MAP55672 Case Study 4
The Multigrid Method

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May 2025

1 Introduction

This report investigates the convergence behavior of the recursive V-cycle multigrid (MG) method for solving the discretised Poisson equation on the unit square. The performance is examined with varying multigrid levels `lmax` and grid sizes `N`, using double-precision arithmetic and a stopping criterion based on the residual norm $\|r\| < 10^{-7}$.

2 Implementation Summary

The implementation follows Algorithm 1 described in the case study specification. Each MG level performs $\nu = 3$ Jacobi smoothing steps with relaxation parameter $\omega = 0.666$. The coarse grid solver applies 50 Jacobi iterations. The residual is computed after each cycle, and iterations continue until the desired tolerance is achieved or the iteration cap is reached.

3 Convergence Results for Fixed Grid Size $N = 128$

The table below shows the performance of the MG solver for different values of `lmax` while keeping grid size $N = 128$ fixed.

lmax	Initial Residual	Iterations	Time (s)
2	7.57×10^{-2}	1175	2.81
3	7.57×10^{-2}	1246	2.57
4	7.59×10^{-2}	1612	3.18
5	7.60×10^{-2}	2053	4.06
6	7.60×10^{-2}	2146	4.29

Table 1: MG convergence for fixed $N = 128$ and varying `lmax`

We observe that increasing `lmax` initially reduces the number of coarse grid solves but beyond a certain point, performance deteriorates due to the computational overhead and diminishing returns from deeper recursion.

4 Performance Comparison: Two-Level vs Max-Level

A comparison between two-level (`lmax = 1`) and max-level (coarsest level $N = 8$) MG setups was conducted across grid sizes:

N	lmax	Iterations	Residual	Time (s)
16	1	46	9.37×10^{-8}	0.0028
16	1	46	9.37×10^{-8}	0.0026
32	1	107	9.56×10^{-8}	0.0246
32	2	154	9.64×10^{-8}	0.0223
64	1	344	9.87×10^{-8}	0.2924
64	3	495	1.00×10^{-7}	0.2329
128	1	1241	1.00×10^{-7}	4.1853
128	4	1612	9.95×10^{-8}	2.9738
256	1	4630	9.99×10^{-8}	63.8233
256	5	5334	9.99×10^{-8}	40.0881

Table 2: Comparison of two-level vs. max-level MG for varying N

For small grids, the two-level MG is faster due to minimal overhead. However, as N increases, the max-level MG becomes more efficient in terms of runtime despite higher iteration counts, due to faster per-iteration convergence and reduced fine-grid work. Best performance was observed with a moderate depth (e.g., `lmax = 4` for $N = 128$). Excessive depth (`lmax = 6`) yields diminishing returns.

5 Conclusion

The multigrid method demonstrates effective convergence for the Poisson problem. Best practice is:

- Use shallow MG (e.g., 2-level) for small N.
- Use max-level MG for larger N, with the coarsest level at $N = 8$.
- Choose *lmax* to balance between residual convergence and overhead.

This balance ensures optimal convergence speed with minimal computational effort.