Categories for the Working Mathematicians. 2nd ed.

Mac Lane. Springer.

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§ 2.6. Comma Categories.

Def.

$\mathcal{C}$: category. $b \in \mathcal{C}$: object.

$(b \downarrow \mathcal{C})$: category of objects under b with

- objects: $\langle f, c \rangle$, where $c \in \mathcal{C}$: obj. and $f : b \to c$ in $\mathcal{C}$.
- arrows: $\langle f, c \rangle \xrightarrow{h} \langle f', c' \rangle$, where $h : c \to c'$ s.t.

  $f' = h \circ f$.

Objects:
$$\begin{tikzcd} b \arrow[d, "f"] \\ c \end{tikzcd}$$

Arrows:
$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ c \arrow[rr, "h"'] & {} & c' \end{tikzcd}$$

//

Exs.

- $*$: 1-point set のとき. $(* \downarrow Set) = Set_*$.

  (∵ Obj. $* \to X$ は. pair $\langle x \in X, X \rangle$ のこと.

  Arr. $\langle x, X \rangle \to \langle x', X' \rangle$ は. base point を保つ map のこと.)

- $(\mathbb{Z} \downarrow Ab) = Ab_*$

  (∵ Obj. $\mathbb{Z} \xrightarrow{f} G$ は. $\langle f(1), G \rangle$ のこと. (base point $f(1)$).

  Arr. $\langle g, G \rangle \to \langle g', G' \rangle$ は. base point を保つ group hom. のこと.)

Def.

$\mathcal{C}$: cat. $a \in \mathcal{C}$: obj.

$(\mathcal{C} \downarrow a)$: category of objects over a with

Objs:
$$\begin{tikzcd} c \arrow[d, "f"] \\ a \end{tikzcd}$$

Arrs:
$$\begin{tikzcd} c \arrow[rr, "h"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\circlearrowright"] & c' \arrow[dl, "f'"] \\ & a & \end{tikzcd}$$

//

Exs.

- $*$: 1-point set. $\Rightarrow$ $(\mathbf{Set} \downarrow *) \underset{\text{cat}}{\cong} \mathbf{Set}$.

(∵ obj. $X \xrightarrow{f} *$ は、$*$: terminal obj. in $\mathbf{Set}$ だから、X に他ならない。
任意の $h: X \longrightarrow X'$ in $\mathbf{Set}$ は、$f = f' \circ h$ を満たす。

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, dashed, "\exists! f"'] & \arrow[d, phantom, "\circlearrowright"] & X' \arrow[dl, dashed, "\exists! f'"] \\ & * & \end{tikzcd}$$

$(f: X \to *,\ f': X' \to *)$.)

- $(\mathbf{Rng} \downarrow \mathbb{Z})$: augmentation $R \longrightarrow \mathbb{Z}$ as objects,
  ring hom preserving augmentations as arrows.

(∵ Obj. $R \xrightarrow{\varepsilon} \mathbb{Z}$ は、augmentation と呼ばれる。(一般論).
Arr $\langle \varepsilon, R \rangle \xrightarrow{h} \langle \varepsilon', R' \rangle$ は、$\varepsilon = \varepsilon' \circ h$ なる ring hom $h: R \to R'$.
i.e., ring hom that preserves augmentations.)

Def.

$S: \mathcal{D} \longrightarrow \mathcal{C}$: functor. $b \in \mathcal{C}$: obj.

$(b \downarrow S)$: category of objects S-under b. with

Obj. $\begin{tikzcd} b \arrow[d, "f"] \\ Sd \end{tikzcd}$ Arr. $\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ Sd \arrow[rr, "Sh"'] & & Sd' \end{tikzcd}$

//

Def.

$T: \mathcal{E} \longrightarrow \mathcal{C}$: functor. $a \in \mathcal{C}$: obj.

$(T \downarrow a)$: category of objects T-over a with

obj. $\begin{tikzcd} Te \arrow[d, "f"] \\ a \end{tikzcd}$ Arr. $\begin{tikzcd} Te \arrow[rr, "Th"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowright"] & Te' \arrow[dl, "f'"] \\ & a & \end{tikzcd}$

//

Ex.

- $U : \mathbf{Grp} \to \mathbf{Set}$: Forgetful functor. $x \in \mathbf{Set}$: obj. とする.

  $(x \downarrow U)$: map $x \xrightarrow{f} Ug$ as obj.

  group hom $h : (f, g) \to (f', g')$ s.t. $f' = h \circ f$ as arr.

Def.

$\mathcal{E} \xrightarrow{T} \mathcal{C} \xleftarrow{S} \mathcal{D}$: functors.

$(T \downarrow S)$: <u>comma category</u> with

Obj.
$$\begin{tikzcd} Te \arrow[d, "f"] \\ Sd \end{tikzcd}$$

Arr.
$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd}$$

//

Rem.

Composites of $(T \downarrow S)$ は well-defined :

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd} , \quad \begin{tikzcd} Te' \arrow[r, "Tk'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowright"] & Te'' \arrow[d, "f''"] \\ Sd' \arrow[r, "Sh'"'] & Sd'' \end{tikzcd}$$

を 2つの arrows とすると.

