NOTES ON HENNION-KAPRANOV

In this note we detail the computation of the (continuous) Lie algebra cohomology of the Lie algebra of vector fields on a smooth manifold, due originally to Gelfand and Fuks. We follow the exposition of Hennion and Kapranov, which uses the local-to-global tools of factorization homology to compute this cohomology.

Let us outline the idea behind the computation. Let M be a smooth manifold.

(i) For $U \subset M$ we may consider the cdga
$$\mathcal{A}(U) := \mathsf{CE}^*(\mathrm{Vect}(U))$$
computing the cohomology of vector fields on U. The assignment $U \mapsto \mathcal{A}(U)$ then defines a factorization algebra $\mathcal{A}$ on M, whose global sections (factorization homology) is the object we would like to calculate. Peculiar to the case of cdgas, factorization algebras are just (homotopy) cosheaves. Furthermore, the cosheaf $\mathcal{A}$ is actually locally constant.

(ii) According to Gelfand and Fuks, the Lie algebra cohomology of vector fields can locally be written in terms of the (Thom-Whitney) cochains of a particular CW complex Y. It follows that the cosheaf $\mathcal{A}$ on M is weakly equivalent to a certain locally constant cosheaf $\mathcal{Y}$ on M associated to Y.

(iii) It now remains to compute the global sections of $\mathcal{Y}$. For this we use nonabelian Poincaré duality, incarnated in this context as a close relationship between locally constant sheaves and cosheaves, which describes the factorization homology $\mathcal{Y}(M)$ as the space of sections of a certain fibration over M with fiber Y.

1. Gelfand-Fuks theory

2. Globalization

Appendix A. The algebraic picture

In the above we have considered smooth vector fields on smooth manifolds. One might ask, however, whether an analog of the Gelfand-Fuks result holds in the setting of (smooth) algebraic varieties. Hennion and Kapranov note that this question dates back to B.L. Feigin in the 80's, but there was apparently no progress on the question, even for the case of curves. The bulk of their paper is concerned with determining the Lie algebra cohomology of regular vector fields in this case. I am not very familiar with algebraic geometry, which is why I decided to focus on the smooth case. I think it is worth outlining the constructions, though, to get a sense of the ideas involved.

The overall argument is similar to the one described above. For X a smooth affine algebraic variety over we wish to determine the Chevalley-Eilenberg complex of the (right derived functor of the) global sections of the tangent sheaf $\mathsf{CE}^*(R\Gamma(X, \mathcal{T}_X))$. Let us write $\mathfrak{l} = R\Gamma(X, \mathcal{T}_X)$ in what follows.

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We start by representing the Lie algebra homology (tor, not ext) of $\mathfrak{l}$ as the factorization homology of, roughly, a certain $\mathcal{D}$-module on the Ran space Ran X. What do we mean by this? Recall that the Ran space of X is the set of all nonempty finite subsets of X, topologized to allow points to collide. In an appropriately general category of algebraic objects, the Ran space can be viewed as the colimit of a certain diagram $X^{\mathcal{S}}$ of powers of X. Here $\mathcal{S}$ is the category of nonempty finite sets and surjections. For $I \twoheadrightarrow J$ the map $X^J \to X^I$ is given by the diagonal embedding corresponding to $I \twoheadrightarrow J$.

Now a **lax $\mathcal{D}^!$-module** on $X^{\mathcal{S}}$ is, roughly, the assignment to each variety in the diagram $X^{\mathcal{S}}$ an object of the derived category of quasicoherent right $\mathcal{D}$-modules in a suitably compatible way. Intuitively, since the Ran space is a colimit, its category of $\mathcal{D}$-modules must be a limit of categories of $\mathcal{D}$-modules on powers of X. The word lax is signifying that the compatibility condition can be strengthened to give a notion of strict $\mathcal{D}^!$-module.

Write $\mathcal{L} = L \otimes_{\mathcal{O}_X} \mathcal{D}_X$, where for us, $L = \mathcal{T}_X$. Then we can define a lax $\mathcal{D}^!$-module $\mathcal{C}_1$ on $X^{\mathcal{S}}$ by

$$\mathcal{C}_1^{(I)} = (\delta_I)_*\mathcal{L},$$

where $\delta_I : X \to X^I$ is the diagonal embedding. Given a surjection $g : I \twoheadrightarrow J$ the corresponding structure map

$$\delta_g : (\delta_g)_*(\delta_J)_*\mathcal{L} \to (\delta_I)_*\mathcal{L}$$

is the isomorphism coming from the equality $\delta_g \circ \delta_J = \delta_I$.

There is a symmetric monoidal product on the category of lax $\mathcal{D}^!$-modules, denoted by $\otimes^*$ (distinct from the "chiral" tensor product). It is defined as follows:

$$(\mathcal{E} \otimes \mathcal{F})^{(I)} = \bigoplus_{I_1 \cup I_2 = I} \mathcal{E}^{(I_1)} \boxtimes \mathcal{F}^{(I_2)}$$

In our case, $\mathcal{C}_1$ becomes a Lie algebra object with respect to this tensor product. Thus we can define

$$\mathcal{C}_* = (\mathrm{Sym}_{\otimes^*}^{*\geq 1}\mathcal{C}_1[1], d_{\mathrm{CE}}).$$

The factorization homology of $\mathcal{C}_*$, which we now define, will compute the Chevalley-Eilenberg homology of $\mathfrak{l}$.

The factorization homology of a lax $\mathcal{D}^!$-module $\mathcal{E}$ on X is

$$\int_X \mathcal{E} = \mathrm{holim}_{I \in \mathcal{S}} R\Gamma_{\mathrm{dR}}(X^I, \mathcal{E}^{(I)}) \in \mathsf{Ind}(\mathsf{Perf}) = \mathsf{Ch},$$

i.e. the global sections of $\mathcal{E}$ on the Ran space. A few words about $R\Gamma_{\mathrm{dR}}$ are in order: this is the global sections functor applied to the de Rham complex of a $\mathcal{D}$-module. Recall that the de Rham complex is $\mathrm{dR}(\mathcal{M}) = \mathcal{M} \otimes^h_{\mathcal{D}_X} \mathcal{O}_X$. With this in mind, Kapranov and Hennion show that

$$\int_X \mathcal{C}_* \simeq \mathrm{CE}_*(\mathfrak{l}).$$

From this, invoking duality, they obtain

$$\mathrm{CE}^*(\mathfrak{l}) \simeq R\Gamma_{\mathrm{dR}}^{[[c]]}(X^{\mathcal{S}}, \psi(\mathcal{C}^\vee)). \tag{1}$$

Here we are taking the compactly supported sections of a certain Verdier dual of $\mathcal{C}$. This maneuver is the analog of non-abelian Poincaré duality above and the close relationship between sheaves and cosheaves.

Now $\psi(\mathcal{C}^\vee)$ satisfies some nice properties, giving us a factorization algebra (in the usual sense) on X_{an},

$$\mathcal{A}(U) = \mathrm{holim}_{I \in \mathcal{S}} R\Gamma_c(U^I, \mathrm{dR}(\psi(\mathcal{C}^\vee)^{(I)})_{\mathrm{an}}),$$

such that its global sections, its factorization homology, on X_{an} is the right-hand side of 1. Thus we obtain an identification

$$\int_{X_{\mathrm{an}}} \mathcal{A} \simeq \mathrm{CE}^*(\mathfrak{l}).$$

The remainder of the calculation is now similar to the local argument given in the topological case, showing that the factorization algebra is locally given as a section space, from which the global identification as a certain section space follows.