Categories for the Working Mathematicians. 2nd ed.

Mac Lane. Springer.

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§2.6. Comma Categories.

Def.

$\mathscr{C}$: category. $b \in \mathscr{C}$: object.

$(b \downarrow \mathscr{C})$: category of objects under b with

- objects: $\langle f, c \rangle$, where $c \in \mathscr{C}$: obj. and $f : b \longrightarrow c$ in $\mathscr{C}$.
- arrows: $\langle f, c \rangle \xrightarrow{h} \langle f', c' \rangle$, where $h : c \longrightarrow c'$ s.t.

  $f' = h \circ f$.

Objects:
$$\begin{tikzcd} b \arrow[d, "f"] \\ c \end{tikzcd}$$

Arrows:
$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ c \arrow[rr, "h"'] & {} & c' \end{tikzcd}$$

Exs.

- $*$: 1-point set のとき. $(* \downarrow \mathbf{Set}) = \mathbf{Set}_*$.

  (Obj. $* \longrightarrow X$ は, pair $\langle x \in X, X \rangle$ のこと.

  Arr. $\langle x, X \rangle \longrightarrow \langle x', X' \rangle$ は, base point を保つ map のこと.)

- $(\mathbb{Z} \downarrow \mathbf{Ab}) = \mathbf{Ab}_*$

  (Obj. $\mathbb{Z} \xrightarrow{f} G$ は, $\langle f(1), G \rangle$ のこと. (base point $f(1)$).

  Arr. $\langle g, G \rangle \longrightarrow \langle g', G' \rangle$ は, base point を保つ group hom. のこと.)

Def.

$\mathscr{C}$: cat. $a \in \mathscr{C}$: obj.

$(\mathscr{C} \downarrow a)$: category of objects over a with

Objs:
$$\begin{tikzcd} c \arrow[d, "f"] \\ a \end{tikzcd}$$

Arrs:
$$\begin{tikzcd} c \arrow[rr, "h"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\circlearrowright"] & c' \arrow[dl, "f'"] \\ & a & \end{tikzcd}$$

Exs.

- $*$: 1-point set. $\Rightarrow$ $(\mathbf{Set}\downarrow *) \underset{\text{cat}}{\cong} \mathbf{Set}$.

  ($\because$ obj. $X \xrightarrow{f} *$ は、$*$: terminal obj. in $\mathbf{Set}$ だから、X に他ならない。
  任意の $h: X \longrightarrow X'$ in $\mathbf{Set}$ は、$f = f' \circ h$ を満たす。

$$
\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, dashed, "\exists ! f"'] & \arrow[d, phantom, "\circlearrowleft"] & X' \arrow[dl, dashed, "\exists ! f'"] \\
& * &
\end{tikzcd}
$$

  $(f: X \longrightarrow *,\ f': X' \longrightarrow *)$.)

- $(\mathbf{Ring} \downarrow \mathbb{Z})$: augmentation $R \longrightarrow \mathbb{Z}$ as objects,
  ring hom preserving augmentations as arrows.

  ($\because$ obj. $R \xrightarrow{\varepsilon} \mathbb{Z}$ は、augmentation と呼ばれる。(一般論).
  Arr. $\langle \varepsilon, R \rangle \xrightarrow{h} \langle \varepsilon', R' \rangle$ は、$\varepsilon = \varepsilon' \circ h$ なる ring hom $h: R \longrightarrow R'$.
  i.e., ring hom that preserves augmentations.)

Def.

$S: \mathcal{D} \longrightarrow \mathcal{E}$: functor. $b \in \mathcal{E}$: obj.

$(b \downarrow S)$: category of objects S-under b. with

Obj.
$$
\begin{tikzcd}
b \arrow[d, "f"] \\
Sd
\end{tikzcd}
$$

Arr.
$$
\begin{tikzcd}
& b \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\
Sd \arrow[rr, "Sh"'] & & Sd'
\end{tikzcd}
$$
//

Def.

$T: \mathcal{E} \longrightarrow \mathcal{C}$: functor. $a \in \mathcal{C}$: obj.

$(T \downarrow a)$: category of objects T-over a with

Obj.
$$
\begin{tikzcd}
Te \arrow[d, "f"] \\
a
\end{tikzcd}
$$

Arr.
$$
\begin{tikzcd}
Te \arrow[rr, "Th"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowright"] & Te' \arrow[dl, "f'"] \\
& a &
\end{tikzcd}
$$
//

Ex.

- $U : \mathrm{Grp} \to \mathrm{Set}$: Forgetful functor. $x \in \mathrm{Set}$: obj. とする.

