Categories for the Working Mathematicians. 2nd ed.

Mac Lane. Springer.

Notes by @naughie (Github)

https://naughie.github.io/maclane_notes

§2.6. Comma Categories.

Def.

$\mathscr{C}$: category. $b \in \mathscr{C}$: object.

$(b \downarrow \mathscr{C})$: category of objects under b with

- objects: $\langle f, c\rangle$, where $c \in \mathscr{C}$: obj. and $f: b \longrightarrow c$ in $\mathscr{C}$.
- arrows: $\langle f, c\rangle \xrightarrow{h} \langle f', c'\rangle$, where $h: c \longrightarrow c'$ s.t. $f' = h \circ f$.

Objects:
$$\begin{tikzcd} b \arrow[d, "f"] \\ c \end{tikzcd}$$

Arrows:
$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ c \arrow[rr, "h"'] & {} & c' \end{tikzcd}$$

Exs.

- $*$: 1-point set のとき. $(* \downarrow Set) = Set_*$.

  (Obj. $* \longrightarrow X$ は. pair $\langle x \in X, X\rangle$ のこと.
  Arr. $\langle x, X\rangle \longrightarrow \langle x', X'\rangle$ は. base point を保つ map のこと.)

- $(\mathbb{Z} \downarrow Ab) = Ab_*$

  (Obj. $\mathbb{Z} \xrightarrow{f} G$ は. $\langle f(1), G\rangle$ のこと. (base point $f(1)$).
  Arr. $\langle g, G\rangle \longrightarrow \langle g', G'\rangle$ は. base point を保つ group hom. のこと.)

Def.

$\mathscr{C}$: cat. $a \in \mathscr{C}$: obj.

$(\mathscr{C} \downarrow a)$: category of objects over a with

Objs:
$$\begin{tikzcd} c \arrow[d, "f"] \\ a \end{tikzcd}$$

Arrs:
$$\begin{tikzcd} c \arrow[rr, "h"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\circlearrowright"] & c' \arrow[dl, "f'"] \\ & a & \end{tikzcd}$$

Exs.

- $*$: 1-point set. $\Rightarrow$ $(\mathbf{Set} \downarrow *) \underset{\text{cat}}{\cong} \mathbf{Set}$.

$\left(\because \right.$ obj. $X \xrightarrow{f} *$ は、$*$: terminal obj. in $\mathbf{Set}$ だから、X に他ならない。
任意の $h: X \longrightarrow X'$ in $\mathbf{Set}$ は、$f = f' \circ h$ を満たす。

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, dashed, "\exists! f"'] & \arrow[d, phantom, "\circlearrowright"] & X' \arrow[dl, dashed, "\exists! f'"] \\ & * & \end{tikzcd}$$

$(f: X \longrightarrow *,\ f': X' \longrightarrow *)$. $\left. \right)$

- $(\mathbf{Ring} \downarrow \mathbb{Z})$: augmentation $R \longrightarrow \mathbb{Z}$ as objects,
ring hom preserving augmentations as arrows.

$\left(\because \right.$ Obj. $R \xrightarrow{\varepsilon} \mathbb{Z}$ は、augmentation と呼ばれる。(一般論).
Arr $\langle \varepsilon, R \rangle \xrightarrow{h} \langle \varepsilon', R' \rangle$ は、$\varepsilon = \varepsilon' \circ h$ なる ring hom $h: R \longrightarrow R'$.
i.e., ring hom that preserves augmentations. $\left. \right)$

Def.

$S: \mathcal{D} \longrightarrow \mathcal{E}$: functor. $b \in \mathcal{E}$: obj.

$(b \downarrow S)$: category of objects S-under b. with

Obj.
$$\begin{tikzcd} b \arrow[d, "f"] \\ Sd \end{tikzcd}$$

Arr.
$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ Sd \arrow[rr, "Sh"'] & {} & Sd' \end{tikzcd}$$

//

Def.

$T: \mathcal{E} \longrightarrow \mathcal{C}$: functor. $a \in \mathcal{C}$: obj.

