Categories for the Working Mathematicians. 2nd ed.

Mac Lane. Springer.

§ 2.6. Comma Categories.

Def.

$\mathcal{C}$: category. $b \in \mathcal{C}$: object.

$(b \downarrow \mathcal{C})$: category of objects under b with

- objects: $\langle f, c \rangle$, where $c \in \mathcal{C}$: obj. and $f : b \longrightarrow c$ in $\mathcal{C}$.
- arrows: $\langle f, c \rangle \xrightarrow{h} \langle f', c' \rangle$, where $h : c \longrightarrow c'$ s.t.
  $f' = h \circ f$.

Objects:
$$\begin{tikzcd} b \arrow[d, "f"] \\ c \end{tikzcd}$$

Arrows:
$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ c \arrow[rr, "h"'] & {} & c' \end{tikzcd}$$

Exs.

- $*$: 1-point set のとき. $(* \downarrow \mathbf{Set}) = \mathbf{Set}_*$.

  (Obj. $* \longrightarrow X$ は. pair $\langle x \in X, X \rangle$ のこと.
  Arr. $\langle x, X \rangle \longrightarrow \langle x', X' \rangle$ は. base point を保つ map のこと.)

- $(\mathbb{Z} \downarrow \mathbf{Ab}) = \mathbf{Ab}_*$

  (Obj. $\mathbb{Z} \xrightarrow{f} G$ は. $\langle f(1), G \rangle$ のこと. (base point $f(1)$).
  Arr. $\langle g, G \rangle \longrightarrow \langle g', G' \rangle$ は. base point を保つ group hom. のこと.)

Def.

$\mathcal{C}$: cat. $a \in \mathcal{C}$: obj.

$(\mathcal{C} \downarrow a)$: category of objects over a with

Objs:
$$\begin{tikzcd} c \arrow[d, "f"] \\ a \end{tikzcd}$$

Arrs:
$$\begin{tikzcd} c \arrow[rr, "h"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\circlearrowleft"] & c' \arrow[dl, "f'"] \\ & a & \end{tikzcd}$$

Exs.

- $*$: 1-point set. $\Longrightarrow$ $(\mathbf{Set} \downarrow *) \underset{\text{cat}}{\cong} \mathbf{Set}$.

(∵ obj. $X \xrightarrow{f} *$ は、$*$: terminal obj. in $\mathbf{Set}$ だから、Xに依らない。
任意の $h: X \longrightarrow X'$ in $\mathbf{Set}$ は、$f = f' \circ h$ を満たす。

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, dashed, "\exists! f"'] & \arrow[d, phantom, "\circlearrowright"] & X' \arrow[dl, dashed, "\exists! f'"] \\ & * & \end{tikzcd}$$

$(f: X \to *,\ f': X' \to *)$.)

- $(\mathbf{Ring} \downarrow \mathbb{Z})$: augmentation $R \longrightarrow \mathbb{Z}$ as objects,
  ring hom preserving augmentations as arrows.

(∵ Obj. $R \xrightarrow{\varepsilon} \mathbb{Z}$ は、augmentation と呼ばれる。(一般論).
Arr $\langle \varepsilon, R \rangle \xrightarrow{h} \langle \varepsilon', R' \rangle$ は、$\varepsilon = \varepsilon' \circ h$ なる ring hom $h: R \longrightarrow R'$.
i.e., ring hom that preserves augmentations.)

Def.

$S: \mathcal{D} \longrightarrow \mathcal{E}$: functor. $b \in \mathcal{E}$: obj.

$(b \downarrow S)$: category of objects S-under b. with

Obj.
$$\begin{tikzcd} b \arrow[d, "f"] \\ Sd \end{tikzcd}$$

Arr.
$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ Sd \arrow[rr, "Sh"'] & {} & Sd' \end{tikzcd}$$

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Def.

$T: \mathcal{E} \longrightarrow \mathcal{C}$: functor. $a \in \mathcal{C}$: obj.

