindblad evolution.

Conclusion

- Continuous measurements :
  - Measurement records $\xrightarrow{\text{maps}}$ Quantum trajectories operators (QTs)
- We use the CP and the unraveling Lindblad master equation as criteria of measurement operators.
- The existing maps do not satisfy the criteria to high-order in Δt.
- We proposed a new map from the original definition of the measurement operator for the qubit bath coupling.
- Our proposed map outperforms other maps both analytics and simulation.