

Categories for the Working Mathematicians. 2nd ed.

Mac Lane. Springer.

Notes by @naughe (Github)

<https://naughe.github.io/macLane-notes>

§ 2.6. Comma Categories

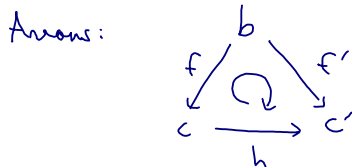
Def.

\mathcal{C} : category. $b \in \mathcal{C}$: object.

$(b \downarrow \mathcal{C})$: category of objects under b with

• objects: $\langle f, c \rangle$, where $c \in \mathcal{C} \text{ obj.}$ and $f: b \rightarrow c$ in \mathcal{C} .

• arrows: $\langle f, c \rangle \xrightarrow{h} \langle f', c' \rangle$, where $h: c \rightarrow c'$ s.t.
 $f' = h \circ f$.



Exs.

• $*$: 1-point set $\text{on } \mathbb{Z}$. $(* \downarrow \text{Set}) = \text{Set}_*$.

(\odot Obj. $* \rightarrow X$ is pair $\langle x \in X, X \rangle$ on \mathbb{Z} .
 Arr. $\langle x, X \rangle \rightarrow \langle x', X' \rangle$ is base point $\in \langle \frac{x}{x'} \rangle$ map on \mathbb{Z} .)

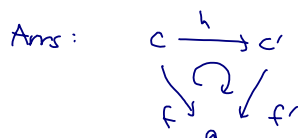
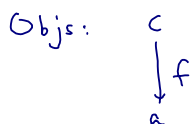
• $(\mathbb{Z} \downarrow \text{Ab}) = \text{Ab}_*$

(\odot Obj. $\mathbb{Z} \xrightarrow{f} G$ is $\langle f(1), G \rangle$ on \mathbb{Z} . (base point $f(1)$).
 Arr. $\langle g, G \rangle \rightarrow \langle g', G' \rangle$ is base point $\in \langle \frac{g}{g'} \rangle$ group hom. on \mathbb{Z} .)

Def.

\mathcal{C} : cat. $a \in \mathcal{C}$: obj.

$(\mathcal{C} \downarrow a)$: category of objects over a with



Exs.

① $*$: 1-point set. $\Rightarrow (\text{Set} \downarrow *) \cong_{\text{cat}} \text{Set}.$

② Obj. $X \xrightarrow{f} *$ is, $*$: terminal obj. in Set だから, X は何なりか...
 任意の $h: X \rightarrow X'$ in Set は, $f = f' \circ h$ を満たす.

$$X \xrightarrow{h} X' \quad (f: X \rightarrow *, f': X' \rightarrow *)$$

③ $(\text{Rng} \downarrow \mathbb{Z})$: augmentation $R \rightarrow \mathbb{Z}$ as objects,
 ring hom preserving augmentations as arrows.

④ Obj. $R \xrightarrow{\varepsilon} \mathbb{Z}$ is, augmentation ε を持つもの. (一般論).
 Arr $\langle \varepsilon, R \rangle \xrightarrow{h} \langle \varepsilon', R' \rangle$ is, $\varepsilon = \varepsilon' \circ h$ である ring hom $h: R \rightarrow R'$.
 i.e., ring hom that preserves augmentations.

Def.

$S: \mathcal{D} \rightarrow \mathcal{E}$: functor. $b \in \mathcal{E}$: obj.

$(b \downarrow S)$: category of objects S -under b . with

Obj. $\begin{array}{c} b \\ \downarrow f \\ Sd \end{array}$ Arr. $\begin{array}{ccc} & b & \\ f \swarrow & & \searrow f' \\ Sd & \xrightarrow{Sh} & Sd' \end{array}$

Def.

$T: \mathcal{E} \rightarrow \mathcal{E}$: functor. $a \in \mathcal{E}$: obj.

