

§3.5. Categories with Finite Products

Def.

\mathcal{C} : category.

\mathcal{C} has finite products

def $\forall \{c_i\} \subset \mathcal{C}$: finite family of obj's. $\in \mathcal{C}$,
 $\exists \prod c_i \in \mathcal{C}$

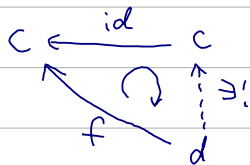
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Rem.

① 0個の object の product とは, terminal object のこと.

$$\begin{array}{c} \exists! \\ t \leftarrow \cdots \cdots c \end{array}$$

② 1個の object の product は, the object itself.



Prop. 3.5.1

\mathcal{C} cat. $\left\{ \begin{array}{l} \exists t \in \mathcal{C} : \text{terminal} \\ \forall a, b \in \mathcal{C}, \exists a \times b \in \mathcal{C} \end{array} \right.$ かつ.

$\Rightarrow \mathcal{C}$ has finite products.

すなわち.

i) $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (は bifunctor).

$$\begin{array}{c} \cup \\ \langle a, b \rangle \mapsto a \times b \end{array}$$

ii) $\exists \alpha_{a,b,c} : (a \times b) \times c \cong a \times (b \times c)$, natural in $a, b, c \in \mathcal{C}$,

iii) $\exists \lambda_a : a \cong t \times a$, $\exists \rho_a : a \cong a \times t$, natural in $a \in \mathcal{C}$

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③ 上の Rem. より, \mathcal{C} が 0, 1, 2 個の object(s) の product を持つことはわかる

$n \geq 3$ には? 証明は,

$$(\cdots ((a_1 \times a_2) \times a_3) \times \cdots) \times a_n$$

が a_1, \dots, a_n の product になる. これは induction on n で示す.

$a := (\dots (a_1 \times a_2) \times \dots) \times a_{n-1}$ が a_1, \dots, a_{n-1} の product とするとき,
 $a \times a_n$ が a_1, \dots, a_n の product とすることも言える

$$\begin{array}{ccccc} a_n & \xleftarrow{p_n} & a \times a_n & \xrightarrow{p} & a & \xrightarrow{p_i} & a_i & (1 \leq i \leq n-1) \\ & \nwarrow f_n & \uparrow \exists! \tilde{f} & \nearrow \exists! \tilde{f} & & & \nearrow f_i \\ & & c & & & & \end{array}$$

$\forall c \in \mathcal{C}, f_i: c \rightarrow a_i \ (1 \leq i \leq n),$

① a の universality から,

$$\exists! \tilde{f}: c \rightarrow a \text{ s.t. } f_i = p_i \circ \tilde{f} \ (\forall i \leq n-1).$$

② $a \times a_n$ の universality から,

$$\exists! \tilde{f}: c \rightarrow a \times a_n \text{ s.t. } \begin{cases} f_n = p_n \circ \tilde{f} \\ \tilde{f} = p \circ \tilde{f} \end{cases}$$

$$\tilde{f} = p \circ \tilde{f} \text{ かつ, } f_i = (p_i \circ p) \circ \tilde{f} \ (\forall i \leq n-1).$$

よって, $a \times a_n$ is a product of a_1, \dots, a_n .

$\therefore \mathcal{C}$ has finite products

i) $\langle f, g \rangle: \langle a, b \rangle \rightarrow \langle a', b' \rangle$ に対して,

$$\exists! f \times g: a \times b \rightarrow a' \times b' \text{ s.t. } \begin{cases} p' \circ (f \times g) = f \circ p \\ q' \circ (f \times g) = g \circ q \end{cases}$$

$$\begin{array}{ccccc} a & \xleftarrow{p} & a \times b & \xrightarrow{q} & b \\ f \downarrow & & \downarrow \exists! f \times g & & \downarrow g \\ a' & \xleftarrow{p'} & a' \times b' & \xrightarrow{q'} & b' \\ f' \downarrow & & \downarrow \exists! f' \times g' & & \downarrow g' \\ a'' & \xleftarrow{p''} & a'' \times b'' & \xrightarrow{q''} & b'' \end{array}$$

さらに, $f \times g$ の uniqueness より,

$$\begin{cases} (f' \times g') \circ (f \times g) = (f' \circ f) \times (g' \circ g) \\ id_a \times id_b = id_{a \times b} \end{cases}$$

も明らか.

