On the distribution of impulse-response functions in macroeconomic shocks

Gabriel Montes-Rojas
IIEP-BAIRES-Universidad de Buenos Aires and CONICET

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- An important way to summarize the dynamics of macroeconomic data is to make use of a vector autoregressive (VAR) model. The VAR approach provides statistical tools for data description, forecasting, and structural inference to study rich dynamics in multivariate time-series models.
- Nevertheless, the use of a constant-coefficient model as representative of time-series models may not be adequate. These models cannot appropriately account for the presence of asymmetric and heterogeneous dynamic responses.
- Of particular interest is the asymmetric business cycle dynamics of economic variables, as the occurrence of asymmetries may call into question the usefulness of models with time invariant structures as means of modeling such series.
- Alternatives: nonlinear models, regimes, structural breaks, multivariate volatility models...
- ...or **quantile regression**.

- Quantile regression (QR) is a statistical method for estimating models of conditional quantile functions. This method offers a systematic strategy for examining how covariates influence the location, scale, and shape of the entire response distribution, thereby exposing a variety of heterogeneity in response dynamics.
- Koenker and Xiao (2006) QAR estimator applies QR models in time-series. Galvao, Montes-Rojas, and Park (2013) interpret the QR time-series framework as modeling the business cycle, where high (low) conditional realizations of a distributed lag model correspond to high (low) quantiles.
  Ex. AR(1) model:

$$E[Y_t|Y_{t-1}] = \alpha + \beta Y_{t-1}$$

$$vs.$$

$$Q_{Y_t}[\tau|Y_{t-1}] = \alpha(\tau) + \beta(\tau)Y_{t-1},\ \tau \in (0,1)$$

  where $Q_{Y_t}[\tau|Y_{t-1}]$ is the conditional quantile of $Y_t|Y_{t-1}$ (i.e., $F^{-1}_{Y_t}(\tau|Y_{t-1})$ for continuous cdf).
- Forecasting expected value vs. the full distribution (through quantiles).

- It is not possible to reproduce all "desirable properties" of scalar quantile regression in higher dimensions, so various proposals focus on achieving different sets of properties.
- Koenker (2005): "search for a satisfactory notion of multivariate quantiles has become something of a quest for the statistical holy grail in recent years."
- Consider a univariate random variable Y with domain in $\mathcal{Y} \subseteq \mathbb{R}$ and distribution function $F_Y(y) := P(Y \leq y)$. Then the τth-quantile for $\tau \in (0,1)$ is defined as $Q_Y(\tau) := inf\{y \in \mathcal{Y} : \tau \leq F_Y(y)\}$. Note that if $F_Y(.)$ is continuous then $Q_Y(\tau) = F^{-1}(\tau)$.
- However, for m-variate random variable Y with domain in $\mathcal{Y} \subseteq \mathbb{R}^m$, $inf\{y \in \mathcal{Y} : \tau \leq F_Y(y)\}$ is not unique.

Directional quantiles

- Hallin, Paindaveine, and Šiman (2010) propose to analyze the distributional and quantile features of multivariate response variables using the directional quantiles notion of Chaudhuri (1996), Koltchinskii (1997), Wei (2008) and others. Further work by Paindaveine and Šiman (2011, 2012) and Fraiman and Pateiro-López (2012). Multivariate quantile analysis should be endowed with a magnitude and a direction.
- Carlier, Chernozhukov, and Galichon (2016) and Chernozhukov, Galichon, Hallin, and Henry (2015) propose a vector quantile regression (linear) model that produces a monotone map, in the sense of being a gradient of convex function.
- Montes-Rojas (2017, Journal of Multivariate Analysis) builds on directional quantiles and consider a model in which the orthonormal basis is fixed, i.e. a set of directions orthogonal to each other that span the domain of the dependent variable.
- The reduced form directional quantiles are defined as a fixed point of a system of directional quantiles.
- The solution maps $\mathcal{X} \times (0,1)^m \mapsto \mathcal{Y}$.

My contribution

- Use directional QR to construct VARQ model for multivariate VAR.
- Construct and discuss forecasting procedures for the multivariate system.
- Introduce the idea of **quantile paths**, i.e., forecasting for different quantile configurations.
- Construct quantile impulse response functions (QIRFs).
- Distribution of QIRFs.
- Empirical application: Evaluate the effect of monetary policy (shock in interest rate) on output and inflation (U.S.).

The model

- Consider a $m-$dimensional process $\boldsymbol{Y}_t = (Y_{1t}, ..., Y_{mt})'$ with domain in $\mathcal{Y} \subseteq \mathbb{R}^m$. (endogeneous variables)
- Consider a $k-$dimensional process $\boldsymbol{X}_t$ with domain in $\mathcal{X} \subseteq \mathbb{R}^k$. (explanatory/control variables)
- Of particular interest is the case of the covariates generated by the σ-field generated by $\{\boldsymbol{Y}_s, s \leq t\}$ and all other information available at time t, denoted by $\mathcal{F}_t$. For that case the model becomes a vector autoregressive quantile (VARQ) model. (Montes-Rojas, 2019, Journal of Time Series Analysis)
- For an autoregressive model of $p-$order then $\boldsymbol{X}_{t-1} = [\boldsymbol{Y}'_{t-1}, \boldsymbol{Y}'_{t-2}, ..., \boldsymbol{Y}'_{t-p}]'$ and $k = m \times p$.

- Quantiles are analyzed in terms of a quantile magnitude and a direction.
- Define $\boldsymbol{\tau} = (\tau_1, \tau_2, ..., \tau_m) \in (0, 1)^m$ be a collection of quantile indexes.
- $\boldsymbol{\tau}$ factorizes into $\boldsymbol{\tau} \equiv \tau \boldsymbol{v}$ where $\tau = \|\boldsymbol{\tau}\| \in (0, 1)$ (magnitude) and $\boldsymbol{v} \in \mathcal{V}^{m-1} \equiv \{\boldsymbol{v} \in \mathbb{R}^m : \|\boldsymbol{v}\| = 1\}$ (direction).
- τ represents a scalar quantile index;
- $\boldsymbol{v}$ is a $m - 1$-directional vector;

Directional quantiles

- Let the vector $\boldsymbol{\tau}$ be an index on the open unit ball in $\mathbb{R}^m$ (deprived of the origin) $\mathcal{T}^m \equiv \{\boldsymbol{\tau} \in \mathbb{R}^m : 0 < \|\boldsymbol{\tau}\| < 1\}$. Our interest lies in defining and estimating

$$Q_{\boldsymbol{Y}_t|\boldsymbol{X}_t}(\boldsymbol{\tau}|\boldsymbol{X}_t) = \boldsymbol{B}(\boldsymbol{\tau})\boldsymbol{X}_t + \boldsymbol{A}(\boldsymbol{\tau}),$$

  where $\boldsymbol{B}(\boldsymbol{\tau})$ is a $m \times k$ matrix of coefficients, $\boldsymbol{A}(\boldsymbol{\tau})$ is a $m \times 1$ vector of coefficients. Let $\boldsymbol{B}(\boldsymbol{\tau}) \equiv [\boldsymbol{B}_1(\boldsymbol{\tau})', \boldsymbol{B}_2(\boldsymbol{\tau})', ..., \boldsymbol{B}_m(\boldsymbol{\tau})']'$ where $\boldsymbol{B}_j(\boldsymbol{\tau})$, $j = 1, 2, ..., m$, are the corresponding $1 \times k$ vector of coefficients of the jth element in $\boldsymbol{Y}$.
- Q is a map $\mathcal{X} \times \mathcal{T}^m \mapsto \mathcal{Y}$ and corresponds to our proposed definition of multivariate quantiles, which we will be defined as vector directional quantiles (VDQ).