$(T \downarrow a)$: category of objects T-over a with

obj.
$$\begin{tikzcd} Te \arrow[d, "f"] \\ a \end{tikzcd}$$

Arr.
$$\begin{tikzcd} Te \arrow[rr, "Th"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowright"] & Te' \arrow[dl, "f'"] \\ & a & \end{tikzcd}$$

//

Ex.

- $U : \mathbf{Grp} \to \mathbf{Set}$: forgetful functor. $x \in \mathbf{Set}$: obj. とする.

  $(x \downarrow U)$: map $x \xrightarrow{f} Ug$ as obj

  group hom $h : \langle f, g \rangle \to \langle f', g' \rangle$ s.t. $f' = h \circ f$ as arr.

Def.

$\mathcal{E} \xrightarrow{T} \mathcal{C} \xleftarrow{S} \mathcal{D}$: functors.

$(T \downarrow S)$: comma category with

Obj.
$$\begin{tikzcd} Te \arrow[d, "f"] \\ Sd \end{tikzcd}$$

Arr.
$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd}$$
//

Rem.

Composites of $(T \downarrow S)$ は well-defined :

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd} , \quad \begin{tikzcd} Te' \arrow[r, "Tk'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowright"] & Te'' \arrow[d, "f''"] \\ Sd' \arrow[r, "Sh'"'] & Sd'' \end{tikzcd}$$
を 2つの arrows とすると.

その composite

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[r, "Tk'"] \arrow[d, "f'"] \arrow[dr, phantom, "\circlearrowright"] & Te'' \arrow[d, "f''"] \\ Sd \arrow[r, "Sh"'] & Sd' \arrow[r, "Sh'"'] & Sd'' \end{tikzcd}$$

は $(T \downarrow S)$ の arrow

i.e., commutative diagram

になっている.

$(T \downarrow S)$ は $(b \downarrow \mathcal{C})$, $(\mathcal{C} \downarrow a)$, $(b \downarrow S)$, $(T \downarrow a)$ の一般化である!

- $(b \downarrow S)$ は, $T = b : \mathbb{1} \to \mathcal{C}$ としたもの.
- $(T \downarrow a)$ は, $S = a : \mathbb{1} \to \mathcal{C}$ としたもの.
- $(b \downarrow \mathcal{C})$ は, $T = b : \mathbb{1} \to \mathcal{C}$, $S = 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ としたもの.
- $(\mathcal{C} \downarrow a)$ は, $S = a : \mathbb{1} \to \mathcal{C}$, $T = 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ としたもの.

Exs.

- $\mathcal{C}$: cat. $\mathcal{C} : \mathcal{C} \longrightarrow \mathcal{C}$: id. functor. のとき.

$(\mathcal{C} \downarrow \mathcal{C}) = \mathcal{C}^2$

$\because$ obj. $c \xrightarrow{f} d$.

Arr.
$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\circlearrowright"] & d \arrow[d, "h"] \\ c' \arrow[r, "f'"'] & d' \end{tikzcd}$$
$\Rightarrow \mathcal{C}^2$ (obj : arrows. arr : commutative squares)

- $T = b,\ S = a : \mathbb{1} \longrightarrow \mathcal{C}$. のとき.

$(b \downarrow a) = \hom_{\mathcal{C}}(b, a)$ (as a discrete cat.).

$\because$ obj. $b \xrightarrow{f} a$.

Arr.
$$\begin{tikzcd} b \arrow[r, "f"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowright"] & a \arrow[d, "id"] \\ b \arrow[r, "f'"'] & a \end{tikzcd}$$
$\Rightarrow f = f'$. i.e., arrow は trivial なもののみ.

↑ "comma category" の名前の由来!

