§3.5. Categories with Finite Products

Def.

$\mathcal{C}$: category.

$\mathcal{C}$ has finite products

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall \{c_i\} \subset \mathcal{C}$: finite family of objs. $\in \mathcal{C}$,

$\exists \prod c_i \in \mathcal{C}$

//

Rem.

◎ 0個の object の product とは、terminal object のこと.

$$\begin{tikzcd} t & c \arrow[l, dashed, "\exists!"'] \end{tikzcd}$$

◎ 1個の object の product は、the object itself.

$$\begin{tikzcd} c & c \arrow[l, "id"'] \\ & d \arrow[ul, "f"] \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

Prop. 3.5.1

$\mathcal{C}$: cat. $\begin{cases} \exists t \in \mathcal{C} : \text{terminal} \\ \forall a, b \in \mathcal{C}, \ \exists a \times b \in \mathcal{C} \end{cases}$ とする.

$\Longrightarrow$ $\mathcal{C}$ has finite products.

さらに、

i) $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ は bifunctor.
$\quad \langle a, b \rangle \longmapsto a \times b$

ii) $\exists \alpha_{a,b,c} : (a \times b) \times c \cong a \times (b \times c)$, natural in $a, b, c \in \mathcal{C}$.

iii) $\exists \lambda_a : a \cong t \times a$, $\exists \rho_a : a \cong a \times t$, natural in $a \in \mathcal{C}$ //

(∵) 上の Rem. より、$\mathcal{C}$ が 0, 1, 2 個の object(s) の product を持つことはわかる

$n \geq 3$ に対しては、

$(\cdots((a_1 \times a_2) \times a_3) \times \cdots) \times a_n$

が $a_1, \cdots, a_n$ の product になる. これを induction on n で示す.

$a := (\cdots(a_1 \times a_2)\times \cdots)\times a_{n-1}$ が $a_1,\cdots,a_{n-1}$ の product とするとき、

$a \times a_n$ が $a_1,\cdots,a_n$ の product であることを言えばよい

$$\begin{tikzcd}
a_n & a \times a_n \arrow[l, "p_n"'] \arrow[r, "p"] & a \arrow[r, "p_i"] & a_i \\
 & c \arrow[ul, "f_n"] \arrow[u, dashed, "\exists!\tilde{f}"'] \arrow[ur, dashed, "\exists!\tilde{f}'"'] \arrow[urr, bend right, "f_i"'] & &
\end{tikzcd} \quad (1 \leqq i \leqq n-1)$$

$\forall c \in \mathcal{C},\ f_i : c \longrightarrow a_i \ (1 \leqq i \leqq n)$,

- a の universality から、
  $\exists!\ \tilde{f}' : c \longrightarrow a$ s.t. $f_i = p_i \circ \tilde{f}'$ $(\forall i \leqq n-1)$.
- $a \times a_n$ の universality から、
  $\exists!\ \tilde{f} : c \longrightarrow a \times a_n$ s.t. $\begin{cases} f_n = p_n \circ \tilde{f} \\ \tilde{f}' = p \circ \tilde{f} \end{cases}$

$\tilde{f}' = p \circ \tilde{f}$ より、$f_i = (p_i \circ p)\circ \tilde{f}$ $(\forall i \leqq n-1)$.

よって、$a \times a_n$ is a product of $a_1,\cdots,a_n$.

$\therefore$ $\mathcal{C}$ has finite products

i) $\langle f, g\rangle : \langle a, b\rangle \longrightarrow \langle a', b'\rangle$ に対して、

$\exists!\ f \times g : a \times b \longrightarrow a' \times b'$ s.t. $\begin{cases} p' \circ (f \times g) = f \circ p \\ q' \circ (f \times g) = g \circ q \end{cases}$

$$\begin{tikzcd}
a \arrow[d, "f"'] & a \times b \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, dashed, "\exists!\, f \times g"] & b \arrow[d, "g"] \\
a' \arrow[d, "f'"'] & a' \times b' \arrow[l, "p'"] \arrow[r, "q'"'] \arrow[d, dashed, "\exists!\, f' \times g'"] & b' \arrow[d, "g'"] \\
a'' & a'' \times b'' \arrow[l, "p''"] \arrow[r, "q''"'] & b''
\end{tikzcd}$$

さらに、$f \times g$ の uniqueness より、

$\begin{cases} (f' \times g')\circ(f \times g) = (f' \circ f)\times(g' \circ g) \\ id_a \times id_b = id_{a \times b} \end{cases}$

も明らか.

