

# Categories for the Working Mathematicians. 2nd ed.

Mac Lane. Springer.

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## § 2.6. Comma Categories

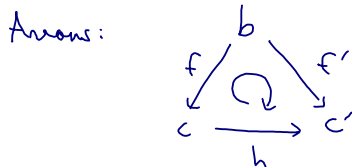
Def.

$\mathcal{C}$ : category.  $b \in \mathcal{C}$ : object.

$(b \downarrow \mathcal{C})$ : category of objects under  $b$  with

• objects:  $\langle f, c \rangle$ , where  $c \in \mathcal{C} \text{ obj.}$  and  $f: b \rightarrow c$  in  $\mathcal{C}$ .

• arrows:  $\langle f, c \rangle \xrightarrow{h} \langle f', c' \rangle$ , where  $h: c \rightarrow c'$  s.t.  
 $f' = h \circ f$ .



Exs.

•  $*$ : 1-point set  $\text{on } \mathbb{Z}$ .  $(* \downarrow \text{Set}) = \text{Set}_*$ .

( $\odot$  Obj.  $* \rightarrow X$  is pair  $\langle x \in X, X \rangle$  on  $\mathbb{Z}$ .  
 Arr.  $\langle x, X \rangle \rightarrow \langle x', X' \rangle$  is base point  $\in \langle \mathbb{Z} \rangle$  map on  $\mathbb{Z}$ .)

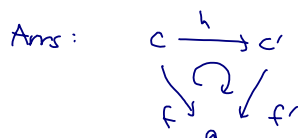
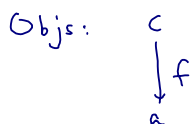
•  $(\mathbb{Z} \downarrow \text{Ab}) = \text{Ab}_*$

( $\odot$  Obj.  $\mathbb{Z} \xrightarrow{f} G$  is  $\langle f(1), G \rangle$  on  $\mathbb{Z}$ . (base point  $f(1)$ ).  
 Arr.  $\langle g, G \rangle \rightarrow \langle g', G' \rangle$  is base point  $\in \langle \mathbb{Z} \rangle$  group hom. on  $\mathbb{Z}$ .)

Def.

$\mathcal{C}$ : cat.  $a \in \mathcal{C}$ : obj.

$(\mathcal{C} \downarrow a)$ : category of objects over  $a$  with



## Exs.

①  $*$ : 1-point set.  $\Rightarrow (\text{Set} \downarrow *) \cong_{\text{cat}} \text{Set}.$

② Obj.  $X \xrightarrow{f} *$  is,  $*$ : terminal obj. in Set だから,  $X$  は何なりか...  
 任意の  $h: X \rightarrow X'$  in Set は,  $f = f' \circ h$  を満たす.  

$$X \xrightarrow{h} X' \quad (f: X \rightarrow *, f': X' \rightarrow *)$$

③  $(\text{Rng} \downarrow \mathbb{Z})$ : augmentation  $R \rightarrow \mathbb{Z}$  as objects,  
 ring hom preserving augmentations as arrows.

④ Obj.  $R \xrightarrow{\varepsilon} \mathbb{Z}$  is, augmentation  $\varepsilon$  に対して. (一般論).  
 An  $\langle \varepsilon, R \rangle \xrightarrow{h} \langle \varepsilon', R' \rangle$  is,  $\varepsilon = \varepsilon' \circ h$  である ring hom  $h: R \rightarrow R'$ .  
 i.e., ring hom that preserves augmentations.

## Def.

$S: \mathcal{D} \rightarrow \mathcal{E}$ : functor.  $b \in \mathcal{E}$ : obj.

$(b \downarrow S)$ : category of objects  $S$ -under  $b$ . with

Obj.  $\begin{array}{c} b \\ \downarrow f \\ Sd \end{array}$      An.  $\begin{array}{ccc} & b & \\ f \swarrow & \curvearrowright & \searrow f' \\ Sd & \xrightarrow{Sh} & Sd' \end{array}$

## Def.

$T: \mathcal{E} \rightarrow \mathcal{E}$ : functor.  $a \in \mathcal{E}$ : obj.

$(T \downarrow a)$ : category of objects  $T$ -over  $a$  with

Obj.  $\begin{array}{c} Tc \\ \downarrow f \\ a \end{array}$      An.  $\begin{array}{ccc} Tc & \xrightarrow{Th} & Tc' \\ f \downarrow & \curvearrowright & \downarrow f' \\ & a & \end{array}$

Ex.

①  $U: \text{Grp} \rightarrow \text{Set}$  : forgetful functor.  $x \in \text{Set} : \text{obj.}$  と  $\{ \}$ .

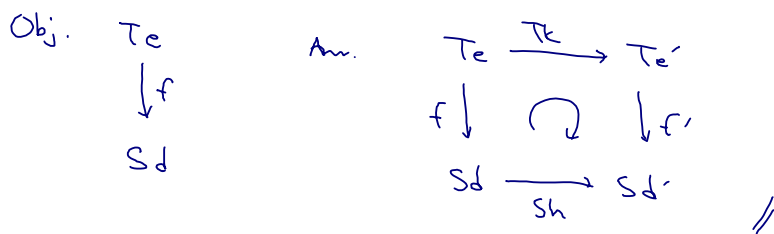
$(x \downarrow U) : \text{map } x \xrightarrow{f} Uq \text{ as obj}$

group hom  $h: \langle f, g \rangle \rightarrow \langle f', g' \rangle$  s.t.  $f' = h \circ f$  as arr.

Def.

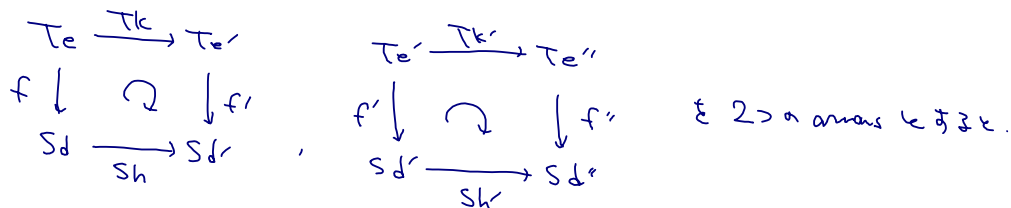
$\mathcal{C} \xrightarrow{T} \mathcal{C} \xleftarrow{S} \mathcal{D}$  : functors.