$(T \downarrow S)$ の普遍性 (cf. Exercise (5)).

$$\begin{tikzcd} & & (T \downarrow S) \arrow[dll, "P"'] \arrow[d, "R"] \arrow[drr, "Q"] & & \\ \mathcal{E} \arrow[r, "T"'] & \mathcal{C} & \mathcal{C}^2 \arrow[l, "\mathcal{C}^{d_0}"] \arrow[r, "\mathcal{C}^{d_1}"'] & \mathcal{C} & \mathcal{D} \arrow[l, "S"] \end{tikzcd}$$

$d_0, d_1 : \mathbb{1} \longrightarrow \mathbb{2}$

P, Q : projections.

$\mathcal{C}^{d_0}$ (resp $\mathcal{C}^{d_1}$) は. arrow $a \xrightarrow{f} b$ を. $a = \mathrm{dom}\, f$ (resp. $b = \mathrm{cod}\, f$) に送る functor. (cf. §2.5).

Exercises.

(1) K: com. ring のとき. $(K \downarrow \mathbf{CRng}) = \mathbf{CAlg}_K$.

∵ obj. A: com ring, $f: K \longrightarrow A$: ring hom.

i.e., A: com. alg. $/K$.

Arr. $(A, f) \xrightarrow{h} (A', f')$: ring hom s.t.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ A \arrow[rr, "h"'] & & A' \end{tikzcd}$$

i.e., scalar 倍 f, f' を保つ ring hom h

i.e., alg. hom. $h: A \longrightarrow A'$.

(2) $\mathcal{C}$: cat. $t \in \mathcal{C}$: terminal $\Longrightarrow (\mathcal{C} \downarrow t) \cong \mathcal{C}$.

∵ obj. $b \xrightarrow{f} t$ は. t: terminal だから.

各 $b \in \mathcal{C}$ に対して one and exactly one 存在.

Arr. $b \xrightarrow{h} b'$ s.t.

$$\begin{tikzcd} b \arrow[rr, "h"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & b' \arrow[dl] \\ & t & \end{tikzcd}$$

t: terminal だから, この diagram は 自動的に commutative.

(4) (S. A. Hug). $T, S: \mathcal{D} \longrightarrow \mathcal{C}$: functors とする.

$\tau: T \xrightarrow{\bullet} S$: nat. transformation は. $\tau: \mathcal{D} \longrightarrow (T \downarrow S)$: functor s.t.

$P\tau = Q\tau = id_{\mathcal{D}}$ に他ならない. (P, Q: projs.)

∵ $\tau: T \xrightarrow{\bullet} S$

$\Longleftrightarrow \forall d \in \mathcal{D}$, $\exists \tau d: Td \longrightarrow Sd$ s.t.

$$\begin{tikzcd} Td \arrow[r, "\tau d"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Sd \arrow[d] \\ Td' \arrow[r, "\tau d'"'] & Sd' \end{tikzcd}$$

$\tau: \mathcal{D} \longrightarrow (T \downarrow S)$ s.t. $P\tau = Q\tau = id_{\mathcal{D}}$

$\Longleftrightarrow$ 各 $d \in \mathcal{D}$ に対して. $\tau d: Te \longrightarrow Sg$ $(\exists e, g \in \mathcal{D})$

各 $d \xrightarrow{f} d'$ に対して

$$\begin{tikzcd} Te \arrow[r, "\tau d"] \arrow[d, "Tk"'] \arrow[dr, phantom, "\circlearrowright"] & Sg \arrow[d, "Sh"] \\ Te' \arrow[r, "\tau d'"'] & Sg' \end{tikzcd}$$

$(\exists k: e \longrightarrow e', h: g \longrightarrow g')$

s.t.

$$\begin{array}{ccccccccccc} e & = & g & = & d & , & k & = & h & = & f \\ \| & & \| & & \| & & \| & & \| & & \| \\ P\tau d & & Q\tau d & & \mathrm{id}_{\mathcal{D}}\, d & & P\tau f & & Q\tau f & & \mathrm{id}_{\mathcal{D}}\, f \end{array} .$$

$\Leftrightarrow$ $\forall d \in \mathcal{D}.\ \exists \tau d : Td \longrightarrow Sd$ s.t.

$$\begin{tikzcd} Td \arrow[r, "\tau d"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\circlearrowright"] & Sd \arrow[d, "Sf"] \\ Td' \arrow[r, "\tau d'"'] & Sd' \end{tikzcd} \qquad (f : d \longrightarrow d').$$

上の2つは同値.