VARQ

- Define the **univariate** QR models for $j = 1, \ldots, m$

$$q_j(\tau_j | \boldsymbol{x}_{t-1}, \boldsymbol{y}_{-jt}) := Q_{Y_{jt}}(\tau_j | \boldsymbol{x}_{t-1}, \boldsymbol{y}_{-jt}) = \boldsymbol{c}_j(\tau_j)^\top \boldsymbol{y}_{-jt} + \boldsymbol{b}_j(\tau_j)^\top \boldsymbol{x}_{t-1} + a_j(\tau_j)$$

  [Note: This corresponds to a particular direction in the space $\mathcal{Y}$.]
- In order to construct the VARQ model define $Q_{\boldsymbol{Y}_t}(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) := \{q_1(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}), \ldots, q_m(\boldsymbol{\tau}|\boldsymbol{x}_{t-1})\}^\top$ from the system of equations below:

$$\begin{cases} q_1(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) & := \boldsymbol{c}_1(\tau_1)^\top q_{-1}(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) + \boldsymbol{b}_1(\tau_1)^\top \boldsymbol{x}_{t-1} + a_1(\tau_1) \\ \vdots & := \vdots \\ q_m(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) & := \boldsymbol{c}_m(\tau_m)^\top q_{-m}(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) + \boldsymbol{b}_m(\tau_m)^\top \boldsymbol{x}_{t-1} + a_m(\tau_m), \end{cases}$$

  where $\{\boldsymbol{c}_j(\tau_j)\}_{j=1}^m$ and $\{\boldsymbol{b}_j(\tau_j)\}_{j=1}^m$ are vectors of dimensions $(m-1) \times 1$ and $k \times 1$, respectively, and $\{a_j(\tau_j)\}_{j=1}^m$ are scalars.
- All the m-directions together correspond to an orthonormal basis. The solution is a fixed point or a simultaneous solution of all m equations.

VARQ

- Consider the following matrices based on the coefficients above: $\boldsymbol{C}(\boldsymbol{\tau}) := \{\boldsymbol{C}_1(\tau_1), \dots, \boldsymbol{C}_m(\tau_m)\}^\top$ is an $m \times m$ matrix in which the $\{\boldsymbol{C}_j(\tau_j)\}_{j=1}^m$ $m \times 1$-dimensional vectors contain all the elements of the $m-1$ vector of coefficients $\{\boldsymbol{c}_j(\tau_j)\}_{j=1}^m$ augmented with a 0 in the corresponding jth component, $\boldsymbol{b}(\boldsymbol{\tau}) = \{\boldsymbol{b}_1(\tau_1), \dots, \boldsymbol{b}_m(\tau_m)\}^\top$ is an $m \times k$ matrix, and $\boldsymbol{a}(\boldsymbol{\tau}) = \{a_1(\tau_1), \dots, a_m(\tau_m)\}^\top$ is an $m \times 1$ vector.
- Then, the VARQ model is defined as

$$Q_{\boldsymbol{Y}_t}(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) = \{\boldsymbol{I}_m - \boldsymbol{C}(\boldsymbol{\tau})\}^{-1}\{\boldsymbol{b}(\boldsymbol{\tau})\boldsymbol{x}_{t-1} + \boldsymbol{a}(\boldsymbol{\tau})\} := \boldsymbol{B}(\boldsymbol{\tau})\boldsymbol{x}_{t-1} + \boldsymbol{A}(\boldsymbol{\tau}),$$

where $\boldsymbol{I}_m$ is the m-dimensional identity matrix, $\boldsymbol{B}(\boldsymbol{\tau}) := \{\boldsymbol{I}_m - \boldsymbol{C}(\boldsymbol{\tau})\}^{-1}\boldsymbol{b}(\boldsymbol{\tau})$ and $\boldsymbol{A}(\boldsymbol{\tau}) := \{\boldsymbol{I}_m - \boldsymbol{C}(\boldsymbol{\tau})\}^{-1}\boldsymbol{a}(\boldsymbol{\tau})$.

Forecasting

One-period ahead forecasting

- The VARQ model implicitly defines a one-period ahead forecasting method for the entire distribution of $\boldsymbol{Y}_{t+1}$ given all the information available at t.

$$Q_{\boldsymbol{Y}_{t+1}}(\boldsymbol{\tau}|\boldsymbol{x}_t) = Q_{\boldsymbol{Y}_{t+1}}(\boldsymbol{\tau}|\{\boldsymbol{y}_t, \boldsymbol{y}_{t-1}, ..., \boldsymbol{y}_{t-p}\}).$$

- Define thus $Q_1(\boldsymbol{\tau}|\boldsymbol{x}_t) = Q_{\boldsymbol{Y}_{t+1}}(\boldsymbol{\tau}|\boldsymbol{x}_t)$ as the one-period ahead forecast given all the information available at time t.
- The distribution of $\boldsymbol{Y}_{t+1}|\boldsymbol{x}_t$ can be found by $Q_1(\boldsymbol{u}|\boldsymbol{x}_t)$ with $\boldsymbol{u} \sim iidU(0,1)^m$ as a random coefficient model.

Forecasting

Two-periods ahead forecasting - quantile paths

- Consider now the two-periods ahead forecast, i.e. $t+2$, at quantiles $\boldsymbol{\tau}_2$.
- Note that this would depend on the response at $t+1$ and the implicit quantile $\boldsymbol{\tau}_1$. In turn then this would depend on both quantiles, $(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1)$. This is defined as a two-periods **quantile path**, where the collection of indexes correspond to a potential path of the system of endogenous variables. Then

$$Q_2\{(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1)|\boldsymbol{x}_t\} := Q[\boldsymbol{\tau}_2|\{Q_1(\boldsymbol{\tau}_1|\boldsymbol{x}_t), \boldsymbol{y}_t, \ldots, \boldsymbol{y}_{t-p+1}\}].$$

- The distribution of $\boldsymbol{Y}_{t+2}|\boldsymbol{x}_t$ can be found by $Q_2(\boldsymbol{u}^{(2)}|\boldsymbol{x}_t)$ with $\boldsymbol{u}^{(2)} = [\boldsymbol{u}_1, \boldsymbol{u}_2]$, $\boldsymbol{u}_i \sim iidU(0,1)^m, i = 1, 2$ as a random coefficient model.

Forecasting

h-periods ahead forecasting - quantile paths

- In general the h-periods ahead forecast can be written as a function of the forecast of the previous quantiles

$$Q_h\{(\boldsymbol{\tau}_h, \ldots, \boldsymbol{\tau}_1)|\boldsymbol{x}_t\}$$

  where $(\boldsymbol{\tau}_h, \ldots, \boldsymbol{\tau}_1)$ is the h-periods **quantile path**.
- The distribution of $\boldsymbol{Y}_{t+h}|\boldsymbol{x}_t$ can be found by $Q_h(\boldsymbol{u}^{(h)}|\boldsymbol{x}_t)$ with $\boldsymbol{u}^{(h)} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_h]$, $\boldsymbol{u}_i \sim iidU(0,1)^m, i = 1, 2, \ldots, h$ as a random coefficient model.

