§3.5. Categories with Finite Products

Def.

$\mathcal{C}$: category.

$\mathcal{C}$ has finite products

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall \{c_i\} \subset \mathcal{C}$: finite family of objs. $\in \mathcal{C}$,

$\exists \prod c_i \in \mathcal{C}$ //

Rem.

- 0個の object の product とは、terminal object のこと.

$$t \xleftarrow{\exists !} c$$

- 1個の object の product は、the object itself.

$$\begin{tikzcd} c & c \arrow[l, "id"'] \\ & d \arrow[ul, "f"] \arrow[u, dashed, "\exists !"'] \end{tikzcd}$$

Prop. 3.5.1

$\mathcal{C}$: cat. $\begin{cases} \exists t \in \mathcal{C} : \text{terminal} \\ \forall a, b \in \mathcal{C}, \exists a \times b \in \mathcal{C} \end{cases}$ とする.

$\Longrightarrow$ $\mathcal{C}$ has finite products.

さらに,

i) $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ は bifunctor.
$\quad \langle a, b \rangle \longmapsto a \times b$

ii) $\exists \alpha_{a,b,c} : (a \times b) \times c \cong a \times (b \times c)$, natural in $a, b, c \in \mathcal{C}$.

iii) $\exists \lambda_a : a \cong t \times a$, $\exists \rho_a : a \cong a \times t$, natural in $a \in \mathcal{C}$ //

(∵) 上の Rem. より, $\mathcal{C}$ が 0, 1, 2 個の object(s) の product を持つことはわかる.

$n \geq 3$ に対しては,

$(\cdots((a_1 \times a_2) \times a_3) \times \cdots) \times a_n$

が $a_1, \cdots, a_n$ の product になる. これを induction on n で示す.

$a := (\cdots(a_1 \times a_2) \times \cdots) \times a_{n-1}$ が $a_1, \cdots, a_{n-1}$ の product とするとき、

$a \times a_n$ が $a_1, \cdots, a_n$ の product であることを言えばよい

$$\begin{tikzcd} a_n & a \times a_n \arrow[l, "p_n"'] \arrow[r, "p"] & a \arrow[r, "p_i"] & a_i \\ & c \arrow[ul, "f_n"] \arrow[u, dashed, "\exists! \tilde{f}"'] \arrow[ur, dashed, "\exists! \tilde{f'}"'] \arrow[urr, bend right, "f_i"'] & & \end{tikzcd} \quad (1 \leq i \leq n-1)$$

$\forall c \in \mathscr{C},\ f_i : c \longrightarrow a_i \ (1 \leq i \leq n)$,

- a の universality から、
  $\exists! \tilde{f'} : c \longrightarrow a$ s.t. $f_i = p_i \circ \tilde{f'}$ $(\forall i \leq n-1)$.
- $a \times a_n$ の universality から、
  $\exists! \tilde{f} : c \longrightarrow a \times a_n$ s.t. $\begin{cases} f_n = p_n \circ \tilde{f} \\ \tilde{f'} = p \circ \tilde{f} \end{cases}$

$\tilde{f'} = p \circ \tilde{f}$ より、$f_i = (p_i \circ p) \circ \tilde{f}$ $(\forall i \leq n-1)$.

よって、$a \times a_n$ is a product of $a_1, \cdots, a_n$.

$\therefore$ $\mathscr{C}$ has finite products

i) $\langle f, g \rangle : \langle a, b \rangle \longrightarrow \langle a', b' \rangle$ に対して、

$\exists! f \times g : a \times b \longrightarrow a' \times b'$ s.t. $\begin{cases} p' \circ (f \times g) = f \circ p \\ q' \circ (f \times g) = g \circ q \end{cases}$

$$\begin{tikzcd} a \arrow[d, "f"] & a \times b \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, dashed, "\exists! f \times g"] & b \arrow[d, "g"] \\ a' \arrow[d, "f'"] & a' \times b' \arrow[l, "p'"] \arrow[r, "q'"'] \arrow[d, dashed, "\exists! f' \times g'"] & b' \arrow[d, "g'"] \\ a'' & a'' \times b'' \arrow[l, "p''"] \arrow[r, "q''"'] & b'' \end{tikzcd}$$

さらに、$f \times g$ の uniqueness より、

$\begin{cases} (f' \times g') \circ (f \times g) = (f' \circ f) \times (g' \circ g) \\ id_a \times id_b = id_{a \times b} \end{cases}$