$(T \downarrow a)$: category of objects T -over a with

Obj. $\begin{array}{c} Tc \\ \downarrow f \\ a \end{array}$ Arr. $\begin{array}{ccc} Tc & \xrightarrow{Th} & Tc' \\ f \downarrow & \curvearrowright & \downarrow f' \\ & a & \end{array}$

Ex.

① $U: \mathbf{Grp} \rightarrow \mathbf{Set}$: forgetful functor. $x \in \mathbf{Set} : \text{obj.} \in \{ \}$.

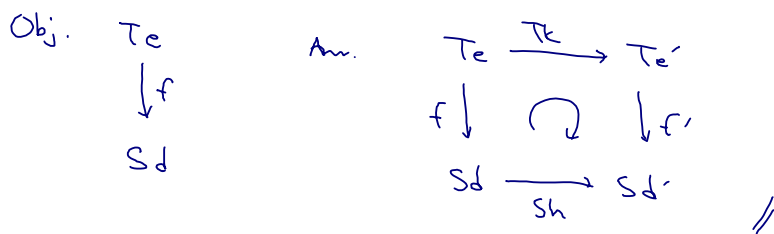
$(x \downarrow U) : \text{map } x \xrightarrow{f} Uq \text{ as obj}$

group hom $h: \langle f, g \rangle \rightarrow \langle f', g' \rangle$ s.t. $f' = h \circ f$ as arr.

Def.

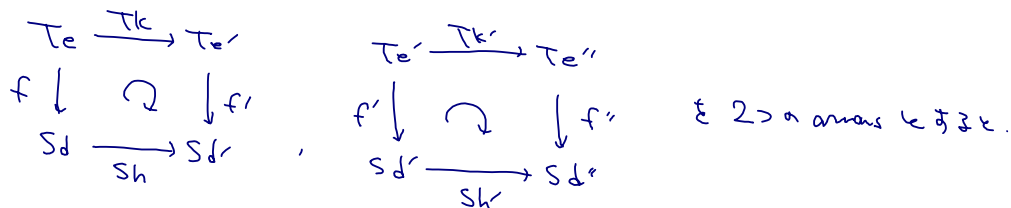
$\mathcal{E} \xrightarrow{T} \mathcal{C} \xleftarrow{S} \mathcal{D}$: functors.

$(T \downarrow S)$: comma category with

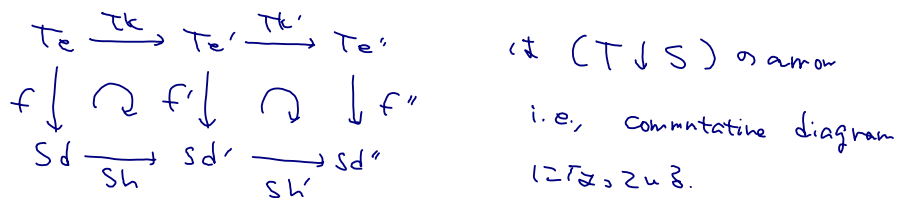


Rem.

Composites of $(T \downarrow S)$ is well-defined:



$\{ \}$ composite



$(T \downarrow S)$ is $(b \downarrow \mathcal{E})$, $(\mathcal{E} \downarrow a)$, $(b \downarrow S)$, $(T \downarrow a)$ の一般化である!

① $(b \downarrow S)$ is. $T = b: \mathbb{1} \rightarrow \mathcal{E}$ と $\mathcal{E} \in \{ \}$.

② $(T \downarrow a)$ is. $S = a: \mathbb{1} \rightarrow \mathcal{D}$ と $\mathcal{D} \in \{ \}$.

③ $(b \downarrow \mathcal{E})$ is. $T = b: \mathbb{1} \rightarrow \mathcal{E}$, $S = 1_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E}$ と $\mathcal{E} \in \{ \}$.

④ $(\mathcal{E} \downarrow a)$ is. $S = a: \mathbb{1} \rightarrow \mathcal{D}$, $T = 1_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E}$ と $\mathcal{E} \in \{ \}$.

Exs.