$(T \downarrow S)$  : comma category with

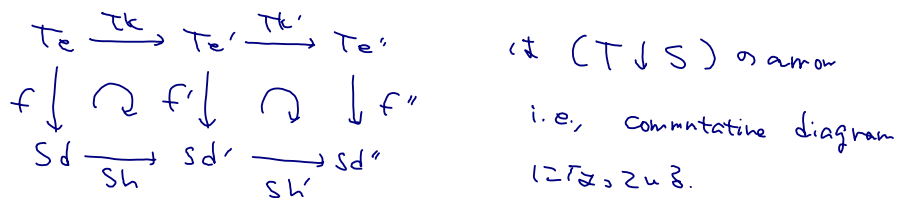


Rem.

Composites of  $(T \downarrow S)$  is well-defined:



$\{ \}$  composite



$(T \downarrow S)$  is  $(b \downarrow \mathcal{C})$ ,  $(\mathcal{C} \downarrow a)$ ,  $(b \downarrow S)$ ,  $(T \downarrow a)$  の一般化である!

①  $(b \downarrow S)$  is.  $T = b: \mathbb{1} \rightarrow \mathcal{C}$  と  $\mathcal{C} \in \mathcal{C}$  と.

②  $(T \downarrow a)$  is.  $S = a: \mathbb{1} \rightarrow \mathcal{C}$  と  $\mathcal{C} \in \mathcal{C}$  と.

③  $(b \downarrow \mathcal{C})$  is.  $T = b: \mathbb{1} \rightarrow \mathcal{C}$ ,  $S = 1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$  と  $\mathcal{C} \in \mathcal{C}$  と.

④  $(\mathcal{C} \downarrow a)$  is.  $S = a: \mathbb{1} \rightarrow \mathcal{C}$ ,  $T = 1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$  と  $\mathcal{C} \in \mathcal{C}$  と.

Exs.

①  $\mathcal{C} : \text{cat.}$   $\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} : \text{id. functor.}$   $\circ \neq \text{id.}$

$$(\mathcal{C} \downarrow \mathcal{C}) = \mathcal{C}^2$$

$$\left( \begin{array}{l} \text{Obj. } c \xrightarrow{f} d. \\ \text{Arr. } \begin{array}{ccc} c & \xrightarrow{f} & d \\ k \downarrow & \curvearrowright & \downarrow h \\ c' & \xrightarrow{f'} & d' \end{array} \end{array} \Rightarrow \mathcal{C}^2 \left( \begin{array}{l} \text{obj: arrows.} \\ \text{arr: commutative squares} \end{array} \right) \right)$$

②  $T = b, S = a : \mathbb{1} \rightarrow \mathcal{C}.$   $\circ \neq \text{id.}$

$$(b \downarrow a) = \text{hom}_{\mathcal{C}}(b, a) \quad (\text{as a discrete cat.}).$$

$$\left( \begin{array}{l} \text{Obj. } b \xrightarrow{f} a. \\ \text{Arr. } \begin{array}{ccc} b & \xrightarrow{f} & a \\ \text{id} \downarrow & \curvearrowright & \downarrow \text{id} \\ b & \xrightarrow{f'} & a \end{array} \end{array} \Rightarrow f = f'. \quad \text{i.e., arrow is trivial to be equal.} \right)$$

↑ "comma category" の名前が由来!

$(T \downarrow S)$  の普遍性 (cf. Exercise (5)).

$$\begin{array}{ccccc} & & (T \downarrow S) & & \\ & \swarrow P & \downarrow R & \searrow Q & \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} & \xleftarrow{\mathcal{C}^{d_0}} & \mathcal{C}^2 & \xrightarrow{\mathcal{C}^{d_1}} & \mathcal{C} & \xleftarrow{S} & \mathcal{D} \end{array}$$

$$d_0, d_1 : \mathbb{1} \rightarrow \mathbb{2}$$

$P, Q$  : projections.

$\mathcal{C}^{d_0}$  (resp  $\mathcal{C}^{d_1}$ ) は, arrow  $a \xrightarrow{f} b \in \mathcal{C}$ ,  $a = \text{dom } f$  (resp.  $b = \text{cod } f$ ) になる functor. (cf. § 2.5).

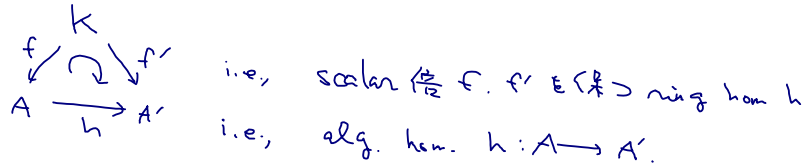
# Exercises.

(1)  $K$ : com. ring  $\alpha \neq 0$ .  $(K \downarrow \mathbb{C}Rng) = \mathbb{C}Alg_K$ .

Obj.  $A$ : com ring,  $f: K \rightarrow A$ : ring hom.

i.e.,  $A$ : com. alg. /  $K$ .

Ans.  $(A, f) \xrightarrow{h} (A', f')$ : ring hom s.t.

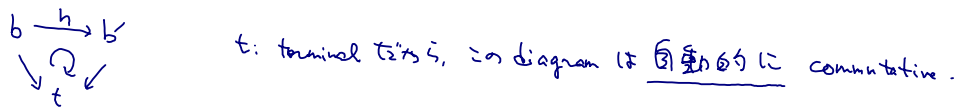


(2)  $\mathcal{C}$ : cat.  $t \in \mathcal{C}$ : terminal  $\Rightarrow (\mathcal{C} \downarrow t) \cong \mathcal{C}$ .

Obj.  $b \xrightarrow{f} t$  is  $t$ : terminal  $t$ 's is.

$\forall b \in \mathcal{C}$  (2  $\nexists$   $t$ ) one and exactly one  $f$  exists.