§2.7. Graphs and Free Categories.

まず. free monoid の場合を思い出す:

X: set から生成される free monoid F とは,

- Word $x_1 \cdots x_n$ $(x_i \in X)$ を元に持ち,
- words の結合 (juxtaposition) を積とし,
- empty word を identity element とする

monoid のこと.

これと equivalent な条件は.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & F \arrow[d, dashed, "\exists ! \tilde{f}"] \\ & M \end{tikzcd}$$

$(F$: monoid, $\iota: X \to F$: map$)$ が free monoid

$\Leftrightarrow \forall (M$: monoid, $f: X \to M$: map$)$,

$\exists ! \tilde{f}: F \to M$: monoid hom. s.t. $f = \tilde{f} \circ \iota$.

<u>これを元に free category を考える.</u>

<u>Def.</u> (§1.2)

G: (directed) graph とは.

- O: set of objects (vertices)
- A: set of arrows (edges).

together with functions

$$A \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} O, \quad \partial_0 f = \mathrm{dom}\, f, \quad \partial_1 f = \mathrm{cod}\, f.$$

$D: G \to G'$: morphism of graphs とは.

- $D_O: O \to O'$
- $D_A: A \to A'$ s.t. $D_O \partial_0 = \partial_0 D_A$, $D_O \partial_1 = \partial_1 D_A$.

//

$\rightsquigarrow$ **Grph**: category of graphs.

Graphs には composite と identity がない.

⤴ diagram scheme や precategory とも呼ぶ!

Graphs obtained from categories

$\mathcal{C}$: cat. に対して $U\mathcal{C}$: graph を

- obj. : objects of $\mathcal{C}$;
- arr. : arrows of $\mathcal{C}$

と定める. $F : \mathcal{C} \longrightarrow \mathcal{C}'$: functor は

graph morphism $UF : U\mathcal{C} \longrightarrow U\mathcal{C}'$ に対応する.

$\therefore U : \mathbf{Cat} \longrightarrow \mathbf{Grph}$: forgetful functor.

Categories obtained from graphs

O : set : fixed.

Def.

O-graph とは graph whose object set is O.

A morphism of O-graph とは O-graphs 間の graph morphism のこと. //

A, B : sets of arrows (regarded as O-graphs) とすると.

$$A \times_0 B := \{ \langle g, f \rangle \mid \partial_0 g = \partial_1 f,\ g \in A\ \ f \in B \}$$

(composable pairs of arrows $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$)

と定める.

$$\partial_0 \langle g, f \rangle := \partial_0 f, \quad \partial_1 \langle g, f \rangle := \partial_1 g$$

とすれば $A \times_0 B$: O-graph になる.

$\times_0$ は associative

($\because$ $(A \times_0 B) \times_0 C \cong A \times_0 (B \times_0 C)$
$\cong \{ \langle h, g, f \rangle \mid \partial_0 h = \partial_1 g,\ \partial_0 g = \partial_1 f,\ h \in A,\ g \in B,\ f \in C \}$)

任意の A: O-graph に対して.

$$\begin{array}{ccccc} A & \cong & A\times_O O & \cong & O\times_O A \\ \in & & \in & & \in \\ f & \longmapsto & \langle f, \partial_0 f\rangle & \longmapsto & \langle \partial_1 f, f\rangle. \end{array}$$

Rem. O を次のように O-graph とみなしている:

- arrows: O itself.
- $\partial_0, \partial_1 = \mathrm{id}_O : O \rightrightarrows O$

$$\begin{tikzcd} O \arrow[r, shift left] \arrow[r, shift right] & O \\ \text{arrows} \arrow[u] & \text{objects} \arrow[u] \end{tikzcd}$$

一方, $\mathcal{C}$: cat. whose object set is O は,

A: O-graph together with morphisms

$c : A\times_O A \longrightarrow A$ (composite)