④ $\mathcal{C} : \text{cat.}$ $\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} : \text{id. functor.}$ $\circ \mathcal{C} \neq$.

$$(\mathcal{C} \downarrow \mathcal{C}) = \mathcal{C}^2$$

$$\left(\begin{array}{l} \text{Obj. } c \xrightarrow{f} d. \\ \text{Arr. } \begin{array}{ccc} c & \xrightarrow{f} & d \\ k \downarrow & \curvearrowright & \downarrow h \\ c' & \xrightarrow{f'} & d' \end{array} \end{array} \Rightarrow \mathcal{C}^2 \left(\begin{array}{l} \text{obj: arrows.} \\ \text{arr: commutative squares} \end{array} \right) \right)$$

⑤ $T = b, S = a : \mathbb{1} \rightarrow \mathcal{C}.$ $\circ \mathcal{C} \neq$.

$$(b \downarrow a) = \text{hom}_{\mathcal{C}}(b, a) \quad (\text{as a discrete cat.}).$$

$$\left(\begin{array}{l} \text{Obj. } b \xrightarrow{f} a. \\ \text{Arr. } \begin{array}{ccc} b & \xrightarrow{f} & a \\ \text{id} \downarrow & \curvearrowright & \downarrow \text{id} \\ b & \xrightarrow{f'} & a \end{array} \end{array} \Rightarrow f = f'. \quad \text{i.e., arrow is trivial to be equal.} \right)$$

↑ "comma category" の名前が由来!

$(T \downarrow S)$ の普遍性 (cf. Exercise (5)).

$$\begin{array}{ccccc} & & (T \downarrow S) & & \\ & \swarrow P & \downarrow R & \searrow Q & \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} & \xleftarrow{\mathcal{C}^{d_0}} & \mathcal{C}^2 & \xrightarrow{\mathcal{C}^{d_1}} & \mathcal{C} & \xleftarrow{S} & \mathcal{D} \end{array}$$

$$d_0, d_1 : \mathbb{1} \rightarrow \mathbb{2}$$

P, Q : projections.

\mathcal{C}^{d_0} (resp \mathcal{C}^{d_1}) は, arrow $a \xrightarrow{f} b \in \mathcal{C}$, $a = \text{dom } f$ (resp. $b = \text{cod } f$) になる functor. (cf. § 2.5).

Exercises.

(1) K : com. ring $\alpha \neq 0$. $(K \downarrow \mathbb{C}Rng) = \mathbb{C}Alg_K$.

Obj. A : com ring, $f: K \rightarrow A$: ring hom.

i.e., A : com. alg. $/K$.

Ans. $(A, f) \xrightarrow{h} (A', f')$: ring hom s.t.

$$\begin{array}{ccc} & K & \\ f \swarrow & \curvearrowright & \searrow f' \\ A & \xrightarrow{h} & A' \end{array} \quad \begin{array}{l} \text{i.e., scalar } \frac{f}{f'} \\ \text{i.e., alg. hom. } h: A \rightarrow A'. \end{array}$$

(2) \mathcal{C} : cat. $t \in \mathcal{C}$: terminal $\Rightarrow (\mathcal{C} \downarrow t) \cong \mathcal{C}$.

Obj. $b \xrightarrow{f} t$ is t : terminal test.

$\forall b \in \mathcal{C}$ (2つある) one and exactly one test.

Ans. $b \xrightarrow{h} b$ s.t.

$$\begin{array}{ccc} b & \xrightarrow{h} & b \\ & \curvearrowright & \\ & t & \end{array}$$

t : terminal test, この diagram は 自動的に commutative.

(4) (S.A. Hug). $T, S: \mathcal{D} \rightarrow \mathcal{C}$: functors $\alpha \neq 0$.

$\tau: T \rightarrow S$: nat. transformation $(\neq, \tau: \mathcal{D} \rightarrow (T \downarrow S)$: functor s.t.

$P_\tau = Q_\tau = id_{\mathcal{D}}$ is satisfied. (P, Q : prjs.)