Ans.  $b \xrightarrow{h} b$  s.t.

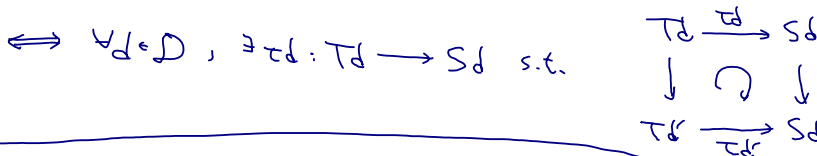


(4) (S.A. Hug).  $T, S: \mathcal{D} \rightarrow \mathcal{C}$ : functors  $\alpha \neq 0$ .

$\tau: T \rightarrow S$ : nat. transformation  $\neq 0$ .  $\tau: \mathcal{D} \rightarrow (T \downarrow S)$ : functor s.t.

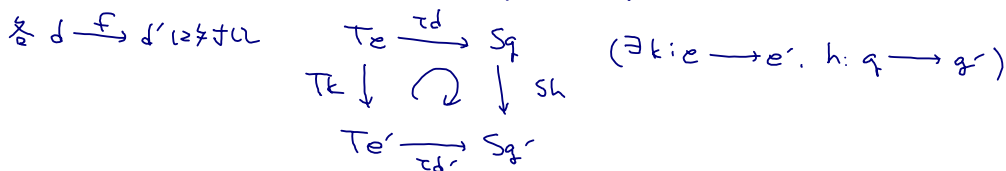
$P_\tau = Q_\tau = id_{\mathcal{D}}$  is satisfied. ( $P, Q$ : prjs.)

Obj.  $\tau: T \rightarrow S$



$\tau: \mathcal{D} \rightarrow (T \downarrow S)$  s.t.  $P_\tau = Q_\tau = id_{\mathcal{D}}$

$\Leftrightarrow \forall d \in \mathcal{D}$  (2  $\nexists$   $t$ )  $\tau_d: T_d \rightarrow S_d$  ( $\exists e, g \in \mathcal{D}$ )



s.t.

$$\begin{array}{ccccc}
 e & = & g & = & d \\
 \parallel & & \parallel & & \parallel \\
 P_{\tau d} & & Q_{\tau d} & & id_D d
 \end{array}, \quad
 \begin{array}{ccccc}
 k & = & h & = & f \\
 \parallel & & \parallel & & \parallel \\
 P_{\tau f} & & Q_{\tau f} & & id_D f
 \end{array}$$

$$\Leftrightarrow \forall d \in \mathcal{D}. \exists \tau_d: T_d \rightarrow S_d \text{ s.t.}$$

$$\begin{array}{ccc}
 T_d & \xrightarrow{\tau_d} & S_d \\
 \tau_f \downarrow & \curvearrowright & \downarrow S_f \\
 T_{d'} & \xrightarrow{\tau_{d'}} & S_{d'}
 \end{array} \quad (f: d \rightarrow d')$$

上の2>は同値.

## §2.7. Graphs and Free Categories.

まず, free monoid の場合を思い出そう.

$X$ : set of symbols 生成される free monoid  $F$  とは,

- Word  $x_1 \dots x_n$  ( $x_i \in X$ ) が元を持ち,
- words の 結合 (juxtaposition) を積とし,
- empty word が identity element となる

monoid のこと.

これは equivalent な条件は,

$$\begin{array}{ccc}
 X & \xrightarrow{\iota} & F \\
 & \searrow f & \downarrow \exists! \tilde{f} \\
 & & M
 \end{array}
 \quad (F: \text{monoid}, \iota: X \rightarrow F: \text{map}) \text{ free monoid}$$

$$\Leftrightarrow \forall (M: \text{monoid}, f: X \rightarrow M: \text{map}), \quad \exists! \tilde{f}: F \rightarrow M: \text{monoid hom. s.t. } f = \tilde{f} \circ \iota.$$

これは  $\pi$  に free category と考えられる.

Def. (§1.2)

$G$ : (directed) graph とは,

- $O$ : set of objects (vertices)
- $A$ : set of arrows (edges),

together with functions

$$A \xrightarrow[\partial_1]{\partial_0} O, \quad \partial_0 f = \text{dom } f, \quad \partial_1 f = \text{cod } f.$$

$D: G \rightarrow G'$ : morphism of graphs とは,

$$\bullet D_0: O \rightarrow O'$$

$$\bullet D_A: A \rightarrow A' \quad \text{s.t.} \quad D_0 \partial_0 = \partial_0 D_A, \quad D_0 \partial_1 = \partial_1 D_A.$$

//

Graph: category of graphs.



Graphs に  $\circ$  (composite) と identity が  $\neq$  ない。

↑ diagram scheme  $\neq$  precategory  $\neq$   $\mathbb{C}$  ではない!

### Graphs obtained from categories

$\mathcal{C}$ : cat.  $\hookrightarrow \mathbb{C}$   $\cup \mathcal{C}$ : graph  $\mathcal{C}$ .

• obj.: objects of  $\mathcal{C}$ ;

• arr.: arrows of  $\mathcal{C}$

$\mathbb{C}$  の  $\mathbb{C}$  である.  $F: \mathcal{C} \rightarrow \mathcal{C}'$ : functor  $\neq$ .

graph morphism  $UF: U\mathcal{C} \rightarrow U\mathcal{C}'$  に  $\neq$  対応する.

$\therefore U: \text{Cat} \rightarrow \text{Graph}$ : forgetful functor.

### Categories obtained from graphs

$O$ : set: fixed.

Def.

$O$ -graph  $\neq$  graph where object set is  $O$ .

A morphism of  $O$ -graph  $\neq$   $O$ -graphs  $\mathbb{C}$  の graph morphism  $\neq$   $\mathbb{C}$ . //

$A, B$ : sets of arrows (regarded as  $O$ -graphs)  $\neq \mathbb{C}$   $\neq$ .

$$A \times_0 B := \{ \langle g, f \rangle \mid \partial_0 g = \partial_1 f, g \in A, f \in B \}$$

(composable pairs of arrows  $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ .)