  $(x \downarrow U)$: map $x \xrightarrow{f} Ug$ as obj

  group hom $h : \langle f, g \rangle \to \langle f', g' \rangle$ s.t. $f' = h \circ f$ as arr.

Def.

$\mathcal{E} \xrightarrow{T} \mathcal{C} \xleftarrow{S} \mathcal{D}$: functors.

$(T \downarrow S)$: comma category with

Obj.
$$\begin{tikzcd} Te \arrow[d, "f"] \\ Sd \end{tikzcd}$$

Arr.
$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd}$$

//

Rem.

Composites of $(T \downarrow S)$ は well-defined:

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd} , \quad \begin{tikzcd} Te' \arrow[r, "Tk'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowright"] & Te'' \arrow[d, "f''"] \\ Sd' \arrow[r, "Sh'"'] & Sd'' \end{tikzcd}$$

を 2つの arrows とすると.

その composite

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Te' \arrow[r, "Tk'"] \arrow[d, "f'"] \arrow[dr, phantom, "\circlearrowright"] & Te'' \arrow[d, "f''"] \\ Sd \arrow[r, "Sh"'] & Sd' \arrow[r, "Sh'"'] & Sd'' \end{tikzcd}$$

は $(T \downarrow S)$ の arrow

i.e., commutative diagram

になっている.

$(T \downarrow S)$ は $(b \downarrow \mathcal{C})$, $(\mathcal{C} \downarrow a)$, $(b \downarrow S)$, $(T \downarrow a)$ の一般化である!

- $(b \downarrow S)$ は, $T = b : \mathbb{1} \to \mathcal{C}$ としたもの.
- $(T \downarrow a)$ は, $S = a : \mathbb{1} \to \mathcal{C}$ としたもの.
- $(b \downarrow \mathcal{C})$ は. $T = b : \mathbb{1} \to \mathcal{C}$, $S = 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ としたもの.
- $(\mathcal{C} \downarrow a)$ は. $S = a : \mathbb{1} \to \mathcal{C}$, $T = 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ としたもの.

Exs.

- $\mathcal{C}$: cat. $\mathcal{C} : \mathcal{C} \longrightarrow \mathcal{C}$: id. functor. のとき.

  $(\mathcal{C} \downarrow \mathcal{C}) = \mathcal{C}^2$

  (∵ obj. $c \xrightarrow{f} d$.

  Arr.
$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\circlearrowright"] & d \arrow[d, "h"] \\ c' \arrow[r, "f'"'] & d' \end{tikzcd}$$
  $\Rightarrow \mathcal{C}^2$ (obj : arrows. arr : commutative squares))

- $T = b,\ S = a : \mathbb{1} \longrightarrow \mathcal{C}$. のとき.

  $(b \downarrow a) = \hom_{\mathcal{C}}(b, a)$ (as a discrete cat.).

  (∵ obj. $b \xrightarrow{f} a$.

  Arr.
$$\begin{tikzcd} b \arrow[r, "f"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowright"] & a \arrow[d, "id"] \\ b \arrow[r, "f'"'] & a \end{tikzcd}$$
  $\Rightarrow f = f'$. i.e., arrow は. trivial なもののみ.)

  ↑ "comma category" の名前の由来!

$(T \downarrow S)$ の普遍性 (cf. Exercise (5)).

$$\begin{tikzcd} & & (T \downarrow S) \arrow[dll, "P"'] \arrow[d, "R"] \arrow[drr, "Q"] & & \\ \mathcal{E} \arrow[r, "T"'] & \mathcal{C} & \mathcal{C}^2 \arrow[l, "\mathcal{C}^{d_0}"] \arrow[r, "\mathcal{C}^{d_1}"'] & \mathcal{C} & \mathcal{D} \arrow[l, "S"] \end{tikzcd}$$

$d_0, d_1 : \mathbb{1} \longrightarrow \mathbb{2}$

P, Q : projections.

$\mathcal{C}^{d_0}$ (resp $\mathcal{C}^{d_1}$) は. arrow $a \xrightarrow{f} b$ を. $a = \mathrm{dom}\, f$ (resp. $b = \mathrm{cod}\, f$) に送る functor. (cf. §2.5).

Exercises.

(1) K: com. ring のとき. $(K \downarrow \mathbb{C}\mathrm{Rng}) = \mathbb{C}\mathrm{Alg}_K$.

(∵) obj. A : com ring, $f : K \longrightarrow A$: ring hom.

i.e., A : com. alg. $/K$.