$(T \downarrow a)$: category of objects T-over a with

obj.
$$\begin{tikzcd} Te \arrow[d, "f"] \\ a \end{tikzcd}$$

Arr.
$$\begin{tikzcd} Te \arrow[rr, "Th"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowright"] & Te' \arrow[dl, "f'"] \\ & a & \end{tikzcd}$$

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Ex.

- $U : \mathbf{Grp} \to \mathbf{Set}$: forgetful functor. $x \in \mathbf{Set}$: obj. とする.

  $(x \downarrow U)$: map $x \xrightarrow{f} Ug$ as obj

  group hom $h : \langle f, g \rangle \to \langle f', g' \rangle$ s.t. $f' = h \circ f$ as arr.

Def.

$\mathcal{E} \xrightarrow{T} \mathcal{C} \xleftarrow{S} \mathcal{D}$: functors.

$(T \downarrow S)$: comma category with

Obj.
$$\begin{tikzcd} Te \arrow[d, "f"] \\ Sd \end{tikzcd}$$

Arr.
$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd}$$

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Rem.

Composites of $(T \downarrow S)$ は well-defined :

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd} , \quad \begin{tikzcd} Te' \arrow[r, "Tk'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowright"] & Te'' \arrow[d, "f''"] \\ Sd' \arrow[r, "Sh'"'] & Sd'' \end{tikzcd}$$

を 2つの arrows とすると.

その composite

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[r, "Tk'"] \arrow[d, "f'"] \arrow[dr, phantom, "\circlearrowright"] & Te'' \arrow[d, "f''"] \\ Sd \arrow[r, "Sh"'] & Sd' \arrow[r, "Sh'"'] & Sd'' \end{tikzcd}$$

は $(T \downarrow S)$ の arrow

i.e., commutative diagram

になっている.

$(T \downarrow S)$ は $(b \downarrow \mathcal{C})$, $(\mathcal{C} \downarrow a)$, $(b \downarrow S)$, $(T \downarrow a)$ の一般化である!

- $(b \downarrow S)$ は, $T = b : \mathbb{1} \to \mathcal{C}$ としたもの.
- $(T \downarrow a)$ は, $S = a : \mathbb{1} \to \mathcal{C}$ としたもの.
- $(b \downarrow \mathcal{C})$ は, $T = b : \mathbb{1} \to \mathcal{C}$, $S = 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ としたもの.
- $(\mathcal{C} \downarrow a)$ は, $S = a : \mathbb{1} \to \mathcal{C}$, $T = 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ としたもの.

Exs.

- $\mathcal{C}$: cat. $\mathcal{C} : \mathcal{C} \longrightarrow \mathcal{C}$: id. functor. のとき.

$(\mathcal{C} \downarrow \mathcal{C}) = \mathcal{C}^2$

☺ obj. $c \xrightarrow{f} d$.

Arr.
$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\circlearrowright"] & d \arrow[d, "h"] \\ c' \arrow[r, "f'"'] & d' \end{tikzcd}$$
$\Rightarrow \mathcal{C}^2$ (obj : arrows. arr : commutative squares)

- $T = b,\ S = a : \mathbb{1} \longrightarrow \mathcal{C}$. のとき.

$(b \downarrow a) = \hom_{\mathcal{C}}(b, a)$ (as a discrete cat.).

☺ obj. $b \xrightarrow{f} a$.

Arr.
$$\begin{tikzcd} b \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\circlearrowright"] & a \arrow[d, "\mathrm{id}"] \\ b \arrow[r, "f'"'] & a \end{tikzcd}$$
$\Rightarrow f = f'$. i.e., arrow は. trivial なもののみ.

"comma category" の名前の由来!

(T↓S) の普遍性 (cf. Exercise (5)).

$$\begin{tikzcd} & & (T \downarrow S) \arrow[dll, "P"'] \arrow[d, "R"] \arrow[drr, "Q"] & & \\ \mathcal{E} \arrow[r, "T"'] & \mathcal{C} & \mathcal{C}^2 \arrow[l, "\mathcal{C}^{d_0}"] \arrow[r, "\mathcal{C}^{d_1}"'] & \mathcal{C} & \mathcal{D} \arrow[l, "S"] \end{tikzcd}$$

$d_0, d_1 : \mathbb{1} \longrightarrow \mathbb{2}$

P, Q : projections.