Forecasting

Quantile paths

- This framework allows for forecasting different **quantile paths**.
- A canonical case is fixing $\boldsymbol{\tau}_i = (0.5, \ldots, 0.5)$ for all $i = 1, \ldots, h$, which corresponds to evaluating future values on the conditional **median** values of the endogenous variables. In general this procedure delivers similar estimates as the mean-based VAR forecasts.
- This procedure can be generalized for any $\boldsymbol{\tau}_i = (\tau, \ldots, \tau)$ for all $i = 1, \ldots, h$. In this case high values of τ correspond to the persistent occurrence of the τ conditional quantile in all endogenous variables.
- Moreover, we do not necessarily need the same τ for all endogenous variables equations. As an example we can consider the 0.1 and 0.9 quantiles of output, while we keep the median for inflation and interest rate. As such, we are constructing a potential quantile path where output is either at the low or high end of the business cycle. See Galvao, Montes-Rojas, and Park (2013) for an interpretation of QR time-series models in terms of the business cycle.

Impulse response functions

- Our interest lies in evaluating the propagation of shocks of the m-variate process. (Identification of shocks comes from elsewhere.)
- We then compute the impulse response function by comparing the multivariate quantiles at $\boldsymbol{x}_t^{\delta} := (\boldsymbol{y}_t + \boldsymbol{\delta}, \boldsymbol{y}_{t-1}, ..., \boldsymbol{y}_{t-p})$ with those at $\boldsymbol{x}_t = (\boldsymbol{y}_t, \boldsymbol{y}_{t-1}, ..., \boldsymbol{y}_{t-p})$.
- Define the $\boldsymbol{\tau}$-quantile impulse response function (QIRF) at $t+1$ for a shock at time t, $\boldsymbol{\delta} \in \mathcal{Y} \subseteq \mathbb{R}^m$, as

$$\mathrm{Qirf}_1(\boldsymbol{\tau}, \delta | \boldsymbol{x}_t) = Q_1(\boldsymbol{\tau} | \boldsymbol{x}_t^{\delta}) - Q_1(\boldsymbol{\tau} | \boldsymbol{x}_t) = \boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau})\delta,$$

  where Q_1 is the one-period ahead forecast.
- The distribution of Qirf_1 can be found by $\mathrm{Qirf}_1(\boldsymbol{u}, \boldsymbol{\delta} | \boldsymbol{x}_t)$ with $\boldsymbol{u} \sim iidU(0,1)^m$..

Impulse response functions

- Consider now the IRF two-periods ahead, i.e. $t+2$, at quantiles $\boldsymbol{\tau}_2$. Note that this would depend on the response at $t+1$ and the implicit quantile $\boldsymbol{\tau}_1$. In turn then this would depend on both quantiles, $(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1)$, defined as a quantile path.

$$
\begin{aligned}
\mathrm{Qirf}_{2(1)}\left\{(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1), \boldsymbol{\delta} | \boldsymbol{x}_t\right\} &= Q_2\left\{(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1) | \boldsymbol{x}_t^{\delta}\right\} - Q_2\left\{(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1) | \boldsymbol{x}_t\right\} \\
&= \begin{cases} (\boldsymbol{B}_{\cdot 2}(\boldsymbol{\tau}_2) + \boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau}_2)\boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau}_1))\boldsymbol{\delta} & p > 1 \\ \boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau}_2)\boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau}_1))\boldsymbol{\delta} & p = 1 \end{cases} .
\end{aligned}
$$

- The distribution of Qirf_2 can be found by $\mathrm{Qirf}_2(\boldsymbol{u}^{(2)}, \boldsymbol{\delta} | \boldsymbol{x}_t)$ with with $\boldsymbol{u}^{(2)} = [\boldsymbol{u}_1, \boldsymbol{u}_2],\ \boldsymbol{u}_i \sim iidU(0,1)^m, i = 1, 2$.

Impulse response functions

- This procedure above can be generalized for h-periods ahead IRFs, by defining

$$\begin{aligned} &\mathrm{Qirf}_h \left\{ (\boldsymbol{\tau}_h, \boldsymbol{\tau}_{h-1}, ..., \boldsymbol{\tau}_1), \boldsymbol{\delta} | \boldsymbol{x}_t \right\} \\ &= Q_h \left\{ (\boldsymbol{\tau}_h, \boldsymbol{\tau}_{h-1}, ..., \boldsymbol{\tau}_1) | \boldsymbol{x}_t^{\delta} \right\} - Q_h \left\{ (\boldsymbol{\tau}_h, \boldsymbol{\tau}_{h-1}, ..., \boldsymbol{\tau}_1) | \boldsymbol{x}_t \right\}, \end{aligned}$$

  for a given *path* of multivariate quantiles $(\boldsymbol{\tau}_h, \boldsymbol{\tau}_{h-1}, ..., \boldsymbol{\tau}_1)$ and shock $\boldsymbol{\delta}$ at time t.
- The distribution of Qirf_h can be found by $\mathrm{Qirf}_h(\boldsymbol{u}^{(h)}, \boldsymbol{\delta}|\boldsymbol{x}_t)$ with with $\boldsymbol{u}^{(h)} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_h]$, $\boldsymbol{u}_i \sim iidU(0,1)^m, i = 1, 2, \ldots, h$.
- In the long run the QIRF for $h \to \infty$ becomes 0 for stationary models.

Accumulated impulse response functions

- Of particular interest is the accumulated QIRF, defined as

$$\mathrm{aQIRF}_h\left\{\boldsymbol{\tau}^{(h)},\boldsymbol{\delta}\right\}\equiv\sum_{j=1}^{h}\mathrm{QIRF}_j\left\{\boldsymbol{\tau}^{(j)},\boldsymbol{\delta}\right\},$$

  where $\boldsymbol{\tau}^{(j)}$ are concatenated with each other $\boldsymbol{\tau}^{(j+1)}=[\boldsymbol{\tau}^{(j)},\boldsymbol{\tau}_{j+1}]$ for $j=1,\ldots,h-1$.
- Thus the main result of interest lies in the simulated distribution of the model above, which can be obtained by

$$\mathrm{aQIRF}_h\left\{\boldsymbol{u}^{(h)},\boldsymbol{\delta}\right\}\equiv\sum_{j=1}^{h}\mathrm{QIRF}_j\left\{\boldsymbol{u}^{(j)},\boldsymbol{\delta}\right\},$$

  where $\boldsymbol{u}^{(h)}$ is defined as above, i.e., $\boldsymbol{u}^{(1)}\sim iid\ U(0,1)^m$ and $\boldsymbol{u}^{(j+1)}=[\boldsymbol{u}^{(j)},\boldsymbol{u}_{j+1}]$ with $\boldsymbol{u}_{j+1}\sim iid\ U(0,1)^m$ for $j=1,\ldots,h-1$.
- The long-run distribution of a given shock can then be obtained by $\lim_{h\to\infty}\mathrm{aQIRF}_h\left\{\boldsymbol{u}^{(h)},\boldsymbol{\delta}\right\}$.

