

Categories for the Working
Mathematicians. 2nd ed.

Mac Lane. Springer.

§ 2.6. Comma Categories

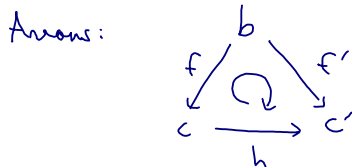
Def.

\mathcal{C} : category. $b \in \mathcal{C}$: object.

$(b \downarrow \mathcal{C})$: category of objects under b with

• objects: $\langle f, c \rangle$, where $c \in \mathcal{C} \text{ obj.}$ and $f: b \rightarrow c$ in \mathcal{C} .

• arrows: $\langle f, c \rangle \xrightarrow{h} \langle f', c' \rangle$, where $h: c \rightarrow c'$ s.t.
 $f' = h \circ f$.



Exs.

• $*$: 1-point set $\text{on } \mathbb{Z}$. $(* \downarrow \text{Set}) = \text{Set}_*$.

(\odot Obj. $* \rightarrow X$ is pair $\langle x \in X, X \rangle$ on \mathbb{Z} .
 Arr. $\langle x, X \rangle \rightarrow \langle x', X' \rangle$ is base point $\in \langle \frac{x}{x'} \rangle$ map on \mathbb{Z} .)

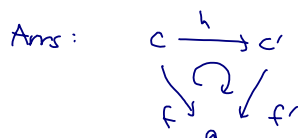
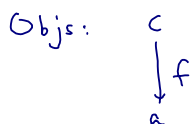
• $(\mathbb{Z} \downarrow \text{Ab}) = \text{Ab}_*$

(\odot Obj. $\mathbb{Z} \xrightarrow{f} G$ is $\langle f(1), G \rangle$ on \mathbb{Z} . (base point $f(1)$).
 Arr. $\langle g, G \rangle \rightarrow \langle g', G' \rangle$ is base point $\in \langle \frac{g}{g'} \rangle$ group hom. on \mathbb{Z} .)

Def.

\mathcal{C} : cat. $a \in \mathcal{C}$: obj.

$(\mathcal{C} \downarrow a)$: category of objects over a with



Exs.

① $*$: 1-point set. $\Rightarrow (\text{Set} \downarrow *) \cong_{\text{cat}} \text{Set}.$

② Obj. $X \xrightarrow{f} *$ is, $*$: terminal obj. in Set だから, X は何なりか...
 任意の $h: X \rightarrow X'$ in Set は, $f = f' \circ h$ を満たす.

$$X \xrightarrow{h} X' \quad (f: X \rightarrow *, f': X' \rightarrow *)$$

③ $(\text{Rng} \downarrow \mathbb{Z})$: augmentation $R \rightarrow \mathbb{Z}$ as objects,
 ring hom preserving augmentations as arrows.

④ Obj. $R \xrightarrow{\varepsilon} \mathbb{Z}$ is, augmentation ε に対して. (一般論).
 An $\langle \varepsilon, R \rangle \xrightarrow{h} \langle \varepsilon', R' \rangle$ is, $\varepsilon = \varepsilon' \circ h$ である ring hom $h: R \rightarrow R'$.
 i.e., ring hom that preserves augmentations.

Def.

$S: \mathcal{D} \rightarrow \mathcal{E}$: functor. $b \in \mathcal{E}$: obj.

$(b \downarrow S)$: category of objects S -under b . with

Obj.
$$\begin{array}{c} b \\ \downarrow f \\ Sd \end{array}$$
 An.
$$\begin{array}{ccc} & b & \\ f \swarrow & \downarrow & \searrow f' \\ Sd & \xrightarrow{Sh} & Sd' \end{array}$$

Def.

$T: \mathcal{E} \rightarrow \mathcal{E}$: functor. $a \in \mathcal{E}$: obj.

$(T \downarrow a)$: category of objects T -over a with

Obj.
$$\begin{array}{c} Tc \\ \downarrow f \\ a \end{array}$$
 An.
$$\begin{array}{ccc} Tc & \xrightarrow{Th} & Tc' \\ f \downarrow & \downarrow & \downarrow f' \\ & a & \end{array}$$

Ex.

① $U: \text{Grp} \rightarrow \text{Set}$: forgetful functor. $x \in \text{Set} : \text{obj.}$ と $\{ \}$.

