Karnataka State Open University

Study Material for BCA / IMCA

Mathematics - Code - BCA 21 / IMCA 21

by

K.S. Srinivasa

Retd. Principal &
Professor of Mathematics
Bangalore

Published by

Sharada Vikas Trust (R)

Bangalore

BCA 21 / IMCA 21 / MATHEMATICS

Syllabus

1. **Matrix Theory :**

   Review of the fundamentals. Solution of linear equations by Cramers' Rule and by Matrix method, Eigen values and Eigen vectors, Cayley Hamilton's Theorem, Diagonalization of matrices, simple problems.

2. **Algebraic Structures**

   Definition of a group, properties of groups, sub groups, permutation groups, simple problems, scalars & vectors, algebra of vectors, scalar & vector products, scalar triple product, simple problems.

3. **Analytical Geometry in three dimensions :**

   Direction cosines and direction ratios, distance formula, section formula, equations to planes and straight lines, angles between two planes & two lines, problems, equation of a sphere, right circular cone and right circular cylinder, simple problems.

4. **Differential Calculus :**

   Limits, continuity and differentiability (definition only), standard derivatives, rules for differentiation, derivatives of function of a function and parametric functions, problems. Successive differentiation, n^{th} derivative of standard functions, statement of Leibnitz's Theorem, problems, statements of Rolle's, Lagrange's, Cauchy's and Taylor's Mean Value Theorems and simple problems, Indeterminate forms, L' Hospital's rule, partial derivatives, definition and simple problems.

5. **Integral Calculus**

   Introduction, standard integrals, integration by substitution and by parts, integration of rational, irrational and trigonometric functions, definite integrals, properties (no proof), simple problems, reduction formulae and simple problems.

6. **Differential Equations of first order**

   Introduction, solution by separation of variables, homogeneous equations, reducible to homogeneous linear equation, Bernoulli's equation, exact differential equations and simple problems.

Text Books

1. *Elementary Engineering Mathematics* by Dr. B.S. Grewal, Khanna Publications
2. *Higher Engineering Mathematics* by B.S. Grewal, Khanna Publications

Reference Books

1. *Differential Calculus* by Shanti Narayan, Publishers S. Chand & Co.
2. *Integral Calculus* by Shanti Narayan, Publishers S. Chand & Co.
3. *Modern Abstract Algebra* by Shanti Narayan, Publishers S. Chand & Co.

CONTENTS

Page Nos.

1. Matrix Theory 01
2. Algebraic Structures 21
3. Analytical Geometry in three dimensions 43
4. Differential Calculus 65
5. Integral Calculus 96
6. Differential Equations 118

MATRIX THEORY

Review of the fundamentals

A rectangular array of mn elements arranged in m rows & n columns is called a **'Matrix'** of a order $m \times n$ matrices are denoted by capital letters of The English Alphabet.

Examples

Matrix of order 3×2 is $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$

Matrix of order 4×3 is $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}$

Matrix of order 3×3 is $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Note :- Elements of Matrices are written in rows and columns with in the bracket ( ) or [ ].

Types of Matrices

(1) **Equivalent Matrices :** Two matrices are said to be equivalent if the order is the same.

(2) **Equal Matrices :** Two matrices are said to be equal if the corresponding elements are equal.

(3) **Rectangular & Square Matrices :** A matrix of order $m \times n$ is said to be rectangular if $m \neq n$, square if $m = n$.

(4) **Row Matrix :** A matrix having only one row is called Row Matrix.

(5) **Column Matrix :** A matrix having only one column is called Column Matrix.

(6) **Null Matrix or Zero Matrix :** A matrix in which all the elements are zeros is called Null Matrix or Zero Matrix denoted as O. [English alphabet O not zero where as elements are zeros]

(7) **Diagonal Matrix :** A diagonal matrix is a square matrix in which all elements except the elements in the principal diagonal are zeros.

Example $\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

are diagonal matrices of order 2 & 3.

(8) **Scalar Matrix :** A diagonal matrix in which all the elements in the principal diagonal are same.

$$\text{Example}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

are Scalar Matrices of order 2 & 3.

(9) Unit Matrix or Identity Matrix : A diagonal matrix in which all the elements in the principal diagonal is 1 is called Unit Matrix or Identity Matrix denoted by I.

$$\text{Example}: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are unit matrices of order 2 & 4.

(10) Transpose of a Matrix : If A is any matrix then the matrix obtained by interchanging the rows & columns of A is called 'Transpose of A and it is written as A' or A^T.

$$\text{Example: If } A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \text{ then } A' \text{ is } \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

A is of order 3×2 but A' is of order 2×3.

Matrix addition

Two matrices can be added or subtracted if their orders are same.

$$\text{Example: If } A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \,\&\, B = \begin{pmatrix} c_1 & d_1 & e_1 \\ c_2 & d_2 & e_2 \end{pmatrix}$$

$$A + B = \begin{pmatrix} a_1 + c_1 & b_1 + d_1 & c_1 + e_1 \\ a_2 + c_2 & b_2 + d_2 & c_2 + e_2 \end{pmatrix}$$

$$A - B = \begin{pmatrix} a_1 - c_1 & b_1 - d_1 & c_1 - e_1 \\ a_2 - c_2 & b_2 - d_2 & c_2 - e_2 \end{pmatrix}$$

Matrix Multiplication

If A is a matrix of order $m \times p$ and B is matrix of order $p \times n$, then the product AB is defined and its order is $m \times n$. (ie. for AB to be defined number of columns of A must be same as number of rows of B)

$$\text{Example: Let } A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \,\&\, B = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix}$$

$$\text{then } AB = \begin{pmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 \end{pmatrix}$$

which is of order 2×2.

Note :- If A is multiplied by A then AA is denoted as A^2, AAA.... as A^3 etc.

Scalar Multiplication of a Matrix

If A is a matrix of any order and K is a scalar (a constant), then KA represent a matrix in which every element of A is multiplied by K.

$$\text{Example: If } A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ then } KA=\begin{pmatrix} Ka_1 & Kb_1 & Kc_1 \\ Ka_2 & Kb_2 & Kc_2 \\ Ka_3 & Kb_3 & Kc_3 \end{pmatrix}$$

Symmetric and Skew Symmetric Matrices

Let A be a matrix of order $n \times n$ an element in i^{th} row and j^{th} column can be denoted as aij. Hence a matrix of order $n \times n$ can be denoted as (aij) or $[aij]$ where $i = 1, 2, ...n$, $j = 1, 2, ...n$

A matrix of order $n \times n$ is said to be Symmetric if $aij = aji$ and Skew Symmetric if $aij = -aji$ or A is symmetric if $A = A^T$ or $A = A'$, skew symmetric if $A = -A^T$ or $A = -A'$ also $A + A'$ is symmetric & $A - A'$ is skew symmetric.

<u>Note</u> :- In a skew symmetric matrix the elements in principal diagonal are all zeros.

$$\text{Example: } A=\begin{bmatrix} 2 & 3 & 5 \\ 3 & 7 & 6 \\ 5 & 6 & 8 \end{bmatrix} \text{ is symmetric where } A=A'$$

$$B=\begin{pmatrix} 0 & -2 & 7 \\ 2 & 0 & 6 \\ -7 & -6 & 0 \end{pmatrix} \text{ is skew symmetric where } B=-B'$$

Determinant

A determinant is defined as a mapping (function) from the set of square matrices to the set of real numbers.

If A is a square matrix its determinant is denoted as $|A|$.

$$\text{Example: Let } A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ then det. } A \text{ or } |A|=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Minors and Co-factors

Let $A=(aij) \quad i=1,2,3 \quad j=1,2,3$

$$\text{ie } A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Consider $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ which is a determinant formed by leaning all the elements of row and column in which all lies. This determinant is called Minor of a_{11}. Thus we can form nine minors. In general if A is matrix of order $n \times n$ then minor of aij is obtained by leaning all the elements in the row and column in which *aij* lies in $|A|$. The order of this minor is $n-1$ where as the order of given determinant is n if this minor is multiplied by $(-1)^{i+j}$ then it is called Co-factors of *aij*.

$$\text{Example: Let } A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\text{Minor of } a_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Co-factor of } a_{11}=(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Minor of } a_{21} \text{ is } \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Co-factor of } a_{21} \text{ is } (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}=-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Value of a determinant

Consider a matrix A of order $n\times n$. Consider all the elements of any row or column and multiply each element by its corresponding co-factor. Then the algebraic sum of the product is the value of the determinant.

$$\text{Example: Let } |A|=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Co-factor of a_1 is b_2

Co-factor of b_1 is $-a_2$

$$\therefore\ |A|=a_1b_2-b_1a_2$$

$$\text{Let } |A|=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Co-factor of } a_1 \text{ is } (-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}=\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$\text{Co-factor of } b_1 \text{ is } (-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}=-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$\text{Co-factor of } c_1 \text{ is } (-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}=\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\therefore\ |A|=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$=a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$$

$$=a_1b_2c_3-a_1b_3c_2)-a_2b_1c_3-a_3b_1c_2+a_2b_3c_1-a_3b_2c_1$$

Properties of determinants

(1) If the elements of any two rows or columns are interchanged then value of the determinant changes only in sign.

(2) If the elements of two rows or columns are identical then the value of the determinant is zero.

(3) If all the elements of any row or column is multipled by a constant *K*, then the value of the determinant is multipled by *K*.

(4) If all the elements of any row or column are written as sum of two elements then the determinant can be written as sum of two determinants.

(5) If all the elements of any row or column are multiplied by a constant and added to the corresponding elements of any other row or column then the value of the determinant donot alter.

Adjoint of a Matrix

$$\text{Let } A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Let us denoted the co-factors of a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 as A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 transpose of matrix of co-factors is called **Adjoint** of the Matrix.

$$\text{Matrix of Co-factors}=\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

$$\text{Adjoint of } A=\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

Theorem $A.\text{adj}.A=|A|I=\text{adj}.A.A$

$$A.\text{adj}.A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

$$=\begin{bmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{bmatrix}$$

Now $a_1A_1+b_1B_1+c_1C_1=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}=\Delta$ The value of the det. *A*.

Similarly $a_2A_2+b_2B_2+c_2C_2=\Delta$

$$a_3A_3+b_3B_3+c_3C_3=\Delta$$

$$a_1A_2+b_1B_2+c_1C_2=-a_1\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+b_1\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}-c_1\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$=-a_1(b_1c_3-b_3c_1)+b_1(a_1c_3-a_3c_1)-c_1(a_1b_3-a_3b_1)$$

$$=-a_1b_1c_3+a_1b_3c_1+a_1b_1c_3-a_3b_1c_1-a_1b_3c_1+a_3b_1c_1=0$$

Similarly the other five elements of A adj.A is zero.

$$\therefore \; A.\text{adj}.A = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix} \text{ where } \Delta = |A|$$

$$= \Delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Delta.I$$

$$\therefore \; A.\text{adj}.A = |A| \cdot I = \text{adj}.A$$

Singular and Non-singular Matrices

A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

Inverse of a Matrix

Two non-singular matrices A & B of the same order is said to be inverse of each other if $AB = I = BA$. Inverse of A is denoted as A^{-1}. Inverse of B is denoted as B^{-1} and further $(AB)^{-1} = B^{-1}A^{-1}$.

To find the inverse of A

$$A.\text{adj}.A = |A| \cdot I$$

multiply by A^{-1}, $AA^{-1}.\text{adj}.A = |A|A^{-1}$

ie $\text{adj}.A = |A|A^{-1} \Rightarrow A^{-1} = \dfrac{\text{adj}.A}{|A|}$

Example : Find the inverse of $\begin{pmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{pmatrix}$

$$\text{Matrix of Co-factors} = \begin{bmatrix} \begin{vmatrix} -5 & 4 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -2 & -5 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 4 & -2 \\ -5 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ -2 & -5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (-5+8) & -(-2-4) & (4+5) \\ -(4-4) & (1+2) & -(-2-4) \\ (16-10) & -(4-4) & (-5+8) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 6 \\ 6 & 0 & 3 \end{bmatrix}$$

$$\therefore \ \text{adj.}A = \begin{bmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 9 & 6 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 1(-5+8)-4(-2-4)-2(4+5) = 3+24-18 = 9$$

$$A^{-1} = \frac{1}{|A|}\text{adj.}A = \frac{1}{9}\begin{pmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 9 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 3/9 & 0 & 6/9 \\ 6/9 & 3/9 & 0 \\ 9/9 & 6/9 & 3/9 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \\ 1 & 2/3 & 1/3 \end{pmatrix}$$

Solutions of Linear equations

Cramer's Rule

To solve the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\text{Consider } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{1}$$

first evaluate & if it is not zero then multiply both sides of (1) by x.

$$\Delta x = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_{3x} & b_3 & c_3 \end{vmatrix}$$

multiply the elements of columns 2 & 3 by y & z and add to elements of column 1.

$$\text{then } \Delta x = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta_1 \text{ (say)} \tag{2}$$

multiply both sides of (1) by y

$$\Delta y = y\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1y & c_1 \\ a_2 & b_2y & c_2 \\ a_3 & b_3y & c_3 \end{vmatrix}$$

multiply the elements of columns 1 & 3 by x & z and add to the elements of column 2.

$$=\begin{vmatrix} a_1 & a_1x+b_1y+c_1z & c_1 \\ a_2 & a_2x+b_2y+c_2z & c_2 \\ a_3 & a_3x+b_3y+c_3z & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \Delta_2 \text{ (say)} \qquad (3)$$

multiply both sides of (1) by z

$$\Delta z = z\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1z \\ a_2 & b_2 & c_2z \\ a_3 & b_3 & c_3z \end{vmatrix}$$

multiply the elements of columns 1 & 2 by x & y and add to the elements of column 3.

$$=\begin{vmatrix} a_1 & b_1 & a_1x+b_1y+c_1z \\ a_2 & b_2 & a_2x+b_2y+c_2z \\ a_3 & b_3 & a_3x+b_3y+c_3z \end{vmatrix}$$

$$=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \Delta_3 \text{ (say)} \qquad (4)$$

then $x = \dfrac{\Delta_1}{\Delta}$ from (2)

$y = \dfrac{\Delta_2}{\Delta}$ from (3)

$z = \dfrac{\Delta_3}{\Delta}$ from (4)

Note :- Verification of values of x, y, z can be done by substituting in the given equations.

Example - 1

Solve $2x+y-z=3$

$x+y+z=1$

$x-2y-3y=4$

$$\text{Let } \Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix} \qquad (1)$$

$$= 2(-3+2)-1(-3-1)-1(-2-1) = -2+4+3=5$$

multiply both sides of (1) by x

$$\text{then } \Delta x = x\begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 2x & 1 & -1 \\ x & 1 & 1 \\ x & -2 & -3 \end{vmatrix}$$

multiply the elements of columns 2 & 3 by y and z and add to the elements of column 1.

$$\Delta x = \begin{vmatrix} 2x+y-z & 1 & -1 \\ x+y+z & 1 & 1 \\ x-2y-3y & -2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix} = 3(-3+2)-1(-3-4)-1(-2-4) = -3+7+6=10$$

$$\therefore \ x = \frac{10}{\Delta} = \frac{10}{5} = 2$$

multiply both sides of (1) by y

$$\Delta y = y\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} == \begin{vmatrix} 2 & y & -1 \\ 1 & y & 1 \\ 1 & -2y & -3 \end{vmatrix}$$

multiply the elements of column 1 by x & 3 by z and to the corresponding elements of column 2.

$$\text{then } \Delta y = \begin{vmatrix} 2 & 2x+y-z & -1 \\ 1 & x+y+z & 1 \\ 1 & x-2y-3z & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = 2(-3-4)-3(-3-1)-1(4-1) = -14+12-3=-5$$

$$\therefore \ y = \frac{-5}{\Delta} = \frac{-5}{5} = -1$$

multiply both sides of (1) by z

$$\Delta z = z\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -z \\ 1 & 1 & z \\ 1 & 4 & -3z \end{vmatrix}$$

multiply the elements of column 1 by x & column 2 by y and to the corresponding elements of column 3.

$$\text{then } \Delta z = \begin{vmatrix} 2 & 1 & 2x+y-z \\ 1 & 1 & x+y+z \\ 1 & -2 & x-2y-3z \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = 2(4+2)-1(4-1)+3(-2-1) = 12-3-9=0$$

$$\therefore \ z = \frac{0}{\Delta} = \frac{0}{5} = 0$$

Thus solution is $x = 2$, $y = -1$ & $z = 0$ which can be verified by substituting in the given equations.

Example - 2

Solve $4x+y=7$

$3y+4z=5$

$5x+3z=2$

$$\Delta = \begin{vmatrix} 4 & 1 & 0 \\ 5 & 3 & 4 \\ 2 & 0 & 3 \end{vmatrix} = 4(9-0)-1(0-20)+0(0-15) = 36+20=56$$

$$\Delta_1 = \begin{vmatrix} 7 & 1 & 0 \\ 5 & 3 & 4 \\ 2 & 0 & 3 \end{vmatrix} = 7(9-0)-1(15-8)+0(0-6) \; = 63-7=56$$

$$\Delta_2 = \begin{vmatrix} 4 & 7 & 0 \\ 0 & 5 & 4 \\ 5 & 2 & 3 \end{vmatrix} = 4(15-8)-7(0-20)+0(0-25) = 28+140=168$$

$$\Delta_3 = \begin{vmatrix} 4 & 1 & 7 \\ 0 & 3 & 5 \\ 5 & 0 & 2 \end{vmatrix} = 4(6-0)-1(0-25)+7(0-15) \; = 24+25-105=-56$$

$$\therefore \; x = \frac{\Delta_1}{\Delta} = \frac{56}{56} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{168}{56} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-56}{56} = -1$$

Solution of Linear equations by Matrix Method

Given $\quad a_1x+b_1y+c_1z=d_1$

$a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$

Consider $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ **&** $B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

then given equations can be written in Matrix form as $AX = B$. If $|A| \neq 0$ solution exists multiply both sides by A^{-1}

$A^{-1}(AX) = A^{-1}B$

$A^{-1}AX = A^{-1}B$

ie $IX = A^{-1}B$

$\therefore \; X = A^{-1}B$

Example

Solve $3x-y+2z=13$

$2x+y-z=3$

$x+3y-5z=-8$

Let $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & -5 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; B = \begin{pmatrix} 13 \\ 3 \\ -8 \end{pmatrix}$

then given equations can be written as $AX = B$

$$\therefore \; X = A^{-1}B \qquad (1)$$

To find A^{-1}

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & -5 \end{vmatrix} = 3(-5+3)+1(-10+1)+2(6-1) = -6-9+10 = -5$$

$$\text{Matrix of Co-factors} = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & -5 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (-5+3) & -(-10+1) & (6-1) \\ -(5-6) & (-15-2) & -(9+1) \\ 1-2 & -(-3-4) & (3+2) \end{bmatrix} = \begin{bmatrix} -2 & 9 & 5 \\ 1 & -17 & -10 \\ -1 & -7 & 5 \end{bmatrix}$$

$$\text{adj.}A = \begin{pmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{pmatrix} \qquad \therefore \; A^{-1} = \frac{1}{|A|}\text{adj.}A = -\frac{1}{5}\begin{pmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{pmatrix}$$

Using this in (1)

$$X = -\frac{1}{5}\begin{pmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{pmatrix}\begin{pmatrix} 13 \\ 3 \\ -8 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -26 & 3 & 8 \\ 117 & -51 & -56 \\ 65 & -30 & -40 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -15 \\ 10 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore$ $x = 3, y = -2, z = 1$ is the solution

Verification : Consider the first equation

$$3x - y + 2z = 9 + 2 + 2 = 13$$

Characteristic equation, Eigen Values & Eigen Vectors

Let A & I be square matrices of same order and λ a scalar then $|A - \lambda I| = 0$ is called **Characteristic equation** and the roots of this equation ie values of λ are called **Eigen Values or Characteristic roots.** The matrix X satisfying $AX = \lambda X$ is called **Eigen Vector.**

Example - 1

Find the eigen roots and eigen vectors of the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ Characteristic equation is $\left|\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$

ie $\begin{vmatrix} 1-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} = 0 \qquad \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow 3 - 3\lambda - \lambda + \lambda^2 - 8 = 0 \quad \Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$$\Rightarrow (\lambda-5)(\lambda+1)=0 \Rightarrow \lambda=-1, 5$$

$\therefore$ Eigen roots are –1 & 5.

To find eigen vector X, corresponding to –1,

$AX = -1$

$$\text{ie } \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x+4y \\ 2x+3y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$$\Rightarrow \begin{matrix} x+4y=-x \\ 2x+3y=-y \end{matrix} \quad \Rightarrow \quad \begin{matrix} ix=-4y \\ \text{ie } x=-2y \end{matrix} \quad \Rightarrow \therefore \frac{x}{-2}=\frac{y}{1}$$

$\therefore$ Eigen vector corresponding to eigen value –1 is (–2, 1)

To find the eigen vector corresponding to 5.

$$\text{ie } \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} x_1+4x_2 \\ 2x_1+3x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} x_1+4x_2=5x_1 \\ 2x_1+3x_2=5x_2 \end{matrix}\right\} \quad \text{ie } x_1=x_2 \quad \text{ie } \frac{x_1}{1}=\frac{x_2}{1}$$

$\therefore$ Eigen vector is (1, 1)

$\therefore$ Eigen vector corresponding to eigen root 5 is (1, 1)

Example - 2

Find the eigen roots and eigen vectors of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ Characteristic equation is $\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$$\text{ie } (6-\lambda)\left[(3-\lambda)^2-1\right]+2\left[-2(3-\lambda)+2\right]+2\left[2-2(3-\lambda)\right]=0$$

$$\text{ie } (6-\lambda)\left[9+\lambda^2-6\lambda-1\right]+2\left[-6+2\lambda+2\right]+2\left[2-6+2\lambda)\right]=0$$

$$\text{ie } (6-\lambda)(\lambda^2-6\lambda+8)+2(2\lambda-4)+2(2\lambda-4)=0$$

$$\text{ie } 6\lambda^2-36\lambda+48-\lambda^3+6\lambda^2-8\lambda+4\lambda-8+4\lambda-8=0$$

$$\text{ie } -\lambda^3+12\lambda^2-36\lambda+32=0 \qquad \text{ie } \lambda^3-12\lambda^2+36\lambda-32=0$$

which is the characteristic equation, by inspection 2 is a root

$\therefore$ dividing $\lambda^3-12\lambda^2+36\lambda-32$ by $\lambda-2$, we have

$$(\lambda-2)(\lambda^2-10\lambda+16)=0$$

$$\text{ie } (\lambda-2)(\lambda-2\lambda)(\lambda-8)=0 \quad \therefore \lambda=2, 2, 8$$

To find eigen vector or $\lambda = 2$

Consider $AX = 2X$

$$\text{ie} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 6x_1 - 2x_2 + 2x_3 &= 2x_1 \\ -2x_1 + 3x_2 - x_3 &= 2x_2 \\ 2x_1 - x_2 + 3x_3 &= 2x_3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + x_2 - x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

the above three equations represent one equation $2x_1 - x_2 + x_3 = 0$.

Let $x_3 = 0$, then $2x_1 = x_2$ ie $\frac{x_1}{1} = \frac{x_2}{2}$ ie $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0}$

$\therefore$ Eigen Vector is (1, 2, 0)

To find the eigen vector for $\lambda = 8$

Consider $AX = 8X$

$$\text{ie} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{pmatrix}$$

$$\text{ie} \quad \begin{aligned} 6x_1 - 2x_2 + 2x_3 &= 8x_1 \\ -2x_1 + 3x_2 - x_3 &= 8x_2 \\ 2x_1 - x_2 + 3x_3 &= 8x_3 \end{aligned} \qquad \text{ie} \quad \begin{aligned} -2x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 - 5x_2 - x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0 \end{aligned}$$

ie $x_1 + x_2 - x_3 = 0$ (1)

$2x_1 + 5x_2 + x_3 = 0$ (2)

$2x_1 - x_2 - 5x_3 = 0$ (3)

adding (1) & (2) we get $3x_1 + 6x_2 = 0$

ie $x_1 + 2x_2 = 0$ ie $x_1 = -2x_2$ $\therefore \frac{x_1}{-2} = \frac{x_2}{1} = K$ (Say)

then $x_1 = -2K$, $x_2 = K$

subsitituting in (1)

$-2K + K - x_3 = 0 \Rightarrow x_3 = -K$

$\therefore x_1 = -2K,\ x_2 = K$ **&** $x_3 = -K$

$\therefore$ Eigen vector is $(-2, 1, -1)$ or $(2, -1, 1)$

$\therefore$ Eigen roots are 2, 2, 8

& Eigen vectors are (1, 2, 0) **&** (2, −1, 1)

Properties of Eigen values

(1) The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

(2) The product of the eigen values of a matrix is equal to the value of its determinant.

(3) If λ is an eigen value of A then $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

Cayley - Hamilton Theorem

Every square matrix satisfies its characteristic equation.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Characteristic equation is $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

which on simplification becomes a quadric equation in λ in the form $\lambda^2 + a_1\lambda + a_2 = 0$ where a_1, a_2 are constants. Cayley Hamilton Theorem states that $A^2 + a_1 A + a_2 I = 0$ where I is a unit matrix of order 2 & 0 is a null matrix of order 2.

If $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

then characteristic equation is $\begin{vmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{vmatrix} = 0$

which on simplification becomes $-\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ which is a cubic equation.

Then as per Cayley Hamilton Theorem $-A^3 + a_1 A^2 + a_2 A + a_3 I = 0$ where I is a unit matrix of order 3 & 0 is a null matrix of order 3.

In general if A is a square matrix of order n then characteristic equation will be of the form

$$(-1)^n \lambda^n + a\lambda^{n-1} + a_2\lambda^{n-z} + \dots\dots + a_n I = 0$$

and by Cayley Hamilton Theorem

$$(-1)^n A^n + aA^{n-1} + a_2 A^{n-z} + \dots\dots + a_n I = 0$$

where I is a unit matrix of order n & 0 is a null matrix of order n.

Note :- If we put $\lambda = 0$ in the characteristic equation then $a_n = |A|$

$\therefore$ If $a_n = 0$, matrix A is singular & $a_n \neq 0$ the matrix A is non-singular & hence inverse exists and we can find the inverse of A using Cayley Hamilton Theorem.

Example - 1

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Characteristic equation is $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

ie $\lambda^2 + a_1\lambda + a_2 = 0$ where a_1, a_2 are constants.

By Cayley Hamilton Theorem

$$A^2 + a_1 A + a_2 I = 0$$

multiply both sides by A^{-1}

then $A^2 A^{-1} + a_1 A A^{-1} + a_2 A^{-1} = 0$

ie $A + a_1 I + a_2 A^{-1} = 0$

$\therefore$ $a_2 A^{-1} = -(A + a_1 I)$

$\therefore$ $A^{-1} = -\dfrac{1}{a_2}(A + a_1 I)$

Example - 2

The characteristic equation of a matrix A of order 2 is $\lambda^2 - 5\lambda + 10 = 0$ find $|A|$.

Solution : put $\lambda = 0$ in C.E. then the constant 10 is $|A|$.

Example - 3

Find the inverse of $\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ using Cayley Hamilton Theorem.

Solution : Let $A = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ C.E. is $\begin{vmatrix} 2-\lambda & -1 \\ -3 & 4-\lambda \end{vmatrix} = 0$

ie $(2-\lambda)(4-\lambda) - 3 = 0$ ie $8 - 4\lambda - 2\lambda + \lambda^2 - 3 = 0$

ie $\lambda^2 - 6\lambda + 5 = 0$

by Cayley Hamilton Theorem

$$A^2 - 6A + 5I = 0$$

multiply both sides by A^{-1}

$$A - 6I + 5A^{-1} = 0$$

$$\therefore\ 5A^{-1} = -A + 6I$$

$$= -\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2+6 & 1+0 \\ 3+0 & -4+6 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

Diagonalisation of Matrices

If A is a square matrix of order n where all the eigen values are linearly independent then a matrix P can be found such that $P^{-1}AP$ is a Diagonal Matrix.

Let A be a square matrix of order 3 and let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values, corresponding to these. Let X_1, X_2, X_3 be three vectors where

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ X_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},\ X_3 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Let $P = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$ Then $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Example - 1

Let $A = \begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$

C.E. is $\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) - 10 = 0$

ie $\lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda+1)(\lambda-6) = 0 \quad \Rightarrow \lambda = -1, 6$ are eigen values

For $\lambda = -1,$ let $AX = -X$

ie $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$

ie $\left.\begin{aligned} x_1 - 2x_2 &= -x_1 \\ -5x_1 + 4x_2 &= -x_2 \end{aligned}\right\} \Rightarrow x_1 = x_2 \quad \therefore \frac{x_1}{1} = \frac{x_2}{1}$ ....eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = 6,$ $AX = 6X$

ie $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6x_1 \\ 6x_2 \end{pmatrix}$

$\left.\begin{aligned} x_1 - 2x_2 &= -x_1 \\ -5x_1 + 4x_2 &= -x_2 \end{aligned}\right\} \Rightarrow 5x_1 = -2x_2$

ie $\frac{x_1}{-2} = \frac{x_2}{5} \quad \therefore$ eigen vector is $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$

Let $P = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$ Then $P^{-1} = \frac{1}{7}\begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$

$$\therefore P^{-1}AP = \frac{1}{7}\begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 5-10 & -10+8 \\ -1-5 & 2+4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -5 & -2 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} -5-2 & 10-10 \\ -6+6 & 12+30 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -7 & 0 \\ 0 & 42 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}$$

Thus $P = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$ diagonalize the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

<u>Example - 2</u>

Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Characteristic equation is $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

ie $(1-\lambda)[(5-\lambda)(1-\lambda)-1] - 1[1-\lambda-3] + 3[1-3(5-\lambda)] = 0$

ie $(1-\lambda)(\lambda^2 - 6\lambda + 4) - (-\lambda - 2) + 3(-14 + 3\lambda) = 0$

ie $\lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 - 42 + 9\lambda = 0$

ie $-\lambda^3 + 7\lambda^2 - 36 = 0$ ie $\lambda^3 - 7\lambda^2 + 36 = 0$

by inspection –2 is a root $\therefore$ λ + 2 is a factor. $\therefore$ equation becomes

$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$

ie $(\lambda+2)(\lambda-3(\lambda-6)=0 \quad \therefore\ \lambda=-2,3,6$

ie characteristic roots are –2, 3, 6.

To find the eigen vector for $\lambda = -2$ Consider $AX_1 = -2X_1$

where $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

ie $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{pmatrix}$

ie $x_1 + x_2 + 3x_3 = -2x_1$ ie $3x_1 + x_2 + 3x_3 = 0$ (1)

$x_1 + 5x_2 + x_3 = -2x_2$ $x_1 + 7x_2 + x_3 = 0$ (2)

$3x_1 + x_2 + x_3 = -2x_3$ $3x_1 + x_2 + 3x_3 = 0$ (3)

(1) & (3) are same.

Put $x_2 = 0$ in (1) or (2), then $x_1 + x_3 = 0$

ie $x_1 = -x_3 \Rightarrow \dfrac{x_1}{-1} = \dfrac{x_3}{1}$ $\therefore$ eigen vector $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Let X_2 be the eigen vector for $\lambda = 3$.

ie $AX_2 = 3X_2$

ie $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3y_1 \\ 3y_2 \\ 3y_3 \end{pmatrix}$ where $X_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

ie $y_1 + y_2 + 3y_3 = 3y_1$ ie $-2y_1 + y_2 + 3y_3 = 0$ (1)

$y_1 + 5y_2 + 3y_3 = 3y_2$ $y_1 + 2y_2 + y_3 = 0$ (2)

$3y_1 + y_2 + 3y_3 = 3y_3$ $3y_1 + y_2 - 2y_3 = 0$ (3)

Let us eliminate y_1 from (1) & (2)

(1)×1 is $-2y_1 + y_2 + 3y_3 = 0$

(2)×2 is $\underline{2y_1 + 4y_2 + 2y_3 = 0}$

adding $5y_2 + 5y_3 = 0$

$\Rightarrow y_2 + y_3 = 0$ ie $y_2 = -y_3 \Rightarrow \dfrac{y_2}{-1} = \dfrac{y_3}{1}$ (4)

Let us eliminate y_2 from (2) & (3)

(2)×1 is $y_1 + 2y_2 + y_3 = 0$

(3)×2 is $\underline{6y_1 + 2y_2 - 4y_3 = 0}$

subtracting $-5y_2 + 5y_3 = 0$

$\Rightarrow 5y_1 = 5y_3 \Rightarrow y_1 = y_3$ $\therefore \dfrac{y_1}{1} = \dfrac{y_3}{1}$ (5)

From (4) & (5) $\frac{y_1}{1} = \frac{y_2}{-1} = \frac{y_3}{1} \quad \therefore \; X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Next, let X_3 be the eigen vector for $\lambda = 6$

ie $AX_3 = 6X_3$

ie $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 6z_1 \\ 6z_2 \\ 6z_3 \end{pmatrix}$ where $X_3 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

ie $z_1 + z_2 + 3z_3 = 6z_1$ ie $-5z_1 + z_2 + 3z_3 = 0$ (1)

$z_1 + 5z_2 + z_3 = 6z_2$ $z_1 - z_2 + z_3 = 0$ (2)

$3z_1 + z_2 + z_3 = 6z_3$ $3z_1 + z_2 - 5z_3 = 0$ (3)

adding (1) & (2), $-4z_1 + 4z_3 = 0$

ie $z_1 = z_3 \;\therefore\; \frac{z_1}{1} = \frac{z_3}{1}$ (4)

Let us eliminate z_3 from (2) & (3)

$(2)\times 5$ is $5z_1 - 5z_2 + 5z_3 = 0$

$(3)\times 1$ is $\underline{3z_1 + z_2 + -5y_3 = 0}$

adding $8z_1 - 4z_2 = 0$

ie $2z_1 = z_2 \Rightarrow \frac{z_1}{1} = \frac{z_2}{2}$ (5)

From (4) & (5), $\frac{z_1}{1} = \frac{z_2}{2} = \frac{z_3}{1} \quad \therefore \; X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Let $P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ Then $P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$\therefore \; P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ diagonalize $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Exercise

1. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}$.

2. Evaluate $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

3. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{vmatrix}$.

4. If $\begin{vmatrix} 2 & 0 & 4 \\ 6 & x & 5 \\ -1 & -3 & 1 \end{vmatrix} = 0$, then find x.

5. Evaluate $\begin{vmatrix} 1 & p & -q \\ -p & 1 & r \\ q & -r & 1 \end{vmatrix}$.

6. Find the adjoint of $\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$.

7. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{pmatrix}$ find the co-factor of 1.

8. If $A = \begin{pmatrix} 2 & 1 & 5 \\ 0 & 3 & 7 \end{pmatrix}$ find AA' & $A'A$.

9. Find the inverse of $\begin{pmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{pmatrix}$.

10. If $A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 0 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 2 \end{pmatrix}$ find $5A - 3B$ and $6B - 7A$.

11. If $3A + B = \begin{pmatrix} 4 & -2 & 5 \\ 3 & 7 & 6 \end{pmatrix}$ and $2B + A = \begin{pmatrix} 2 & 0 & 4 \\ -1 & 2 & 3 \end{pmatrix}$ find A and B.

12. If $\begin{pmatrix} 1+x & x \\ 5 & x \end{pmatrix} + \begin{pmatrix} x & y \\ 7 & -y \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 12 & 4x \end{pmatrix}$ find x and y.

13. If $A = \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}$ verify that $(A')' = A$.

14. If $A = \begin{pmatrix} -3 & 4 & 5 \\ 6 & -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -4 & 3 \\ 3 & -2 \end{pmatrix}$ find $A + B'$ and $A' - B$.

15. If $A = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$ prove that $A^2 - 10A + I = 0$.

16. Find the inverse of $\begin{pmatrix} 6 & 5 \\ 3 & 2 \end{pmatrix}$.

17. Find the characteristic equation of $\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$.

18. Find the eigen values of $\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$.

19. Solve $2x+z=-1$, $2y+x=5$, $z-y=-2$ by Cramer's Rule.

20. Solve
$$5x-y+4z=5$$
$$2x+3y+5z=2$$
$$7x-2y+6z=5$$
by matrix method.

21. Find the characteristic roots of $\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$.

22. Find the characteristic roots of $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$.

23. Verify Cayley - Hamilton Theorem for the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

24. Verify Cayley - Hamilton Theorem for the matrix $\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$.

25. Find the eigen vectors for the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

26. Find the eigen values and eigen vectors for the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

ALGEBRAIC STRUCTURES

Abreviations used

N : represent set of natural numbers.

Z or I : represent set of +ve and –ve integers including zero.

Z^+ : represent non-negative integers ie. +ve integers including zero.

Q : represent set of rational numbers.

R : represent set of real numbers.

C : represent set of complex numbers.

$Z_n = \{0, 1, 2, 3, ... n-1\}$ ie. Z_n represent set of integers modulo n.

Q^+ : represent set of +ve rational numbers.

$z - \{o\}$: represent set of integers except 0.

$Q - \{o\}$: represent set of rational numbers except zero.

$R - \{o\}$: set of real numbers except zero.

A set in general is denoted by S.

$\forall$: for all

$\in$: belongs to

Binary Operation

If S is a non-empty set then a mapping (function) from $S \times S$ to S is defined as Binary Operation (in short B.O.) and denoted by $*$ (read as star). ie. : $S \times S \to S$ (Star maps S cross S to S)

Another Definition

If S is non-empty set then $*$ (star) is said to be a Binary operation if $\forall\ a, b \in S, a * b \in S$.

Examples

(1) on N + and $\times$ (ie addition & multiplication) are B.O.

$2+3=5 \in N \qquad 2\times 3 = 6 \in N$

(2) on Z, +, – & $\times$ are B.O.

$4+5 \in Z,\ 3-4=1 \in Z, \qquad 4-3=1 \in Z,\ 5\times 6 = 30 \in Z$

(3) On Q & R +, – & $\times$ are B.O. but $\div$ is not a B.O. on Q & R $\because$ for 0, $\frac{a}{0} \notin Q \ \& \ R$ but on $Q - \{o\}$ & $R - \{o\}$ $\div$ is a B.O.

(4) on C, + and $\times$ are B.O.

$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \in C$

$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \in C$

Definitions

(1) A non-empty set S with one or more binary operations is called an **'Algebraic Structure'.**
$(N, +)$, $(Z, +, \times)$, $(Q, +, \times)$ are all algebraic structures.

(2) **Closure Law :** A set S is said to be closed under a B.O. $*$ if $\forall\, a, b \in S$, $a * b \in S$.

(3) **Associative Law :** A B.O. $*$ is said to be associative on S if $\forall\, a, b, c \in S$
$(a * b) * c = a * (b * c)$

(4) **Commutative Law :** A B.O. $*$ is said to be commutative on S if $\forall\, a, b \in S$, $a * b = b * a$.

(5) **Identity Law :** An element $e \in S$ satisfying
$a * e = a = e * a$. $\forall\, a \in S$ is called an identity element for the B.O. $*$ on S.

<u>Examples</u>

(i) $+$ and $\times$ (addition and multiplication) are associative and commutative on N, Z, Q & R.

(ii) B.O. – (subtraction) is not associative & commutative.

(iii) 1 is an identity for B.O. $\times$ on N but $+$ has no identity on N. Where as O is an identity on Z, Q and R for the B.O. $+$.

(iv) If S is a set of 2×2 matrices and B.O. is matrix multiplication then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity element.

Group

A non-empty set G together with a B.O. $*$ ie $(G, *)$ is said to form a group if the following axioms are satisfied.

G_1. **Closure Law :** $\forall\, a, b \in G$, $a * b \in G$

G_2. **Associative Law :** $\forall\, a, b, c \in G$, $(a * b) * c = a * (b * c)$

G_3. **Identity Law :** There exists an element $e \in G$ such that $\forall\, a \in G$, $a * e = a = e * a$.

G_4. **Inverse :** $\forall\, a \in G$, there exists an element b such that $a * b = e = b * a$. This b is called inverse of a and usually denoted as a^{-1}

ie. $a * a^{-1} = e = a^{-1} * a$.

In addition to the above four axions if $\forall\, a, b \in G$, $a * b = b * a$. Then $(G, *)$ is called an **'abelian group'** or **'commutative group'.**

Note (1) If for $(G, *)$ only G_1 is satisfied it is called a **'groupoid'.**

(2) If for $(G, *)$ G_1 & G_2 are satisfied it is called a **'semi-group'.**

(3) If for $(G, *)$ G_1, G_2 & G_3 are satisfied it is called **'Monoid'.**

<u>Examples</u>

(i) $(N, +)$ is a groupoid and semigroup.

(ii) $(N, \times)$ is a groupoid, semigroup and Monoid (identity for $\times$ is 1)

(iii) $(Z, +)$ is a group (identity is O and $a^{-1} = -a$) ie. $a + (-a) = 0 = -a + a$.

<u>Note</u> :- Every group is a monoid but the converse is not true, $(Z, +)$ is a group and also a monoid but $(N, \times)$ is a monoid but not a group.

Properties of Groups

1. Cancellation laws are valid in a group

   ie if $(G, *)$ is a group then $\forall\, a, b, c \in G,$

   (i) $a*b = a*c \Rightarrow b = c$ (left cancellation law)

   (ii) $b*a = c*a \Rightarrow b = c$ (right cancellation law)

   Proof :- $a*b = a*c,$ as $a \in G,$ $a^{-1} \in G$

   $\therefore a^{-1}*(a*b) = a^{-1}*(a*c)$

   ie $(a^{-1}*a)*b = (a^{-1}*a)*c$

   ie $e*b = e*c$ where e is the identity.

   $\Rightarrow b = c$

   Similarly by considering

   $(b*a)*a^{-1} = (c*a)*a^{-1}$

   we get $b = c$

2. In a group G, the equations $a*x = b$ and $y*a = b$ have unique solutions, $\forall a, b \in G.$

   Proof :- $a*x = b$

   Operating on both sides by a^{-1}

   $a^{-1}*(a*x) = a^{-1}*b$

   ie $(a^{-1}*a)*x = a^{-1}*b$

   ie $e*x = a^{-1}*b$

   $\therefore$ $x = a^{-1}*b$

   To prove that the solution is unique, let x_1 & x_2 be two solutions of $a*x = b.$

   ie $a*x_1 = b$ **&** $a*x_2 = b$

   $\Rightarrow a*x_1 = a*x_2$

   Operating on both sides by a^{-1}

   We get $a^{-1}*(a*x_1) = a^{-1}*(a*x_2)$

   ie $(a^{-1}*a)*x_1 = (a^{-1}*a)*x_2$

   ie $e*x_1 = e*x_2$

   $\Rightarrow x_1 = x_2$

   $\therefore$ solution is unique.