よって、$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \langle a, b\rangle & \longmapsto & a \times b \end{array}$ は bifunctor を定める。

ii) 上と同様の議論より、$(a \times b)\times c \cong a \times (b \times c)$ $(\cong a \times b \times c)$ は分かる.

これが natural であること、i.e.,

$$\begin{tikzcd}
(a \times b)\times c \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "(f \times g)\times h"'] \arrow[dr, phantom, "\circlearrowright"] & a \times (b \times c) \arrow[d, "f \times (g \times h)"] \\
(a' \times b')\times c' \arrow[r, "\alpha_{a',b',c'}"'] & a' \times (b' \times c')
\end{tikzcd} \qquad \forall \langle f, g, h\rangle : \langle a, b, c\rangle \longrightarrow \langle a', b', c'\rangle$$

は、各 map の uniqueness から従う.

iii) λ_a について示す. (ρ_a も同様.)

$$\begin{tikzcd}
t & t \times a \arrow[l, "q"'] \arrow[r, "p"] & a \\
t \arrow[u, phantom] & t \times a \arrow[ul, "q"] \arrow[u, dashed, "\exists !"'] \arrow[ur, "p"'] \arrow[l, "q"'] \arrow[r, "p"] & a \\
& a \arrow[ul, dotted, "\exists ! f"] \arrow[u, dashed, "\exists ! \lambda_a"'] \arrow[ur, "id"'] &
\end{tikzcd}$$

・ $t \times a$ の universality より,

$\exists \lambda_a : a \longrightarrow t \times a$ s.t.

$$\begin{cases} p \circ \lambda_a = id_a \\ q \circ \lambda_a = f \end{cases}$$

・ t : terminal obj. より, $q = f \circ p$.

・ $\begin{cases} p \circ (\lambda_a \circ p) = id_a \circ p = p, & q \circ (\lambda_a \circ p) = f \circ p = q, \\ p \circ id_{t \times a} = p & q \circ id_{t \times a} = q \end{cases}$ より,

$id_{t \times a} = \lambda_a \circ p$.

よって, $p \circ \lambda_a = id_a$, $\lambda_a \circ p = id_{t \times a}$ より,

$$a \overset{\lambda_a}{\cong} t \times a.$$

この naturality は, 各 map の uniqueness から従う. //

Coproducts に関する dual prop. も同様に成り立つ.

$\mathcal{C}$ has both finite products & finite coproducts とする.

$a_1, \cdots, a_n \in \mathcal{C}$ に対して,

$f : \coprod a_i \longrightarrow c$ は,

$f_j := f \circ \iota_j : a_j \longrightarrow c$ $(1 \leq j \leq n)$ によって unique に定まる.

特に,

$f : \coprod_{i=1}^{m} a_i \longrightarrow \prod_{i=1}^{n} b_i$ は,

$f_{jk} := \pi_k \circ f \circ \iota_j : a_j \longrightarrow b_k$ $\begin{pmatrix} 1 \leq j \leq m \\ 1 \leq k \leq n \end{pmatrix}$ によって unique に定まる.

さらに, $\exists z \in \mathcal{C}$: null とすると,

$g : \coprod_{i=1}^{n} a_i \longrightarrow \prod_{i=1}^{n} a_i$ を,

$g_{jk} = \begin{cases} id & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ として定めることができる.

$(0 : a_j \longrightarrow z \longrightarrow a_k)$

Exs.

・ $\mathcal{C} = \mathbf{Ab}, {}_R\mathbf{Mod}$, etc., $\Longrightarrow$ g : iso.

・ $\mathcal{C} = \mathbf{Set}_*, \mathbf{Top}_*$ $\Longrightarrow$ g : proper monic.

・ $\mathcal{C} = \mathbf{Grp}$ $\Longrightarrow$ g : proper epi.

Exercises

(1) $\delta_c : c \longrightarrow c \times c$: diagonal map

$$\begin{tikzcd} c & c\times c \arrow[l, "p_1"'] \arrow[r, "p_2"] & c \\ & c \arrow[ul, "id"] \arrow[u, dashed, "\delta_c"'] \arrow[ur, "id"'] & \end{tikzcd}$$

は natural in c.

∵ $c \xrightarrow{f} c'$ とすると、

$$\begin{tikzcd} & c' \arrow[dl, "id"'] \arrow[d, dashed, "\exists!\delta_{c'}"] \arrow[dr, "id"] & \\ c' & c' \times c' \arrow[l, "p_1'"'] \arrow[r, "p_2'"] & c' \\ c \arrow[u, "f"] & c \times c \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[u, dashed, "\exists! f\times f"'] & c \arrow[u, "f"'] \\ & c \arrow[ul, "id"] \arrow[u, dashed, "\exists!\delta_c"'] \arrow[ur, "id"'] \arrow[uuu, bend right=90, "f"'] & \end{tikzcd}$$

から従う。 //

(5) $\mathcal{B}$ has finite products $\Longrightarrow$ so does $\mathcal{B}^{\mathcal{C}}$.