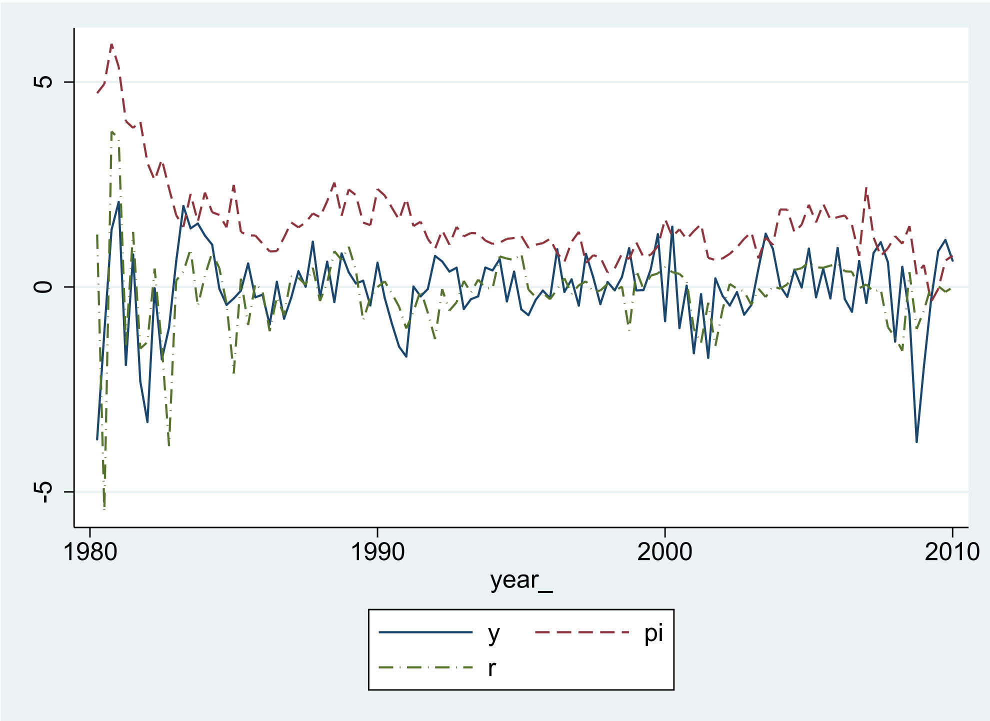
Effect of monetary policy

- We estimate a three-variable (output gap, inflation, Fed Funds rate) VAR(1) model using U.S. quarterly data from 1980q1 to 2010q1 (121 quarters). This simple framework corresponds to the three-variable framework of New Keynesian model rational expectations model of Cho and Moreno (2004, 2006) and Jordà (2005), among others.
- The output gap is generated by the first-difference of the Hodrick-Prescott linear filter with linear trend, using the logarithm of the Gross National Product, 1996 constant prices (source: Federal Reserve Bank of St. Louis), denoted y_t.
- The inflation rate is the log first-difference of the GDP deflator, seasonally adjusted (source: Federal Reserve Bank of St. Louis), denoted π_t.
- The Fed Funds rate is the monetary policy instrument (source: Board of Governors of the Federal Reserve System), denoted r_t, and corresponds to the first-difference of the 3-months Treasury Bill rate (end of the quarter). The reason we use the first-difference of the interest rate is that over the period of analysis it shows a negative trend and we cannot reject it has a unit root.
- For this case then $\boldsymbol{Y}_t = (y_t, \pi_t, r_t)$.

Effect of monetary policy

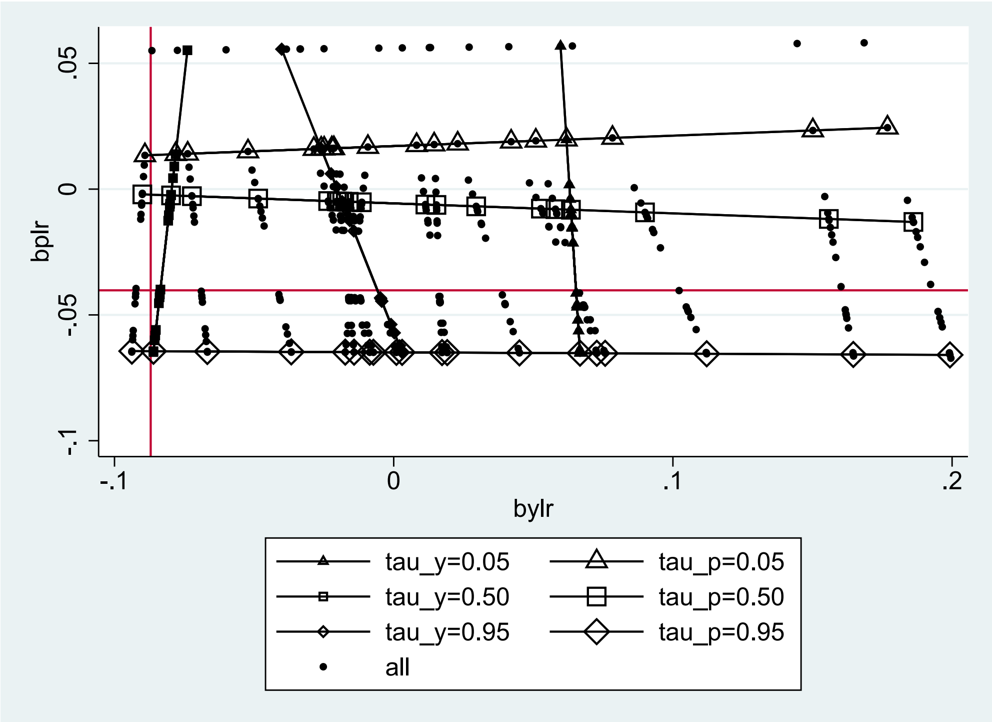
- We follow the Cholesky identification procedure in Christiano, Eichenbaum, and Evans (1996), using the residuals from a VAR model where we assume the standard ordering:
  - r has no contemporaneous effect on y and π;
  - π has an effect on r but not on y; and
  - y affects both π and r. This implies that shocks to the Fed Funds rate has no contemporaneous effect on the other economic variables.
- Then we evaluate the effect of a shock in r, calculated as the standard deviation of this structural shock, on output gap and inflation (also standardized by the standard deviation of their corresponding structural shocks).

Figure: Series 1980q1-2010q1



Notes: Output gap, inflation and interest rate (first diff.) series. In the figures all the variables are standardized by the sample standard deviation.

Figure: VARQ coefficients for $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$ and $\tau_r = 0.50$



Notes: Vertical and horizontal lines correspond to the mean-based VAR effects.

Table: VAR system stability - Modulus of eigenvalues of $\boldsymbol{B}(\boldsymbol{\tau})$

Model	eigen 1	eigen 2	eigen 3
$VAR-OLS$	0.853	0.152	0.067
$VARQ(\tau_y = 0.5, \tau_\pi = 0.1, \tau_r = 0.5)$	0.669	0.131	0.131
$VARQ(\tau_y = 0.1, \tau_\pi = 0.5, \tau_r = 0.5)$	0.813	0.535	0.054
$VARQ(\tau_y = 0.5, \tau_\pi = 0.5, \tau_r = 0.5)$	0.818	0.145	0.145
$VARQ(\tau_y = 0.5, \tau_\pi = 0.9, \tau_r = 0.5)$	0.984	0.153	0.153
$VARQ(\tau_y = 0.9, \tau_\pi = 0.5, \tau_r = 0.5)$	0.820	0.285	0.058

Figure: Accumulated IRF and QIRF

Output gap (y)