3. In a group the identity element and inverse of an element are unique.

   Proof :- To prove identity is unique. If possible let e_1 & e_2 are two identities then

   $\forall a \in G,$ $a*e_1 = a = e_1*a$ (1)

   & $a*e_2 = a = e_2*a$ (2)

   From LHS of (1), $a*e_1 = a = e_2*a$ (using (2))

   ie $a*e_1 = e_2*a = a*e_2$ (using LHS of (2)) by left cancellation, $e_1 = e_2$. Thus the identity is unique.

To prove that inverse of an element is unique.

Let $a \in G$, if possible let b & c are inverses of a ie $b^{-1} = a$ and $c^{-1} = a$

Now, $a * b = e$ **&** $a * c = e$

where e is the identity of the group

$\therefore\ a * b = a * c$

by left cancellation law $b = c$. Thus inverse of an element is unique.

4. In a group $G, \left(a^{-1}\right)^{-1} = a\ \forall a \in G$

Proof :- As a^{-1} is the inverse of a

We have $a * a^{-1} = e = a^{-1} * a$

it can be easily seen from above relation that inverse of a^{-1} is a ie $\left(a^{-1}\right)^{-1} = a$.

Note :- If b & c are elements of G, such that $b * c = e = c * b$ then each is the inverse of the other.

5. In a group $(G, *)$

$\forall a, b \in G,\ (a * b)^{-1} = b^{-1} * a^{-1}$

Proof :- Consider $(a * b) * (b^{-1} * a^{-1})$

$$= a * (b * b^{-1}) * a^{-1} = a * e * a^{-1} = a * a^{-1} = e$$

$\therefore\ (a * b)^{-1} = b^{-1} * a^{-1}$

6. A group of order 3 is abelian.

Proof :- Order of group means the number of elements in a group. If a group G has n elements. The order of G is n, which is denoted as $O(G) = n$.

If n is finite it is called finite group and n is Infinite then it is called Infinite group.

Let $G = \{e, a, b\}$ be a group with a binary operation $*$.

e is the identity by definition of identity it commutes with every element $\because a * e = a = e * a$

So we have to prove that $a * b = b * a$

Let $a * b = a \Rightarrow (a^{-1} * a) * b = a^{-1} * a \Rightarrow e * b = e \Rightarrow b = e$

which is not possible.

Let $a * b = b$

$\Rightarrow (a * b) * b^{-1} = b * b^{-1} \Rightarrow a * e = e \Rightarrow a = e$

which is not possible.

$a * b$ cannote be equal to a or b $\therefore\ a * b = e$

Similarly we can prove that

$b * a = e$

$\therefore\ a * b = b * a\ \therefore\ (G, *)$ is abelian.

Subgroups

A non-empty subset H of a group G is said to form a subgroup with respect to the same binary operation $*$ if $(H, *)$ is a group.

Eg. (1) $(z, +)$ is a subgroup of $(Q, +)$

(2) $H = \{0, 2, 4\}$ is a subgroup of $G = \{0, 1, 2, 3, 4, 5\}$ with B.O. $+\mathbf{mod}6$ ie $\oplus_6$

(3) $H = \{-1, 1\}$ is a subgroup of $G = \{1, -1, i, -i\}$ with respect to the B.O. multiplication.

Theorem

A non-empty subset H of a group G is a subgroup of G if and only if $\forall a, b \in H, ab^{-1} \in H.$

Proof :- **case (i)** Let H be a subgroup of G then H is a group

$\therefore\ \forall a, b \in G, b^{-1} \in G$ (inverse axion) & $ab^{-1} \in G$ (closure law)

$\therefore$ condition is satisfied.

case (ii) Let H be a non-empty subset of G with the property $\forall a, b \in H, ab^{-1} \in H.$

We have to prove that H is a subgroup.

Let $a \in H, a, a^{-1} \in H \Rightarrow aa^{-1} = e \in H$ **&** $e, a \in H \Rightarrow ea^{-1} \in H$ ie $a^{-1} \in H$

Since all elements of H are elements of G, associative property is satisfied.

Let $b \in H, b^{-1} \in H$ **&** for $a \in H, a\left(b^{-1}\right)^{-1}$ ie $ab \in H$ ie closure property is satisfied.

Hence H is a group and hence a subgroup.

Permutation group

Let $S = \{a_1, a_2, a_3, \ldots\ldots\ldots\, a_n\}$

Then a one-one and onto mapping or function from S onto itself is called a **Permutation.**

Permutation is denoted as $\begin{pmatrix} a_1 & a_2 & a_3 & \ldots\ldots\ldots\, a_n \\ f(a_1) & f(a_2) & f(a_3) & \ldots\ldots\ldots\, f(a_n) \end{pmatrix}$

There will be n ! ie $\angle n$ permutations the set of permutations is denoted by S_n.

Let $f, g \in S_n$. There is a composite mapping for f & g denoted as $f \circ g$, this can be taken as binary operation. Then the set S_n with binary operation 'O' (ie composite mapping) will form a group. For convenience $f \circ g$ is denoted as gf.

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ **&** $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \in S_4$ the B.O. composite function is given by

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \mathbf{?} & \mathbf{?} & \mathbf{?} & \mathbf{?} \end{pmatrix}$$

to fill up the second row, following is the procedure.

in g **:** $g(1) = 2,\ g(2) = 3,\ g(3) = 4,\ g(4) = 1$

& in f **:** $f(1) = 3,\ f(2) = 1,\ f(3) = 4,\ f(4) = 2$

Now $f \circ g(1) = f[g(1)] = f(2) = 1$

$f \circ g(2) = f[g(2)] = f(3) = 4$

$f \circ g(3) = f[g(3)] = f(4) = 2$

$f \circ g(4) = f[g(4)] = f(1) = 3$

$$\therefore \; f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

for convenience $f \circ g$ is written as gf

$$ie \;\; f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$\& \;\; gf = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

the composite function is also called product function.

Eg. (1) Let $S = \{1, 2\}$

then $S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$

Let B.O. be product permutation

then $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

closure law is satisfied, associative law can be easily verified. inverse $e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$\therefore S_2$ forms a group.

Eg. (2) Let $S = \{1, 2, 3\}$

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

Let us denote the elements as f_1, f_2, f_3, f_4, f_5 **&** f_6 respectively.

ie $S_3 = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

The following is the multiplication table.

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_4	f_3	f_6	f_5
f_3	f_3	f_5	f_1	f_6	f_2	f_4
f_4	f_4	f_6	f_2	f_5	f_1	f_3
f_5	f_5	f_3	f_6	f_1	f_4	f_2
f_6	f_6	f_4	f_5	f_2	f_3	f_1

It can be seen from the table that closure law is satisfied.

For associative law

Consider $(f_3 f_4) f_5 = f_6 f_5 = f_3$

$f_3(f_4 f_5) = f_3 f_1 = f_3$

$\therefore$ $(f_3 f_4) f_5 = f_3(f_4 f_5)$ hence associative law is satisfied.

identity $e = f_1$

$f_1^{-1} = f_1$, $f_2^{-1} = f_2$ $f_3^{-1} = f_3$, $f_4^{-1} = f_5$ $f_5^{-1} = f_4$ **&** $f_6^{-1} = f_6$

inverse of all elements exists.

$\therefore$ S_3 forms a group, but it is not an abelian group

$\because$ $f_3 f_4 = f_6$ but $f_4 f_3 = f_2$ $\therefore$ $f_3 f_4 \neq f_4 f_3$.

Examples

(1) Show that the set R - {o} with B.O. $\times$ forms a group.

Solution : For any elements $a, b \in R$ - {o}. $a * b \in R$ - {o}

2, 3 $\in R$ - {o}, $2 \times 3 = 6 \in R$ - {o}

$\therefore$ closure law is satisfied.

For any three elements $a, b, c \in R$ - {o}

$(a \times b) \times c = a \times (b \times c)$

$(-3 \times 4) \times 5 = -12 \times 5 = -60$.

$-3 \times (4 \times 5) = -3 \times 20 = -60$.

$\therefore$ associative law is satisfied.

Identity element is 1,

ie. $\forall\ a \in R$ - {o}, $a \times 1 = a = 1 \times a$.

Let $a \in R$ - {o} then there exists

$\frac{1}{a} \in R$ - {o} such that $a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$

$\therefore$ inverse exists for all elements R - {o}.

$\therefore$ (R - {o}, $\times$) forms an abelian group.

(2) Show that $(Z_5, \oplus_5)$ forms an abelian group.

Solution : Let us construct table for $(Z_5, \oplus_5)$

$\oplus_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

From the above table it can be easily seen that closure law is satisfied.

$2 \oplus_5 (3 \oplus_5 4) = 2 \oplus_5 2 = 4$

$(2 \oplus_5 3) \oplus_5 4 = 0 \oplus_5 4 = 4$

ie $2 \oplus_5 (3 \oplus_5 4) = (2 \oplus_5 3) \oplus_5 4$

$\therefore$ associative law can be versified.

identity element is 0.

inverse of 0 is 0

inverse of 1 is 4

inverse of 2 is 3

inverse of 3 is 2

inverse of 4 is 1

$\therefore$ $(Z_5, \oplus_5)$ form a group, further it can be seen from the table that for any two element $a, b \in Z_5$

$a \oplus_5 b = b \oplus_5 a$

$\therefore$ $(Z_5, \oplus_5)$ is an abelian group.

(3) Show that $G = \{1, 2, 3, 4\}$ with B.O. multiplication mod 5 ie $\otimes_5$ is an abelian group.

Solution : The following in the relevant table for elements

$\otimes_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

From the above table it can be seen that closure law is satisfied.

$(2 \otimes_5 3) \otimes_5 4 = 1 \otimes_5 4 = 4$

$2 \otimes_5 (3 \otimes_5 4) = 2 \otimes_5 2 = 4$

$\therefore$ associate law is satisfied.

identity is 1.

inverse of 1 is 1

inverse of 2 is 3

inverse of 3 is 2

inverse of 4 is 4

$\therefore$ $(G, \otimes_5)$ forms a group and it can be seen from the table that it forms an abelian group.

4. If R is the set of real numbers and $*$ is a binary operation defined on R as $x * y = 1 + xy, \forall\, x, y \in R.$ Show that $*$ is commutative but not associative.

Commutative property $a * b = b * a, \forall\, a, b \in R$

$x * y = 1 + xy$

$y * x = 1 + yx \quad \therefore \ x * y = y * x$

Associative property $(x * y) * z = x * (y * z)$

LHS $= (1+xy) * z = P * z = 1 + pz = 1 + (1+xy)z = 1 + z + xyz$

RHS $= x * (1+yz) = x * Q = 1 + xQ = 1 + x(1+yz) = 1 + x + xyz$

$\therefore$ LHS $\neq$ RHS $\therefore$ $*$ is not a associative

5. Show that set of integers with $a * b = a - b$ where $a, b \in I$ is not a group

$a * b = a - b \in I$ $\therefore$ closure axiom is satisfied

$(a * b) * c \neq a * (b * c)$, $a * b = a - b \in I$, $\forall\, a, b \in I$.

LHS $= (a - b) * c = a - b - c$

RHS $= a * (b - c) = a - b + c$ $\therefore$ associative axiom is not satisfied

$\therefore$ $(I, *)$ is not a group.

When one of the axiom is not satisfied, it is not a group. Hence, we need not have to check the rest of the axioms.

6. In a group $(G, *), a * b = \dfrac{ab}{2}$. Find the identity element, inverse of 4 and solve $4 * x = 5$

To find $e: a * e = a = e * a$

$a * e = \dfrac{ae}{2} = a \ \therefore e = 2$ i.e. identity element is 2.

$a * a^{-1} = e = a^{-1} * a$

$a * a^{-1} = 2 \Rightarrow \dfrac{aa^{-1}}{2} = 2 \ \therefore a^{-1} = \dfrac{4}{a}$

$4^{-1} = \dfrac{4}{4} = 1.$ $\therefore$ inverse of 4 is 1.

$4 * x = 5 \Rightarrow \dfrac{4x}{2} = 5 \ \therefore x = \dfrac{10}{4} = \dfrac{5}{2}$

7. Prove that $G = \left\{\cos\theta + i\sin\theta \mid \theta \text{ is real}\right\}$ is an abelian group under multiplication

$G = \left\{\cos\theta + i\sin\theta \mid \theta \text{ is real}\right\}$

Let x, y, z be any three elements of G.

Take $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$.

where α, β, γ are real numbers.

i) $xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta) \in G \ \because (\alpha+\beta)$ is real

$\therefore$ Closure axiom is satisfied

ii) $(xy)z = [(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)](\cos\gamma + i\sin\gamma)$

$= \{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\}(\cos\gamma + i\sin\gamma)$

$= \cos\{(\alpha+\beta)+\gamma\} + i\sin\{(\alpha+\beta)+\gamma\}$

$= \cos\{\alpha+(\beta+\gamma)\} + i\sin\{\alpha+(\beta+\gamma)\}$

$= x\,(yz)$ $\because$ multiplication is associative on R $\therefore$ Associative axiom is satisfied.

iii) $1 = \cos 0 + i\sin 0 \in G$ is the identity element

iv) $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1 \Rightarrow \cos\theta - i\sin\theta$ is the multiplicative inverse of $\cos\theta + i\sin\theta$.

v) $xy = (\cos\theta + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

$= \cos(\beta+\alpha) + i\sin(\beta+\alpha)$ $\because$ is commutative on R

$= yx$

$\therefore$ Commutative law is satisfied.

So all the axioms are satisfied. Hence, G is an abelian group under multiplication.

8. Show that the cube roots of unity form an abelian group under multiplication

We know that the cube roots of unity are $1, \omega$ and ω^2. Let $G = \{1, \omega, \omega^2\}$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

here $\omega^3 = 1$

& $\omega^4 = \omega^3 \cdot \omega = \omega$.

i) All the entries in the table are the same as the elements of the set. This means the closure law is satisfied.

ii) Consider $1\cdot(\omega\cdot\omega^2) = 1\cdot 1 = 1$

$(1\cdot\omega)\cdot\omega^2 = \omega\cdot\omega^2 = 1$

$\therefore$ $1\cdot(\omega\cdot\omega^2) = (1\cdot\omega)\cdot\omega^2$. Associative law is satisfied.

iii) The row heading 1 is the same as the topmost row.

$\therefore$ 1 is the identity element.

iv) Inverse of 1 is 1, inverse of ω is ω^2 and inverse of ω^2 is ω.

Every element has a inverse.

v) The table is symmetrical about the principal diagonal

$\therefore$ commutative law holds good.

So G is an abelian group under multiplication.

9. Show that the four matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ form an abelian group under matrix multiplication.

Take $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = A, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = C;\ G = \{1, A, B, C\}$

$IA = AI = A,\ IB = BI = B,\ IC = CI = C$

$$AB = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1+0 & 0+0 \\ 0+0 & 0-1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = C$$

Similarly, it can be shown that $BA = C,\ AC = CA = B,\ BC = CB = A,\ A\cdot A = I$ etc.

The composition table is

.	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

i) The entries in the table are the same as the elements of the set G. $\therefore$ Closure law is satisfied.

ii) $A(BC) = A(A) = I;\ (AB)C = (C)C = I$ $\therefore$ Associative law is satisfied.

iii) I is the identity element.

iv) Inverses of I, A, B are respectively $I, A, B, C.$ $\therefore G$ is a group under matrix multiplication $\therefore (G, \cdot)$ is a group.

v) Since the entries on either side of the leading diagonal are symmetric, $(G, \cdot)$ is an abelian group.

10. If every element of a group G has its own inverse, show that G is abelian

Given $a^{-1} = a,\ \forall\, a, b \in G$ (1)

we know that, $\forall\, a, b \in G,\ (ab)^{-1} = b^{-1}a^{-1}$ (2)

from (1), $a^{-1} = a,\ b^{-1} = b$ and $(ab)^{-1} = ab.$ Using these in (2), $ab = ba,\ \forall\, a, b \in G$
$\therefore$ the commutative law is satisfied, so G is abelian.

11. In a group $(G, \cdot)$ if $(ab)^2 = a^2b^2, \forall\, a, b \in G.$ Prove that $(G, \cdot)$ is abelian and conversely.

$(ab)^2 = (ab)(ab) = (aa)(bb)$

$\Rightarrow a[b(ab)] = a[a(bb)]$ using L.C.L.

$\Rightarrow b(ab) = a(bb) \Rightarrow (ba)b = (ab)b$ using RCL $\Rightarrow ba = ab$ $\therefore$ it is abelian

Conversely

If $ab = ba$ pre operating by a we get, $a(ab) = a(ba) \Rightarrow (a \cdot a)b = (ab)a$

post operating by b we get $[(a \cdot a) \cdot b] = [(ab)a]b$

$\Rightarrow (a \cdot a)(b \cdot b) = (ab)(ab) \Rightarrow a^2b^2 = (ab)^2$

12. Given Q_0, the set of non zero rational numbers is a multiplicative group and $H = \{2^n \mid n \in Z\}$ show that H is a subgroup of Q_0 under multiplication.

$H = \{2^n \mid n \in Z\} = \{..2^{-2}, 2^{-1}, 2^{0,} 2^1.....\}$

i) $2^m, 2^n \in H,\ 2^m \cdot 2^n = 2^{m+n} \in H$ $\therefore$ closure law is satisfied

ii) $(2^m \cdot 2^n)2^r = 2^m \cdot (2^n \cdot 2^r),\ \forall\, m, n, r \in z$ ie $(2^{m+n})2^r = 2^m(2^{n+r})$

$2^{(m+n)+r} = 2^{m+(n+r)}$ i.e. $2^{m+n+r} = 2^{m+n+r}$ $\therefore$ Associative law is satisfied.

iii) 2^0 is the identity element

iv) $\forall 2^m$; there exist 2^{-m} such that $2^m \cdot 2^{-m} = 2^0$ $\therefore$ Inverse of 2^m is 2^{-m}.

$\therefore H$ is a group under multiplication and $H \subset Q_0$ ie H is a subgroup of Q_0 under multiplication.

Exercise

1. If $N = \{1, 2, 3, ...\}$, which of the following are binary operation of N.

 (1) $a*b = a+2b$ (2) $a*b = 3a-4b$ (5) $a*b = \frac{a}{b}$

2. Which of the following operations on the given set are binary

 (1) on I, the set of integers, $a*b = 3a-4b$

 (2) on $R, a*b = \sqrt{a^2-b^2}$

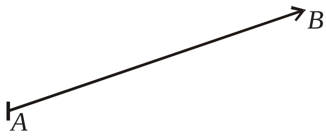
 (3) on $R, a*b = \sqrt{ab}$

3. If $*$ is given by $a*b = a^b$, show that $*$ is not a binary operation of Z.
4. Why the set of rationals does not form a group w.r.t. multiplication ?
5. If x, y, z are any three elements of a group G, find $(xyz)^{-1}$.
6. In a group $G, \forall\, a, b \in G$, find $\left(a^{-1}b^{-1}\right)^{-1}$.
7. If the binary operation $*$ on the set Z is defined by $a*b = a+b+5$, find the identity element.
8. In the group of non zero integers mod 5. Find the multiplicative inverse of 4.
9. If $f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ are permutations in S_3, find $f \circ g$.
10. If $S = \{1, 2, 3, 4, 5, 6\}$ w.r.t. multiplication (mod 7), solve the equation $3x = 5$ in S.
11. Show that $S = \{1, 2, 3\}$ under multiplication (mod 4) is not a group.
12. If $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ find gf.
13. The binary operation $*$ is defined by $a*b = \frac{ab}{7}$, on set of rational numbers, show that $*$ is associative.
14. If $*$ is defined by $a*b = \frac{ab}{2}$, on the set of real numbers, show that $*$ is both commutative and associative.
15. If $*$ is defined by $a*b = \sqrt{a^2+b^2}$, show that $*$ is associative.
16. On the set of real numbers, $R, a*b = a+b-1, \forall\, a, b \in R$. Show that $*$ is associative.
17. On the set of real numbers, R, $*$ is defined by $a*b = 2a-3b+ab$, examine whether $*$ is commutative and associative.
18. In the set of rationals except 1, binary operation $*$ is defined by $a*b = a+b-ab$. Find the identity and inverse of 2.
19. On the set of positive rational numbers $Q^+, a*b = \frac{ab}{4}, \forall\, a, b \in Q^+$. Find the identity element and the inverse of 8.
20. In a group of integers, an operation $*$ is defined by $a*b = a+b-1$. Find the identity element.

Vectors and Scalars

Vector : A physical quantity which has both direction and magnitude is called a 'Vector'.

Eg. Velocity, acceleration, force, weight etc.



Scalar : A physical quantity which has only magnitude and no direction is called a 'Scalar'.

Eg. speed, volume, mass, density, temperature etc.

Vectors are represented by directed line segments. Let AB be the line segment. Vector from A to B is denoted by $\overrightarrow{AB}$ and vector from B to A is denoted by $\overrightarrow{BA}$. $\overrightarrow{AB}$ can also be represented by $\vec{a}$. The length of $\overrightarrow{AB}$ is magnitude of the vector denoted as $\left|\overrightarrow{AB}\right|$ or $\left|\vec{a}\right|$ or simply a. For $\overrightarrow{AB}$, A is the initial point and B the terminal point.

Scalar multiplication of a vector

Let λ be a scalar and $\vec{a}$ a vector then $\lambda\vec{a}$ represents a vector whose magnitude is λ times the magnitude of $\vec{a}$ and the direction is same as that of $\vec{a}$ if λ is positive but opposite to that of $\vec{a}$ if λ is negative. If $\lambda = 0$ it represents a null vector denoted by '$\vec{0}$' ie a **null vector** is a vector of magnitude zero but its direction is arbitrary.

A vector whose magnitude is 1 is called a **unit vector** and a unit vector in the direction of $\vec{a}$ is written as

$\dfrac{\vec{a}}{\left|\vec{a}\right|}$ or $\dfrac{\vec{a}}{a}$ or $\hat{a}$ (read as a cap)

Like and unlike vectors

Vectors having same direction are called 'like vectors' and those having the opposite direction are called 'unlike vectors'.

Co-initial vectors : Vectors having the same initial point are called co-initial vectors.

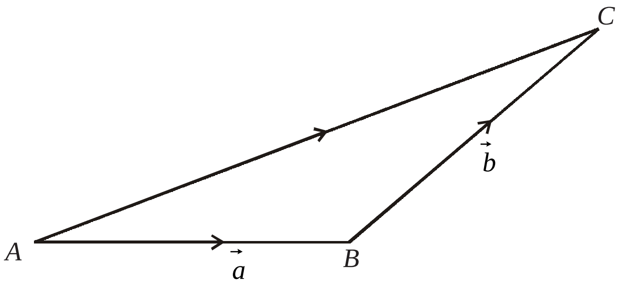
Coplanar vectors : Vectors in the same plane are called 'coplanar vectors'.

Parallel vectors : Vectors having same direction but different initial points are called 'parallel vectors'.

Triangle Law for addition of vectors

Let $\overrightarrow{AB}$ & $\overrightarrow{BC}$ represents two vectors then $\overrightarrow{AC}$ represents

$\overrightarrow{AB} + \overrightarrow{BC}$ ie $\vec{a} + \vec{b}$



Parallelogram Law

Let $\overrightarrow{OA} = \vec{a}$ **&** $\overrightarrow{OB} = \vec{b}$

Complete the parallelogram $OACB$.

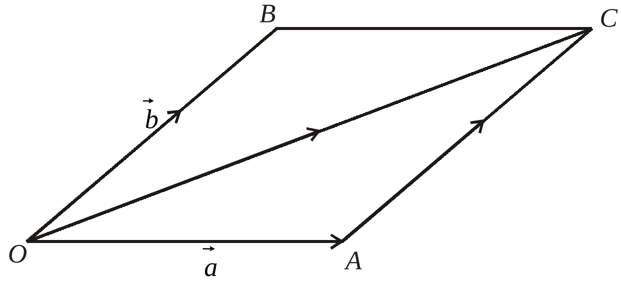
then $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ by triangle law

but $\overrightarrow{AC} = \overrightarrow{OB}$ (parallel vectors)

$\therefore\ \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = \vec{a} + \vec{b}$

Note : - $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

& $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$



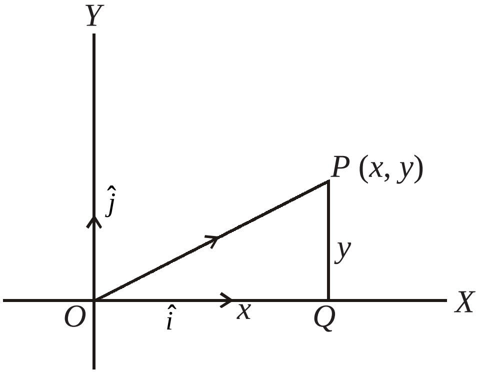
Properties

(i) Vector addition is commutative ie $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

(ii) Vector addition is associative ie $\vec{a}+\left(\vec{b}+\vec{c}\right)=\left(\vec{a}+\vec{b}\right)+\vec{c}$.

(iii) Set of vectors V, with binary operation vector addition will form a **'Group'**. The identity being $\vec{0}$ (null vector) and inverse of $\vec{a}$ is $-\vec{a}$.

Position vectors



(i) Let P be a point in a plane where O is the origin and OX & OY are co-ordinate axes. $\overrightarrow{OP}$ is called position vectors of P.

Draw $PQ \perp^r$ to OX, then $OQ = x$ **&** $QP = y$

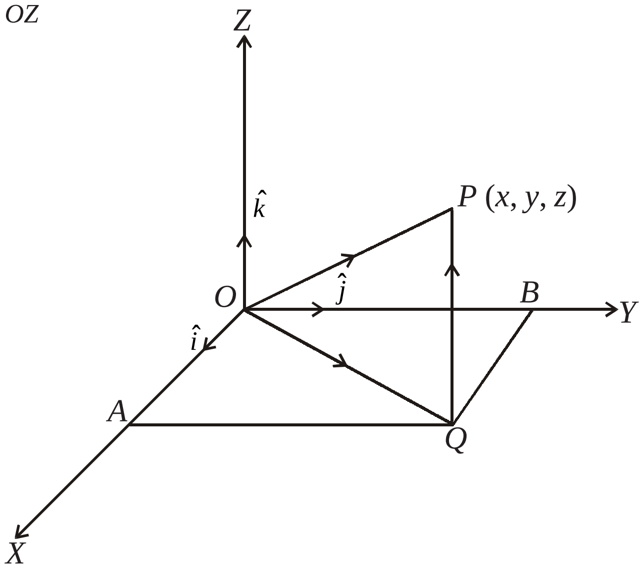
Let $\hat{i}$ **&** $\hat{j}$ represents unit vectors in the direction of OX & OY. Then $\overrightarrow{OQ} = x\hat{i}$, $\overrightarrow{QP} = y\hat{j}$

$\therefore\ \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP} = x\hat{i} + y\hat{j}$

$\left|\overrightarrow{OP}\right| = \sqrt{x^2 + y^2}$

Note :- A plane vector is an ordered pair of real numbers and the distance between O & P is the magnitude of $\overrightarrow{OP}$.

(ii) Let P be a point in three dimensional space where OX, OY & OZ are co-ordinate axes. Let (x, y, z) be the co-ordinates of P. Draw $PQ \perp^r$ to the plane XOY & QA & QB parallel to OY & OX respectives to meet OX at A & OY at B.



Let $\hat{i}$, $\hat{j}$ **&** $\hat{k}$ be unit vectors in the direction of OX, OY & OZ

Then $\overrightarrow{OA} = x\hat{i}$, $\overrightarrow{OB} = y\hat{j}$, $\overrightarrow{QP} = z\hat{k}$

$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{OB}$ (by parallelogram law)

$= x\hat{i} + y\hat{j}$

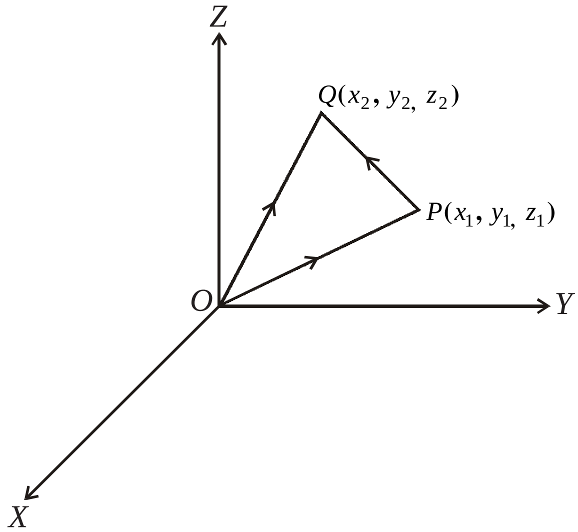
$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$ (by triangle law)

$= x\hat{i} + y\hat{j} + z\hat{k}$

$\left|\overrightarrow{OP}\right| = \sqrt{x^2 + y^2 + z^2}$

This position vector $\overrightarrow{OP}$ of the point P is usually denoted by $\vec{r}$ ie $\overrightarrow{OP} = \vec{r}$

unit vector in the direction of $\overrightarrow{OP}$ is given by $\dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$



Let $P(x_1, y_1, z_1)$ & $P(x_2, y_2, z_2)$ be any two points in 3-space to find $\overrightarrow{PQ}$, from the triangle OPQ

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

$$\therefore\ \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k} - x_1\hat{i} - y_1\hat{j} - z_1\hat{k}$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

and unit vector in the direction of

$$\overrightarrow{PQ} \text{ is } \frac{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Scalar product of two vectors

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two non-zero vectors, then $a_1b_1 + a_2b_2 + a_3b_3$ is defined as **'scalar product'** of two vectors $\vec{a}$ & $\vec{b}$, denoted as $\vec{a}\cdot\vec{b}$ also known as **'dot product'.**

Vector product of two vectors

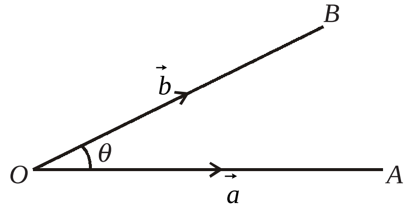
If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two non-zero vectors, then a vector

$$(a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

ie $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ is defined as 'vector product' of two vector $\vec{a}$ & $\vec{b}$, denoted as $\vec{a}\times\vec{b}$ also known as **cross product.**

$$\text{ie } \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \sum (a_2b_3 - a_3b_2)\hat{i}$$

<u>Geometrical Meaning</u> of $\vec{a}\cdot\vec{b}$ and $\vec{a}\times\vec{b}$



Let $\overrightarrow{OA} = \vec{a}$ & $\overrightarrow{OB} = \vec{b}$ and let $A\hat{O}B = \theta$

then $\vec{a}\cdot\vec{b} = ab\cos\theta$

where $a = |\vec{a}|$ & $b = |\vec{b}|$

when $\theta = 0°$, $\vec{a}\cdot\vec{b} = ab\cos 0° = ab$

when $\theta = 90°$, $\vec{a}\cdot\vec{b} = ab\cos 90° = 0$

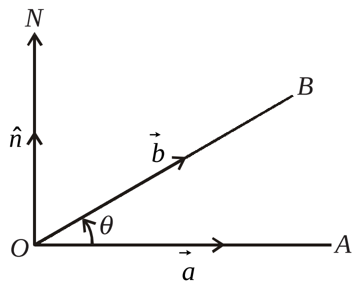
hence two vectors are said to be **'orthogonal'** if $\vec{a}\cdot\vec{b} = 0$.

Let ON represent a line which is $\perp^r$ to both OA & OB. ie ON is $\perp^r$ to the plane OAB. Let $\hat{n}$ represent a unit vector in the direction of ON.

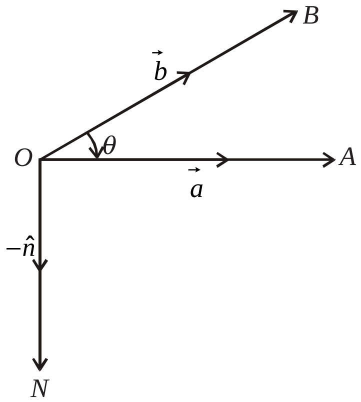
Then $ab\sin\theta\,\hat{n}$ is $\vec{a}\times\vec{b}$ and $-ab\sin\theta\,\hat{n}$ is $\vec{b}\times\vec{a}$ if $\theta = 0°$, then $\sin\theta = 0$

$\therefore\ \vec{a}\times\vec{b}$ or $-\vec{b}\times\vec{a}$ is a null vector, hence two vectors are said to be parallel or coincident if $\vec{a}\times\vec{b} = \vec{0}$ (null vector)

Note :-



$\vec{a} \times \vec{b} = ab\,\mathbf{sin}\,\theta\hat{n}$ represent anti - clockwise rotation.

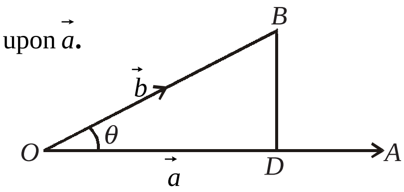


$\vec{b} \times \vec{a} = -ab\,\mathbf{sin}\,\theta\hat{n}$ represent clockwise rotation.

also for unit vectors $\hat{i}, \hat{j}$ **&** $\hat{k}$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{k} \times \hat{i} = \hat{j},\ \hat{j} \times \hat{k} = \hat{i}\ \ \&\ \ \hat{j} \times \hat{i} = -\hat{k},\ \hat{i} \times \hat{k} = -\hat{j},\ \hat{k} \times \hat{j} = -\hat{i}.$$

Projection of $\vec{b}$ upon $\vec{a}$



Let $\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b}$ **&** $\hat{A}OB = \theta$**.** Draw $BD \perp^r$ to OA then OD is the projection of $\vec{b}$ upon $\vec{a}$**.**

Since $\vec{a} \cdot \vec{b} = ab\,\mathbf{cos}\,\theta,\ \mathbf{cos}\,\theta = \dfrac{\vec{a} \cdot \vec{b}}{ab}$

from the Δ^{le} OBD $\mathbf{cos}\,\theta = \dfrac{OD}{OB} = \dfrac{OD}{b}$

$$\therefore\ OD = b\,\mathbf{cos}\,\theta = b \cdot \frac{\vec{a} \cdot \vec{b}}{ab}$$

$$= \frac{\vec{a}}{a} \cdot \vec{b} = \hat{a} \cdot \vec{b}$$

further area of parallelogram whose adjacent sides are OA & OB is given by

$$OA \cdot BD = ab\,\mathbf{sin}\,\theta\ \ \because \frac{BD}{OB} = \mathbf{sin}\,\theta\ \text{ but }\ \left|\vec{a} \times \vec{b}\right| = ab\,\mathbf{sin}\,\theta.$$

$\therefore$ Area of parallelogram whose co-terminus edges are $\vec{a}$ **&** $\vec{b}$ is given by $\left|\vec{a} \times \vec{b}\right|$

and therefore area of Δ^{le} $OAB = \dfrac{1}{2}\left|\vec{a} \times \vec{b}\right|$.

Also area of the parallelogram whose diagonals are $\overrightarrow{d_1}$ **&** $\overrightarrow{d_2}$ is $\dfrac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right|$

Scalar Triple Product

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\ \ \&\ \ \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Then $\vec{a}\cdot(\vec{b}\times\vec{c})$ or $(\vec{a}\times\vec{b})\cdot\vec{c}$ is called 'Scalar Triple Product'

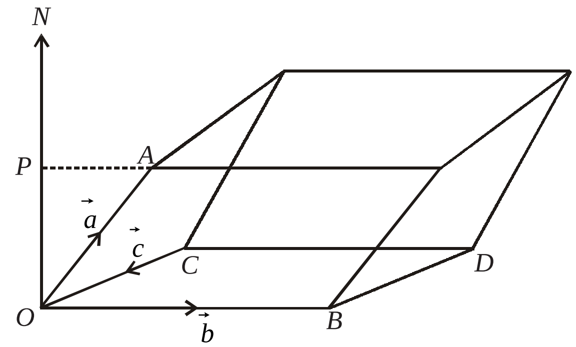
both on computation gives $a_1b_2c_3 - a_1b_3c_1 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$

which is $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Thus $\vec{a}\cdot(\vec{b}\times\vec{c})$ or $(\vec{a}\times\vec{b})\cdot\vec{c}$

which is **'Scalar Triple Product'** which is usually denoted as $[\,\vec{a}\,\vec{b}\,\vec{c}\,]$ also called **'Box-Product'.**

Geometrical Meaning of $[\,\vec{a}\,\vec{b}\,\vec{c}\,]$



Consider a parallelopiped whose co-terminus edges are $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ ie $\vec{a}, \vec{b}, \vec{c}$

Area of parallelogram $OBDC$ is $|\vec{b}\times\vec{c}|$

Let OP be the projection of $\overrightarrow{OA}$ ie $\vec{a}$ upon ON which is $\perp^r$ to the plane $OBDC$.

$$\therefore\ OP = \frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}\cdot\vec{a} = \frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}$$

Volume of parallelopiped = area of parallelogram $\times\ OP$

$$= |\vec{b}\times\vec{c}|\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}$$

$$= \vec{a}\cdot(\vec{b}\times\vec{c}) = [\,\vec{a}\,\vec{b}\,\vec{c}\,]$$

$\therefore$ Volume of parallelopiped whose coterminus edges are given by $\vec{a}, \vec{b}, \vec{c}$ is

$$[\,\vec{a}\,\vec{b}\,\vec{c}\,] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If $[\,\vec{a}\,\vec{b}\,\vec{c}\,] = 0$ then the vectors $\vec{a}, \vec{b}$ **&** $\vec{c}$ are coplanar.

Vector Triple Product

If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors then

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

also $$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$$

Examples

1. If $\vec{a} = (2, 5), \vec{b} = (-2, 3)$ find $\vec{a} \cdot \vec{b}$.

Solution : $\vec{a} \cdot \vec{b} = -4 + 15 = 11$

2. If $\vec{\alpha} = \hat{i} + 2\hat{j} - 3\hat{k},\ \vec{\beta} = 3\hat{i} - \hat{j} - 2\hat{k}$, find $\left(2\vec{\alpha} + \vec{\beta}\right) \cdot \left(\vec{\alpha} + 2\vec{\beta}\right)$

Solution : $\left(2\vec{\alpha} + \vec{\beta}\right) \cdot \left(\vec{\alpha} + 2\vec{\beta}\right) = (5\hat{i} + 3\hat{j} - 8\hat{k}) \cdot (7\hat{i} - 7\hat{k}) = 35 + 0 + 56 = 91$

3. Prove that the vectors $\vec{a} = 3\hat{i} - 2\hat{j} - \hat{k},\ \vec{b} = 2\hat{i} + \hat{j} - 4\hat{k}$ are perpendicular to each other.

Solution : $\vec{a} \cdot \vec{b} = 6 - 2 - 4 = 0 \Rightarrow \vec{a}$ **&** $\vec{b}$ are perpendicular to each other.

4. Find m such that $3\hat{i} + m\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} - 8\hat{k}$ are orthogonal.

Solution : $(3\hat{i} + m\hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} - 8\hat{k}) = 0 \Rightarrow 6 - m - 8 = 0 \Rightarrow -m = -2$

5. If $\left|\vec{a} + \vec{b}\right| = \left|\vec{a} - \vec{b}\right|$. Show that $\vec{a}$ and $\vec{b}$ are orthogonal to each other.

Solution : Given $\left|\vec{a} + \vec{b}\right| = \left|\vec{a} - \vec{b}\right|$

Squaring both the sides, $\left(\vec{a} + \vec{b}\right) \cdot \left(\vec{a} + \vec{b}\right) = \left(\vec{a} - \vec{b}\right) \cdot \left(\vec{a} - \vec{b}\right)$

$\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$ i.e. $2\left(\vec{a} \cdot \vec{b}\right) = 0$

$\therefore\ \vec{a} \cdot \vec{b} = 0\ \therefore\ \vec{a}$ is perpendicular to $\vec{b}$.

6. Find the projection of $\vec{a} = \hat{i} + 3\hat{j} + 5\hat{k}$ on $\vec{b} = -3\hat{i} + \hat{j} + \hat{k}$.

Solution : Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{\left|\vec{b}\right|} = \dfrac{-3 + 3 + 5}{\sqrt{9 + 1 + 1}} = \dfrac{5}{\sqrt{11}}$.

7. If $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} + 2\hat{k}$, find the projection of $\vec{a} + \vec{c}$ on $\vec{b}$.