$i : O \longrightarrow A$ (identity) s.t.

$$\begin{tikzcd} (A\times_O A)\times_O A \cong A\times_O (A\times_O A) \arrow[r, "1\times c"] \arrow[d, "c\times 1"'] \arrow[dr, phantom, "\circlearrowright"] & A\times_O A \arrow[d, "c"] \\ A\times_O A \arrow[r, "c"'] & A \end{tikzcd}$$

(associativity)

$$\begin{tikzcd} O\times_O A \arrow[r, "i\times 1"] \arrow[dr, "\wr"'] & A\times_O A \arrow[d] & A\times_O O \arrow[l, "1\times i"'] \arrow[dl, "\sim"] \\ & A & \end{tikzcd}$$

(unit law).

G: O-graph から $\mathcal{C} = \mathcal{C}(G)$: cat を作る.

大雑把に言えば, G の composable pair of arrows をすべて"繋げた"もの.

厳密に言えば,

Thm. 2.1.

$G = \{A \rightrightarrows O\}$: small graph.

$\Rightarrow \exists\, \mathcal{C} = \mathcal{C}_G$: small cat. $P : G \longrightarrow U\mathcal{C}$: morphism of graphs s.t.

$\forall\,(\mathcal{B}$: cat., $D : G \longrightarrow U\mathcal{B}$: morphism of graphs$)$,

$\exists !\, \tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$: functor with $(U\tilde{D}) \circ P = D$.

$$\begin{tikzcd} \mathcal{C} \arrow[d, dashed, "\exists ! \tilde{D}"'] \\ \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "P"] \arrow[dr, "D"'] & U\mathcal{C} \arrow[d, dashed, "U\tilde{D}"] \arrow[dl, phantom, "\circlearrowright"] \\ & U\mathcal{B} \end{tikzcd}$$

("任意の graph morphism $D : G \longrightarrow U\mathcal{B}$ を、

functor $\tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$ に拡張できる!")

特に、object set of $\mathcal{B} = O$, D : O-graph morphism

$\Longrightarrow \tilde{D}|_{\mathrm{obj.}} = \mathrm{id}$.

$P : G \longrightarrow U\mathcal{C}$ は $(G \downarrow U)$ の initial object である.

$\therefore$ unique up to isomorphism.

$\mathcal{C}$ を free category generated by G と呼ぶ.

Proof.

次のように、$\mathcal{C}$ を構成する.

- obj.: objects of G i.e., O.
- arr.: finite strings $\langle a_1, f_1, a_2, \dots, f_{n-1}, a_n \rangle \overset{\text{denote}}{\equiv} a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} a_n$

  where $a_i \in O$, $f_i \in A$, $\partial_0 f_i = a_i$, $\partial_1 f_i = a_{i+1}$.
- composite: juxtaposition of strings (i.e., "concatenation").
- identity: $\langle a \rangle \overset{\text{denotes}}{\equiv} a : a \longrightarrow a$

Then, every $\underbrace{\text{arrow of } \mathcal{C}}_{\text{(of length} > 1)}$ is a composite of (finitely many) strings of length 2:

$\langle a_1, f_1, \dots, f_{n-1}, a_n \rangle = \langle a_{n-1}, f_{n-1}, a_n \rangle \circ \cdots \circ \langle a_1, f_1, a_2 \rangle \quad (n \geqq 2)$

Define graph morphism $P : G \longrightarrow U\mathcal{C}$ as

- on obj. : identity $O \longrightarrow O$;
- on arr. : $A \ni f \longmapsto \langle \partial_0 f, f, \partial_1 f \rangle$.

この $(\mathcal{C}, P)$ が条件を満たすことを示す.

$\mathcal{B}$: cat., $D : G \longrightarrow U\mathcal{B}$: graph morphism を任意に取る.

If $\exists \tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$: functor s.t. $U\tilde{D} \circ P = D$,

$\Longrightarrow$ $\tilde{D}$ must be

- on obj. : $\underline{\tilde{D}a = Da}$;
- on arr. : $\tilde{D}\langle a_1, f_1, a_2 \rangle = Df_1$.