よって, $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ は bifunctor を定める.

$$\begin{array}{ccc} \mathcal{C} & \times & \mathcal{C} \\ \downarrow & & \downarrow \\ \langle a, b \rangle & \mapsto & a \times b \end{array}$$

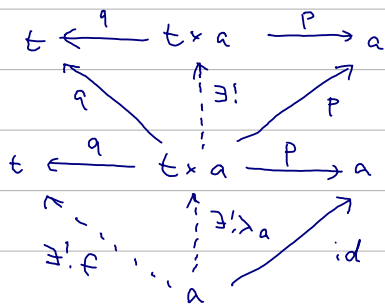
ii) 上と同様のモーション, $(a \times b) \times c \cong a \times (b \times c) \ (\cong a \times b \times c)$ は分かる.

これが natural であること, i.e.,

$$\begin{array}{ccccc} (a \times b) \times c & \xrightarrow{\alpha_{a,b,c}} & a \times (b \times c) & & \forall \langle f, g, h \rangle: \langle a, b, c \rangle \\ (f \times g) \times h \downarrow & \searrow & \downarrow f \times (g \times h) & & \longrightarrow \langle a', b', c' \rangle \\ (a' \times b') \times c' & \xrightarrow{\alpha_{a',b',c'}} & a' \times (b' \times c') & & \end{array}$$

は, 各 map の uniqueness から従う.

iii) λ_a について示す. (ρ_a も同様.)



• $t \times a$ の universality $\nexists!$,

$\exists \lambda_a : a \rightarrow t \times a$ s.t.

$$\begin{cases} p \circ \lambda_a = id_a \\ q \circ \lambda_a = f \end{cases}$$

• t : terminal obj. $\nexists!$, $q = f \circ p$.

$$\begin{cases} p \circ (\lambda_a \circ p) = id_a \circ p = p, & q \circ (\lambda_a \circ p) = f \circ p = q, \\ p \circ id_{t \times a} = p & q \circ id_{t \times a} = q \end{cases} \nexists!$$

$$id_{t \times a} = \lambda_a \circ p.$$

$$\nexists! \lambda_a, p \circ \lambda_a = id_a, \lambda_a \circ p = id_{t \times a} \nexists!$$

$$a \xrightarrow{\lambda_a} t \times a.$$

この naturality (≠) 各 map の uniqueness から従う.

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Coproducts に関する dual prop. も同様に成り立つ.

\mathcal{C} has both finite products & finite coproducts $\nexists!$.

$$a_1, \dots, a_n \in \mathcal{C} \nexists! \perp \mathbb{Z},$$

$$f: \sqcup a_i \rightarrow c \nexists!$$

$$f_j := f \circ \iota_j : a_j \rightarrow c \quad (1 \leq j \leq n) \nexists! \text{ unique に定まる.}$$

特に,

$$f: \bigsqcup_{i=1}^m a_i \rightarrow \prod_{i=1}^n b_i \nexists!$$

$$f_{jk} := \pi_k \circ f \circ \iota_j : a_j \rightarrow b_k \quad \left(\begin{matrix} 1 \leq j \leq m \\ 1 \leq k \leq n \end{matrix} \right) \nexists! \text{ unique に定まる.}$$

さらに, $\exists z \in \mathcal{C} : \text{null}$ $\nexists!$ と.

$$g: \bigsqcup_{i=1}^n a_i \rightarrow \prod_{i=1}^n a_i \nexists!$$

$$g_{jk} = \begin{cases} id & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases} \quad \nexists! \text{ 定まることとは } z \nexists!$$

$$(0: a_j \rightarrow z \rightarrow a_k)$$

Exs.