Solution : Projection of $\left(\vec{a} + \vec{c}\right)$ on $\vec{b} = \dfrac{\left(\vec{a} + \vec{c}\right) \cdot \vec{b}}{\left|\vec{b}\right|} = \dfrac{\left(5\hat{i} - 3\hat{j} + 3\hat{k}\right) \cdot \left(\hat{i} - 2\hat{j} + 2\hat{k}\right)}{\sqrt{1 + 4 + 4}} = \dfrac{5 + 6 + 6}{3} = \dfrac{17}{3}$

8. Find the cosine of the angle between vectors $\vec{a} = 4\hat{i} - 3\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$

Solution: $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|} = \dfrac{8 - 3 - 3}{\sqrt{16 + 9 + 9}\sqrt{4 + 1 + 1}} = \dfrac{2}{\sqrt{34}\sqrt{6}}$

9. Find the cross product of vectors $\vec{a} = 9\hat{i} - \hat{j} + 4\hat{k},\ \vec{b} = 8\hat{i} - \hat{j} + \hat{k}$

Solution: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 9 & -1 & 4 \\ 8 & -1 & 1 \end{vmatrix} = \hat{i}(-1 + 4) - \hat{j}(9 - 32) + \hat{k}(-9 + 8) = 3\hat{i} + 23\hat{j} - \hat{k}$

10. If $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, then find $\left|\vec{a} \times \vec{b}\right|$

Solution : $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 2\\ 2 & 1 & -2\end{vmatrix}=\hat{i}(-4-2)-\hat{j}(-2-4)+\hat{k}(1-4)=-6\hat{i}+6\hat{j}-3\hat{k}$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{36+36+9}=\sqrt{81}=9$$

11. Find the unit vector perpendicular to the pair of vectors $\vec{a}=6\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b}=3\hat{i}+\hat{j}-2\hat{k}$

Solution : Vector perpendicular to $\vec{a}$ **&** $\vec{b}$ is $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 6 & -2 & 1\\ 3 & 1 & -2\end{vmatrix}=\hat{i}(4-1)-\hat{j}(-12-3)+\hat{k}(6+6)=3\hat{i}+15\hat{j}+12\hat{k}$

$$\hat{n}=\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}=\frac{3\hat{i}+15\hat{j}+12\hat{k}}{\sqrt{9+225+144}}=\frac{3\hat{i}+15\hat{j}+12\hat{k}}{\sqrt{378}}$$

12. Prove that $\left(2\vec{a}+\vec{b}\right)\times\left(\vec{a}+2\vec{b}\right)=3\left(\vec{a}\times\vec{b}\right)$

Solution : $\left(2\vec{a}+\vec{b}\right)\times\left(\vec{a}+2\vec{b}\right)=2\left(\vec{a}\times\vec{a}\right)+4\left(\vec{a}\times\vec{b}\right)+\vec{b}\times\vec{a}+2\left(\vec{b}\times\vec{b}\right)=4\left(\vec{a}\times\vec{b}\right)+\vec{b}\times\vec{a}=3\left(\vec{a}\times\vec{b}\right).$

13. If $\vec{a},\vec{b},\vec{c}$ are non zero vectors prove that $\vec{a}\times\left(\vec{b}+\vec{c}\right)+\vec{b}\times\left(\vec{c}+\vec{a}\right)+\vec{c}\times\left(\vec{a}+\vec{b}\right)=\vec{0}$

Solution : $\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}=\vec{0}.$

14. Find the sine of the angle between the vectors $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}-3\hat{j}+2\hat{k}$

Solution : $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 2 & -3 & 2\end{vmatrix}=\hat{i}(2+3)-\hat{j}(2-2)+\hat{k}(-3-2)=5\hat{i}-5\hat{k},\ \left|\vec{a}\times\vec{b}\right|=\sqrt{5^2+5^2},$

$$\therefore\ \mathbf{sin}\,\theta=\frac{\sqrt{5^2+5^2}}{\sqrt{1+1+1}\sqrt{4+9+4}}=\frac{5\sqrt{2}}{\sqrt{3}\sqrt{17}}$$

15. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then show that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

Solution : Given $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\vec{a}\times\left(\vec{a}+\vec{b}+\vec{c}\right)=\vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=\vec{0}\ \therefore\vec{a}\times\vec{b}=-\left(\vec{a}\times\vec{c}\right)\ [\because\ \vec{a}\times\vec{a}=\vec{0}\ ]\ \therefore\vec{a}\times\vec{b}=\vec{c}\times\vec{a}$

Similarly $\vec{b}\times\vec{a}=\vec{c}\times\vec{b}$ or $\vec{c}\times\vec{b}=-\vec{a}\times\vec{b}$ or $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}$

Similarly $\vec{c}\times\vec{a}=\vec{b}\times\vec{c}\ \therefore\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

16. Find the area of the triangle whose sides are $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-3\hat{j}+5\hat{k}$

Solution : $A=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -2 & 1\\ 1 & -3 & 5\end{vmatrix}=\hat{i}(-10+3)-\hat{j}(15-1)+\hat{k}(-9+2)=-7\hat{i}-14\hat{j}-7\hat{k}$$

$$\text{area} = \frac{\sqrt{49+196+49}}{2} = \frac{\sqrt{294}}{2}\text{sq.units}$$

17. Position vectors of the points A, B and C are respectively $\hat{i}-\hat{j}+\hat{k}$, $2\hat{i}+\hat{j}-\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$. Find the area of triangle ABC.

Solution : $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\hat{i}+\hat{j}-\hat{k})-(\hat{i}-\hat{j}+\hat{k}) = \hat{i}+2\hat{j}-2\hat{k}.$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 3\hat{i}-2\hat{j}+\hat{k}-(\hat{i}-\hat{j}+\hat{k}) = 2\hat{i}-\hat{j}.$$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 2 & -1 & 0 \end{vmatrix} = \hat{i}(0-12-\hat{j}(4)+\hat{k}(-1-4) = -2\hat{i}-4\hat{j}-5\hat{k}.$$

$$\text{area} = \frac{1}{2}\sqrt{4+16+25} = \frac{\sqrt{45}}{2}\text{sq.units}$$

18. Find the area of a parallelogram whose adjacent sides are $\vec{a} = 3\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$.

Solution: $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i}(6+2)-\hat{j}(9+1)+\hat{k}(6-2) = 8\hat{i}-10\hat{j}+4\hat{k}$

$$A = \sqrt{64+100+16} = \sqrt{180} \text{ sq.units}$$

19. Find the area of a parallelogram whose diagonals are $\overrightarrow{d_1} = 3\hat{i}+\hat{j}+2\hat{k}$ and $\overrightarrow{d_2} = \hat{i}-3\hat{j}+4\hat{k}$.

Solution : $A = \frac{1}{2}\left|\overrightarrow{d_1}\times\overrightarrow{d_2}\right|$

$$\overrightarrow{d_1}\times\overrightarrow{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & -3 & 4 \end{vmatrix} = \hat{i}(4+6)-\hat{j}(12-2)+\hat{k}(-9-1) = 10\hat{i}-10\hat{j}-10\hat{k}$$

$$\left|\overrightarrow{d_1}\times\overrightarrow{d_2}\right| = \sqrt{100+100+100};\ \frac{1}{2}\left|\overrightarrow{d_1}\times\overrightarrow{d_2}\right| = \frac{1}{2}\sqrt{300}\text{ sq.units} = 5\sqrt{3}\text{ sq.units.}$$

20. Find the scalar triple product of vectors $\vec{a} = 2\hat{i}-\hat{j}+3\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}-\hat{k}$.

Solution: $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & -1 \end{vmatrix} = 2(-2-3)+1(-1-9)+3(1-6) = -10-10-15 = -35.$

21. Evaluate $[\,\hat{i}-\hat{j},\ \hat{j}-\hat{k},\ \hat{k}-\hat{i}\,]$

Solution: $\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 1(1)+1(-1)+0 = 0$

22. Find the volume of parallelopiped whose coterminus edges are

$\vec{a} = \hat{i}-\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}-4\hat{j}+5\hat{k}$, $\vec{c} = 3\hat{i} = 5\hat{j}+\hat{k}$.

Solution: $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}1 & -1 & 1\\ 2 & -4 & 5\\ 3 & -5 & 1\end{vmatrix}=1(-4+25)+1(2-15)+1(-10+12)=21-13+2=10$ cubic units

23. Find λ if the vectors $\vec{a}=(2,-3\ 4),\ \vec{b}=(1,2,-1)$ and $\vec{c}=(\lambda,-1,2)$ are coplanar.

Solution: $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix}2 & -3 & 4\\ 1 & 2 & -1\\ \lambda & -1 & 2\end{vmatrix}=0$

i.e., $2(4-1)+3(2+\lambda)+4(-1-2\lambda)=0$

$$6+6+3\lambda-4-8\lambda=0\Rightarrow 5\lambda=8;\ \lambda=\frac{8}{5}.$$

24. Show that the points $A(4,5,1),\ B(0,-1,-1),\ C(3,9,4)$ and $D(-4,4,4)$ are coplanar.

Solution: $\overrightarrow{OA}=4\hat{i}+5\hat{j}+\hat{k},\ \overrightarrow{OB}=-\hat{j}-\hat{k},\ \overrightarrow{OC}=3\hat{i}+9\hat{j}+4\hat{k},\ \overrightarrow{OD}=-4\hat{i}+4\hat{j}+4\hat{k}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-4\hat{i}-6\hat{j}-2\hat{k},\ \overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=-\hat{i}+4\hat{j}+3\hat{k},\ \overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=-8\hat{i}-\hat{j}+3\hat{k}$

Consider

$$\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)=\begin{vmatrix}-4 & -6 & -2\\ -1 & 4 & 3\\ -8 & -1 & 3\end{vmatrix}=-4(12+3)+6(-3+24)-2(1+32)=-60+126-66=0$$

$\therefore$ A, B, C and D are coplanar.

Exercise

1. The position vectors of the points A, B and C are respectively $\vec{a},\vec{b}$ and $2\vec{a}-3\vec{b}.$ Express vectors $\overrightarrow{BC},\overrightarrow{AC},\overrightarrow{AB}$ in terms of $\vec{a}$ and $\vec{b}.$

2. Position vectors are A, B, C and D are $2\vec{a}+4\vec{c},\ 5\vec{a}+3\sqrt{3}\,\vec{b}+4\vec{c},\ -2\sqrt{3}\,\vec{b}+\vec{c}$ and $2\vec{a}+\vec{c}$ respectively.

   Show that $AB \parallel CD$ and $\overrightarrow{AB}=\frac{3}{2}\overrightarrow{CD}$

3. If $\vec{a}=3\hat{i}-4\hat{j},\ \vec{b}=2\hat{i}-3\hat{j},\ \vec{c}=\hat{i}+\hat{j},$ find (i) $2\vec{b}-\vec{c}+2\vec{a}$ (ii) $2\vec{a}-3\vec{c}+2\vec{b}$

4. Find the unit vector in the direction of $\vec{a}+\vec{b}+\vec{c}$ where $\vec{a}=\hat{i}+4\hat{j}+2\hat{k},\ \vec{b}=3\hat{i}-3\hat{j}-2\hat{k}$ and $\vec{c}=-2\hat{i}+2\hat{j}+6\hat{k}.$

5. If $\vec{a}=2\hat{i}+3\hat{j}$ and $\vec{b}=-\hat{i}+2\hat{j},$ find $\vec{a}\cdot\vec{b}.$

6. Show that the vectors (–1, 2, 3) and (2, –5, 4) are orthogonal.

7. Find the values of λ such that $\lambda\hat{i}-3\hat{j}+\hat{k}$ and $\lambda\hat{i}+\lambda\hat{j}+2\hat{k}$ may be orthogonal.

8. If $\vec{a}$ and $\vec{b}$ are unit vector show that the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are orthogonal.

9. If $\vec{a}=(1,-1,3)$ and $\vec{b}=(2,1,1),$ find $\vec{a}\times\vec{b}.$

10. Find the projection of $\vec{a}$ on $\vec{b}$ where $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=-2\hat{i}+3\hat{j}.$

11. If $\vec{a}=2\hat{i}+\hat{j}-3\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}.$ Find the projection of $\vec{b}$ on $\vec{a}.$

12. If $\vec{a} = (2, 3)$ and $\vec{b} = (8, m)$ and $\vec{a} \times \vec{b}$ is a null vector, find m.

13. If $\vec{a} = (2, -1, 3)$, $\vec{b} = (-2, 1, 4)$ and $\vec{c} \equiv (2, 1, -7)$, find the unit vector in the direction of $\vec{a} + \vec{b} + \vec{c}$.

14. Show that $\cos\theta \sin\phi\, \hat{i} + \sin\theta\, \hat{j} + \cos\phi\, \hat{k}$ is a unit vector.

15. Show that points A (3, –2, 4), B (1, 1, 1) and C (–1, 4, –2) are collinear.

16. Show that points A (1, 1, 1), B (7, 2, 3), C (2, –1, 1) form a triangle.

17. Show that points A, B and C whose position vectors are $2\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively form a right-angled triangle.

18. Find the cosine of the angle between the vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 2\hat{k}$.

19. Find the sine of the angle between the vectors $2\hat{i} - 3\hat{j} + \hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$.

20. Find the unit vector perpendicular to the vectors $3\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} - 2\hat{j} + 4\hat{k}$.

21. Find the volume of the parallelopiped whose co-terminal edges are represented by $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$; $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$.

22. Show that vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} - 4\hat{j} - 5\hat{k}$ are coplanar.

23. If the vectors $\hat{i} + 2\hat{j} + 5\hat{k}$, $2\hat{i} + x\hat{j} - 10\hat{k}$ and $3\hat{i} + 9\hat{j} - 2\hat{k}$ are coplanar. find x.

24. Show that points A (2, 3, –1), B (1, –2, 3), C (3, 4, –2) and D (1, –6, 6) are coplanar.

25. Find the scalar triple product of, $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j} + \hat{k}$.

26. Prove that $\left(2\vec{a} + 3\vec{b}\right) \times \left(\vec{a} + 4\vec{b}\right) = 5\left(\vec{a} \times \vec{b}\right)$

27. Find the area of the triangle whose 2 sides represented by $\vec{a} = \hat{i} - 3\hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

28. Find the area of the triangle whose vertices are $\hat{i} - \hat{j} + 2\hat{k}$, $2\hat{j} + \hat{k}$ and $\hat{j} + 3\hat{k}$.

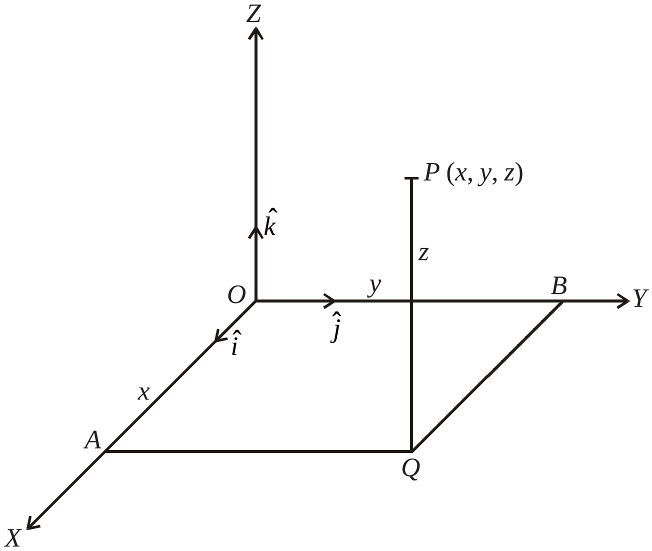
29. Find the area of parallelogram whose diagonals are $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$.

30. If $\vec{a} = (1, -1, 2)$, $\vec{b} = (1, 2, 3)$ & $\vec{c} = (3, -2, 4)$. Find $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$ and $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$.

ANALYTICAL GEOMETRY IN THREE DIMENSIONS OR SOLID GEOMETRY

Co-ordinates of a point in 3-space

Consider three mutually perpendicular lines OX, OY & OZ in 3-space with O as origin. Let P be a point in 3-space. Draw $PQ \perp^{r}$ to the plane XOY.



Draw QA & QB parallel to OY & OX to meet OX at A & OY at B.

Let $OA = x,\ OB = y\ \ \&\ \ QP = z$

Then (x, y, z) are taken as coordinates the point P. Three mutually $\perp^{r}$ lines OX, OY & OZ divide 3-space into eight equal parts called Octants. Co-ordinates of any point in the plane XOY will be of the form (x, y, o), in XOZ plane (x, o, z) in YOZ plane (o, y, z) co-ordinates of any point on x-axis are (x, o, o) on y-axis, (o, y, o) and z-axis (o, o, z). Co-ordinates of origin (o, o, o). The position vector of P is

$\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ is also denoted by $\vec{r}$.

Distance between two points

$P(x_1, y_1, z_1)\ \ \&\ \ Q(x_2, y_2, z_2)$

using vectors, $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\ \ \&\ \ \overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

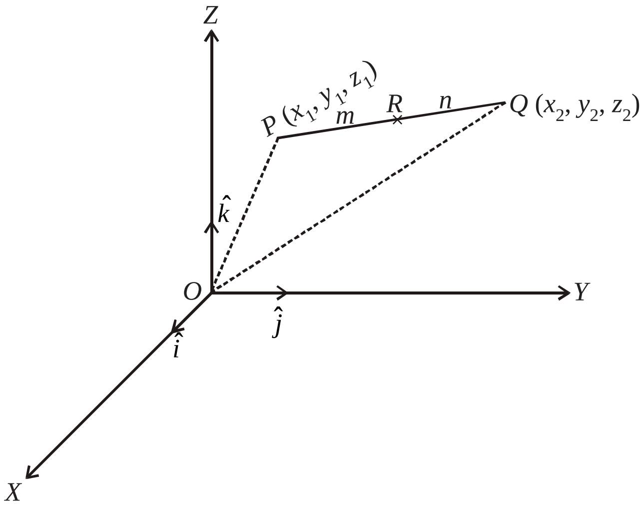
$$\therefore\ \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\therefore\ \left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$\therefore$ Distance between two points $(x_1, y_1, z_1)\ \ \&\ \ Q(x_2, y_2, z_2)$ is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Section Formula

Let R divide PQ in the ratio $m : n$



then $\dfrac{PR}{RQ} = \dfrac{m}{n}$

ie $n\,PR = m\,RQ$

$\therefore\ n\,\overrightarrow{PR} = m\,\overrightarrow{RQ}$

ie $n\left(\overrightarrow{OR} - \overrightarrow{OP}\right) = m\left(\overrightarrow{OQ} - \overrightarrow{OR}\right)$

ie $n\,\overrightarrow{OR} - n\,\overrightarrow{OP} = m\,\overrightarrow{OQ} - m\,\overrightarrow{OR}$

$\therefore\ (m + n)\,\overrightarrow{OR} = m\,\overrightarrow{OQ} + n\,\overrightarrow{OP}$

$$\therefore \ \overrightarrow{OR} = \frac{m\overrightarrow{OQ} + n\overrightarrow{OP}}{m+n}$$

OR represents the position vector of R which divides P & Q in the ratio $m : n$.

If R is the mid - point, then $\overrightarrow{OR} = \dfrac{\overrightarrow{OQ} + \overrightarrow{OP}}{2}$

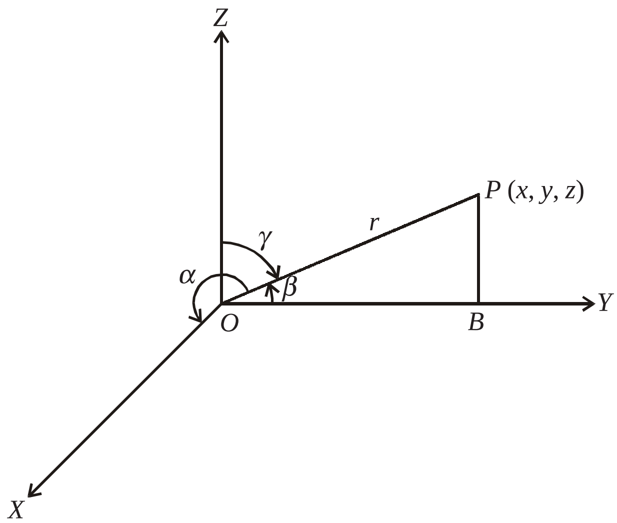
Cartesian co - ordinates of R are given by $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}, \dfrac{mz_2 + nz_1}{m+n}\right)$

if R lie out segment PQ.

Then co - ordinates of R are $\left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}, \dfrac{mz_2 - nz_1}{m-n}\right)$

Direction cosines of a line

Let P be any point in plane, let the line OP make angles α, β, γ with co - ordinate axes then $\cos\alpha, \cos\beta$ & $\cos\gamma$ are defined as directions cosines and usually denoted as l, m, n.



To prove that $l^2 + m^2 + n^2 = 1$. Draw $PB \perp^r$ to OY,

then $\cos\beta = \dfrac{y}{r}$ where $OP = r$.

$$\therefore \ y = r\cos\beta = mr$$

Similarly by droping $\perp^{rs}$ to OX & OZ,

we can show that $x = r\cos\alpha = lr$ & $z = r\cos\gamma = nr$

Squaring and adding, we get

$$x^2 + y^2 + z^2 = l^2r^2 + m^2r^2 + n^2r^2$$

$$= (l^2 + m^2 + n^2)r^2$$

but $x^2 + y^2 + z^2 = r^2$

$$\therefore \ l^2 + m^2 + n^2 = 1$$

Direction of ratios of a line

If the direction cosines of a line are proportional to a, b, c then a, b, c are called **Direction Ratios** of the line.

ie $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c} = k$ (say)

then $l = ak$, $m = bk$, $n = ck$

$$\therefore \ l^2 + m^2 + n^2 = a^2k^2 + b^2k^2 + c^2k^2$$

but $l^2 + m^2 + n^2 = 1$

$$\therefore \ (a^2 + b^2 + c^2)k^2 = 1$$

ie $k^2 = \dfrac{1}{a^2 + b^2 + c^2} \quad \therefore \ k = \dfrac{1}{\sqrt{a^2 + b^2 + c^2}}$

$\therefore$ If a, b, c are the direction ratios of any line then direction cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are any two points in 3-space. The direction ratios of PQ are given by $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ and hence the direction cosines are

$$\frac{x_2-x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}, \frac{y_2-y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}, \frac{z_2-z_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

Note :- Co-ordinates of a unit vector represents the direction cosines of the line of the vector. For any other vectors the co-ordinates represent the direction ratios.

Examples

1. Find the distance between the points (4, 3, –6) & (–2, 1, –3).

Solution : Let $P = (4, 3, -6)$, $Q = (-2, 1, -3)$

$$PQ = \sqrt{(4+2)^2 + (3-1)^2 + (-6+3)^2}$$

$$= \sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{36+4+9} = 7$$

2. Show that the points (–2, 3, 5), (1, 2, 3) & (7, 0, –1) are colinear.

Solution : Let $A = (-2, 3, 5)$, $B = (1, 2, 3)$, $C = (7, 0, -1)$

$$AB = \sqrt{(-2-1)^2 + (3-2)^2 + (5-3)^2} = \sqrt{9+1+4} = \sqrt{14}$$

$$BC = \sqrt{(1-7)^2 + (2-0)^2 + (3+1)^2} = \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$$

$$AC = \sqrt{(-2-7)^2 + (3-0)^2 + (5+1)^2} = \sqrt{81+9+36} = \sqrt{126} = 3\sqrt{14}$$

$$AB + BC = \sqrt{14} + 2\sqrt{14} = 3\sqrt{14} = AC$$

$\therefore$ The points A, B, C are colinear.

<u>Alternate Method</u>

Direction ratios AB are – 2 – 1, 3 – 2, 5 – 3 ie –3, 1, 2

Direction ratios of BC are 1 – 7, 2 – 0, 3 + 1, ie – 6, 2, 4 ie – 3, 1, 2

Direction ratios of AB & BC are same. $\therefore$ A, B & C are colinear.

3. Show that the points (3, 2, 2), (–1, 1, 3), (0, 5, 6), (2, 1, 2) lie on a sphere whose centre is (1, 3, 4). Find also the radius of the sphere.

Solution : Let the given points be $P = (3, 2, 2)$, $Q = (-1, 1, 3)$, $R = (0, 5, 6)$ **&** $S = (2, 1, 2)$. Let $C = (1, 3, 4)$ be the centre.

$$CP = \sqrt{(1-3)^2 + (3-2)^2 + (4-2)^2} = \sqrt{4+1+4} = 3$$

$$CQ = \sqrt{(1+1)^2 + (3-1)^2 + (4-3)^2} = \sqrt{4+4+1} = 3$$

$$CR = \sqrt{(0-1)^2 + (5-3)^2 + (6-4)^2} = \sqrt{1+4+4} = 3$$

$$CS = \sqrt{(2-1)^2 + (1-3)^2 + (2-4)^2} = \sqrt{1+4+4} = 3$$

$\therefore \; CP = CQ = CR = CS = 3$

$\therefore \; P, Q, R$ **&** S lie on a sphere & radius is 3.

4. Find the co-ordinates of the point which divide the line joining the points (2, –4, 3) & (–4, 5, –6) in the ratio 2 : 1.

Solution : Let $\vec{a} = (2, -4, 3), \; \vec{b} = (-4, 5, -6)$

Point of section is given by $\dfrac{2\vec{b} + \vec{a}}{2+1}$

$$= \frac{2(-4, 5, -6) + 1(2, -4, 3)}{2+1}$$

$$= \frac{(-8+2, \; 10-4, \; -12+3)}{3}$$

$$= \frac{(-6, 6, -9)}{3} = (-2, 2, -3)$$

5. Find the co-ordinates of the point which divide the line joining (3, 2, 1) & (1, 3, 2) in the ratio – 2 : 1.

Solution : Let $\vec{a} = (3, 2, 1), \; \vec{b} = (1, 3, 2)$

Point of section is given by $\dfrac{-2\vec{b} + 1 \cdot \vec{a}}{-2+1}$

$$= \frac{-2(1, 3, 2) + 1(3, 2, 1)}{-1}$$

$$= \frac{(-2, -6, -4) + (3, 2, 1)}{-1}$$

$$= \frac{(1, -4, -3)}{-1} = (-1, 4, 3)$$

6. Find the ratios in which *XY* plane divides the join of (– 3, 4, – 8) & (5, – 6, 4).
Also obtain the coordinates of the point of section.

Solution : Let $\vec{a} = (-3, 4, -8), \; \vec{b} = (5, -6, 4)$

Let k : 1 be the ratio.

Point of section is given by $\dfrac{k\vec{b} + \vec{a}}{k+1} = \left(\dfrac{5k-3}{k+1}, \; \dfrac{-6k+4}{k+1}, \; \dfrac{4k-8}{k+1} \right)$

This point lies on *XY* plane

$$\therefore \; \frac{4k-8}{k+1} = 0 \Rightarrow 4k = 8 \Rightarrow k = 2$$

$\therefore$ ratio is 2 : 1 and co-ordinates of the point are $\left(\dfrac{7}{3}, \; -\dfrac{8}{3}, \; 0 \right)$

7. Direction ratios of a line are 6, 2, 3. Find the direction cosines.

Solution : Now $\sqrt{6^2 + 2^2 + 3^2} = \sqrt{36+4+9} = 7$

$\therefore$ direction cosines are $\dfrac{6}{7}, \; \dfrac{2}{7}, \; \dfrac{3}{7}.$

8. Find the direction cosines of the line joining the points (1, 4, – 3) & (4, 7, – 6).

Solution : Direction ratios of the line joining the points (1, 4, – 3) & (4, 7, – 6) are 4 – 1, 7 – 4, – 6 + 3

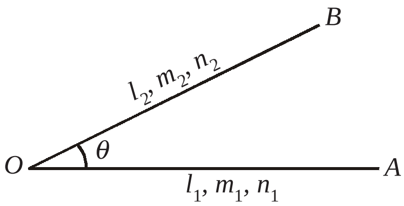
ie 3, 3, – 3.

$$\therefore \text{ direction cosines are } \frac{3}{\sqrt{9+9+9}}, \frac{3}{\sqrt{9+9+9}}, \frac{-3}{\sqrt{9+9+9}}$$

$$\text{ie } \frac{3}{3\sqrt{3}}, \frac{3}{3\sqrt{3}}, \frac{-3}{3\sqrt{3}} \quad \text{ie } \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}.$$

To find the angle between two lines in 3-space

Let OA & OB be two lines whose direction cosines are l_1, m_1, n_1 **&** l_2, m_2, n_2.



Let $A\hat{O}B = \theta$

Let $\vec{a} = \overrightarrow{OA} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$, $\vec{b} = \overrightarrow{OB} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$

$\vec{a} \cdot \vec{b} = l_1 l_2 + m_1 m_2 + n_1 n_2$

but $\vec{a} \cdot \vec{b} = ab\cos\theta$

$$\therefore \cos\theta = \frac{\vec{a} \cdot \vec{b}}{ab} = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$= l_1 l_2 + m_1 m_2 + n_1 n_2$$

If $\theta = \frac{\pi}{2}$, $\cos\frac{\pi}{2} = 0 \;\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\therefore$ condition for two lines to be $\perp^r$ is $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

If a_1, b_1, c_1 **&** a_2, b_2, c_2 are direction ratios of two lines. Then angle θ between them is given by

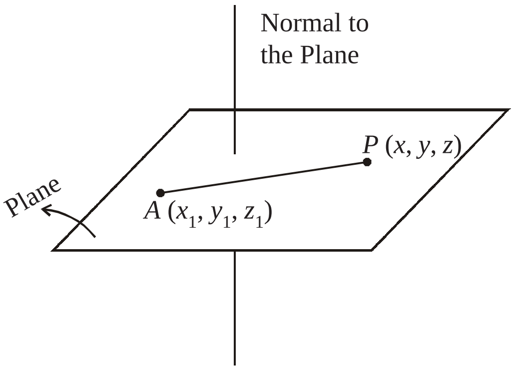
$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

lines will be perpendicular if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.

Note :- Two lines will be parallel if direction cosines of the two lines are same or if the direction ratios are proportional.

The Plane

To find the equation of the plane passing through the point (x_1, y_1, z_1) and having *a, b, c* as the direction ratios of the normal to the plane.



Let $A(x_1, y_1, z_1)$ be a point in the plane.

Let $P(x, y, z)$ be any point in the plane then direction ratios of AP are given by $x - x_1$, $y - y_1$, $z - z_1$ as this is perpendicular to the normal to the plane, we have

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$\text{ie } ax + by + cz - (ax_1 + by_1 + cz_1) = 0$$

which is a first degree equation in *x, y* & *z*.

Note :- (1) $z = 0$ represents the equation to the *XOY* plane.

(2) $y = 0$ represents the equation of the *ZOX* plane.

(3) $x = 0$ represents the equation of *YOZ* plane.

To prove that *ax + by + cy + d = 0* a first degree equation in *x, y, z* represents a plane.

Let $P(x_1, y_1, z_1)$ **&** $Q(x_2, y_2, z_2)$ be any two points satisfying $ax+by+cz+d=0$.

Then $ax_1+by_1+cz_1+d=0$ (1)

$ax_2+by_2+cz_2+d=0$ (2)

multiply (2) by k & add to (1)

$k(ax_2+by_2+cz_2+d)+(ax_1+by_1+cz_1+d)=0$

ie $a(kx_2+x_1)+b(ky_2+y_1)+c(kz_2+z_1)+d(k+1)=0$

divide through out by $k+1\,(k \neq -1)$

$$a\frac{(kx_2+x_1)}{k+1}+b\frac{(ky_2+y_1)}{k+1}+c\frac{(kz_2+z_1)}{k+1}+d=0$$

This shows that the point whose coordinates are $\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$

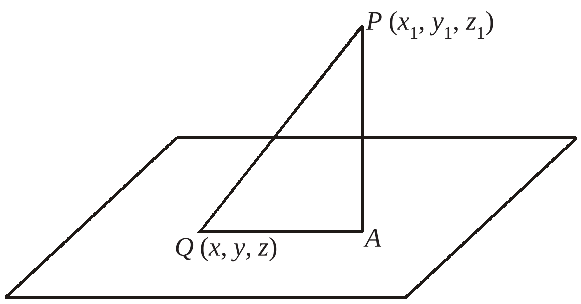
satisfy the equation $ax+by+cz+d=0$.

Thus every point on the line joining P & Q lie on the locus. $\therefore$ The equation represents the plane.

To find the length of the perpendicular from (x_1, y_1, z_1) upon the plane *ax + by + cy + d = 0.*

Let $P(x_1, y_1, z_1)$ be the given point & *PA* is the length of the perpendicular on the plane $ax+by+cz+d=0$.

Let $Q(x, y, z)$ be any point in the plane, *AP* is the projection of $\overrightarrow{QP}$ upon the normal to the plane if $\hat{n}$ is the unit normal vector of the plane then



$$\hat{n}=\frac{a\hat{i}+b\hat{j}+c\hat{k}}{\sqrt{a^2+b^2+c^2}}$$

$$AP=\overrightarrow{QP}\cdot\hat{n}=\left((x_1-x)\hat{i}+(y_1-y)\hat{j}+(z_1-z)\hat{k}\right)\cdot\hat{n}$$

$$=\frac{a(x_1-x)+b(y_1-y)+c(z_1-z)}{\sqrt{a^2+b^2+c^2}}$$

$$=\frac{ax_1+by_1+cz_1-(ax+by+cz)}{\sqrt{a^2+b^2+c^2}} \quad =\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$$

$\because \; ax+by+cz=-d$

$\therefore$ length of the perpendicular $AP == \dfrac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$

Note :- Two points (x_1, y_1, z_1) **&** (x_2, y_2, z_2) lie on the same side or on opposite sides of the plane $ax+by+cz+d=0$ according as $ax_1+by_1+cz_1+d$ **&** $ax_2+by_2+cz_2+d$ are of same sign or of opposite signs.

Normal form of the equation of the plane

Let p represents length of the $\perp^r$ from the origin upon the plane $ax+by+cz+d=0$

then $p=\dfrac{d}{\sqrt{a^2+b^2+c^2}}$

$lx+my+nz=p$ where l, m, n are the direction cosines of the normal to the plane, is taken as the normal equation of the plane.

To find the angle between two planes

$a_1x+b_1y+c_1z+d_1=0$ & $a_2x+b_2y+c_2z+d_2=0$

The two planes intersect along a line, the angle between two planes is nothing but angle between two normals of the plane. $\therefore$ if θ is the angle between two planes

then $\cos\theta=\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$ & planes will be $\perp^r$ if $a_1a_2+b_1b_2+c_1c_2=0$

Intercept form of the equation if the plane

Let a, b & c be the intercept made by the plane on the co-ordinate axes ; ie the plane passes through the points (a, o, o), (o, b, o) & (o, o, c)

Let the equation to the plane be $\alpha x+\beta y+\gamma z+d=0$.

(a, o, o) lies on the plane $\therefore \alpha a+d=0 \ \therefore \alpha=-\dfrac{d}{a}$

(o, b, o) lies on the plane $\therefore \beta b+d=0 \Rightarrow \beta=-\dfrac{d}{b}$

(o, o, c) lies on the plane $\therefore \gamma c+d=0 \Rightarrow \gamma=-\dfrac{d}{c}$

$\therefore$ equation of the plane is $-\dfrac{d}{a}x-\dfrac{d}{b}y-\dfrac{d}{c}z+d=0$

ie $-\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{z}{c}+1=0$

ie $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ is the equation of the plane in the intercept form.

Examples

(1) Find the equation of the plane passing through the point (1, 2, 3) and having the vector $\vec{n}=2\hat{i}-\hat{j}+3\hat{k}$ as normal.

Solution : Direction ratios of the normal to the planes are 2, –1, 3

$\therefore$ equation to the plane is

$2(x-1)-1(y-2)+3(z-3)=0$

ie $2x-2-y+2+3z-9=0$

ie $2x-y+3z-9=0$

(2) Find the equation of the plane through the point (2, 1, 0) and perpendicular to the planes $2x-y-z=5$ and $x+2y-3y=5$

Solution : Equation of the plane through the point $(2,1,0)$ is $a(x-2)+b(y-1)+c(z-0)=0$

This plane is $\perp^r$ to $2x-y-z=5$

$$\therefore\ 2a-b-c=0 \qquad (1)$$

the plane is also $\perp^r$ to $x+2y-3z=5$

$$\therefore\ a+2b-3x=0 \qquad (2)$$

Let us eliminate c between (1) & (2)

$(1)\times 3$ is $6a-3b-3c=0$

$(2)\times 1$ is $\underline{a+2b-3c=0}$

Subtracting $5a-5b=0 \Rightarrow a=b$

from (1), $2a-a-c=0 \Rightarrow a=c$

$\therefore\ a=b=c$

$\therefore$ Equation of the plane is $a(x-2)+b(y-1)+c(z-0)=0$

ie $x-2+y-1+z+0=0$

ie $x+y+z=3.$

(3) Find the equation of the plane through (1, 2, 3) and parallel to the plane $4x+5y-3z=7$.

Solution : Equation of the plane parallel to $4x+5y-3z=7$ can be taken as $4x+5y-3z=k$ where k is a constant. The plane passes through (1, 2, 3)

$\therefore\ 4(1)+5(2)-3(3)=k$ ie $4+10-9=k$ $\therefore\ k=5$

$\therefore$ Equation of the required plane is $4x+5y-3z=5.$

(4) Find the $\perp^r$ distance of the point (3, 2, 1) from the plane passing through the points (1, 1, 0), (3, – 1, 1) & (–1, 0, 2)

Solution : Let $P=(3,2,1)$ & $A=(1,1,0)$, $B=(3,-1,1)$, $C=(-1,0,2)$

$\overrightarrow{AB}=2\hat{i}-2\hat{j}+\hat{k}$

$\overrightarrow{AC}=-2\hat{i}-\hat{j}+2\hat{k}$

$$\therefore\ \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -2 & 1\\ -2 & -1 & 2\end{vmatrix}$$

$$=\hat{i}(-4+1)-\hat{j}(4+2)+\hat{k}(-2-4)=-3\hat{i}-6\hat{j}-6\hat{k}$$

$\therefore$ direction ratios of the normal to the plane are –3, –6, –6

Equation of the plane is given by $-3(x-1)-6(y-1)-6(z-0)=0$

ie $-3x-6y-6z+3+6=0$

ie $x+2y+2z-3=0$

$\therefore$ length of the $\perp^r$ from $(3,2,1)$ upon the the plane is $\dfrac{3+4+2-3}{\sqrt{1+4+4}}=\dfrac{6}{3}=2$ units.

(5) Show that the four points (0, –1, 0), (2, 1, –1), (1, 1, 1) & (3, 3, 0) are coplanar. Find the equation of the plane.

Solution : Let $A=(0,-1,0),\ B=(2,1,-1),\ C=(1,1,1)\ \&\ D=(3,3,0)$

Let us find the equation of the plane passing through A, B & C.

The plane passing through (0, –1, 0) is given by $a(x-0)+b(y+1)+c(z-o)=0$

ie $ax+by+cz+b=0$

$(2,1,-1)$ lies on it

$\therefore\ 2a+b-c+b=0$

ie $2a+2b-c=0$ (1)

(1, 1, 1) lies on it

$\therefore\ a+2b+c=0$ (2)

from (1) & (2) $\dfrac{a}{2+2}=\dfrac{b}{-1-2}=\dfrac{c}{4-2}$

ie $\dfrac{a}{4}=\dfrac{b}{-3}=\dfrac{c}{2}$

$\therefore$ Equation of the plane is $4(x-0)-3(y+1)+2(z-0)=0$

ie $4x-3y+2z-3=0.$

Consider $D=(3,3,0)$

it can be seen that it lies on the plane. $\therefore$ the given points are coplanar.

(6) Find the distance between the parallel planes $2x-2y+z+3=0\ \&\ 4x-4y+2z+5=0.$

Solution : Now $(0,0,-3)$ is a point on $2x-2y+z+3=0.$

$\perp^r$ distance from $(0,0,-3)$ upon the plane $4x-4y+2z+5=0$

is $\dfrac{4(0)-4(0)+2(-3)+5}{\sqrt{16+16+4}}$

$=\left|\dfrac{-6+5}{6}\right|=\left|\dfrac{-1}{6}\right|=\dfrac{1}{6}$

$\therefore$ distance between parallel planes is $\dfrac{1}{6}$

(7) Find the angle between the planes $x+2y-3z-2=0$ and $2x+y+z+3=0.$

Solution : If θ is the angle between two planes, we have

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$$

$$=\frac{2+2-3}{\sqrt{1+4+9}\ \sqrt{4+1+1}}=\frac{1}{\sqrt{14}\ \sqrt{6}}=\frac{1}{\sqrt{7\times2}\ \sqrt{2\times3}}=\frac{1}{2\sqrt{21}}$$

$$\therefore\ \theta=\cos^{-1}\left(\frac{1}{2\sqrt{21}}\right)$$

(8) Find the equation of the plane passing through the line of intersection of the planes $x+y+z=1$ and $2x+3y-z+4=0$ and perpendicular to the plane $2y-3z=4$

Solution : Equation of the plane passing through the line of intersection of the given two planes can be taken as

$$(x+y+z-1)+\lambda(2x+3y-z+4)=0$$

where λ is a constant.

ie $(1+2\lambda)x+(1+3\lambda)y+(1-\lambda)z-1+4\lambda=0$

this plane is $\perp^r$ to $2y-3z=4$

$\therefore\ 0(1+2\lambda)+2(1+3\lambda)-3(1-\lambda)=0$

ie $2+6\lambda-3+3\lambda=0$

ie $9\lambda=1 \Rightarrow \lambda=\frac{1}{9}$

$\therefore$ Equation of plane is $(x+y+z+1)+\frac{1}{9}(2x+3y-z+4)=0$

ie $9x+9y+9z-9+2x+3y-z+4=0$

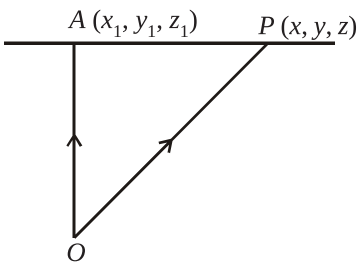
ie $11x+12y+8z-5=0$

is the equation of required plane.

The Straight Line

1. Two planes intersect along a straight line, therefore $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ taken together represent a line. These are called General equations of the line.

2. If a line passes through the point $A(x_1, y_1, z_1)$ **&** have l, m, n as direction cosines then



Let $P(x, y, z)$ be any point on the line

$\overrightarrow{AP}=(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}$ and direction cosines are l, m, n

$$\therefore\ \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

represent two equations of the line. This is called **Symmetric form** of the equation of the line.

3. If a, b, c are the direction ratios of the line passing through the point (x_1, y_1, z_1), then equations of the line are

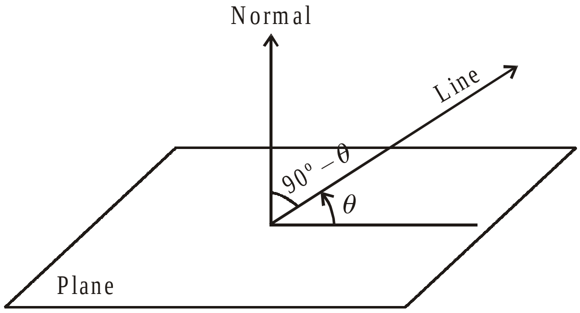
$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

If the line passes through the points (x_1, y_1, z_1) **&** (x_2, y_2, z_2), then the direction ratios of the line are x_2-x_1, y_2-y_1, z_2-z_1

$\therefore$ its equations are $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

4. To find the angle between the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$.

Solution : If θ is the normal angle between the line and the plane $90°-\theta$ is the angle between the line and the normal to the plane.



$$\therefore \cos(90°-\theta)=\frac{la+mb+nc}{\sqrt{l^2+m^2+n^2}\,\sqrt{a^2+b^2+c^2}}$$

$$\therefore \sin\theta=\frac{la+mb+nc}{\sqrt{a^2+b^2+c^2}}.$$

If the line is parallel to the plane then $\theta=0°$

$\therefore\ la+mb+nc=0.$

If the line is perpendicular to the plane then $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$

Examples

(1) Show that the line $\frac{x-1}{3}=\frac{y+2}{-2}=\frac{3-1}{2}$ is parallel to the plane $2x+2y-z=6$ and find the distance between them.

Solution : Direction ratios of the normal to the plane $2x+2y-z=6$ are $2, 2, -1$

Direction ratios of the given line are 3, –2, 2

Now 2(3) + 2(– 2) – 1(2) = 6 – 4 – 2 = 0.