And, by the composition of a functor,

$$\underline{\tilde{D}\langle a_1, f_1, \dots, f_{n-1}, a_n \rangle = Df_{n-1} \circ \cdots \circ Df_1}.$$

$\therefore$ $\tilde{D}$ is <u>automatically unique</u>.

よって, $\tilde{D}$ が存在することを示せば, 証明は完了する.

上のように $\tilde{D}$ を construct したとき,

$\tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$ が functor で,

$U\tilde{D} \circ P = D$ を満たすことは明らか. //

<u>Exs.</u>

- $G = \begin{tikzcd} \bullet \arrow[loop above, "f"] \end{tikzcd}$ $(O = \{\bullet\},\ A = \{f\})$ のとき.

  $\mathcal{C}_G$ の arrows は. $f, f^2, f^3, \dots$

● $G = \bullet \xrightarrow{g} \bullet$ 　（$O = \{0, 1\}$. $A = \{g\}$）のとき.

$\mathcal{C}_G$ の arrows は. $id_0 : 0 \longrightarrow 0$,

$id_1 : 1 \longrightarrow 1$,

$g : 0 \longrightarrow 1$

のみ. （g と juxtapose できる arrow がない！）

● $G = \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ 　（$O = \{0, 1, 2\}$. $A = \{f, g\}$）のとき.

$\mathcal{C}_G$ の arrows は. $id_i : i \longrightarrow i$ 　($i = 0, 1, 2$) および

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & \bullet \end{tikzcd}$$

のみ. （composable pair は $\langle f, g \rangle$ しかない！）
　　　　$\parallel$
　juxtaposable

Cor. 2.2.

$\forall X$: set. $\exists M$: monoid, $p : X \longrightarrow UM$: map,

where $U : \mathbf{Mon} \longrightarrow \mathbf{Set}$: forgetful functor, s.t.

$$\begin{tikzcd} M \arrow[d, dashed, "\exists! \tilde{h}"'] \\ L \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "h"'] & UM \arrow[d, dashed, "U\tilde{h}"] \arrow[dl, phantom, "\circlearrowright"] \\ & UL \end{tikzcd}$$

$\forall$ (L : monoid, $h : X \longrightarrow UL$: map)

$\exists! \tilde{h} : M \longrightarrow L$; monoid hom. s.t.

$h = U\tilde{h} \circ p$.

//

Proof of Cor 2.2.

Thm. 2.1 において $O = \{\bullet\}$, $A = X$ とすると,

$\exists M$: cat. whose object set is $O = \{\bullet\}$

（M : monoid と見なせる！）　s.t.

$U : \mathbf{Cat} \longrightarrow \mathbf{Graph}$

$\forall L$: monoid (regarded as a category) and $h : G \longrightarrow UL$: graph morphism,

（$h : X \longrightarrow UL$: map と見なせる！）,

$U : \mathbf{Mon} \longrightarrow \mathbf{Set}$

$\exists!\ \tilde{h} : M \longrightarrow L$: functor (regarded as a monoid hom.) s.t.

$h = U\tilde{h} \circ p$

$U : \mathbf{Cat} \longrightarrow \mathbf{Grph}$ (regarded as $\mathbf{Mon} \longrightarrow \mathbf{Set}$).

$\mathbf{Mon} \subset \mathbf{Cat}$

set maps (regarded as graph morphisms).

//

Graphs as diagrams

Def.

G : graph. $\mathcal{B}$: cat.

A diagram of shape G in $\mathcal{B}$ とは

graph morphism $D : G \longrightarrow U\mathcal{B}$ のこと.

//

By Thm. 2.1, D は $\tilde{D} : \mathcal{C}_G \longrightarrow \mathcal{B}$: functor に他ならない.

$$\mathbf{Cat}(\mathcal{C}_G, \mathcal{B}) \underset{\text{bij.}}{\cong} \mathbf{Grph}(G, U\mathcal{B})$$

is natural in G and $\mathcal{B}$.