その composite

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[r, "Tk'"] \arrow[d, "f'"] \arrow[dr, phantom, "\circlearrowright"] & Te'' \arrow[d, "f''"] \\ Sd \arrow[r, "Sh"'] & Sd' \arrow[r, "Sh'"'] & Sd'' \end{tikzcd}$$

は $(T \downarrow S)$ の arrow

i.e., commutative diagram

になっている.

<u>$(T \downarrow S)$ は $(b \downarrow \mathcal{C})$, $(\mathcal{C} \downarrow a)$, $(b \downarrow S)$, $(T \downarrow a)$ の一般化である!</u>

- $(b \downarrow S)$ は, $T = b : \mathbb{1} \to \mathcal{C}$ としたもの.
- $(T \downarrow a)$ は, $S = a : \mathbb{1} \to \mathcal{C}$ としたもの.
- $(b \downarrow \mathcal{C})$ は, $T = b : \mathbb{1} \to \mathcal{C}$, $S = 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ としたもの.
- $(\mathcal{C} \downarrow a)$ は, $S = a : \mathbb{1} \to \mathcal{C}$, $T = 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ としたもの.

Exs.

- $\mathcal{C}$: cat. $\mathcal{C} : \mathcal{C} \longrightarrow \mathcal{C}$: id. functor. のとき.

$(\mathcal{C} \downarrow \mathcal{C}) = \mathcal{C}^2$

$\left(\begin{array}{l} \text{obj. } c \xrightarrow{f} d. \\ \text{Arr. } \begin{array}{ccc} c & \xrightarrow{f} & d \\ {\scriptstyle k}\downarrow & \circlearrowright & \downarrow{\scriptstyle h} \\ c' & \xrightarrow[f']{} & d' \end{array} \Rightarrow \mathcal{C}^2 \left(\begin{array}{l} \text{obj : arrows.} \\ \text{arr : commutative squares} \end{array} \right) \end{array} \right)$

- $T = b,\ S = a : \mathbb{1} \longrightarrow \mathcal{C}$. のとき.

$(b \downarrow a) = \hom_{\mathcal{C}}(b, a)$ (as a discrete cat.).

$\left(\begin{array}{l} \text{obj. } b \xrightarrow{f} a. \\ \text{Arr. } \begin{array}{ccc} b & \xrightarrow{f} & a \\ {\scriptstyle id}\downarrow & \circlearrowright & \downarrow{\scriptstyle id} \\ b & \xrightarrow[f']{} & a \end{array} \Rightarrow f = f'. \quad \text{i.e., arrow は. trivial なもののみ.} \end{array} \right)$

"comma category" の名前の由来!

$(T \downarrow S)$ の普遍性 (cf. Exercise (5)).

$$\begin{tikzcd}
 & & (T \downarrow S) \arrow[dll, "P"'] \arrow[d, "R"] \arrow[drr, "Q"] & & \\
\mathcal{E} \arrow[r, "T"'] & \mathcal{C} & \mathcal{C}^2 \arrow[l, "\mathcal{C}^{d_0}"] \arrow[r, "\mathcal{C}^{d_1}"'] & \mathcal{C} & \mathcal{D} \arrow[l, "S"]
\end{tikzcd}$$

$d_0, d_1 : \mathbb{1} \longrightarrow \mathbb{2}$

P, Q : projections.

$\mathcal{C}^{d_0}$ (resp $\mathcal{C}^{d_1}$) は. arrow $a \xrightarrow{f} b$ を. $a = \operatorname{dom} f$ (resp. $b = \operatorname{cod} f$) に送る functor. (cf. §2.5).

Exercises.

(1) K: com. ring のとき. $(K \downarrow \mathbf{CRng}) = \mathbf{CAlg}_K$.

$\because$ obj. A: com ring, $f: K \longrightarrow A$: ring hom.

i.e., A: com. alg. $/K$.

Arr. $(A, f) \xrightarrow{h} (A', f')$: ring hom s.t.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ A \arrow[rr, "h"'] & & A' \end{tikzcd}$$

i.e., scalar 倍 f, f' を保つ ring hom h

i.e., alg. hom. $h: A \longrightarrow A'$.

(2) $\mathcal{C}$: cat. $t \in \mathcal{C}$: terminal $\Longrightarrow (\mathcal{C} \downarrow t) \cong \mathcal{C}$.

$\because$ obj. $b \xrightarrow{f} t$ は. t: terminal だから.

各 $b \in \mathcal{C}$ に対して one and exactly one 存在.

Arr. $b \xrightarrow{h} b'$ s.t.

$$\begin{tikzcd} b \arrow[r, "h"] \arrow[dr] \arrow[r, phantom, shift right=3, "\circlearrowright"] & b' \arrow[d] \\ & t \end{tikzcd}$$

t: terminal だから, この diagram は <u>自動的に</u> commutative.

(4) (S.A. Hung). $T, S: \mathcal{D} \longrightarrow \mathcal{C}$: functors とする.

$\tau: T \xrightarrow{\bullet} S$: nat. transformation は. $\tau: \mathcal{D} \longrightarrow (T \downarrow S)$: functor s.t.

$P\tau = Q\tau = \mathrm{id}_{\mathcal{D}}$ に他ならない. (P, Q: projs.)