$\therefore$ line is $\perp^r$ to the normal to the plane. $\therefore$ line is parallel to the plane.

$(1,-2,1)$ is a point on the given line. $\therefore \perp^r$ distance of $(1,-2,1)$ upon the plane $2x+2y-z=6$

is $\left|\frac{2(1)+2(-2)-1-6}{\sqrt{4+4+1}}\right|=\left|\frac{2-4-7}{3}\right|=\frac{9}{3}=3.$

(2) Find the equation of the line through (1, 2, – 1) perpendicular to each of the lines

$\frac{x}{1}=\frac{y}{0}=\frac{z}{-1}$ and $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}.$

Solution : Let a, b, c be the direction ratios of the required line.

Then $a+o-c=0$ & $3a+4b+5c=0.$

$$\therefore \frac{a}{0+4}=\frac{b}{-3-5}=\frac{c}{4-0}$$

ie $\frac{a}{4}=\frac{b}{-8}=\frac{c}{4}$ ie $\frac{a}{1}=\frac{b}{-2}=\frac{c}{1}$

$\therefore$ equations of the required line are $\frac{x-1}{1}=\frac{y-2}{-2}=\frac{3+1}{1}.$

(3) Find the angle between the line $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ and the plane $3x+y+z=7.$

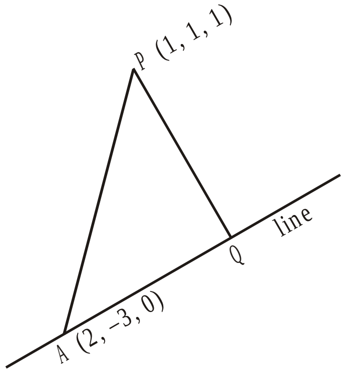
Solution : If θ is the angle between line and the plane, then

$$\sin\theta = \frac{(2)(3)+3(1)+6(1)}{\sqrt{4+9+36}\ \sqrt{9+1+1}} = \frac{6+3+6}{7\sqrt{11}} = \frac{15}{7\sqrt{11}}$$

$$\therefore\ \theta = \sin^{-1}\frac{15}{7\sqrt{11}}$$

(4) Find the perpendicular distance of the point (1, 1, 1) from the line $\dfrac{x-2}{2} = \dfrac{y+3}{2} = \dfrac{z}{-1}$

Solution : Let $A(2,-3,0)$ is a point on the line & $P(1,1,1)$ be the given point



$$\therefore\ AP = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = -\hat{i} + 4\hat{j} + \hat{k}$$

$$\text{Projection of } \overrightarrow{AP} \text{ on the line} = AQ = (-\hat{i}+4\hat{j}+\hat{k})\cdot\frac{(2\hat{i}+2\hat{j}-\hat{k})}{\sqrt{4+4+1}}$$

$$= \frac{-2+8-1}{3} = \frac{5}{3}$$

$$\therefore\ PQ^2 = AP^2 - AQ^2 = 18 - \frac{25}{9} = \frac{162-25}{9} = \frac{137}{9}$$

$$\therefore\ PQ = \frac{\sqrt{137}}{3}$$

<u>Alternate Method</u>

Let $\dfrac{x-2}{2} = \dfrac{y+3}{2} = \dfrac{z}{-1} = r$

any point on the line is $(2r+2,\ 2r-3,\ -r)$

direction ratios of the line joining this point & the point $(1,1,1)$ are $2r+1,\ 2r-4,\ -r-1$

This line will be $\perp^r$ to the given line if $2(2r+1)+2(2r-4)-1(-r-1)=0$

ie $4r+2+4r-8+r+1=0$

ie $9r-5=0$ ie $r=\dfrac{5}{9}$

$\therefore$ Co-ordinates of the foot of the perpendicular are $\left(\dfrac{10}{9}+2,\ \dfrac{10}{9}-3,\ \dfrac{-5}{9}\right)$ ie $\left(\dfrac{28}{9},\ \dfrac{-17}{9},\ \dfrac{-5}{9}\right)$

Distance between $\left(\dfrac{28}{9},\ \dfrac{-17}{9},\ \dfrac{-5}{9}\right)$ & $(1,1,1)$ is $\sqrt{\left(\dfrac{28}{9}-1\right)^2+\left(\dfrac{-17}{9}-1\right)^2+\left(\dfrac{-5}{9}-1\right)^2}$

$$= \frac{\sqrt{19^2+26^2+14^2}}{9} = \frac{\sqrt{9\times 137}}{9} = \frac{\sqrt{137}}{3}$$

(5) Find the image of the point (1, 3, 4) in the plane $2x-y+z+3=0$

Solution : Let P be the given point and Q be its image on the plane $2x-y+z+3=0$

Equations to PQ are $\dfrac{x-1}{2} = \dfrac{y-3}{-1} = \dfrac{z-4}{1} = r$ (say)

Then co-ordinates of Q are $(2r+1,\ -r+3,\ r+4)$

Mid-point of PQ is given by

$$\left(\frac{2r+1+1}{2},\ \frac{-r+3+3}{2},\ \frac{r+4+4}{2}\right)$$

ie $\left(r+1,\ \frac{-r+6}{2},\ \frac{r+8}{2}\right)$ this point lie on the plane

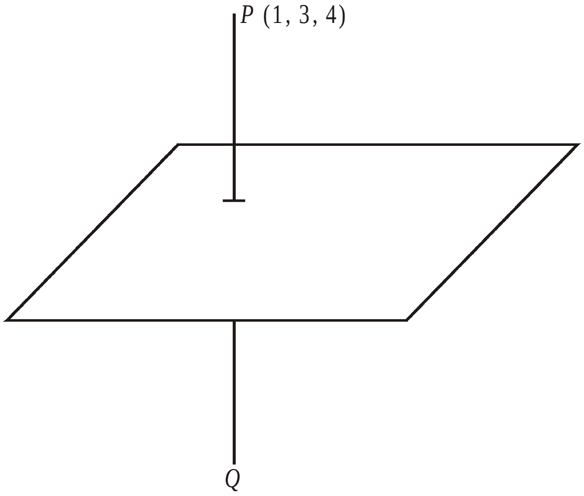
$\therefore\ 2(r+1)-\left(\frac{-r+6}{2}\right)+\frac{r+8}{2}+3=0$

ie $4r+4+r-6+r+8+6=0$

ie $6r=-12 \Rightarrow r=-2$

$\therefore\ Q=(-4+1,\ 2+3,\ -2+4)=(-3,5,2)$

$\therefore$ Image of $(1,3,4)$ in the given plane is $(-3,5,2)$



(6) To find the condition for two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ to intersect (or to be coplanar) and to find the equation of the plane.

Solution : The equation of the plane containing the first line is given by $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ (1)

where $al_1+bm_1+cn_1=0$ (2)

the second line will lie in the plane if $al_2+bm_2+cn_2=0$ (3)

and (x_2, y_2, z_2) satisfy (1)

ie $a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$ (4)

eliminating a, b, c from (2), (3) & (4)

We get $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

which is the required condition for coplanarity of two lines.

Equation to the plane is by eliminating a, b, c from (1), (2) & (3)

ie $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

is the required equation of the plane.

<u>Exercise</u>

1. Find the equation of the plane passing through the point (– 2, 2, 2) and containing the line joining the points (1, 1, 1) & (1, – 1, 2).

(Ans: $x-3y-6z+8=0$)

2. Show that the points (– 6, 3, 2), (3, – 2, 4), (5, 7, 3) and (– 13, 17, – 1) are coplanar.

3. Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x+6y+6z=9$

(Ans: $3x+4y-5z=0$)

4. Find the equation of the plane through the points whose position vectors are $3\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$

(Ans: $2x+2y+z=5$)

5. Find the equation of the plane through the points (2, 2, 1), (1, – 2, 3) and parallel to the joining the points (2, 1, – 3), (– 1, 5, – 8)

(Ans: $12x-11y-16z+14=0$)

6. Find the equation of the plane which contains the line $\dfrac{x-1}{2}=\dfrac{y-3}{1}=\dfrac{z-13}{2}$ and is perpendicular to the plane $x+y+z=3$.

(Ans: $x-z+12=0$)

7. Find the equation of the plane through the line $\dfrac{x-1}{3}=\dfrac{y-4}{2}=\dfrac{z-4}{-2}$ and parallel to the line $\dfrac{x+1}{3}=\dfrac{y-1}{-4}=\dfrac{z+2}{1}$

(Ans: $2x+3y+6z=38$)

8. Show that the lines $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$ and $\dfrac{x-2}{3}=\dfrac{y-3}{4}=\dfrac{z-4}{5}$ are coplanar and find the equation of the plane containing them.

(Ans: $x-2y+z=0$)

9. Show that the lines $\dfrac{x+3}{2}=\dfrac{y+5}{3}=\dfrac{z-7}{-3}$ and $\dfrac{x+1}{4}=\dfrac{y+1}{5}=\dfrac{z+1}{-1}$ are coplanar and find the equation of the plane containing them.

(Ans: $6x-5y-z=0$)

10. Show that the line $x+10=\dfrac{8-y}{2}=z$ lies in the plane $x+2y+3z=6$.

11. Find the co-ordinates of the foot of the perpendicular drawn from the point (–1, –3, 2) upon the plane $3x+4y+5z=5$.

$\left[\text{Ans}: \left(-\dfrac{2}{5},\ -\dfrac{11}{5},\ 3\right)\right]$

12. Prove that the lines $\dfrac{x-4}{1}=\dfrac{y+3}{-4}=\dfrac{z+1}{7}$ and $\dfrac{x-1}{2}=\dfrac{y+1}{-3}=\dfrac{z+10}{8}$ intersect and find the co-ordinates of the point of intersection.

$[\text{Ans}: (5,-7,6)]$

Sphere

Definition :- A Sphere is the locus of a point which remains at a constant distance from a fixed point in three dimension.

The fixed point is the centre and constant distance is called **radius.**

Equation of the sphere whose centre is at (x_1, y_1, z_1) and radius r is given by

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2$$

1. To show that $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a sphere.

The expression can be written as

$$(x^2+2ux)+(y^2+2vy)+(z^2+2wz)+d=0$$

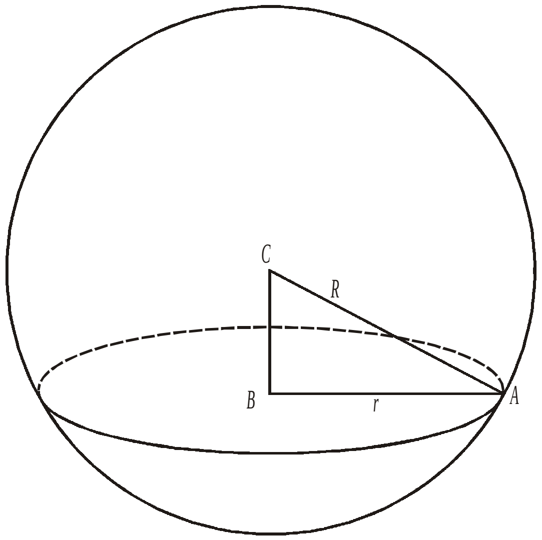
ie $(x+u)^2-u^2+(y+v)^2-v^2+(z+w)^2-w^2+d=0$

ie $(x+u)^2+(y+v)^2+(z+w)^2=u^2+v^2+w^2-d$

which represents a sphere whose centre is $(-u, -v, -w)$ and radius is $\sqrt{u^2+v^2+w^2-d}$

2. **Plane section of a sphere**

Plane section of a sphere is always a circle. If the plane passes through the centre of the sphere it is called a great circle, otherwise a small circle.



Let C be the centre of the sphere and A be a point on the sphere and on the plane which intersects the sphere. Let B be the centre of the small circle then BC is $\perp^r$ to AB.

$\therefore\ AC^2=AB^2+BC^2$

$BC=p$ **&** $AC=R$ (radius of the sphere)

Then $AB^2=AC^2-BC^2=R^2-p^2$

$\therefore$ radius of the small circle is $AB=\sqrt{R^2-p^2}$

Let $S:\ x^2+y^2+z^2+2ux+2vy+2wz+d=0$ be the equation of the sphere

and $P:\ ax+by+cz+d=0$ be the plane.

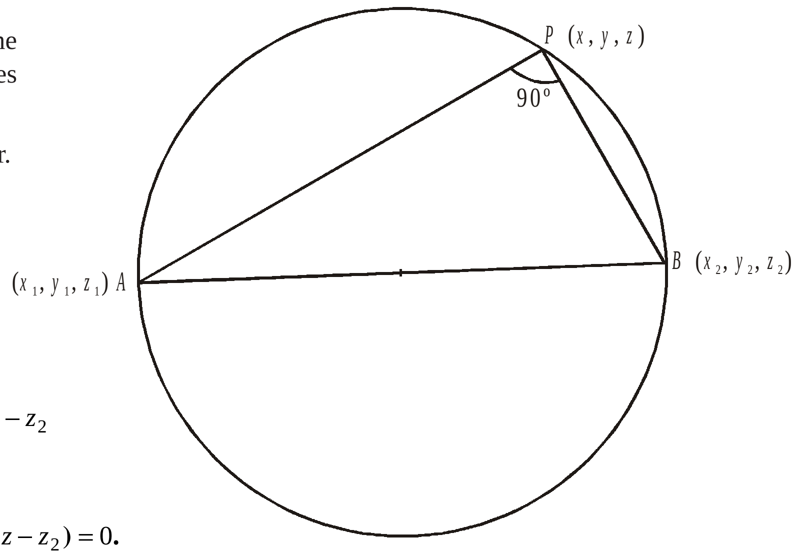
Then the equation of the sphere passing through the circle is $S+\lambda P=0$ where λ is a constant.

If $S_1=0$ ie $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ **&** $S_2=0$ ie $x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$

represents two spheres then the equation of the circle common to $S_1=0$ **&** $S_2=0$ is $S_1-S_2=0$

$\therefore$ The equation of the sphere through the circle is given by $S_1+\lambda(S_1-S_2)=0$.

3. To find the equation of the sphere having the (x_1, y_1, z_1) **&** (x_2, y_2, z_2) as the extremities of a diameter.



Let A & B be the extremities of the diameter.

Let $P(x, y, z)$ be a point on the sphere.

AP is $\perp^r$ to PB.

Direction ratios of AP are

$x-x_1,\ y-y_1,\ z-z_1$

Direction ratios of BP are $x-x_2,\ y-y_2,\ z-z_2$

Since AP is $\perp^r$ to PB,

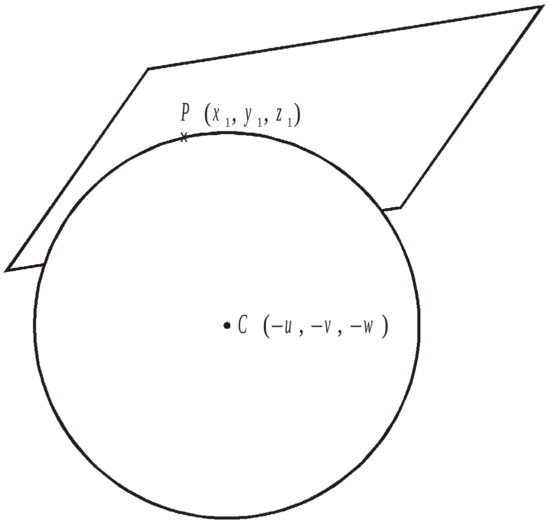
$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$.

4. To find the equation of the tangent plane at (x_1, y_1, z_1) on the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Let $Q(x, y, z)$ be any point in the tangent plane then direction ratios of PQ are $x-x_1,\ y-y_1,\ z-z_1$ and direction ratios of PC are $x_1+u,\ y_1+v,\ z_1+w$ where C is the centre of the sphere.

As PQ is $\perp^r$ to PC, we have

$$(x-x_1)(x_1+u)+(y-y_1)(y_1+v)+(z-z_1)(z_1+w)=0$$



ie $xx_1 - x_1^2 + ux - ux_1 + yy_1 - y_1^2 + vy - vy_1$

$$+ zz_1 - z_1^2 + wz - wz_1 = 0$$

ie $xx_1 + yy_1 + zz_1 + ux - ux_1 + vy - vy_1 + wz - zw_1$

$$-x_1^2 - y_1^2 - z_1^2 = 0$$

Since (x_1, y_1, z_1) is a point on the sphere

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

$\therefore\ -x_1^2 - y_1^2 - z_1^2 = 2ux_1 + 2vy_1 + 2wz_1 + d$

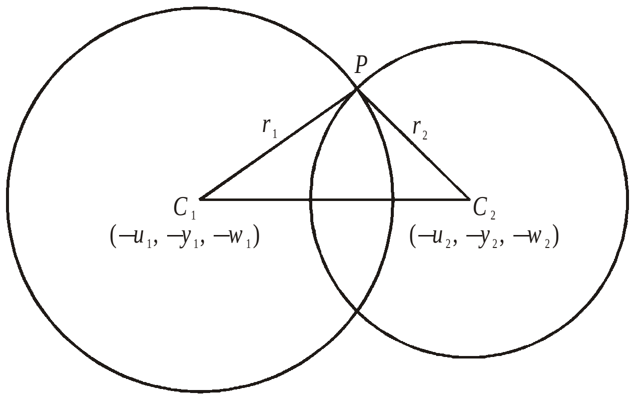
substituting this in (1) and simplifying, we have

$$xx_1 + yy_1 + zz_1 + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0$$

is the equation to the tangent plane.

5. To find the condition for two spheres to cut orthogonally (ie. the tangents plane at a point of intersection are at right angles).

Let C_1 **&** C_2 be the centres of two spheres whose radii are r_1 **&** r_2. P is point of intersection.



Let the spheres be

$$S_1:\ x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$S_1:\ x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

Now $C_1\hat{P}C_2 = 90^\circ \quad \therefore\ C_1C_1^2 = r_1^2 + r_2^2$

ie $(-u_1+u_2)^2 + (-v_1+v_2)^2 + (-w_1+w_2)^2$

$$= \left(\sqrt{u_1^2 + v_1 + w_1^2 - d_1}\right)^2 + \left(\sqrt{u_2^2 + v_2 + w_2^2 - d_2}\right)^2$$

ie $u_1^2 + u_2^2 - 2u_1u_2 + v_1^2 + v_2^2 - 2v_1v_2 + w_1^2 + w_2^2 - 2w_1w_2 = u_1^2 + v_1^2 + w_1^2 - d_1 + u_2^2 + v_2^2 + w_2^2 - d_2$

ie $-2u_1u_2 - 2v_1v_2 - 2w_1w_2 = -d_1 - d_2$

ie $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

is the required condition.

Note :- If $ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0$ be the given equation then it can be reduced to the standard form as

$$x^2 + y^2 + z^2 + \frac{2u}{a}x + \frac{2v}{a}y + \frac{2w}{a}z + \frac{d}{a} = 0$$

Examples

1. Find the equation of the sphere whose centre is at (2, –3, 4) and radius 3 units.

Solution: The required equation is $(x-2)^2 + (y+3)^2 + (z-4)^2 = 3^2$

ie $x^2 - 4x + 4 + y^2 + 6y + 9 + z^2 - 8y + 16 - 9 = 0$

ie $x^2 + y^2 + z^2 - 4x + 6y - 8z + 20 = 0$

2. Find the centre and radius of the sphere $x^2+y^2+z^2-6x+4y-3z-\frac{3}{4}=0$

Solution : Comparing with standard equation $2u=-6,\ 2v=4,\ 2w=-3,\ d=-\frac{3}{4}$

$$\therefore\ u=-3,\ v=2,\ w=-\frac{3}{2}$$

$$\therefore\ \text{Centre}=(-u,-v,-w)=\left(3,-2,\frac{3}{2}\right)$$

$$\text{radius } r=\sqrt{9+4+\frac{9}{4}+\frac{3}{4}}=\sqrt{13+\frac{9+3}{4}}=\sqrt{16}=4.$$

3. Find the equation of the sphere whose diameter is the line joining the points (4, 0, –2) and (0, 3, 1).

Solution : Required equation is of the form $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

$\therefore$ Required equation is $(x-4)(x-0)+(y-0)(y-3)+(z+2)(z-1)=0$

ie $x^2-4x+y^2-3y+z^2+z-2=0$

ie $x^2+y^2+z^2-4x-3y+z-2=0$

4. Find the equation of the sphere through the circle $x^2+y^2+z^2=9,\ 2x+3y+4z=5$ and the point (1, 2, 3).

Solution: The required equation is $x^2+y^2+z^2-9+\lambda(2x+3y+4z-5)=0$, (1, 2, 3) lie on it

$\therefore\ 1+4+9-9+\lambda(2+6+12-5)=0$

ie $5+\lambda(15)=0\Rightarrow\lambda=-\frac{1}{3}$

$\therefore$ Equation of the sphere is $(x^2+y^2+z^2-9)-\frac{1}{3}(2x+3y+4z-5)$

ie $3x^2+3y^2+3z^2-27-2x-3y-4z+5=0$

ie $3x^2+3y^2+3z^2-2x-3y-4z-22=0$

5. Find the equation of the sphere for which the circle $x^2+y^2+z^2-3x+4y-2z-5=0,\ 5x-2y+4z+7=0$ is a great circle.

Solution: Required equation of the sphere is $x^2+y^2+z^2-3x+4y-2z-5+\lambda(5x-2y+4z+7)=0$

Centre of this sphere is $\left[-\frac{1}{2}(-3+5\lambda),\ -\frac{1}{2}(4-2\lambda),\ -\frac{1}{2}(-2+4\lambda)\right]$

Circle will be a great circle if the centre lies on the plane.

$\therefore\ -\frac{5}{2}(-3+5\lambda)+(4-2\lambda)-2(-2+4\lambda)+7=0$

ie $\frac{15}{2}-\frac{25}{2}\lambda+4-2\lambda+4-8\lambda+7=0$

ie $-\frac{45}{2}\lambda+\frac{45}{2}=0\Rightarrow\lambda=1$

$\therefore$ Equation to the required sphere is $x^2+y^2+z^2-3x+4y-2z-5+5x-2y+4z+7=0$

ie $x^2+y^2+z^2+2x+2y+2z+2=0.$

6. Find the equation of the tangent plane to the sphere $3(x^2+y^2+z^2)-2x-3y-4z-22=0$ at the point $(1,2,3)$

Solution : Equation of the sphere is $x^2+y^2+z^2-\frac{2}{3}x-y-\frac{4}{3}z-\frac{22}{3}=0$

Tangent plane at $(1,2,3)$ is $x(1)+y(2)+z(3)-\frac{1}{3}(x+1)-\frac{1}{2}(y+2)-\frac{2}{3}(z+3)-\frac{22}{3}=0$

ie $x+2y+3z-\frac{1}{3}x-\frac{1}{3}-\frac{y}{2}-1-\frac{2z}{3}-2-\frac{22}{3}=0$

multiply through out by 6

$6x+12y+18z-2x-2-3y-6-4z-12-44=0$

ie $4x+9y+14z-64=0$

7. Show that the spheres $x^2+y^2+z^2+6y+14z+8=0$ and $x^2+y^2+z^2+6x+4z+20=0$ intersect at right angles.

Solution : Condition for two spheres to be orthogonal is $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

For the first sphere $u_1=0,\ v_1=3,\ w_1=7,\ d_1=8$

For the second sphere $u_2=3,\ v_2=0,\ w_2=2,\ d_2=20$

$\therefore\ 2u_1u_2+2v_1v_2+2w_1w_2=0+0+28=28$

$d_1+d_2=8+20=28$

$\therefore\ 2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

$\therefore$ The two spheres intersect orthogonally.

8. Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$ and find the point of contact.

Solution : Let C be the centre of the sphere, then C = (1, 2, –1).

Length of the $\perp^r$ from C upon the plane

$2x-2y+z+12=0$ is $\frac{2-4-1+12}{\sqrt{4+4+1}}=\frac{9}{3}=3.$

Let P be the point of Contact then CP is $\perp^r$ to the plane.

$\therefore$ Equation to CP can be taken as

$$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}=r\ \text{(say)}$$

$\therefore$ Co-ordinates of P can be taken as $(2r+1,\ -2r+2,\ r-1)$

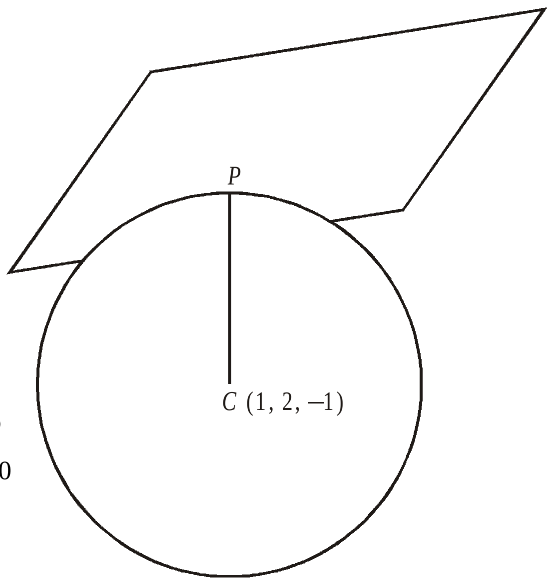
This will lie in the plane if $2(2r+1)-2(-2r+2)(r-1)+12=0$

ie $4r+2+4r-4+r-1+12=0$

$9r+9=0 \Rightarrow r=-1$

$\therefore$ Co-ordinates of P are $(-2+1,\ 2+2,\ -1-1)=(-1,4,-2)$

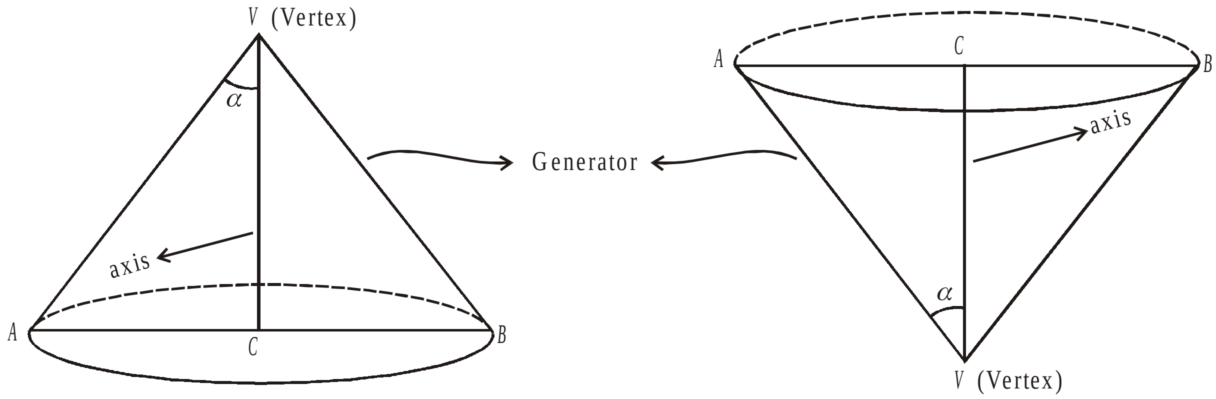
$\therefore$ Point of contact is $(-1,4,-2)$



Right Circular Cone

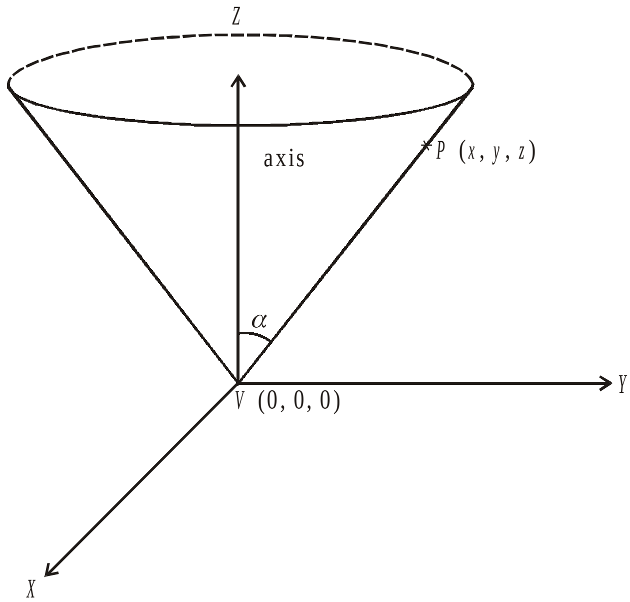
Definition : A right circular cone is a surface genereated by a straight line which passes through a fixed point (called **Vertex**) and makes a constant angle with a fixed line (called **axis**).

The constant angle is called semi-vertical angle of the cone.



Examples

(1) Find the equation of the right circular cone whose vertex is at the origin, semi-vertical angle is α and having axis of Z as its axis.



Solution : Let $P(x, y, z)$ be any point on the generator, OZ is the axis whose direction cosines are 0, 0, 1.

Direction ratios of OP are x, y, z

$$\therefore \ \mathbf{cos}\,\alpha = \frac{x\cdot 0 + y\cdot 0 + z\cdot 1}{\sqrt{x^2+y^2+z^2}\cdot 1} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Squaring both sides

$$\mathbf{cos}^2\,\alpha = \frac{z^2}{x^2+y^2+z^2}$$

$$\therefore \ x^2+y^2+z^2 = z^2\,\mathbf{sec}^2\,\alpha$$

ie $x^2+y^2 = z^2(\mathbf{sec}^2\,\alpha - 1) = z^2\,\mathbf{tan}^2\,\alpha$

$\therefore$ Equation of the cone is $x^2+y^2 = z^2\,\mathbf{tan}^2\,\alpha$

(2) Find the equation of the right circular cone with semi-vertical angle 30º, vertex at the point (2, 1, –3) and the direction ratios of whose axis are 3, 4, –1.

Solution : Let $P(x, y, z)$ be any point on the generator and vectex $V = (2, 1, -3)$

Direction ratios of PV are $x-2,\ y-1,\ z+3$

Direction ratios of axis are $3, 4, -2$ **&** $\alpha = 30°$

$$\therefore \ \mathbf{cos}\,30° = \frac{3(x-2)+4(y-1)-1(z+3)}{\sqrt{(x-2)^2+(y-1)^2+(z+3)^2}\ \sqrt{9+16+1}}$$

ie $$\frac{\sqrt{3}}{2} = \frac{3x+4y-z-6-4-3}{\sqrt{(x-2)^2+(y-1)^2+(z+3)^2}\ \sqrt{26}}$$

Cross-multiplying and squaring, we have

$$3\times 26[(x-2)^2+(y-1)^2+(z+3)^2]=4(3x+4y-z-13)^2$$

ie $78[x^2-4x+4+y^2-2y+1+z^2+6z+9]=4[(9x^2+16y^2+(z+13)^2+24xy-8y(z+13)-6x(z+13)]$

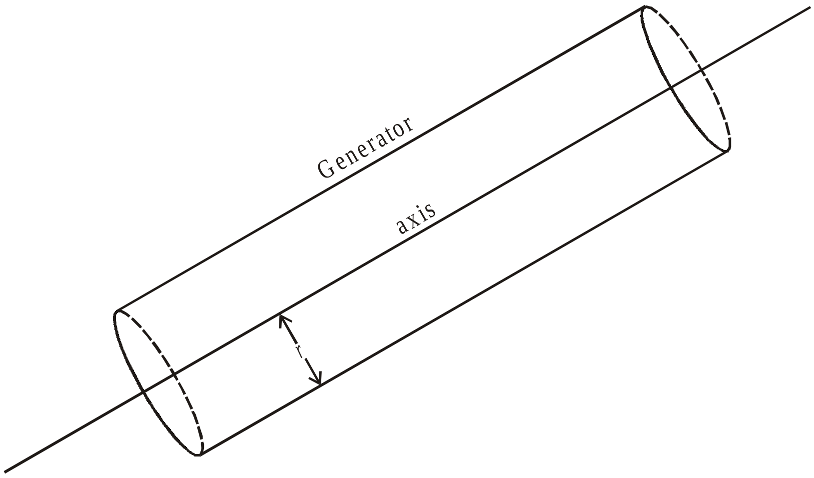
ie $39(x^2+y^2+z^2-4x-2y+6z+14)=2[9x^2+16y^2+z^2+26z+169+24xy-8yz-104y-6xz-78x]$

ie $(39-18)x^2+(39-32)y^2+(39-2)z^2-48xy+16yz+12zx+130y+182z+208=0$

ie $21x^2+7y^2+37z^2-48xy+16yz+12zx+130y+182z+208=0$

Right Circular Cylinder

Definition : A right circular cylinder is a surface generated be a straight line which is parallel to a fixed line (called axis) and is at a constant distance from it (called radius).



Examples

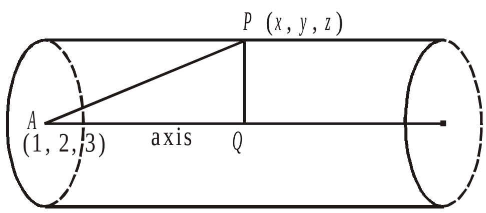
(1) Find the equation of the right circular cylinder of radius 2 units, whose axis passes through (1, 2, 3) and has direction ratios 2, –3, 6.

Solution : Let $P(x, y, z)$ be any point on the generator and $A(1, 2, 3)$ be the given point on the axis.

Draw $PQ\perp^r$ to the axis, then AQ is the projection of $\overrightarrow{AP}$ upon the axis

$$\therefore\ AQ=\frac{2(x-1)-3(y-2)+6(z-3)}{\sqrt{4+9+36}}$$

$$=\frac{2x-3y+6z-2+6-18}{\sqrt{49}}=\frac{2x-3y+6z-14}{7}$$



$PQ=2$ units & $AP=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$

Now $AP^2=AQ^2+PQ^2$

$$\therefore\ (x-1)^2+(y-2)^2+(z-3)^2=\left(\frac{2x-3y+6z-14}{7}\right)^2+4$$

ie $49[(x-1)^2+(y-2)^2+(z-3)^2]=[2x-3y+(6z-14)]^2+196$

ie $49[x^2+y^2+z^2-2x-4y-6z+14]=4x^2+9y^2+(6z-14)^2-12xy-6y(6z-14)+4x(6z-14)+196$

$$=4x^2+9y^2+36z^2-168z+196-12xy-36yz+84y+24xz-56x+196$$

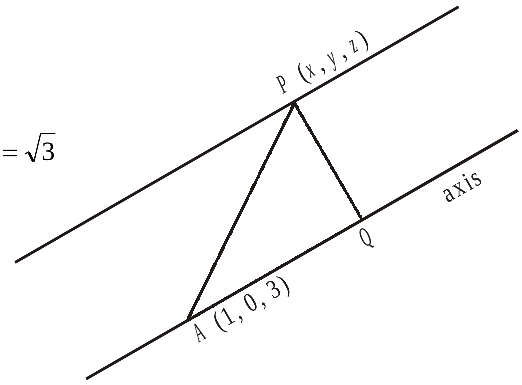
$$\therefore\ 45x^2+40y^2+13z^2+12xy+36yz-24xz-42x-280y-126z+294=0$$

(2) Find the equation of the right circular cylinder of radius $\sqrt{3}$ units and axis given by $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$

Solution : Let $A(1, 0, 3)$ be a point on the axis, Let $P(x, y, z)$ be a point on the generator

$\overrightarrow{AP}=(x-1)\hat{i}+(y-0)\hat{j}+(z-3)\hat{k}$

Draw $PQ \perp^r$ to the axis



$AQ =$ Projection of $\overrightarrow{AP}$ on $AQ = \frac{2(x-1)+3y+1(z-3)}{\sqrt{4+9+1}}$ **&** $PQ=\sqrt{3}$

Now $AP^2 = AQ^2 + PQ^2$

$\therefore\ (x-1)^2+y^2+(z-3)^2=\frac{(2x+3y+z-5)^2}{14}+3$

ie $14[(x-1)^2+y^2+(z-3)^2=(2x+3y+z-5)^2+42$

ie $14[x^2+y^2+z^2-2x-6z+10]=4x^2+9y^2+(z-5)^2+12xy+6y(z-5)+4x(z-5)+42$

$=4x^2+9y^2+z^2-10z+25+12xy+6yz-30y+4xz-20x+42$

ie $10x^2+5y^2+13z^2-12xy-6yz-4xz-8x+30y-74z+73=0$

is the equation of the right circular cone.

Exercise

(1) Find the equation of the sphere through the circle $x^2+y^2+z^2+2x+3y+6=0,\ x-2y+4z=9$ and passing through the centre of $x^2+y^2+z^2-2x+4y-6z+5=0$

(Ans: $x^2+y^2+z^2+7y-8z+24=0$)

(2) Obtain the equation of the sphere having the circle $x^2+y^2+z^2+10y-4z-8=0,\ x+y+z=3$ as the great cirlce.

(Ans: $x^2+y^2+z^2-4x+6y-8z+4=0$)

(3) Find the equation of the sphere passing through the circle $x^2+y^2+z^2-2x-3y+4z+8=0,\ x-2y+z=8$ and having its centre on the plane $4x-5y-z=3$

(Ans: $13(x^2+y^2+z^2)-35x-21y+43z+176=0$)

(4) Obtain the equation of the sphere whose centre is on the line $2x-3y=0=5y+2z$ and passing through two points (0, –2, –4), (2, –1, –1)

(Ans: $x^2+y^2+z^2-6x-4y+10z+12=0$)

(5) Find the value of a for which the plane $x+y+z=a\sqrt{3}$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$

(Ans: $\sqrt{3}\pm 3$)

(6) Find the equations of the spheres passing through the circle $x^2+y^2+z^2-6x-2z+5=0,\ y=0$ and touching the plane $3y+4z+5=0$

(Ans: $4(x^2+y^2+z^2)-24x-11y+8z+20=0$ and $x^2+y^2+z^2-6x-4y-2z+5=0$)

(7) Find the equations of two spheres which pass through the circle $x^2+y^2+z^2-2x+2y+4z-3=0,\ 2x+y+z=4$ and touch the plane $3x+4y=14$

(Ans: $x^2+y^2+z^2-2x+2y+4z-3=0$ and $x^2+y^2+z^2+2x+4y+6z-11=0$)

(8) Find the equation if the right circular cone which passes through the point $(1,1,2)$ and has its vertex at the origin and the line $\frac{x}{2}=\frac{y}{-4}=\frac{z}{3}$ as axis.

(Ans: $4x^2+40y^2+19z^2-48xy-72yz+36xz=0$)

(9) Find the equation of the right circular cone with the vertex (1, –2, –1), semi-vertical angle 60º and the line $\frac{x-1}{3}=\frac{y+2}{-4}=\frac{z+1}{5}$ as its axis.

(Ans: $7x^2-7y^2-25z^2+48xy+80yz-60zx+22x+4y+170z+78=0$)

(10) Find the equation of the circular cylinder having for its base the circle $x^2+y^2+z^2=9,\ x-y+z=3.$

(Ans: $x^2+y^2+z^2+xy+yz-zx=9$)

DIFFERENTIAL CALCULUS

Limits of functions

Consider $y = f(x) = \frac{x^2-1}{x-1}$ the function is defined for all values of x except for $x = 1$.

$\therefore$ for $x = 1$, $f(x) = \frac{1-1}{1-1} = \frac{0}{0}$ which is indeterminate.

Let us consider the values of $f(x)$ as x approaches 1

x	$f(x) = \frac{x^2-1}{x-1}$
.9	1.9
.99	1.99
.999	1.999
1.01	2.01
1.001	2.001
1.0001	2.0001

It can be seen from the above values that as x approaches 1, $\frac{x^2-1}{x-1}$ approaches 2.

This value 2 is called "Limit of $\frac{x^2-1}{x-1}$ as x approaches 1" which can be written as $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$ or $\underset{x\to 1}{Lt}\frac{x^2-1}{x-1} = 2$.

In general the limit of a function $f(x)$ as x approaches a is denoted as l and which is written as

$$\lim_{x\to a} f(x) = l \quad \text{or} \quad \underset{x\to a}{Lt}\, f(x) = l$$

Properties

(1) $\lim\limits_{x\to a}\left[f(x) \pm g(x)\right] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

(2) $\lim\limits_{x\to a}\left[f(x)g(x)\right] = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$ in particular $\lim\limits_{x\to a} kf(x) = k \cdot \lim\limits_{x\to a} f(x)$ where k is a contant

(3) $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ provided $\lim\limits_{x\to a} g(x) \neq 0$.

Standard Limits

(1) $\lim\limits_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$

(2) $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$ (θ in radians) also $\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta}=1$

(3) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=e\quad 2<e<3$ or $\lim\limits_{x\to 0}(1+x)^{1/x}=e$

(4) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a\quad (a>0)$ in particular $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1.$

Examples

(1) $\lim\limits_{x\to 0}\dfrac{x^2+4x+3}{x^2-5x+4}=\dfrac{0+0+3}{0-0+4}=\dfrac{3}{4}$

(2) $\lim\limits_{x\to\infty}\dfrac{2x^2-3x+4}{3x^2+2x+1}$ (dividing Nr & Dr by x^2) $=\lim\limits_{x\to\infty}\dfrac{2-\dfrac{3}{x}+\dfrac{4}{x^2}}{3+\dfrac{2}{x}+\dfrac{1}{x^2}}=\dfrac{2-0+0}{3+0+0}=\dfrac{2}{3}$

(3) $\lim\limits_{x\to 3}\dfrac{x^4-81}{x-3}=\lim\limits_{x\to 3}\dfrac{x^4-3^4}{x-3}=4(3)^3=108$

(4) $\lim\limits_{x\to -a}\dfrac{x^7+a^7}{x^5+a^5}=\lim\limits_{x\to -a}\dfrac{\dfrac{x^7+a^7}{x+a}}{\dfrac{x^5+a^5}{x+a}}=\lim\limits_{x\to -a}\dfrac{\dfrac{x^7-(-a)^7}{x-(-a)}}{\dfrac{x^5-(-a)^5}{x-(-a)}}=\dfrac{7(-a)^6}{5-(-a)^4}=\dfrac{7}{5}a^2$

(5) $\lim\limits_{x\to 0}\dfrac{\sin 7x}{x}=\lim\limits_{x\to 0}\dfrac{\sin 7x}{x}\times 7=1\times 7=7$

(6) $\lim\limits_{\theta\to 0}\dfrac{1-\cos 2\theta}{\theta^2}=\lim\limits_{\theta\to 0}\dfrac{2\sin^2\theta}{\theta^2}=2$

(7) $\lim\limits_{x\to 0}\dfrac{\tan 3x-x}{3x-\sin x}$ (dividing Nr & Dr by x) $=\lim\limits_{x\to 0}\dfrac{\dfrac{\tan 3x}{x}-1}{3-\dfrac{\sin x}{x}}=\dfrac{3-1}{3-1}=1$

(8) $\lim\limits_{x\to 0}(1+ax)^{b/x}=\lim\limits_{x\to 0}\left[(1+ax)^{\frac{1}{ax}}\right]^{ab}=e^{ab}$

(9) $\lim\limits_{n\to\infty}\left(1+\dfrac{3}{n}\right)^n=\lim\limits_{n\to\infty}\left[\left(1+\dfrac{3}{n}\right)^{n/3}\right]^3=e^3$

(10) $\lim\limits_{x\to 0}(1-2x)^{1/x}=\lim\limits_{x\to 0}\left[(1-2x)^{-\frac{1}{2x}}\right]^{-2}=e^{-2}$

(11) $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x}=\lim\limits_{x\to 0}\dfrac{(a^x-x)-(b^x-x)}{x}=\lim\limits_{x\to 0}\dfrac{a^x-x}{x}-\lim\limits_{x\to 0}\dfrac{b^x-x}{x}=\log_e a-\log_e b=\log_e\dfrac{a}{b}$

(12) $\lim\limits_{x\to 0}\dfrac{2^x-1}{\sin x}=\lim\limits_{x\to 0}\dfrac{\frac{2^x-1}{x}}{\frac{\sin x}{x}}=\dfrac{\log_e 2}{1}=\log_e 2$

Continuity of a function

A function $f(x)$ is said to be continuous at $x=a$ if $\lim\limits_{x\to a} f(x)=f(a)$

$\therefore$ A function $f(x)$ is said to be continuous if $\lim\limits_{x\to 0} f(x)$ exists, $f(a)$ exists and they are equal.