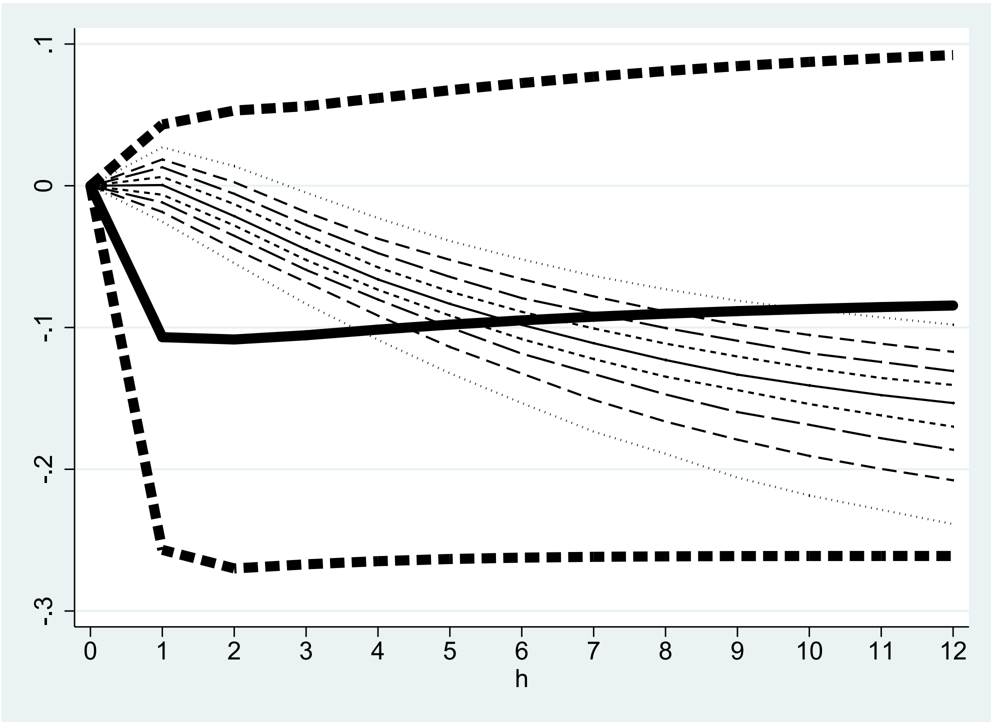


Figure: Accumulated IRF and QIRF

Inflation (π)

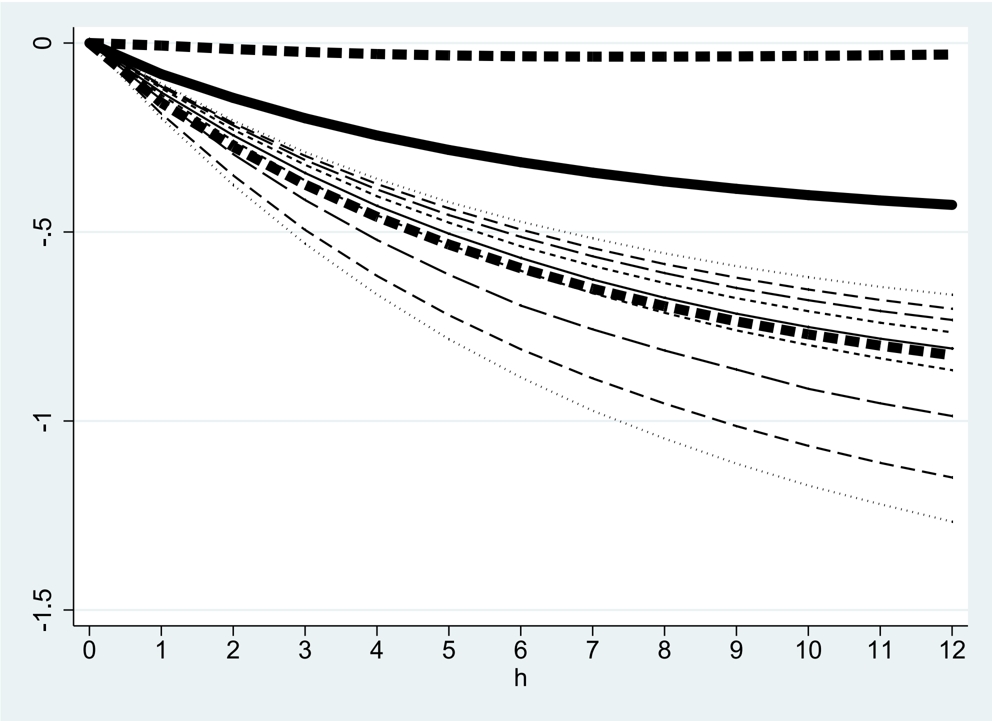


Figure: Accumulated IRF and QIRF

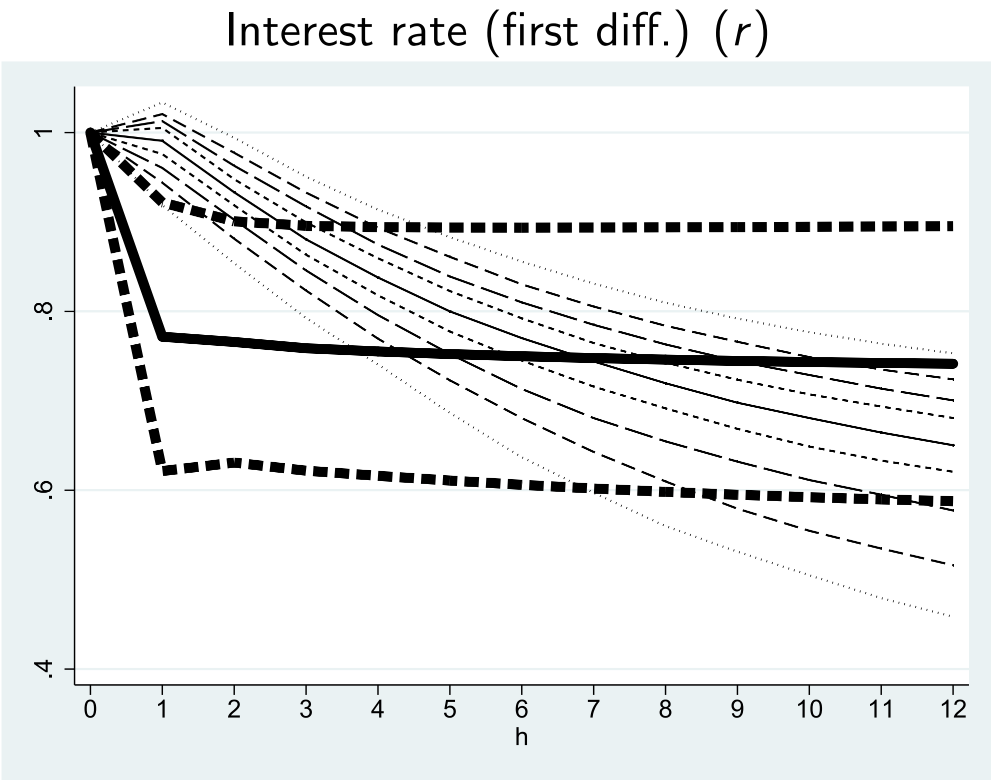


Figure: Accumulated QIRF for different *h*

Output gap (y)