も明らか。

よって、
$$\begin{array}{ccc} \mathscr{C} \times \mathscr{C} & \longrightarrow & \mathscr{C} \\ \cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid \\ \langle a, b \rangle & \longmapsto & a \times b \end{array}$$
は bifunctor を定める。

ii) 上と同様のギロンより、$(a \times b) \times c \cong a \times (b \times c)$ $(\cong a \times b \times c)$ は分かる。

これが natural であること、i.e.,

$$\begin{tikzcd} (a \times b) \times c \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "(f \times g) \times h"'] \arrow[dr, phantom, "\circlearrowright"] & a \times (b \times c) \arrow[d, "f \times (g \times h)"] \\ (a' \times b') \times c' \arrow[r, "\alpha_{a',b',c'}"'] & a' \times (b' \times c') \end{tikzcd} \quad \forall \langle f, g, h \rangle : \langle a, b, c \rangle \longrightarrow \langle a', b', c' \rangle$$

は、各 map の uniqueness から従う。

iii) λ_a について示す。(ρ_a も同様。)

$$\begin{tikzcd}
t & t\times a \arrow[l, "q"'] \arrow[r, "p"] & a \\
t & t\times a \arrow[l, "q"'] \arrow[r, "p"] \arrow[ul, "q"] \arrow[u, dashed, "\exists!"] \arrow[ur, "p"'] & a \\
& a \arrow[ul, dotted, "\exists! f"] \arrow[u, dashed, "\exists!\lambda_a"] \arrow[ur, "id"'] &
\end{tikzcd}$$

- $t\times a$ の universality より，

  $\exists \lambda_a : a \longrightarrow t\times a$ s.t.
  $$\begin{cases} p\circ\lambda_a = id_a \\ q\circ\lambda_a = f \end{cases}$$

- t : terminal obj. より，$q = f\circ p$.

- $\begin{cases} p\circ(\lambda_a\circ p) = id_a\circ p = p, & q\circ(\lambda_a\circ p) = f\circ p = q, \\ p\circ id_{t\times a} = p & q\circ id_{t\times a} = q \end{cases}$ より，

  $id_{t\times a} = \lambda_a\circ p$.

よって，$p\circ\lambda_a = id_a$, $\lambda_a\circ p = id_{t\times a}$ より，

$$a \overset{\lambda_a}{\cong} t\times a.$$

この naturality は，各 map の uniqueness から従う。 //

Coproducts に関する dual prop. も同様に成り立つ。

$\mathscr{C}$ has both finite products & finite coproducts とする。

$a_1, \cdots, a_n \in \mathscr{C}$ に対して，

$f : \coprod a_i \longrightarrow c$ は，

$f_j := f\circ\iota_j : a_j \longrightarrow c \quad (1\leq j\leq n)$ によって <u>unique</u> に定まる。

特に，

$f : \coprod_{i=1}^{m} a_i \longrightarrow \prod_{i=1}^{n} b_i$ は，

$f_{jk} := \pi_k\circ f\circ\iota_j : a_j \longrightarrow b_k \quad \begin{pmatrix} 1\leq j\leq m \\ 1\leq k\leq n \end{pmatrix}$ によって <u>unique</u> に定まる。

さらに，$\exists z\in\mathscr{C}$: null とすると，

$g : \coprod_{i=1}^{n} a_i \longrightarrow \prod_{i=1}^{n} a_i$ を，

$g_{jk} = \begin{cases} id & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases}$ として定めることができる。

$(0 : a_j \longrightarrow z \longrightarrow a_k)$

<u>Exs.</u>

- $\mathscr{C} = Ab$, ${}_R Mod$, etc. $\Longrightarrow$ g : iso.
- $\mathscr{C} = Set_*$, Top_* $\Longrightarrow$ g : proper monic.
- $\mathscr{C} = Grp$ $\Longrightarrow$ g : proper epi.

Exercises

(1) $\delta_c : c \longrightarrow c \times c$: diagonal map

$$\begin{tikzcd} c & c\times c \arrow[l, "p_1"'] \arrow[r, "p_2"] & c \\ & c \arrow[ul, "id"] \arrow[u, dashed, "\delta_c"'] \arrow[ur, "id"'] & \end{tikzcd}$$

は natural in c.