If these donot happen then the function is said to be not continuous or discontinuous.

A function $f(x)$ is said to be continuous in an interval if it is continuous at all points in the interval.

Examples

(1) $f(x)=\dfrac{x^2-1}{x-1}$ is not continuous at $x=1$ $\because \lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}=2$ exists but $f(0)=\dfrac{0}{0}$ donot exists.

where as it is continuous at all other values of x.

(2) Discuss the continuity of function $f(x)=\begin{cases}4x+3 & \text{for } x\ge 4\\ 3x+7 & \text{for } <4\end{cases}$ at $x=4$.

Solution : While finding the limit of a function $f(x)$ as x approaches a, if we consider the limit of the function as x approaches a from left hand side, the limit is called **'Left Hand Limit' (LHL)** and if x approaches a from right hand side the limit is called **'Right Hand Limit' (RHL)** and the limit of the function is said to exists if both LHL & RHL exists and are equal, for convenience LHL & RHL are denoted as $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ and further

LHL $=\lim\limits_{x\to a^-} f(x)=\lim\limits_{h\to 0} f(a-h)$ **&** RHL $=\lim\limits_{x\to a^+} f(x)=\lim\limits_{h\to 0} f(a+h)$

For the given problem

LHL at $x=4$ is $\lim\limits_{x\to 4^-} f(x)=\lim\limits_{h\to 0} f(4-h)=\lim\limits_{h\to 0} 3(4-h)+7=19$

RHL at $x=4$ is $\lim\limits_{x\to 4^+} f(x)=\lim\limits_{h\to 0} f(4+h)=\lim\limits_{h\to 0} 4(4+h)+3=19$

and $f(4)=19$

$\therefore$ The function is continuous at $x=4$

(3) Examine the continuity of $f(x)=\begin{cases}\dfrac{\sin x}{x} & \text{for } x\ne 0\\ 2 & \text{for } x=0\end{cases}$ at $x=0$.

Solution : $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$, but $f(0)=2$

$\therefore$ The function is discontinuous at $x=0$.

(4) Examine the continuity of the function $f(x)=\begin{cases}5x-4 & \text{for } 0<x\le 1\\ 4x^2-2x & \text{for } 1<x<2\\ 4x+4 & \text{for } x\ge 2\end{cases}$

at $x=1$ and $x=2$.

at $x=1$, LHL $= \lim\limits_{x\to1^-} f(x) = \lim\limits_{h\to0} f(1-h) = \lim\limits_{h\to0} 5(1-h)-4 = 1$

RHL $= \lim\limits_{h\to0} f(1+h) = \lim\limits_{h\to0} 4(1+h)^2 - 2(1+h) = 4-2 = 2$

LHL $\neq$ RHL at $x=1$

$\therefore$ The function is discontinuous at $x=1$.

at $x=2$, LHL $= \lim\limits_{h\to0} f(2-h) = \lim\limits_{h\to0} 4(2-h)^2 - 2(2-h) = 16-4 = 12$

RHL $= \lim\limits_{h\to0} f(2+h) = \lim\limits_{h\to0} 4(2+h)+4 = 12$

and $f(2) = 4\times2+4 = 12$

$\therefore \lim\limits_{x\to2} f(x) = f(2)$

$\therefore$ The function is continuous at $x=2$.

Differentiability of a function

A function $f(x)$ is said to be differentiable at a point a if $\lim\limits_{h\to0} \dfrac{f(a+h)-f(a)}{h}$ exists and the derivative is denoted as $f'(a)$.

A function $f(x)$ is said to be differentiable at x if $\lim\limits_{\delta x\to0} \dfrac{f(x+\delta x)-f(x)}{\delta x}$ exists and the derivative is denoted as $f'(x)$.

[δx is called the increment in x which is very very small].

If $y = f(x)$, the derivative is denoted by $\dfrac{dy}{dx}$ or $f'(x)$

Note :- A function which is differentiable is always continuous but the converse is not always true.

eg. $y = f(x) = |x| = \begin{cases} -x & \text{for } x<0 \\ x & \text{for } x\geq 0 \end{cases}$ is continuous for all x but not differentiable at $x = 0$.

To find the derivates of x^n, $\log_e x$, a^x, $\sin x$, $\cos x$ and a constant C with respect to x.

(1) Let $y = x^n$, $\dfrac{dy}{dx} = \lim\limits_{\delta x\to0} \dfrac{(x+\delta x)^n - x^n}{x+\delta x - x} = nx^{n-1}$

Eg. (i) $\dfrac{d}{dx}\left(x^7\right) = 7x^6$, (ii) $\dfrac{d}{dx}\left(x^{-4}\right) = -4x^{-5}$ (iii) $\dfrac{d}{dx}\left(x^{9/4}\right) = \dfrac{9}{4}x^{9/4-1} = \dfrac{9}{4}x^{5/4}$

(iv) $\dfrac{d}{dx}\left(x^{-5/3}\right) = -\dfrac{5}{3}x^{-5/3-1} = -\dfrac{5}{3}x^{-8/3}$ (v) $\dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{1}{2\sqrt{x}}$

(2) Let $y = \log_e x$ then $\dfrac{dy}{dx} = \lim\limits_{\delta x\to0} \dfrac{\log_e(x+\delta x) - \log_e x}{\delta x}$

$$= \lim_{\delta x\to0} \frac{1}{x}\cdot\frac{x}{\delta x}\log\left(\frac{x+\delta x}{x}\right) = \lim_{\delta x\to0} \frac{1}{x}\cdot\log\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} = \frac{1}{x}\log_e e = \frac{1}{x}.$$

(3) Let $y = a^x$

$\dfrac{dy}{dx} = \lim\limits_{\delta x\to0} \dfrac{a^{x+\delta x} - a^x}{\delta x} = \lim\limits_{\delta x\to0} a^x \dfrac{(a^{\delta x}-1)}{\delta x} = a^x \log_e a$ $(a>0)$ in particular $\dfrac{d}{dx}(e^x) = e^x$

(4) Let $y = \sin x$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\frac{x+\delta x+x}{2}\sin\frac{x+\delta x-x}{2}}{\delta x} = \lim_{\delta x \to 0} \cos\left(x+\frac{\delta x}{2}\right)\cdot\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}} = \cos(x+0)\cdot 1 = \sin x.$$

(5) Let $y = \cos x$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\cos(x+\delta x) - \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} -\sin\left(x+\frac{\delta x}{2}\right)\cdot\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}} = -\sin(x+0)\cdot 1 = -\sin x.$$

(6) Let $y = c$ (a constant) $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{c-c}{\delta x} = 0$

Thus derivative of a constant is zero.

Thus we have the following standard derivatives

Function $y = f(x)$	**Derivative**
x^n	nx^{n-1}
$\log x$	$\frac{1}{x}$
a^x	$a^x \log_e a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
constant	zero

Rules for differentiation

I **<u>Sum Rule</u>**

If $y = u + v$ whose u **&** v are functions of x then to find $\frac{dy}{dx}$

Give a small increment δx to x, let the corresponding increments in u, v **&** y be δu, δv **&** δy respectively.

Then $y + \delta y = u + \delta u + v + \delta v$

Subtracting

$$y + \delta y - y = u + \delta u + v + \delta v - u - v \quad \text{ie} \quad \delta y = \delta u + \delta v$$

divide through out by δx then $\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}$

take limits on both sides as $\delta x \to 0$.

Then $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + \lim_{\delta x \to 0} \frac{\delta v}{\delta x}$

ie $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Note:- If $y = u + v - w$ then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$

Eg. (1) If $y = x^6 + \mathbf{sin}\,x - \mathbf{cos}\,x$ then $\frac{dy}{dx} = \frac{d}{dx}(x^6) + \frac{d}{dx}(\mathbf{sin}\,x) - \frac{d}{dx}(\mathbf{cos}\,x)$

$= 6x^5 + \mathbf{cos}\,x - (-\mathbf{sin}\,x) = 6x^5 + \mathbf{cos}\,x + \mathbf{sin}\,x.$

(2) If $y = 3^x - \mathbf{log}_e\,x + c$ then $\frac{dy}{dx} = 3^x\,\mathbf{log}_e\,3 - \frac{1}{x} + 0.$

II Product Rule

If $y = uv$ where u & v are functions of x then to find $\frac{dy}{dx}$.

Give a small increment δx to x, let the corresponding increments in u, v & y be δu, δv & δy respectively.

then $y + \delta y = (u + \delta u)(v + \delta v) = uv + u\delta v + v\delta u + \delta u \cdot \delta v$

$\therefore\ \delta y = y + \delta y - y = u\delta v + v\delta u + \delta u \cdot \delta v$

divide through out by δx,

$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u \cdot \frac{\delta v}{\delta x}$$

take limit on both sides as $\delta x \to 0$,

then $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = u \lim\limits_{\delta x \to 0} \frac{\delta v}{\delta x} + v \lim\limits_{\delta x \to 0} \frac{\delta u}{\delta x} + \lim\limits_{\delta x \to 0} \delta u \cdot \frac{\delta v}{\delta x}$

ie $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + o\frac{dv}{dx}$

ie $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Note (1) If $y = kv$ where k is a constant then $\frac{dy}{dx} = k\frac{dv}{dx}$

(2) If $y = uvw$, then $\frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

Eg. (1) If $y = x^{3/2}\,\mathbf{sin}\,x$ then $\frac{dy}{dx} = x^{3/2}\frac{d}{dx}(\mathbf{sin}\,x) + \mathbf{sin}\,x\frac{d}{dx}\left(x^{3/2}\right)$

$= x^{3/2}\,\mathbf{cos}\,x + \mathbf{sin}\,x \cdot \frac{3}{2}x^{1/2}$

(2) If $y = 8\,\mathbf{cos}\,x$ then $\frac{dy}{dx} = -8\,\mathbf{sin}\,x$

(3) If $y = e^x\,\mathbf{sin}\,x \cdot \mathbf{log}\,x$ then $\frac{dy}{dx} = e^x\,\mathbf{sin}\,x\frac{d}{dx}(\mathbf{log}\,x) + e^x\,\mathbf{log}\,x \cdot \frac{d}{dx}(\mathbf{sin}\,x) + \mathbf{sin}\,x \cdot \mathbf{log}\,x\frac{d}{dx}(e^x)$

$= e^x\,\mathbf{sin}\,x \cdot \frac{1}{x} + e^x\,\mathbf{log}\,x \cdot \mathbf{cos}\,x + \mathbf{sin}\,x \cdot \mathbf{log}\,x \cdot e^x.$

III **Quotient Rule**

If $y=\dfrac{u}{v}$ where u **&** v are functions of x then to find $\dfrac{dy}{dx}$.

Give a small increment δx to x, let the corresponding increments in u, v **&** y be $\delta u, \delta v$ **&** δy respectively.

then $y+\delta y=\dfrac{u+\delta u}{v+\delta v}$

$$\therefore\ \delta y=\frac{u+\delta u}{v+\delta v}-\frac{u}{v}=\frac{v(u+\delta u)-u(v+\delta v)}{v(v+\delta v)}=\frac{vu+v\delta u-uv-u\delta v}{v(v+\delta v)}$$

divide through out by δx

$$\frac{\delta y}{\delta x}=\frac{v\dfrac{\delta u}{\delta x}-u\dfrac{\delta v}{\delta x}}{v(v+\delta v)}$$

take limit on both sides as $\delta x\to 0$

$$\text{ie}\ \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{v\dfrac{\delta u}{\delta x}-u\dfrac{\delta v}{\delta x}}{v(v+\delta v)}$$

$$\text{ie}\ \frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$$

This rule can be easily remembered in the following manner

If $y=\dfrac{u}{v}=\dfrac{Nr}{Dr}$ (say)

Then $\dfrac{dy}{dx}=\dfrac{Dr\ (\text{derivative of } Nr)-Nr(\text{derivative of } Dr)}{(Dr)^2}$

Note :- If $y=\dfrac{k}{v}$ where k is a constant then $\dfrac{dy}{dx}=-\dfrac{k}{v^2}\dfrac{dv}{dx}$

Eg. (1) If $y=\mathbf{tan}x=\dfrac{\mathbf{sin}\,x}{\mathbf{cos}\,x}$

then $\dfrac{dy}{dx}=\dfrac{\mathbf{cos}\,x\dfrac{d}{dx}(\mathbf{sin}\,x)-\mathbf{sin}\,x\dfrac{d}{dx}(\mathbf{cos}\,x)}{\mathbf{cos}^2\,x}$

$=\dfrac{\mathbf{cos}\,x\cdot\mathbf{cos}\,x-\mathbf{sin}\,x(-\mathbf{sin}\,x)}{\mathbf{cos}^2\,x}=\dfrac{\mathbf{cos}^2\,x+\mathbf{sin}^2\,x}{\mathbf{cos}^2\,x}=\dfrac{1}{\mathbf{cos}^2\,x}=\mathbf{sec}^2\,x.$

$\therefore$ If $y=\mathbf{tan}x$, then $\dfrac{dy}{dx}=\mathbf{sec}^2\,x.$

(2) If $y=\mathbf{cot}\,x=\dfrac{\mathbf{cos}\,x}{\mathbf{sin}\,x}$

then $\dfrac{dy}{dx}=\dfrac{\mathbf{sin}\,x(-\mathbf{sin}\,x)-\mathbf{cos}\,x(\mathbf{cos}\,x)}{\mathbf{sin}^2\,x}$

$=\dfrac{-\mathbf{sin}^2\,x-\mathbf{cos}^2\,x}{\mathbf{sin}^2\,x}=\dfrac{-1}{\mathbf{sin}^2\,x}=-\text{cosec}^2 x.$

$\therefore$ If $y = \mathbf{cot}\, x$, then $\frac{dy}{dx} - \text{cosec}^2 x$.

(3) If $y = \mathbf{sec}\, x = \frac{1}{\mathbf{cos}\, x}$

then $\frac{dy}{dx} = -\frac{1}{\mathbf{cos}^2 x} \cdot \frac{d}{dx}(\mathbf{cos}\, x)$

$= -\frac{1}{\mathbf{cos}^2 x}(-\mathbf{sin}\, x) = \frac{1}{\mathbf{cos}\, x}(\mathbf{sin}\, x) = \frac{1}{\mathbf{cos}\, x} \cdot \frac{\mathbf{sin}\, x}{\mathbf{cos}\, x} = \mathbf{sec}\, x\, \mathbf{tan} x.$

$\therefore$ If $y = \mathbf{sec}\, x$, then $\frac{dy}{dx} = \mathbf{sec}\, x\, \mathbf{tan} x.$

(4) If $y = \text{cosec}\, x = \frac{1}{\mathbf{sin}\, x}$

then $\frac{dy}{dx} = -\frac{1}{\mathbf{sin}^2 x} \cdot \mathbf{cos}\, x$

$= -\frac{1}{\mathbf{sin}\, x} \cdot \frac{\mathbf{cos}\, x}{\mathbf{sin}\, x} = -\text{cosec}\, x \cdot \mathbf{cot}\, x.$

$\therefore$ If $y = \text{cosec}\, x$, then $\frac{dy}{dx} = -\text{cosec}\, x \cdot \mathbf{cot}\, x.$

IV Chain Rule or function of a function rule

If $y = f(u)$ where $u = g(x)$ to find $\frac{dy}{dx}$, consider $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

ie $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ also if $y = f(x), u = g(v)$ & $v = h(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

Eg. (1) If $y = (ax^2 + bx + c)^n$ then put $u = ax^2 + bx + c$

$y = u^n \quad \therefore \frac{dy}{du} = nu^{n-1}$

$u = ax^2 + bx + c$

$\frac{du}{dx} = 2ax + b$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (ax^2 + bx + c)^n (2ax + b).$

(2) If $y = \mathbf{log}(x^3 - 2x^2 + 7)$

then $\frac{dy}{dx} = \frac{1}{(x^3 - 2x^2 + 7)} \times \frac{d}{dx}(x^3 - 2x^2 + 7) = \frac{3x^2 - 4x}{x^3 - 2x^2 + 7}$

(3) If $y = \mathbf{sin}(2x^2 + 4x - 3)$

then $\frac{dy}{dx} = \mathbf{cos}(2x^2 + 4x - 3)(4x + 4) = 4(x+1)\,\mathbf{cos}(2x^2 + 4x - 3)$

(4) If $y = \tan\left(\frac{x^2-1}{x^2+1}\right)$

then $\frac{dy}{dx} = \sec^2\left(\frac{x^2-1}{x^2+1}\right)\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$

$= \sec^2\left(\frac{x^2-1}{x^2+1}\right)\times\frac{(x^2+1)(2x)-(x^2-1)2x}{(x^2+1)^2} = \sec^2\left(\frac{x^2-1}{x^2+1}\right)\times\frac{2x(x^2+1-x^2+1)}{(x^2+1)^2}$

$= \sec^2\left(\frac{x^2-1}{x^2+1}\right)\times\frac{4x}{(x^2+1)^2}$

Derivative of Hyperbolic functions

(1) If $y = \sinh x$,

then $\frac{dy}{dx} = \frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^x-e^{-x}}{2} = \cosh x.$

(2) If $y = \cosh x$,

then $\frac{dy}{dx} = \frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right) = \frac{e^x+e^{-x}}{2} = \sinh x.$

(3) If $y = \tanh x = \frac{\sinh x}{\cosh x}$,

then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$

$= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \text{sech}^2 x.$

(4) If $y = \coth x = \frac{\cosh x}{\sinh x}$,

then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right)$

$= \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\text{cosech}^2 x.$

(5) If $y = \text{sech}\, x = \frac{1}{\cosh x}$,

then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\cosh x}\right) = -\frac{1}{\cosh^2 x}\frac{d}{dx}(\cosh x)$

$= \frac{-1}{\cosh^2 x}\cdot\sinh x = \frac{-1}{\text{sech}\, x}\cdot\frac{\sinh x}{\cosh x} = -\text{sech}\, x \tanh x.$

(6) If $y = \text{cosech}\, x = \dfrac{1}{\mathbf{sinh}\, x}$,

then $\dfrac{dy}{dx} = -\dfrac{-1}{\mathbf{sinh}^2 x}\dfrac{d}{dx}(\mathbf{sinh}\, x)$

$= \dfrac{-1}{\mathbf{sinh}^2 x}\cdot \mathbf{cosh}\, x = \dfrac{-1}{\mathbf{sinh}\, x}\cdot\dfrac{\mathbf{cosh}\, x}{\mathbf{sinh}\, x} = -\text{cosech}\, x \cdot \mathbf{coth}\, x.$

Implicit Functions

Function of the type $f(x, y) = 0$ is called Implicit function.

To find $\dfrac{dy}{dx}$ treat y as a function of x **&** use chain rule.

Eg. (1) If $ax^2 + 2hxy + by^2 = 0$

then $2ax + 2h\left(x\dfrac{dy}{dx} + y\right) + 2by\dfrac{dy}{dx} = 0$

ie $(2hx + 2by)\dfrac{dy}{dx} = -2ax - 2hy$

$\therefore \dfrac{dy}{dx} = \dfrac{-2(ax+hy)}{2(hx+by)} = \dfrac{-(ax+hy)}{hx+by}$

Eg. (2) If $x\,\mathbf{sin}\, y + y\,\mathbf{sin}\, x = 10$

differentiating w.r.t. x

$x\,\mathbf{cos}\, y\dfrac{dy}{dx} + \mathbf{sin}\, y + y\,\mathbf{cos}\, x + \mathbf{sin}\, x\dfrac{dy}{dx} = 0$

ie $(x\,\mathbf{cos}\, y + \mathbf{sin}\, x)\dfrac{dy}{dx} = -\mathbf{sin}\, y - y\,\mathbf{cos}\, x$

$\therefore \dfrac{dy}{dx} = \dfrac{-(\mathbf{sin}\, y + y\,\mathbf{cos}\, x)}{(x\,\mathbf{cos}\, y + \mathbf{sin}\, x)}$

Parametric functions

Functions of the type $x = f(t)$, $y = g(t)$ taken together is called **Parametric function,** where t is the paramter. Parametric functions are also denoted as $x = f(\theta)$, $y = f(\theta)$ where θ is the parameter.

To find $\dfrac{dy}{dx}$, consider $\dfrac{dx}{dt}$ **&** $\dfrac{dy}{dt}$ or $\dfrac{dx}{d\theta}$ & $\dfrac{dy}{d\theta}$

then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ or $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

Eg. (1) If $x = a\,\mathbf{cos}^3 t$, $y = a\,\mathbf{sin}^3 t$, find $\dfrac{dy}{dx}$

Solution : $x = a\,\mathbf{cos}^3 t$, $\dfrac{dx}{dt} = -3a\,\mathbf{cos}^2 t\,\mathbf{sin}\, t$, $y = a\,\mathbf{sin}^3 t$, $\dfrac{dy}{dt} = 3a\,\mathbf{sin}^2 t\,\mathbf{cos}\, t$

$\therefore \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3a\,\mathbf{sin}^2 t\,\mathbf{cos}\, t}{-3a\,\mathbf{cos}^2 t\,\mathbf{sin}\, t} = -\mathbf{tan}\, t$

(2) If $x = a(3\cos\theta - 4\sin^3\theta)$ & $y = a(3\sin\theta - 4\cos^3\theta)$, find $\dfrac{dy}{dx}$

Solution : $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a(3\cos\theta + 12\cos^2\theta\sin\theta)}{a(-3\sin\theta - 12\sin^2\cos\theta)}$

$$= \frac{3\cos\theta(1+4\sin\theta\cos\theta)}{-3\sin\theta(1+4\sin\theta\cos\theta)} = -\cot\theta$$

Differentiation using Logarithms

If $y = f(x)^{g(x)}$

use logarithms on both sides

$$\log y = \log f(x)^{g(x)} = g(x)\log f(x)$$

differentiating w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = g(x)\frac{1}{f(x)}f'(x) + g'(x)\log f(x) = \frac{g(x)f'(x)}{f(x)} + g'(x)\log f(x)$$

$$\therefore \frac{dy}{dx} = f(x)^{g(x)}\left[\frac{g(x)f'(x)}{f(x)} + g'(x)\log f(x)\right]$$

Eg. (1) Find $\dfrac{dy}{dx}$ if $y = x^x$

Solution : $\log y = x\log x$

$$\therefore \frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x\cdot 1$$

$$\text{ie } \frac{dy}{dx} = x^x[1+\log x]$$

(2) If $y = x^{\sin x} + (\cos x)^{\tan x}$, find $\dfrac{dy}{dx}$.

Solution: Let $y = u + v$ where $u = x^{\sin x}$ & $v = (\cos x)^{\tan x}$

$$\log u = \sin x\log x$$

$$\therefore \frac{1}{u}\frac{du}{dx} = \frac{\sin x}{x} + \cos x\log x$$

$$\text{ie } \frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x\log x\right]$$

$$v = (\cos x)^{\tan x}$$

$$\log v = \tan x\log\cos x$$

$$\therefore \frac{1}{v}\frac{dv}{dx} = \tan x\frac{1(-\sin x)}{\cos x} + \sec^2 x\log\cos x$$

$$\therefore \frac{dv}{dx} = (\cos x)^{\tan x}[-\tan^2 x + \sec^2 x\log\cos x]$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x\log x\right] + (\cos x)^{\tan x}[-\tan^2 x + \sec^2 x\log\cos x]$$

Derivatives of inverse Trigonometric functions

(1) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ for $|x| < 1$

(2) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ for $|x| < 1$

(3) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(4) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(5) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ for $|x| > 1$

(6) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$ for $|x| > 1$

Derivatives of inverse hyperbolic functions

(1) $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$ $\forall x$

(2) $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$ for $|x| > 1$

(3) $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ for $|x| < 1$

(4) $\frac{d}{dx}(\coth^{-1} x) = \frac{-1}{x^2-1}$ for $|x| > 1$

(5) $\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$ for $|x| < 1$

(6) $\frac{d}{dx}(\operatorname{cosech}^{-1} x) = \frac{-1}{x\sqrt{1+x^2}}$

Exercise

Find $\frac{dy}{dx}$ of the following

1. $y = x^2 + a^2$
2. $y = \log_e(3x+2)$
3. $y = \log_e \cos x$
4. $y = \frac{x+1}{x-1}$
5. $y = x^2 e^x$
6. $y = (2x+3)^2$
7. $y = ax^2 + bx + c$
8. $y = e^{\sqrt{x}}$
9. $y = (3x+5)^{1/3}$
10. $y = \sin x + \log_e x + x^2 + e^x$
11. $y = \operatorname{cosec} x - \tan x + x^5$
12. $y = \sinh^{-1} x$
13. $y = \sinh x \sinh^{-1} x$
14. $y = \sin x \cdot \sin^{-1} x$
15. $y = \sec^{-1} x + \operatorname{cosec}^{-1} x$

16. $y=(1+x^2)\tan^{-1}x$

17. $y=(1+x^2)\sin x$

18. $y=\dfrac{x^2+x+1}{\sqrt{x}}$

19. $y=(x+1)^2(x+2)$

20. $y=5e^x+4\log_e x$

21. $y=e^x(\sin x+\cos x)$

22. $y=\dfrac{e^x}{x^2}$

23. $y=\dfrac{e^x+e^{-x}}{2}$

24. $y=\sin x\cdot\sin 2x$

25. $y=(x+a)(x+b)(x+c)$

26. $y=\sin^{-1}\dfrac{2x}{1+x^2}$

27. $y=\tan^{-1}\left(\dfrac{x+a}{1-ax}\right)$

28. $y=\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

29. $y=e^{\sinh x}$

30. $y=\log_e\sqrt{\sin x}$

31. $y=\log_e(x^2\tan x)$

32. $y=(1-x^2)\cos^{-1}x$

33. $y=\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

34. $y=\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

35. $xy=c^2$

36. $\sin^{-1}x+\sin^{-1}y=0$

37. $x^2+y^2=a^2$

38. $x^{2/3}+y^{2/3}=a^{2/3}$

39. $x=at^2,\ y=2at$

40. $x=\sqrt{t},\ y=\dfrac{1}{\sqrt{t}}$

41. $x=4\cosh t,\ y=4\sinh t$

42. $x=a\cos t,\ y=b\sin t$

43. $x=\log_e\sec t,\ y=\tan^2 t$

44. $y=x\cos xy$

45. $y=\sin xy$

46. $y=\dfrac{(x^2+1)e^x}{\log_e x\cdot\operatorname{cosec}x}$

47. $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

48. $y=(1+x^2)\tan x$

49. $y=\dfrac{xe^x}{1+\cos x}$

50. $y=\dfrac{x^2+e^x}{1-x\log_e x}$

51. $y=\dfrac{\cos^{-1}x}{\log_e x}$

52. $y=\dfrac{x^3e^x}{\cos^{-1}x}$

53. $y=\sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)$

54. $y=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

55. $y=\dfrac{5x^3\cosh x}{\operatorname{cosech}x}$

56. $y=\sqrt{\dfrac{1+\sin x}{1-\sin x}}$

57. $y=\dfrac{x+3}{x-1}\sec x$

58. $y=(x-1)^2\operatorname{cosec}^{-1}(x-1)$

59. $\sin(x+y)=\log_e(x+y)$

60. $y\sin x+x\sin y=0$

61. $x=e^t(\cos t+\sin t),\ y=e^t(\cos t-\sin t)$

62. $x=a\cos^4 t,\ y=a\sin^4 t$

63. $x=\log_e\sec t,\ y=\tan^2 t$

64. $x=\dfrac{a}{1+\cos t},\ y=\dfrac{a}{1-\cos t}$

65. $y=\sin^n x\cdot\sin nx$

66. $y=(\sin^{-1}x)^{\tan x}$

67. $y=(\sin x)^{\sin x}$

68. $y=(\sin x)^{\tan^{-1}x}$

69. $y=(\log_x x)^{\log_e x}$

70. $e^{x+y}=e^x+e^y$

Successive Differentiation

If $y = f(x)$ then $\dfrac{dy}{dx} = f'(x)$ is also a function of x, hence further derivatives can be obtained.

The second derivative is denoted by $\dfrac{d^2y}{dx^2}$ or $f''(x)$ or y_2 or D^2y.

The third derivative is denoted by $\dfrac{d^3y}{dx^3}$ or $f'''(x)$ or y_3 or D^3y.

In general n^{th} derivative is denoted as $\dfrac{d^ny}{dx^n}$ or $f^{(n)}(x)$ or y_n or D^ny.

Examples

1. If $y = (1+x^2)\tan^{-1}x$ find $\dfrac{d^2y}{dx^2}$ at $x = 1$

Solution: $y = (1+x^2)\tan^{-1}x$

$$\frac{dy}{dx} = (1+x^2)\times\frac{1}{1+x^2} + \tan^{-1}x\cdot(2x) = 1 + 2x\tan^{-1}x.$$

$$\frac{d^2y}{dx^2} = 0 + 2x\times\frac{1}{1+x^2} + 2\tan^{-1}x$$

$$\text{at } x = 1, \left(\frac{d^2y}{dx^2}\right)_{x=1} = \frac{2}{1+1} + 2\times\frac{\pi}{4} = 1 + \frac{\pi}{2}.$$

2. If $y^2 = 4ax$, show that $\dfrac{d^2y}{dx^2} = \dfrac{-4a}{y^3}$

Solution: $y^2 = 4ax$

differentiating w.r.t. x

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

again differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{2a}{y^2}\left(\frac{2a}{y}\right) = \frac{-4a^2}{y^3}$$

3. If $x = a\left(\cos t + \log\tan\dfrac{t}{2}\right)$ & $y = a\sin t$, then show that $\dfrac{d^2y}{dx^2} = \dfrac{\sin t}{a\cos^4 t}$

Solution: $x = \left(\cos t + \log\tan\dfrac{t}{2}\right)$

$$\therefore \frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\frac{1}{2}}\frac{1}{2}\sec^2\frac{t}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\tan\frac{1}{2}\cos^2\frac{t}{2}}\right] = a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right] = a\left[-\sin t + \frac{1}{\sin t}\right] = a\left[\frac{-\sin^2 t + 1}{\sin t}\right] = \frac{a\cos^2 t}{\sin t}$$

$$y = a\sin t,\ \therefore \frac{dy}{dt} = a\cos t$$

$$\therefore\ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\cos^2 t}\times \sin t = \tan t$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \sec^2 t \times \frac{1}{dx/dt} = \frac{\sec^2 t}{\frac{a\cos^2 t}{\sin t}}$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{\sin t}{a\cos^4 t}$$

4. If $y = \sin^2 x$, find $\dfrac{d^2y}{dx^2}$

Solution: $\dfrac{dy}{dx} = 2\sin x\cos x = \sin 2x$

$$\frac{d^2y}{dx^2} = \cos 2x \cdot 2 = 2\cos 2x$$

5. If $y = x^2 \log_e x$ find $\dfrac{d^2y}{dx^2}$

Solution: $\dfrac{dy}{dx} = x^2 \cdot \dfrac{1}{x} + \log_e x \cdot 2x = x(2\log_e x + 1)$

$$\frac{d^2y}{dx^2} = x\left(\frac{2}{x}\right) + (2\log_e x + 1) = 3 + 2\log_e x$$

6. If $y = e^{ax}\cdot\sin(bx+c)$ find $\dfrac{d^2y}{dx^2}$

Solution: $\dfrac{dy}{dx} = e^{ax}\cdot\cos(bx+c)\cdot b + \sin(bx+c)\cdot ae^{ax}$

$$\frac{dy}{dx} = e^{ax}\left\{b\cos(bx+c) + a\sin(bx+c)\right\}$$

$$\frac{d^2y}{dx^2} = e^{ax}\left\{-b^2\sin(bx+c) + ab\cos(bx+c)\right\} + ae^{ax}\left\{b\cos(bx+c) + a\sin(bx+c)\right\}$$

$$= e^{ax}\left\{2ab\cos(bx+c) + (a^2 - b^2)\sin(bx+c)\right\}$$

7. If $x = a(\theta - \sin\theta),\ y = b(1-\cos\theta)$ find $\dfrac{d^2y}{dx^2}$

Solution: $\dfrac{dx}{d\theta} = a(1-\cos\theta),\ \dfrac{dy}{d\theta} = b\sin\theta$

$$\frac{dy}{dx} = \frac{b\sin\theta}{a(1-\cos\theta)} = \frac{b\cdot 2\sin(\theta/2)\cos(\theta/2)}{a\cdot 2\sin^2(\theta/2)} = \frac{b}{a}\cot(\theta/2)$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\text{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\cdot\frac{d\theta}{dx} = -\frac{b}{2a}\text{cosec}^2\frac{\theta}{2}\cdot\frac{1}{a(1-\cos\theta)} = -\frac{b}{4a^2}\text{cosec}^4(\theta/2).$$

8. If $y = e^{m\sin^{-1}x}$, then prove that $(1-x^2)y_2 - xy_1 - m^2y = 0$

Solution : $\frac{dy}{dx} = e^{m\sin^{-1}x}\cdot\frac{m}{\sqrt{1-x^2}}$

cross multiplying & squaring, we have

$$(1-x^2)y_1^2 = m^2y^2$$

differentiating w.r.t. x

$$(1-x^2)\cdot 2y_1y_2 + y_1^2(-2x) = m^2\cdot 2yy_1$$

Dividing throughout by $2y_1$, we get $(1-x^2)y_2 - xy_1 - m^2y = 0$

9. If $y = \sin(m\sin^{-1}x)$, then prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$

Solution : $y_1 = \cos(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}}$

$$y_1 = \frac{m\sqrt{1-y^2}}{\sqrt{1-x^2}} \Rightarrow (1-x^2)y_1^2 = m^2(1-y^2)$$

differentiating w.r.t. x

$$(1-x^2)2y_1y_2 + y_1^2(-2x) = m^2(-2yy_1)$$

Dividing throughtout by $2y_1$ we get $(1-x^2)y_2 - xy_1 + m^2y = 0.$

10. If $y = e^x\tan^{-1}x$, prove that $(1+x^2)y_2 - 2(1-x+x^2)y_1 + (1-x)^2y = 0$

Solution : $\frac{dy}{dx} = e^x\cdot\frac{1}{1+x^2} + e^x\tan^{-1}x$

$$(1+x^2)[y_1 - y] = e^x$$

differentiating w.r.t. x

$$(1+x^2)(y_2 - y_1) + 2x(y_1 - y) = e^x$$

$$(1+x^2)y_2 - (1+x^2-2x)y_1 - 2xy = (1+x^2)y_1 - (1+x^2)y$$

$$\Rightarrow (1+x^2)y_2 - 2(1+x^2-x)y_1 + (1-x)^2y = 0$$

Exercise

(1) If $4x^2 + 9y^2 = 36$ show that $\frac{d^2y}{dx^2} = -\frac{16}{9y^2}$

(2) If $x^2 + 2xy + 3y^2 = 1$ prove that $\frac{d^2y}{dx^2} = \frac{-2}{(x+3y)^3}$

(3) If $x = a\cos^3\theta$, $y = a\sin^3\theta$ find $\frac{d^2y}{dx^2}$

(4) If $x = a\tan\theta$, $y = \frac{1}{2}a\sin 2\theta$ find $\frac{d^2y}{dx^2}$

(5) If $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$ show that $\dfrac{d^2y}{dx^2} = \dfrac{\sec^3\theta}{a\theta}$

(6) If $y = \sin^{-1} 2x,$ then find $\dfrac{d^2y}{dx^2}$

(7) If $y = x\log_e x,$ then find $\dfrac{d^2y}{dx^2}$

(8) If $y = e^{4x}\cdot\sec 3x,$ then find $\dfrac{d^2y}{dx^2}$

(9) If $y = a^x,$ then find $\dfrac{d^2y}{dx^2}$

(10) If $x = at^2,\ y = 2at,$ then find $\dfrac{d^2y}{dx^2}$

(11) If $x^3y^3 = a^x,$ then find $\dfrac{d^2y}{dx^2}$

(12) If $y = a\cos mx + b\sin mx,$ prove that $\dfrac{d^2y}{dx^2} + m^2y = 0$

(13) If $y = \left(x + \sqrt{x^2+1}\right)^m,$ prove that $(x^2+1)y_2 + xy_1 - m^2y = 0$

(14) If $y = ax^{n+1} + bx^{-n},$ prove that $x^2y_2 - n(n+1)y = 0$

(15) If $x = \sin t,\ y = \sin pt,$ prove that $(1-x^2)y_2 - xy_1 + p^2y = 0$

(16) If $x^2 + xy + y^2 = 1,$ then prove that $(x+2y)^3y_2 + 6 = 0$

(17) If $y = e^{ax}\sin bx$ prove that $y_2 - 2ay_1 + (a^2+b^2)y = 0$

(18) If $y = ax + \dfrac{b}{x^2}$ then show that $x^2y_2 + 2(xy_1 - y) = 0$

(19) If $y = ax^{n+1} + \dfrac{b}{x^n}$ then show that $x^2y_2 = n(n+1)y$

(20) If $x = a\cos nt + b\sin nt$ then show that $\dfrac{d^2x}{dt^2} + n^2x = 0$

n^{th} derivative of Standard functions

1. If $y = (ax+b)^m$ to find y_n

Solution: $y_1 = m(ax+b)^{m-1}a\ ;\ y_2 = m(m-1)(ax+b)^{m-2}a^2$

differentiating n times, we have

$$y_n = m(m-1)\cdots(m-n+1)(ax+b)^{m-n}a^n$$

in particular if $m = -1,$ ie $y = \dfrac{1}{ax+b}$

$$y_n = (-1)(-2)(-3)\cdots(-n)(ax+b)^{-1-n}a^n$$

ie $y_n = \dfrac{(-1)^n n!a^n}{(ax+b)^{n+1}}$

2. If $y = \log(ax+b)$ to find y_n

Solution : $y_1 = \dfrac{1}{ax+b}\cdot a$

differentiating $(n-1)$ times

$$y_n = \frac{(-1)^{n-1} a^{n-1}}{(ax+b)^n} \cdot a = \frac{(-1)^{n-1} a^n}{(ax+b)^n}$$

3. If $y = a^{mx}$ to find y_n

Solution: $y_1 = a^{mx} \cdot m(\mathbf{log}\, a)$; $y_2 = a^{mx} \cdot m^2 (\mathbf{log}\, a)^2$

$\therefore$ In general, $y_n = m^n (\mathbf{log}\, a)^n a^{mx}$

4. If $y = \mathbf{sin}(ax+b)$ to find y_n

Solution: $y_1 = a\,\mathbf{cos}(ax+b) = a\,\mathbf{sin}\left(ax+b+\frac{\pi}{2}\right)$

again differentiating w.r.t. x

$$y_2 = a^2\,\mathbf{cos}\left(ax+b+\frac{\pi}{2}\right) = a^2\,\mathbf{sin}\left(ax+b+2\cdot\frac{\pi}{2}\right)$$

$$y_3 = a^3\,\mathbf{cos}\left(ax+b+2\cdot\frac{\pi}{2}\right) = a^3\,\mathbf{sin}\left(ax+b+3\cdot\frac{\pi}{2}\right)$$

$\therefore$ In general, $y_n = a^n\,\mathbf{sin}\left(ax+b+n\cdot\frac{\pi}{2}\right)$

5. If $y = \mathbf{cos}(ax+b)$ to find y_n

Solution: $y_1 = -a\,\mathbf{sin}(ax+b) = a\,\mathbf{cos}\left(ax+b+\frac{\pi}{2}\right)$

again differentiating w.r.t. x

$$y_2 = -a^2\,\mathbf{sin}\left(ax+b+\frac{\pi}{2}\right) = a^2\,\mathbf{cos}\left(ax+b+2\cdot\frac{\pi}{2}\right)$$

$\therefore$ In general, $y_n = a^n\,\mathbf{cos}\left(ax+b+n\cdot\frac{\pi}{2}\right)$

6. If $y = e^{ax}\,\mathbf{cos}(bx+c)$ to find y_n

Solution: differentiating w.r.t. x

$$y_1 = ae^{ax}\,\mathbf{cos}(bx+c) - be^{ax}\,\mathbf{sin}(bx+c)$$

put $a = r\,\mathbf{cos}\,\alpha$, $b = r\,\mathbf{sin}\,\alpha$

then $y_1 = re^{ax}\,\mathbf{cos}(bx+c)\,\mathbf{cos}\,\alpha - re^{ax}\,\mathbf{sin}(bx+c)\,\mathbf{sin}\,\alpha$

$$= re^{ax}[\mathbf{cos}(bx+c)\,\mathbf{cos}\,\alpha - \mathbf{sin}(bx+c)\,\mathbf{sin}\,\alpha] = re^{ax}\,\mathbf{cos}(bx+c+\alpha)$$

again differentiating w.r.t. x **&** simplifying, we get

$$y_2 = r^2 e^{ax}\,\mathbf{cos}(bx+c+2\alpha)$$

In general, $y_n = r^n e^{ax}\,\mathbf{cos}(bx+c+n\alpha)$ where $r = \sqrt{a^2+b^2}$ **&** $\alpha = \mathbf{tan}^{-1}\frac{b}{a}$

7. If $y = e^{ax}\sin(bx+c)$ to find y_n

Solution : differentiating w.r.t. x

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$

put $a = r\cos\alpha,\ b = r\sin\alpha$

then $y_1 = re^{ax}[\sin(bx+c)\cos\alpha + \cos(bx+c)\sin\alpha] = re^{ax}\sin(bx+c+\alpha)$

again differentiating w.r.t. x **&** simplifying, we get

$$y_2 = r^2 e^{ax}\sin(bx+c+2\alpha)$$

In general, $y_n = r^n e^{ax}\sin(bx+c+n\alpha)$ where $r = \sqrt{a^2+b^2}$ **&** $\alpha = \tan^{-1}\dfrac{b}{a}$

8. Statement of Leibmitz's Theorem on n^{th} derivative of a product

If u **&** v are functions of x, the n^{th} derivative of the product uv is given by

$$(uv)_n = u_n v + nc_1 u_{n-1} v_2 + nc_2 u_{n-2} v_2 + \cdots\cdots + nc_n u v_n$$

where suffixes of u **&** v represent order of the derivatives of u **&** v.