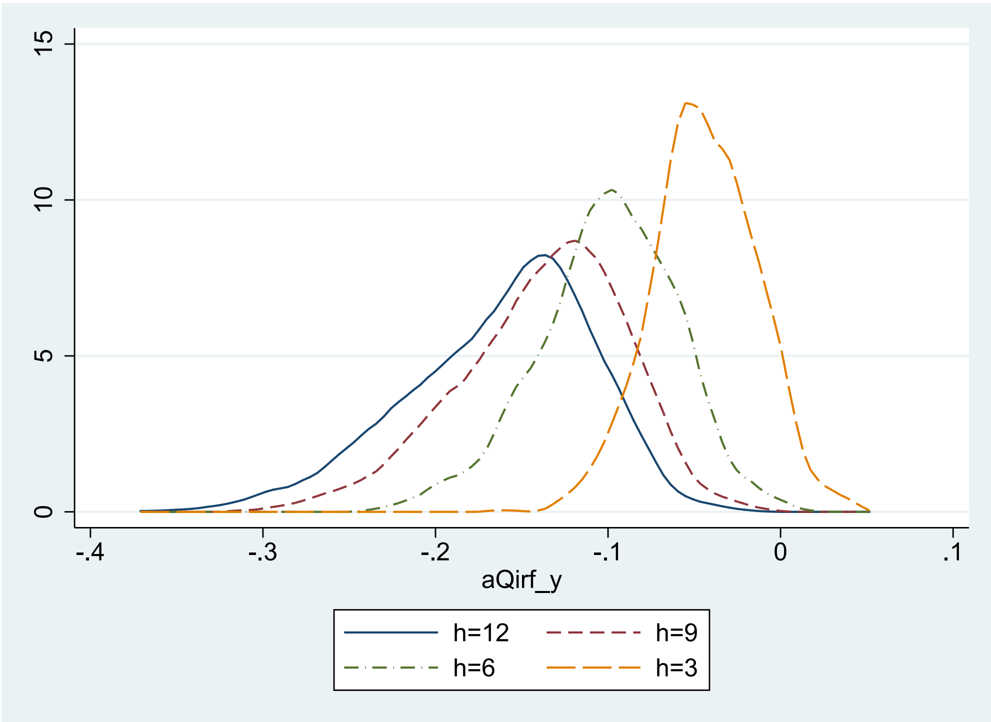


Figure: Accumulated QIRF for different h

Inflation (π)

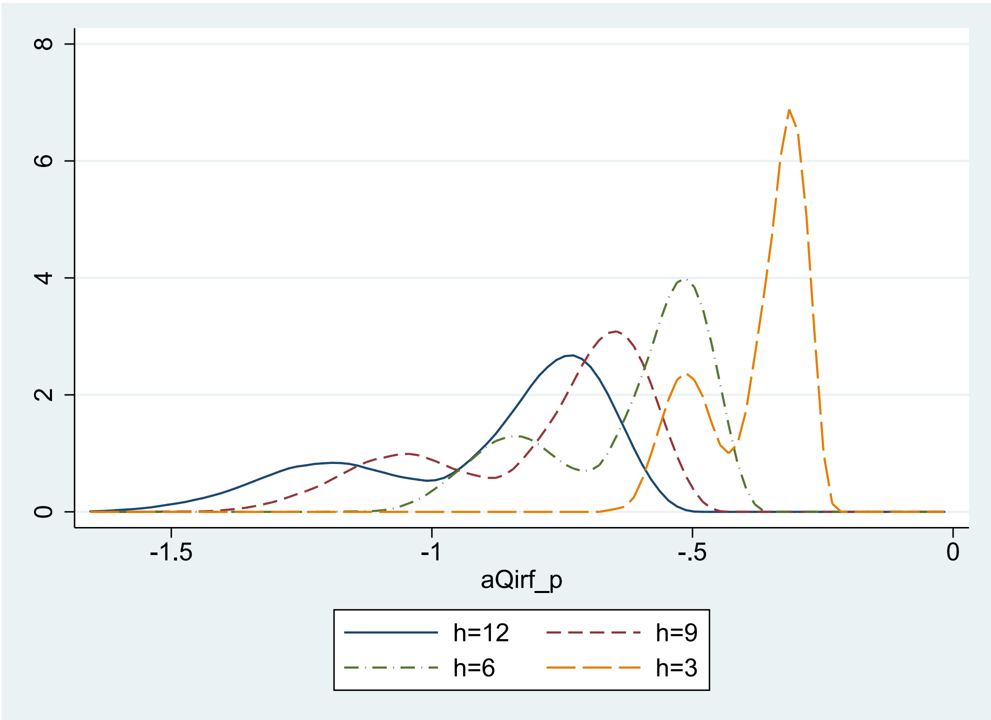


Figure: Accumulated QIRF for different h

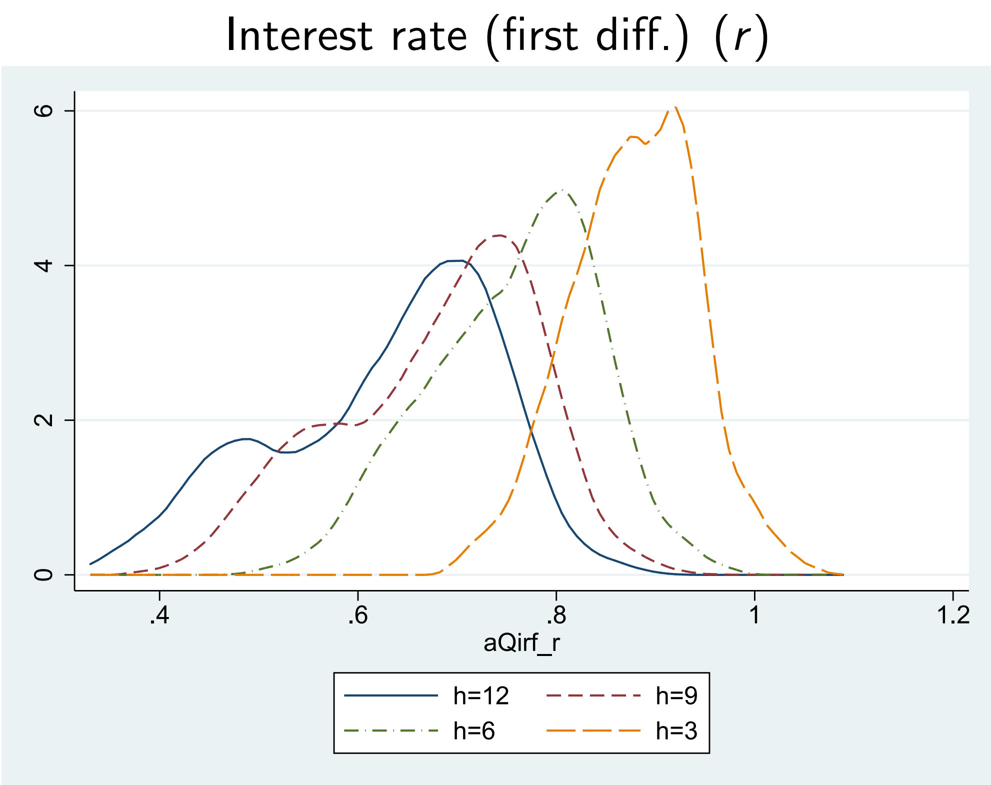


Figure: Accumulated QIRF for $h = 12$, scatter plot

Output gap (y) and Inflation (π)