Arr. $(A, f) \xrightarrow{h} (A', f')$: ring hom s.t.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowright"] & \\ A \arrow[rr, "h"'] & & A' \end{tikzcd}$$

i.e., scalar 倍 f. f' を保つ ring hom h

i.e., alg. hom. $h : A \longrightarrow A'$.

(2) $\mathcal{C}$: cat. $t \in \mathcal{C}$: terminal $\Longrightarrow (\mathcal{C} \downarrow t) \cong \mathcal{C}$.

(∵) obj. $b \xrightarrow{f} t$ は. t : terminal だから.

各 $b \in \mathcal{C}$ に対して one and exactly one 存在.

Arr. $b \xrightarrow{h} b'$ s.t.

$$\begin{tikzcd} b \arrow[rr, "h"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & b' \arrow[dl] \\ & t & \end{tikzcd}$$

t : terminal だから, この diagram は <u>自動的に</u> commutative.

(4) (S.A. Hug). $T, S : \mathcal{D} \longrightarrow \mathcal{C}$: functors とする.

$\tau : T \xrightarrow{\bullet} S$: nat. transformation は. $\tau : \mathcal{D} \longrightarrow (T \downarrow S)$: functor s.t.

$P\tau = Q\tau = \mathrm{id}_{\mathcal{D}}$ に他ならない. (P, Q : projs.)

(∵) $\tau : T \xrightarrow{\bullet} S$

$\Longleftrightarrow \forall d \in \mathcal{D}$, $\exists \tau d : Td \longrightarrow Sd$ s.t.

$$\begin{tikzcd} Td \arrow[r, "\tau d"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Sd \arrow[d] \\ Td' \arrow[r, "\tau d'"'] & Sd' \end{tikzcd}$$

$\tau : \mathcal{D} \longrightarrow (T \downarrow S)$ s.t. $P\tau = Q\tau = \mathrm{id}_{\mathcal{D}}$

$\Longleftrightarrow$ 各 $d \in \mathcal{D}$ に対して. $\tau d : Te \longrightarrow Sg$ ($\exists e, g \in \mathcal{D}$)

各 $d \xrightarrow{f} d'$ に対して

$$\begin{tikzcd} Te \arrow[r, "\tau d"] \arrow[d, "Tk"'] \arrow[dr, phantom, "\circlearrowright"] & Sg \arrow[d, "Sh"] \\ Te' \arrow[r, "\tau d'"'] & Sg' \end{tikzcd}$$

($\exists k : e \longrightarrow e'$, $h : g \longrightarrow g'$)

s.t.

$$e = g = d \quad , \quad k = h = f$$

$$e = P\tau d,\quad g = Q\tau d,\quad d = \mathrm{id}_{\mathcal{D}} d,\qquad k = P\tau f,\quad h = Q\tau f,\quad f = \mathrm{id}_{\mathcal{D}} f .$$

$\Leftrightarrow$ $\forall d \in \mathcal{D}.\ {}^{\exists}\tau d : Td \longrightarrow Sd$ s.t.

$$\begin{tikzcd} Td \arrow[r, "\tau d"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\circlearrowright"] & Sd \arrow[d, "Sf"] \\ Td' \arrow[r, "\tau d'"'] & Sd' \end{tikzcd} \qquad (f : d \longrightarrow d').$$

上の2つは同値.

§2.7. Graphs and Free Categories.

まず. free monoid の場合を思い出す:

X: set から生成される free monoid F とは,

- Word $x_1 \cdots x_n$ $(x_i \in X)$ を元に持ち,
- words の結合 (juxtaposition) を積とし,
- empty word を identity element とする

monoid のこと.

これと equivalent な条件は.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & F \arrow[d, dashed, "\exists ! \tilde{f}"] \\ & M \end{tikzcd}$$

$(F$: monoid, $\iota : X \longrightarrow F$: map$)$ が free monoid

$\Longleftrightarrow$ $\forall\,(M$: monoid, $f: X \longrightarrow M$: map$)$,

$\exists !\ \tilde{f} : F \longrightarrow M$: monoid hom. s.t. $f = \tilde{f} \circ \iota$.

これを元に free category を考える.

Def. (§1.2)

G: (directed) graph とは.

- O: set of objects (vertices)
- A: set of arrows (edges).

together with functions

$$A \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} O, \quad \partial_0 f = \mathrm{dom}\, f, \quad \partial_1 f = \mathrm{cod}\, f.$$

$D : G \longrightarrow G'$: morphism of graphs とは.

- $D_O : O \longrightarrow O'$
- $D_A : A \longrightarrow A'$ s.t. $D_O \partial_0 = \partial_0 D_A$, $D_O \partial_1 = \partial_1 D_A$. //

$\rightsquigarrow$ **Grph** : category of graphs.