Examples

1. Find the n^{th} derivative of $\dfrac{1}{x^2-6x+8}$

Solution : Let $y = \dfrac{1}{x^2-6x+8} = \dfrac{1}{(x-2)(x-4)} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-4)}$ (Say)

multiplying throughout by $(x-2)(x-4)$

$$1 = A(x-4) + B(x-2)$$

put $x = 2,\ 1 = A(-2) \Rightarrow A = -\dfrac{1}{2}$

put $x = 4,\ 1 = 2B \Rightarrow B = \dfrac{1}{2}$

$$\therefore\ y = \frac{-\frac{1}{2}}{(x-2)} + \frac{\frac{1}{2}}{(x-4)}$$

differentiating n times, we have $y_n = \dfrac{-\frac{1}{2}(-1)^n n!}{(x-2)^{n+1}} + \dfrac{\frac{1}{2}(-1)^n n!}{(x-4)^{n+1}}$

2. Find the n^{th} derivative of $\sin^2 x\cos^3 x$

Solution : Let $y = \sin^2 x\cos^3 x = \dfrac{(1-\cos 2x)}{2} \times \dfrac{\cos 3x + 3\cos x}{4}$

ie $y = \dfrac{1}{8}[\cos 3x + 3\cos x - \cos 3x\cos 2x - 3\cos 2x\cos x]$

$$= \frac{1}{8}\left[\cos 3x + 3\cos x - \frac{1}{2}(\cos 5x + \cos x) - \frac{3}{2}(\cos 3x + \cos x)\right]$$

$$= \frac{1}{8}\left[\cos 3x + 3\cos x - \frac{1}{2}\cos 5x - \frac{1}{2}\cos x - \frac{3}{2}\cos 3x - \frac{3}{2}\cos x)\right]$$

$$\therefore\ y=\frac{1}{8}\left[\cos x-\frac{1}{2}\cos 3x-\frac{1}{2}\cos 5x\right]$$

differentiating n times, we have $y_n=\frac{1}{8}\left[\cos\left(x+\frac{\pi}{2}\right)-\frac{1}{2}\cdot 3^n\cos\left(3x+n\frac{\pi}{2}\right)-\frac{1}{2}\cdot 5^n\cos\left(5x+n\frac{\pi}{2}\right)\right]$

3. If $y=e^{2x}\sin^2 x$ to find y_n

Solution: $y=\frac{1}{2}e^{2x}(1-\cos 2x)\qquad y=\frac{1}{2}e^{2x}-\frac{1}{2}e^{2x}\cos 2x$

differentiating n times, we have $y_n=\frac{1}{2}2^n e^{2x}-\frac{1}{2}\cdot r^n e^{2x}\cos(2x+n-\alpha)$

where $r=\sqrt{4+4}=\sqrt{8},\ \alpha=\tan^{-1}\frac{2}{2}=\tan^{-1}1=\frac{\pi}{4}$

$$\therefore\ y_n=2^{n-1}e^{2x}-\frac{1}{2}\left(\sqrt{8}\right)^n\cos\left(2x+n\cdot\frac{\pi}{2}\right)$$

4. If $y=e^{4x}\sin 5x\cos 3x$ find y_n

Solution: $y=\frac{1}{2}e^{4x}[\sin 8x+\sin 2x]$ ie $y=\frac{1}{2}e^{4x}\sin 8x+\frac{1}{2}e^{4x}\sin 2x$

differentiating n times, we have $y_n=\frac{1}{2}\left(\sqrt{16+64}\right)^n e^{4x}\sin\left(8x+n\tan^{-1}\frac{8}{4}\right)+\frac{1}{2}\left(\sqrt{16+4}\right)^n e^{4x}\sin\left(2x+n\tan^{-1}\frac{2}{4}\right)$

ie $y_n=\frac{1}{2}\left(\sqrt{80}\right)^n e^{4x}\sin(8x+n\tan^{-1}2)+\frac{1}{2}\left(\sqrt{20}\right)^n e^{4x}\sin\left(2x+n\tan^{-1}\frac{1}{2}\right)$

5. If $y=x^2\log 3x$ find y_n

Solution: $y=x^2\log 3x$

Let $u=\log 3x=\log 3+\log x$

$$\therefore\ u_n=\frac{(-1)^{n-1}}{x^n},\ v=x^2$$

differentiating n times, using Leibnitz's Theorem,

$$y_n=(uv)_n=\frac{(-1)^{n-1}}{x^n}\cdot x^2+nC_1\frac{(-1)^{n-2}}{x^{n-1}}2x+nC_2\frac{(-1)^{n-3}}{x^{n-2}}\cdot 2$$

$$=\frac{(-1)^{n-3}}{x^{n-2}}\left[(-1)^2+(-1)^1 2n+n(n-1)\right]=\frac{(-1)^{n-3}}{x^{n-2}}\left[1-2n+n^2-n\right]$$

ie $y_n=\frac{(-1)^{n-3}}{x^{n-2}}\left[n^2-3n+1\right]$

6. If $y=a\cos(\log x)+b\sin(\log x)$ show that $x^2y_{n+2}+(2n+1)^x y_{n+1}+(n^2+1)y_n=0$

Solution: Let $y=a\cos(\log x)+b\sin(\log x)$

differentiating w.r.t. x

$$y_1=-a\sin(\log x)\frac{1}{x}+b\cos(\log x)\frac{1}{x}$$

ie $xy_1 = -a\,\mathbf{sin}(\mathbf{log}\,x) + b\,\mathbf{cos}(\mathbf{log}\,x)$

differentiating w.r.t. x again

$$xy_2 + y_1 = -a\,\mathbf{cos}(\mathbf{log}\,x)\frac{1}{x} - b\,\mathbf{sin}(\mathbf{log}\,x)\frac{1}{x}$$

ie $x^2y_2 + xy_1 = -a\,\mathbf{cos}(\mathbf{log}\,x) - b\,\mathbf{sin}(\mathbf{log}\,x) = -y$

ie $x^2y_2 + xy_1 + y = 0$

differentiating n times, using Leibnitz's Theorem, we have

$$\begin{aligned} x^2y_{n+2} + nC_1 2xy_{n+1} + nC_2 \cdot 2y_n & \\ + xy_{n+1} + nC_1 \cdot 1y_n & \\ + y_n = 0 & \end{aligned}$$

adding, $x^2y_{n+2} + (2n+1)xy_{n+1} + [n(n-1) + n + 1]y_n = 0$

ie $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

7. If $y = e^{m\,\mathbf{cos}^{-1}x}$**,** prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

Solution : $y = e^{m\,\mathbf{cos}^{-1}x}$

differentiating w.r.t. x

$$y_1 = e^{m\,\mathbf{cos}\,x} \cdot \frac{-m}{\sqrt{1-x^2}} \qquad \text{ie } \sqrt{1-x^2}\,y_1 = -my$$

squaring both sides $(1-x^2)y_1^2 = m^2y^2$

differentiating again w.r.t. x**,**

$(1-x^2)2y_1y_2 + y_1^2(-2x) = m^2 2yy_1$

dividing by $2y_1$**,** we have

$(1-x^2)y_2 - xy_1 - m^2y = 0$

differentiating w.r.t. x**,** n times using Leibnitz's Theorem,

$$\begin{aligned} (1-x^2)y_{n+2} + nC_1y_{n+1}(-2x) + nC_2y_n(-2) & \\ - xy_{n+1} + nC_1y_n(-1) & \\ - m^2y_n = 0 & \end{aligned}$$

adding, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - n + n + m^2)y_n = 0$

ie $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

8. If $y^{1/m} + y^{-1/m} = 2x$**,** prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Solution: $y^{1/m} + y^{-1/m} = 2x$

ie $\left(y^{1/m}\right)^2 + 1 = 2xy^{1/m}$

$\therefore \left(y^{1/m}\right)^2 - 2xy^{1/m} + 1 = 0$

which is a quadratic equation in $y^{1/m}$ $\therefore$ $y^{1/m} = \dfrac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$

Consider, $y^{1/m} = x + \sqrt{x^2-1}$ $\therefore$ $y = \left(x + \sqrt{x^2-1}\right)^m$

differentiating w.r.t. x

$$y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right) = m\left(x+\sqrt{x^2-1}\right)^{m-1}\frac{\left(\sqrt{x^2-1}+x\right)}{\sqrt{x^2-1}}$$

ie $\sqrt{x^2-1}\,y_1 = my$

Squaring both sides $(x^2-1)y_1^2 = m^2y^2$

differentiating w.r.t. x

$(x^2-1)2y_1y_2 + y_1^2(2x) = 2m^2yy_1$

dividing throughout by $2y_1$, we have

$(x^2-1)y_2 + xy_1 - m^2y = 0$

differentiating n times using Leibnitz's Theorem

$$\begin{aligned}(x^2-1)y_{n+2} + nC_1y_{n+1}2x + nC_2y_n2 & \\ + xy_{n+1} + nC_1y_n1 & \\ - m^2y_n = 0 & \end{aligned}$$

adding, $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-n+n-m^2)y_n = 0$

ie $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

We obtain the same result if $y^{1/m} = \left(x-\sqrt{x^2-1}\right)$ ie $y = \left(x-\sqrt{x^2-1}\right)^m$

Exercise

1. Find the n^{th} derivative of $\dfrac{1}{x^2-5x+6}$
2. Find the n^{th} derivative of (i) $\sin^3 x$ (ii) $\cos^3 x$ (iii) $\sin 4x \cos 3x$ (iv) $\sin 8x \sin 4x$ (v) $\cos 5x \cos x$.
3. Find the n^{th} derivative of (i) $e^{3x}\sin^2 x$ (ii) $e^{2x}\cos^2 x$ (iii) $e^x \sin 5x \cos 2x$
4. If $y = \sin(m\sin^{-1}x)$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$
5. If $y = e^{a\sin^{-1}x}$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$
6. If $y = (x^2-1)^n$, prove that $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

MEAN VALUE THEOREMS

Rolle's Theorem

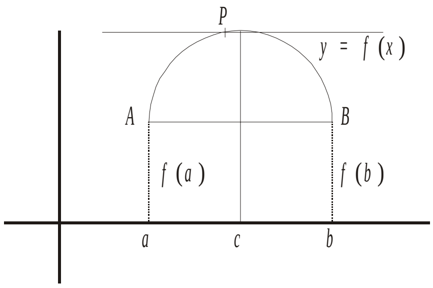
Statement : If $f(x)$ is a function

(i) Continuous in the closed interval $a \le x \le b$

(ii) differentiable in the open interval $a < x < b$ and

(iii) $f(a) = f(b)$ then there exists at least one value c of x such that $f'(c) = 0$ for $a < c < b$.

Geometrical Meaning



A & B are points on the curve such that $f(a) = f(b)$. Tangent at P is parallel to AB such that the slope of the tangent is $f'(c) = 0$

Lagrange's Mean Value Theorem

Statement : If $f(x)$ is a function

(i) Continuous in the closed interval $a \le x \le b$

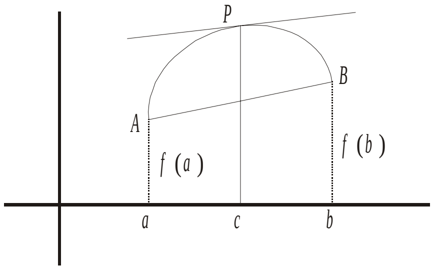
(ii) differentiable in the open interval $a < x < b$

then there exists at least one value c of x in the interval such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$

Note : - If the interval is $(a, a+h)$ then $f(a+h) - f(a) = f'(c)$

ie $f(a+h) = f(a) + hf'(c)$

Geometrical Meaning



A & B are points on the curve corresponding to $x = a$ & $x = b$. Join AB There will be a tangent at P, parallel to AB so that scope of the tangent is $\dfrac{f(b) - f(a)}{b - a}$ which is $f'(c)$

Cauchy's Mean Value Theorem

Statement : If $f(x)$ & $g(x)$ are two functions which are

(i) continuous in the closed interval $[a, b]$

(ii) differentiable in the open interval (a, b) and

(iii) $g'(x) \ne 0$ for any value of x in (a, b)

then there exists at least one value c of x in (a, b) such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$

Taylor's Theorem

If $f(x)$ is a function such that (i) $f(x)$ and its $(n-1)$ derivatives are continuous in the closed interval $[a, a+h]$

ie $a \le x \le a+h$ and (ii) n^{th} derivative $f^{(n)}(x)$ exists in the open interval $(a, a+h)$.

Then there exists at least one number θ $(0<\theta<1)$ such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{21}f''(a)+\cdots+\cdots+\frac{h^n}{n!}f^{(n)}(a+\theta h)$$

Note:- put $a=0$ & $h=x$, then $f(x)=f(0)+xf'(0)+\frac{x^2}{21}f''(0)+\cdots+\cdots+\frac{x^n}{n!}f^{(n)}(\theta x)$, where $0<\theta<1$

This expression for $f(x)$ is called **Maclaurin's Expansion** and further if $\frac{x^n}{n!}f(\theta x)$ tends to zero as $n\to\infty$.

Then $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots\cdots$ to ∞. This is called **Maclaurin's Series** for $f(x)$.

Examples

(1) Verify Rolle's Theorem for $f(x)=(x+2)^3(x-3)^4$ in $(-2, 3)$ and find c.

Solution: $f(x)$ is continuous in the closed interval $[-2, 3]$ & differentiable in $(-2, 3)$ and further $f(-2)=0$, $f(3)=0$

ie $f(-2)=f(3)$

$\therefore$ There exists a value c in $(-2, 3)$ such that $f'(c)=0$

$$f'(x)=3(x+2)^2(x-3)^4+4(x+2)^3(x-3)^3$$

$$f'(c)=3(c+2)^2(c-3)^4+4(c+2)^3(c-3)^3=(c+2)^2(c-3)^3[3(c-3)+4(c+2)]=(c+2)^2(c-3)^2(7c-1)$$

$$f'(c)=0\Rightarrow 7c-1=0\Rightarrow c=\frac{1}{7}$$

(2) Find 'c' of the Lagrange's Mean Value Theorem for the function $f(x)=(x-1)(x-2)(x-3)$ in $(0, 4)$

Solution: $f(0)=(-1)(-2)(-3)=-6$, $f(4)=3\times2\times1=6$

$$f'(x)=(x-2)(x-3)+(x-1)(x-3)+(x-1)(x-2)$$

$$f'(c)=(c-2)(c-3)+(c-1)(c-3)+(c-1)(c-2)=c^2-5c+b+c^2-4c+3+c^2-3c+2=3c^2-12c+11$$

By Lagrange's Theorem $\frac{f(b)-f(a)}{b-a}=f'(c)$

$$\therefore\ \frac{f(4)-f(0)}{4-0}=f'(c)$$

$$\therefore\ \frac{6+6}{4}=3c^2-12c+11$$

ie $12c^2-48c+44=12$

ie $12c^2-48c+32=0\Rightarrow 3c^2-12c+8=0$

$$\therefore\ c=\frac{12\pm\sqrt{144-96}}{6}=\frac{12\pm\sqrt{48}}{6}=\frac{12\pm4\sqrt{3}}{6}$$

$$\therefore\ c=2\pm\frac{2}{\sqrt{3}}\ \text{ ie } c=2-\frac{2}{\sqrt{3}}\ \&\ 2+\frac{2}{\sqrt{3}}$$

(3) Using Maclaurin's Series Express $\sin x$ & $\cos x$ as an infinite series.

Solution : Let $f(x) = \sin x$

differentiating n times $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$

put $x = 0$, $\therefore$ $f^{(n)}(0) = \sin\frac{n\pi}{2}$

$\therefore$ $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^{\text{iv}}(0) = 0$

$\therefore$ $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{\text{iv}}(0) + \cdots\cdots = 0 + x + 0 + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}0$

$\therefore$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}\cdots\cdots$

& let $g(x) = \cos x$, $g^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$

put $x = 0$, $g(0) = 1$, $g^{(n)}(0) = \cos\frac{n\pi}{2}$

$\therefore$ $g(x) = g(0) + xg_2'(0) + \frac{x^2}{2}g''(0)\cdots\cdots$

$\therefore$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\cdots\cdots$

(4) Find the Maclaurin's Series for e^x.

Solution: Let $f(x) = e^x$, $f^{(n)}(x) = e^x$

put $x = 0$, $f^{(n)}(0) = e^0 = 1$

$f(0) = 1$, $f'(0) = 1$, $f''(0) = 1, \cdots\cdots$

$\therefore$ $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots\cdots$

ie $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$

Exercise

1. Examine the application of Rolle's Theorem for $f(x) = 2 + (x+1)^{2/3}$ in the interval $(0, 2)$

(Answer : Rolle's Theorem donot apply because $f'(1)$ donot exists.)

2. Find 'c' of the Mean Value Theorem for $f(x) = x(x-1)(x-2)$ $a = 0$, $b = \frac{1}{2}$

$\left(\text{Answer: } c = 1 - \frac{\sqrt{21}}{6}\right)$

3. Show that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\cdots\cdots$ using Maclaurin's Series.

Indeterminate Forms

While evaluating $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ or $\lim\limits_{x\to a}[f(x)-g(x)]$ or $\lim\limits_{x\to a} f(x)^{g(x)}$ when it takes the forms $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\infty-\infty$, 1^{∞}, ∞°, 0° they are called Indeterminate forms and to evaluate such forms the following rule known as **L' Hospital's Rule** is used.

L' Hospital's Rule

If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ again if this is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ then

$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f''(x)}{g''(x)}$ whenever it is of the $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ This rule can be applied.

To evaluate $\lim\limits_{x\to a}[f(x)-g(x)]$ when it is of the form $\infty-\infty$, then

Consider $\lim\limits_{x\to a}[f(x)-g(x)]=\lim\limits_{x\to a}\dfrac{\dfrac{1}{g(x)}-\dfrac{1}{f(x)}}{\dfrac{1}{g(x)f(x)}}$ which is of the form $\dfrac{0}{0}$ **&** hence L' Hospital's Rule can be applied.

Consider $\lim\limits_{x\to a} f(x)^{g(x)}=y$ (say)

then $\log y=\lim\limits_{x\to a}\log f(x)^{g(x)}=\lim\limits_{x\to a} g(x)\log f(x)=\lim\limits_{x\to a}\dfrac{\log f(x)}{\dfrac{1}{g(x)}}$

which is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and hence L' Hospital's rule can be applied.

Examples

(1) Evaluate $\lim\limits_{x\to 1}\dfrac{\log x}{x^2-3x+2}$

Solution : $\lim\limits_{x\to 1}\dfrac{\log x}{x^2-3x+2}$ this is of the form $\dfrac{0}{0}$ $\therefore$ using L' Hospital's rule

$$=\lim_{x\to 1}\frac{\frac{1}{x}}{2x-3}=\frac{1}{2-3}=-1$$

(2) Evaluate $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin 2x}{x^3}$

Solution : $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin 2x}{x^3}$ This is of the form $\dfrac{0}{0}$ $\therefore$ applying L' Hospital's rule

$=\lim\limits_{x\to 0}\dfrac{2\cos x-2\cos 2x}{3x^2}$ again it is of the form $\dfrac{0}{0}$ $\therefore$ applying L' Hospital's rule again

$$=\lim_{x\to 0}\frac{-2\sin x+4\sin 2x}{6x}=\lim_{x\to 0}\left[-\frac{2}{6}\cdot\frac{\sin x}{x}+\frac{4}{3}\frac{\sin 2x}{2x}\right]=-\frac{2}{6}\times 1+\frac{4}{3}\times 1 \quad \because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

$$=-\frac{1}{3}+\frac{4}{3}=1$$

(3) Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^2 \tan x}$

Solution : $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^2\tan x} = \lim\limits_{x\to 0}\dfrac{\tan x - x}{x^3}\times\dfrac{x}{\tan x} = \lim\limits_{x\to 0}\dfrac{\tan x - x}{x^3} \quad \because \lim\limits_{x\to 0}\dfrac{x}{\tan x} = 1$

This is of the form $\dfrac{0}{0}$ $\therefore$ using L' Hospital's rule

$$= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \frac{0}{0} \quad \therefore \text{ using the rule again}$$

$$= \lim_{x\to 0}\frac{2\sec x\cdot\sec x\tan x}{6x} = \lim_{x\to 0}\frac{2\sec^2 x}{6}\times\frac{\tan x}{x} = \frac{2}{6}\times 1 = \frac{1}{3}$$

(4) Evaluate $\lim\limits_{x\to 0}\left[\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right]$

Solution : This is of the form $\infty - \infty$

$$\therefore \lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{e^x - 1}\right] = \lim_{x\to 0}\frac{(e^x - 1) - x}{(e^x - 1)x} \text{ which is of the form } \frac{0}{0} \quad \therefore \text{ using L' Hospital's rule}$$

$$= \lim_{x\to 0}\frac{e^x - 1}{e^x - 1 + xe^x}$$

which is again of the form $\dfrac{0}{0}$ $\therefore$ using the rule again

$$= \lim_{x\to 0}\frac{e^x}{e^x + xe^x + e^x} = \frac{1}{1+0+1} = \frac{1}{2}$$

(5) Evaluate $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1} - \dfrac{1}{\log x}\right)$

Solution : This of the form $\infty - \infty$

$$\therefore \lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\log x}\right) = \lim_{x\to 1}\frac{\log x - \dfrac{x-1}{x}}{\log x\left(\dfrac{x-1}{x}\right)}$$

This is of the form $\dfrac{0}{0}$ $\therefore$ using L' Hospital's rule

$$= \lim_{x\to 1}\frac{\dfrac{1}{x} - \dfrac{1}{x^2}}{\log x\times\left(\dfrac{1}{x^2}\right) + \dfrac{x-1}{x}\times\dfrac{1}{x}} = \lim_{x\to 1}\frac{x-1}{\log x + x - 1} \text{ is } \frac{0}{0} \text{ form again applying the L' Hospital's rule}$$

$$= \lim_{x\to 1}\frac{1}{\dfrac{1}{x} + 1} = \frac{1}{2}$$

(6) Evaluate $\lim\limits_{x\to \frac{\pi}{2}}(\tan x)^{\cot x}$

Solution : This is of the form ∞^0

Let $y = \lim\limits_{x \to \frac{\pi}{2}} (\tan x)^{\cot x}$

$$\log y = \lim_{x \to \frac{\pi}{2}} \log(\tan x)^{\cot x} = \lim_{x \to \frac{\pi}{2}} \cot x \log \tan x = \lim_{x \to \frac{\pi}{2}} \frac{\log \tan x}{\tan x} \quad \text{using L' Hospital's rule}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{1}{\tan x} = \lim_{x \to \frac{\pi}{2}} \cot x = 0$$

$\therefore \; y = e^0 = 1$

(7) Evaluate $\lim\limits_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x}$

Solution : This is of the form 1^∞

Let $y = \lim\limits_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x}$

$$\log y = \lim_{x \to 0} \frac{1}{x} \log\left(\frac{\sin x}{x} \right) = \lim_{x \to 0} \frac{\log\left(\frac{\sin x}{x} \right)}{x}$$

This is of the form $\frac{0}{0}$ $\therefore$ applying the L'Hospital's rule

$$= \lim_{x \to 0} \frac{\frac{x}{\sin x} \times \frac{x \cos x - \sin x}{x^2}}{1} = \lim_{x \to 0} \frac{x}{\sin x} \times \frac{x \cos x - \sin x}{x^2} = 1 \times \lim_{x \to 0} \frac{x \cos x - \sin x}{x^2}$$

This is of the form $\frac{0}{0}$ $\therefore$ again applying the L'Hospital's rule

$$= \lim_{x \to 0} \frac{\cos x - x \sin x - \cos x}{2x} = \lim_{x \to 0} \frac{-x \sin x}{2x} = \lim_{x \to 0} \frac{-\sin x}{2} = 0$$

ie $\log y = 0 \;\; \therefore y = 1$

(8) If $\lim\limits_{x \to 0} \frac{a \sin x - \sin 2x}{\tan^3 x}$ is finite, find the value of a and the limit.

Solution : The given limit is of the form $\frac{0}{0}$ $\therefore$ applying the L'Hospital's rule

$$= \lim_{x \to 0} \frac{a \cos x - 2 \cos 2x}{3 \tan^2 x \cdot \sec^2 x}$$

limit exists if this is of the form $\frac{0}{0}$

$\therefore \; a \cos x - 2 \cos 2x = 0$ for $x = 0 \;\; \therefore \; a = 2$

$$= \lim_{x \to 0} \frac{2 \cos x - 2 \cos 2x}{3x^2} \times \frac{x^2}{\tan^2 x} \times \frac{1}{\sec^2 x} = \lim_{x \to 0} \frac{3 \cos x - 2 \cos 2x}{3x^2} \times 1 \times 1 \text{ is of form } \frac{0}{0}$$

$\therefore$ using L'Hospital's rule

$$= \lim_{x\to 0}\frac{-2\sin x + 4\sin 2x}{6x} = \lim_{x\to 0}\frac{-2\sin x}{6x} + \frac{4}{3}\times\frac{\sin 2x}{2x} = -\frac{2}{6}+\frac{4}{3} = -\frac{1}{3}+\frac{4}{3} = 1$$

$\therefore\ a = 2$ and the limit is 1.

Exercise

Evaluate the following

(1) $\lim_{x\to 0}\frac{\cosh x - \cos x}{x\sin x}$

(2) $\lim_{x\to 0}\frac{e^x - e^{\sin x}}{x - \sin x}$

(3) $\lim_{x\to 0}\frac{e^x - \sin x - 1}{\log(1+x)}$

(4) $\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$

(5) $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$

(6) $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

(7) $\lim_{x\to 0}(\cos x)^{1/x^2}$

(8) $\lim_{x\to 1}(x)^{\frac{1}{1-x}}$

(9) $\lim_{x\to 0}(\cot x)^{\frac{1}{\log x}}$

(10) $\lim_{x\to 0}\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}}$

Answers : (1) 1 (2) 1 (3) 2 (4) $\frac{3}{2}$ (5) $-\frac{1}{3}$ (6) 0 (7) $\frac{1}{\sqrt{e}}$ (8) $\frac{1}{e}$ (9) $\frac{1}{e}$ (10) $(abc)^{1/3}$

Partial Derivatives

A function of two independent variables and a dependent variable is denoted as $z = f(x, y)$ which is explicit function where x & y are independent variables and z a dependent variable. Implicit function is denoted by $\phi(x, y, z) = C$

If $\lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x, y) - f(x, y)}{\delta x}$ exists then it is called Partial derivative of z or f w.r.t. x and denoted by $\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial f}{\partial x}$

If $\lim\limits_{\delta y \to 0} \dfrac{f(x, y+\delta y) - f(x, y)}{\delta y}$ exists then it is called Partial derivative of z or f w.r.t. y and denoted by $\dfrac{\partial z}{\partial y}$ or $\dfrac{\partial f}{\partial y}$

while obtaining the derivative $\dfrac{\partial z}{\partial x}$ differentiate the given function w.r.t. x treating y as a constant and while finding $\dfrac{\partial z}{\partial y}$ differentiate the given function with respect to y, treating x as a constant.

Eg. (1) If $z = x^2 + xy - y^2$ then $\dfrac{\partial z}{\partial x} = 2x + y + 0 = 2x + y$ **&** $\dfrac{\partial z}{\partial y} = x - 2y$

(2) If $z = x^2 y - x\,\mathbf{sin}\,xy$ then $\dfrac{\partial z}{\partial x} = 2xy - x\,\mathbf{cos}\,xy \cdot y - \mathbf{sin}\,xy$ **&** $\dfrac{\partial z}{\partial y} = x^2 - x\,\mathbf{cos}\,xy \cdot x = x^2(1 - \mathbf{cos}\,xy)$

(3) If $z = \mathbf{tan}^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$ then $\dfrac{\partial z}{\partial x} = \dfrac{1}{1 + \dfrac{4x^2y^2}{\left(x^2 - y^2\right)^2}} \times \dfrac{(x^2 - y^2)2y - 2xy \cdot 2x}{\left(x^2 - y^2\right)^2}$

$$= \frac{\left(x^2 - y^2\right)^2}{\left(x^2 - y^2\right)^2 + 4x^2y^2} \times \frac{2x^2y - 2y^3 - 4x^2y}{\left(x^2 - y^2\right)^2} = \frac{-2y^3 - 2x^2y}{\left(x^2 + y^2\right)^2} = \frac{-2y(y^2 + x^2)}{\left(x^2 + y^2\right)^2} = \frac{-2y}{x^2 + y^2}$$

$$\textbf{\&} \quad \frac{\partial z}{\partial y} = \frac{1}{1 + \dfrac{4x^2y^2}{\left(x^2 - y^2\right)^2}} \times \frac{(x^2 - y^2)(2x) - 2xy(-2y)}{\left(x^2 - y^2\right)^2}$$

$$= \frac{\left(x^2 - y^2\right)^2}{\left(x^2 - y^2\right)^2 + 4x^2y^2} \times \frac{2x^3y - 2xy^2 + 4xy^2}{\left(x^2 - y^2\right)^2} = \frac{2x^3 + 2xy^2}{\left(x^2 + y^2\right)^2} = \frac{2x(x^2 + y^2)}{\left(x^2 + y^2\right)^2} = \frac{2x}{x^2 + y^2}$$

Successive derivatives

For the function $z = f(x, y)$ $\dfrac{\partial z}{\partial x}$ & $\dfrac{\partial z}{\partial y}$ are first order partial derivatives, the second order partial derivatives are

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right), \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right), \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right), \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)$ which are denoted as $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial x \partial y}, \dfrac{\partial^2 z}{\partial y \partial x}, \dfrac{\partial^2 z}{\partial y^2}$ but in general $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$.

In example (1) $\dfrac{\partial^2 z}{\partial x^2} = 2,\ \dfrac{\partial^2 z}{\partial y^2} = -2,\ \dfrac{\partial^2 z}{\partial x \partial y} = 1$ & $\dfrac{\partial^2 z}{\partial y \partial x} = 1$

In example (2) $\dfrac{\partial^2 z}{\partial y \partial x} = 2x + x^2 y \sin xy - x\cos xy - x\cos xy$ & $\dfrac{\partial^2 z}{\partial x \partial y} = 2x + x^2 y \sin xy - 2x\cos xy$

$$\therefore\ \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

In example (3) $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{-2y}{x^2+y^2}\right) = \dfrac{(x^2+y^2)(-2)+2y\cdot 2y}{\left(x^2+y^2\right)^2} = \dfrac{-2x^2-2y^2+4y^2}{\left(x^2+y^2\right)^2} = \dfrac{-2(x^2-y^2)}{\left(x^2+y^2\right)^2}$

and $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{2x}{x^2+y^2}\right) = \dfrac{(x^2+y^2)2-2x\cdot 2x}{\left(x^2+y^2\right)^2} = \dfrac{2y^2-2x^2}{\left(x^2+y^2\right)^2} = \dfrac{-2(x^2-y^2)}{\left(x^2+y^2\right)^2}$

Thus in general, always $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$

Exercise

(1) Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ for $z = \log(x^2+y^2)$ and show that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$

(2) If $x = f(x+ct)+\phi(x-ct)$ show that $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$ where c is a constant.

(3) If $z = e^{ax+by} f(ax-by)$ then show that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$

(4) If $u = \dfrac{x^2+y^2}{x+y}$ then show that $\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)^2 = 4\left(1-\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)$

(5) If $u = \sin^{-1}\left(\dfrac{y}{x}\right)$ then show that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

INTEGRAL CALCULUS

Given $\frac{dy}{dx} = f(x)$, the process of finding y is called **'Integration'** and the resulting function is called **'Integral'.** If $g(x)$ is the integral then $\int f(x)dx = g(x)$ is the notation used to represent the process.

In the above notation $f(x)$ is called **'Integrand'** and further $\frac{d}{dx}[g(x)] = f(x)$.

But $\frac{d}{dx}[g(x)+c] = g'(x)$ when c is a constant $\therefore \int f(x)dx = g(x)+c$

Thus integral of a function is not unique and two integrals always differ by a constant.

Properties

(1) $\int [f(x) \pm \phi(x)]dx = \int f(x)dx \pm \int \varphi(x)dx$

(2) $\int Kf(x)dx = K\int f(x)dx$ where K is a constant

(3) $\int 0dx = c$ (a constant)

Standard Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \ (n \neq -1) \ \because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}+c\right) = x^n$

2. $\int \frac{1}{x}dx = \mathbf{log}_e\, x + c \ \because \frac{d}{dx}(\mathbf{log}\, x + c) = \frac{1}{x}$

3. $\int a^x dx = \frac{a^x}{\mathbf{log}\, a} + c \ \because \frac{d}{dx}\left(\frac{a^x}{\mathbf{log}\, a}+c\right) = a^x$ in particular $\int e^x dx = e^x + c$

4. $\int \mathbf{sin}\, x\, dx = -\mathbf{cos}\, x + c$

5. $\int \mathbf{cos}\, x\, dx = \mathbf{sin}\, x + c$

6. $\int \mathbf{sec}\, x\, \mathbf{tan} x\, dx = \mathbf{sec}\, x + 1$

7. $\int \mathrm{cosec}\, x\, \mathbf{cot}\, x\, dx = -\mathrm{cosec}\, x + c$

8. $\int \mathbf{sinh} x\, dx = \mathbf{cosh}\, x + c$

9. $\int \mathbf{cosh}\, x\, dx = \mathbf{sinh} x + c$

10. $\int \mathrm{sech}\, x\, \mathbf{tanh} x\, dx = -\mathrm{sech}\, x + c$

11. $\int \text{cosech}\, x \,\mathbf{coth}\, x\, dx = -\text{cosesh}\, x + c$

12. $\int \frac{1}{1+x^2}\, dx = \mathbf{tan}^{-1} x + c$ or $-\mathbf{cot}^{-1} x + c$

13. $\int \frac{1}{\sqrt{1-x^2}}\, dx = \mathbf{sin}^{-1} x + c$ or $-\mathbf{cos}^{-1} x + c$

14. $\int \frac{1}{\sqrt{1+x^2}}\, dx = \mathbf{sinh}^{-1} x + c$

15. $\int \frac{1}{\sqrt{x^2-1}}\, dx = \mathbf{cosh}^{-1} x + c$

16. $\int \frac{1}{x\sqrt{x^2-1}}\, dx = \mathbf{sec}^{-1} x + c$ or $-\text{cosec}^{-1} x + c$

17. $\int \frac{1}{x\sqrt{1-x^2}}\, dx = -\text{sech}^{-1} x + c$

18. $\int \frac{1}{x\sqrt{1+x^2}}\, dx = -\text{cosech}^{-1} x + c$

Methods of Integration

There are two methods (1) Integration by substitution & (2) Integration by parts.

1. Integration by substitution

Consider $\int f(x)\, dx$ put $x = \phi(t)$ then $\frac{dx}{dt} = \phi'(t)$ ie $dx = \phi'(t)\, dt$

$\therefore \int f(x)\, dx = \int f\,[\phi(t)]\phi'(t)\, dt$

now for the new integrand, we can use the standard forms, ie. we have to make a proper substitution so that the given integrand reduced to a standard one.

Examples

1. $\int \mathbf{tan}\, x\, dx = \int \frac{\mathbf{sin}\, x}{\mathbf{cos}\, x}\, dx$

put $\mathbf{cos}\, x = t$, then $-\mathbf{sin}\, x = \frac{dt}{dx}$ ie $\mathbf{sin}\, x\, dx = -dt$

$\therefore \int \mathbf{tan}\, x\, dx = \int -\frac{dt}{t} = -\mathbf{log}\, t = -\mathbf{log}\,\mathbf{cos}\, x = \mathbf{log}\,\mathbf{sec}\, x + c$

or $\int \mathbf{tan}\, x\, dx = \int \frac{\mathbf{tan}\, x\, \mathbf{sec}\, x}{\mathbf{sec}\, x}\, dx$

put $\mathbf{sec}\, x = t$, differentiating w.r.t. x

$\mathbf{sec}\, x\, \mathbf{tan}\, x = \frac{dt}{dx}$ $\quad \therefore\ \mathbf{sec}\, x\, \mathbf{tan}\, x\, dx = dt$

$$\therefore \int \tan x\, dx = \int \frac{dt}{t} = \log t = \log \sec x + c$$

2. $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx$

put $\sin x = t$, differentiating w.r.t. x $\quad \cos x = \frac{dt}{dx}$ ie $\cos x\, dx = dt$

$$\therefore \int \cot x\, dx = \int \frac{dt}{t} = \log t = \log \sin x + c$$

3. $\int \tanh x\, dx = \log \cosh x + c$

4. $\int \coth x\, dx = \log \sinh x + c$

5. $\int \sec x\, dx = \log(\sec x + \tan x) + c$

6. $\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x) + c$

In general $\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$ also $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c (n \neq -1)$

2. Integration by parts

If u & v are functions of x, we know that, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\therefore$ By definition of Integration

$$uv = \int\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx \text{ using property (1)}$$

$$\therefore \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

The result can be used as the standard result. Out of the two functions of the product, one has to be taken as u & another $\frac{dv}{dx}$ then the RHS after evaluation gives the integral or if both functions have taken as u & v then the result is as follows

$$\int uv\, dx = u\int v\, dx - \int\left(\frac{du}{dx}\cdot\int v\, dx\right)dx$$

any one form can be used depending on convenience. The first one can also be written as $\int uv'\, dx = uv - \int u'v\, dx$

Examples

1. $\int xe^x\, dx$ put $u = v$, $v' = e^x$, $u' = 1$, $v = e^x$

$$\therefore \int xe^x\, dx = uv - \int u'v\, dx = xe^x - \int 1\cdot e^x\, dx = xe^x - e^x + c$$

2. $\int x\sin x\, dx = x\int \sin x\, du - \int 1\cdot \int \sin x\, dx = -x\cos x - \int -\cos x\, dx = -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + c$

3. $\int \log x\, dx$ put $u = \log x$, $v' = 1$, $u' = \frac{1}{x}$, $v = x$

$\therefore \int uv'\,dx = uv - \int u'v\,dx$

ie $\int \mathbf{log}\, x\,dx = x\,\mathbf{log}\, x - \int \frac{1}{x}\cdot x\,dx = x\,\mathbf{log}\, x - \int 1\cdot dx = x\,\mathbf{log}\, x - x + c$

4. $\int \mathbf{sin}^{-1} x\,dx$ put $u = \mathbf{sin}^{-1} x$, $v' = 1$, $u' = \frac{1}{\sqrt{1-x^2}}$, $v = x$

$\therefore \int \mathbf{sin}^{-1} x\,dx = x\,\mathbf{sin}^{-1} x - \int \frac{1}{\sqrt{1-x^2}}\, x\,dx$

to evaluate $\int \frac{-x}{\sqrt{1-x^2}}\,dx$ put $1 - x^2 = t^2$ differentiating w.r.t. x

$-2x\,dx = 2t\,dt \Rightarrow -x\,dx = t\,dt$

$\therefore \int \frac{-x\,dx}{\sqrt{1-x^2}} = \int \frac{t\,dt}{\sqrt{t^2}} = \int 1\cdot dt = t = \sqrt{1-x^2}$

$\therefore \int \mathbf{sin}^{-1} x\,dx = x\,\mathbf{sin}^{-1} x - \sqrt{1-x^2} + c$

Special Types of Integrals

Type I

(1) $\int \frac{dx}{a^2+x^2}$, (2) $\int \frac{dx}{x^2-a^2}$, (3) $\int \frac{dx}{a^2-x^2}$ **&** (4) $\int \frac{dx}{Ax^2+Bx+C}$

to evaluate (1) put $x = at$, $dx = a\,dt$

$\therefore \int \frac{dx}{a^2+x^2} = \int \frac{a\,dt}{a^2+a^2t^2} = \frac{1}{a}\int \frac{dt}{1+t^2} = \frac{1}{a}\,\mathbf{tan}^{-1} t = \frac{1}{a}\,\mathbf{tan}^{-1} \frac{x}{a} + c$

to evaluate (2) & (3) use partial fractions

$\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a}$ (Say)

multiply throughout by $x^2 - a^2$

$1 = A(x-a) + B(x+a)$

put $x = a$, $1 = 0 + B\cdot 2a$ $\therefore B = \frac{1}{2a}$

put $x = -a$, $1 = A(-2a) + 0$ $\therefore A = \frac{-1}{2a}$

$\therefore \frac{1}{x^2-a^2} = \frac{\frac{-1}{2a}}{x+a} + \frac{\frac{1}{2a}}{x-a}$

$\therefore \int \frac{1}{x^2-a^2}\,dx = -\frac{1}{2a}\int \frac{dx}{x+a} + \frac{1}{2a}\int \frac{dx}{x-a} = -\frac{1}{2a}\,\mathbf{log}(x+a) + \frac{1}{2a}\,\mathbf{log}(x-a) = \frac{1}{2a}\,\mathbf{log}\,\frac{x-a}{x+a}$

$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\,\mathbf{log}\,\frac{x-a}{x+a} + c$

next, $\dfrac{1}{x^2-a^2}=\dfrac{1}{(a+x)(a-x)}=\dfrac{A}{a+x}+\dfrac{B}{a-x}$ (Say)

multiplying throughout by a^2-x^2, then

$$1=A(a-x)+B(a+x)$$

put $x=a,\ 1=0+B(2a)\Rightarrow B=\dfrac{1}{2a}$

put $x=-a,\ A(2a)+0\Rightarrow A=\dfrac{1}{2a}$

$$\therefore\ \frac{1}{a^2-x^2}=\frac{\frac{1}{2a}}{a+x}+\frac{\frac{1}{2a}}{a-x}$$

$$\therefore\ \int\frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\int\frac{1}{a+x}\,dx+\frac{1}{2a}\int\frac{1}{a-x}\,dx=\frac{1}{2a}\mathbf{log}(a+x)-\frac{1}{2a}\mathbf{log}(a-x)=\frac{1}{2a}\mathbf{log}\left(\frac{a+x}{a-x}\right)$$

$$\therefore\ \int\frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\mathbf{log}\left(\frac{a+x}{a-x}\right)+c$$

to evaluate (4) $\displaystyle\int\frac{dx}{Ax^2+Bx+c}$

$$\text{G.I.}=\frac{1}{A}\int\frac{dx}{x^2+\frac{B}{A}x+\frac{C}{A}}=\frac{1}{A}\int\frac{dx}{\left(x+\frac{B}{2A}\right)^2-\frac{B^2}{4A^2}+\frac{C}{A}}=\frac{1}{A}\int\frac{dx}{\left(x+\frac{B}{2A}\right)^2-\frac{B^2-4AC}{4A^2}}$$

This integral will take any one of (1), (2) or (3) and hence can be evaluated.

