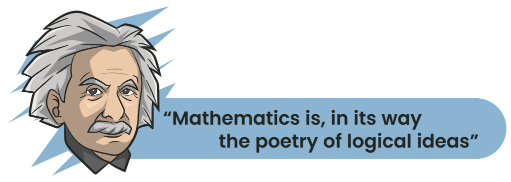

RELATIONS

(1) Types of Relations

1. Empty Relation
A relation in which no element of A is related to any other element of A, i.e., R=φ ⊂ A×A.

2. Universal Relation
A relation in which each element of A is related to every element of A, i.e., R=A×A.

3. Identity Relation
A relation in which each element is related to itself only. $I=\{(a,a), a\in A\}$

4. Reflexive Relation:
$(a, a)\in R$, for every $a \in A$.

5. Symmetric Relation:
$(a_1, a_2)\in R$ implies that $(a_2, a_1)\in R$, for all $a_1, a_2\in A$.

6. Transitive Relation:
$(a_1, a_2)\in R$ & $(a_2, a_3)\in R$ implies that $(a_1, a_3)\in R$, for all $a_1, a_2, a_3\in A$.

7. Equivalence Relation :
A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric & transitive.

8. Inverse Relation
Inverse relation of R from A to B, denoted by R^{-1}, is a relation from B to A is defined by $R^{-1} = \{(b, a) : (a, b)\in R\}$.

9. Asymmetric Relation
$(x, y)\in R \Rightarrow (y, x)\notin R$

10. Antisymmetric: A relation is antisymmetric if:
- For all $x, y\in X[(x, y)\in R \;\&\; (y, x)\in R] \Rightarrow x = y$
- For all $x, y\in X[(x, y)\in R \;\&\; x\neq y] \Rightarrow (y, x)\notin R$

11. Irreflexive
R is irreflexive iff
$\forall a\in A, ((a, a)\notin R)$

12. Partial order relation
R is a partial order, if R is Reflexive, Antisymmetric and Transitive.

2. EXAMPLE:

A = {1, 2, 3, 4}. Identify the properties of relations.

$R_1 = \{(1,1),(2,2),(3,3),(2,1),(4,3),(4,1),(3,2)\}$

$R_2 = A\times A,\ R_3 = \phi,\ R_4 = \{(1,1),(2,2),(3,3),(4,4)\}$

$R_5 = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(4,3),(3,4)\}$

Relation	Reflexive	Symmetric	Asymmetric	Antisymmetric	Irreflexive	Transitive
R_1	✗	✗	✗	✓	✗	✗
R_2	✓	✓	✗	✗	✗	✓
R_3	✗	✓	✓	✓	✓	✓
R_4	✓	✓	✗	✓	✗	✓
R_5	✓	✓	✗	✗	✗	✓

NOTE
If A={1,2}, a relation R={(1,2)} on A is a transitive relation.
using the similar argument a relation R = {(x, y) : x is wife of y} is transitive, where as R = {(x, y) : x is father of y} is not transitive.

3. PROPERTIES

1. R is not reflexive does not imply R is irreflexive. Counter example: $A=\{1,2,3\}, R=\{(1,1)\}$

2. R is asymmetric implies that R is irreflexive. By definition, for all $a, b\in A, (a, b)\in R$ and $(b, a)\notin R$ This implies that for all $(a, b)\in R, a\neq b$ Thus, for all $a\in A, (a, a)\notin R$ Therefore, R is irreflexive.

3. R is not symmetric does not imply R is antisymmetric. Counter example: $A=\{1,2,3\}, R=\{(1,2),(2,3),(3,2)\}$

4. R is not symmetric does not imply R is asymmetric. Counter example: $A=\{1,2,3\}, R=\{(1,2),(2,2)\}$

5. R is not antisymmetric does not imply R is symmetric. Counter example: $A=\{1,2,3\}, R=\{(1,2),(2,3),(3,2)\}$

6. R is reflexive implies that R is not asymmetric. By definition, for all $a\in A, (a, a)\in R$ This implies that, both (a, b) and (b, a) are in R when a = b. Thus, R is not asymmetric.