Graphs には composite と identity がない．

↳ diagram scheme や precategory とも呼ぶ！

Graphs obtained from categories

$\mathcal{C}$: cat. に対して $U\mathcal{C}$: graph を

- obj. : objects of $\mathcal{C}$;
- arr. : arrows of $\mathcal{C}$

と定める．$F : \mathcal{C} \longrightarrow \mathcal{C}'$: functor は

graph morphism $UF : U\mathcal{C} \longrightarrow U\mathcal{C}'$ に対応する．

$\therefore$ U : $\mathbf{Cat} \longrightarrow \mathbf{Grph}$: forgetful functor.

Categories obtained from graphs

O : set : fixed.

Def.

O-graph とは graph whose object set is O.

A morphism of O-graph とは O-graphs 間の graph morphism のこと．//

A, B : sets of arrows (regarded as O-graphs) とすると

$$A \times_O B := \{ \langle g, f \rangle \mid \partial_0 g = \partial_1 f,\ g \in A\ \ f \in B \}$$

(composable pairs of arrows $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$)

と定める．

$$\partial_0 \langle g, f \rangle := \partial_0 f, \quad \partial_1 \langle g, f \rangle := \partial_1 g$$

とすれば $A \times_O B$: O-graph になる．

$\times_O$ は associative

($\because$ $(A \times_O B) \times_O C \cong A \times_O (B \times_O C)$

$\cong \{ \langle h, g, f \rangle \mid \partial_0 h = \partial_1 g,\ \partial_0 g = \partial_1 f,\ h \in A,\ g \in B,\ f \in C \}$)

任意の A : O-graph に対して.

$$
\begin{array}{ccccc}
A & \cong & A \times_O O & \cong & O \times_O A \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
f & \longmapsto & \langle f, \partial_0 f \rangle & \longmapsto & \langle \partial_1 f, f \rangle .
\end{array}
$$

Rem. O を次のように O-graph とみなしている:

- arrows : O itself.
- $\partial_0, \partial_1 = \mathrm{id}_O : O \rightrightarrows O$ (arrows → objects)

一方. $\mathcal{E}$: cat. whose object set is O は.

A : O-graph together with morphisms

$c : A \times_O A \longrightarrow A$ (composite)

$i : O \longrightarrow A$ (identity) s.t.

$$
\begin{tikzcd}
(A \times_O A) \times_O A \cong A \times_O (A \times_O A) \arrow[r, "1 \times c"] \arrow[d, "c \times 1"'] \arrow[dr, phantom, "\circlearrowright"] & A \times_O A \arrow[d, "c"] \\
A \times_O A \arrow[r, "c"'] & A
\end{tikzcd}
$$

(associativity)

$$
\begin{tikzcd}
O \times_O A \arrow[r, "i \times 1"] \arrow[dr, "\wr"'] & A \times_O A \arrow[d] & A \times_O O \arrow[l, "1 \times i"'] \arrow[dl, "\sim"] \\
& A &
\end{tikzcd}
$$

(unit law).

G : O-graph から $\mathcal{E} = \mathcal{E}(G)$: cat を作る.

大雑把に言えば, G の composable pair of arrows をすべて "繋げた" もの.

厳密に言えば,

Thm. 2.1.

$G = \{ A \rightrightarrows O \}$: small graph.

$\Longrightarrow \exists\, \mathcal{C} = \mathcal{C}_G$: small cat. $P : G \longrightarrow U\mathcal{C}$: morphism of graphs s.t.

$\forall (\mathcal{B} :$ cat., $D : G \longrightarrow U\mathcal{B}$: morphism of graphs$)$,

$\exists !\, \tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$: functor with $(U\tilde{D}) \circ P = D$.

$$
\begin{tikzcd}
\mathcal{C} \arrow[d, dashed, "\exists ! \tilde{D}"'] \\
\mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "P"] \arrow[dr, "D"'] & U\mathcal{C} \arrow[d, dashed, "U\tilde{D}"] \\
& U\mathcal{B}
\end{tikzcd}
$$

//

(“任意の graph morphism $D : G \longrightarrow U\mathcal{B}$ を、
functor $\tilde{D} : \mathcal{C} \longrightarrow \mathcal{B}$ に拡張できる!”)

特に、object set of $\mathcal{B} = O$, D : O-graph morphism

$\Longrightarrow \tilde{D}|_{\mathrm{obj.}} = \mathrm{id}$.

$P : G \longrightarrow U\mathcal{C}$ は、$(G \downarrow U)$ の initial object である.

$\therefore$ unique up to isomorphism.

Proof.