∵ Prop. 3.5.1 を利用する。

◎ $\mathcal{B}^{\mathcal{C}}$ has a terminal obj.

∵ $\mathcal{B}$ has finite products より、$\exists t \in \mathcal{B}$: terminal.
$T : \mathcal{C} \longrightarrow \mathcal{B}$: functor を、$Tc := t$ によって定めれば、
$\forall F : \mathcal{C} \longrightarrow \mathcal{B}$: functor, $\exists \tau_c : Fc \longrightarrow Tc = t$ $(\forall c \in \mathcal{C})$.
τ の naturality は、t : terminal より従う。
∴ T は $\mathcal{B}^{\mathcal{C}}$ の terminal obj. //

◎ $\forall F, G : \mathcal{C} \longrightarrow \mathcal{B}$: functors, $\exists F \times G \in \mathcal{B}^{\mathcal{C}}$.

∵ $(F \times G)c := Fc \times Gc$,
$(F \times G)f := Ff \times Gf$ $(f : c \longrightarrow c')$ とすると、
$F \times G : \mathcal{C} \longrightarrow \mathcal{B}$: functor.
$\pi : F \times G \xrightarrow{\cdot} F$, $\rho : F \times G \xrightarrow{\cdot} G$ を、
$\pi_c = p_{Fc} : Fc \times Gc \longrightarrow Fc$
$\rho_c = q_{Gc} : Fc \times Gc \longrightarrow Gc$ (projections)
によって定める。これは natural transformations である。

実際, $\forall f : c \longrightarrow c'$,

$$
\begin{tikzcd}
Fc \arrow[d, "Ff"'] & Fc \times Gc \arrow[l, "\pi_c"'] \arrow[r, "p_c"] \arrow[d, dashed, "\exists!\, Ff \times Gf"] & Gc \arrow[d, "Gf"] \\
Fc' & Fc' \times Gc' \arrow[l, "\pi_{c'}"] \arrow[r, "p_{c'}"'] & Gc'
\end{tikzcd}
$$

からわかる.

$F \times G$: product of F, G を見る.

$\forall \sigma : H \overset{\cdot}{\longrightarrow} F$, $\tau : H \overset{\cdot}{\longrightarrow} G$,

$u : H \overset{\cdot}{\longrightarrow} F \times G$ を,

$$
\begin{tikzcd}
Fc & Fc \times Gc \arrow[l, "\pi_c"'] \arrow[r, "p_c"] & Gc \\
 & Hc \arrow[ul, "\sigma_c"] \arrow[u, dashed, "\exists!\, u_c"'] \arrow[ur, "\tau_c"'] &
\end{tikzcd}
$$

によって定める.

これは natural transformation である. 実際, $\forall f : c \longrightarrow c'$,

$$
\begin{tikzcd}
 & Hc' \arrow[dl, "\sigma_{c'}"'] \arrow[d, dashed, "u_{c'}"] \arrow[dr, "\tau_{c'}"] & \\
Fc' & Fc' \times Gc' \arrow[l, "\pi_{c'}"'] \arrow[r, "p_{c'}"] & Gc' \\
Fc \arrow[u, "Ff"] & Fc \times Gc \arrow[l, "\pi_c"'] \arrow[r, "p_c"] \arrow[u, dashed, "Ff \times Gf"'] & Gc \arrow[u, "Gf"'] \\
 & Hc \arrow[ul, "\sigma_c"] \arrow[u, dashed, "\exists!\, u_c"'] \arrow[ur, "\tau_c"'] \arrow[uuu, bend right=80, "Hf"'] &
\end{tikzcd}
$$

$$
\begin{cases}
\pi_{c'} \circ (u_{c'} \circ Hf) = \sigma_{c'} \circ Hf = Ff \circ \sigma_c, \\
p_{c'} \circ (u_{c'} \circ Hf) = \tau_{c'} \circ Hf = Gf \circ \tau_{c'}
\end{cases}
$$

より,

$(Ff \times Gf) \circ u_c = u_{c'} \circ Hf.$ //

以上と Prop. 3.5.1 より,

$\mathscr{B}^{\mathscr{C}}$ has finite products. //