$\mathbb{C}$  の  $\mathbb{C}$  である.

$$\partial_0 \langle g, f \rangle := \partial_0 f, \quad \partial_1 \langle g, f \rangle := \partial_1 g$$

$\neq$   $\mathbb{C}$   $\neq$   $\mathbb{C}$ :  $A \times_0 B$ :  $O$ -graph  $\neq \mathbb{C}$ .

$\times_0$   $\neq$  associative

$$\left( \begin{aligned} \therefore (A \times_0 B) \times_0 C &\cong A \times_0 (B \times_0 C) \\ &\cong \{ \langle h, g, f \rangle \mid \partial_0 h = \partial_1 g, \partial_0 g = \partial_1 f, h \in A, g \in B, f \in C \} \end{aligned} \right)$$

任意の  $A: \mathcal{O}\text{-graph}$  に対して.

$$\begin{array}{ccccc} A & \cong & A \times_{\mathcal{O}} \mathcal{O} & \cong & \mathcal{O} \times_{\mathcal{O}} A \\ \downarrow & & \downarrow & & \downarrow \\ f & \mapsto & \langle f, \partial_0 f \rangle & \mapsto & \langle \partial_1 f, f \rangle. \end{array}$$

Rem.  $\mathcal{O}$  は次のように  $\mathcal{O}\text{-graph}$  とみなされる:

• arrows:  $\mathcal{O}$  itself.

$$\begin{array}{ccc} \partial_0, \partial_1 = \text{id}_{\mathcal{O}} : \mathcal{O} & \rightrightarrows & \mathcal{O} \\ \uparrow & & \uparrow \\ \text{arrows} & & \text{objects} \end{array}$$

-b.  $\mathcal{C}: \text{cat.}$  whose object set is  $\mathcal{O}$  は.

$A: \mathcal{O}\text{-graph}$  together with morphisms

$$c: A \times_{\mathcal{O}} A \longrightarrow A \quad (\text{composite})$$

$$i: \mathcal{O} \longrightarrow A \quad (\text{identity}) \quad \text{s.t.}$$

$$\begin{array}{ccc} (A \times_{\mathcal{O}} A) \times_{\mathcal{O}} A & \cong & A \times_{\mathcal{O}} (A \times_{\mathcal{O}} A) \xrightarrow{1 \times c} A \times_{\mathcal{O}} A \\ \downarrow c \times 1 & \curvearrowright & \downarrow c \\ A \times_{\mathcal{O}} A & \xrightarrow{c} & A \end{array} \quad (\text{associativity})$$

$$\begin{array}{ccccc} \mathcal{O} \times_{\mathcal{O}} A & \xrightarrow{i \times 1} & A \times_{\mathcal{O}} A & \xleftarrow{1 \times i} & A \times_{\mathcal{O}} \mathcal{O} \\ & \searrow \sim & \downarrow & \swarrow \sim & \\ & & A & & \end{array} \quad (\text{unit law}).$$

$G: \mathcal{O}\text{-graph}$  から  $\mathcal{C} = \mathcal{C}(G): \text{cat}$  を作る.

大雑把に言えば,  $G$  の composable pair of arrows をすべて "繋げた" もの.

厳密に言えば,

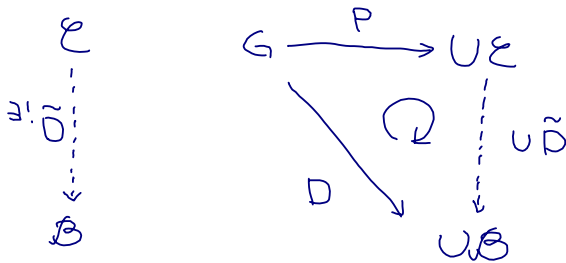
Thm. 2.1.

$G = \{A \rightrightarrows O\}$  : small graph.

$\Rightarrow \exists \mathcal{C} = \mathcal{C}_G$  : small cat.  $P: G \rightarrow U\mathcal{C}$  : morphism of graphs s.t.

$\forall (\mathcal{B}$  : cat.,  $D: G \rightarrow U\mathcal{B}$  : morphism of graphs),

$\exists! \tilde{D}: \mathcal{C} \rightarrow \mathcal{B}$  : functor with  $(U\tilde{D}) \circ P = D$ .



("任意の graph morphism  $D: G \rightarrow U\mathcal{B}$  を.

functor  $\tilde{D}: \mathcal{C} \rightarrow \mathcal{B}$  に拡張できる!")

特に. object set of  $\mathcal{B} = O$ ,  $D: O$ -graph morphism

$\Rightarrow \tilde{D}|_{\text{obj.}} = \text{id.}$

$P: G \rightarrow U\mathcal{C}$  は.  $(G \downarrow U)$  の initial object である.

$\therefore$  unique up to isomorphism.

Proof.

$\mathcal{C}$  is free category generated by  $G$  と呼ぶ.

次のように  $\mathcal{C}$  を構成する.

● obj.: objects of  $G$  i.e.,  $O$ .

●  $ar.$ : finite strings  $\langle a_1, f_1, a_2, \dots, f_{n-1}, a_n \rangle \equiv a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} a_n$

where  $a_i \in O$ ,  $f_i \in A$ ,  $\partial_0 f_i = a_i$ ,  $\partial_1 f_i = a_{i+1}$ .

● composite: juxtaposition of strings (i.e., "concatenation").

● identity:  $\langle a \rangle \equiv a: a \rightarrow a$

Then, every arrow of  $\mathcal{C}$  is a composite of (finitely many) strings of length 2: of length  $> 1$

$$\langle a_1, f_1, \dots, f_{n-1}, a_n \rangle = \langle a_{n-1}, f_{n-1}, a_n \rangle \circ \dots \circ \langle a_1, f_1, a_2 \rangle. \quad (n \geq 2)$$

Define graph morphism  $P: G \longrightarrow U\mathcal{C}$  as

• on obj.: identity  $0 \longrightarrow 0$ ;

• on arr.:  $A \ni f \longmapsto \langle \partial_0 f, f, \partial_1 f \rangle$ .

この  $(\mathcal{C}, P)$  が条件を満たすことを示す.