$\because$ $\tau: T \xrightarrow{\bullet} S$

$\Longleftrightarrow$ $\forall d \in \mathcal{D}$, $\exists \tau d: Td \longrightarrow Sd$ s.t.

$$\begin{tikzcd} Td \arrow[r, "\tau d"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Sd \arrow[d] \\ Td' \arrow[r, "\tau d'"'] & Sd' \end{tikzcd}$$

$\tau: \mathcal{D} \longrightarrow (T \downarrow S)$ s.t. $P\tau = Q\tau = \mathrm{id}_{\mathcal{D}}$

$\Longleftrightarrow$ 各 $d \in \mathcal{D}$ に対して. $\tau d: Te \longrightarrow Sg$ ($\exists e, g \in \mathcal{D}$)

各 $d \xrightarrow{f} d'$ に対して

$$\begin{tikzcd} Te \arrow[r, "\tau d"] \arrow[d, "Tk"'] \arrow[dr, phantom, "\circlearrowright"] & Sg \arrow[d, "Sh"] \\ Te' \arrow[r, "\tau d'"'] & Sg' \end{tikzcd}$$

($\exists k: e \longrightarrow e'$, $h: g \longrightarrow g'$)

s.t.

$$\begin{array}{ccccccccccc} e & = & g & = & d & , & k & = & h & = & f \\ \| & & \| & & \| & & \| & & \| & & \| \\ P\tau d & & Q\tau d & & id_{\mathcal{D}} d & & P\tau f & & Q\tau f & & id_{\mathcal{D}} f \end{array}.$$

$\Leftrightarrow$ $\forall d \in \mathcal{D}.$ ${}^{\exists}\tau d : Td \longrightarrow Sd$ s.t.

$$\begin{tikzcd} Td \arrow[r, "\tau d"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\curvearrowright"] & Sd \arrow[d, "Sf"] \\ Td' \arrow[r, "\tau d'"'] & Sd' \end{tikzcd} \quad (f : d \longrightarrow d').$$

上の2つは同値.

§2.7. Graphs and Free Categories.

まず. free monoid の場合を思い出す:

X : set から生成される free monoid F とは.

- Word $x_1 \cdots x_n$ $(x_i \in X)$ を元に持ち,
- words の結合 (juxtaposition) を積とし,
- empty word を identity element とする

monoid のこと.

これと equivalent な条件は.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & F \arrow[d, dashed, "\exists ! \tilde{f}"] \\ & M \end{tikzcd}$$

$(F : \text{monoid}, \ \iota : X \longrightarrow F : \text{map})$ が free monoid

$\Leftrightarrow \forall (M : \text{monoid}, \ f : X \longrightarrow M : \text{map}),$

$\exists ! \ \tilde{f} : F \longrightarrow M : \text{monoid hom. s.t. } f = \tilde{f} \circ \iota.$

これを元に free category を考える.

Def. (§1.2)

G : (directed) graph とは.

- O : set of objects (vertices)
- A : set of arrows (edges).

together with functions

$$A \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} O \ , \quad \partial_0 f = \operatorname{dom} f, \quad \partial_1 f = \operatorname{cod} f.$$

$D : G \longrightarrow G'$: morphism of graphs とは.

- $D_O : O \longrightarrow O'$
- $D_A : A \longrightarrow A'$ s.t. $D_O \partial_0 = \partial_0 D_A$, $D_O \partial_1 = \partial_1 D_A$.

//

$\rightsquigarrow$ **Grph** : category of graphs.

Graphs には. composite と identity がない.

↳ diagram scheme や. precategory とも呼ぶ!

Graphs obtained from categories

$\mathcal{C}$: cat. に対して. $U\mathcal{C}$: graph を.

- obj. : objects of $\mathcal{C}$;
- arr. : arrows of $\mathcal{C}$

と定める. $F : \mathcal{C} \longrightarrow \mathcal{C}'$: functor は.

graph morphism $UF : U\mathcal{C} \longrightarrow U\mathcal{C}'$ に対応する.

$\therefore U : \mathbf{Cat} \longrightarrow \mathbf{Grph}$: forgetful functor.

Categories obtained from graphs

O : set : fixed.

Def.

O-graph とは. graph whose object set is O.

A morphism of O-graph とは. O-graphs 間の graph morphism のこと. //

A, B : sets of arrows (regarded as O-graphs) とすると.

$$A \times_O B := \{ \langle g, f \rangle \mid \partial_0 g = \partial_1 f,\ g \in A\ \ f \in B \}$$

(composable pairs of arrows $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$)

と定める.

$$\partial_0 \langle g, f \rangle := \partial_0 f, \quad \partial_1 \langle g, f \rangle := \partial_1 g$$

とすれば. $A \times_O B$: O-graph になる.