Obj. $\tau: T \rightarrow S$

$\Leftrightarrow \forall d \in \mathcal{D}, \exists \tau_d: T_d \rightarrow S_d$ s.t.

$$\begin{array}{ccc} T_d & \xrightarrow{\tau_d} & S_d \\ \downarrow & \curvearrowright & \downarrow \\ T_{d'} & \xrightarrow{\tau_{d'}} & S_{d'} \end{array}$$

$\tau: \mathcal{D} \rightarrow (T \downarrow S)$ s.t. $P_\tau = Q_\tau = id_{\mathcal{D}}$

$\Leftrightarrow \forall d \in \mathcal{D}$ (2つある) $\tau_d: T_d \rightarrow S_d$ ($\exists e, g \in \mathcal{D}$)

$\forall d \xrightarrow{f} d'$ (2つある)

$$\begin{array}{ccc} T_e & \xrightarrow{\tau_e} & S_e \\ T_k \downarrow & \curvearrowright & \downarrow S_h \\ T_{e'} & \xrightarrow{\tau_{e'}} & S_{e'} \end{array} \quad (\exists k: e \rightarrow e', h: g \rightarrow g')$$

s.t.

$$\begin{array}{ccccc}
 e & = & g & = & d \\
 \parallel & & \parallel & & \parallel \\
 P_{\tau d} & & Q_{\tau d} & & id_D d
 \end{array}, \quad
 \begin{array}{ccccc}
 k & = & h & = & f \\
 \parallel & & \parallel & & \parallel \\
 P_{\tau f} & & Q_{\tau f} & & id_D f
 \end{array}$$

$$\Leftrightarrow \forall d \in D. \exists \tau_d: T_d \rightarrow S_d \text{ s.t.}$$

$$\begin{array}{ccc}
 T_d & \xrightarrow{\tau_d} & S_d \\
 \tau_f \downarrow & \curvearrowright & \downarrow S_f \\
 T_{d'} & \xrightarrow{\tau_{d'}} & S_{d'}
 \end{array} \quad (f: d \rightarrow d')$$

上の2>は同値.

§2.7. Graphs and Free Categories.

まず, free monoid の場合を思い出そう.

X : set of symbols 生成される free monoid F とは,

- Word $x_1 \dots x_n$ ($x_i \in X$) が元を持ち,
- words の 結合 (juxtaposition) を積とし,
- empty word & identity element を持つ

monoid のこと.

これは equivalent な条件は,

$$\begin{array}{ccc}
 X & \xrightarrow{\iota} & F \\
 & \searrow f & \downarrow \exists! \tilde{f} \\
 & & M
 \end{array}
 \quad
 \begin{array}{l}
 (F: \text{monoid}, \iota: X \rightarrow F: \text{map}) \text{ is free monoid} \\
 \Leftrightarrow \forall (M: \text{monoid}, f: X \rightarrow M: \text{map}), \\
 \exists! \tilde{f}: F \rightarrow M: \text{monoid hom. s.t. } f = \tilde{f} \circ \iota.
 \end{array}$$

これは元は free category と考えられる.

Def. (§1.2)

G : (directed) graph とは,

- O : set of objects (vertices)
- A : set of arrows (edges),

together with functions

$$A \xrightarrow[\partial_1]{\partial_0} O, \quad \partial_0 f = \text{dom } f, \quad \partial_1 f = \text{cod } f.$$

$D: G \rightarrow G'$: morphism of graphs とは,

• $D_0: O \rightarrow O'$

• $D_A: A \rightarrow A'$ s.t. $D_0 \partial_0 = \partial_0 D_A, \quad D_0 \partial_1 = \partial_1 D_A.$

//

Graph: category of graphs.

Graphs に \circ (composite) と identity が \neq ない。

↑ diagram scheme \neq precategory \neq \mathbb{C} ではない!

Graphs obtained from categories

\mathcal{C} : cat. $\hookrightarrow \mathbb{C}$ $\cup \mathcal{C}$: graph \mathcal{C} .