$\mathcal{B}: \text{cat.}$ ,  $D: G \longrightarrow U\mathcal{B}$ : graph morphism を任意に取る.

If  $\exists \tilde{D}: \mathcal{C} \longrightarrow \mathcal{B}$ : functor s.t.  $U\tilde{D} \circ P = D$ ,

$\Rightarrow \tilde{D}$  must be

• on obj.:  $\tilde{D}a = Da$ ;

• on arr.:  $\tilde{D}\langle a_1, f_1, a_2 \rangle = Df_1$ .

And, by the composition of a functor,

$$\underline{\tilde{D}\langle a_1, f_1, \dots, f_{n-1}, a_n \rangle = Df_{n-1} \circ \dots \circ Df_1.}$$

$\therefore \tilde{D}$  is automatically unique.

よて,  $\tilde{D}$  が存在すること示せば, 証明は完了する.

上の通りに  $\tilde{D}$  を construct してやる.

$\tilde{D}: \mathcal{C} \longrightarrow \mathcal{B}$  を "functor".

$U\tilde{D} \circ P = D$  を満たすことは明らか.

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Exs.

•  $G = \begin{array}{c} \text{ } \\ \text{ } \end{array} \xrightarrow{f} \begin{array}{c} \text{ } \\ \text{ } \end{array}$  ( $0 = \{0\}$ ,  $A = \{f\}$ ) のとき.

$\mathcal{C}_G$  の arrs は,  $f, f^2, f^3, \dots$

●  $G = \bullet \xrightarrow{g} \bullet$  ( $O = \{0, 1\}$ ,  $A = \{g\}$ ) のとき.

$\mathcal{E}_G$  の arrows は.  $\text{id}_0: 0 \rightarrow 0$ ,

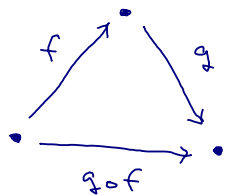
$\text{id}_1: 1 \rightarrow 1$ ,

$g: 0 \rightarrow 1$

の対. ( $g$  と juxtapose できる arrow は 1 個!!)

●  $G = \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$  ( $O = \{0, 1, 2\}$ ,  $A = \{f, g\}$ ) のとき.

$\mathcal{E}_G$  の arrows は.  $\text{id}_i: i \rightarrow i$  ( $i=0, 1, 2$ ) と  $f, g$

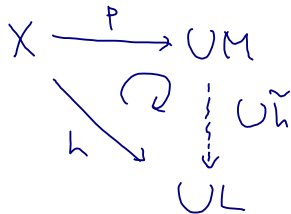


の対. (composable pair は  $\langle f, g \rangle$   
 " juxtaposable  
 (1 個!!))

Cor. 2.2.

$\forall X: \text{set}. \exists M: \text{monoid}, p: X \rightarrow UM: \text{map},$

where  $U: \text{Mon} \rightarrow \text{Set}$ : forgetful functor, s.t.



$\forall (L: \text{monoid}, h: X \rightarrow UL: \text{map})$   
 $\exists! \tilde{h}: M \rightarrow L: \text{monoid hom. s.t.}$   
 $h = U\tilde{h} \circ p.$

Proof of Cor 2.2.

//

Thm. 2.1 にあいて  $O = \{\bullet\}$ ,  $A = X$  とする.

$\exists M: \text{cat.}$  whose object set is  $O = \{\bullet\}$

( $M: \text{monoid}$  と見なせる!) s.t.

$U: \text{Cat} \rightarrow \text{Graph}$

$\forall L: \text{monoid}$  (regarded as a category) and  $h: G \rightarrow UL: \text{graph morphism},$

( $h: X \rightarrow UL: \text{map}$  と見なせる!),

$\hookleftarrow U: \text{Mon} \rightarrow \text{Set}$

$\exists! \tilde{h}: M \rightarrow L$  : functor (regarded as a monoid hom.) s.t.

$$h = U \tilde{h} \circ p$$

$\swarrow \quad \searrow$   
 $U: \text{Cat} \rightarrow \text{Graph}$  (regarded as  $\text{Mon} \rightarrow \text{Set}$ ).  
 $\quad \quad \quad \cap$   
 $\quad \quad \quad \text{Cat}$

$\nearrow$   
 Set maps (regarded as graph morphisms).

## Graphs as diagrams

Def.

$G$ : graph.  $\mathcal{B}$ : cat.

A diagram of shape  $G$  in  $\mathcal{B}$  is

graph morphism  $D: G \rightarrow U\mathcal{B}$  or  $\mathcal{D}$ . //

By Thm. 2.1,  $D \mapsto \tilde{D}: \mathcal{E}_G \rightarrow \mathcal{B}$  : functor  $= \{ \tilde{D} t_1, \tilde{D} t_2, \dots \}$ .

$$\text{Cat}(\mathcal{E}_G, \mathcal{B}) \cong_{\text{bij.}} \text{Graph}(G, U\mathcal{B})$$

is natural in  $G$  and  $\mathcal{B}$ .

$\therefore \mathcal{E}: \text{Graph} \rightarrow \text{Cat}$  is left adjoint to  $U: \text{Cat} \rightarrow \text{Graph}$ .  
(cf. chapter 4).

## Exercise

(2) Every finite ordinal is a free category.

( $\because$ ) An finite ordinal  $n$  is a free category  
generated by  $G = 0 \rightarrow 1 \rightarrow \dots \rightarrow n-1$ .

## §8. Quotient Categories

Def.

$\mathcal{L} : \text{cat.}$

R: congruence on  $\mathcal{E}$  iff

i)  $\forall a, b \in E \Rightarrow \exists! L2$ .  $R_{a,b}$ : equivalence relation on  $E(a,b)$ ;

$$i_i) \ a \xrightleftharpoons[f']{f} b, \ f R_{a,b} f' \Rightarrow \forall g: a' \rightarrow a, \ h: b \rightarrow b', \\ (hfg) R_{a',b'} (hfg').$$

Prop. 2.1.