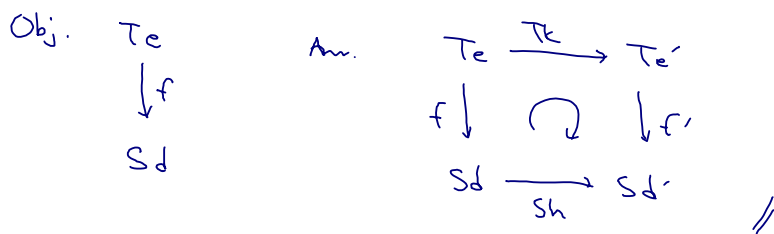
$(x \downarrow U) : \text{map } x \xrightarrow{f} Uq \text{ as obj}$

group hom $h: \langle f, g \rangle \rightarrow \langle f', g' \rangle$ s.t. $f' = h \circ f$ as arr.

Def.

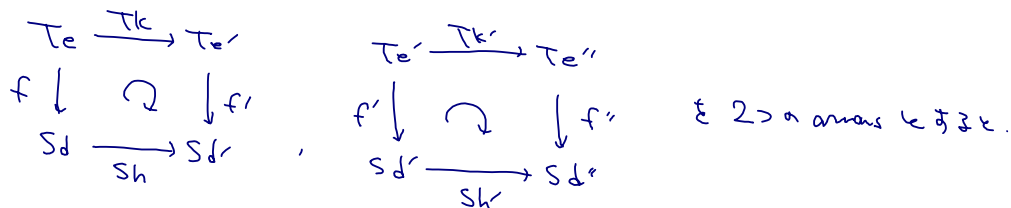
$\mathcal{C} \xrightarrow{T} \mathcal{C} \xleftarrow{S} \mathcal{D}$: functors.

$(T \downarrow S)$: comma category with

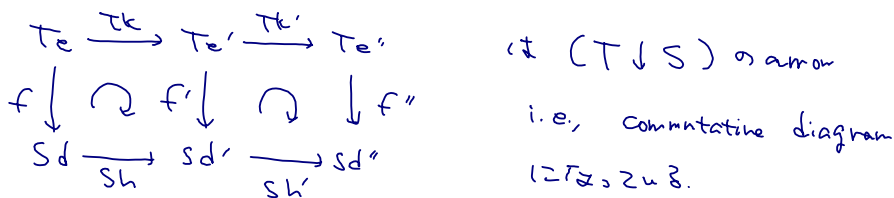


Rem.

Composites of $(T \downarrow S)$ is well-defined:



$\{ \}$ composite



$(T \downarrow S)$ is $(b \downarrow \mathcal{C})$, $(\mathcal{C} \downarrow a)$, $(b \downarrow S)$, $(T \downarrow a)$ の一般化である!

① $(b \downarrow S)$ is. $T = b: \mathbb{1} \rightarrow \mathcal{C}$ と $\mathcal{C} \in \mathcal{C}$ である.

② $(T \downarrow a)$ is. $S = a: \mathbb{1} \rightarrow \mathcal{C}$ と $\mathcal{C} \in \mathcal{C}$ である.

③ $(b \downarrow \mathcal{C})$ is. $T = b: \mathbb{1} \rightarrow \mathcal{C}$, $S = 1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ と $\mathcal{C} \in \mathcal{C}$ である.

④ $(\mathcal{C} \downarrow a)$ is. $S = a: \mathbb{1} \rightarrow \mathcal{C}$, $T = 1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ と $\mathcal{C} \in \mathcal{C}$ である.

Exs.

① $\mathcal{C} : \text{cat.}$ $\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} : \text{id. functor.}$ $\circ \neq \text{id.}$

$$(\mathcal{C} \downarrow \mathcal{C}) = \mathcal{C}^2$$

$$\left(\begin{array}{l} \text{Obj. } c \xrightarrow{f} d. \\ \text{Arr. } \begin{array}{ccc} c & \xrightarrow{f} & d \\ k \downarrow & \curvearrowright & \downarrow h \\ c' & \xrightarrow{f'} & d' \end{array} \end{array} \Rightarrow \mathcal{C}^2 \left(\begin{array}{l} \text{obj: arrows.} \\ \text{arr: commutative squares} \end{array} \right) \right)$$

② $T = b, S = a : \mathbb{1} \rightarrow \mathcal{C}.$ $\circ \neq \text{id.}$

$$(b \downarrow a) = \text{hom}_{\mathcal{C}}(b, a) \quad (\text{as a discrete cat.}).$$

$$\left(\begin{array}{l} \text{Obj. } b \xrightarrow{f} a. \\ \text{Arr. } \begin{array}{ccc} b & \xrightarrow{f} & a \\ \text{id} \downarrow & \curvearrowright & \downarrow \text{id} \\ b & \xrightarrow{f'} & a \end{array} \end{array} \Rightarrow f = f'. \quad \text{i.e., arrow is trivial to be equal.} \right)$$

↑ "comma category" の名前が由来!