• obj.: objects of \mathcal{C} ;

• arr.: arrows of \mathcal{C}

\mathbb{C} の \mathbb{C} の \mathcal{C} : $F: \mathcal{C} \rightarrow \mathcal{C}'$: functor \neq .

graph morphism $UF: U\mathcal{C} \rightarrow U\mathcal{C}'$ に \neq 対応する。

$\therefore U: \text{Cat} \rightarrow \text{Graph}$: forgetful functor.

Categories obtained from graphs

O : set: fixed.

Def.

O -graph \neq graph where object set is O .

A morphism of O -graph \neq O -graphs \mathbb{C} の graph morphism \neq \mathbb{C} . //

A, B : sets of arrows (regarded as O -graphs) $\neq \mathbb{C}$ \neq .

$$A \times_0 B := \{ \langle g, f \rangle \mid \partial_0 g = \partial_1 f, g \in A, f \in B \}$$

(composable pairs of arrows $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$.)

\mathbb{C} の \mathbb{C} .

$$\partial_0 \langle g, f \rangle := \partial_0 f, \quad \partial_1 \langle g, f \rangle := \partial_1 g$$

\mathbb{C} の \mathbb{C} : $A \times_0 B$: O -graph $\neq \mathbb{C}$.

\times_0 is associative

$$\left(\begin{aligned} \therefore (A \times_0 B) \times_0 C &\cong A \times_0 (B \times_0 C) \\ &\cong \{ \langle h, g, f \rangle \mid \partial_0 h = \partial_1 g, \partial_0 g = \partial_1 f, h \in A, g \in B, f \in C \} \end{aligned} \right)$$

任意の $A: \mathcal{O}\text{-graph}$ に対して.

$$\begin{array}{ccccc} A & \cong & A \times_{\mathcal{O}} \mathcal{O} & \cong & \mathcal{O} \times_{\mathcal{O}} A \\ \downarrow & & \downarrow & & \downarrow \\ f & \mapsto & \langle f, \partial_0 f \rangle & \mapsto & \langle \partial_1 f, f \rangle. \end{array}$$

Rem. \mathcal{O} は次のように $\mathcal{O}\text{-graph}$ とみなされる:

• arrows: \mathcal{O} itself.

$$\begin{array}{ccc} \partial_0, \partial_1 = \text{id}_{\mathcal{O}} : \mathcal{O} & \rightrightarrows & \mathcal{O} \\ \uparrow & & \uparrow \\ \text{arrows} & & \text{objects} \end{array}$$

-b. $\mathcal{C}: \text{cat.}$ whose object set is \mathcal{O} は.

$A: \mathcal{O}\text{-graph}$ together with morphisms

$$c: A \times_{\mathcal{O}} A \longrightarrow A \quad (\text{composite})$$

$$i: \mathcal{O} \longrightarrow A \quad (\text{identity}) \quad \text{s.t.}$$

$$\begin{array}{ccc} (A \times_{\mathcal{O}} A) \times_{\mathcal{O}} A & \cong & A \times_{\mathcal{O}} (A \times_{\mathcal{O}} A) \xrightarrow{1 \times c} A \times_{\mathcal{O}} A \\ \downarrow c \times 1 & \curvearrowright & \downarrow c \\ A \times_{\mathcal{O}} A & \xrightarrow{c} & A \end{array} \quad (\text{associativity})$$

$$\begin{array}{ccccc} \mathcal{O} \times_{\mathcal{O}} A & \xrightarrow{i \times 1} & A \times_{\mathcal{O}} A & \xleftarrow{1 \times i} & A \times_{\mathcal{O}} \mathcal{O} \\ & \searrow \sim & \downarrow & \swarrow \sim & \\ & & A & & \end{array} \quad (\text{unit law}).$$

$G: \mathcal{O}\text{-graph}$ から $\mathcal{C} = \mathcal{C}(G): \text{cat}$ を作る.

大雑把に言えば, G の composable pair of arrows をすべて "繋げた" もの.