$\mathcal{E} : \text{cat.}$   $\sum a, b \in \mathcal{E} (= \mathbb{Z} \cup \mathbb{Z})$ .  $R_{a,b}$  : binary relation on  $\mathcal{E}(a, b) \times r$

定数,  $2, 1, 3$  とする.

$$\Rightarrow \exists \mathcal{C}/R : \text{cat. together with } Q = Q_R : \mathcal{C} \rightarrow \mathcal{C}/R : \text{functor, s.t.}$$
$$i) f R_{a,b} f' \Rightarrow Qf = Qf';$$

i.)  $H: \mathcal{E} \rightarrow \mathcal{D}$ : functor s.t.  $f R_{a,b} f'$  implies  $Hf = Hf'$

$$\Rightarrow \exists ! H' : \mathcal{E}/_R \longrightarrow \mathcal{D} : \text{functor with } H' \circ Q_R = H.$$

↳ 51.  $Q_R$  : bij. on obj.

要するに「 $Q$  は  $FR_{a,b} f'$  ならば  $Qf = Qf'$  なる universal functor である!」

$\mathcal{C} = \mathcal{C}_G$  : cat. generated by a graph  $G \in \mathcal{C}^{\#}$ .

$\mathcal{C}_R$ : category with generators  $G$  and relations  $R$  (1.17)

Ex 12  $a, b \in \mathcal{C}$   $\mathbb{Z} \neq \mathbb{L} \mathbb{Z}$   $R_{a,b}, S_{a,b}$  : binary relations 时定弄, 2.12.

$$R_{a,b} \subset S_{a,b} \quad \text{if } a \neq b. \quad R \subset S \quad \text{if } a = b.$$

\_\_\_\_\_

2.  $\exists R'$ : least congruence on  $\mathcal{C} \supset R$  s.t.  $\{$

i.e., i)  $R'$ : congruence on  $E$ ,  $R \subset R'$ .

(i)  $S$ : congruence on  $\mathcal{E}$ ,  $R \subset S \Rightarrow R' \subset S$ .

~~Indeed,  $\forall a, b \in \mathcal{C} \Rightarrow f(a) \leq f(b)$ .~~

$$\cancel{R''_{a,b}} := \{ (f, f') \in \mathcal{E}(a, b) \times \mathcal{E}(a, b) \mid f \cancel{R_{a,b}} f' \text{ or } f' \cancel{R_{a,b}} f \}$$

~~$$R''_{a,b} := \{ (f, f') \in \mathcal{E}(a,b) \times \mathcal{E}(a,b) \mid \exists f_0, \dots, f_n \in \mathcal{E}(a,b) \text{ s.t.} \\ f_0 = f, f_0 R''_{a,b} f_1, f_1 R''_{a,b} f_2, \dots, f_{n-1} R''_{a,b} f_n, f_n = f' \}$$~~

$\gamma \neq \gamma$ .  $R_{a,b}'' \nsubseteq R_{a,b}''' \nsubseteq$  equivalence relation.

さらに.

$$R''_{a,b} := \left\{ (f, f') \in \mathcal{E}(a, b) \times \mathcal{E}(a, b) \mid \right.$$

$$\left. \begin{array}{c} \exists \begin{array}{ccc} a' & \xrightarrow{f_0} & b' \\ \uparrow g & \curvearrowright & \downarrow h \\ a & \xrightarrow{f} & b \end{array} \quad \exists \begin{array}{ccc} a' & \xrightarrow{f'_0} & b' \\ \uparrow g & \curvearrowright & \downarrow h \\ a & \xrightarrow{f'} & b \end{array} \end{array} \right\} \quad \text{s.t.} \quad f_0 R''_{a,b} f'_0$$

~~$$R'_{a,b} = \{ (f, f') \in \mathcal{E}(a,b) \times \mathcal{E}(a,b) \mid \exists f_0, \dots, f_n \in \mathcal{E}(a,b) \text{ s.t.} \\ f_0 = f, f_0 R'''_{a,b} f_1, f_1 R'''_{a,b} f_2, \dots, f_{n-2} R'''_{a,b} f_{n-1}, f_{n-1} = f' \}$$~~

Ex 11.11.  $R'$ : congruence on  $\mathbb{Z}$  s.t.

$\mathcal{C}/R$  : cat.  $E$  の上  $R$  に  $\{k\}$  :

Obj.: obj. of  $\mathcal{L}$ .

Ans.:  $(\mathcal{C}/R)(a, b) := \mathcal{C}(a, b) / R'_{a,b}.$

$$Q = Q_R : \mathcal{C} \longrightarrow \mathcal{C}/R : \text{canonical projection } \mathcal{C} \rightarrow \mathcal{C}/R.$$

$\mathcal{A}''$  の congruence on  $E \supset R$   
 の intersection (≠  $\mathcal{A}$ )  
 congruence on  $E \supset R$  に  
 なす、ということから従う。



$R'$ : congruence  $\vdash$ .  $\mathcal{C}/R$  is category.  $Q$  is functor  $\mathcal{C} \rightarrow \mathcal{C}/R$ .

$\odot$   $f \in \mathcal{C}(a, b)$  の equivalence class  $\bar{f} \in \mathcal{C}/R(a, b)$   $\in \frac{\#}{\#} \in \mathbb{N}$ .

$$a \xrightarrow[f']{f} b \xrightarrow[g']{g} c, \quad \bar{f} = \bar{f'}, \quad \bar{g} = \bar{g'}$$

$$\left. \begin{array}{l} f R'_{a,b} f' \rightsquigarrow g f R'_{a,c} g f' \\ g R'_{b,c} g' \rightsquigarrow g f' R'_{a,c} g' f' \end{array} \right\} \rightsquigarrow g f R_{a,c} g' f'$$

$$\therefore \overline{g \circ f} = \overline{g' \circ f'}$$

i.e.,  $\bar{g} \circ \bar{f} := \overline{g \circ f}$  is well-defined.

is  $\mathcal{C}/R$  associative?  $\bar{id}_a$  is identity?  $\bar{a} \bar{b} = \bar{a \circ b}$ ,

$\exists \mathcal{C}, \mathcal{C}/R$ : category  $\mathcal{C}$  あり  $\mathcal{C}/R$  は  $\mathcal{C}$  同値.