$$\bullet \mathcal{C} = \text{Ab}, \text{RMod}, \text{etc.} \Rightarrow g: \text{iso.}$$

$$\bullet \mathcal{C} = \text{Set}_*, \text{Top}_* \Rightarrow g: \text{proper monic.}$$

$$\bullet \mathcal{C} = \text{Grp} \Rightarrow g: \text{proper epi.}$$

Exercises

(1) $\delta_c : c \rightarrow c \times c$: diagonal map
(\neq natural in c .)

$$\begin{array}{ccccc} c & \xleftarrow{p_1} & c \times c & \xrightarrow{p_2} & c \\ & \nwarrow id & \uparrow \delta_c & \nearrow id & \\ & & c & & \end{array}$$

⊙ $c \xrightarrow{f} c' \quad c \ni \exists \delta_c$

$$\begin{array}{ccccc} & & c' & & \\ & \swarrow id & \downarrow \exists! \delta_{c'} & \searrow id & \\ c' & \xleftarrow{p'_1} & c' \times c' & \xrightarrow{p'_2} & c' \\ \uparrow f & & \uparrow \exists! f \times f & & \uparrow f \\ c & \xleftarrow{p_1} & c \times c & \xrightarrow{p_2} & c \\ & \nwarrow id & \uparrow \exists! \delta_c & \nearrow id & \\ & & c & & \end{array}$$

から従う。

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(5) \mathcal{B} has finite products \Rightarrow so does $\mathcal{B}^{\mathcal{C}}$.

⊙ Prop. 3.5.1 を利用する。

⊙ $\mathcal{B}^{\mathcal{C}}$ has a terminal obj.

⊙ \mathcal{B} has finite products $\& \&$, $\exists t \in \mathcal{B}$: terminal.

$T : \mathcal{C} \rightarrow \mathcal{B}$: functor $\&$. $T_c = t$ によ、2 定めらる。

$\forall F : \mathcal{C} \rightarrow \mathcal{B}$: functor, $\exists \tau_c : F_c \rightarrow T_c = t \quad (\forall c \in \mathcal{C})$.

τ の naturality ($\&$, t : terminal $\&$) 従う。

$\therefore T$ は $\mathcal{B}^{\mathcal{C}}$ の terminal obj.

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⊙ $\forall F, G : \mathcal{C} \rightarrow \mathcal{B}$: functors, $\exists F \times G \in \mathcal{B}^{\mathcal{C}}$.

⊙ $(F \times G)_c := F_c \times G_c$,

$(F \times G)_f := F_f \times G_f \quad (F : c \rightarrow c')$ とおき。

$F \times G : \mathcal{C} \rightarrow \mathcal{B}$: functor.

$\pi : F \times G \rightarrow F, \quad \rho : F \times G \rightarrow G \quad \&$,

$\pi_c = \rho_{F_c} : F_c \times G_c \rightarrow F_c$ (projections)

$\rho_c = \rho_{G_c} : F_c \times G_c \rightarrow G_c$

によ、2 定めらる。こは natural transformations $\&$ 。

実際, $\forall f: c \rightarrow c'$,

$$\begin{array}{ccccc} F_c & \xleftarrow{\pi_c} & F_c \times G_c & \xrightarrow{p_c} & G_c \\ Ff \downarrow & & \downarrow \exists! Ff \times Gf & & \downarrow Gf \\ F_{c'} & \xleftarrow{\pi_{c'}} & F_{c'} \times G_{c'} & \xrightarrow{p_{c'}} & G_{c'} \end{array}$$

からわかる.

$F \times G$: product of F, G も見る.

$\forall \sigma: H \rightarrow F, \tau: H \rightarrow G,$

$U: H \rightarrow F \times G$ と,

$$\begin{array}{ccccc} F_c & \xleftarrow{\pi_c} & F_c \times G_c & \xrightarrow{p_c} & G_c \\ & \nwarrow \sigma_c & \uparrow \exists! U_c & \nearrow \tau_c & \\ & H_c & & & \end{array}$$

に $F \times G$ を定める.