☺ $c \xrightarrow{f} c'$ とすると，

$$\begin{tikzcd} & c' \arrow[dl, "id"'] \arrow[d, dashed, "\exists!\delta_{c'}"] \arrow[dr, "id"] & \\ c' & c'\times c' \arrow[l, "p_1'"'] \arrow[r, "p_2'"] & c' \\ c \arrow[u, "f"] & c\times c \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[u, dashed, "\exists! f\times f"'] & c \arrow[u, "f"'] \\ & c \arrow[ul, "id"] \arrow[u, dashed, "\exists!\delta_c"'] \arrow[ur, "id"'] \arrow[uuu, bend right=90, "f"'] & \end{tikzcd}$$

から従う. //

(5) $\mathcal{B}$ has finite products $\Longrightarrow$ so does $\mathcal{B}^{\mathcal{C}}$.

☺ Prop. 3.5.1 を利用する.

- $\mathcal{B}^{\mathcal{C}}$ has a terminal obj.

  ☺ $\mathcal{B}$ has finite products より，$\exists t \in \mathcal{B}$: terminal.
  $T : \mathcal{C} \longrightarrow \mathcal{B}$: functor を，$Tc := t$ によって定めれば，
  $\forall F : \mathcal{C} \longrightarrow \mathcal{B}$: functor, $\exists \tau_c : Fc \longrightarrow Tc = t$ $(\forall c \in \mathcal{C})$.
  τ の naturality は，t : terminal より従う.
  $\therefore$ T は $\mathcal{B}^{\mathcal{C}}$ の terminal obj. //

- $\forall F, G : \mathcal{C} \longrightarrow \mathcal{B}$: functors, $\exists F \times G \in \mathcal{B}^{\mathcal{C}}$.

  ☺ $(F \times G)c := Fc \times Gc$,
  $(F \times G) f := Ff \times Gf$ $(f : c \longrightarrow c')$ とすると，
  $F \times G : \mathcal{C} \longrightarrow \mathcal{B}$: functor.
  $\pi : F \times G \overset{\cdot}{\longrightarrow} F$, $\rho : F \times G \overset{\cdot}{\longrightarrow} G$ を，
  $\pi_c = p_{Fc} : Fc \times Gc \longrightarrow Fc$
  $\rho_c = q_{Gc} : Fc \times Gc \longrightarrow Gc$ (projections)
  によって定める. これは natural transformations である.

実際, $\forall f: c \longrightarrow c'$,

$$\begin{tikzcd}
Fc \arrow[d, "Ff"'] & Fc \times Gc \arrow[l, "\pi_c"'] \arrow[r, "p_c"] \arrow[d, dashed, "\exists ! Ff \times Gf"] & Gc \arrow[d, "Gf"] \\
Fc' & Fc' \times Gc' \arrow[l, "\pi_{c'}"] \arrow[r, "p_{c'}"'] & Gc'
\end{tikzcd}$$

からわかる.

$F \times G$: product of F, G を見る.

$\forall \sigma: H \overset{\cdot}{\longrightarrow} F$, $\tau: H \overset{\cdot}{\longrightarrow} G$,

$u: H \overset{\cdot}{\longrightarrow} F \times G$ を,

$$\begin{tikzcd}
Fc & Fc \times Gc \arrow[l, "\pi_c"'] \arrow[r, "p_c"] & Gc \\
& Hc \arrow[ul, "\sigma_c"] \arrow[u, dashed, "\exists ! u_c"'] \arrow[ur, "\tau_c"'] &
\end{tikzcd}$$

によって定める.

これは natural transformation である. 実際, $\forall f: c \longrightarrow c'$,

$$\begin{tikzcd}
& Hc' \arrow[dl, "\sigma_{c'}"'] \arrow[d, dashed, "u_{c'}"] \arrow[dr, "\tau_{c'}"] & \\
Fc' & Fc' \times Gc' \arrow[l, "\pi_{c'}"'] \arrow[r, "p_{c'}"] & Gc' \\
Fc \arrow[u, "Ff"] & Fc \times Gc \arrow[l, "\pi_c"'] \arrow[r, "p_c"] \arrow[u, dashed, "Ff \times Gf"'] & Gc \arrow[u, "Gf"'] \\
& Hc \arrow[ul, "\sigma_c"] \arrow[u, dashed, "\exists ! u_c"'] \arrow[ur, "\tau_c"'] \arrow[uuu, bend right=90, "Hf"'] &
\end{tikzcd}$$

$$\begin{cases} \pi_{c'} \circ (u_{c'} \circ Hf) = \sigma_{c'} \circ Hf = Ff \circ \sigma_c, \\ p_{c'} \circ (u_{c'} \circ Hf) = \tau_{c'} \circ Hf = Gf \circ \tau_{c'} \end{cases}$$ より,

$$(Ff \times Gf) \circ u_c = u_{c'} \circ Hf.$$

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以上と Prop. 3.5.7 より,

$\mathscr{B}^{\mathscr{C}}$ has finite products.