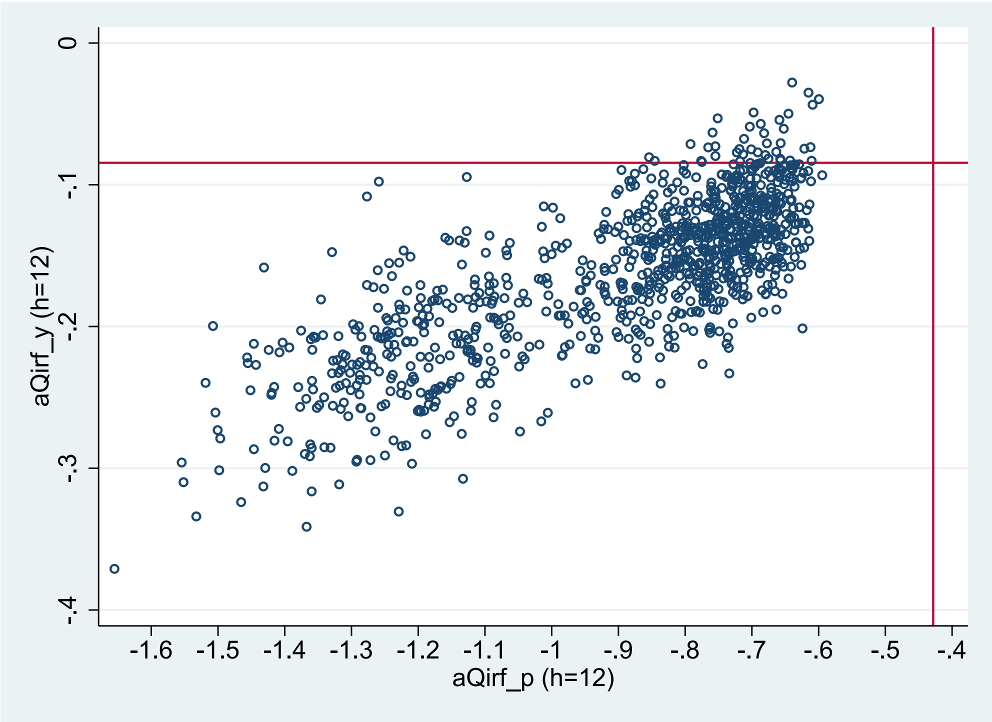


Figure: Accumulated QIRF for $h = 12$, scatter plot

Output gap (y) and Interest rate (first diff.) (r)

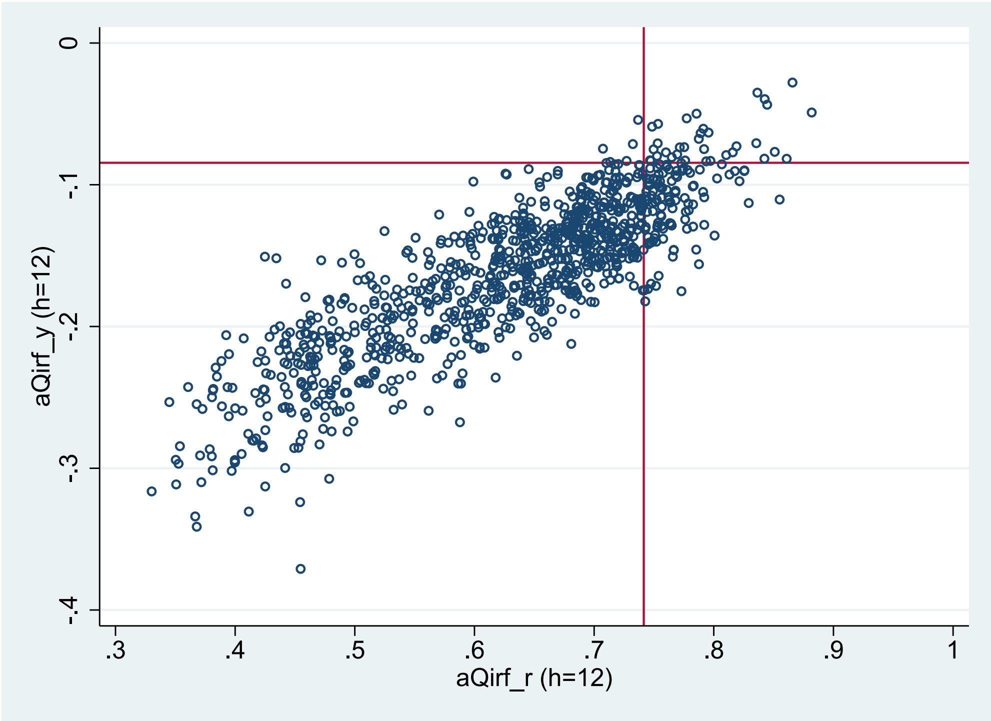


Figure: Accumulated QIRF for $h = 12$, scatter plot

Output gap (y) and Interest rate (first diff.) (r)