(4) COUNTING OF RELATION

- Number of relations from set A to B = 2^{mn}, where $|A|=m, |B|=n$
- Number of Identity relation on a set with 'n' elements = 1
- Number of reflexive relation set on a set with 'n' elements = $2^{n(n-1)}$
- Number of Symmetric relation set on a set with 'n' elements = $2^{n(n+1)/2}$
- The number of antisymmetric binary relations possible on A is $2^n\cdot 3^{(n^2-n)/2}$
- The number of binary relation on A which are both symmetric and antisymmetric is 2^n.
- The number of binary relation on A which are both symmetric and asymmetric is 1.
- The number of binary relation which are both reflexive and antisymmetric on the set A is $3^{(n^2-n)/2}$
- The number of asymmetric binary relation possible on the set A is $3^{(n^2-n)/2}$
- There are at least 2^n transitive relations (lower bound) and at most $2^{n^2}-2^{\frac{n^2-n}{2}}+1$ (upper bound)

5. OPERATION ON RELATIONS:

1. $R_1 - R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \notin R_2\}$
2. $R_2 - R_1 = \{(a,b) \mid (a,b) \in R_2 \text{ and } (a,b) \notin R_1\}$
3. $R_1 \cup R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ or } (a,b) \in R_2\}$
4. $R_1 \cap R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \in R_2\}$

PROPERTIES

1) If R_1 and R_2 are reflexive, and symmetric, then $R_1 \cup R_2$ is reflexive, and symmetric.

2) If R_1 is transitive and R_2 is transitive, then $R_1 \cup R_2$ need not be transitive.

counter example: Let $A = \{1,2\}$ such that $R_1 = \{(1,2)\}$ and $R_2 = \{(2,1)\}$. $R_1 \cup R_2 = \{(1,2),(2,1)\}$ and $(1,1) \notin R_1 \cup R_2$ implies that $R_1 \cup R_2$ is not transitive.

3) If R_1 and R_2 are equivalence relations, then $R_1 \cap R_2$ is an equivalence relation.

4) If R_1 and R_2 are equivalence relations on A,

- $R_1 - R_2$ is not an equivalence relation (reflexivity fails).
- $R_1 - R_2$ is not a partial order (since $R_1 - R_2$ is not reflexive).
- $R_1 \oplus R_2 = R_1 \cup R_2 - (R_1 \cap R_2)$ is neither equivalence relation nor partial order (reflexivity fails)

5) The union of two equivalence relation on a set is not necessarily an equivalence reation on the set.

6) The inverse of a equivalence relation R is an equivalence relation.

6. COMPOSITON OF RELATIONS

Let $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, Composition of R_2 on R_1, denoted as $R_1 \; R_2$ or simply R_1R_2 is

$$R_1 \; R_2 = \{(a,c) \mid a \in A, c \in C \wedge \exists b \in B \text{ such that } ((a,b) \in R_1, (b,c) \in R_2)\}$$

NOTE

$R_1 \, (R_2 \cap R_3) \subset R_1 \, R_2 \cap R_1 \, R_3$

$R_1 \, (R_2 \cup R_3) = R_1 \, R_2 \cup R_1R_3$

$R_1 \subseteq A \times B, R_2 \subseteq B \times C, R_3 \subseteq C \times D. (R_1R_2)R_3 = R_1(R_2R_3)$

$(R_1 o R_2)^{-1} = R_2^{-1} o R_1^{-1}$

7. EQUIVALENCE CLASS

Equivalence class of $a \in A$ is defined as $[a] = \{x \mid (x,a) \in R\}$, that is all the elements related to a under the relation R.

Example

E=Even integers, O=odd integers.

(i) All elements of E are related to each other and all elements of O are related to each other.

(ii) No element of E is related to any element of O and vice-versa.

(iii) E and O are disjoint and $\mathbf{Z} = E \cup O$

The subset E is called the equivalence class containing zero and is denoted by [0].

Properties: consider an equivalence relation R defiend on a set A.

1. $\bigcup_{\forall a \in A} [a] = A$

2. For every a,b∈A such that $a \in [b], a \neq b$ it follows that $[a] = [b]$

3. $\sum_{\forall x \in A} |[x]| = |R|$

4. For any two equivalence class $[a]$ and $[b]$, either $[a] = [b]$ or $[a] \cap [b] = \phi$

5. For all $a, b \in A$, if $a \in [b]$ then $b \in [a]$

6. For all $a, b, c \in A$, if $a \in [b]$ and $b \in [c]$, then $a \in [c]$

7. For all $a \in A, [a] \neq \phi$

Congruence modulo n given by a ≡ b (mod n) if and only if n divides (a – b).

8. BINARY OPERATIONS

Let S be a non-empty set. A function f : S × S → S is called a binary opertion on set S.

Note

Number of binary operations on a set containing n elements is n^{n^2}