Examples

(1) Evaluate $\displaystyle\int\frac{dx}{3x^2-2x+4}$

Solution: $\displaystyle\int\frac{dx}{3x^2-2x+4}=\frac{1}{3}\int\frac{dx}{x^2-\frac{2}{3}x+\frac{4}{3}}=\frac{1}{3}\int\frac{dx}{\left(x-\frac{1}{3}\right)^2-\frac{4}{9}+\frac{4}{3}}=\frac{1}{3}\int\frac{dx}{\left(x-\frac{1}{3}\right)^2+\frac{-4+12}{9}}$

$$=\frac{1}{3}\int\frac{dx}{\left(\frac{\sqrt{8}}{3}\right)^2+\left(x-\frac{1}{3}\right)^2}=\frac{1}{3}\int\frac{dx}{\left(\frac{2\sqrt{2}}{3}\right)^2+\left(x-\frac{1}{3}\right)^2}=\frac{1}{3}\times\frac{1}{\frac{2\sqrt{2}}{3}}\mathbf{tan}^{-1}\frac{x-\frac{1}{3}}{\frac{2\sqrt{2}}{3}}=\frac{1}{2\sqrt{2}}\mathbf{tan}^{-1}\left(\frac{3x-1}{2\sqrt{2}}\right)+c$$

(2) Evaluate $\displaystyle\int\frac{dx}{x^2-10x+21}$

Solution: $\displaystyle\int\frac{dx}{x^2-10x+21}=\int\frac{dx}{(x-5)^2-25+21}=\int\frac{dx}{(x-5)^2-2^2}=\frac{1}{2\times2}\mathbf{log}\frac{x-5-2}{x-5+2}=\frac{1}{4}\mathbf{log}\frac{x-7}{x-3}+c$

(3) Evaluate $\displaystyle\int\frac{dx}{6-4x-2x^2}$

Solution : $\int \frac{dx}{6-4x-2x^2} = \frac{1}{2}\int \frac{dx}{3-(x^2+2x)} = \frac{1}{2}\int \frac{dx}{3-(x+1)^2+1} = \frac{1}{2}\int \frac{dx}{2^2-(x+1)^2}$

$$= \frac{1}{2}\times\frac{1}{2\times 2}\log\frac{2+(x+1)}{2-(x+1)} = \frac{1}{8}\log\left(\frac{3+x}{1-x}\right)+c$$

Type II

(1) $\int \frac{dx}{\sqrt{a^2-x^2}}$, (2) $\int \frac{dx}{\sqrt{a^2+x^2}}$, (3) $\int \frac{dx}{\sqrt{x^2-a^2}}$, (4) $\int \frac{dx}{\sqrt{Ax^2+Bx+C}}$

to evaluate $\int \frac{dx}{\sqrt{a^2-x^2}}$ put $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int 1\cdot d\theta = \theta = \sin^{-1}\frac{x}{a}$$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$$

to evaluate $\int \frac{dx}{\sqrt{a^2+x^2}}$ put $x = a\sinh\theta$, $dx = a\cosh\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\cosh\theta\, d\theta}{\sqrt{a^2+a^2\sinh^2\theta}} = \int \frac{a\cosh\theta\, d\theta}{a\cosh\theta} = \int 1\cdot d\theta = \theta = \sinh^{-1}\frac{x}{a}$$

$$\therefore \int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}+c$$

to evaluate $\int \frac{dx}{\sqrt{x^2-a^2}}$ put $x = a\cosh\theta$, $dx = a\sinh\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}+c$$

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh\theta\, d\theta}{\sqrt{a^2\cosh^2\theta-a^2}} = \int \frac{a\sinh\theta\, d\theta}{a\sinh\theta} = \int 1\cdot d\theta = \theta = \cosh^{-1}\frac{x}{a}$$

to evaluate $\int \frac{dx}{\sqrt{Ax^2+Bx+C}}$

$$\text{G.I.} = \frac{1}{\sqrt{A}}\int \frac{dx}{\sqrt{x^2+\frac{B}{A}x+\frac{C}{A}}} = \frac{1}{\sqrt{A}}\int \frac{dx}{\sqrt{\left(x+\frac{B}{2A}\right)^2-\frac{B^2}{4A^2}+\frac{C}{A}}} = \frac{1}{\sqrt{A}}\int \frac{dx}{\sqrt{\left(x+\frac{B}{2A}\right)^2-\frac{B^2-4AC}{4A^2}}}$$

This will reduce to any one of (1), (2) & (3) and hence can be evaluated.

Examples

(1) Evaluate $\int \frac{dx}{\sqrt{2x-5x^2}}$

Solution : $\int \frac{dx}{\frac{1}{\sqrt{5}}\sqrt{\frac{2}{5}x-x^2}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{-\left(x^2-\frac{2}{5}x\right)}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{\left(\frac{1}{5}\right)^2-\left(x-\frac{1}{5}\right)^2}} = \frac{1}{\sqrt{5}}\mathbf{sin}^{-1}\frac{x-\frac{1}{5}}{\frac{1}{5}} = \frac{1}{\sqrt{5}}\times\mathbf{sin}^{-1}(5x-1)+c$

(2) Evaluate $\int \frac{dx}{\sqrt{x^2-2x+5}}$

Solution : $\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x-1)^2+2^2}} = \int \frac{dx}{\sqrt{2^2+(x-1)^2}} = \mathbf{sinh}^{-1}\frac{x-1}{2}+c$

(3) Evaluate $\int \frac{dx}{\sqrt{4x^2-12x+8}}$

Solution : G.I. $= \frac{1}{\sqrt{4}}\int \frac{dx}{\sqrt{x^2-3x+2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{9-8}{4}}}$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}} = \frac{1}{2}\mathbf{cosh}^{-1}\frac{x-\frac{3}{2}}{\frac{1}{2}} = \frac{1}{2}\mathbf{cosh}^{-1}(2x-3)+c$$

Type III

$\int \frac{px+q}{Ax^2+Bx+C}dx$ and $\int \frac{px+q}{\sqrt{Ax^2+Bx+C}}$

to evaluate put $px+q = l$ (derivative of $Ax^2+Bx+C)+m = l(2Ax+B)+m$

where l & m are the constants to be found out by equating the co-efficients of corresponding terms on both sides. ie. to solve for m & n from the equations

$2Al = p$ and $lB+m = q$

then $\int \frac{px+q}{Ax^2+Bx+c}dx = l\int \frac{2Ax+B}{Ax^2+Bx+C}dx + m\int \frac{dx}{Ax^2+Bx+C} = l\cdot\mathbf{log}(Ax^2+Bx+C)+m\int \frac{dx}{Ax^2+Bx+C}$

the second integral in RHS is Type I and hence can be evaluated.

$$\int \frac{px+q}{\sqrt{Ax^2+Bx+C}}dx = l\int \frac{px+q}{\sqrt{Ax^2+Bx+C}}dx + m\int \frac{dx}{\sqrt{Ax^2+Bx+C}} = 2l\sqrt{Ax^2+Bx+C} + m\int \frac{dl}{\sqrt{Ax^2+Bx+C}}$$

the second integral in RHS is Type II and hence can be evaluated.

Examples

(1) Evaluate $\int \frac{2x+3}{3x^2-4x+5}dx$

Solution: Put $2x+3 = l(6x-4)+m = 6lx-4l+m$

$\therefore\ 6l = 2 \Rightarrow l = \frac{1}{3},\ -4l+m = 3$ ie $m = 3+\frac{4}{3} = \frac{13}{3}$

$$\therefore \int \frac{2x+3}{3x^2-4x+5}dx = \frac{1}{3}\int \frac{6x-4}{3x^2-4x+5}dx + \frac{13}{3}\int \frac{dx}{3x^2-4x+5} = \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{9}\int \frac{dx}{x^2-\frac{4}{3}x+\frac{5}{3}}$$

$$= \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{9}\int \frac{dx}{\left(x-\frac{2}{3}\right)^2-\frac{4}{9}+\frac{5}{3}} = \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{9}\int \frac{dx}{\left(x-\frac{2}{3}\right)^2+\left(\frac{\sqrt{11}}{3}\right)^2}$$

$$= \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{9}\times\frac{3}{\sqrt{11}}\mathbf{tan}^{-1}\left(\frac{x-\frac{2}{3}}{\frac{\sqrt{11}}{3}}\right) = \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{3\sqrt{11}}\mathbf{tan}^{-1}\left(\frac{3x-2}{\sqrt{11}}\right) + c$$

(2) Evaluate $\int \frac{5x-7}{\sqrt{3x-x^2-2}}dx$

Solution: Put $5x-7 = l(3-2x)+m = -2lx+3l+m$

$$\therefore\ -2l = 5 \Rightarrow l = -\frac{5}{2}\ \ \&\ \ 3l+m=-7 \Rightarrow m = -7-3l = -7+\frac{15}{2} = \frac{1}{2}$$

$$\therefore \int \frac{5x-7}{\sqrt{3x-x^2-2}}dx = -\frac{5}{2}\int \frac{3-2x}{\sqrt{3x-x^2-2}}dx + \frac{1}{2}\int \frac{dx}{\sqrt{-2-(x^2-3x)}}$$

$$= -\frac{5}{2}\cdot 2\sqrt{3x-x^2-2} + \frac{1}{2}\int \frac{dx}{\sqrt{-2-\left(x-\frac{3}{2}\right)^2+\frac{9}{4}}} = -5\sqrt{3x-x^2-2} + \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}$$

$$= -5\sqrt{3x-x^2-2} + \frac{1}{2}\times\mathbf{sin}^{-1}\left(\frac{x-\frac{3}{2}}{\frac{1}{2}}\right) = -5\sqrt{3x-x^2-2} + \frac{1}{2}\mathbf{sin}^{-1}(2x-3) + c$$

Type IV

$$\int \frac{dx}{a\,\mathbf{cos}\,x + b\,\mathbf{sin}\,x + c}$$

to evalute put $\mathbf{tan}\frac{x}{2} = t$ then differentiating w.r.t. x $\frac{1}{2}\mathbf{sec}^2\frac{x}{2} = \frac{dt}{dx}$

ie $dx = \frac{2dt}{\mathbf{sec}^2\frac{x}{2}} = \frac{2dt}{1+\mathbf{tan}^2\frac{x}{2}} = \frac{2dt}{1+t^2}$ & $\mathbf{cos}\,x = \mathbf{cos}^2\frac{x}{2} - \mathbf{sin}^2\frac{x}{2} = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$

& $\mathbf{sin}\,x = 2\,\mathbf{sin}\frac{x}{2}\mathbf{cos}\frac{x}{2} = 2\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$

$\therefore$ when $\mathbf{tan}\frac{x}{2} = t$, $dx = \frac{2dt}{1+t^2}$, $\mathbf{cos}\,x = \frac{1-t^2}{1+t^2}$ & $\mathbf{sin}\,x = \frac{2t}{1+t^2}$

$$\therefore \int \frac{dx}{a\cos x + b\sin x + c} = \int \frac{\frac{2dt}{(1+t^2)}}{a\frac{(1-t^2)}{(1+t^2)} + b\frac{2t}{(1+t^2)} + c} = \int \frac{2dt}{a(1-t^2) + 2bt + c(1+t^2)} = \int \frac{2dt}{(c-a)t^2 + 2bt + a + c}$$

which is Type I and hence can be evaluated.

Examples

(1) Evaluate $\int \frac{dx}{2\cos x - 3\sin x + 5}$

Solution: Put $\tan\frac{x}{2} = t$, then $dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ & $\sin x = \frac{2t}{1+t^2}$

$$\text{G.I.} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2(1-t^2)}{1+t^2} - \frac{3\times 2t}{1+t^2} + 5} = \int \frac{2dt}{2(1-t^2) - 6t + 5(1+t^2)} = \int \frac{2dt}{3t^2 - 6t + 7} = \frac{2}{3}\int \frac{dt}{t^2 - 2t + \frac{7}{3}}$$

$$= \frac{2}{3}\int \frac{dt}{(t-1)^2 - 1 + \frac{7}{3}} = \frac{2}{3}\int \frac{dt}{(t-1)^2 + \left(\frac{2}{\sqrt{3}}\right)^2} = \frac{2}{3}\times\frac{1}{\frac{2}{\sqrt{3}}}\tan^{-1}\left(\frac{t-1}{\frac{2}{\sqrt{3}}}\right) = \frac{1}{\sqrt{3}}\tan^{-1}\frac{\sqrt{3}}{2}(t-1)$$

$$\therefore \int \frac{dx}{2\cos x - 3\sin x + 5} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{\sqrt{3}}{2}\left(\tan\frac{x}{2} - 1\right) + c$$

(2) Evaluate $\int \frac{dx}{3 - 5\cos x}$

Solution: Put $\tan\frac{x}{2} = t$, then $dx = \frac{2dt}{1+t^2}$ & $\cos x = \frac{1-t^2}{1+t^2}$

$$\therefore \int \frac{dx}{3 - 5\cos x} = \int \frac{\frac{2dt}{(1+t^2)}}{3 - \frac{5(1-t^2)}{1+t^2}} = \int \frac{2dt}{3(1+t^2) - 5(1-t^2)} = \int \frac{2dt}{8t^2 - 2} = \frac{2}{8}\int \frac{dt}{t^2 - \frac{2}{8}} = \frac{1}{4}\int \frac{dt}{t^2 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{4}\times\frac{1}{2\times\frac{1}{2}}\log\frac{t - \frac{1}{2}}{t + \frac{1}{2}} = \frac{1}{4}\log\frac{\tan\frac{x}{2} - \frac{1}{2}}{\tan\frac{x}{2} + \frac{1}{2}}$$

$$\therefore \int \frac{dx}{3 - 5\cos x} = \frac{1}{4}\log\frac{2\tan\frac{x}{2} - 1}{2\tan\frac{x}{2} + 1} + c$$

(3) Evaluate $\int \frac{dx}{3 + 2\sin x}$

Solution: Put $\tan\frac{x}{2} = t$, then $dx = \frac{2dt}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$

$$\therefore \int \frac{dx}{3+2\sin x} = \int \frac{\frac{2dt}{1+t^2}}{3+\frac{4t}{1+t^2}} = \int \frac{2dt}{3(1+t^2)+4t} = \frac{2}{3}\int \frac{dt}{1+t^2+\frac{4}{3}t} = \frac{2}{3}\int \frac{dt}{\left(t+\frac{2}{3}\right)^2-\frac{4}{9}+1}$$

$$= \frac{2}{3}\int \frac{dt}{\left(t+\frac{2}{3}\right)^2+\left(\frac{\sqrt{5}}{3}\right)^2} = \frac{2}{3}\times\frac{1}{\frac{\sqrt{5}}{3}}\tan^{-1}\frac{t+\frac{2}{3}}{\frac{\sqrt{5}}{3}} = \frac{2}{\sqrt{5}}\tan^{-1}\frac{3t+2}{\sqrt{5}}$$

$$\therefore \int \frac{dx}{3+2\sin x} = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{3\tan\frac{x}{2}+2}{\sqrt{5}}\right)+c$$

Type V

$$\int \frac{a\cos x+b\sin x}{c\sin x+e\cos x}dx$$

Solution: to evaluate put $a\cos x+b\sin x = l(\text{Denominator})+m(\text{derivative of denominator})$

ie $a\cos x+b\sin x = l(c\sin x+e\cos x)+m(c\cos x-e\sin x)$

where l **&** m are constants to be foundout by equating the co - efficients of $\sin x$ **&** $\cos x$ separately.

ie from the equations $lc-me=b$ **&** $le+mc=a$

then $\int \frac{a\cos x+b\sin x}{c\sin x+e\cos x}dx = l\int \frac{c\sin x+e\cos x}{c\sin x+e\cos x}dx + m\int \frac{c\cos x-e\sin x}{c\sin x+e\cos x}dx = lx+m\log(c\sin x+e\cos x)+c$

Examples

Evaluate $\int \frac{3\cos x-2\sin x}{4\sin x+\cos x}dx$

Solution: Put $3\cos x-2\sin x = l(4\sin x+\cos x)+m(4\cos x-\sin x)$

$\therefore\ 4l-m=-2$ (1)

$l+4m=3$ (2)

$(1)\times 4 \quad 16l-4m=-8$

$(2)\times 1 \quad l+4m=3$

adding $17l=-5 \Rightarrow l=-\frac{5}{17}$

from (1), $m=4l+2=-\frac{20}{17}+2=\frac{-20+34}{17}=\frac{14}{17}$

$$\therefore \int \frac{3\cos x-2\sin x}{4\sin x+\cos x}dx = -\frac{5}{17}\int \frac{4\sin x+\cos x}{4\sin x+\cos x}dx+\frac{14}{17}\int \frac{4\cos x-\sin x}{4\sin x+\cos x} = -\frac{5}{17}\int 1\cdot dx+\frac{14}{17}\log(4\sin x+\cos x)$$

$$= -\frac{5}{17}x+\frac{14}{17}\log(4\sin x+\cos x)+c$$

Type VI

$\int f(x) e^x \, dx$ where $f(x) = \phi(x) + \phi'(x)$

Solution: $\int f(x) e^x \, dx = \int \phi(x) e^x \, dx + \int \phi'(x) e^x \, dx$ (1)

Consider $\int \phi(x) e^x \, dx$

put $u = \phi(x),\ v' = e^x,\ u' = \phi(x),\ v = e^x$

$\therefore \int \phi(x) e^x \, dx = \phi(x) e^x - \int \phi'(x) e^x \, dx$

substituting this in (1), we have

$$\int f(x) e^x \, dx = \phi(x) e^x - \int \phi'(x) e^x \, dx + \int \phi'(x) e^x \, dx = \phi(x) e^x + c$$

Examples

(1) Evaluate $\int \frac{xe^x}{(1+x)^2} dx$

Solution: $\int \frac{xe^x}{(1+x)^2} dx = \int \frac{(1+x-1)e^x}{(1+x)^2} dx = \int \frac{(1+x)e^x}{(1+x)^2} dx - \int \frac{e^x}{(1+x)^2} dx = \int \frac{e^x}{1+x} dx - \int \frac{e^x}{(1+x)^2} dx$ (1)

Consider $\int \frac{e^x}{(1+x)} dx$

put $u = \frac{1}{1+x},\ v' = e^x,\ u' = \frac{-1}{(1+x)^2},\ v = e^x$

$\therefore \int \frac{e^x}{(1+x)} dx = \frac{e^x}{(1+x)} - \int \frac{-e^x}{(1+x)^2} dx = \frac{e^x}{(1+x)} + \int \frac{e^x}{(1+x)^2} dx$

substituting in (1)

$$\int \frac{xe^2}{(1+x)^2} dx = \frac{e^x}{(1+x)} + \int \frac{e^x}{(1+x)^2} dx - \int \frac{e^x}{(1+x)^2} dx = \frac{e^x}{1+x} + c$$

(2) Evaluate $\int \frac{x - \sin x}{1 - \cos x} dx$

Solution: $\int \frac{x - \sin x}{1 - \cos x} dx = \int \frac{x}{1 - \cos x} dx - \int \frac{\sin x}{1 - \cos x} dx = \int \frac{x}{2\sin^2 \frac{x}{2}} dx - \int \frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx$

$$= \int \frac{1}{2} x \operatorname{cosec}^2 \frac{x}{2} dx - \int \cot \frac{x}{2} dx \quad (1)$$

Consider $\int \frac{1}{2} x \operatorname{cosec}^2 x \, dx$

put $u = x,\ v' = \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2},\ u' = 1,\ v = -\cot \frac{x}{2}$

$$\int \frac{1}{2} x \operatorname{cosec}^2 \frac{x}{2}\, dx = -x \cot\frac{x}{2} + \int \cot\frac{x}{2}\, dx$$

substituting in (1)

$$\int \frac{x - \sin x}{1 - \cos x}\, dx = -x \cot\frac{x}{2} + \int \cot\frac{x}{2}\, dx - \int \cot\frac{x}{2}\, dx = -x \cot\frac{x}{2} + c$$

Other examples

(1) Evaluate $\int \frac{\sin x \cos x}{1 + \sin^4 x}\, dx$

Solution: Put $\sin^2 x = t$, then $2 \sin x \cos x\, dx = dt$

$$\therefore \int \frac{\sin x \cos x}{1 + \sin^4 x}\, dx = \int \frac{\frac{1}{2} dt}{1 + t^2} = \frac{1}{2} \tan^{-1} t = \frac{1}{2} \tan^{-1}(\sin^2 x) + c$$

(2) Evaluate $\int \frac{x^2 + 1}{(x+1)(x^2+2)}\, dx$

Solution: Let $\frac{x^2+1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 2}$

multiplying throughout by $(x+1)(x^2+2)$

then $x^2 + 1 = A(x^2 + 2) + (Bx + C)(x + 1)$

put $x = -1$, $2 = A(1+2) + 0 \Rightarrow A = \frac{2}{3}$

put $x = 0$, $1 = 2A + C \Rightarrow C = 1 - \frac{4}{3} = -\frac{1}{3}$

Equating co-efficient of x^2 on both sides

$$A + B = 1 \Rightarrow B = 1 - A = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \frac{x^2+1}{(x+1)(x^2+2)} = \frac{\frac{2}{3}}{x+1} + \frac{\frac{1}{3}x - \frac{1}{3}}{x^2+2}$$

$$\therefore \int \frac{x^2+1}{(x+1)(x^2+2)}\, dx = \frac{2}{3}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{x-1}{x^2+2}\, dx = \frac{2}{3}\int \frac{dx}{x+1} + \frac{1}{6}\int \frac{2x}{x^2+2}\, dx - \frac{1}{3}\int \frac{dx}{x^2+2}$$

$$= \frac{2}{3} \log(x+1) + \frac{1}{6} \log(x^2+2) - \frac{1}{3\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + c$$

Type VII

(1) $\int \sqrt{a^2 - x^2}\, dx$ (2) $\int \sqrt{a^2 + x^2}\, dx$ (3) $\int \sqrt{x^2 - a^2}\, dx$ (4) $\int \sqrt{Ax^2 + Bx + C}\, dx$

(5) $\int (px + q)\sqrt{Ax^2 + Bx + C}\, dx$

(1) To evaluate $\int \sqrt{a^2 - x^2}\, dx$ put $x = a\,\mathbf{sin}\,\theta$, $dx = a\,\mathbf{cos}\,\theta\, d\theta$

$$\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - a^2\,\mathbf{sin}^2\theta} \cdot a\,\mathbf{cos}\,\theta\, d\theta = \int a\,\mathbf{cos}\,\theta \cdot a\,\mathbf{cos}\,\theta\, d\theta = \int a^2\,\mathbf{cos}^2\theta\, d\theta = \frac{a^2}{2}\int (1 + \mathbf{cos}\,2\theta)\, d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2} \times \frac{\mathbf{sin}\,2\theta}{2} = \frac{a^2}{2}\theta + \frac{a^2}{2}\mathbf{sin}\,\theta\,\mathbf{cos}\,\theta = \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot \mathbf{sin}\,\theta \cdot \sqrt{1 - \mathbf{sin}^2\theta}$$

$$= \frac{a^2}{2}\mathbf{sin}^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}} = \frac{a^2}{2}\mathbf{sin}^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a^2}\sqrt{a^2 - x^2}$$

$$\therefore \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\mathbf{sin}^{-1}\frac{x}{a} + c$$

(2) To evaluate $\int \sqrt{a^2 + x^2}\, dx$ put $x = a\,\mathbf{sinh}\theta$, then $dx = a\,\mathbf{cosh}\theta\, d\theta$

$$\int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2 + a^2\,\mathbf{sinh}^2\theta} \cdot a\,\mathbf{cosh}\theta\, d\theta = \int a^2\,\mathbf{cosh}^2\theta\, d\theta = \frac{a^2}{2}\int (1 + \mathbf{cosh}\,2\theta)\, d\theta$$

$$= \frac{a^2}{2}\int 1 \cdot d\theta + \frac{a^2}{2}\int \mathbf{cosh}\,2\theta\, d\theta = \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\frac{\mathbf{sinh}2\theta}{2} = \frac{a^2}{2}\theta + \frac{a^2}{2} \times \mathbf{sinh}\theta\,\mathbf{cosh}\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\mathbf{sinh}\theta\sqrt{1 + \mathbf{sinh}^2\theta} = \frac{a^2}{2}\mathbf{sinh}^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{1 + \frac{x^2}{a^2}} = \frac{a^2}{2}\mathbf{sinh}^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 + x^2}$$

$$\therefore \int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\mathbf{sinh}^{-1}\frac{x}{a} + c$$

(3) To evaluate $\int \sqrt{x^2 - a^2}\, dx$ put $x = a\,\mathbf{cosh}\theta$, then $dx = a\,\mathbf{sinh}\, d\theta$

$$\int \sqrt{x^2 - a^2}\, dx = \int \sqrt{a^2\,\mathbf{cosh}^2\theta - a^2} \cdot a\,\mathbf{sinh}\theta\, d\theta = \int a\,\mathbf{sinh}\theta \cdot a\,\mathbf{sinh}\theta\, d\theta = \int a^2\,\mathbf{sinh}^2\theta\, d\theta = \frac{a^2}{2}\int (\mathbf{cosh}\,2\theta - 1)\, d\theta$$

$$= \frac{a^2}{2}\int \mathbf{cosh}\,2\theta\, d\theta - \frac{a^2}{2}\int 1 \cdot d\theta = \frac{a^2}{2}\cdot\frac{\mathbf{sinh}2\theta}{2} - \frac{a^2}{2}\theta = \frac{a^2}{2}\mathbf{sinh}\theta\,\mathbf{cosh}\theta - \frac{a^2}{2}\mathbf{cosh}^{-1}\frac{x}{a}$$

$$= \frac{a^2}{2}\sqrt{\mathbf{cosh}^2\theta - 1}\,\mathbf{cosh}\theta - \frac{a^2}{2}\mathbf{cosh}^{-1}\frac{x}{a} = \frac{a^2}{2}\sqrt{\frac{x^2}{a^2} - 1}\cdot\frac{x}{a} - \frac{a^2}{2}\mathbf{cosh}^{-1}\frac{x}{2} = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\mathbf{cosh}^{-1}\frac{x}{a}$$

$$\therefore \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\mathbf{cosh}^{-1}\frac{x}{a} + c$$

(4) To evaluate $\int \sqrt{Ax^2 + Bx + C}\, dx = \sqrt{A}\int \sqrt{x^2 + \frac{B}{A}x + \frac{C}{A}}\, dx$. This will take the form (1), (2) or (3) and hence can be evaluated.

(5) To evaluate $\int (px + q)\sqrt{Ax^2 + Bx + C}\, dx$

put $px + q = l(\text{derivative of } Ax^2 + Bx + C) + m = l(2Ax + B) + m$

where l **&** m are constants to be found out,

then, $\int (px + q)\sqrt{Ax^2 + Bx + C}\, dx = l\int (2Ax + B)\sqrt{Ax^2 + Bx + C}\, dx + m\int \sqrt{Ax^2 + Bx + C}$

$$= \frac{2l}{3}(Ax^2 + Bx + C)^{3/2} + m\sqrt{A}\int\sqrt{x^2 + \frac{B}{A}x + \frac{C}{A}}\,dx$$

the second integral reduces to (1), (2) or (3) and hence can be evaluated.

Type VIII

$$\int \frac{dx}{(px+q)\sqrt{Ax^2 + Bx + C}}$$

put $px + q = \frac{1}{t}$ then $p\,dx = \frac{-1}{t^2}dt$ **&** $x = \frac{1}{p}\left(\frac{1}{t} - q\right)$

$$\therefore \int \frac{dx}{(px+q)\sqrt{Ax^2+Bx+C}} = \int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\frac{A}{p^2}\left(\frac{1}{t}-q\right)^2 + \frac{B}{p}\left(\frac{1}{t}-q\right) + C}} = \int \frac{-dt}{\sqrt{\frac{A}{p^2}(1-tq)^2 + \frac{B}{p}(t - qt^2) + Ct^2}}$$

This integral reduces to any one of Type II and hence can be solved.

Type IX

$$\int e^{ax}\,\mathbf{cos}(bx+c)\,dx \text{ and } \int e^{ax}\,\mathbf{sin}(bx+c)\,dx$$

To evaluate we have to use integration by parts.

Let $C = \int e^{ax}\,\mathbf{cos}(bx+c)\,dx$ **&** $S = \int e^{ax}\,\mathbf{sin}(bx+c)\,dx$

Consider $C = \int e^{ax}\,\mathbf{cos}(bx+c)\,dx$

put $u = e^{ax}$, $u' = ae^{ax}$, $v' = \mathbf{cos}(bx+c)$, $v = \frac{\mathbf{sin}(bx+c)}{b}$

$$C = \frac{e^{ax}\,\mathbf{sin}(bx+c)}{b} - \frac{a}{b}\int e^{ax}\,\mathbf{sin}(bx+c)\,dx$$

$bC = e^{ax}\,\mathbf{sin}(bx+c) - aS$

$\therefore\ aS + bC = e^{ax}\,\mathbf{sin}(bx+c)$ (1)

Consider $S = \int e^{ax}\,\mathbf{sin}(bx+c)\,dx$

put $u = e^{ax}$, $u' = ae^{ax}$, $v' = \mathbf{sin}(bx+c)$, $v = \frac{-\mathbf{cos}(bx+c)}{b}$

$$\therefore\ S = \frac{-e^{ax}\,\mathbf{cos}(bx+c)}{b} + \frac{a}{b}\int e^{ax}\,\mathbf{cos}(bx+c)\,dx$$

$bS = -ea^{ax}\,\mathbf{cos}(bx+c) + aC$

ie $bS - aC = -e^{ax}\,\mathbf{cos}(bx+c)$ (2)

$(1)\times a \quad a^2S + abC = ae^{ax}\,\mathbf{sin}(bx+c)$

$(2)\times b \quad b^2S - abC = -be^{ax}\,\mathbf{cos}(bx+c)$

adding $\quad (a^2 + b^2)S = e^{ax}\left[a\,\mathbf{sin}(bx+c) - b\,\mathbf{cos}(bx+c)\right]$

$$\therefore\ S=\frac{e^{ax}\left[a\,\mathbf{sin}(bx+c)-b\,\mathbf{cos}(bx+c)\right]}{(a^2+b^2)}$$

$(1)\times b \qquad abS+b^2C=be^{ax}\,\mathbf{sin}(bx+c)$

$(2)\times a \qquad abS-a^2C=-ae^{ax}\,\mathbf{cos}(bx+c)$

subtracting $(a^2+b^2)C=e^{ax}\left[b\,\mathbf{sin}(bx+c)+a\,\mathbf{cos}(bx+c)\right]$

$$\therefore\ C=\frac{e^{ax}\left[a\,\mathbf{cos}(bx+c)+b\,\mathbf{sin}(bx+c)\right]}{(a^2+b^2)}$$

Examples

(1) $$\int e^{2x}\,\mathbf{sin}\,3x\,\mathbf{cos}\,2x\,dx=\frac{1}{2}\int e^{2x}[\mathbf{sin}\,5x+\mathbf{sin}\,x]\,dx=\frac{1}{2}\int e^{2x}\,\mathbf{sin}\,5x\,dx+\frac{1}{2}\int e^{2x}\,\mathbf{sin}\,dx$$

$$=\frac{1}{2}e^{2x}\,\frac{(2\,\mathbf{sin}\,5x-5\,\mathbf{cos}\,5x)}{29}+\frac{1}{2}e^{2x}\,\frac{(2\,\mathbf{sin}\,x-\mathbf{cos}\,x)}{5}+c$$

(2) $$\int e^{3x}\,\mathbf{cos}^2\,x\,dx=\frac{1}{2}\int e^{3x}(1+\mathbf{cos}\,2x)\,dx=\frac{1}{2}\int e^{3x}\,dx+\frac{1}{2}\int e^{3x}\,\mathbf{cos}\,2x\,dx=\frac{1}{2}\times\frac{e^{3x}}{3}+\frac{1}{2}e^{3x}\,\frac{(3\,\mathbf{cos}\,2x+2\,\mathbf{sin}\,2x)}{9+4}$$

$$=\frac{e^{3x}}{6}+\frac{e^{3x}}{26}(3\,\mathbf{cos}\,2x+2\,\mathbf{sin}\,2x)+c$$

Exercise

Integrate the following w.r.t. x

(1) $\sqrt{\dfrac{\mathbf{sin}^{-1}\,x}{1-x^2}}$ (2) $\dfrac{1}{x\,\mathbf{cos}^2(\mathbf{log}\,x)}$ (3) $\dfrac{e^{\mathbf{sin}^{-1}x}}{\sqrt{1-x^2}}$

(4) $\dfrac{\mathbf{sec}^2\,x}{\mathbf{tan}\,x(2+\mathbf{tan}\,x)}$ (5) $\dfrac{3x-2}{(x+1)^2(x+3)}$ (6) $\dfrac{x}{(x-1)(x^2+4)}$

(7) $\dfrac{x^3-x-2}{x^2-1}$ (8) $\dfrac{4x+5}{x^2+22x+2}$ (9) $\dfrac{4x+1}{\sqrt{x^2-6x+18}}$

(10) $\dfrac{(1+x)}{(2+x)^2}e^x$ (11) $\dfrac{1}{2+\mathbf{cos}\,x-\mathbf{sin}\,x}$ (12) $\dfrac{1}{3+4\,\mathbf{cos}\,x}$

(13) $\dfrac{2\,\mathbf{sin}\,x+3\,\mathbf{cos}\,x}{4\,\mathbf{sin}\,x+5\,\mathbf{cos}\,x}$ (14) $\dfrac{1}{4\,\mathbf{cos}^2\,x+9\,\mathbf{sin}^2\,x}$ (15) $\dfrac{(1+\mathbf{sin}\,x)}{(1+\mathbf{cos}\,x)}e^x$

(16) $\dfrac{1}{x(x^n+1)}$ (17) $\sqrt{6-4x-2x^2}$ (18) $(2x-5)\sqrt{x^2-3x+2}$

(19) $\dfrac{1}{(x+1)\sqrt{2x^2+3x+4}}$ (20) $\dfrac{1}{(x+1)\sqrt{x^2-1}}$ (21) $e^{2x}\,\mathbf{sin}\,4x\,\mathbf{sin}\,2x$

(22) $e^{2x}\,\mathbf{cos}\,3x\,\mathbf{cos}\,x$ (23) $e^{3x}\,\mathbf{cos}^3\,x$ (24) $e^{4x}\,\mathbf{sin}^3\,x$

Definite Integrals

Let $f(x)$ be a function defined in the interval (a,b) and $\int f(x)\,dx = g(x)+c$

The value of the integral at $x=b$ minus the value of the integral at $x=a$ ie $[g(b)+c]-[g(a)+c]$

ie $g(b)-g(a)$ is defined as Definite integral and denoted as $\int_a^b f(x)\,dx$

ie If $\int f(x)\,dx = g(x)$ then $\int_a^b f(x)\,dx = g(b)-g(a)$

b is called upper limit and a is called lower limit.

Examples

(1) Evaluate $\int_1^2 (x^3-2x^2+3)\,dx$

Solution : $\int_1^2 (x^3-2x^2+3)\,dx = \left[\frac{x^4}{4}-2\times\frac{x^3}{3}+3x\right]_1^2 = \left(\frac{2^4}{4}-2\times\frac{2^3}{3}+3\cdot 2\right)-\left(\frac{1}{4}-\frac{2}{3}+3\right)^1 = 4 = \frac{16}{3}+6-\frac{1}{4}+\frac{2}{3}-3$

$$= 7-\frac{16}{3}-\frac{1}{4}+\frac{2}{3} = \frac{84-64-3+8}{12} = \frac{25}{12}$$

(2) Evaluate $\int_0^1 \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}\,dx$

Solution : Put $\sin^{-1}x = t$ then $\frac{1}{\sqrt{1-x^2}}\,dx = dt$

when $x=0,\ t=\sin^{-1}0=0$ when $x=1,\ t=\sin^{-1}1=\frac{\pi}{2}$

$$\therefore \int_0^1 \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} t^2\,dt = \left.\frac{t^3}{3}\right]_0^{\pi/2} = \frac{1}{3}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{24}$$

(3) Evaluate $\int_{-1}^1 \frac{dx}{x^2+2x+5}dx$

Solution : $\int_{-1}^1 \frac{dx}{x^2+2x+5}dx = \int_{-1}^1 \frac{dx}{(x+1)^2+2^2} = \left.\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^1 = \frac{1}{2}\tan^{-1}\left(\frac{1+1}{2}\right)-\frac{1}{2}\tan^{-1}\left(\frac{-1+1}{2}\right)$

$$= \frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0 = \frac{1}{2}\frac{\pi}{4}-0 = \frac{\pi}{8}$$

(4) Evaluate $\int_0^{\pi} \frac{dx}{4+3\cos x}$

Solution : Put $\tan\frac{x}{2} = t$ then $dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

when $x=0,\ t=\tan 0=0$ when $x=\pi,\ t=\tan\frac{\pi}{2}=\infty$

$$\int_0^{\pi} \frac{dx}{4+3\cos x} = \int_0^{\infty} \frac{\frac{2dt}{1+t^2}}{4+\frac{3(1-t^2)}{(1+t^2)}} = \int_0^{\infty} \frac{2dt}{4(1+t^2)+3(1-t^2)} = \int_0^{\infty} \frac{2dt}{t^2+\left(\sqrt{7}\right)^2} = \frac{1}{\sqrt{7}} \tan^{-1} \frac{t}{\sqrt{7}}\Bigg]_0^{\infty}$$

$$= \frac{1}{\sqrt{7}} \tan^{-1} \infty - \frac{1}{\sqrt{7}} \tan^{-1} 0 = \frac{1}{\sqrt{7}} \times \frac{\pi}{2} = \frac{\pi}{2\sqrt{7}}$$

Properties of Definite Integrals

1. $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ also $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

5. $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is an even function} \\ 0 & \text{if } f(x) \text{ is an odd function} \end{cases}$$

6. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

Examples

(1) Evaluate $\int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx$

Solution: Let $I = \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx = \int_0^{\pi/2} \frac{\sin^n\left(\frac{\pi}{2}-x\right)}{\cos^n\left(\frac{\pi}{2}-x\right)+\sin^n\left(\frac{\pi}{2}-x\right)}\,dx$ using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$= \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx$$

$$\therefore\ 2I = \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx + \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx = \int_0^{\pi/2} \frac{(\sin^n x + \cos^n x)}{(\cos^n x + \sin^n x)}\,dx = \int_0^{\pi/2} 1 \cdot dx = x\Big]_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore\ I = \frac{\pi}{4}$$

(2) Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$

Solution: Let $I = \int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}\,dx$ using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

ie $I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x} dx$

$\therefore\ 2I = \int_0^{\pi} \frac{x\sin x}{1+\sin x} dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x} dx = \int_0^{\pi} \frac{x\sin x + (\pi - x)\sin x}{1+\sin x} dx = \int_0^{\pi} \frac{\pi \sin x}{1+\sin x} dx = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}$

$= \pi \int_0^{\pi} \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)} dx = \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x} dx = \pi \int_0^{\pi} \sec x \tan x\, dx - \pi \int_0^{\pi} \tan^2 x\, dx$

$= \pi \left[\int_0^{\pi} \sec x \tan x - \int_0^{\pi} \sec^2 x\, dx + \int_0^{\pi} 1\, dx \right] = \pi [\sec x - \tan x + x]_0^{\pi} = \pi [\sec \pi - \tan \pi + \pi] - \pi [\sec 0 - \tan 0 + 0]$

$= \pi[-1+\pi] - \pi[1] = \pi[-2+\pi]$

$\therefore\ I = \frac{\pi}{2}(\pi - 2)$

(3) Evaluate $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

Solution: Let $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}dx}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)}$ using $\int_a^b f(x)\, dx = \int_a^b f(a-x)\, dx$

$\therefore\ I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

$\therefore\ 2I = \int_{\pi/6}^{\pi/3} \left[\frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}} \right] dx = \int_{\pi/6}^{\pi/3} \left(\frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx = \int_{\pi/6}^{\pi/3} 1 \cdot dx = x \Big]_{\pi/6}^{\pi/3}$

$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$\therefore\ I = \frac{\pi}{12}$

Exercise

Evaluate the following

(1)	$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$	(2)	$\int_1^e \log x\, dx$	(3)	$\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x} dx$
(4)	$\int_{-1/2}^{1/2} \frac{dx}{9 - x^2}$	(5)	$\int_0^1 x \tan^{-1} x\, dx$	(6)	$\int_{-1}^1 x e^{-x}\, dx$
(7)	$\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$	(8)	$\int_0^{\infty} \frac{x\, dx}{(x+1)(x^2+1)}$	(9)	$\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$
(10)	$\int_0^{2a} \frac{f(x)}{f(x) + f(2a-x)} dx$	(11)	$\int_0^{\pi/2} \frac{x\, dx}{\sin x + \cos x}$	(12)	$\int_0^1 \frac{\log(1+x)}{(1+x^2)} dx$

Answers: (1) $\frac{\pi}{4}$, (2) 1, (3) $e-1$, (4) $\frac{1}{3}\log\frac{7}{5}$, (5) $\frac{\pi}{4} - \frac{1}{2}$, (6) $-\frac{2}{e}$, (7) $\frac{\pi}{2ab}$, (8) $\frac{\pi}{4}$,

(9) $\frac{\pi}{8}\log 2$ (10) a (11) $\frac{\pi}{2\sqrt{2}}\log\left(\sqrt{2}+1\right)$ (12) $\frac{\pi}{8}\log 2$

Reduction formulae

I. To obtain the reduction formula for $I_n = \int \sin^n x\,dx$ and hence to evaluate $\int_0^{\pi/2} \sin^n x\,dx$

Solution: $I_n = \int \sin^{n-1} x \cdot \sin x\,dx$

put $u = \sin^{n-1} x$ & $v' = \sin x$, $u' = (n-1)\sin^{n-2} x\cos x$ & $v = -\cos x$

$$\therefore\ I_n = -\sin^{n-1} x\cos x + \int (n-1)\sin^{n-2} x\cos^2 x\,dx = -\sin^{n-1}\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx = -\sin^{n-1}\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore\ I_n + (n-1)I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

ie $(1+n-1)I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$

ie $I_n = \dfrac{-\sin^{n-1} x\cos x}{n} + \dfrac{n-1}{n} I_{n-2}$

the ultimate integral is I_0 or I_1 according as n is even or odd

If n is even $I_0 = \int 1\,dx = x$ If n is odd $I_1 = \int \sin x\,dx = -\cos x$

If $I_n = \int_0^{\pi/2} \sin^n x\,dx$ then

$$I_n = \left.\frac{-\sin^{n-1} x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2} = 0 + \frac{n-1}{n} I_{n-2}$$

$$\therefore\ I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}\times\frac{n-3}{n-2} I_{n-4} = \frac{n-1}{n}\times\frac{n-3}{n-2}\times\frac{n-5}{n-4} I_{n-6}$$

in general

$$I_n = \begin{cases} \dfrac{n-1}{n}\times\dfrac{n-3}{n-2}\times\cdots\cdots\times\dfrac{1}{2}\times\dfrac{\pi}{2} & \text{if } n \text{ is even.} \\[2ex] \dfrac{n-1}{n}\times\dfrac{n-3}{n-2}\times\cdots\cdots\times\dfrac{2}{3}\times 1 & \text{if } n \text{ is odd.} \end{cases}$$

Eg. (1) $I_6 = \int_0^{\pi/2} \sin^6 x\,dx = \dfrac{5}{6}\times\dfrac{3}{4}\times\dfrac{1}{2}\times\dfrac{\pi}{2} = \dfrac{5\pi}{32}$

(2) $I_5 = \int_0^{\pi/2} \sin^5 x\,dx = \dfrac{4}{5}\times\dfrac{2}{3}\times 1 = \dfrac{8}{15}$

II. To obtain the reduction formula for $I_n = \int \cos^n dx$ and to evaluate $\int_0^{\pi/2} \cos^n x\,dx$

Solution: using integration by parts as in I we obtain

$$I_n = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n} I_n$$

and further if $I_n = \int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \cos^n\left(\dfrac{\pi}{2} - x\right)dx$ using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$= \int_0^{\pi/2} \mathbf{sin}^n x\,dx \quad \text{which is } I$$

$$\therefore \text{ Eg. (1) } \int_0^{\pi/2} \mathbf{cos}^7 x\,dx = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1 = \frac{16}{35}$$

$$\text{(2) } \int_0^{\pi/2} \mathbf{cos}^8 x\,dx = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{35\pi}{256}$$

III. To obtain the reduction formula for $I_n = \int \mathbf{tan}^n x\,dx$

Solution: $I_n = \int \mathbf{tan}^{n-2} x \cdot \mathbf{tan}^2 x\,dx = \int \mathbf{tan}^{n-2} x \cdot (\mathbf{sec}^2 x - 1)\,dx = \int \mathbf{tan}^{n-2} x \cdot \mathbf{sec}^2 x\,dx - \int \mathbf{tan}^{n-2} x\,dx$

$I_n = \dfrac{\mathbf{tan}^{n-1} x}{n-1} - I_{n-2}$ which is the required formula.