これは natural transformation である. 実際, $\forall f: c \rightarrow c'$,

$$\begin{array}{ccccc} & & H_{c'} & & \\ & \swarrow \sigma_{c'} & \downarrow U_{c'} & \searrow \tau_{c'} & \\ F_{c'} & \xleftarrow{\pi_{c'}} & F_{c'} \times G_{c'} & \xrightarrow{p_{c'}} & G_{c'} \\ Ff \uparrow & & \uparrow Ff \times Gf & & \uparrow Gf \\ F_c & \xleftarrow{\pi_c} & F_c \times G_c & \xrightarrow{p_c} & G_c \\ & \nwarrow \sigma_c & \uparrow \exists! U_c & \nearrow \tau_c & \\ & H_c & & & \end{array}$$

Hf

$$\begin{cases} \pi_{c'} \circ (U_{c'} \circ Hf) = \sigma_{c'} \circ Hf = Ff \circ \sigma_c, \\ p_{c'} \circ (U_{c'} \circ Hf) = \tau_{c'} \circ Hf = Gf \circ \tau_c, \\ (Ff \times Gf) \circ U_c = U_{c'} \circ Hf. \end{cases} \quad \text{f'ly,}$$

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↓ \mathcal{C} is Prop. 3.5.1 f'ly.

$\mathcal{B}^{\mathcal{C}}$ has finite products.

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§3.6 Groups in Categories

Def.

\mathcal{C} : category with finite products, $t \in \mathcal{C}$: terminal. $c \in \mathcal{C} \iff \exists$.

c is monoid in \mathcal{C} iff $\exists \mu, \eta$.

$\exists \mu: c \times c \rightarrow c, \eta: t \rightarrow c$ s.t.

$$\begin{array}{ccccc} (c \times c) \times c & \xrightarrow{\sim} & c \times (c \times c) & \xrightarrow{1 \times \mu} & c \times c \\ \mu \times 1 \downarrow & & \downarrow \mu & & \downarrow \mu \\ c \times c & \xrightarrow{\mu} & c & & c \end{array}$$

$$\begin{array}{ccccc} t \times c & \xrightarrow{\eta \times 1} & c \times c & \xleftarrow{1 \times \eta} & c \times t \\ & \searrow \eta & \downarrow \mu & \swarrow \eta & \\ & c & & c & \end{array}$$

c is group in \mathcal{C} iff $\exists \zeta$ s.t. c is monoid and $\exists \zeta$.

$\exists \zeta: c \rightarrow c$ s.t.

$$\begin{array}{ccccc} & c \times c & & & \\ \zeta \times 1 \swarrow & \uparrow \delta_c & \searrow 1 \times \zeta & & \\ c \times c & & c \times c & & \\ \swarrow \mu & \downarrow \eta & \searrow \mu & & \\ & c & & & \end{array}$$

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Prop. 3.6.1

\mathcal{C} : category with finite products, $c \in \mathcal{C} \iff \exists$.

c is monoid in $\mathcal{C} \iff \mathcal{C}(\cdot, c)$ is monoid in $\mathbf{Set}^{\text{cop}}$.
(resp. group) (resp. group)