厳密に言えば,

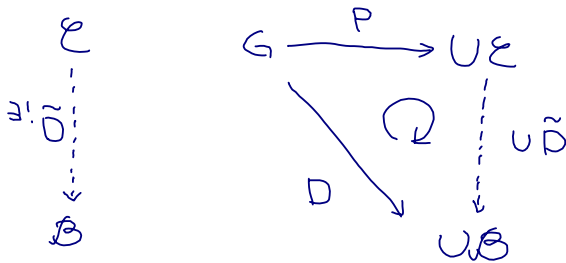
Thm. 2.1.

$G = \{A \rightrightarrows O\}$: small graph.

$\Rightarrow \exists \mathcal{C} = \mathcal{C}_G$: small cat. $P: G \rightarrow U\mathcal{C}$: morphism of graphs s.t.

$\forall (\mathcal{B}$: cat., $D: G \rightarrow U\mathcal{B}$: morphism of graphs),

$\exists! \tilde{D}: \mathcal{C} \rightarrow \mathcal{B}$: functor with $(U\tilde{D}) \circ P = D$.



("任意の graph morphism $D: G \rightarrow U\mathcal{B}$ を.

functor $\tilde{D}: \mathcal{C} \rightarrow \mathcal{B}$ に拡張できる!")

特に. object set of $\mathcal{B} = O$, $D: O$ -graph morphism

$\Rightarrow \tilde{D}|_{\text{obj.}} = \text{id.}$

$P: G \rightarrow U\mathcal{C}$ は. $(G \downarrow U)$ の initial object である.

\therefore unique up to isomorphism.

Proof.

\mathcal{C} is free category generated by G と呼ぶ.

次のように \mathcal{C} を構成する.

● obj.: objects of G i.e., O .

● $ar.$: finite strings $\langle a_1, f_1, a_2, \dots, f_{n-1}, a_n \rangle \equiv a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} a_n$

where $a_i \in O$, $f_i \in A$, $\partial_0 f_i = a_i$, $\partial_1 f_i = a_{i+1}$.

● composite: juxtaposition of strings (i.e., "concatenation").

● identity: $\langle a \rangle \equiv a: a \rightarrow a$

Then, every arrow of \mathcal{C} is a composite of (finitely many) strings of length 2: of length > 1

$$\langle a_1, f_1, \dots, f_{n-1}, a_n \rangle = \langle a_{n-1}, f_{n-1}, a_n \rangle \circ \dots \circ \langle a_1, f_1, a_2 \rangle. \quad (n \geq 2)$$

Define graph morphism $P: G \longrightarrow U\mathcal{C}$ as

• on obj.: identity $0 \longrightarrow 0$;

• on arr.: $A \ni f \longmapsto \langle \partial_0 f, f, \partial_1 f \rangle$.

この (\mathcal{C}, P) が条件を満たすことを示す.

$\mathcal{B}: \text{cat.}$, $D: G \longrightarrow U\mathcal{B}$: graph morphism を任意に取る.

If $\exists \tilde{D}: \mathcal{C} \longrightarrow \mathcal{B}$: functor s.t. $U\tilde{D} \circ P = D$,

$\Rightarrow \tilde{D}$ must be

• on obj.: $\tilde{D}a = Da$;

• on arr.: $\tilde{D}\langle a_1, f_1, a_2 \rangle = Df_1$.

And, by the composition of a functor,

$$\underline{\tilde{D}\langle a_1, f_1, \dots, f_{n-1}, a_n \rangle = Df_{n-1} \circ \dots \circ Df_1.}$$

$\therefore \tilde{D}$ is automatically unique.

よて, \tilde{D} が存在すること示せば, 証明は完了する.

上の通りに \tilde{D} を construct してやる.

$\tilde{D}: \mathcal{C} \longrightarrow \mathcal{B}$ を "functor".

$U\tilde{D} \circ P = D$ を満たすことは明らか.

//

Exs.