$\therefore\ \mathcal{C} : \mathbf{Grph} \longrightarrow \mathbf{Cat}$ is left adjoint to $U : \mathbf{Cat} \longrightarrow \mathbf{Grph}$.

(cf. chapter 4).

Exercise

(2) Every finite ordinal is a free category.

($\because$ An finite ordinal n is a free category generated by $G = 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-1$.)

§8. Quoitient Categories

Def.

$\mathscr{C}$: cat.

R : congruence on $\mathscr{C}$ とは.

i) 各 $a, b \in \mathscr{C}$ に対して. $R_{a,b}$: equivalence relation on $\mathscr{C}(a,b)$;

ii) $a \underset{f'}{\overset{f}{\rightrightarrows}} b$, $f R_{a,b} f' \Longrightarrow \forall g : a' \longrightarrow a,\ h : b \longrightarrow b'$,
$(hfg) R_{a',b'} (hf'g)$.

//

Prop. 2.1.

$\mathscr{C}$: cat. 各 $a, b \in \mathscr{C}$ に対して. $R_{a,b}$: binary relation on $\mathscr{C}(a,b)$ が定まっているとする.

$\Longrightarrow \exists\, \mathscr{C}/R$: cat. together with $Q = Q_R : \mathscr{C} \longrightarrow \mathscr{C}/R$: functor, s.t.

i) $f R_{a,b} f' \Longrightarrow Qf = Qf'$;

ii) $H : \mathscr{C} \longrightarrow \mathscr{D}$: functor s.t. $f R_{a,b} f'$ implies $Hf = Hf'$;

$\Longrightarrow \exists!\, H' : \mathscr{C}/R \longrightarrow \mathscr{D}$: functor with $H' \circ Q_R = H$.

さらに, Q_R : bij. on obj.

//

短かく言えば. Q は $f R_{a,b} f'$ ならば $Qf = Qf'$ なる universal functor である!

$\mathscr{C} = \mathscr{C}_G$: cat. generated by a graph G のとき.

$\mathscr{C}/R$: category with generators G and relations R という.

各 $a, b \in \mathscr{C}$ に対して $R_{a,b}$, $S_{a,b}$: binary relations が定まっていて, $R_{a,b} \subset S_{a,b}$ なるとき, $R \subset S$ と書く.

Proof of Prop. 2.1.

まず. $\exists R'$: least congruence on $\mathcal{C} \supset R$ である.

i.e., i) R': congruence on $\mathcal{C}$, $R \subset R'$.

ii) S: congruence on $\mathcal{C}$, $R \subset S \Rightarrow R' \subset S$.

Indeed, 各 $a, b \in \mathcal{C}$ に対して,

$$R''_{a,b} := \{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \mid f R_{a,b} f' \text{ or } f' R_{a,b} f \}$$

$$R'''_{a,b} := \{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \mid \exists f_0, \dots, f_n \in \mathcal{C}(a,b) \text{ s.t. } f_0 = f,\ f_0 R''_{a,b} f_1,\ f_1 R''_{a,b} f_2, \dots, f_{n-1} R''_{a,b} f_n,\ f_n = f' \}$$

とおくと, $R'''_{a,b}$ は. $R_{a,b} \subset R'''_{a,b}$ なる equivalence relation.

さらに.

$$R''''_{a,b} := \{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \mid$$

$$\exists \begin{array}{ccc} a' & \xrightarrow{f_0} & b' \\ {\scriptstyle g}\uparrow & \circlearrowright & \downarrow{\scriptstyle h} \\ a & \xrightarrow[f]{} & b \end{array} \quad \exists \begin{array}{ccc} a' & \xrightarrow{f_0'} & b' \\ {\scriptstyle g}\uparrow & \circlearrowright & \downarrow{\scriptstyle h} \\ a & \xrightarrow[f']{} & b \end{array} \quad \text{s.t. } f_0 R'''_{a',b'} f_0' \}$$

$$R'_{a,b} := \{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \mid \exists f_0, \dots, f_n \in \mathcal{C}(a,b) \text{ s.t. } f_0 = f,\ f_0 R''''_{a,b} f_1,\ f_1 R''''_{a,b} f_2, \dots, f_{n-1} R''''_{a,b} f_n,\ f_n = f' \}.$$

とすれば. R': congruence on $\mathcal{C}$ である.