$\times_O$ は associative

(∵ $(A \times_O B) \times_O C \cong A \times_O (B \times_O C)$

$\cong \{ \langle h, g, f \rangle \mid \partial_0 h = \partial_1 g,\ \partial_0 g = \partial_1 f,\ h \in A,\ g \in B,\ f \in C \}$)

任意の A : O-graph に対して.

$$
\begin{array}{ccccc}
A & \cong & A \times_O O & \cong & O \times_O A \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
f & \longmapsto & \langle f, \partial_0 f \rangle & \longmapsto & \langle \partial_1 f, f \rangle.
\end{array}
$$

Rem. O を次のように O-graph とみなしている:

- arrows : O itself.
- $\partial_0, \partial_1 = \mathrm{id}_O : O \rightrightarrows O$

$$
\begin{array}{ccc}
O & \rightrightarrows & O \\
\uparrow & & \uparrow \\
\text{arrows} & & \text{objects}
\end{array}
$$

一方, $\mathcal{C}$: cat. whose object set is O は.

A : O-graph together with morphisms

$c : A \times_O A \longrightarrow A$ (composite)

$i : O \longrightarrow A$ (identity) s.t.

$$
\begin{tikzcd}
(A \times_O A) \times_O A \cong A \times_O (A \times_O A) \arrow[r, "1 \times c"] \arrow[d, "c \times 1"'] \arrow[dr, phantom, "\circlearrowright"] & A \times_O A \arrow[d, "c"] \\
A \times_O A \arrow[r, "c"'] & A
\end{tikzcd}
$$

(associativity)

$$
\begin{tikzcd}
O \times_O A \arrow[r, "i \times 1"] \arrow[dr, "\wr"'] & A \times_O A \arrow[d] & A \times_O O \arrow[l, "1 \times i"'] \arrow[dl, "\sim"] \\
& A &
\end{tikzcd}
$$

(unit law).

G : O-graph から $\mathcal{C} = \mathcal{C}(G)$:cat を作る.

大雑把に言えば, G の composable pair of arrows をすべて"繋げた"もの.

厳密に言えば,

Thm. 2.1.

$G = \{A \rightrightarrows O\}$: small graph.

$\Rightarrow \exists\, \mathcal{C} = \mathcal{C}_G$: small cat. $P : G \longrightarrow U\mathcal{C}$: morphism of graphs s.t.

$\forall\, (\mathcal{B}$: cat., $D : G \longrightarrow U\mathcal{B}$: morphism of graphs$)$,

$\exists!\, \tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$: functor with $(U\tilde{D}) \circ P = D$.

$$
\begin{tikzcd}
\mathcal{C} \arrow[d, dashed, "\exists! \tilde{D}"'] \\
\mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "P"] \arrow[dr, "D"'] & U\mathcal{C} \arrow[d, dashed, "U\tilde{D}"] \\
& U\mathcal{B}
\end{tikzcd}
$$

("任意の graph morphism $D : G \longrightarrow U\mathcal{B}$ を、

functor $\tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$ に拡張できる!")

特に、object set of $\mathcal{B} = O$, D : O-graph morphism

$\Rightarrow \tilde{D}|_{obj.} = id$.

$P : G \longrightarrow U\mathcal{C}$ は、$(G \downarrow U)$ の initial object である.

∴ unique up to isomorphism.

$\mathcal{C}$ を free category generated by G と呼ぶ.

Proof.

次のように、$\mathcal{C}$ を構成する.

- obj.: objects of G i.e., O.
- arr.: finite strings $\langle a_1, f_1, a_2, \dots, f_{n-1}, a_n \rangle \overset{\text{denote}}{\equiv} a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} a_n$

  where $a_i \in O$, $f_i \in A$, $\partial_0 f_i = a_i$, $\partial_1 f_i = a_{i+1}$.
- composite: juxtaposition of strings (i.e., "concatenation").
- identity: $\langle a \rangle \overset{\text{denotes}}{\equiv} a : a \longrightarrow a$

Then, every $\underbrace{\text{arrow of } \mathcal{C}}_{\text{of length} > 1}$ is a composite of (finitely many) strings of length 2:

$\langle a_1, f_1, \cdots, f_{n-1}, a_n \rangle = \langle a_{n-1}, f_{n-1}, a_n \rangle \circ \cdots \circ \langle a_1, f_1, a_2 \rangle \quad (n \geq 2)$

Define graph morphism $P : G \longrightarrow U\mathcal{C}$ as

- on obj. : identity $O \longrightarrow O$;
- on arr. : $A \ni f \longmapsto \langle \partial_0 f, f, \partial_1 f \rangle$.

この $(\mathcal{C}, P)$ が条件を満たすことを示す.

$\mathcal{B}$: cat., $D : G \longrightarrow U\mathcal{B}$: graph morphism を任意に取る.

If $\exists \tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$: functor s.t. $U\tilde{D} \circ P = D$,

$\Longrightarrow$ $\tilde{D}$ must be

- on obj. : $\underline{\tilde{D}a = Da}$;
- on arr. : $\tilde{D}\langle a_1, f_1, a_2 \rangle = Df_1$.

And, by the composition of a functor,

$$\underline{\tilde{D}\langle a_1, f_1, \cdots, f_{n-1}, a_n \rangle = Df_{n-1} \circ \cdots \circ Df_1}.$$

$\therefore$ $\tilde{D}$ is <u>automatically unique</u>.

よって. $\tilde{D}$ が存在することを示せば. 証明は完了する.

上のように $\tilde{D}$ を construct したとき.

$\tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$ が functor で.

$U\tilde{D} \circ P = D$ を満たすことは明らか.