IV. To obtain the reduction formula for $I_n = \int \mathbf{cot}^n x\,dx$

Solution: $I_n = \int \mathbf{cot}^{n-2} x \cdot \mathbf{cot}^2 x\,dx = \int \mathbf{cot}^{n-2} x \cdot (\mathrm{cosec}^2 x - 1)\,dx = \int \mathbf{cot}^{n-2} x\,\mathrm{cosec}^2 x\,dx - \int \mathbf{cot}^{n-2} x\,dx$

$$I_n = \frac{-\mathbf{cot}^{n-1} x}{n-1} - I_{n-2}$$

V. To obtain the reduction formula for $I_n = \int \mathbf{sec}^n x\,dx$

Solution: $I_n = \int \mathbf{sec}^{n-2} x \cdot \mathbf{sec}^2 x\,dx$

put $u = \mathbf{sec}^{n-2} x$ **&** $v' = \mathbf{sec}^2 x$, $u' = (n-2)\,\mathbf{sec}^{n-3} x \cdot \mathbf{sec}\,x\,\mathbf{tan}x$ **&** $v = \mathbf{tan}x$

$\therefore\ I_n = \sec^{n-2} \tan x - \int (n-2)\sec^{n-2} x \cdot \tan^2 x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2}(\sec^2 x - 1)\,dx$

$= \mathbf{sec}^{n-2} x\,\mathbf{tan}x - (n-2)\int \mathbf{sec}^n x\,dx + (n-2)\int \mathbf{sec}^{n-2} x\,dx$

$\therefore\ I_n + (n-2)I_n = \mathbf{sec}^{n-2} x\,\mathbf{tan}x + (n-2)I_{n-2}$

$(1+n-2)I_n = \mathbf{sec}^{n-2} x\,\mathbf{tan}x + (n-2)I_{n-2}$

$\therefore\ I_n = \dfrac{\mathbf{sec}^{n-2} x\,\mathbf{tan}x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$ which is the required reduction formula.

VI. To obtain the reduction formula for $I_n = \int \mathrm{cosec}^n x\,dx$

Solution: $I_n = \int \mathrm{cosec}^{n-2} x \cdot \mathrm{cosec}^2 dx$

put $u = \mathrm{cosec}^{n-2} x$ **&** $v' = \mathrm{cosec}^2 x$, $u' = -(n-2)\mathrm{cosec}^{n-3} x \cdot \mathrm{cosec}\,x\,\mathbf{cot}\,x$ **&** $v = -\mathbf{cot}\,x$

$\therefore\ I_n = -\mathrm{cosec}^{n-2} x\,\mathbf{cot}\,x - \int (n-2)\mathrm{cosec}^{n-2} x \cdot \mathbf{cot}^2 x\,dx = -\mathrm{cosec}^{n-2} x\,\mathbf{cot}\,x - (n-2)\int \mathrm{cosec}^{n-2} x(\mathrm{cosec}^2 x - 1)\,dx$

$= -\mathrm{cosec}^{n-2} x\,\mathbf{cot}\,x - (n-2)\int \mathrm{cosec}^n x\,dx + (n-2)\int \mathrm{cosec}^{n-2} x\,dx = -\mathrm{cosec}^{n-2} x\,\mathbf{cot}\,x - (n-2)I_n + (n-2)I_{n-2}$

ie $I_n + (n-2)I_n = -\text{cosec}^{n-2} x \,\mathbf{cot}\, x + (n-2)I_{n-2}$

ie $(n-1)I_n = -\text{cosec}^{n-2} x \,\mathbf{cot}\, x + (n-2)I_{n-2}$

ie $I_n = \dfrac{-\text{cosec}^{n-2} x \,\mathbf{cot}\, x}{(n-1)} + \dfrac{(n-2)}{(n-1)} I_{n-2}$ which is the required reduction formula.

VII. To obtain the reduction formula of $I_{m,n} = \int \mathbf{sin}^m x \,\mathbf{cos}^n x \, dx$ and hence to evaluate $\int_0^{\pi/2} \mathbf{sin}^m x \,\mathbf{cos}^n x \, dx$

Solution: $I_{m,n} = \int \mathbf{sin}^m x \,\mathbf{cos}^{n-1} x \cdot \mathbf{cos}\, x \, dx$

put $u = \mathbf{sin}^m x \,\mathbf{cos}^{n-1} x$ **&** $v' = \mathbf{cos}\, x$, $u' = m\,\mathbf{sin}^{m-1} x \cdot \mathbf{cos}^n x - (n-1)\,\mathbf{sin}^{m+1} x \cdot \mathbf{cos}^{n-2} x$ **&** $v = \mathbf{sin}\, x$

$$\therefore\ I_{m,n} = \mathbf{sin}^m x \,\mathbf{cos}^{n-1} x \cdot \mathbf{sin}\, x - \int \left(m\,\mathbf{sin}^{m-1} x \,\mathbf{cos}^n x - (n-1)\,\mathbf{sin}^{m+1} \mathbf{cos}^{n-2} x \right) \mathbf{sin}\, x \, dx$$

$$= \mathbf{sin}^{m+1} x \,\mathbf{cos}^{n-1} x - m \int \mathbf{sin}^m x \,\mathbf{cos}^n x \, dx + (n-1) \int \mathbf{sin}^{m+2} x \cdot \mathbf{cos}^{n-2} x \, dx$$

$$= \mathbf{sin}^{m+1} x \,\mathbf{cos}^{n-1} x - mI_{m,n} + (n-1) \int \mathbf{sin}^m x (1 - \mathbf{cos}^2 x) \,\mathbf{cos}^{n-2} x \, dx$$

$$= \mathbf{sin}^{m+1} x \,\mathbf{cos}^{n-1} x - mI_{m,n} + (n-1) \int (\mathbf{sin}^m x \,\mathbf{cos}^{n-2} x - \mathbf{sin}^m x \,\mathbf{cos}^n x) \, dx$$

$$= \mathbf{sin}^{m+1} x \,\mathbf{cos}^{n-1} x - mI_{m,n} + (n-1)I_{m,m-2} - (n-1)I_{m,n}$$

ie $I_{m,n} + mI_{m,n} + (n-1)I_{m,n} = \mathbf{sin}^{m+1} x \,\mathbf{cos}\, x^{n-1} x + (n-1)I_{m,n-2}$

ie $I_{m,n} = \dfrac{\mathbf{sin}^{m+1} x \,\mathbf{cos}^{n-1} x}{(m+n)} + \dfrac{(n-1)}{m+n} I_{m,n-2}$

which is the required reduction formula, if $I_{m,n} = \int_0^{\pi/2} \mathbf{sin}^m x \,\mathbf{cos}^n x \, dx$

then $I_{m,n} = \left. \dfrac{\mathbf{sin}^{m+1} x \,\mathbf{cos}^{n-1} x}{(m+n)} \right]_0^{\pi/2} + \dfrac{n-1}{m+n} I_{m,n-2} = 0 + \dfrac{n-1}{m+n} I_{m,n-2} \quad \therefore\ I_{m,n} = \dfrac{n-1}{m+n} I_{m,n-2}$

applying this reduction formula continuously, we have

$$I_{m,n} = \begin{cases} \dfrac{n-1}{m+n} \times \dfrac{n-3}{m+n-2} \times \cdots\cdots \times \dfrac{2}{m+3} \times \dfrac{1}{m+1} & \text{if } n \text{ is odd \& } m \text{ odd or even} \\[2ex] \dfrac{n-1}{m+n} \times \dfrac{n-3}{m+n-2} \times \cdots\cdots \times \dfrac{1}{m+2} \times \dfrac{m-1}{m} \times \dfrac{m-3}{m-2} \times \cdots\cdots \times \dfrac{2}{3} \times 1 & \text{if } n \text{ is even \& } m \text{ is odd} \\[2ex] \dfrac{n-1}{m+n} \times \dfrac{n-3}{m+n-2} \times \cdots\cdots \times \dfrac{1}{m+2} \times \dfrac{m-1}{m} \times \dfrac{m-3}{m-2} \times \cdots\cdots \times \dfrac{1}{2} \times \dfrac{\pi}{2} & \text{if } n \text{ is even \& } m \text{ is even} \end{cases}$$

Examples

(1) $\quad I_{5,5} = \int_0^{\pi/2} \mathbf{sin}^5 x \,\mathbf{cos}^5 x \, dx = \dfrac{4}{10} \times \dfrac{2}{8} \times \dfrac{1}{6} = \dfrac{1}{60}$

(2) $\quad I_{6,5} = \int_0^{\pi/2} \mathbf{sin}^6 x \,\mathbf{cos}^5 x \, dx = \dfrac{4}{11} \times \dfrac{2}{9} \times \dfrac{1}{7} = \dfrac{8}{693}$

(3) $I_{7,4} = \int_0^{\pi/2} \sin^7 x \cos^4 x\, dx = \frac{3}{11}\times\frac{1}{9}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1 = \frac{48}{3465}$

(4) $I_{6,6} = \int_0^{\pi/2} \sin^6 x \cos^6 x\, dx = \frac{5}{12}\times\frac{3}{10}\times\frac{1}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{5\pi}{2048}$

(5) Evaluate $\int_0^1 \frac{x^9}{\sqrt{1-x^2}}\, dx$

Solution: put $x = \sin\theta,\ dx = \cos\theta\, d\theta$ when $x = 0,\ \theta = 0$ when $x = 1,\ \theta = \frac{\pi}{2}$

$$\int_0^1 \frac{x^9}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/2} \frac{\sin^9\theta \cdot \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\pi/2} \frac{\sin^9\theta \cdot \cos\theta\, d\theta}{\cos\theta} = \int_0^{\pi/2} \sin^9\theta\, d\theta = \frac{8}{9}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1 = \frac{128}{315}$$

(6) Evaluate $\int_0^{2a} \frac{x^3\, dx}{\sqrt{2ax-x^2}}$

Solution: $\int_0^{2a} \frac{x^3\, dx}{\sqrt{2ax-x^2}} = \int_0^{2a} \frac{x^3\, dx}{\sqrt{a^2-(x-a)^2}}$ put $x-a = a\sin\theta,\ dx = a\cos\theta\, d\theta$

when $x = 0,\ \sin\theta = -1 \Rightarrow \theta = -\frac{\pi}{2}$ when $x = 2a,\ \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \text{ G.I.} = \int_{-\pi/2}^{\pi/2} \frac{(a+a\sin\theta)^3 \cdot a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int_{-\pi/2}^{\pi/2} \frac{a^3(1+\sin\theta)^3 a\cos\theta\, d\theta}{a\cos\theta} = a^3\int_{-\pi/2}^{\pi/2} (1+\sin^3\theta + 3\sin\theta + 3\sin^2\theta)\, d\theta$$

$$= a^3\left[\int_{-\pi/2}^{\pi/2} 1 d\theta + \int_{-\pi/2}^{\pi/2} \sin^3\theta\, d\theta + 3\int_{-\pi/2}^{\pi/2} \sin\theta\, d\theta + 3\int_{-\pi/2}^{\pi/2} \sin^2\theta\, d\theta\right] = a^3\left[\theta\Big|_{-\pi/2}^{\pi/2} + 0 + 0 + 6\int_0^{\pi/2} \sin^2\theta\, d\theta\right]$$

$$= a^3\left[\frac{\pi}{2} - \left(\frac{\pi}{2}\right) + 6\times\frac{1}{2}\times\frac{\pi}{2}\right] = a^3\left(\pi + \frac{3\pi}{2}\right) = \frac{5\pi}{2}a^3$$

Exercise

Evaluate the following

(1) $\int_0^{\pi/6} \sin^5 3\theta\, d\theta$ (2) $\int_0^{\pi} x\sin^7 x\, dx$ (3) $\int_0^1 x^4\left(1-x^2\right)^{3/2} dx$

(4) $\int_0^{\infty} \frac{dx}{\left(1+x^2\right)^{7/2}}$ (5) $\int_0^1 x^6\sqrt{1-x^2}\, dx$ (6) $\int_0^2 x^{5/2}\sqrt{2-x}\, dx$

(7) $\int_{\pi/4}^{\pi/2} \cot^4 x\, dx$ (8) $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^5 x\, dx$ (9) $\int_0^{\pi} \sin^2\theta \frac{\sqrt{1-\cos\theta}}{1+\cos\theta}\, d\theta$

(10) $\int_0^1 \frac{x^7}{\sqrt{1-x^4}}\, dx$ (11) $\int_0^1 \frac{x^3}{\left(1+x^2\right)^4}\, dx$ (12) $\int_0^a \frac{x^7\, dx}{\sqrt{a^2-x^2}}$

Answers: (1) $\frac{8}{45}$, (2) $\frac{16\pi}{35}$, (3) $\frac{3\pi}{256}$, (4) $\frac{8}{15}$, (5) $\frac{5\pi}{256}$, (6) $\frac{5\pi}{8}$, (7) $\frac{3\pi-8}{12}$, (8) $\frac{11\sqrt{3}}{4} + \frac{3}{8}\log\left(2+\sqrt{3}\right)$, (9) $\frac{8\sqrt{2}}{3}$,

(10) $\frac{1}{3}$, (11) $\frac{1}{24}$, (12) $\frac{16a^7}{35}$

DIFFERENTIAL EQUATIONS

An equation which consists of one dependent variable and its derivatives with respect to one or more independent variables is called a **'Differential Equation'.** A differential equation of one dependent and one independent variable is called **'Ordinary Differential Equation'.** A differential equation having one dependent and more than one independent variable is called **'Partial Differential Equation'.**

Examples of ordinary differential equation

1.	$\frac{dy}{dx} - \frac{y}{x} = 0$	2.	$x\,dx - y\,dx = 0$
3.	$ax\,dx + by\,dy = 0$	4.	$\frac{d^2y}{dx^2} = 0$
5.	$\frac{dy}{dx} = \frac{2x+3y-7}{3x-y+4}$	6.	$\frac{dy}{dx} = \frac{2x-3y+4}{4x-6y+1}$
7.	$a\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}$	8.	$y\,dx + x\,dy = 0$
9.	$\frac{d^2y}{dx^2} - 4\left(\frac{dy}{dx}\right)^2 + 3y = 0$	10.	$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 3y = 0$

Order and Degree of a differential equation

Order

The order of the highest derivative occurring in a differential equation is called **'Order'** of the differential equation.

Degree

The highest degree of the highest order derivative occurring in a differential equation (after removing the radicals if any) is called **'Degree'** of the differential equation.

In the examples given above (1), (2), (3), (5), (6) & (8) are of order one and degree one, (4), (9) & (10) are of order two and degree one where as (7) is of order two and degree two after removing the radicals.

Formation of Differential Equation

Differential equations are formed by eliminating the arbitrary constants. Arbitrary constants are eliminated by differentiation.

Examples

(1) Form the differential equation by eliminating the constant a from $x^2 + y^2 = a^2$

Solution: $x^2 + y^2 = a^2$

differentiating w.r.t. x, we have

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow x\,dx + y\,dy = 0$ which is the differential equation.

(2) Form the differential equation by eliminating 'm' **&** 'c' from $y = mx + c$

Solution : $y = mx + c$

differentiating w.r.t. x, we have $\frac{dy}{dx} = m$

again differentiating w.r.t. x, we have

$\frac{d^2y}{dx^2} = 0$ is the required differential equation.

(3) Obtain the differential equation by eliminating 'a' **&** 'b' from $y = a\cos 3x + b\sin 3x$

Solution : $y = a\cos 3x + b\sin 3x$

differentiating w.r.t. x, we have

$$\frac{dy}{dx} = -3a\sin 3x + 3b\cos 3x$$

again differentiating w.r.t. x, we have

$$\frac{d^2y}{dx^2} = -9a\cos 3x - 9b\sin 3x = -9y$$

ie $\frac{d^2y}{dx^2} + 9y = 0$ which is the required differential equation.

Note :- It can be seen from the above examples that the order of the differential equation depends on the number of arbitrary constants in the equation. ie. if arbitrary constant is one then order is one and if the arbitrary constants are two then the order is two.

Solution of equations of first order and first degree

I. Variable Separable

If the given differential equation can be written as $f(x)\,dx + g(y)\,dy = 0$ then this type is called 'Variable separable', solution is obtained by integration

ie $\int f(x)\,dx + \int g(y)\,dy = \text{Constant}$

Eg. 1. Solve $y\sqrt{1-x^2}\;dy + x\sqrt{1-y^2}\;dx = 0$

Solution : divide throughout by $\sqrt{1-x^2}\;\sqrt{1-y^2}$

equation becomes $\frac{y\,dy}{\sqrt{1-y^2}} + \frac{x\,dx}{\sqrt{1-x^2}} = 0$

integrating $\int \frac{y\,dy}{\sqrt{1-y^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}} = \text{Constant}$

ie $-\sqrt{1-y^2} - \sqrt{1-x^2} = -C$

ie $\sqrt{1-y^2} + \sqrt{1-x^2} = C$ is the solution.

Eg. 2. Solve $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

Solution : dividing throughout by $\tan y(1-e^x)$, we have

$$\frac{3e^x}{1-e^x}dx + \frac{\mathbf{sec}^2 y}{\mathbf{tan} y}dy = 0$$

integrating, $\int \frac{3e^x}{1-e^x}dx + \int \frac{\mathbf{sec}^2 y}{\mathbf{tan} y}dy = \text{Constant}$

$$-3\,\mathbf{log}(1-e^x) + \mathbf{log}\,\mathbf{tan}\,y = \mathbf{log}\,C$$

ie $\mathbf{log}\frac{\mathbf{tan}\,y}{(1-e^x)^3} = \mathbf{log}\,C$

$\therefore$ $\mathbf{tan}\,y = C(1-e^x)^3$ is the solution.

Eg. 3. Solve $xy\frac{dy}{dx} = 1 + x + y + xy$

Solution : given equation is $xy\frac{dy}{dx} = (1+x)(1+y)$

ie $xy\,dy = (1+x)(1+y)\,dx$

divide throughout by $x(1+y)$

then $\frac{y\,dy}{1+y} = \frac{(1+x)\,dx}{x}$

ie $\left(\frac{1+y-1}{1+y}\right)dy = \left(\frac{1+x}{x}\right)dx$

ie $\left(1-\frac{1}{1+y}\right)dy = \left(\frac{1}{x}+1\right)dx$

integrating, $y - \mathbf{log}(1+y) = \mathbf{log}\,x + x + c$ is the soluton

Eg. 4. Solve $(x-y)^2\frac{dy}{dx} = a^2$

Solution: put $x - y = u$

then $1-\frac{dy}{dx} = \frac{du}{dx}$ ie $1-\frac{du}{dx} = \frac{dy}{dx}$

given equation becomes $u^2\left(1-\frac{du}{dx}\right) = a^2$

ie $-u^2\frac{du}{dx} = a^2 - u^2$

ie $\frac{u^2 du}{u^2 - a^2} = dx$

ie $\left(\frac{u^2 - a^2 + a^2}{u^2 - a^2}\right)du = dx$

ie $\left(1 + \frac{a^2}{u^2 - a^2}\right)du = dx$

integrating $u + \frac{a^2}{2a}\mathbf{log}\frac{u-a}{u+a} = x + c$

ie $x-y+\frac{a}{2}\log\frac{x-y-a}{x-y+a}=x+c$

ie $\frac{a}{2}\log\frac{x-y-a}{x-y+a}=y+c$ is the solution.

Eg. 5. Solve $\frac{dy}{dx}=xe^{y-x^2}$ given $y=0$ when $x=0$

Solution : given equation is $\frac{dy}{dx}=xe^{y}\cdot e^{-x^2}$

ie $e^{-y}\,dy=xe^{-x^2}\,dx$

integrating, $-e^{-y}=-\frac{1}{2}e^{-x^2}+c$

when $x=0,\ y=0$ $\therefore\ -1=-\frac{1}{2}+c \Rightarrow c=-\frac{1}{2}$

$\therefore$ solution is $-e^{-y}=-\frac{1}{2}e^{-x^2}-\frac{1}{2}$ ie $2e^{-y}=e^{-x^2}+1$

II. Homogeneous Equation

Equation of the type $\frac{dy}{dx}=\frac{f(x,y)}{f(x,y)}$ or $f(x,y)\,dx+g(x,y)\,dy=0$ where $f(x,y)$ **&** $g(x,y)$ are homogeneous expressions in x & y of same degree is called a 'Homogeneous Equation'.

To solve put $y=vx$

then $\frac{dy}{dx}=v+x\frac{dv}{dx}$ or $dy=v\,dx+x\,dv$

by substituting this, the given equation reduces to variable separable form and hence can be solved.

Eg. (1) Solve $2xy\frac{dy}{dx}=3y^2+x^2$

Solution : put $y=vx$ then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

given equation becomes $2x^2v\left[v+x\frac{dv}{dx}\right]=3x^2v^2+x^2$

divide throughout by x^2

then $2v\left(v+x\frac{dv}{dx}\right)=3v^2+1$

ie $2v^2+2vx\frac{dv}{dx}=3v^2+1$

ie $2vx\frac{dv}{dx}=v^2+1$ ie $\frac{2v\,dv}{v^2+1}=\frac{dx}{x}$

integrating, we have $\log(v^2+1)=\log x+\log c$

ie $\log\left(\frac{y^2}{x^2}+1\right)=\log x+\log c$

ie $\log\frac{(y^2+x^2)}{x^2}=\log x+\log c$

ie $\log(y^2+x^2)-\log x^2=\log x+\log c$

$\therefore\ \log(y^2+x^2)=3\log x+\log c=\log cx^3$

$\therefore$ solution is $y^2+x^2=cx^3$

Eg. (2) Solve $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$

Solution: put $x=vy$ then $dx=v\,dy+y\,dv$

equation becomes $(1+e^v)(v\,dy+y\,dv)+e^v(1-v)\,dy=0$

ie $v\,dy+y\,dv+ve^v\,dy+ye^v\,dv+e^v\,dy-ve^v\,dy=0$

ie $(v+ve^v+e^v-e^v)\,dy+y(1+e^v)\,dv=0$

ie $(v+e^v)\,dy+y(1+e^v)\,dv=0$

divide throughout by $y(v+e^v)$

$\therefore\ \frac{dy}{y}+\frac{(1+e^v)\,dv}{v+e^v}=0$

integrating, $\log y+\log(v+e^v)=\log c$

ie $\log y(v+e^v)=\log c$

ie $y(v+e^v)=c$

$y\left(\frac{x}{y}+e^{x/y}\right)=c$

ie $x+ye^{x/y}=c$ is the solution

Equations reducible to homogeneous equatins

Give $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ or $(a_1x+b_1y+c_1)\,dx+(a_2x+b_2y+c_2)\,dy=0$

Case (i) If $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ put $a_1x+b_1y=t$ then it reduces to homogeneous equation and hence can be solved.

Case (ii) If $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ put $x=X+h$ **&** $y=Y+k$

where h **&** k are constants to be found out such that $a_1h+b_1k+c_1=0$ **&** $a_2h+b_2k+c_2=0$

then given equation reduces to

$\frac{dY}{dX}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$ or $(a_1X+b_1Y)\,dX+(a_2X+b_2Y)\,dY=0$ which is homogeneous and hence can be solved.

Eg. (1) Solve $\frac{dy}{dx} = \frac{x+y-1}{2x+2y+3}$

Solution: put $x+y=t$ then $1+\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ given equation becomes $\frac{dt}{dx} - 1 = \frac{t-1}{2t+3}$

$\frac{dt}{dx} = \frac{t-1}{2t+3} + 1 = \frac{t-1+2t+3}{2t+3} = \frac{3t+2}{2t+3}$

ie $\frac{2t+3}{3t+2}dt = dx$

integrating $\int \frac{2t+3}{3t+2}dt = x+c$

put $2t+3 = l(3t+2)+m = 3lt+3l+m$

$\therefore$ $3l = 2 \Rightarrow l = \frac{2}{3}$ $\qquad 2l+m=3 \Rightarrow m = 3-2l = 3-\frac{4}{3} = \frac{5}{3}$

$\therefore$ $x+c = \frac{2}{3}\int \frac{3t+2}{3t+2}dt + \frac{5}{3}\int \frac{dt}{3t+2} = \frac{2}{3}\int 1dt + \frac{5}{9}\int \frac{dt}{t+\frac{2}{3}} = \frac{2}{3}t + \frac{5}{9}\log\left(t+\frac{2}{3}\right)$

$\therefore$ $x+c = \frac{2}{3}(x+y) + \frac{5}{9}\log\left(x+y+\frac{2}{3}\right)$

ie $\frac{1}{3}x + c = \frac{2}{3}y + \frac{5}{9}\log\left(x+y+\frac{2}{3}\right)$ is the solution

Eg. (2) Solve $(2x+y-3)\,dy = (x+2y-3)\,dx$

Solution: put $x = X+h,\ y = Y+k$

choose h **&** k such that

$2h+k=3$ (1)

$h+2k=3$ (2)

$(1)\times 2$ $\qquad 4h+2k=6$

$(2)\times 1$ $\qquad \underline{h+2k=3}$

subtracting $\qquad 3h=3 \Rightarrow h=1,\ k=3-2h=3-2=1$

The given equation reduces to $(2X+Y)\,dY = (X+2Y)\,dX$

put $Y = vX$ then $dY = v\,dX + X\,dv$

$\therefore$ $(2X+vX)(v\,dX + X\,dv) = (X+2vX)\,dX$

divide throughout by X

then $(2+v)v\,dX + 2(2+v)X\,dv = (1+2v)\,dX$

$\therefore$ $(2v+v^2-1-2v)\,dX + (2+v)X\,dv = 0$

ie $(v^2-1)\,dX + (v+2)X\,dv = 0$

ie $\frac{dX}{X} + \frac{(v+2)\,dv}{v^2-1} = 0$

integrating, $\log X + \int \frac{(v+2)\,dv}{v^2-1} = \text{Constant}$

ie $\log X + \frac{1}{2}\int \frac{2v\,dv}{v^2-1} + 2\int \frac{dv}{v^2-1} = \text{Constant}$

ie $\log X + \frac{1}{2}\log(v^2-1) + 2\times\frac{1}{2}\log\frac{v-1}{v+1} = \log c$

ie $2\log X + \log\left(\frac{Y^2}{X^2}-1\right) + 2\log\frac{\frac{Y}{X}-1}{\frac{Y}{X}+1} = 2\log c$

ie $2\log X + \log(Y^2 - X^2) - 2\log X + 2\log(Y-X) - 2\log(Y+X) = 2\log c$

ie $\log(Y+X) + \log(Y-X) + 2\log(Y-X) - 2\log(Y+X) = 2\log c$

$3\log(Y-X) - \log(Y+X) = 2\log c$

ie $\log\frac{(Y-X)^3}{Y+X} = \log c^2$

$\therefore\ (Y-X)^3 = c^2(Y+X)$ but $X = x-1,\ Y = y-1$

$\therefore\ (y-x)^3 = c^2(y+x-2)$ is the solution.

III. Linear Equation (Leibnitz's Equation)

Equation of the type $\frac{dy}{dx} + Py = Q$ where P & Q are functions of x is called a 'Linear Equation'.

To find the solution multiply both sides of the given equation by $e^{\int P\,dx}$

then $\left(\frac{dy}{dx} + Py\right)e^{\int P\,dx} = Qe^{\int P\,dx}$

ie $\frac{dy}{dx}e^{\int P\,dx} + yPe^{\int P\,dx} = Qe^{\int P\,dx}$

ie $\frac{d}{dx}\left[ye^{\int P\,dx}\right] = Qe^{\int P\,dx}$

$\therefore\ ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$ is the required solution.

Note:- $\frac{dx}{dy} + Px = Q$ where P & Q are function of y is also a linear equation and its solution is $xe^{\int P\,dy} = \int Qe^{\int P\,dy}\,dy + c$

Eg. (1) Solve the equation $\frac{dy}{dx} + y\tan x = \cos x$

Solution: Comparing with the standard equation

$P = \tan x$ & $Q = \cos x$

$\int P\,dx = \int \tan x\,dx = \log\sec x$

$e^{\int P\,dx} = e^{\log\sec x} = \sec x$

$\therefore$ Solution is $ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$

ie $y\sec x = \int \cos x \cdot \sec x\,dx + c = \int 1\,dx + c = x + c$

Eg. (2) $(1+y^2)\,dx + (x - e^{-\tan^{-1}y})\,dy = 0$

Solution: divide throughout by $(1+y^2)\,dy$

$\therefore \dfrac{dx}{dy} + \dfrac{x - e^{-\tan^{-1}y}}{(1+y^2)} = 0$

ie $\dfrac{dx}{dy} + \dfrac{x}{y^2+1} = \dfrac{e^{-\tan^{-1}y}}{y^2+1}$

Comparing with standard equation

$P = \dfrac{1}{y^2+1}$ $Q = \dfrac{e^{-\tan^{-1}y}}{y^2+1}$

$\int P\,dy = \tan^{-1}y$ $\therefore$ Solution is $xe^{\int P\,dy} = \int Qe^{\int P\,dy}\,dy + c$

ie $xe^{\tan^{-1}y} = \int \dfrac{e^{-\tan^{-1}y}}{(y^2+1)} e^{\tan^{-1}y}\,dy + c$

ie $xe^{\tan^{-1}y} = \int \dfrac{dy}{y^2+1} + c = \tan^{-1}y + c$

$\therefore$ Solution is $xe^{\tan^{-1}y} = \tan^{-1}y + c$

Bernoulli's Equation

The equation $\dfrac{dy}{dx} + Py = Qy^n$ where P & Q are functions of x is called Bernoulli's Equation.

To find the solution divide by y^n then $\dfrac{1}{y^n}\dfrac{dy}{dx} + \dfrac{P}{y^{n-1}} = Q$

put $\dfrac{1}{y^{n-1}} = z$ ie $y^{-n+1} = z$

differentiating w.r.t. x

$(-n+1)y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$ ie $\dfrac{1}{y^n}\dfrac{dy}{dx} = \dfrac{1}{(-n+1)}\dfrac{dz}{dx}$ $(n \neq -1)$

$\therefore$ equation becomes $\dfrac{1}{(-n+1)}\dfrac{dz}{dx} + Pz = Q$

ie $\dfrac{dz}{dx} + (-n+1)Pz = (-n+1)Q$ which is a linear equation and hence can be solved.

Note :- $\dfrac{dx}{dy} + Px = Qx^n$ is also Bernoulli's equation, whose solution is given by $\dfrac{dz}{dy} + (-n+1)Pz = (-n+1)Q$

where P & Q are functions of y

Eg. (1) Solve $\frac{dy}{dx} + y\,\mathbf{tan}x = y^2\,\mathbf{sec}\,x$

Solution: divide throughout by y^2

then $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\mathbf{tan}x = \mathbf{sec}\,x$

put $\frac{1}{y} = z$ then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ equation becomes $-\frac{dz}{dx} + \mathbf{tan}x \cdot z = \mathbf{sec}\,x$

ie $\frac{dz}{dx} - \mathbf{tan}x \cdot z = -\mathbf{sec}\,x$

which is linear where $P = -\mathbf{tan}x,\; Q = -\mathbf{sec}\,x$

$\int P\,dx = \int -\mathbf{tan}x\,dx = \mathbf{log}\,\mathbf{cos}\,x$

$e^{\int p\,dx} = e^{\mathbf{log}\,\mathbf{cos}\,x} = \mathbf{cos}\,x$

$\therefore$ Solution is $z\,\mathbf{cos}\,x = \int -\mathbf{sec}\,x \cdot \mathbf{cos}\,x\,dx + c = \int -1\,dx + c = -x + c$

$\therefore$ Solution is $\frac{\mathbf{cos}\,x}{y} = -x + c$

Eg. (2) Solve $\frac{dy}{dx} - \frac{\mathbf{tan}\,y}{(1+x)} = (1+x)\,e^x\,\mathbf{sec}\,y$

Solution: the given equation is $\mathbf{cos}\,y\frac{dy}{dx} - \frac{\mathbf{sin}\,y}{(1+x)} = (1+x)\,e^x$

put $\mathbf{sin}\,y = z$, then $\mathbf{cos}\,y\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ equation becomes $\frac{dz}{dx} - \frac{z}{1+x} = (1+x)\,e^x$ which is a linear equation where

$P = -\frac{1}{1+x},\; Q = (1+x)\,e^x$

$\int P\,dx = \int -\frac{1}{(1+x)}\,dx = -\mathbf{log}(1+x)$

$\therefore$ $e^{\int P\,dx} = e^{-\mathbf{log}(1+x)} = e^{\mathbf{log}\frac{1}{1+x}} = \frac{1}{1+x}$

$\therefore$ Solution is $z \cdot \frac{1}{1+x} = \int (1+x)\,e^x\,\frac{1}{(1+x)}\,dx = \int e^x\,dx = e^x + c$

$\therefore$ Solution is $\frac{\mathbf{sin}\,y}{1+x} = e^x + c$

IV. Exact Differential Equation

Let $M\,dx + N\,dy = 0$ be the differential equation where M **&** N are functions of x **&** y, the equation is said to 'Exact' if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and to find the solution

Consider $\int M\,dx$ (1)

where the integration is done w.r.t. x treating y as a constant and take $\int N\,dy$ (2)

here the integration is done w.r.t. y omitting the terms containing x in N

then the soltuion is $(1)+(2)$ is Constant.

Eg. (1) Solve $(2x^3 - xy^2 - 2y + 3)\,dx - (x^2 y + 2x)\,dy = 0$

Solution: Comparing with standard equation

$$M = 2x^3 - xy^2 - 2y + 3, \quad N = -x^2 y - 2x$$

$$\frac{\partial M}{\partial y} = -2xy - 2, \quad \frac{\partial N}{\partial x} = -2xy - 2 \quad \therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{hence equation is exact}$$

$$\int M\,dx = \int (2x^3 - xy^2 - 2y + 3)\,dx = 2\cdot\frac{x^4}{4} - y^2\cdot\frac{x^2}{2} - 2xy + 3x = \frac{x^4}{2} - \frac{x^2y^2}{2} - 2xy + 3x \quad (1)$$

$$\int N\,dy = \int (-x^2 y - 2x)\,dy = \int 0\cdot dy \quad \text{(omitting the terms which contain } x\text{)}$$

$$= \text{Constant}$$

$$\therefore \text{ solution is } \frac{x^4}{2} - \frac{x^2y^2}{2} - 2xy + 3x = \frac{c}{2}$$

ie $x^4 - x^2y^2 - 4xy + 6x = c$

Eg. (2) Solve $(x^2 + y^2 - a^2)x\,dx + (x^2 - y^2 - b^2)y\,dy = 0$

Solution: Comparing with standard equation

$$M = x^3 + xy^2 - a^2x, \quad N = x^2y - y^3 - b^2y$$

$$\frac{\partial M}{\partial y} = 2xy, \quad \frac{\partial N}{\partial x} = 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{equation is exact}$$

for solution, consider $\int M\,dx = \int (x^3 + xy^2 - a^2x)\,dx = \frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{a^2x^2}{2}$ (treating y as a constant)

$\int N\,dy = \int (x^2y - y^3 - b^2y)\,dy = \int (-y^3 - b^2y)\,dy$ omitting the terms which contain x

$$= -\frac{y^4}{4} - \frac{b^2y^2}{2}$$

$$\therefore \text{ solution is } \frac{x^4}{4} + \frac{x^2 - y^2}{2} - \frac{a^2x^2}{2} - \frac{y^4}{4} - \frac{b^2y^2}{2} = \frac{c}{4}$$

ie $x^4 + 2x^2y^2 - 2a^2x^2 - y^4 - 2b^2y^2 = c$

Exercise

Solve the following

(1) $x^2(1-y)\dfrac{dy}{dx}+y^2(1+x)=0$

(2) $\dfrac{dy}{dx}=e^{2x-3y}+4x^2e^{-3y}$

(3) $\dfrac{dy}{dx}=\dfrac{x(2\log x+1)}{(\sin y+y\cos y)}$

(4) $\cos(x+y)\,dy=dx$

(5) $y-x\dfrac{dy}{dx}=a\left(y^2+\dfrac{dy}{dx}\right)$

(6) $(x+1)\dfrac{dy}{dx}+1=e^{-y}$

(7) $x\,dy-y\,dx=\sqrt{x^2+y^2}\,dx$

(8) $\dfrac{dy}{dx}=\dfrac{y}{x}+\sin\dfrac{y}{x}$

(9) $x^2y\,dx-(x^3+y^3)\,dy=0$

(10) $(2x+5y+1)\,dx-(5x+2y-1)\,dy=0$

(11) $(4x-6y-1)\,dx+(3y-2x-2)\,dy=0$

(12) $\dfrac{dy}{dx}=\dfrac{2y-x-4}{y-3x+3}$

(13) $x\log x\dfrac{dy}{dx}+y=(\log x)^2$

(14) $\dfrac{dy}{dx}=x^3-2xy$ if $y=2$ when $x=1$

(15) $\dfrac{dy}{dx}+y\cos x=y^3\sin 2x$

(16) $x\dfrac{dy}{dx}+y=x^3y^6$

(17) $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy$

(18) $(2xy+y-\tan y)\,dx+(x^2-x\tan^2 y+\sec^2 y)\,dy=0$

(19) $\dfrac{2x}{y^3}dx+\dfrac{y^2-3x^2}{y^4}dy=0$

(20) $\left[y\left(1+\dfrac{1}{x}\right)+\cos y\right]dy+(x+\log x-x\sin y)\,dy=0$

Answers

(1) $\log\dfrac{x}{y}-\dfrac{1}{x}-\dfrac{1}{y}=c$ (2) $3e^{2x}-2e^{3y}+8x^3=c$ (3) $y\sin y=x^2\log x+c$ (4) $y=\tan\dfrac{x+y}{2}+c$ (5) $(x+1)(2-e^y)=c$

(6) $(x+1)(1-e^y)=c$ (7) $y+\sqrt{x^2+y^2}=cx^2$ (8) $y=2x\tan^{-1}(cx)$ (9) $\left(\dfrac{x}{y}\right)^3=3\log cy$ (10) $(x+y)^7=c\left(x-y-\dfrac{2}{3}\right)^3$

(11) $(2x-y)+\dfrac{5}{4}\log(7-8x+12y)=c$ (12) $(X^2-XY+Y^2)=c\left\{\dfrac{2Y+(5+\sqrt{21})X}{2Y-(5-\sqrt{21})X}\right\}^{\frac{1}{\sqrt{21}}}$ where $X=x-2,\ Y=y-3$

(13) $y\log x=\dfrac{1}{3}(\log x)^3+c$ (14) $2y-x^2+1=4e^{1-x^2}$ (15) $y^{-2}=ce^{2\sin x}+2\sin x+1$ (16) $x^3y^5\left(\dfrac{5}{2}+cx^2\right)=1$

(17) $x=\tan^{-1}y-1+ce^{-\tan^{-1}y}$ (18) $x^2y+xy-x\tan y+\tan y=c$ (19) $x^2-y^2=cy^3$ (20) $y(x+\log x)+x\cos y=c$