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§3.6 Groups in Categories

Def.

$\mathscr{C}$: category with finite products, $t \in \mathscr{C}$: terminal. $c \in \mathscr{C}$ とする.

c が monoid in $\mathscr{C}$ であるとは.

$\exists \mu : c \times c \longrightarrow c$, $\eta : t \longrightarrow c$ s.t.

$$\begin{tikzcd}
(c\times c)\times c \arrow[r, "\sim"] \arrow[d, "\mu\times 1"'] & c\times(c\times c) \arrow[r, "1\times\mu"] \arrow[d, phantom, "\circlearrowright"] & c\times c \arrow[d, "\mu"] \\
c\times c \arrow[rr, "\mu"'] & & c
\end{tikzcd}$$

$$\begin{tikzcd}
t\times c \arrow[r, "\eta\times 1"] \arrow[dr, "\wr"'] & c\times c \arrow[d, "\mu"] & c\times t \arrow[l, "1\times\eta"'] \arrow[dl, "\sim"] \\
& c &
\end{tikzcd}$$

c が group in $\mathscr{C}$ であるとは. c : monoid であって.

$\exists \zeta : c \longrightarrow c$ s.t.

$$\begin{tikzcd}
& c\times c \arrow[dl, "\zeta\times 1"'] \arrow[dr, "1\times\zeta"] & \\
c\times c \arrow[ddr, "\mu"'] \arrow[r, phantom, "\circlearrowright"] & c \arrow[u, "\delta_c"'] \arrow[d] \arrow[r, phantom, "\circlearrowright"] & c\times c \arrow[ddl, "\mu"] \\
& t \arrow[d, "\eta"] & \\
& c &
\end{tikzcd}$$

//

Prop. 3.6.1

$\mathscr{C}$: category with finite products, $c \in \mathscr{C}$ とする.

c : monoid in $\mathscr{C}$ $\Longleftrightarrow$ $\mathscr{C}(\cdot, c)$: monoid in $\mathbf{Set}^{\mathscr{C}^{op}}$.

(resp. group) (resp. group)

//

∵) $\Longrightarrow$

$\bar{\mu} : \mathscr{C}(\cdot, c)\times\mathscr{C}(\cdot, c) \xrightarrow[\sim]{\Theta} \mathscr{C}(\cdot, c\times c) \xrightarrow{\mu_*} \mathscr{C}(\cdot, c)$,

$\bar{\eta} : \mathscr{C}(\cdot, t) \xrightarrow{\eta_*} \mathscr{C}(\cdot, c)$,

$\bar{\zeta} : \mathscr{C}(\cdot, c) \xrightarrow{\zeta_*} \mathscr{C}(\cdot, c)$ により定める. (cf. Exercise 3.6.5)

これが monoid または group であることを示すには.

$$
\begin{tikzcd}
(\mathcal{C}(d,c)\times\mathcal{C}(d,c))\times\mathcal{C}(d,c) \arrow[d, "\bar{\mu}\times 1"'] \arrow[r, "\wr"] & \mathcal{C}(d,c)\times(\mathcal{C}(d,c)\times\mathcal{C}(d,c)) \arrow[d, "1\times\bar{\mu}"] \\
\mathcal{C}(d,c)\times\mathcal{C}(d,c) \arrow[dr, "\bar{\mu}"'] \arrow[r, phantom, "\circlearrowright"] & \mathcal{C}(d,c)\times\mathcal{C}(d,c) \arrow[d, "\bar{\mu}"] \\
& \mathcal{C}(d,c)
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{C}(d,c)\times\mathcal{C}(d,c) \arrow[d, "\bar{\mu}"] & \\
\mathcal{C}(d,t)\times\mathcal{C}(d,c) \arrow[ur, "\bar{\eta}\times 1"] \arrow[r, "\wr"'] \arrow[r, phantom, "\circlearrowright", shift left=3] & \mathcal{C}(d,c) & \mathcal{C}(d,c)\times\mathcal{C}(d,t) \arrow[ul, "1\times\bar{\eta}"'] \arrow[l, "\sim"] \arrow[l, phantom, "\circlearrowright", shift right=3]
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{C}(d,c)\times\mathcal{C}(d,c) \arrow[dl, "\bar{\xi}\times 1"'] \arrow[dr, "1\times\bar{\xi}"] & \\
\mathcal{C}(d,c)\times\mathcal{C}(d,c) \arrow[ddr, "\bar{\mu}"'] \arrow[r, phantom, "\circlearrowright"] & \mathcal{C}(d,c) \arrow[u, "\delta_{\mathcal{C}(d,c)}"'] \arrow[d] \arrow[r, phantom, "\circlearrowright"] & \mathcal{C}(d,c)\times\mathcal{C}(d,c) \arrow[ddl, "\bar{\mu}"] \\
& \mathcal{C}(d,t) \arrow[d, "\bar{\eta}"] & \\
& \mathcal{C}(d,c) &
\end{tikzcd}
$$