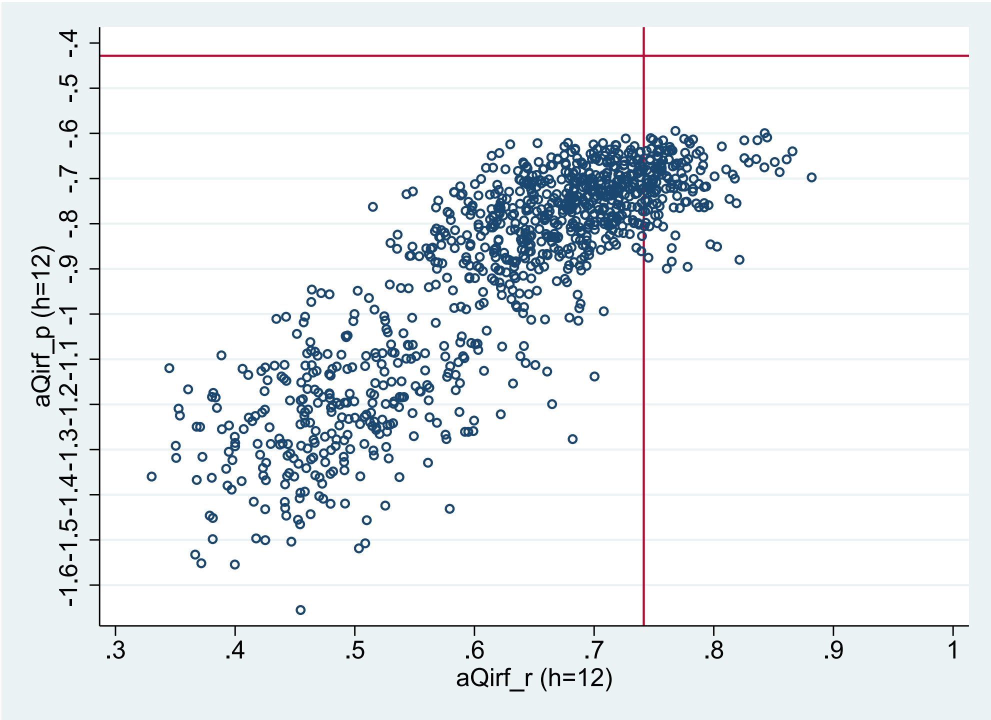


Figure: Accumulated QIRF for $h = 12$, bivariate density plot

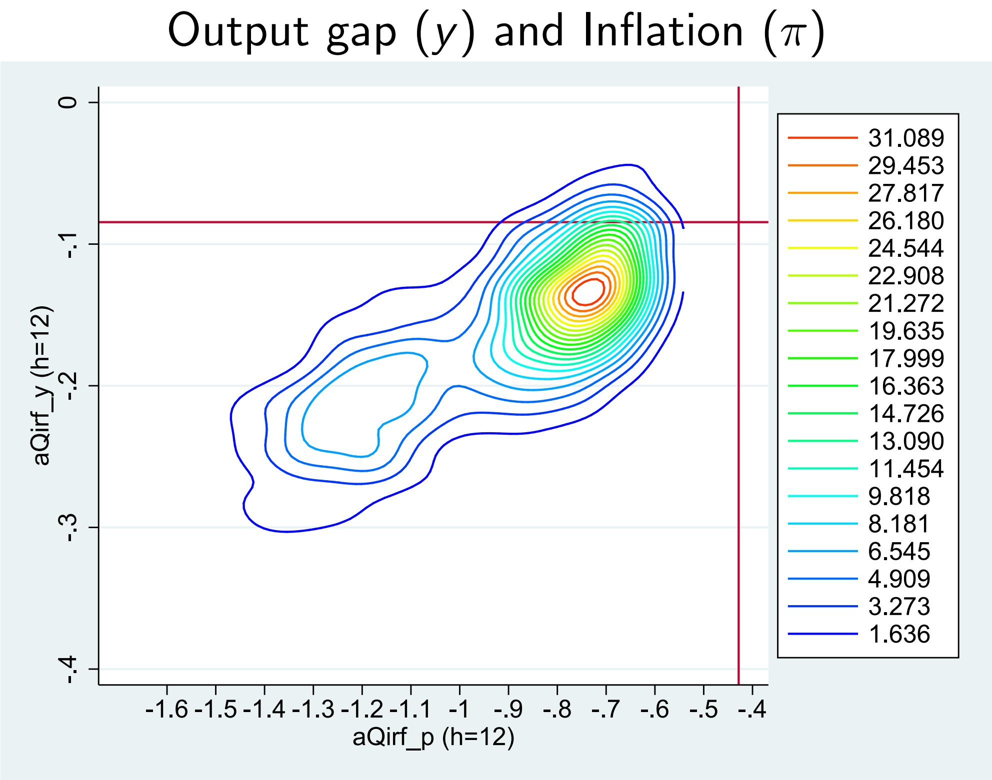
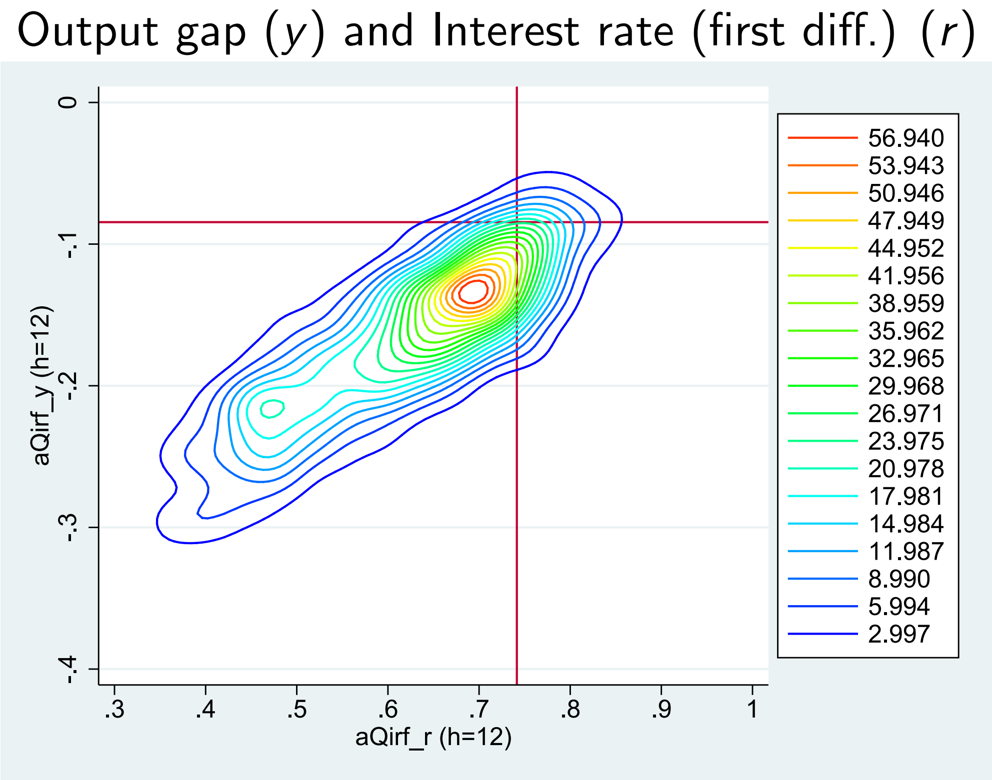


Figure: Accumulated QIRF for $h = 12$, bivariate density plot



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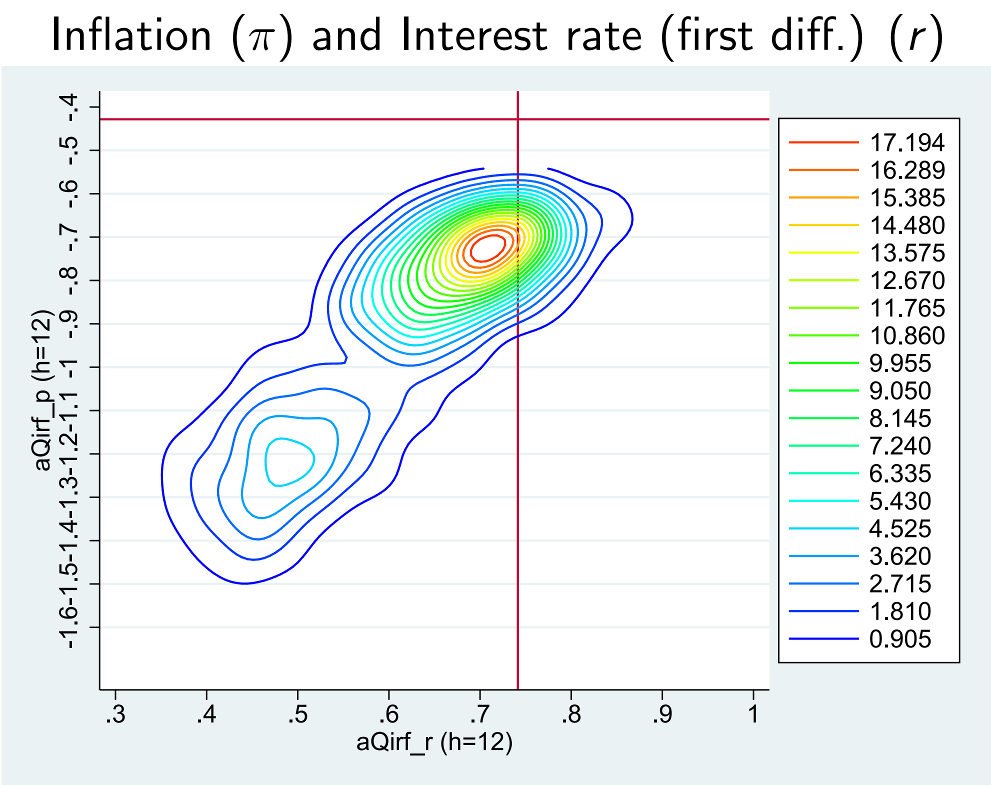

Further research ideas

- The model can be extended to nonlinear QR models. Fix point solution to a nonlinear system.
- Multivariate density forecasting. Consider a grid of G m-quantile indexes $\{\boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_G}\}$, then $\{Q_{\mathbf{Y}_t}(\boldsymbol{\tau_1}|\mathbf{X}_{t-1}), \ldots, Q_{\mathbf{Y}_t}(\boldsymbol{\tau_G}|\mathbf{X}_{t-1})\}$ can be used to construct $\{f_{\mathbf{Y}_t}(\boldsymbol{\tau_1}|\mathbf{X}_{t-1}), \ldots, f_{\mathbf{Y}_t}(\boldsymbol{\tau_G}|\mathbf{X}_{t-1})\}$ density points for $f : \mathcal{Y} \to \mathbb{R}$.
- Structural VARQ. Cholesky decomposition or other identification strategies for different quantile indexes.
- Quantile path analysis as an alternative to structural breaks.

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