• $G = \begin{array}{c} \text{ } \\ \text{ } \end{array} \xrightarrow{f} \begin{array}{c} \text{ } \\ \text{ } \end{array}$ ($0 = \{0\}$, $A = \{f\}$) のとき.

\mathcal{C}_G の arrs は, f, f^2, f^3, \dots

● $G = \bullet \xrightarrow{g} \bullet$ ($O = \{0, 1\}$, $A = \{g\}$) のとき.

\mathcal{E}_G の arrows は. $\text{id}_0: 0 \rightarrow 0$,

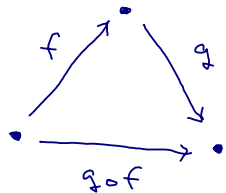
$\text{id}_1: 1 \rightarrow 1$,

$g: 0 \rightarrow 1$

の対. (g と juxtapose できる arrow は 1 個!!)

● $G = \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ ($O = \{0, 1, 2\}$, $A = \{f, g\}$) のとき.

\mathcal{E}_G の arrows は. $\text{id}_i: i \rightarrow i$ ($i=0, 1, 2$) と f, g

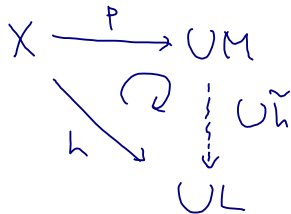


の対. (composable pair は $\langle f, g \rangle$
 " juxtaposable
 (1 個!!))

Cor. 2.2.

$\forall X: \text{set}. \exists M: \text{monoid}, p: X \rightarrow UM: \text{map},$

where $U: \text{Mon} \rightarrow \text{Set}$: forgetful functor, s.t.



$\forall (L: \text{monoid}, h: X \rightarrow UL: \text{map})$
 $\exists! \tilde{h}: M \rightarrow L: \text{monoid hom. s.t.}$
 $h = U\tilde{h} \circ p.$

Proof of Cor 2.2.

//

Thm. 2.1 にあいて $O = \{\bullet\}$, $A = X$ とする.

$\exists M: \text{cat.}$ whose object set is $O = \{\bullet\}$

($M: \text{monoid}$ と見なせる!) s.t.

$U: \text{Cat} \rightarrow \text{Graph}$

$\forall L: \text{monoid}$ (regarded as a category) and $h: G \rightarrow UL: \text{graph morphism},$

($h: X \rightarrow UL: \text{map}$ と見なせる!),

$\hookleftarrow U: \text{Mon} \rightarrow \text{Set}$

$\exists! \tilde{h}: M \rightarrow L$: functor (regarded as a monoid hom.) s.t.

$$h = U \tilde{h} \circ p$$

$\swarrow \quad \searrow$
 $U: \text{Cat} \rightarrow \text{Graph}$ (regarded as $\text{Mon} \rightarrow \text{Set}$).
 $\quad \quad \quad \cap$
 $\quad \quad \quad \text{Cat}$

$\nwarrow \quad \nearrow$
 Set maps (regarded as graph morphisms).

Graphs as diagrams

Def.

G : graph. \mathcal{B} : cat.

A diagram of shape G in \mathcal{B} is

graph morphism $D: G \rightarrow U\mathcal{B}$ or \mathcal{D} . //

By Thm. 2.1, $D \mapsto \tilde{D}: \mathcal{E}_G \rightarrow \mathcal{B}$: functor $= \{ \tilde{D} t_1, \tilde{D} t_2, \dots \}$.

$$\text{Cat}(\mathcal{E}_G, \mathcal{B}) \cong_{\text{bij.}} \text{Graph}(G, U\mathcal{B})$$

is natural in G and \mathcal{B} .

$\therefore \mathcal{E}: \text{Graph} \rightarrow \text{Cat}$ is left adjoint to $U: \text{Cat} \rightarrow \text{Graph}$.
(cf. chapter 4).

Exercise

(2) Every finite ordinal is a free category.

(\because) An finite ordinal n is a free category
generated by $G = 0 \rightarrow 1 \rightarrow \dots \rightarrow n-1$.