//

Exs.

- $G =$  $(O = \{\bullet\},\ A = \{f\})$ のとき.

$\mathcal{C}_G$ の arrows は. $f, f^2, f^3, \cdots$

- $G = \bullet \xrightarrow{g} \bullet$ （$O=\{0,1\}$. $A=\{g\}$）のとき.

$\mathcal{C}_G$ の arrows は. $id_0 : 0 \longrightarrow 0$.

$id_1 : 1 \longrightarrow 1$.

$g : 0 \longrightarrow 1$

のみ.（g と juxtapose できる arrow がない！）

- $G = \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ （$O=\{0,1,2\}$. $A=\{f, g\}$）のとき.

$\mathcal{C}_G$ の arrows は. $id_i : i \longrightarrow i$ （$i = 0,1,2$）および

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & \bullet \end{tikzcd}$$

のみ.（composable pair は $\langle f, g\rangle$ しかない！）
$\quad\;\;\underset{\text{juxtaposable}}{\overset{\shortparallel}{}}$

Cor. 2.2.

$\forall X$: set. $\exists M$: monoid, $p : X \longrightarrow UM$: map,

where $U : \mathbf{Mon} \longrightarrow \mathbf{Set}$: forgetful functor, s.t.

$$\begin{tikzcd} M \arrow[d, dashed, "\exists!\tilde{h}"'] \\ L \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "h"'] & UM \arrow[d, dashed, "U\tilde{h}"] \arrow[dl, phantom, "\circlearrowright"] \\ & UL \end{tikzcd}$$

$\forall$ (L : monoid, $h : X \longrightarrow UL$: map)

$\exists! \tilde{h} : M \longrightarrow L$; monoid hom. s.t.

$h = U\tilde{h} \circ p$.

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Proof of Cor 2.2.

Thm. 2.1 において $O = \{\bullet\}$, $A = X$ とすると.

$\exists M$: cat. whose object set is $O = \{\bullet\}$

（M : monoid と見なせる！） s.t.

$U : \mathbf{Cat} \longrightarrow \mathbf{Graph}$

$\forall L$: monoid (regarded as a category) and $h : G \longrightarrow UL$: graph morphism,

（$h : X \longrightarrow UL$: map と見なせる！）,

$U : \mathbf{Mon} \longrightarrow \mathbf{Set}$

$\exists! \ \tilde{h} : M \longrightarrow L$: functor (regarded as a monoid hom.) s.t.

$U : \mathbf{Cat} \longrightarrow \mathbf{Grph}$ (regarded as $\mathbf{Mon} \longrightarrow \mathbf{Set}$).

$\mathbf{Mon} \cap \mathbf{Cat}$

$$h = U\tilde{h} \circ p$$

set maps (regarded as graph morphisms).

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Graphs as diagrams

Def.

G : graph. $\mathcal{B}$: cat.

A *diagram of shape G in $\mathcal{B}$* とは

graph morphism $D : G \longrightarrow U\mathcal{B}$ のこと.

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By Thm. 2.1, D は. $\tilde{D} : \mathcal{C}_G \longrightarrow \mathcal{B}$: functor に他ならない.

$$\mathbf{Cat}(\mathcal{C}_G, \mathcal{B}) \underset{\text{bij.}}{\cong} \mathbf{Grph}(G, U\mathcal{B})$$

is natural in G and $\mathcal{B}$.

$\therefore \ \mathcal{C} : \mathbf{Grph} \longrightarrow \mathbf{Cat}$ is left adjoint to $U : \mathbf{Cat} \longrightarrow \mathbf{Grph}$.

(cf. chapter 4).

Exercise

(2) Every finite ordinal is a free category.

($\because$ An finite ordinal n is a free category

generated by $G = 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-1$.)

§8. Quotient Categories

Def.

$\mathcal{C}$: cat.

R : congruence on $\mathcal{C}$ とは.

i) 各 $a, b \in \mathcal{C}$ に対して. $R_{a,b}$: equivalence relation on $\mathcal{C}(a,b)$;

ii) $a \underset{f'}{\overset{f}{\rightrightarrows}} b$, $f R_{a,b} f' \Rightarrow \forall g : a' \to a,\ h : b \to b'$,
$(hfg) R_{a',b'} (hf'g)$.

//

Prop. 2.1.

$\mathcal{C}$: cat. 各 $a, b \in \mathcal{C}$ に対して. $R_{a,b}$: binary relation on $\mathcal{C}(a,b)$ が定まっているとする.

$\Rightarrow \exists\, \mathcal{C}/R$: cat. together with $Q = Q_R : \mathcal{C} \to \mathcal{C}/R$: functor, s.t.

i) $f R_{a,b} f' \Rightarrow Qf = Qf'$;

ii) $H : \mathcal{C} \to \mathcal{D}$: functor s.t. $f R_{a,b} f'$ implies $Hf = Hf'$,

$\Rightarrow \exists!\, H' : \mathcal{C}/R \to \mathcal{D}$: functor with $H' \circ Q_R = H$.

さらに, Q_R : bij. on obj.

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短かく言えば. Q は $f R_{a,b} f'$ ならば $Qf = Qf'$ なる universal functor である!