すべての congruence on $\mathcal{C} \supset R$ の intersection は また congruence on $\mathcal{C} \supset R$ になっていることから従う.

$\mathcal{C}/R$: cat. を次のように作る:

Obj.: obj. of $\mathcal{C}$.

Arr.: $(\mathcal{C}/R)(a,b) := \mathcal{C}(a,b) / R'_{a,b}$.

$Q = Q_R : \mathcal{C} \longrightarrow \mathcal{C}/R$: canonical projection とする.

R': congruence より. $\mathcal{C}/R$ は category で. Q は functor になる.

∵ $f \in \mathcal{C}(a,b)$ の equivalence class を $\overline{f} \in \mathcal{C}/R\,(a,b)$ と表せば.

$$a \underset{f'}{\overset{f}{\rightrightarrows}} b \underset{g'}{\overset{g}{\rightrightarrows}} c, \quad \overline{f} = \overline{f'}, \quad \overline{g} = \overline{g'}$$

$$\Rightarrow \left.\begin{array}{l} f R'_{a,b} f' \rightsquigarrow gf R'_{a,c} gf' \\ g R'_{b,c} g' \rightsquigarrow gf' R'_{a,c} g'f' \end{array}\right\} \rightsquigarrow gf R_{a,c} g'f'.$$

$$\therefore \overline{g \circ f} = \overline{g' \circ f'}.$$

i.e., $\overline{g} \circ \overline{f} := \overline{g \circ f}$ は well-defined.

この合成が associative で. $\overline{id_a}$ が identity になること,

従って $\mathcal{C}/R$: category であることは明らか.

$Q : \mathcal{C} \longrightarrow \mathcal{C}/R$: functor も. $\mathcal{C}/R$: category からすぐに分かる.

$H : \mathcal{C} \longrightarrow \mathcal{D}$: functor s.t.

$$f R_{a,b} f' \Rightarrow Hf = Hf'$$

を任意に取る.

$$S_{a,b} := \{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \mid Hf = Hf' \} \quad (a, b \in \mathcal{C})$$

とおけば. S : congruence on $\mathcal{C}$ かつ. $R_{a,b} \subset S_{a,b}$.

$$\therefore R'_{a,b} \subset S_{a,b}$$

よって. $H = H' \circ Q_R$ と分解される. //

Exs.

- $\mathcal{C} = \mathbf{Top}$. $f R_{a,b} f' \overset{\text{def}}{\iff} f \underset{\text{homotopic}}{\simeq} f'$ とすると、

  $\mathcal{C}/R = \mathbf{Toph}$.

- $\mathbf{3} =$
$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & \arrow[u, phantom, "\circlearrowleft"] & \bullet \end{tikzcd}$$
  は.

$$G = \begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] & & 2 \end{tikzcd}$$
を generator とし、

$h = g \circ f$ を relation とした quotient category.

Exercises

(1)
$$G = \begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "f"'] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r, "g'"'] & \bullet \end{tikzcd}$$
で生成され、

relation $g'f = f'g$ で割られた quotient category $\mathcal{C}$ は

commutative square with a diagonal
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\circlearrowleft", shift left=3ex] \arrow[dr, phantom, "\circlearrowleft"', shift right=3ex] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$
である.

(2) G: group (regarded as a cat.), R: congruence on G (as a cat.)

$\Rightarrow \exists N \triangleleft G$ s.t. fRg iff $g^{-1}f \in N$.

$\because$ $N := \{ g^{-1}f \mid f, g \in G,\ fRg \}$

$= \{ f \in G \mid fR1 \}$ とおけばよい. //