$(T \downarrow S)$ の普遍性 (cf. Exercise (5)).

$$\begin{array}{ccccc} & & (T \downarrow S) & & \\ & \swarrow P & \downarrow R & \searrow Q & \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} & \xleftarrow{\mathcal{C}^{d_0}} & \mathcal{C}^2 & \xrightarrow{\mathcal{C}^{d_1}} & \mathcal{C} & \xleftarrow{S} & \mathcal{D} \end{array}$$

$$d_0, d_1 : \mathbb{1} \rightarrow \mathbb{2}$$

P, Q : projections.

\mathcal{C}^{d_0} (resp \mathcal{C}^{d_1}) は, arrow $a \xrightarrow{f} b \in \mathcal{C}$, $a = \text{dom } f$ (resp. $b = \text{cod } f$) になる functor. (cf. § 2.5).

Exercises.

(1) K : com. ring $\alpha \neq 0$. $(K \downarrow \mathbb{C}Rng) = \mathbb{C}Alg_K$.

Obj. A : com ring, $f: K \rightarrow A$: ring hom.

i.e., A : com. alg. $/K$.

Ans. $(A, f) \xrightarrow{h} (A', f')$: ring hom s.t.

$$\begin{array}{ccc} & K & \\ f \swarrow & \curvearrowright & \searrow f' \\ A & \xrightarrow{h} & A' \end{array} \quad \begin{array}{l} \text{i.e., scalar } \frac{f}{f'} \\ \text{i.e., alg. hom. } h: A \rightarrow A'. \end{array}$$

(2) \mathcal{C} : cat. $t \in \mathcal{C}$: terminal $\Rightarrow (\mathcal{C} \downarrow t) \cong \mathcal{C}$.

Obj. $b \xrightarrow{f} t$ is t : terminal t 's is.

$\forall b \in \mathcal{C}$ ($\exists! f$) one and exactly one f 's is.

Ans. $b \xrightarrow{h} b$ s.t.

$$\begin{array}{ccc} b & \xrightarrow{h} & b \\ & \curvearrowright & \\ & t & \end{array} \quad t: \text{terminal } t \text{'s is, } \text{この diagram は自動的に commutative.}$$

(4) (S.A. Hug). $T, S: \mathcal{D} \rightarrow \mathcal{C}$: functors $\alpha \neq 0$.

$\tau: T \rightarrow S$: nat. transformation (\neq , $\tau: \mathcal{D} \rightarrow (T \downarrow S)$: functor s.t.

$P_\tau = Q_\tau = id_{\mathcal{D}}$ is satisfied. (P, Q : prjs.)

Obj. $\tau: T \rightarrow S$

$$\Leftrightarrow \forall d \in \mathcal{D}, \exists \tau_d: T_d \rightarrow S_d \text{ s.t.} \quad \begin{array}{ccc} T_d & \xrightarrow{\tau_d} & S_d \\ \downarrow & \curvearrowright & \downarrow \\ T_{d'} & \xrightarrow{\tau_{d'}} & S_{d'} \end{array}$$

$\tau: \mathcal{D} \rightarrow (T \downarrow S)$ s.t. $P_\tau = Q_\tau = id_{\mathcal{D}}$

$\Leftrightarrow \forall d \in \mathcal{D}$ ($\exists! f$) $\tau_d: T_d \rightarrow S_d$ ($\exists e, g \in \mathcal{D}$)

$$\begin{array}{ccc} \forall d \xrightarrow{f} d' & \tau_d \xrightarrow{\tau_{d'}} & S_d \\ T_d \downarrow & \curvearrowright & \downarrow S_{d'} \\ T_{d'} & \xrightarrow{\tau_{d'}} & S_{d'} \end{array} \quad (\exists k: e \rightarrow e', h: g \rightarrow g')$$

s.t.

$$\begin{array}{ccccc}
 e & = & g & = & d \\
 \parallel & & \parallel & & \parallel \\
 P_{\tau d} & & Q_{\tau d} & & id_D d
 \end{array}, \quad
 \begin{array}{ccccc}
 k & = & h & = & f \\
 \parallel & & \parallel & & \parallel \\
 P_{\tau f} & & Q_{\tau f} & & id_D f
 \end{array}$$

$$\Leftrightarrow \forall d \in D. \exists \tau_d: T_d \rightarrow S_d \text{ s.t.}$$

$$\begin{array}{ccc}
 T_d & \xrightarrow{\tau_d} & S_d \\
 \tau_f \downarrow & \curvearrowright & \downarrow S_f \\
 T_{d'} & \xrightarrow{\tau_{d'}} & S_{d'}
 \end{array} \quad (f: d \rightarrow d')$$

上の2>は同値.