$Q: \mathcal{C} \rightarrow \mathcal{C}/R$ : functor.  $\mathcal{C}/R$ : category  $\mathcal{C}$  同値.

$H: \mathcal{C} \rightarrow \mathcal{D}$ : functor s.t.

$$f R_{a,b} f' \Rightarrow Hf = Hf'$$

任意に取る.

$$S_{a,b} := \{ (f, f') \in \mathcal{C}(a, b) \times \mathcal{C}(a, b) \mid Hf = Hf' \} \quad (a, b \in \mathcal{C})$$

よって  $S$ : congruence on  $\mathcal{C}$   $\Rightarrow R_{a,b} \subset S_{a,b}$ .

$$\therefore R'_{a,b} \subset S_{a,b}$$

$\exists \mathcal{C}. H = H' \circ Q_R$  と分解できる.

//

Exs.

$$\bullet \mathcal{C} = \mathcal{T}_{\text{op}}. \quad f R_{\text{a,b}} f' \iff f \underset{\text{homotopic}}{\simeq} f' \quad \text{etc.}$$

$$\mathcal{C}/R = \mathcal{T}_{\text{oph}}.$$

$$\bullet \mathcal{B} = \begin{array}{ccc} & \bullet & \\ \nearrow & \curvearrowright & \searrow \\ \bullet & \xrightarrow{\quad} & \bullet \end{array} \quad \text{is.}$$

$$G = \begin{array}{ccc} & 1 & \\ f \nearrow & & \searrow g \\ 0 & \xrightarrow{h} & 2 \end{array} \quad \text{is generator etc.}$$

$$h = g \circ f \quad \text{is relation etc. quotient category.}$$

Exercises

$$(1) \quad G = \begin{array}{ccc} & & g \\ f \downarrow & \square & \downarrow f' \\ & & g' \end{array} \quad \text{etc.}$$

relation  $g'f = f'g$  etc. quotient category  $\mathcal{C}$  is

commutative square with a diagonal  $\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ \downarrow & \searrow \curvearrowright & \downarrow \\ \bullet & \xrightarrow{\quad} & \bullet \end{array}$  etc.

(2)  $G$ : group. (regarded as a cat.),  $R$ : congruence on  $G$  (as a cat.)

$$\Rightarrow \exists N \triangleleft G \text{ s.t. } f R g \text{ iff } g^{-1}f \in N.$$

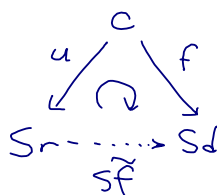
$$\begin{aligned} \textcircled{!} N &:= \{ g^{-1}f \mid f, g \in G, f R g \} \\ &= \{ f \in G \mid f R 1 \} \quad \text{etc.} \end{aligned} \quad //$$

### §3.1. Universal arrows

Def.

$S: \mathcal{D} \rightarrow \mathcal{E}$ : functor.  $c \in \mathcal{E}$ .

$\langle r \in \mathcal{D}, u: c \rightarrow S_r \rangle$ : universal arrow from  $c$  to  $S$   $\Leftrightarrow$



$\forall \langle d \in \mathcal{D}, f: c \rightarrow S_d \rangle,$

$\exists! \tilde{f}: r \rightarrow d$  s.t.

$$f = S\tilde{f} \circ u.$$

//

$c \xrightarrow{u} S_r$ : universal arrow

$\Leftrightarrow c \xrightarrow{u} S_r$ : initial obj. in  $(c \downarrow S)$

2"ある.  $\nexists$  3"に. universal arrow is unique up to iso.

Exs.

1)  $F$  2)  $U: \mathcal{D} \rightarrow \mathcal{E}$ : forgetful functor  $\Leftrightarrow$  3).

①  $k$ : field.  $X$ : set.  $X \hookrightarrow \text{span}_k X$  (it. universal arrow from  $X$  to  $U: \text{Vect}_k \rightarrow \text{Set}$ .)

②  $G$ : graph.  $P: G \rightarrow \mathcal{U}\mathcal{E}_G$  (it. universal arrow from  $G$  to  $U: \text{Grph} \rightarrow \text{Cat}$ .)

③  $X$ : set.  $\langle X \rangle$ : free group generated by  $X$ .  $U: \text{Grp} \rightarrow \text{Set}$ .

$X \hookrightarrow U\langle X \rangle$  (it. universal arrow from  $X$  to  $U$ .)

④  $\text{Dom}_m$ : category with obj. = integral domain.

mor. = monomorphism  $\Leftrightarrow$  5).

$U: \text{Fld} \rightarrow \text{Dom}_m$  (fields  $\mathbb{Q}$  の ring map は 単射 2"あることに注意).

$D \in \text{Dom}_m$ .  $\text{Frac } D$ : field of quotients of  $D$ .

$D \hookrightarrow U\text{Frac } D$  (it. universal arrow from  $D$  to  $U$ .)

Rem.  $\text{Dom}$  と  $\text{Dom}$  に  $\mathbb{Q}$  を換えては行けない!

Indeed,

$F: \text{Field}, f: D \rightarrow F: \text{ring map}$  と

$\tilde{f}: \text{Frac } D \rightarrow F: \text{ring map}$  に拡張するときは、

$$\tilde{f}\left(\frac{1}{r}\right) := \frac{1}{f(r)} \quad \text{でなければならぬ.}$$

$z$  のときは、

$$r \in D \setminus \{0\} \Rightarrow f(r) \in F \setminus \{0\}$$

i.e.,  $\ker f = \{0\}$

でなければならぬ.

③  $\text{Met}$ : category with obj.: metric space.

ar.: map preserving metric.

$\text{CMet} \subset \text{Met}$ : full subcat. where obj. are complete.

$U: \text{CMet} \rightarrow \text{Met}, X$ : metric space,  $\bar{X}$ : completion of  $X$ .

$X \rightarrow \bar{X}$  is universal arrow from  $X$  to  $U$ .

Def.

$H: D \rightarrow \text{Set}$ : functor.