$\mathcal{C} = \mathcal{C}_G$: cat. generated by a graph G のとき.

$\mathcal{C}/R$: category with generators G and relations R という.

各 $a, b \in \mathcal{C}$ に対して $R_{a,b}$, $S_{a,b}$: binary relations が定まっていて,
$R_{a,b} \subset S_{a,b}$ なるとき. $R \subset S$ と書く.

Proof of Prop. 2.1.

まず. $\exists R'$: least congruence on $\mathcal{C} \supset R$ である.

i.e., i) R': congruence on $\mathcal{C}$, $R \subset R'$.

ii) S: congruence on $\mathcal{C}$, $R \subset S \Rightarrow R' \subset S$.

Indeed, 各 $a, b \in \mathcal{C}$ に対して,

$$R''_{a,b} := \{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \mid f R_{a,b} f' \text{ or } f' R_{a,b} f \}$$

$$R'''_{a,b} := \{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \mid \exists f_0, \dots, f_n \in \mathcal{C}(a,b) \text{ s.t. } f_0 = f,\ f_0 R''_{a,b} f_1,\ f_1 R''_{a,b} f_2, \dots, f_{n-1} R''_{a,b} f_n,\ f_n = f' \}$$

とおくと. $R'''_{a,b}$ は. $R_{a,b} \subset R'''_{a,b}$ なる equivalence relation.

さらに.

$$R''''_{a,b} := \left\{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \;\middle|\; \exists \begin{tikzcd} a' \arrow[r, "f_0"] & b' \arrow[d, "h"] \\ a \arrow[u, "g"] \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowleft"] & b \end{tikzcd} \quad \exists \begin{tikzcd} a' \arrow[r, "f'_0"] & b' \arrow[d, "h"] \\ a \arrow[u, "g"] \arrow[r, "f'"'] \arrow[ur, phantom, "\circlearrowleft"] & b \end{tikzcd} \text{ s.t. } f_0 R'''_{a',b'} f'_0 \right\}$$

$$R'_{a,b} := \{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \mid \exists f_0, \dots, f_n \in \mathcal{C}(a,b) \text{ s.t. } f_0 = f,\ f_0 R''''_{a,b} f_1,\ f_1 R''''_{a,b} f_2, \dots, f_{n-1} R''''_{a,b} f_n,\ f_n = f' \}.$$

とすれば. R' : congruence on $\mathcal{C}$ である.

全ての congruence on $\mathcal{C} \supset R$ の intersection はまた congruence on $\mathcal{C} \supset R$ になっていることから従う.

$\mathcal{C}/R$: cat. を次のように作る:

Obj.: obj. of $\mathcal{C}$.

Arr.: $(\mathcal{C}/R)(a,b) := \mathcal{C}(a,b)/R'_{a,b}$.

$Q = Q_R : \mathcal{C} \longrightarrow \mathcal{C}/R$: canonical projection とする.

R': congruence より. $\mathcal{C}/R$ は category で. Q は functor になる.

(∵ $f \in \mathcal{C}(a,b)$ の equivalence class を $\bar{f} \in \mathcal{C}/R\,(a,b)$ と表せば.

$a \overset{f}{\underset{f'}{\rightrightarrows}} b \overset{g}{\underset{g'}{\rightrightarrows}} c$, $\bar{f} = \bar{f'}$, $\bar{g} = \bar{g'}$

$\Rightarrow$ $f R'_{a,b} f' \rightsquigarrow g f R'_{a,c} g f'$

$g R'_{b,c} g' \rightsquigarrow g f' R'_{a,c} g' f'$

$\rightsquigarrow g f R_{a,c} g' f'$.

$\therefore$ $\overline{g \circ f} = \overline{g' \circ f'}$.

i.e., $\bar{g} \circ \bar{f} := \overline{g \circ f}$ は well-defined.

この合成が associative で. $\overline{id_a}$ が identity になること,

従って $\mathcal{C}/R$: category であることは明らか.

$Q : \mathcal{C} \longrightarrow \mathcal{C}/R$: functor も. $\mathcal{C}/R$: category からすぐにわかる.)

$H : \mathcal{C} \longrightarrow \mathcal{D}$: functor s.t.

$f R_{a,b} f' \Rightarrow Hf = Hf'$

を任意に取る.

$$S_{a,b} := \{ (f, f') \in \mathcal{C}(a,b) \times \mathcal{C}(a,b) \mid Hf = Hf' \} \quad (a, b \in \mathcal{C})$$

とおけば, S : congruence on $\mathcal{C}$ かつ. $R_{a,b} \subset S_{a,b}$.

$\therefore$ $R'_{a,b} \subset S_{a,b}$

よって. $H = H' \circ Q_R$ と分解される. //

Exs.

- $\mathcal{C} = \mathbf{Top}$. $\quad f R_{a,b} f' \overset{\text{def.}}{\iff} \underset{\text{homotopic}}{f \simeq f'}$ とすると、

$$\mathcal{C}/R = \mathbf{Toph}.$$

- $$\mathbf{3} = \begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3, "\circlearrowleft"] & & \bullet \end{tikzcd}$$ は、

$$G = \begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] & & 2 \end{tikzcd}$$ を generator とし、

$h = g \circ f$ を relation とした quotient category.