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\Rightarrow

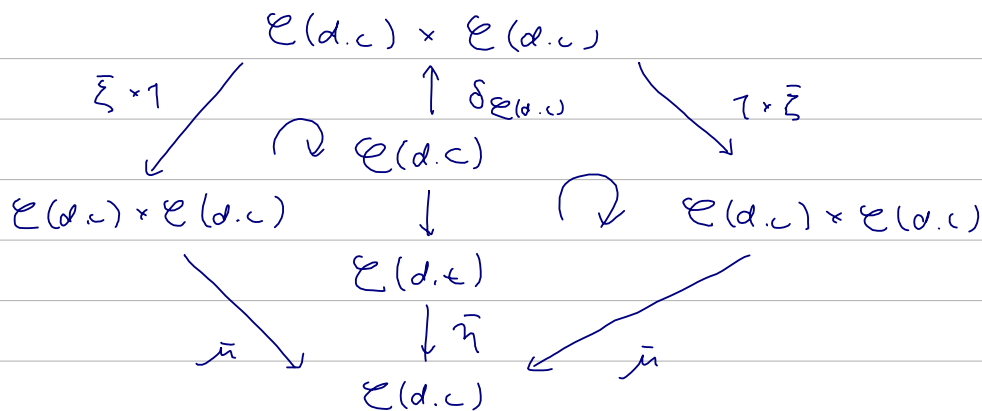
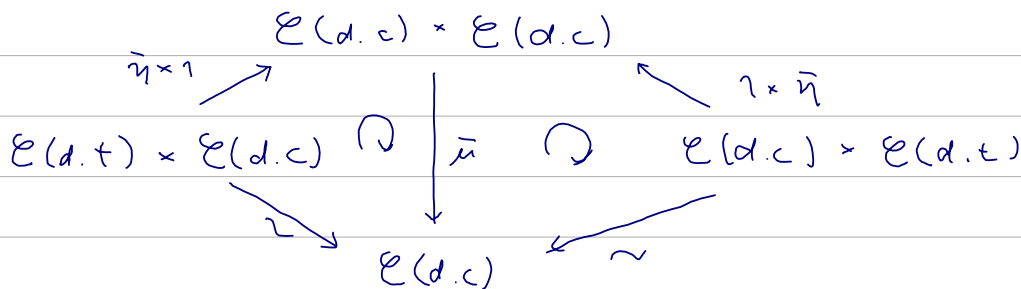
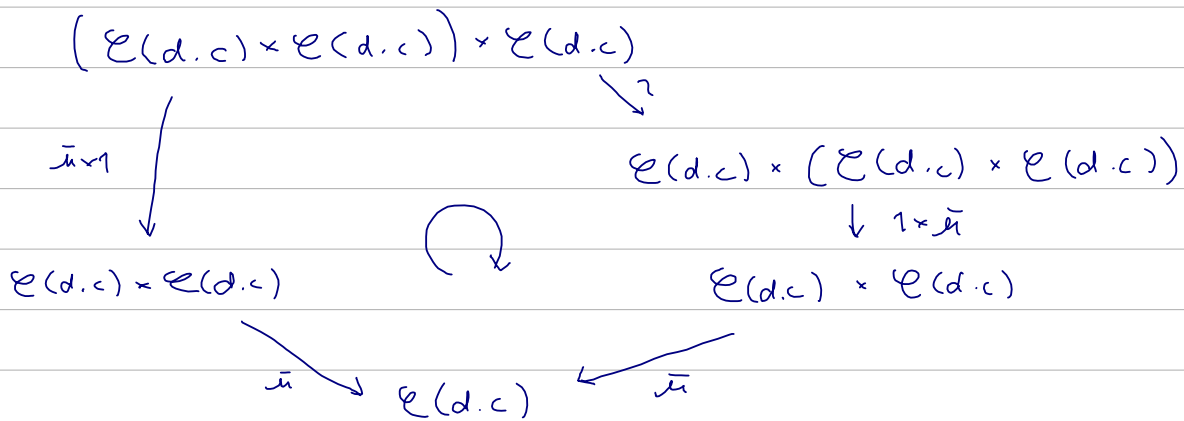
$$\bar{\mu}: \mathcal{C}(\cdot, c) \times \mathcal{C}(\cdot, c) \xrightarrow{\sim} \mathcal{C}(\cdot, c \times c) \xrightarrow{\mu^*} \mathcal{C}(\cdot, c),$$

$$\bar{\eta}: \mathcal{C}(\cdot, t) \xrightarrow{\eta^*} \mathcal{C}(\cdot, c),$$

$$\bar{\zeta}: \mathcal{C}(\cdot, c) \xrightarrow{\zeta^*} \mathcal{C}(\cdot, c)$$

(cf. Exercise 3.6.5)

この monoid は unit group であることを示すには、



の可換性を示すのは「お、か」。これは $\mathcal{E}(\cdot, c)$ が functor であることを、 μ, η, ξ に関する diagrams から示す。

$\therefore \mathcal{E}(\cdot, c) : \text{monoid (or group) in } \mathbf{Set}^{\mathcal{E}^{\text{op}}}$