$\mathcal{C}^{d_0}$ (resp $\mathcal{C}^{d_1}$) は. arrow $a \xrightarrow{f} b$ を. $a = \mathrm{dom}\, f$ (resp. $b = \mathrm{cod}\, f$) に送る functor. (cf. §2.5).

Exercises.

(1) K: com. ring のとき. $(K \downarrow \mathbb{C}\mathrm{Rng}) = \mathbb{C}\mathrm{Alg}_K$.

∵ obj. A : com ring, $f: K \longrightarrow A$: ring hom.

i.e., A : com. alg. $/K$.

Arr. $(A, f) \xrightarrow{h} (A', f')$: ring hom s.t.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ A \arrow[rr, "h"'] & & A' \end{tikzcd}$$

i.e., scalar 倍 f, f' を保つ ring hom h

i.e., alg. hom. $h: A \longrightarrow A'$.

(2) $\mathcal{C}$: cat. $t \in \mathcal{C}$: terminal $\Longrightarrow (\mathcal{C} \downarrow t) \cong \mathcal{C}$.

∵ obj. $b \xrightarrow{f} t$ は. t : terminal だから.

各 $b \in \mathcal{C}$ に対して one and exactly one 存在.

Arr. $b \xrightarrow{h} b'$ s.t.

$$\begin{tikzcd} b \arrow[rr, "h"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & b' \arrow[dl] \\ & t & \end{tikzcd}$$

t : terminal だから, この diagram は <u>自動的に</u> commutative.

(4) (S. A. Hug). $T, S: \mathcal{D} \longrightarrow \mathcal{C}$: functors とする.

$\tau: T \xrightarrow{\bullet} S$: nat. transformation は. $\tau: \mathcal{D} \longrightarrow (T \downarrow S)$: functor s.t.

$P\tau = Q\tau = id_{\mathcal{D}}$ に他ならない. (P, Q : projs.)

∵ $\tau: T \xrightarrow{\bullet} S$

$\Longleftrightarrow \forall d \in \mathcal{D}$, $\exists \tau d: Td \longrightarrow Sd$ s.t.

$$\begin{tikzcd} Td \arrow[r, "\tau d"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Sd \arrow[d] \\ Td' \arrow[r, "\tau d'"'] & Sd' \end{tikzcd}$$

$\tau: \mathcal{D} \longrightarrow (T \downarrow S)$ s.t. $P\tau = Q\tau = id_{\mathcal{D}}$

$\Longleftrightarrow$ 各 $d \in \mathcal{D}$ に対して. $\tau d : Te \longrightarrow Sg$ ($\exists e, g \in \mathcal{D}$)

各 $d \xrightarrow{f} d'$ に対して

$$\begin{tikzcd} Te \arrow[r, "\tau d"] \arrow[d, "Tk"'] \arrow[dr, phantom, "\circlearrowright"] & Sg \arrow[d, "Sh"] \\ Te' \arrow[r, "\tau d'"'] & Sg' \end{tikzcd}$$

($\exists k: e \longrightarrow e'$, $h: g \longrightarrow g'$)

s.t.

$$
\begin{array}{ccccc}
e & = & g & = & d \\
\| & & \| & & \| \\
P\tau d & & Q\tau d & & \mathrm{id}_{\mathcal{D}}\, d
\end{array}
\quad , \quad
\begin{array}{ccccc}
k & = & h & = & f \\
\| & & \| & & \| \\
P\tau f & & Q\tau f & & \mathrm{id}_{\mathcal{D}}\, f
\end{array}
\; .
$$

$\Leftrightarrow$ $\forall d \in \mathcal{D}.\ \exists \tau d : Td \longrightarrow Sd$ s.t.

$$
\begin{tikzcd}
Td \arrow[r, "\tau d"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\curvearrowright"] & Sd \arrow[d, "Sf"] \\
Td' \arrow[r, "\tau d'"'] & Sd'
\end{tikzcd}
\qquad (f : d \longrightarrow d') .
$$

上の2つは同値.