Exercises

(1)
$$G = \begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "f"'] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r, "g'"'] & \bullet \end{tikzcd}$$ で生成され、

relation $g'f = f'g$ で割られた quotient category $\mathcal{C}$ は

commutative square with a diagonal
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$ である。

(2) G: group. (regarded as a cat.), R: congruence on G (as a cat.)

$\Longrightarrow \exists N \lhd G$ s.t. $\quad fRg$ iff $g^{-1}f \in N$.

$\because$ $N := \{ g^{-1}f \mid f, g \in G,\ fRg \}$

$= \{ f \in G \mid fR1 \}$ とおけばよい。 //

§3.1. Universal arrows

Def.

$S : \mathcal{D} \longrightarrow \mathcal{C}$: functor. $c \in \mathcal{C}$.

$\langle r \in \mathcal{D},\ u : c \longrightarrow Sr \rangle$: universal arrow from c to S とは.

$$\begin{tikzcd} & c \arrow[dl, "u"'] \arrow[dr, "f"] \arrow[d, phantom, "\circlearrowright"] & \\ Sr \arrow[rr, dashed, "S\tilde{f}"'] & & Sd \end{tikzcd} \qquad \begin{tikzcd} r \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ d \end{tikzcd}$$

$\forall \langle d \in \mathcal{D},\ f : c \longrightarrow Sd \rangle$,

$\exists!\ \tilde{f} : r \longrightarrow d$ s.t.

$f = S\tilde{f} \circ u$.

//

$c \xrightarrow{u} Sr$: universal arrow

$\Longleftrightarrow$ $c \xrightarrow{u} Sr$: initial obj. in $(c \downarrow S)$

である. 特に, universal arrow は unique up to iso.

Exs.

以下では, $U : \mathcal{D} \longrightarrow \mathcal{C}$: forgetful functor とする.

- k : Field. X : set. $X \hookrightarrow \mathrm{span}_k X$ は. universal arrow from X to $U : \mathbf{Vect}_k \longrightarrow \mathbf{Set}$.
- G : graph. $P : G \longrightarrow U\mathcal{C}_G$ は universal arrow from G to $U : \mathbf{Grph} \longrightarrow \mathbf{Cat}$.
- X : set. $\langle X \rangle$: free group generated by X. $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$.
  $X \hookrightarrow U\langle X \rangle$ は. universal arrow from X to U.
- $\mathbf{Dom}_m$: category with obj. = integral domain,
  arr. = monomorphism とする.

  $U : \mathbf{Fld} \longrightarrow \mathbf{Dom}_m$ (fields 間の ring map は単射であることに注意).

  $D \in \mathbf{Dom}_m$, $\mathrm{Frac}\, D$: field of quotients of D.

  $D \hookrightarrow U\,\mathrm{Frac}\, D$ は universal arrow from D to U.

Rem. Domm を Dom に置き換えてはいけない!

Indeed,

F: Field, $f: D \longrightarrow F$: ring map を

$\tilde{f}: \mathrm{Frac}\, D \longrightarrow F$: ring map に拡張するためには,

$\tilde{f}\left(\frac{1}{r}\right) := \frac{1}{f(r)}$ でなければならない.

そのためには,

$$r \in D \setminus \{0\} \Longrightarrow f(r) \in F \setminus \{0\}$$

i.e., $\ker f = \{0\}$

でなければならない.

Ex. $\mathbf{Met}$: category with obj.: metric space.

arr.: map preserving metric.

$\mathbf{CMet} \subset \mathbf{Met}$: full subcat. whose objs. are complete.

$U: \mathbf{CMet} \longrightarrow \mathbf{Met}$. X: metric space. $\tilde{X}$: completion of X.

$X \longrightarrow \tilde{X}$ は universal arrow from X to U.

Def.

$H: \mathcal{D} \longrightarrow \mathbf{Set}$: functor.

$\langle r \in \mathcal{D}, e \in Hr \rangle$: universal element of H とは,

$\forall \langle d \in \mathcal{D}, x \in Hd \rangle$, $\exists ! f: r \longrightarrow d$ s.t. $(Hf)e = x$. //

Rem.

特殊な状況では, universal arrows と universal elements は同じもの.

Ex. $H: \mathcal{D} \longrightarrow \mathbf{Set}$: functor. $\langle r, e \rangle$: universal element とは,

$* \xrightarrow{e} Hr$ in $\mathbf{Ens}$ と見たときの $\langle r, e \rangle$: universal arrow である.

$$\begin{tikzcd}
 & * \arrow[dl, "e"'] \arrow[dr, "x"] \arrow[d, phantom, "\circlearrowleft"] & \\
Hr \arrow[rr, dashed, "Hf"'] & & Hd
\end{tikzcd}
\qquad
\begin{tikzcd}
r \arrow[d, dashed, "\exists ! f"] \\
d
\end{tikzcd}$$

◎逆に，$\mathcal{E}$: small cat. $S : \mathcal{D} \longrightarrow \mathcal{E}$: functor. $c \in \mathcal{E}$ のとき，

$\langle r, u : c \longrightarrow Sr \rangle$: universal arrow とは，

$H = \mathcal{E}(c, S\cdot) : \mathcal{D} \longrightarrow \mathbf{Set}$ の universal element である．

<u>Exs.</u>

◎ S : set. $E \subset S \times S$: equiv. relation. $\pi : S \twoheadrightarrow S/_E$.