⇐ 逆に、 $\mathcal{E}(\cdot, c) : \text{monoid (or group) in } \mathbf{Set}^{\mathcal{E}^{\text{op}}}$ 。

$$\bar{\mu} : \mathcal{E}(\cdot, c) \times \mathcal{E}(\cdot, c) \longrightarrow \mathcal{E}(\cdot, c)$$

$$\bar{\eta} : \mathcal{E}(\cdot, +) \longrightarrow \mathcal{E}(\cdot, c)$$

$$\bar{\xi} : \mathcal{E}(\cdot, c) \longrightarrow \mathcal{E}(\cdot, c)$$

を構造成射とする。

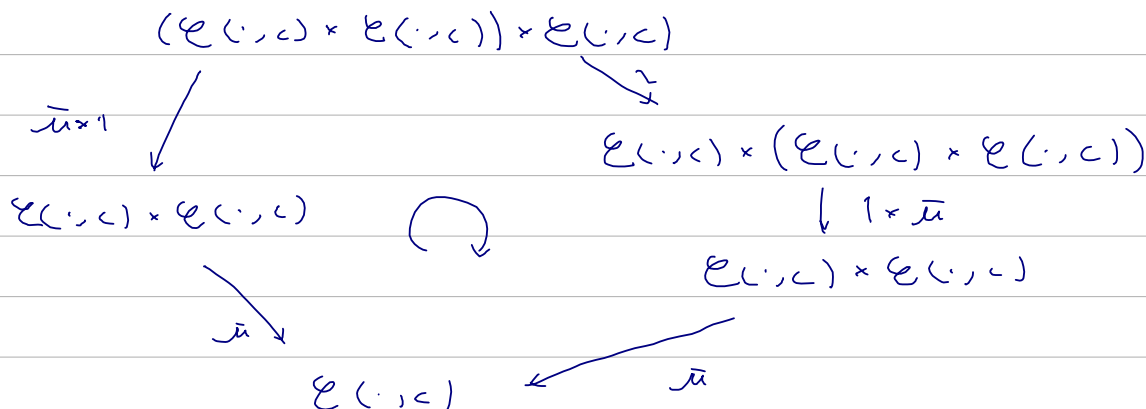
Yoneda Lemma に於ける natural iso. ξ γ へ, $\gamma \in \mathcal{C}$.

$$\mu := \gamma(\bar{\mu} \circ \theta^\gamma) = \bar{\mu}_{c \times c} \cdot \theta_{1c}^\gamma 1_{c \times c}$$

$$\eta := \gamma(\bar{\eta}) = \bar{\eta}_c 1_c$$

$$\xi := \gamma(\bar{\xi}) = \bar{\xi}_c 1_c$$

と置く. このとき,



に γ を適用して.

$$\gamma(\downarrow) = \mu \circ (\mu \times 1)$$

$$\gamma(\downarrow) = \mu \circ (1 \times \mu)$$

を得. γ は bijective であるから, μ の associativity が成り立つ.

η, ξ に関する公理も同様.

$\therefore c$ は monoid (or group) in \mathcal{C} .

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Exercises

\mathcal{C} : category with finite products, $t \in \mathcal{C}$: terminal object.

(1) \mathcal{G} : category with

• obj.: monoid in \mathcal{C}

• arr.: $f: G \rightarrow H$ in \mathcal{C} s.t. $f \mu_G = \mu_H (f \times f)$

とする. $t \in \mathcal{G}$ は, $\mathcal{G} \neq \emptyset$.

\mathcal{G} は 実際 category である. (\odot $1_G: G \rightarrow G$ は monoid map).

このとき, \mathcal{G} has finite products.

\odot $G, H \in \mathcal{G} \Rightarrow G \times H \in \mathcal{G}$ を示すのは "易".

これは monoid map である. \odot $c = t$ である.

$\alpha_c: (c \times c) \times c \xrightarrow{\sim} c \times (c \times c)$ とする.

$G \times H \in \mathcal{C}_I$ を示すには. $\mu_G \times \mu_H$ の associativity と unit law を見れば"よ"か.
これは products の universality から従う. //