の可換性を確かめればよいが、これは $\mathcal{C}(\cdot,c)$ が functor であることと、

μ, η, ξ に関する diagrams から従う.

$\therefore\ \mathcal{C}(\cdot,c)$: monoid (or group) in $\mathbf{Set}^{\mathcal{C}^{op}}$.

$\Leftarrow$ 逆に、$\mathcal{C}(\cdot,c)$; monoid (or group) in $\mathbf{Set}^{\mathcal{C}^{op}}$ で、

$\bar{\mu} : \mathcal{C}(\cdot,c)\times\mathcal{C}(\cdot,c) \overset{\cdot}{\longrightarrow} \mathcal{C}(\cdot,c)$

$\bar{\eta} : \mathcal{C}(\cdot,t) \overset{\cdot}{\longrightarrow} \mathcal{C}(\cdot,c)$,

$\bar{\xi} : \mathcal{C}(\cdot,c) \overset{\cdot}{\longrightarrow} \mathcal{C}(\cdot,c)$

を構造射とする.

Yoneda Lemma における natural iso. を Y として、

$$\mu := Y(\bar{\mu} \circ \theta^{-1}) = \bar{\mu}_{c\times c} \cdot \theta^{-1}_{c\times c}\, 1_{c\times c}$$
$$\eta := Y(\bar{\eta}) = \bar{\eta}_t\, 1_t$$
$$\xi := Y(\bar{\xi}) = \bar{\xi}_c\, 1_c$$

とおく。このとき、

$$\begin{tikzcd}
(\mathcal{C}(\cdot,c)\times\mathcal{C}(\cdot,c))\times\mathcal{C}(\cdot,c) \arrow[d, "\bar{\mu}\times 1"'] \arrow[r, "\cong"] & \mathcal{C}(\cdot,c)\times(\mathcal{C}(\cdot,c)\times\mathcal{C}(\cdot,c)) \arrow[d, "1\times\bar{\mu}"] \\
\mathcal{C}(\cdot,c)\times\mathcal{C}(\cdot,c) \arrow[dr, "\bar{\mu}"'] \arrow[r, phantom, "\circlearrowright"] & \mathcal{C}(\cdot,c)\times\mathcal{C}(\cdot,c) \arrow[d, "\bar{\mu}"] \\
& \mathcal{C}(\cdot,c)
\end{tikzcd}$$

に Y を適用して、

$Y(\langle) = \mu \circ (\mu \times 1)$

$Y(\swarrow) = \mu \circ (1 \times \mu)$

を得、Y が bijective であることから、μ の associativity が分かる。

η, ξ に関する公理も同様。

$\therefore$ c は monoid (or group) in $\mathcal{C}$. //

Exercises $\mathcal{C}$: category with finite products, $t \in \mathcal{C}$: terminal とする。

(1) $\mathcal{G}$: category with

- obj. : monoid in $\mathcal{C}$
- arr. : $f: G \longrightarrow H$ in $\mathcal{C}$ s.t. $f\mu_G = \mu_H(f\times f)$

これを monoid map と呼ぶことにする。

とする。$t \in \mathcal{G}$ より、$\mathcal{G} \neq \phi$.

$\mathcal{G}$ は実際に category である。($\because$ $1_G : G \longrightarrow G$ は monoid map).

このとき、$\mathcal{G}$ has finite products.

($\because$) $G, H \in \mathcal{G} \Rightarrow G\times H \in \mathcal{G}$ を示せばよい。

$\alpha_c : (c \times c) \times c \xrightarrow{\sim} c \times (c \times c)$ とする.

$G \times H \in \mathcal{G}$ を示すには. $\mu_G \times \mu_H$ の associativity と unit law を見ればよいが.
これらは products の universality から従う.

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