$\langle r \in D, e \in Hr \rangle$ : universal element of  $H$  とは、

$\forall \langle d \in D, x \in Hd \rangle, \exists ! f: r \rightarrow d \text{ s.t. } (Hf)e = x.$  //

Rem.

特異な状況では、universal arrows と universal elements は別物.

④  $H: D \rightarrow \text{Set}$ : functor,  $\langle r, e \rangle$ : universal element とは、

$* \xrightarrow{e} Hr$  in  $\mathbb{E}ns$  と見ると、 $\langle r, e \rangle$ : universal arrow と見る.

$$\begin{array}{ccc} & r & \\ e \swarrow & \times & \searrow z \\ & \Omega & \\ Hr & \xrightarrow{\quad} & Hd \\ & Hf & \end{array} \quad \begin{array}{c} r \\ \vdots \exists ! f \\ d \end{array}$$

⑧ 逆に,  $\mathcal{C}$ : small cat.  $S: \mathcal{D} \rightarrow \mathcal{C}$ : functor.  $c \in \mathcal{C}$  へ  $\hookrightarrow$ .

$\langle r, u: c \rightarrow Sr \rangle$ : universal arrow  $\hookrightarrow$  だ.

$H = \mathcal{C}(c, S \cdot): \mathcal{D} \rightarrow \mathbf{Set}$  の universal element  $\hookrightarrow$  だ.

Exs.

⑧  $S$ : set.  $E \subset S \times S$ : equiv. relation.  $\pi: S \twoheadrightarrow S/E$ .

$\langle S/E, \pi \rangle$  は universal element of  $H: \mathbf{Set} \rightarrow \mathbf{Set}$ ,

where  $HX := \{ f: S \rightarrow X \mid s \in s' \Rightarrow fs = fs' \}$ .

$$\begin{array}{ccc} Hg: HX & \longrightarrow & HY \quad (g: X \rightarrow Y) \\ \downarrow & & \downarrow \\ f & \longmapsto & g \circ f \end{array}$$

⑧  $G$ : group  $N \triangleleft G$ .  $\pi: G \twoheadrightarrow G/N$ .

$\langle G/N, \pi \rangle$  は universal element of  $H: \mathbf{Grp} \rightarrow \mathbf{Set}$ ,

where  $HG' := \{ f: G \rightarrow G' : \text{group hom.} \mid \ker f \subset N \}$ .

$$\begin{array}{ccc} Hg: HG' & \longrightarrow & HG'' \quad (g: G' \rightarrow G'') \\ \downarrow & & \downarrow \\ f & \longmapsto & g \circ f \end{array}$$

⑧  $V_1, V_2$ : vect. sps.  $/_K$ .  $H: \mathbf{Vect}_K \rightarrow \mathbf{Set}$ ,

where  $HW := \mathbf{Bilin}(V_1, V_2; W) := \{ f: V_1 \times V_2 \rightarrow W : \text{bilinear} \}$ ,

$$\begin{array}{ccc} Hg: HW & \longrightarrow & HW' \quad (g: W \rightarrow W') \\ \downarrow & & \downarrow \\ f & \longmapsto & g \circ f \end{array}$$

$\langle V_1 \otimes_K V_2, \otimes \rangle$  は universal element of  $H$ .

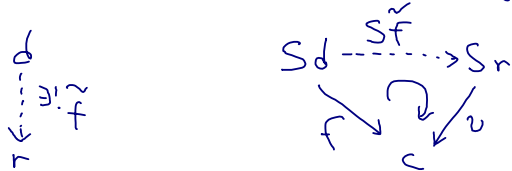
$$\begin{array}{ccc} \otimes: V_1 \times V_2 & \longrightarrow & V_1 \otimes_K V_2 \\ \downarrow & & \downarrow \\ (x, y) & \longmapsto & x \otimes y \end{array}$$

$\mathbf{Vect}_K \cong \mathbf{Mod}_K \cong \mathbf{U}(K)$ .

Def.  $S: D \rightarrow \mathcal{C}$ ,  $c \in \mathcal{C}$ .

$\langle r \in D, v: S_r \rightarrow c \rangle$  is universal arrow from  $S$  to  $c$  iff.

$\langle r, v \rangle$  or  $(S \downarrow c)$  is terminal obj. iff  $\exists! f = \tilde{f}$ .



//

Exs.  $\mathcal{C} = \text{Grp}, \text{Set}, \text{Cat}, \text{Top}, \text{Vect}_K$ , etc.

(category where the direct product of two objs. can be defined.)

$a, b \in \mathcal{C}$ ,  $p: a \times b \rightarrow a$ ,  $q: a \times b \rightarrow b$ .

$\langle p, q \rangle$  is universal arrow from  $\Delta: \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$  to  $\langle a, b \rangle$ ,

where  $\Delta c := \langle c, c \rangle$ .

$\Delta f := \langle f, f \rangle$  (diagonal functor)

$\exists! \langle f_1 \times f_2, f_1 \times f_2 \rangle$   
 $\langle c_1, c_2 \rangle \dashrightarrow \langle a \times b, a \times b \rangle$



### Exercises

(2) The universal element of  $\mathcal{P}: \text{Set}^{\mathcal{P}} \rightarrow \text{Set}$  (power set) is  $\langle \{0, 1\}, 1 \in \{0, 1\} \rangle$ .

(3)  $G \in \text{Grp}$  (or  $\in \text{Ab}$ ),  $X \in \text{Set}$ .

The universal arrow from  $G, G, X, X$ , resp. to the following commutator group functions are:

•  $U: \text{Ab} \rightarrow \text{Grp} \Rightarrow \langle G/[G, G], \pi: G \rightarrow G/[G, G] \rangle$

•  $U: \text{Rng} \rightarrow \text{Ab} \Rightarrow \langle R[G], \iota: G \hookrightarrow R[G] \rangle$  (group ring).

•  $U: \text{Top} \rightarrow \text{Set} \Rightarrow \langle (X, 2^X), \text{id}: X \rightarrow X \rangle$  (discrete topology)

•  $U: \text{Set}_K \rightarrow \text{Set} \Rightarrow \langle X \cup \{X\}, \iota: X \hookrightarrow X \cup \{X\} \rangle$ .