$\langle S/_E, \pi \rangle$ は universal element of $H : \mathbf{Set} \longrightarrow \mathbf{Set}$,

where $HX := \{ f : S \longrightarrow X \mid sEs' \Rightarrow fs = fs' \}$.

$$\begin{array}{ccccc} Hg : & HX & \longrightarrow & HY & (g : X \longrightarrow Y). \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & f & \longmapsto & g \circ f & \end{array}$$

◎ G : group $N \triangleleft G$. $\pi : G \twoheadrightarrow G/_N$.

$\langle G/_N, \pi \rangle$ は，universal element of $H : \mathbf{Grp} \longrightarrow \mathbf{Set}$,

where $HG' := \{ f : G \longrightarrow G' : \text{group hom.} \mid \ker f \subset N \}$.

$$\begin{array}{ccccc} Hg : & HG' & \longrightarrow & HG'' & (g : G' \longrightarrow G'') \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & f & \longmapsto & g \circ f & \end{array}$$

◎ V_1, V_2 : vect. sps. $/_k$. $H : \mathbf{Vect}_k \longrightarrow \mathbf{Set}$,

where $HW := \mathrm{Bilin}(V_1, V_2 ; W) := \{ f : V_1 \times V_2 \longrightarrow W : \text{bilinear} \}$.

$$\begin{array}{ccccc} Hg : & HW & \longrightarrow & HW' & (g : W \longrightarrow W') \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & f & \longmapsto & g \circ f & \end{array}$$

$\langle V_1 \otimes_k V_2, \otimes \rangle$ は universal element of H.

$$\begin{array}{cccc} \otimes : & V_1 \times V_2 & \longrightarrow & V_1 \otimes_k V_2 \\ & \rotatebox{90}{$\in$} & & \\ & (x, y) & \longmapsto & x \otimes y. \end{array}$$

$\mathbf{Vect}_k$ ではなく，$\mathbf{Mod}_R$ でも OK!

Def. $S: \mathcal{D} \longrightarrow \mathcal{C}$, $c \in \mathcal{C}$.

$\langle r \in \mathcal{D},\ v: Sr \longrightarrow c \rangle$ が <u>universal arrow from S to c</u> とは、

$\langle r, v \rangle$ が $(S \downarrow c)$ の terminal obj. であること.

$$\begin{tikzcd} d \arrow[d, dashed, "\exists! \tilde{f}"] \\ r \end{tikzcd} \qquad \begin{tikzcd} Sd \arrow[rr, dashed, "S\tilde{f}"] \arrow[dr, "f"'] & & Sr \arrow[dl, "v"] \\ & c & \end{tikzcd}$$

//

Exs. $\mathcal{C} = \mathbf{Grp}, \mathbf{Set}, \mathbf{Cat}, \mathbf{Top}, \mathbf{Vect}_k$, etc.

(category where the direct product of two objs. can be defined.)

$a, b \in \mathcal{C}$, $p: a \times b \twoheadrightarrow a$, $q: a \times b \twoheadrightarrow b$.

$\langle p, q \rangle$ は universal arrow from $\Delta: \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$ to $\langle a, b \rangle$,

where $\Delta c := \langle c, c \rangle$, (<u>diagonal functor</u>)

$\Delta f := \langle f, f \rangle$

$$\begin{tikzcd} \langle c_1, c_2 \rangle \arrow[rr, dashed, "\exists! \langle f_1 \times f_2,\, f_1 \times f_2 \rangle"] \arrow[dr, "\langle f_1, f_2 \rangle"'] & & \langle a \times b, a \times b \rangle \arrow[dl, "\langle p, q \rangle"] \\ & \langle a, b \rangle & \end{tikzcd}$$

Exercises

(2) The universal element of $\mathcal{P}: \mathbf{Set}^{op} \longrightarrow \mathbf{Set}$ (power set)

is $\langle \{0, 1\},\ 1 \in \{0, 1\} \rangle$.

(3) $G \in \mathbf{Grp}$ (or $\in \mathbf{Ab}$), $X \in \mathbf{Set}$.

The universal arrow form G, G, X, X, resp. to the following functors are:

- $U: \mathbf{Ab} \longrightarrow \mathbf{Grp} \Rightarrow \langle G/\underbrace{[G,G]}_{\text{commutator group}},\ \pi: G \twoheadrightarrow G/[G,G] \rangle$
- $U: \mathbf{Rng} \longrightarrow \mathbf{Ab} \Rightarrow \langle R[G],\ \iota: G \hookrightarrow R[G] \rangle$ (group ring).
- $U: \mathbf{Top} \longrightarrow \mathbf{Set} \Rightarrow \langle (X, 2^X),\ id: X \longrightarrow X \rangle$ (discrete topology)
- $U: \mathbf{Set}_* \longrightarrow \mathbf{Set} \Rightarrow \langle X \sqcup \{x\},\ \iota: X \hookrightarrow X \sqcup \{x\} \rangle$.